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The Stability of Generalized Volterra Equations

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In this paper, a particular type of a system of generalized Volterra equations [1], whose solutions are assured to be nonnegative for arbitrary nonnegative initial values, is considered. The extended stability theorem of LaSalle is used for deriving conditions for a nonnegative equilibrium point to be stable with respect to a certain subset of the Euclidean space. The obtained stability theorem has a close relation with Lyapunov's stability condition for linear systems with constant coefficients and is generally less restrictive than conditions known so far.

1. INTRODUCTION

In this paper, a particular type of a system of generalized Volterra equations [1], whose solutions are assured to be nonnegative for arbitrary nonnegative initial values, is considered. The system arises frequently in the fields of ecology, economics, etc.

Many papers, for example, ones by MacArthur [2], Kilmer [3], Aiken and Lapidus [4], Goel *et al.* [5], etc., have been published on the stability of a positive equilibrium point of the generalized Volterra equation. In [2, 4, 5], a characteristic matrix of the quadratic term is restricted to be symmetric or skew-symmetric. Kilmer's stability condition is valid for the very restraint matrix [3].

The purpose of this paper is to provide sufficient conditions for a nonnegative equilibrium point of the generalized Volterra equation to be stable with respect to a certain subset of the Euclidean space, without imposing such restrictions on the matrix as in [2–5]. It is accomplished by using LaSalle's extended stability theorem [6]. Further, a necessary and sufficient condition is gained under a certain restriction to the matrix.

2. DESCRIPTION OF SYSTEM AND DEFINITION OF STABILITY

2.1. *Description of System and Notations*

Let us consider an n-dimensional differential equation,

$$\frac{d}{dt} x_i = x_i\left(b_i - \sum_{j=1}^{n} a_{ij}x_j\right), \qquad i = 1, 2,..., n. \tag{2.1}$$

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Here, x_i is a variable, b_i and a_{ij} are real constant parameters. In this paper, system (2.1) is called a generalized Volterra equation after the famous Volterra–Lotka equation [5]. From (2.1), a rate of change per unit quantity $(=\dot{x}_i/x_i)$ is a linear combination of n variables x_j $(j = 1, 2,..., n)$, where the symbol "·" means the time derivative.

In vector form, system (2.1) is

$$(d/dt)\, x = X(b - Ax) \tag{2.2}$$

where $x = (x_1, x_2, ..., x_n)^t$ is an n-vector, $X = \operatorname{diag}(x_1, x_2, ..., x_n)$ is an $n \times n$ real diagonal matrix, $b = (b_1, b_2, ..., b_n)^t$ is an n-dimensional real constant vector, and $A = (a_{ij})$ is an $n \times n$ real nonsingular matrix.

Unless otherwise specified, the following notations are employed. If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ real matrices, $A \geqq B$ and $A > B$ mean that all the elements of them satisfy $a_{ij} \geqslant b_{ij}$ and $a_{ij} > b_{ij}$, respectively. $A \geqslant B$ implies $A \geqq B$ but $A \neq B$. Inequalities $\geqq$, $>$, and $\geqslant$ between two vectors are similarly defined. A vector x is said to be positive (nonnegative) if $x > 0$ $(x \geqq 0)$. A^t (x^t) is the transpose of A (x). $\operatorname{Diag} A = \operatorname{diag}(a_{11}, a_{22}, ..., a_{nn})$. Off-diag $A = A - \operatorname{Diag} A$.

R^n represents the n-dimensional Euclidean space. $R^n_{+0} = \{x \mid x \in R^n, x \geqq 0\}$. $R_+{}^n = \{x \mid x \in R^n, x > 0\}$. $R_*{}^n = \{x \mid x \in R^n, x_i \geqslant 0, x_j \geqslant 0, ..., x_k \geqslant 0, x_m > 0, m \neq i, j, ..., k\}$ defines the set which corresponds to the nonnegative equilibrium point $x^* = (x_1{}^*, x_2{}^*, ..., x_n{}^*)^t$, where $x_i{}^* = x_j{}^* = \cdots = x_k{}^* = 0$, $x_m{}^* > 0$ for $m \neq i, j, ..., k$. Therefore, if $x^* > 0$, $R_*{}^n$ means $R_+{}^n$, and if $x^* = 0$, $R_*{}^n$ implies R^n_{+0}.

2.2. *Definition of Stability*

For system (2.1),

$$\frac{d}{dt} x_i \Big|_{x_i=0} = x_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right) \Big|_{x_i=0} = 0 \tag{2.3}$$

holds for every i, so solutions $x(t)$ of (2.1) are positive, namely, $x(t) \in R_+{}^n$ for all $t \geqslant t_0$, for arbitrary positive initial values $x(t_0) = x^0 \in R_+{}^n$, because the solution is uniquely determined. This property of solutions has a physical meaning, since a variable $x(t)$ of system (2.1) represents populations, chemical, or biochemical concentrations, etc. That is, it makes no sense to speak of negative populations or chemical concentrations. Therefore, only nonnegative solutions are considered in this paper.

Now, define the stability of system (2.1) in consideration of the nonnegative property of the solution.

DEFINITION 1. A nonnegative equilibrium point x^* of system (2.1) is said to be asymptotically stable in the large with respect to the set $R_*{}^n$, if and only if

(1) the equilibrium point $x^* \geqslant 0$ is stable with respect to R_*^n, namely, if for every $\epsilon > 0$ there exists $\delta(\epsilon; t_0)$ such that if $| x^0 - x^* | < \delta$ and the solution $x(t; t_0, x^0)$ is in R_*^n, then $| x(t; t_0, x^0) - x^* | < \epsilon$ for $t \geqslant t_0$,

(2) and every solution converges to x^* as $t \to +\infty$, if $x^0 \in R_*^n$.

As can be seen in the above definition, only the nonnegative set R_*^n is considered in connection with the stability. The nonnegative equilibrium point is said to be stable briefly hereafter, if it is stable in the sense of Definition 1.

