1.1 Relations

We take for granted usual notions and results of set theory, such as the empty set and subsets, the union, intersection and difference of sets, relations, functions, injectivity, surjectivity, natural numbers... (see e.g. [Kri71]). Unless otherwise stated, relations are binary relations. We denote by $|\mathcal{S}|$ the cardinal of a set $\mathcal{S}$.

1.1.1 Definitions & notations

Definition 1 (Relations)

- We denote the composition of relations by $\cdot$, the identity relation by Id, and the inverse of a relation by $^{-1}$, all defined below:

  Let $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{R}' : \mathcal{B} \longrightarrow \mathcal{C}$.

  - Composition
    $\mathcal{R} \cdot \mathcal{R}' : \mathcal{A} \longrightarrow \mathcal{C}$ is defined as follows: given $M \in \mathcal{A}$ and $N \in \mathcal{C}$, $M(\mathcal{R} \cdot \mathcal{R}')N$ if there exists $P \in \mathcal{B}$ such that $M\mathcal{R}P$ and $P\mathcal{R}'N$. Sometimes we also use the notation $\mathcal{R}' \circ \mathcal{R}$ for $\mathcal{R} \cdot \mathcal{R}'$.
  - Identity
    $\mathsf{Id}[\mathcal{A}] : \mathcal{A} \longrightarrow \mathcal{A}$ is defined as follows:
    given $M \in \mathcal{A}$ and $N \in \mathcal{A}$, $M \mathsf{Id}_{\mathcal{A}} N$ if $M = N$.
  - Inverse
    $\mathcal{R}^{-1} : \mathcal{B} \longrightarrow \mathcal{A}$ is defined as follows:
    given $M \in \mathcal{B}$ and $N \in \mathcal{A}$, $M\mathcal{R}^{-1}N$ if $N\mathcal{R}M$.

- If $\mathcal{D} \subseteq \mathcal{A}$, we write $\mathcal{R}(\mathcal{D})$ for $\{M \in \mathcal{B} |\ \exists N \in \mathcal{D}, N\mathcal{R}M\}$, or equivalently $\bigcup_{N \in \mathcal{D}} \{M \in \mathcal{B} |\ N\mathcal{R}M\}$. When $\mathcal{D}$ is the singleton $\{M\}$, we write $\mathcal{R}(M)$ for $\mathcal{R}(\{M\})$.

- Now when $\mathcal{A} = \mathcal{B}$ we define the *relation induced by $\mathcal{R}$ through $\mathcal{R}'$*, written $\mathcal{R}'[\mathcal{R}]$, as $\mathcal{R}'^{-1} \cdot \mathcal{R} \cdot \mathcal{R}' : \mathcal{C} \longrightarrow \mathcal{C}$.

- We say that a relation $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{B}$ is *total* if $\mathcal{R}^{-1}(\mathcal{B}) = \mathcal{A}$.

- If $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{A}' \subseteq \mathcal{A}$ then $\mathcal{R}_{|\mathcal{A}'} : \mathcal{A}' \longrightarrow \mathcal{B}$ is the restriction of $\mathcal{R}$ to $\mathcal{A}'$, i.e. those pairs of $\mathcal{R}$ whose first components are in $\mathcal{A}'$.

- All those notions and notations can be used in the particular case when $\mathcal{R}$ is a function, that is, if $\forall M \in \mathcal{A}$, $\mathcal{R}(M)$ is of the form $\{N\}$ (which we simply write $\mathcal{R}(M) = N$).

- A total function is called a *mapping* (also called an *encoding*, a *translation* or an *interpretation*).

- An injective mapping is called an *embedding*.

Remark 1 Notice that composition is associative, and identity relations are neutral for the composition operation.

Computation in a calculus is described by the notion of reduction relation, defined as follows.

Definition 2 (Reduction relation)

- A *reduction relation* on $\mathcal{A}$ is a relation from $\mathcal{A}$ to $\mathcal{A}$ (i.e. a subset of $\mathcal{A}\times\mathcal{A}$), which we often write as $\rightarrow$.

- Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we define the set of $\rightarrow$-*reducible forms* (or just *reducible forms* when the relation is clear) as $\mathsf{rf}^{\rightarrow} := \{M \in \mathcal{A} |\ \exists N \in \rightarrow(M)\}$. We define the set of *normal forms* as $\mathsf{nf}^{\rightarrow} := \{M \in \mathcal{A} |\ \rightarrow(M) = \emptyset\}$. In other words,
$$\begin{array}{rcl}\mathsf{rf}^{\rightarrow} & := & \{M \in \mathcal{A} |\ \exists N \in \mathcal{A}, M \rightarrow N\}\\ \mathsf{nf}^{\rightarrow} & := & \{M \in \mathcal{A} |\ \nexists N \in \mathcal{A}, M \rightarrow N\}\end{array}$$

- Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we write $\leftarrow$ for $\rightarrow^{-1}$, and we define $\rightarrow^n$ by induction on the natural number n as follows:
  $\rightarrow^0 := \mathsf{Id}$
  $\rightarrow^{n+1} := \rightarrow \cdot \rightarrow^n (= \rightarrow^n \cdot \rightarrow)$
  $\rightarrow^+$ denotes the transitive closure of $\rightarrow$ (i.e. $\rightarrow^+ := \bigcup_{n\geq 1} \rightarrow^n$).
  $\rightarrow^*$ denotes the transitive and reflexive closure of $\rightarrow$ (i.e. $\rightarrow^* := \bigcup_{n\geq 0} \rightarrow^n$).
  $\leftrightarrow$ denotes the symmetric closure of $\rightarrow$ (i.e. $\leftrightarrow := \leftarrow \cup \rightarrow$).
  $\leftrightarrow^*$ denotes the transitive, reflexive and symmetric closure of $\rightarrow$.