3. Stability Conditions

Denote the equilibrium point by $x^+ = (x_1^+, x_2^+, ..., x_n^+)^t$ of system (2.1), and x^+ satisfies

$$X^+(b - Ax^+) = 0, \tag{3.1}$$

where

$$X^+ = \operatorname{diag}(x_1^+, x_2^+, ..., x_n^+). \tag{3.2}$$

There are 2^n solutions of Eq. (3.1), because A is nonsingular. Assume the equilibrium point x^* which satisfies

$$b - Ax^* = 0, \tag{3.3}$$

to be nonnegative, that is,

$$x^* = A^{-1}b \geqslant 0. \tag{3.4}$$

In this paper, let us consider the stability for x^*.

Substituting (3.3) into (2.2),

$$\frac{d}{dt} x = -XA(x - x^*), \tag{3.5}$$

that is,

$$\frac{d}{dt} x_i = -x_i \sum_{j=1}^{n} a_{ij}(x_j - x_j^*), \qquad i = 1, 2, ..., n. \tag{3.6}$$

The following theorem gives a sufficient condition for x^* to be stable.

Theorem 1. *Assume the existence of the nonnegative equilibrium point $x^* = A^{-1}b$ for system* (2.1). *Then x^* is stable if there exists a positive definite diagonal matrix W such that a matrix*

$$WA + A^tW \tag{3.7}$$

is positive definite.

Proof. Assume for simplicity that the rth component of the nonnegative equilibrium point x^* is equal to zero and others are positive, that is,

$$x_r^* = 0, \qquad x_i^* > 0 \qquad \text{for } i \neq r. \tag{3.8}$$

The solution of (3.6) is positive if the initial value is positive, so, it is possible to define a transformation such that

$$\begin{aligned} y_i &= \ln(x_i/x_i^*) \qquad \text{for any } i \neq r, \\ y_r &= x_r , \end{aligned} \tag{3.9}$$

that is,

$$\begin{aligned} x_i &= x_i^* \exp(y_i) \qquad \text{for any } i \neq r, \\ x_r &= y_r . \end{aligned} \tag{3.10}$$

By this transformation, (3.6) becomes

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ \vdots \\ y_r \\ \vdots \\ y_n \end{pmatrix}$$

$$= -\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & y_r & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} A \begin{pmatrix} x_1^* & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & x_n^* \end{pmatrix} \begin{pmatrix} \exp(y_1) - 1 \\ \vdots \\ y_r \\ \vdots \\ \exp(y_n) - 1 \end{pmatrix}. \tag{3.11}$$

Obviously, by the transformation (3.10), x^* is transformed into the equilibrium point $y = 0$ of (3.11).

Now, define a continuously differentiable function $V(y)$,

$$V(y) = \sum_{\substack{i=1 \\ i \neq r}}^{n} x_i^* w_i (\exp(y_i) - y_i - 1) + w_r y_r \tag{3.12}$$

$$= (w_1, \ldots, w_r, \ldots, w_n) \begin{pmatrix} x_1^* & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & x_n^* \end{pmatrix} \begin{pmatrix} \exp(y_1) - y_1 - 1 \\ \vdots \\ y_r \\ \vdots \\ \exp(y_n) - y_n - 1 \end{pmatrix}, \tag{3.13}$$

where w_i is the ith component of W, that is,

$$W = \operatorname{diag}(w_1, w_2, \ldots, w_n), \qquad w_i > 0 \qquad \text{for any } i. \tag{3.14}$$

Further, define a bounded and closed (compact) set $\Omega = \{y \mid y_r \geqslant 0, V(y) \leqslant L(y(t_0))\}$. Here $L(y(t_0))$ is a positive constant number which depends on an initial value $y(t_0)$, and it satisfies $L(y(t_0)) \geqslant V(y(t_0))$.

The following properties for $V(y)$ are easily obtained:

$$\begin{aligned} &(1) \quad V(y) \geqslant 0 \quad \text{in} \quad \Omega, \\ &(2) \quad V(y) = 0 \qquad \text{holds only for } y = 0 \quad \text{in} \quad \Omega. \end{aligned} \tag{3.15}$$

The time derivative of $V(y(t))$ along a solution of (3.11) is

$$\begin{aligned} \frac{d}{dt} V(y(t)) \Big|_{(3.11)} &= -(\exp(y_1) - 1, \ldots, y_r, \ldots, \exp(y_n) - 1) \begin{pmatrix} x_1^* & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & x_n^* \end{pmatrix} \\ &\quad \times \begin{pmatrix} w_1 & & & & 0 \\ & \ddots & & & \\ & & w_r & & \\ & & & \ddots & \\ 0 & & & & w_n \end{pmatrix} A \begin{pmatrix} x_1^* & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & x_n^* \end{pmatrix} \begin{pmatrix} \exp(y_1) - 1 \\ \vdots \\ y_r \\ \vdots \\ \exp(y_n) - 1 \end{pmatrix} \\ &= -z^t W A z \\ &= -\tfrac{1}{2} z^t (WA + A^t W) z, \end{aligned} \tag{3.16}$$

where

$$z = \begin{pmatrix} x_1^* & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & x_n^* \end{pmatrix} \begin{pmatrix} \exp(y_1) - 1 \\ \vdots \\ y_r \\ \vdots \\ \exp(y_n) - 1 \end{pmatrix} = \begin{pmatrix} x_1^*(\exp(y_1) - 1) \\ \vdots \\ y_r \\ \vdots \\ x_n^*(\exp(y_n) - 1) \end{pmatrix}. \tag{3.17}$$

From the assumption, the right-hand side of Eq. (3.16) is negative definite. By Eq. (3.11) and the uniqueness of the solution, $y_r(t) = 0$ for $t \geqslant t_0$ if $y_r(t_0) = 0$. So, $y_r(t) \geqslant 0$ for $t \geqslant t_0$ if $y_r(t_0) \geqslant 0$. Therefore, every solution of (3.11) remains in Ω for all $t \geqslant t_0$, if the initial value belongs to Ω. Accordingly, all the solutions starting in Ω approach the origin $y = 0$ as $t \to +\infty$ by the extended stability theorem of LaSalle [6]. Further, the origin is stable with respect to Ω, since every solution initially in Ω remains in Ω.

The set Ω approaches the set $\{y \mid y_r \geqslant 0, y_i \in R^1 \text{ for any } i \neq r\}$ as $L(y(t_0)) \to +\infty$. This set corresponds to the set $R_*{}^n = \{x \mid x \in R^n, x_r \geqslant 0, x_i > 0 \text{ for any } i \neq r\}$ by (3.10). Therefore, the equilibrium point described by (3.8) is stable with respect to $R_*{}^n$ in the sense of Definition 1.