- An *equivalence relation* on $\mathcal{A}$ is a transitive, reflexive and symmetric reduction relation on $\mathcal{A}$, i.e. a relation $\rightarrow\, = \,\leftrightarrow^*$, hence denoted more often by $\sim, \equiv \ldots$

- Given a reduction relation $\rightarrow$ on $\mathcal{A}$ and a subset $\mathcal{B} \subseteq \mathcal{A}$, the *closure of* $\mathcal{B}$ *under* $\rightarrow$ is $\rightarrow^*(\mathcal{B})$.

Definition 3 (Finitely branching relation) A reduction relation $\rightarrow$ on $\mathcal{A}$ is *finitely branching* if $\forall M \in \mathcal{A},\ \rightarrow(M)$ is finite.

Definition 4 (Stability) Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we say that a subset $\mathcal{T}$ of $\mathcal{A}$ is $\rightarrow$-*stable* (or *stable under* $\rightarrow$) if $\rightarrow(\mathcal{T}) \subseteq \mathcal{T}$ (in other words, if $\mathcal{T}$ is equal to its closure under $\rightarrow$).

Definition 5 (Reduction modulo) Let $\sim$ be an equivalence relation on a set $\mathcal{A}$, let $\to$ be a reduction relation on $\mathcal{A}$. The *reduction relation* $\to$ *modulo* $\sim$ on $\mathcal{A}$, denoted $\to_\sim$, is $\sim \cdot \to \cdot \sim$. It provides a reduction relation on the $\sim$-equivalence classes of $\mathcal{A}$. If $\to'$ is a reduction relation $\to$ modulo $\sim$, $\to$ alone is called the *basic* reduction relation and denoted $\to'_b$.[1]

We now present the notion of simulation. We shall use it for two kinds of results: confluence (below) and strong normalisation (section 1.2). While simulation is often presented using an mapping from one calculus to another, we provide here a useful generalised version for an arbitrary relation between two calculi.

Definition 6 (Strong and weak simulation)
Let $\mathcal{R}$ be a relation between two sets $\mathcal{A}$ and $\mathcal{B}$, respectively equipped with the reduction relations $\to_\mathcal{A}$ and $\to_\mathcal{B}$.

- $\to_\mathcal{B}$ *strongly simulates* $\to_\mathcal{A}$ *through* $\mathcal{R}$ if $(\mathcal{R}^{-1} \cdot \to_\mathcal{A}) \subseteq (\to_\mathcal{B}^+ \cdot \mathcal{R}^{-1})$.

  In other words, for all $M, M' \in \mathcal{A}$ and for all $N \in \mathcal{B}$, if $M \mathcal{R} N$ and $M \to_\mathcal{A} M'$ then there is $N' \in \mathcal{B}$ such that $M' \mathcal{R} N'$ and $N \to_\mathcal{B}^+ N'$.

  Notice that when $\mathcal{R}$ is a function, this implies $\mathcal{R}[\to_\mathcal{A}] \subseteq \to_\mathcal{B}^+$.
  If it is a mapping, then $\to_\mathcal{A} \subseteq \mathcal{R}^{-1}[\to_\mathcal{B}^+]$.

- $\to_\mathcal{B}$ *weakly simulates* $\to_\mathcal{A}$ *through* $\mathcal{R}$ if $(\mathcal{R}^{-1} \cdot \to_\mathcal{A}) \subseteq (\to_\mathcal{B}^* \cdot \mathcal{R}^{-1})$.

  In other words, for all $M, M' \in \mathcal{A}$ and for all $N \in \mathcal{B}$, if $M \mathcal{R} N$ and $M \to_\mathcal{A} M'$ then there is $N' \in \mathcal{B}$ such that $M' \mathcal{R} N'$ and $N \to_\mathcal{B}^* N'$.

  Notice that when $\mathcal{R}$ is a function, this implies $\mathcal{R}[\to_\mathcal{A}] \subseteq \to_\mathcal{B}^*$.
  If it is a mapping, then $\to_\mathcal{A} \subseteq \mathcal{R}^{-1}[\to_\mathcal{B}^*]$.

The notions are illustrated in Fig. 1.1.

$$
\begin{tikzcd}
M \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}"] & N \arrow[d, "\mathcal{B}+"] \\
M' \arrow[r, Rightarrow, "\mathcal{R}"] & N'
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}"] & N \arrow[d, "\mathcal{B}*"] \\
M' \arrow[r, Rightarrow, "\mathcal{R}"] & N'
\end{tikzcd}
$$

Strong simulation Weak simulation

Figure 1.1: Strong and weak simulation

[1]This is not a functional notation that only depends on a reduction relation $\to'$ on $\sim$-equivalence classes of $\mathcal{A}$, but a notation that depends on the construction of $\to'$ as a reduction relation modulo $\sim$.

Remark 2

1. If $\rightarrow_{\mathcal{B}}$ strongly (resp. weakly) simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, and if $\rightarrow_{\mathcal{B}}\subseteq\rightarrow'_{\mathcal{B}}$ and $\rightarrow'_{\mathcal{A}}\subseteq\rightarrow_{\mathcal{A}}$, then $\rightarrow'_{\mathcal{B}}$ strongly (resp. weakly) simulates $\rightarrow'_{\mathcal{A}}$ through $\mathcal{R}$.
2. If $\rightarrow_{\mathcal{B}}$ strongly (resp. weakly) simulates $\rightarrow_{\mathcal{A}}$ and $\rightarrow'_{\mathcal{A}}$ through $\mathcal{R}$, then it also strongly (resp. weakly) simulates $\rightarrow_{\mathcal{A}} \cdot \rightarrow'_{\mathcal{A}}$ through $\mathcal{R}$.
3. Hence, if $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, then it also strongly simulates $\rightarrow^{+}_{\mathcal{A}}$ through $\mathcal{R}$.
   If $\rightarrow_{\mathcal{B}}$ strongly or weakly simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, then it also weakly simulates $\rightarrow^{+}_{\mathcal{A}}$ and $\rightarrow^{*}_{\mathcal{A}}$ through $\mathcal{R}$.

We now define some more elaborate notions based on simulation, such as equational correspondence [SF93], Galois connection and reflection [MSS86].

Definition 7 (Galois connection, reflection & related notions)
Let $\mathcal{A}$ and $\mathcal{B}$ be sets respectively equipped with the reduction relations $\rightarrow_{\mathcal{A}}$ and $\rightarrow_{\mathcal{B}}$. Consider two mappings $f : \mathcal{A} \longrightarrow \mathcal{B}$ and $g : \mathcal{B} \longrightarrow \mathcal{A}$.

- f and g form an *equational correspondence* between $\mathcal{A}$ and $\mathcal{B}$ if the following holds:
  - $f[\leftrightarrow_{\mathcal{A}}] \subseteq\leftrightarrow_{\mathcal{B}}$
  - $g[\leftrightarrow_{\mathcal{B}}] \subseteq\leftrightarrow_{\mathcal{A}}$
  - $f \cdot g \subseteq\leftrightarrow_{\mathcal{A}}$
  - $g \cdot f \subseteq\leftrightarrow_{\mathcal{B}}$
- f and g form a *Galois connection* from $\mathcal{A}$ to $\mathcal{B}$ if the following holds:
  - $\rightarrow_{\mathcal{B}}$ weakly simulates $\rightarrow_{\mathcal{A}}$ through f
  - $\rightarrow_{\mathcal{A}}$ weakly simulates $\rightarrow_{\mathcal{B}}$ through g
  - $f \cdot g \subseteq\rightarrow^{*}_{\mathcal{A}}$
  - $g \cdot f \subseteq\leftarrow^{*}_{\mathcal{B}}$
- f and g form a *pre-Galois connection* from $\mathcal{A}$ to $\mathcal{B}$ if in the four conditions above we remove the last one.
- f and g form a *reflection* in $\mathcal{A}$ of $\mathcal{B}$ if the following holds:
  - $\rightarrow_{\mathcal{B}}$ weakly simulates $\rightarrow_{\mathcal{A}}$ through f
  - $\rightarrow_{\mathcal{A}}$ weakly simulates $\rightarrow_{\mathcal{B}}$ through g
  - $f \cdot g \subseteq\rightarrow^{*}_{\mathcal{A}}$
  - $g \cdot f = \mathsf{Id}_{\mathcal{B}}$

Remark 3

1. Note that saying that f and g form an equational correspondence between $\mathcal{A}$ and $\mathcal{B}$ only means that f and g extend to a bijection between $\leftrightarrow_{\mathcal{A}}$-equivalence classes of $\mathcal{A}$ and $\leftrightarrow_{\mathcal{B}}$-equivalence classes of $\mathcal{B}$. If f and g form an equational correspondence, so do g and f; it is a symmetric relation, unlike (pre-)Galois connections and reflections.

2. A Galois connection forms both an equational correspondence and a pre-Galois connection. A reflection forms a Galois connection. Also note that if f and g form a reflection then g and f form a pre-Galois connection.

3. If f and g form an equational correspondence between $\mathcal{A}$ and $\mathcal{B}$ (resp. a pre-Galois connection from $\mathcal{A}$ to $\mathcal{B}$, a Galois connection from $\mathcal{A}$ to $\mathcal{B}$, a reflection in $\mathcal{A}$ of $\mathcal{B}$),
and f' and g' form an equational correspondence between $\mathcal{B}$ and $\mathcal{C}$ (resp. a pre-Galois connection from $\mathcal{B}$ and $\mathcal{C}$, a Galois connection from $\mathcal{B}$ and $\mathcal{C}$, a reflection in $\mathcal{B}$ of $\mathcal{C}$),
then $f \cdot f'$ and $g \cdot g'$ form an equational correspondence between $\mathcal{A}$ and $\mathcal{C}$ (resp. a pre-Galois connection from $\mathcal{A}$ and $\mathcal{C}$, a Galois connection from $\mathcal{A}$ and $\mathcal{C}$, a reflection in $\mathcal{A}$ of $\mathcal{C}$).