With regard to the nonnegative equilibrium point $x^* = A^{-1}b$ whose zero components are more than one, or with regard to the positive equilibrium point, it can be proved similarly that x^* is stable. Q.E.D.

When W is not a diagonal matrix, but is positive definite, the stability condition given in Theorem 1 is the famous Lyapunov necessary and sufficient condition for linear systems with constant coefficients. However, the necessary and sufficient condition for the existence of a positive definite diagonal matrix W of Theorem 1 is not known yet.

Let us consider sufficient conditions and necessary conditions for the existence of W given in Theorem 1.

DEFINITION 2 [7]. Let $B = (b_{ij})$ be an $n \times n$ real matrix. Then B is said to have a positive dominant diagonal if and only if there is a set of n positive numbers $\pi_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$b_{ii}\pi_i > \sum_{j \neq i}^{n} |b_{ij}| \pi_j, \qquad i = 1, 2, \ldots, n. \tag{3.18}$$

DEFINITION 3 [8]. Let $B = (b_{ij})$ be an $n \times n$ real matrix. Then B is said to be an M-matrix if and only if the off-diagonal elements are all nonpositive and the principal minors are all positive.

Define a matrix $\bar{A} = (\bar{a}_{ij})$ for A such that

$$\begin{aligned} \bar{a}_{ii} &= a_{ii}, \qquad & i &= 1, 2, \ldots, n, \\ \bar{a}_{ij} &= -|a_{ij}|, \qquad & i &\neq j. \end{aligned} \tag{3.19}$$

Obviously, the off-diagonal elements of $\bar{A}$ are all nonpositive. By Definition 2, A has a positive dominant diagonal if and only if there is a positive vector $\pi = (\pi_1, \pi_2, \ldots, \pi_n)^t > 0$ such that $\bar{A}\pi > 0$. This is equivalent to that $\bar{A}$ is an M-matrix [8]. Therefore, A has a positive dominant diagonal if and only if $\bar{A}$ is an M-matrix.

On the other hand, if $\bar{A}$ is an M-matrix, then there is a positive definite diagonal matrix W such that $W\bar{A} + \bar{A}^t W$ is positive definite [8].

COROLLARY 1.1. *Assume the existence of the nonnegative equilibrium point $x^* = A^{-1}b$ for system* (2.1). *Then x^* is stable if A has a positive dominant diagonal.*

Proof. In the proof of Theorem 1, Eq. (3.16) is

$$\begin{aligned}\frac{d}{dt} V(y(t)) \Big|_{(3.11)} &= -\frac{1}{2} z^t(WA + A^tW) z \\ &= -\sum_{i=1}^{n} w_i a_{ii} z_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{n} w_i a_{ij} z_i z_j \\ &\leqslant -\sum_{i=1}^{n} w_i a_{ii} z_i^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n} w_i \mid a_{ij} \mid \mid z_i \mid \mid z_j \mid \\ &= -\tfrac{1}{2}(\mid z_1 \mid, \mid z_2 \mid, \dots, \mid z_n \mid)(W\bar{A} + \bar{A}^tW) \begin{pmatrix} \mid z_1 \mid \\ \mid z_2 \mid \\ \vdots \\ \mid z_n \mid \end{pmatrix}, \end{aligned} \tag{3.20}$$

where $\bar{A}$ is defined by (3.19).

A has a positive dominant diagonal if and only if $\bar{A}$ is an M-matrix. Therefore, there is a positive definite diagonal matrix W such that $W\bar{A} + \bar{A}^tW$ is positive definite. So, the remainder can be proved similarly to the proof of Theorem 1. Q.E.D.

By inequality (3.20), it is clear that, if A has a positive dominant diagonal, then there exists a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite. Therefore, the condition given in Corollary 1.1 is sufficient for the condition given in Theorem 1 to hold.

DEFINITION 4 [7]. Let $B = (b_{ij})$ be an $n \times n$ real matrix and u be an n-dimensional real vector. Then B is said to be a positive quasi-definite matrix if and only if the quadratic form u^tBu is positive definite.

By definition 4, A is positive quasi-definite if and only if $(A + A^t)/2$ is positive definite. Therefore, if A is positive quasi-definite, then there is a positive definite diagonal matrix W (an $n \times n$ unit matrx) such that $WA + A^tW$ is positive definite, so x^* is stable.

COROLLARY 1.2. *Assume the existence of the nonnegative equilibrium point $x^* = A^{-1}b$ for system* (2.1). *Then x^* is stable if A is positive quasi-definite.*

Next, consider necessary conditions for the stability condition given in Theorem 1 to hold.

DEFINITION 5 [7]. Let $B = (b_{ij})$ be an $n \times n$ real matrix. Then B is said to be a P-matrix if and only if the principal minors of B are all positive.

THEOREM 2. *If there exists a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite, then A is a P-matrix and the real parts of the eigenvalues of A are all positive.*

Proof. It is obvious by Lyapunov's theorem that $-A$ is stable, so the real parts of the eigenvalues of A are all positive.

Further, if $WA + A^tW$ is positive definite, then the matrix WA is positive quasi-definite by Definition 4. If WA is positive quasi-definite, then the principal minors of WA are all positive, namely, WA is a P-matrix [7]. The principal minor of WA is equal to the product of the corresponding principal minors of A and W, since W is diagonal. Therefore, every principal minor of A is positive, that is, A is a P-matrix, since W is positive definite. This proves Theorem 2.

Q.E.D.

Conditions given in Theorem 1 and Corollaries 1.1 and 1.2 are sufficient for x^* to be stable. The following theorem gives a necessary and sufficient condition.

THEOREM 3. *Let A of* (2.1) *be an $n \times n$ matrix with nonpositive off-diagonal elements, namely,*

$$\text{off-diag } A \geqq 0, \tag{3.21}$$

and assume the existence of the nonnegative equilibrium point $x^* = A^{-1}b$ for (2.1). *Then x^* is stable if and only if the principal minors of A are all positive.*

The following lemma is used for the proof of the necessity.