1.1.2 Confluence

Definition 8 (Confluence & Church-Rosser)

- A reduction relation $\rightarrow$ on $\mathcal{A}$ is *confluent* if $\leftarrow^* \cdot \rightarrow^* \subseteq \rightarrow^* \cdot \leftarrow^*$
- A reduction relation $\rightarrow$ on $\mathcal{A}$ is *Church-Rosser* if $\leftrightarrow^* \subseteq \rightarrow^* \cdot \leftarrow^*$

Theorem 4 (Confluence is equivalent to Church-Rosser)
A reduction relation $\rightarrow$ is confluent if and only if it is Church-Rosser.

Proof:

- *if*: it suffices to note that $\leftarrow^* \cdot \rightarrow^* \subseteq \leftrightarrow^*$.
- *only if*: we prove $\leftrightarrow^n \subseteq \rightarrow^* \cdot \leftarrow^*$ by induction on n. For $n = 0$ it trivially holds. Suppose it holds for $\leftrightarrow^n$.

$$\begin{array}{rcll} \leftrightarrow^{n+1} & = & \leftrightarrow^n \cdot (\leftarrow \cup \rightarrow) & \\ & \subseteq & \rightarrow^* \cdot \leftarrow^* \cdot (\leftarrow \cup \rightarrow) & \text{by i.h.} \\ & = & (\rightarrow^* \cdot \leftarrow^*) \cup (\rightarrow^* \cdot \leftarrow^* \cdot \rightarrow) & \\ & \subseteq & \rightarrow^* \cdot \leftarrow^* & \text{by assumption} \end{array}$$

We can illustrate in Fig. 1.2 the right-hand side case of the union $\cup$.

□

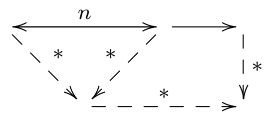

Figure 1.2: Confluence implies Church-Rosser

Theorem 5 (Confluence by simulation) *If f and g form a pre-Galois connection from $\mathcal{A}$ to $\mathcal{B}$ and $\to_{\mathcal{B}}$ is confluent, then $\to_{\mathcal{A}}$ is confluent.*

Proof:

$$\begin{array}{rcll}
\leftarrow^*_{\mathcal{A}} \cdot \to^*_{\mathcal{A}} & \subseteq & f^{-1}[\leftarrow^*_{\mathcal{B}} \cdot \to^*_{\mathcal{B}}] & \text{weak simulation} \\
& \subseteq & f^{-1}[\to^*_{\mathcal{B}} \cdot \leftarrow^*_{\mathcal{B}}] & \text{confluence of } \to_{\mathcal{B}} \\
& = & f \cdot \to^*_{\mathcal{B}} \cdot \leftarrow^*_{\mathcal{B}} \cdot f^{-1} & \\
& \subseteq & f \cdot g^{-1}[\to^*_{\mathcal{A}} \cdot \leftarrow^*_{\mathcal{A}}] \cdot f^{-1} & \text{weak simulation} \\
& = & f \cdot g \cdot \to^*_{\mathcal{A}} \cdot \leftarrow^*_{\mathcal{A}} g^{-1} \cdot f^{-1} & \text{weak simulation} \\
& \subseteq & \to^*_{\mathcal{A}} \cdot \leftarrow^*_{\mathcal{A}} & \text{by assumption}
\end{array}$$

This proof can be graphically represented in Fig. 1.3. □

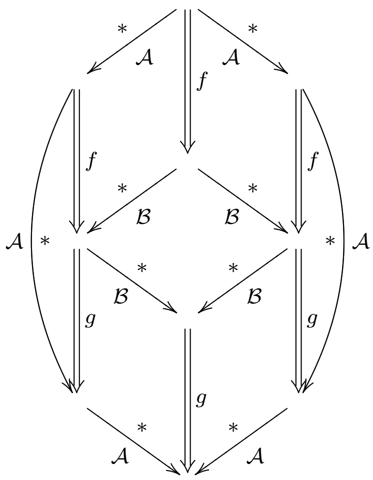

Figure 1.3: Confluence by simulation

1.1.3 Inference & derivations

We take for granted the notion of (labelled) tree, the notions of node, internal node and leaf, see e.g. [CDG+97]. In particular, the *height* of a tree is the length of its longest branch (e.g. the height of a tree with only one node is 1), and its size is its number of nodes.

We now introduce the notions of *inference structure* and *derivations*. The former are used to inductively define atomic predicates, which can be seen as particular sets (for predicates with one argument), or as particular n-ary relations (for predicates with n-arguments). The definitions are more readable if we only consider sets (rather than arbitrary n-ary relations), but are not less general: indeed, a n-ary relation is but a set of n-tuples.

Definition 9 (Inference structure) Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be sets whose elements are called *judgements*. An *inference structure* is a set of non-empty tuples of judgements, usually denoted

$$\frac{M_1 \ldots M_n}{M}$$

instead of $(M, M_1, \ldots, M_n)$. M is called the *conclusion* of the tuple and $M_1 \ldots M_n$ are called the *premisses*.

Definition 10 (Derivations)

- A *derivation* in an inference structure (sometimes called *full derivation*[2]) is a tree whose nodes are labelled with judgements, and such that if a node is labelled with M and has n sons ($n \geq 0$) respectively labelled with $M_1, \ldots, M_n$ then $(M, M_1, \ldots, M_n)$ is in the inference structure.[3]

- A *partial derivation*[4] is a tree whose nodes are labelled with judgements, together with a subset of its leaves whose elements are called *open leaves*, and such that if a node is not an open leaf, is labelled with M and has n sons ($n \geq 0$) respectively labelled with $M_1, \ldots, M_n$ then $(M, M_1, \ldots, M_n)$ is in the inference structure.[5]

- The judgement at the root of a (partial or full) derivation is called the *conclusion of the derivation*. The latter is said to *conclude* this judgement.