LEMMA 1. *Consider the initial value problem of the autonomous system of an n-dimensional differential equation;*

$$\frac{d}{dt} x = f(x), \qquad x(t_0) = x^0. \tag{L.1}$$

Assume that there exists a nonempty subset Ω of R^n_{+0} and a scalar function $V(x)$ with a continuous first derivative in Ω such that

(1) *the solution $x(t)$ remains in the set $\Omega \cup \partial\Omega$ for any $t \geqslant t_0$ if the initial value $x^0 \in \Omega$, where $\partial\Omega$ is a boundary of Ω,*

(2) $V(x) > 0$ *and* $dV(x)/dt\,|_{(\mathrm{L}.1)} \geqslant 0$ *in* Ω,

(3) $V(x) = 0$ *holds for only* $x = x^*$, *where* $f(x^*) = 0$,

(4) $x^* \in \partial\Omega$.

Then the equilibrium point x^* is unstable.

The proof of Lemma 1 is given in Appendix 2.

Proof of Theorem 3. Sufficiency.

Since A is an M-matrix, there is a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite. Therefore, x^* is stable by Theorem 1.

Necessity.

When off-diag $A \leqq 0$, A is an M-matrix if and only if there is a vector $y \geqslant 0$ such that $A^ty > 0$ [8]. By a theorem for a linear inequality (see Lemma 2 of Appendix 2), there is no vector $z \geqslant 0$ such that $Az \leqq 0$ if and only if there is a vector $y \geqslant 0$ such that $A^ty > 0$. Therefore, it is enough to prove that x^* is unstable, if there is a vector $z \geqslant 0$ such that $Az \leqq 0$. Choose a vector z as $x - x^*$, and it is sufficient to prove that x^* is unstable, if there is a vector $x - x^* \geqslant 0$ such that $A(x - x^*) = Ax - b \leqq 0$.

First consider condition (1) of Lemma 1. There is a nonempty set $\Omega = \{x \mid b - Ax \geqq 0, x \geqslant x^*\}$ by the assumption. Define a set $\Omega_1 = \{x \mid x \geqslant x^*\}$.

By a transformation:

$$y = x - x^*, \tag{3.22}$$

system (2.1) and Ω_1 are

$$\begin{aligned} \frac{d}{dt} y &= -(X^* + Y) Ay, \\ \Omega_{1y} &= \{y \mid y \geqslant 0\}, \end{aligned} \tag{3.23}$$

where $Y = \operatorname{diag}(y_1, y_2, \ldots, y_n)$, $X^* = \operatorname{diag}(x_1^*, x_2^*, \ldots, x_n^*)$. For each $i \neq j$,

$$\begin{aligned} \frac{d}{dt} y_i \Big|_{\substack{y_i=0 \\ y_j \geqslant 0}} &= -(x_i^* + y_i) \sum_{j=1}^{n} a_{ij} y_j \Big|_{\substack{y_i=0 \\ y_j \geqslant 0}} \\ &= -x_i^* \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} y_j \Big|_{y_j \geqslant 0} \geqslant 0 \end{aligned}$$

holds, since $a_{ij} \leqslant 0$ for any $i \neq j$ and $x_i^* \geqslant 0$. Therefore, every trajectory $y(t)$ of (3.23) initially in Ω_{1y} remains in Ω_{1y} for any $t \geqslant t_0$ by Appendix 1, so every trajectory $x(t)$ of (2.1) starting in Ω_1 remains in Ω_1 for any $t \geqslant t_0$.

Next, define a set $\Omega_2 = \{x \mid b - Ax \geqq 0\}$. By a transformation:

$$z = b - Ax, \tag{3.24}$$

system (2.1) and Ω_2 are

$$\begin{aligned} \frac{d}{dt} z &= AZA^{-1}(z - b), \\ \Omega_{2z} &= \{z \mid z \geqq 0\}, \end{aligned} \tag{3.25}$$

where $Z = \operatorname{diag}(z_1, z_2, \ldots, z_n)$. Every trajectory $z(t)$ remains in a set Ω_{3z}, where

$$\Omega_{3z} = \{z \mid -A^{-1}(z-b) \geqq 0\} = \{x \mid x \geqq 0\} = R^n_{+0}, \tag{3.26}$$

if the initial value $z(t_0) \in \Omega_{3z}$, since the solution $x(t)$ starting in R^n_{+0} remains in R^n_{+0}. For every trajectory $z(t)$ initially in $\Omega_{2z} \cap \Omega_{3z}$,

$$\frac{d}{dt} z_i \Big|_{\substack{z_i=0 \\ z_j \geqslant 0}} = a_i{}^t Z A^{-1}(z-b) \Big|_{\substack{z_i=0 \\ z_j \geqslant 0}} \geqslant 0 \tag{3.27}$$

holds for each $i \neq j$, since $a_{ij} \leqslant 0$ for any $i \neq j$, where $a_i{}^t = (a_{i1}, a_{i2}, \ldots, a_{in})$. Theorefore, $z(t)$ remains in $\Omega_{2z} \cap \Omega_{3z}$, namely, every trajectory of (2.1) starting in $\Omega_2 \cap \Omega_3$ remains in $\Omega_2 \cap \Omega_3$ for $t \geqslant t_0$.

Accordingly, any trajectory of (2.1) initially in $\Omega_1 \cap \Omega_2 \cap \Omega_3$ remains in $\Omega_1 \cap \Omega_2 \cap \Omega_3$ for $t \geqslant t_0$. Since $\Omega_1 \subseteq \Omega_3$, any trajectory remains in $\Omega_1 \cap \Omega_2 (= \Omega)$. This shows condition (1) of Lemma 1.

Define a scalar function:

$$\begin{aligned} V(x) &= \sum_{j=1}^{n} (x_j - x_j{}^*)^2 \\ &= (x - x^*)^t (x - x^*). \end{aligned} \tag{3.28}$$

Then $V(x) > 0$ in Ω. The time derivative of $V(x(t))$ along a solution of (2.1) is

$$\frac{d}{dt} V(x(t)) \Big|_{(2.1)} = 2(x - x^*)^t X(b - Ax) \geqslant 0 \quad \text{in} \quad \Omega. \tag{3.29}$$

Therefore, condition (2) of Lemma 1 holds. Obviously, conditions (3) and (4) of Lemma 1 hold. Accordingly, x^* is unstable. Q.E.D.