- A *derivation of a judgement* is a (full) derivation concluding this judgement. The latter is said to be *derivable*.

- A *derivation from a set $\mathcal{A}$ to a judgement M* is a partial derivation concluding M and whose open leaves are labelled with judgements in $\mathcal{A}$.

- Derivations inherit from their tree structures a notion of *sub-derivation*, *height* and *size*. We sometimes say that we prove a statement "by induction on a derivation" when we mean "on the height of a derivation".

- A *derivation step* is a partial derivation of height 1, i.e. a node and its sons (i.e. an element of the inference structure).

[2]Some authors also call them *complete derivations* or *categorical derivations*
[3]Leaves of the tree are such that $n = 0$, with (M) belonging to the inference structure.
[4]Some authors also call them *complete derivations* or *hypothetical derivations*
[5]Note that no condition is imposed on open leaves.

- We write down derivations by composing with itself the notation with one horizontal bar that we use for inference steps, as shown in Example 1.

Example 1 (Inference structure & derivation) Consider the following inference structure:

$$\frac{d \quad b}{c} \qquad \frac{c \quad b}{a} \qquad \frac{}{d}$$

The following derivation is from $\{b\}$ to a, has height 3 and size 5.

$$\frac{\dfrac{\dfrac{}{d} \quad b}{c} \qquad b}{a}$$

Note the different status of the leaves labelled with b and d, the former being open and the latter being not.

Definition 11 (Derivability & admissibility)

- A tuple of judgements $\dfrac{M_1 \dots M_n}{M}$ is *derivable* in an inference system if there is a derivation from the set $\{M_1, \dots, M_n\}$ to M.

  In this case we write $\dfrac{M_1 \dots M_n}{\overline{\overline{}}\,M}$.[6]

- A tuple of judgements $\dfrac{M_1 \dots M_n}{M}$ is *admissible* in an inference system if for all derivations of $M_1 \dots M_n$ there is a derivation of M.

  In this case we write $\begin{array}{c} M_1 \dots M_n \\ \cdots\cdots\cdots\cdots \\ M \end{array}$.

- A tuple of judgements $\dfrac{M_1 \dots M_n}{M}$ is *height-preserving admissible* in an inference system if for all derivations of $M_1 \dots M_n$ with heights at most $h \in \mathbb{N}$ there exists a derivation of M with height at most h.

  In this case we write $\begin{array}{c} M_1 \dots M_n \\ - - - - - \\ M \end{array}$.[7]

[6]Note that our notation for derivability, using a double line, is used by some authors for invertibility. Our notation is based on Kleene's [Kle52], with the rationale that the double line evokes several inference steps.

[7]The rationale of our notation for height-preserving admissibility is that we can bound the height of a derivation with fake steps of height-preserving admissibility just by not counting these steps, since the conclusion in such a step can be derived with a height no greater than that of some premiss.

- A tuple of judgements $\dfrac{M_1 \dots M_n}{M}$ is *invertible* in an inference system if it is derivable and if for all derivations of M there are derivations of $M_1 \dots M_n$. In this case we write $\genfrac{}{}{0pt}{}{M_1 \dots M_n}{\wr\wr\wr\wr\wr\wr\wr\wr\wr\wr \atop M}$.[8]

We shall use these notations within derivations: when writing a derivation we can use derivable and admissible tuples as *fake inference steps*. Proofs of derivability and admissibility then provide the real derivations that are denoted with fake steps: a fake step of derivability stands in fact for a sequence of real steps, while a fake step of admissibility requires its premisses to be derivable (i.e. with full derivations rather than partial ones).

Remark 6 If $\dfrac{M_1 \dots M_n}{M}$ is derivable then it is admissible (we can plug the derivations of $M_1, \dots, M_n$ into the derivation from $M_1, \dots, M_n$ to M). Note that the reverse is not true: in order to build a derivation of M knowing derivations of $M_1, \dots, M_n$ we could use another construction than the one above, potentially without the existence of a derivation from $M_1, \dots, M_n$ to M.

Remark 7 Note that a reduction relation is a particular inference structure made of pairs.

Definition 12 (Reduction sequence)

- A *reduction sequence* is a partial derivation in a reduction relation $\to$, and we often write $M \to \cdots \to N$ instead of $\dfrac{\dfrac{M}{\vdots}}{N}$. In that case we also say *reduction step* instead of *inference step*. Note that $M \to^* N$ is then the same as $\genfrac{}{}{0pt}{}{M}{= \atop N}$.

- The height of a reduction sequence is also called its *length*.

Remark 8 Note that $M \to^n N$ if and only if there is a reduction sequence of length n from M to N.

[8] The rationale of our notation for invertibility is that derivability of the conclusion is *equivalent* to the derivability of the premisses.

1.2 A constructive theory of normalisation

1.2.1 Normalisation & induction

Proving a universally quantified property by induction consists of verifying that the set of elements having the property is stable, in some sense similar to —yet more subtle than— that of Definition 4. Leading to different induction principles, we define two such notions of stability property: being *patriarchal* and being *paternal*.

Definition 13 (Patriarchal, paternal) Given a reduction relation $\to$ on $\mathcal{A}$, we say that

- a subset $\mathcal{T}$ of $\mathcal{A}$ is $\to$-*patriarchal* (or just *patriarchal* when the relation is clear) if $\forall N \in \mathcal{A}, \to(N) \subseteq \mathcal{T} \Rightarrow N \in \mathcal{T}$.
- a subset $\mathcal{T}$ of $\mathcal{A}$ is $\to$-*paternal* (or just *paternal* when the relation is clear) if it contains $\mathsf{nf}^{\to}$ and is stable under $\to^{-1}$.
- a predicate P on $\mathcal{A}$ is *patriarchal* (resp. *paternal*) if $\{M \in \mathcal{A} | \ P(M)\}$ is *patriarchal* (resp. *paternal*).

Lemma 9 *Suppose that for any N in $\mathcal{A}$, $N \in \mathsf{rf}^{\to}$ or $N \in \mathsf{nf}^{\to}$ and suppose $\mathcal{T} \subseteq \mathcal{A}$. If $\mathcal{T}$ is paternal, then it is patriarchal.*

Proof: In order to prove $\forall N \in \mathcal{A}, \to(N) \subseteq \mathcal{T} \Rightarrow N \in \mathcal{T}$, a case analysis is needed: either $N \in \mathsf{rf}^{\to}$ or $N \in \mathsf{nf}^{\to}$. In both cases $N \in \mathcal{T}$ because $\mathcal{T}$ is paternal. $\square$

Remark 10 Notice that we can obtain from classical logic the hypothesis for all N in $\mathcal{A}$, $N \in \mathsf{rf}^{\to}$ or $N \in \mathsf{nf}^{\to}$, because it is an instance of the Law of Excluded Middle. In intuitionistic logic, assuming this amounts to saying that being reducible is decidable, which might not always be true.

We would not require this hypothesis if we defined that $\mathcal{T}$ is paternal whenever $\forall N \in \mathcal{A}, N \in \mathcal{T} \vee (N \in \mathsf{rf}^{\to} \wedge (\to(N) \cap \mathcal{T} = \emptyset))$. This is classically equivalent to the definition above, but this definition also has some disadvantages as we shall see later.

Typically, if we want to prove that a predicate P on on some set $\mathcal{A}$ holds throughout $\mathcal{A}$, we actually prove that P is patriarchal or paternal, depending on the induction principle we use.

Hence, we define normalisation so that normalising elements are those captured by an induction principle, which should hold for every predicate satisfying the corresponding stability property. We thus get two notions of normalisation: the *strongly* (resp. *weakly*) *normalising* elements are those in every patriarchal (resp. paternal) set.

Definition 14 (Normalising elements) Given a reduction relation $\rightarrow$ on $\mathcal{A}$:

- The set of $\rightarrow$-*strongly normalising* elements is

$$\mathsf{SN}^{\rightarrow} := \bigcap_{\mathcal{T} \text{ is patriarchal}} \mathcal{T}$$

- The set of $\rightarrow$-*weakly normalising* elements is

$$\mathsf{WN}^{\rightarrow} := \bigcap_{\mathcal{T} \text{ is paternal}} \mathcal{T}$$

Remark 11 Interestingly enough, $\mathsf{WN}^{\rightarrow}$ can also be captured by an inductive definition:

$$\mathsf{WN}^{\rightarrow} = \bigcup_{n \geq 0} \mathsf{WN}_n^{\rightarrow}$$

where $\mathsf{WN}_n^{\rightarrow}$ is defined by induction on the natural number n as follows:

$$\begin{array}{lcl} \mathsf{WN}_0^{\rightarrow} & := & \mathsf{nf}^{\rightarrow} \\ \mathsf{WN}_{n+1}^{\rightarrow} & := & \{M \in \mathcal{A} |\ \exists n' \leq n, M \in \rightarrow^{-1}(\mathsf{WN}_{n'}^{\rightarrow})\} \end{array}$$

With the alternative definition of paternal suggested in Remark 10, the inclusion $\mathsf{WN}^{\rightarrow} \subseteq \bigcup_n \mathsf{WN}_n^{\rightarrow}$ would require the assumption that being reducible by $\rightarrow$ is decidable. We therefore preferred the first definition because we can then extract from a term M in $\mathsf{WN}^{\rightarrow}$ a natural number n such that $M \in \mathsf{WN}_n^{\rightarrow}$, without the hypothesis of decidability.

Such a characterisation gives us the possibility to prove that all weakly normalising elements satisfy some property by induction on natural numbers. On the other hand, trying to do so with strong normalisation leads to a different notion, as we shall see below. Hence, we lack for $\mathsf{SN}^{\rightarrow}$ an induction principle based on natural numbers, which is the reason why we built a specific induction principle into the definition of $\mathsf{SN}^{\rightarrow}$.