Let us consider a special type of system (2.1),

$$\begin{aligned} \frac{d}{dt} x_i &= -x_i \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, 2, \ldots, n, \\ \frac{d}{dt} x &= -XAx. \end{aligned} \tag{3.30}$$

A scalar function $V(x)$ corresponding to (3.13) is

$$V(x) = \sum_{i=1}^{n} w_i x_i, \qquad w_i > 0, \qquad i = 1, 2, \ldots, n, \tag{3.31}$$

and the time derivative of $V(x(t))$ along a solution of (3.30) is

$$\frac{d}{dt} V(x(t)) \Big|_{(3.30)} = -\frac{1}{2} x^t(WA + A^tW)\,x. \tag{3.32}$$

Define a bounded and closed set $\tilde{\Omega} = \{x \mid x \geqq 0,\ V(x) \leqslant L(x(t_0))\}$, where $L(x(t_0)) \geqslant V(x(t_0))$, $x(t_0) \geqslant 0$.

Therefore, it can be shown that Theorem 1 and Corollaries 1.1 and 1.2 give sufficient conditions for an equilibrium point $x^* = 0$ of (3.30) to be stable with respect to the set R^n_{+0}.

It is also obvious that Theorem 3 gives a necessary and sufficient condition for the equilibrium point $x^* = 0$ of (3.30) to be stable.

The following theorem shows that, if a matrix $\tilde{A}$, instead of A, satisfies the condition given in Theorem 1, then $x^* = 0$ of (3.30) is stable. Here $\tilde{A}$ is defined by

$$\begin{aligned} \tilde{a}_{ii} &= a_{ii}, && i = 1, 2, \ldots\ n, \\ \tilde{a}_{ij} &= \min(a_{ij}, 0) && \text{for any } i \neq j. \end{aligned} \tag{3.33}$$

THEOREM 4. *Let $\tilde{A} = (\tilde{a}_{ij})$ be an $n \times n$ matrix defined by* (3.33). *Then the equilibrium point $x^* = 0$ of* (3.30) *is stable, if there exists a positive definite diagonal matrix W such that a matrix*

$$W\tilde{A} + \tilde{A}^tW \tag{3.34}$$

is positive definite.

Proof. The matrix A of (3.30) can be expressed as

$$A = \tilde{A} + D, \tag{3.35}$$

$$\begin{aligned} D &= (d_{ij}); \quad d_{ii} = 0, && i = 1, 2, \ldots, n, \\ d_{ij} &= \max(a_{ij}, 0) && \text{for any } i \neq j, \end{aligned} \tag{3.36}$$

by the definition of $\tilde{A}$.

The time derivative of $V(x(t))$ defined by (3.31) along a solution of (3.30) is

$$\begin{aligned} \frac{d}{dt} V(x(t)) \Big|_{(3.30)} &= -\frac{1}{2} x^t(WA + A^tW)\,x \\ &= -\frac{1}{2} x^t(W\tilde{A} + \tilde{A}^tW)\,x - x^tWDx. \end{aligned} \tag{3.37}$$

The right-hand side of (3.37) is nonpositive in $\tilde{\Omega}$, namely,

$$\frac{d}{dt} V(x(t)) \Big|_{(3.30)} \leqslant 0, \tag{3.38}$$

because $\tilde{A}$ satisfies the assumption of the theorem and D is a nonnegative matrix ($D \geqslant 0$). The equality of (3.38) holds only for $x = 0$ in $\tilde{\Omega}$. So, it can be proved similarly to the proof of Theorem 1 that $x^* = 0$ is stable. Q.E.D.

The sufficient condition given in Theorem 4 is equivalent to the condition "$\tilde{A}$ is an M-matrix," since the off-diagonal elements of $\tilde{A}$ are all nonpositive. By a property of the M-matrix, this is equivalent to the condition, "the leading principal minors of $\tilde{A}$ are all positive," that is,

$$\det \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1i} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{i1} & \tilde{a}_{i2} & \cdots & \tilde{a}_{ii} \end{vmatrix} > 0 \qquad \text{for } i = 1, 2,..., n. \tag{3.39}$$

A has a positive dominant diagonal if and only if there exists a positive vector $\pi > 0$ such that $\tilde{A}\pi > 0$, since the off-diagonal elements of $\tilde{A}$ are all nonpositive. So, $\tilde{A}$ has a positive dominant diagonal if and only if $\tilde{A}$ is an M-matrix. Therefore, the stability condition given in Theorem 4 is also equivalent to the condition "$\tilde{A}$ has a positive dominant diagonal."

Corollary 4.1. *Let $\tilde{A} = (\tilde{a}_{ij})$ be an $n \times n$ matrix defined by* (3.33). *Then the equilibrium point $x^* = 0$ of* (3.30) *is stable, if the leading principal minors of $\tilde{A}$ are all positive, or if $\tilde{A}$ has a positive dominant diagonal.*

If A has a positive dominant diagonal, then $\tilde{A}$ has a positive dominant diagonal, since $A \geqq \tilde{A}$ by the definition (3.33). Therefore, with regard to the stability of the equilibrium point $x^* = 0$ of (3.30), sufficient conditions given in Theorem 4 and Corollary 4.1 include those given in Theorem 1 and Corollary 1.1.

Further, if $\tilde{A}$ is a positive quasi-definite matrix, the $n \times n$ unit matrix can be chosen for a positive definite diagonal matrix W such that $W\tilde{A} + \tilde{A}^t W$ is positive definite.

Corollary 4.2. *Let $\tilde{A} = (\tilde{a}_{ij})$ be an $n \times n$ matrix defined by* (3.33). *Then the equilibrium point $x^* = 0$ of* (3.30) *is stable, if $\tilde{A}$ is a positive quasi-definite matrix.*

A necessary and sufficient condition for $x^* = 0$ of (3.30) to be stable is not known yet, when the off-diagonal elements of A are not nonpositive. However, for the case of $n = 2$, the following theorem is obtained.

Theorem 5. *Consider system* (3.30) *of $n = 2$. Then the equilibrium point $x^* = 0$ of* (3.30) *is stable, if and only if*

(1) $a_{ii} > 0$, $i = 1, 2$,

(2) *there exists a vector $\zeta = (\zeta_1, \zeta_2)^t > 0$ such that $A\zeta > 0$.*

4. Comparison of Stability Conditions

In this section, the stability conditions given in Theorem 1 and Corollaries 1.1 and 1.2 are compared with those given by MacArthur and Kilmer. Further, relations among sufficient conditions obtained in Section 3 are discussed.

4.1. *Relations to MacArthur's and Kilmer's Conditions*

MacArthur's result can be stated as follows [2]. "If A is positive definite, then the positive equilibrium point $x^* = A^{-1}b$ (if it exists) of (2.1) is stable with respect to R_+^n." If A is positive definite, then the matrix $A + A^t$ is also positive definite. Since a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite, can be chosen as a unit matrix, the condition given in Theorem 1 includes MacArthur's condition. But the inverse is not true. Therefore, the stability condition given in Theorem 1 is less restrictive than the one given by MacArthur. The most important difference between them is that Theorem 1 does not require the symmetry of the matrix A.

Kilmer showed the stability condition as follows [3]. "If A of (2.1) satisfies the following conditions:

$$(1) \quad a_{ii} > 0 \qquad \text{for } i = 1, 2, \ldots, n, \tag{4.1}$$

$$(2) \quad \min_i |a_{ii}| > \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|, \tag{4.2}$$

then the positive equilibrium point $x^* = A^{-1}b$ (if it exists) of (2.1) is stable with respect to R_+^n."

If these conditions are satisfied, then an inequality,

$$\begin{aligned} a_{ii} = |a_{ii}| &\geqslant \min_i |a_{ii}| \\ &> \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \qquad i = 1, 2, \ldots, n, \end{aligned} \tag{4.3}$$

holds. This inequality implies that A has a positive dominant diagonal if conditions (4.1) and (4.2) are satisfied. Therefore, the stability condition given in Corollary 1.1 is less restrictive than the one given by Kilmer.

The stability conditions obtained in Section 3 are not only for a positive equilibrium point $x^* = A^{-1}b > 0$ but also for a nonnegative equilibrium point $x^* = A^{-1}b \geqslant 0$ and their stability region R_*^n includes R_+^n of MacArthur and Kilmer.

4.2. *Relations of Stability Conditions*

Next, consider the relations among the conditions provided in Section 3. In Section 3, the following relations were obtained.

(1) If A is an M-matrix (Theorem 3), then A has a positive dominant diagonal (Corollary 1.1).

(2) If A has a positive dominant diagonal (Corollary 1.1), then there is a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite (Theorem 1).

The inverse of (1) is not true, since the off-diagonal elements of A which has a positive dominant diagonal are not necessarily nonpositive. The inverse of (2) also is not true. For example,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

Choose as

$$W = \mathrm{diag}(1, 2, 1),$$

then

$$WA + A^tW = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 12 & 4 \\ 1 & 4 & 2 \end{pmatrix}$$

is positive definite. On the other hand, $\bar{A}$ defined by (3.19),

$$\bar{A} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & -1 \\ 0 & -2 & 1 \end{pmatrix}$$

is not an M-matrix. Therefore, A does not have a positive dominant diagonal.

With respect to a positive quasi-definite matrix, the following relations were obtained in Section 3.

(3) If A is positive quasi-definite (Corollary 1.2), then there exists a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite (Theorem 1).

(4) If A is positive quasi-definite (Corollary 1.2), then A is a P-matrix and the real parts of the eigenvalues of A are all positive (Theorem 2).

If A is an M-matrix, then there exists a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite, but A is not necessarily a positive quasi-definite matrix. Therefore, the inverse of (3) also is not true. For example,

$$A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

is an M-matrix, but is not positive quasi-definite, since $\det | A + A^t | = 0$. Similarly, if A is an M-matrix, then A is a P-matrix and $-A$ is a stable matrix,

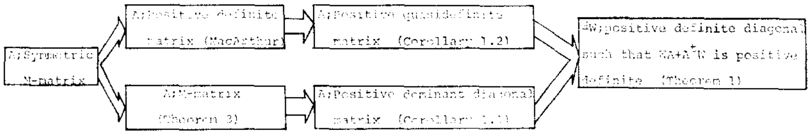

FIG. 1. Relations of the stability conditions given in Section 3. "$A \Rightarrow B$" means that condition A is included in condition B. See Section 4 for details.

but A is not necessarily positive quasi-definite as can be seen in the above example.

Accordingly, there are relations shown in Fig. 1 among the stability conditions provided in Section 3.

If there exists a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite, then A is a P-matrix and the real parts of the eigenvalues of A are all positive by Theorem 2. However, it has not been proved that it is also a sufficient condition for the existence of such a matrix W. However, when A is a 2×2 matrix, it can be shown that it is a necessary and sufficient condition. The 2×2 matrix A is a P-matrix if and only if

$$\begin{gathered} a_{11} > 0, \qquad a_{22} > 0, \\ a_{11}a_{22} - a_{12}a_{21} > 0. \end{gathered} \tag{4.4}$$

THEOREM 6. *Let $A = (a_{ij})$ be a 2×2 matrix. Then there exists a positive definite diagonal matrix W such that $WA + A^tW$ is positive definite, if and only if A is a P-matrix.*

Proof. Necessity is obvious by Theorem 2. Denote

$$C = WA + A^tW = \begin{pmatrix} 2a_{11}w_1 & a_{12}w_1 + a_{21}w_2 \\ a_{12}w_1 + a_{21}w_2 & 2a_{22}w_2 \end{pmatrix}. \tag{4.5}$$

When $a_{12} \neq 0$,

$$\det|C| = -a_{12}^2\left(w_1 - \frac{2a_{11}a_{22} - a_{12}a_{21}}{a_{12}^2}w_2\right)^2 + \frac{4a_{11}a_{22}(a_{11}a_{22} - a_{12}a_{21})}{a_{12}^2}w_2{}^2$$

holds. Define

$$w_1 = \frac{2a_{11}a_{22} - a_{12}a_{21}}{a_{12}^2 w_2} > 0,$$

then $\det|C| > 0$, that is, C is positive definite, if conditions (4.4) are satisfied.

When $a_{12} = 0$, $a_{21} \neq 0$,

$$\det | C | = -a_{21}^2 \left(w_2 - \frac{2a_{11}a_{22}}{a_{21}^2} w_1 \right)^2 + \frac{4a_{11}^2 a_{22}^2}{a_{21}^2} w_1{}^2$$

holds. Define

$$w_2 = \frac{2a_{11}a_{22}}{a_{21}^2 w_1} > 0,$$

then $\det | C | > 0$.