Definition 15 (Bounded elements) The set of $\rightarrow$-*bounded* elements is defined as

$$\mathsf{BN}^{\rightarrow} := \bigcup_{n \geq 0} \mathsf{BN}_n^{\rightarrow}$$

where $\mathsf{BN}_n^{\rightarrow}$ is defined by induction on the natural number n as follows:

$$\begin{array}{lcl} \mathsf{BN}_0^{\rightarrow} & := & \mathsf{nf}^{\rightarrow} \\ \mathsf{BN}_{n+1}^{\rightarrow} & := & \{M \in \mathcal{A} |\ \exists n' \leq n,\ \rightarrow(M) \subseteq \mathsf{BN}_{n'}^{\rightarrow}\} \end{array}$$

But we have the following fact:

$$\begin{tikzcd}
 & & M \arrow[dll] \arrow[dl] \arrow[d, dotted] \arrow[dr] \arrow[drr, dotted] & & \\
M_{1,1} & M_{2,1} \arrow[d] & \cdots & M_{i,1} \arrow[d] & \cdots \\
 & M_{2,2} & \cdots & M_{i,2} \arrow[d] & \cdots \\
 & & \cdots & M_{i,i} & \cdots
\end{tikzcd}$$

Figure 1.4: $M \in \mathsf{SN}^{\rightarrow}$ but $M \notin \mathsf{BN}^{\rightarrow}$

Remark 12 For some reduction relations $\rightarrow$, $\mathsf{SN}^{\rightarrow} \neq \mathsf{BN}^{\rightarrow}$. For instance, Fig. 1.4 shows a term M and relation $\rightarrow$ such that $M \in \mathsf{SN}^{\rightarrow}$ but $M \notin \mathsf{BN}^{\rightarrow}$.

Lemma 13 *However, if $\rightarrow$ is finitely branching, then $\mathsf{BN}^{\rightarrow}$ is patriarchal. As a consequence, $\mathsf{BN}^{\rightarrow} = \mathsf{SN}^{\rightarrow}$ (the counter-example above could not be finitely branching).*

Proof: Suppose $\rightarrow(M) \subseteq \mathsf{BN}^{\rightarrow}$. Because $\rightarrow$ is finitely branching, there exists a natural number n such that $\rightarrow(M) \subseteq \mathsf{BN}^{\rightarrow}_{n}$. Clearly, $M \in \mathsf{BN}^{\rightarrow}_{n+1} \subseteq \mathsf{BN}^{\rightarrow}$. □

Remark 14 As a trivial example, all the natural numbers are $>$-bounded. Indeed, any natural number n is in $\mathsf{BN}^{>}_{n}$, which can be proved by induction.

A canonical way of proving a statement $\forall M \in \mathsf{BN}^{\rightarrow}, P(M)$ is to prove by induction on the natural number n that $\forall M \in \mathsf{BN}^{\rightarrow}_{n}, P(M)$. Although we can exhibit no such natural number on which a statement $\forall M \in \mathsf{SN}^{\rightarrow}, P(M)$ can be proved by induction, the following induction principles hold by definition of normalisation:

Remark 15 Given a predicate P on $\mathcal{A}$ and an element $M \in \mathcal{A}$,

1. If P is patriarchal and $M \in \mathsf{SN}^{\rightarrow}$, then $P(M)$.

2. If P is paternal and $M \in \mathsf{WN}^{\rightarrow}$, then $P(M)$.

When we use this remark to prove $\forall M \in \mathsf{SN}^{\rightarrow}, P(M)$ (resp. $\forall M \in \mathsf{WN}^{\rightarrow}, P(M)$), we say that we prove it *by raw induction in $\mathsf{SN}^{\rightarrow}$* (resp. *in $\mathsf{WN}^{\rightarrow}$*).

Definition 16 (Strongly & weakly normalising relations) Given a reduction relation $\rightarrow$ on $\mathcal{A}$ and a subset $\mathcal{T} \subseteq \mathcal{A}$, we say that the reduction relation is *strongly normalising* or *terminating* on $\mathcal{T}$ (or it *terminates* on $\mathcal{T}$) if $\mathcal{T} \subseteq \mathsf{SN}^{\rightarrow}$. We say that it is *weakly normalising* on $\mathcal{T}$ if $\mathcal{T} \subseteq \mathsf{WN}^{\rightarrow}$. If we do not specify $\mathcal{T}$, it means that we take $\mathcal{T} = \mathcal{A}$.

Lemma 16

1. *If $n < n'$ then $\mathit{BN}_n^{\rightarrow} \subseteq \mathit{BN}_{n'}^{\rightarrow} \subseteq \mathit{BN}^{\rightarrow}$. In particular, $\mathit{nf}^{\rightarrow} \subseteq \mathit{BN}_n^{\rightarrow} \subseteq \mathit{BN}^{\rightarrow}$.*

2. *$\mathit{BN}^{\rightarrow} \subseteq \mathit{SN}^{\rightarrow}$ and $\mathit{BN}^{\rightarrow} \subseteq \mathit{WN}^{\rightarrow}$.*
   Hence, all natural numbers are in $\mathit{SN}^{>}$ and $\mathit{WN}^{>}$.