When $a_{12} = a_{21} = 0$, w_1 and w_2 are chosen arbitrarily, since $\det | C | = 4a_{11}a_{22}w_1w_2 > 0$. Q.E.D.

5. Applications

In this section, let us consider how the obtained stability conditions can be explained in the context of multispecies' community models in biology.

In population dynamics, a generalized Volterra equation (2.1) is interpreted as a model for the dynamics of species i (x_i) in an n species' interacting community. The first term ($b_i x_i$) of (2.1) represents the behavior of species i in the absence of others; b_i may be positive, negative, or zero, corresponding to the sign of the difference in the rate of births over deaths. The quadratic terms $(-\sum_{j=1}^{n} a_{ij}x_i x_j)$ of (2.1) imply interactions of species i with itself and other species. The diagonal elements of the matrix $-A = -(a_{ij})$ describe the interactions with itself through the density dependence, so $-a_{ii}$ is less than zero. This means the intraspecies' competition for limited resources lumped in the evironment. The off-diagonal elements of $-A$ express the interactions of the one species with others. When $a_{ij} = -a_{ji}$ for $i \neq j$, the interaction with others is the relation of predator–prey; the species i lost (or gained) per unit time are equal to the j's gained (or lost), which was studied by Aiken and Lapidus [4] and Goel *et. al* [5]. When $a_{ij} = a_{ji} \geqslant 0$ for $i \neq j$, it is the relation of competition, which was studied by MacArthur [2].

In Section 3, Theorem 3 gives a necessary and sufficient condition for the nonnegative equilibrium point $x^* = A^{-1}b$ of (2.1) to be stable. The premise "off-diag $A \leqq 0$" of Theorem 3 can be considered to mean that interactions among species are symbiotic. On the other hand, an M-matrix implies that the diagonal element of A is larger than the sum of absolute values of the off-diagonal elements in a certain sense. Therefore, the interpretation of Theorem 3 can be that the symbiotic community is stable if and only if the gain by symbiotic interactions is less than the loss by density dependences.

No assumptions are made on the signs of the off-diagonal elements of A in Theorem 1 and Corollaries 1.1 and 1.2, so they give sufficient conditions for the system with "mixed competitive predator–prey symbiotic" interactions among species to be stable.

EXAMPLE. Let us consider a chain system which is shown in Fig. 2. For this chain system, the matrix A is

$$A = \begin{pmatrix} \alpha_1 & \gamma_2 & 0 & 0 & \cdot & \cdot & 0 \\ \beta_1 & \alpha_2 & \gamma_3 & 0 & \cdot & \cdot & 0 \\ 0 & \beta_2 & \alpha_3 & \gamma_4 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \beta_{n-2} & \alpha_{n-1} & \gamma_n \\ 0 & \cdot & \cdot & 0 & 0 & \beta_{n-1} & \alpha_n \end{pmatrix}. \tag{5.1}$$

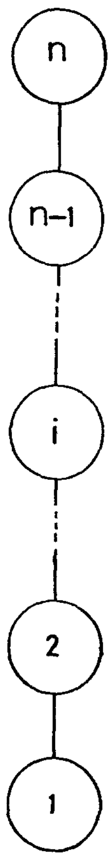

FIG. 2. A chain system used in Section 5. If species i either feeds on species j or eaten by j, a bond connects circles i and j. See Section 5 for details.

Assume that α_i, β_i, and γ_i satisfy the following relations:

$$\begin{aligned} \alpha_i > 0, &\qquad i = 1, 2,..., n, \\ \beta_i \gamma_{i+1} < 0, &\qquad i = 1, 2,..., n-1. \end{aligned} \tag{5.2}$$

If there exists a nonnegative equilibrium point $x^* = A^{-1}b$, then (2.1) with the matrix A defined by (5.1) is

$$\frac{d}{dt} x_i = -x_i(\beta_{i-1}(x_{i-1} - x_{i-1}^*) + \alpha_i(x_i - x_i^*) + \gamma_{i+1}(x_{i+1} - x_{i+1}^*)), \qquad i = 1, 2,..., n, \tag{5.3}$$

where $\beta_0 = \gamma_{n+1} = 0$.

Define the positive definite diagonal matrix W in order to satisfy the relations,

$$w_i \gamma_{i+1} + w_{i+1} \beta_i = 0, \qquad i = 1, 2,..., n-1, \tag{5.4}$$

that is,

$$w_{i+1} = -(\gamma_{i+1}/\beta_i)\, w_i > 0, \qquad i = 1, 2,..., n-1. \tag{5.5}$$

Using these relations, w_i $(i = 1, 2,..., n)$ is determined as follows,

$$w_i = \prod_{j=0}^{i-1} \Delta_j > 0, \qquad i = 1, 2,..., n, \tag{5.6}$$

where $\Delta_0 = 1$, $\Delta_j = -\gamma_{j+1}/\beta_j > 0$.

If w_i is determined by (5.6), then

$$WA + A^t W = \mathrm{diag}(w_1 \alpha_1, w_2 \alpha_2, ..., w_n \alpha_n) \tag{5.7}$$

is positive definite, since $\alpha_i > 0$, $w_i > 0$ for $i = 1, 2,..., n$. Therefore, this chain system with A defined by (5.1) is stable by Theorem 1.

6. Conclusions

The stability of the system of the generalized Volterra equations, whose solutions are nonnegative for arbitrary nonnegative initial values, was considered.

Using the extended stability theorem of LaSalle, it was shown that the nonnegative equilibrium point $x^* = A^{-1}b$ is stable (if such an equilibrium point exists), if there exists a positive definite diagonal matrix W such that $WA + A^t W$ is positive definite (Theorem 1). This condition includes the stability conditions obtained by MacArthur and Kilmer.

If the matrix A has a positive dominant diagonal (Corollary 1.1), or if A is positive quasi-definite (Corollary 1.2), then A satisfies the condition given in Theorem 1.

Further, when the off-diagonal elements of A are all nonpositive, $x^* = A^{-1}b$ is stable if and only if the principal minors of A are all positive (Theorem 3).

For the system described by only the quadratic terms, the equilibrium point $x^* = 0$ is stable if the matrix $\tilde{A}$ defined by (3.33), instead of A, satisfies conditions given in Theorem 1 and Corollaries 1.1 and 1.2 (Theorem 4 and Corollaries 4.1 and 4.2).

In the last, relations among stability conditions were considered.

Appendix 1

THEOREM. *Consider the initial value problem of a system of an n-dimensional autonomous differential equation,*

$$\frac{d}{dt} x = f(x), \qquad x(0) = \eta, \tag{A.1}$$

in $J = [0, T]$, $0 < T < +\infty$. *Assume that*

(1) *there is a solution of* (A.1) *in* J *and it is uniquely determined,*

(2) $x(0) = \eta \geqslant 0$,

(3) *for any* $i = 1, 2, \ldots, n$, *if* $x_i = 0$ *and* $x_j \geqslant 0$ $(j \neq i)$, *then* $f_i(x) \geqslant 0$.

Then the solutions of system (A.1) *are all nonnegative.*

Proof. Consider the initial value problem,

$$\begin{aligned} \frac{d}{dt} \xi &= f(\xi) + \epsilon, \\ \xi(0) &= \eta + \delta, \end{aligned} \tag{A.2}$$

where ϵ and δ are positive vectors. Since $x(t)$ is a solution of the initial value problem (A.1) which exists in J and is uniquely determined by the assumption (1), if $|\epsilon|$, $|\delta|$ are sufficiently small, then all the solutions $\xi(t)$ of (A.2) exist in J. Further, $\xi(t)$ converges to $x(t)$ uniformly in J as $\epsilon \to 0$, $\delta \to 0$ [9]. By assumptions (2), (3) and $\epsilon > 0$, $\delta > 0$,

$$\begin{aligned} &\xi(0) = \eta + \delta > 0, \\ &\qquad f_i(\xi) + \epsilon_i > 0 \qquad \text{for } \xi_i = 0, \quad \xi_j \geqslant 0 \quad (j \neq i). \end{aligned}$$

Therefore, all the solutions $\xi(t)$ are positive in J. Since $\xi(t) > 0$ converges to $x(t)$ as $\epsilon \to 0$, $\delta \to 0$, all the solutions $x(t)$ are nonnegative in J. This proves the Theorem. Q.E.D.

Appendix 2

Lemma 1. *Consider the initial value problem of the autonomous system of an n-dimensional differential equation,*

$$\frac{d}{dt} x = f(x), \qquad x(t_0) = x^0. \tag{L.1}$$

Assume that there exists a nonempty subset Ω of R^n_{+0} and a scalar function $V(x)$ with a continuous first derivative in Ω such that

(1) *the solution $x(t)$ remains in the set $\Omega \cup \partial\Omega$ for any $t \geqslant t_0$ if the initial value $x^0 \in \Omega$, where $\partial\Omega$ is a boundary of Ω,*

(2) *$V(x) > 0$ and $dV(x)/dt\,|_{(\mathrm{L}.1)} \geqslant 0$ in Ω,*

(3) *$V(x) = 0$ holds for only $x = x^*$, where $f(x^*) = 0$,*

(4) *$x^* \in \partial\Omega$.*

Then the equilibrium point x^* is unstable.

Proof. $V(x(t))$ is a nondecreasing function of t along a solution of system (L.1) since $\dot{V}(x)|_{(\mathrm{L}.1)} \geqslant 0$ in Ω. There is a subset $\Omega_1 = \{x \mid x \in \Omega, 0 \leqslant V(x) \leqslant L\}$ of Ω such that $x^* \in \partial\Omega_1$ since $V(x^*) = 0$ and $V(x) > 0$ in Ω. Here, L is a positive constant number. Define a set $\Omega_2 = \{x \mid x \in \Omega,\ V(x) > L\}$ (that is, $\Omega_1 \cup \Omega_2 = \Omega$). Solutions of system (L.1) initialy in Ω_2 remain in $\Omega_2 \cup \partial\Omega_2$ since $V(x)$ is nondecreasing along solutions and they remain in $\Omega \cup \partial\Omega$. Therefore, they never approach a boundary point x^* of Ω_1, since $\Omega_2 \cup \partial\Omega_2 \not\ni x^*$. This proves Lemma 1.

Q.E.D.

Lemma 2. *Let A be $p \times n$ matrix. Then*

(1) *there exists a solution $x \in R^n$ such that $Ax \leqq 0$, $x \geqslant 0$, or*

(2) *there exists a solution $y \in R^p$ such that $A^t y > 0$, $y \geqslant 0$,*

but both do not hold simultaneously.

Proof. Tucker's theorem for linear inequalities can be stated as follows [10]. "Let B, C, and D be $p^1 \times n$, $p^2 \times n$, $p^3 \times n$ matrices, and B not a zero matrix. Then

(1) there exists a solution $x \in R^n$ such that $Bx \geqslant 0$, $Cx \geqq 0$, $Dx = 0$, or

(2) there exist solutions $y_2 \in R^{p^1}$, $y_3 \in R^{p^2}$, and $y_4 \in R^{p^3}$ such that $B^t y_2 + C^t y_3 + D^t y_4 = 0$, $y_2 > 0$ and $y_3 \geqq 0$,

but both do not hold simultaneously."

Let $B = (\delta_{ij})$ be an $n \times n$ unit matrix, $C = -A$ be a $p \times n$ matrix, and $D = 0$ be a zero matrix, and Propositions (1), (2) of Tucker's theorem can be stated as follows:

(1)′ There exists a solution $x \in R^n$ such that $x \geqslant 0$, $Ax \leqq 0$.

(2)′ There exist solutions $y_2 \in R^n$, $y_3 \in R^p$ such that $y_2 - A^t y_3 = 0$, $y_2 > 0$, and $y_3 \geqq 0$.

Since $A^t y_3 = y_2 > 0$, Proposition (2)′ is equivalent to

(2)″ there exists a solution $y_3 \in R^p$ such that $A^t y_3 > 0$, $y_3 \geqq 0$.

It is obvious that if $y_3 = 0$, then $A^t y_3 = 0$, so (2)″ is equivalent to (2)‴ there exists a solution $y_3 \in R^p$ such that $A^t y_3 > 0$, $y_3 \geqslant 0$.

This proves Lemma 2. Q.E.D.

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