3. *If being reducible is decidable (or if we work in classical logic), then $\mathit{SN}^{\rightarrow} \subseteq \mathit{WN}^{\rightarrow}$.*

Proof:

1. By definition.

2. Both facts can be proved for all $\mathsf{BN}_n^{\rightarrow}$ by induction on n.

3. This comes from Remark 9 and thus requires either classical logic or the particular instance of the Law of Excluded Middle stating that for all N,

$$N \in \mathsf{rf}^{\rightarrow} \vee N \in \mathsf{nf}^{\rightarrow}$$

$\square$

Lemma 17

1. *$\mathit{SN}^{\rightarrow}$ is patriarchal, $\mathit{WN}^{\rightarrow}$ is paternal.*

2. *If $M \in \mathit{BN}^{\rightarrow}$ then $\rightarrow(M) \subseteq \mathit{BN}^{\rightarrow}$.*
   If $M \in \mathit{SN}^{\rightarrow}$ then $\rightarrow(M) \subseteq \mathit{SN}^{\rightarrow}$.
   If $M \in \mathit{WN}^{\rightarrow}$ then either $M \in \mathit{nf}^{\rightarrow}$ or $M \in \rightarrow^{-1}(\mathit{WN}^{\rightarrow})$
   (which implies $M \in \mathit{rf}^{\rightarrow} \Rightarrow M \in \rightarrow^{-1}(\mathit{WN}^{\rightarrow})$).

Proof:

1. For the first statement, let $M \in \mathcal{A}$ such that $\rightarrow(M) \subseteq \mathsf{SN}^{\rightarrow}$ and let $\mathcal{T}$ be patriarchal. We want to prove that $M \in \mathcal{T}$. It suffices to prove that $\rightarrow(M) \subseteq \mathcal{T}$. This is the case, because $\rightarrow(M) \subseteq \mathsf{SN}^{\rightarrow} \subseteq \mathcal{T}$.
   For the second statement, first notice that $\mathsf{nf}^{\rightarrow} \subseteq \mathsf{WN}^{\rightarrow}$. Now let $M, N \in \mathcal{A}$ such that $M \rightarrow N$ and $N \in \mathsf{WN}^{\rightarrow}$, and let $\mathcal{T}$ be paternal. We want to prove that $M \in \mathcal{T}$. This is the case because $N \in \mathcal{T}$ and $\mathcal{T}$ is paternal.

2. The first statement is straightforward.
   For the second, we show that $\mathcal{T} = \{P \in \mathcal{A} | \rightarrow(P) \subseteq \mathsf{SN}^{\rightarrow}\}$ is patriarchal:
   Let $P \in \mathcal{A}$ such that $\rightarrow(P) \subseteq \mathcal{T}$, that is, $\forall R \in \rightarrow(P), \rightarrow(R) \subseteq \mathsf{SN}^{\rightarrow}$.
   Because $\mathsf{SN}^{\rightarrow}$ is patriarchal, $\forall R \in \rightarrow(P), R \in \mathsf{SN}^{\rightarrow}$.
   Hence, $\rightarrow(P) \subseteq \mathsf{SN}^{\rightarrow}$, that is, $P \in \mathcal{T}$ as required.
   Now by definition of $\mathsf{SN}^{\rightarrow}$, we get $M \in \mathcal{T}$.

For the third statement, we prove that $\mathcal{T} = \mathsf{nf}^{\rightarrow} \cup \rightarrow^{-1}(\mathsf{WN}^{\rightarrow})$ is paternal: Clearly, it suffices to prove that it is stable under $\rightarrow^{-1}$. Let $P, Q \in \mathcal{A}$ such that $P \rightarrow Q$ and $Q \in \mathcal{T}$. If $Q \in \mathsf{nf}^{\rightarrow} \subseteq \mathsf{WN}^{\rightarrow}$, then $P \in \rightarrow^{-1}(\mathsf{WN}^{\rightarrow}) \subseteq \mathcal{T}$. If $Q \in \rightarrow^{-1}(\mathsf{WN}^{\rightarrow})$, then, because $\mathsf{WN}^{\rightarrow}$ is paternal, we get $Q \in \mathsf{WN}^{\rightarrow}$, so that $P \in \rightarrow^{-1}(\mathsf{WN}^{\rightarrow}) \subseteq \mathcal{T}$ as required.
Now by definition of $M \in \mathsf{WN}^{\rightarrow}$, we get $M \in \mathcal{T}$.

□

Notice that this lemma gives the well-known characterisation of $\mathsf{SN}^{\rightarrow}$:
$M \in \mathsf{SN}^{\rightarrow}$ if and only if $\forall N \in \rightarrow(M), N \in \mathsf{SN}^{\rightarrow}$.

Now we refine the induction principle immediately contained in the definition of normalisation by relaxing the requirement that the predicate should be patriarchal or paternal:

Theorem 18 (Induction principle) *Given a predicate P on $\mathcal{A}$,*

1. *Suppose $\forall M \in \mathsf{SN}^{\rightarrow}, (\forall N \in \rightarrow(M), P(N)) \Rightarrow P(M)$.*
   Then $\forall M \in \mathsf{SN}^{\rightarrow}, P(M)$.
2. *Suppose $\forall M \in \mathsf{WN}^{\rightarrow}, (M \in \mathsf{nf}^{\rightarrow} \vee \exists N \in \rightarrow(M), P(N)) \Rightarrow P(M)$.*
   Then $\forall M \in \mathsf{WN}^{\rightarrow}, P(M)$.

When we use this theorem to prove a statement $P(M)$ for all M in $\mathsf{SN}^{\rightarrow}$ (resp. $\mathsf{WN}^{\rightarrow}$), we just add $(\forall N \in \rightarrow(M), P(N))$ (resp. $M \in \mathsf{nf}^{\rightarrow} \vee \exists N \in \rightarrow(M), P(N)$) to the assumptions, which we call the induction hypothesis.

We say that we prove the statement by induction in $\mathsf{SN}^{\rightarrow}$ *(resp.* in $\mathsf{WN}^{\rightarrow}$*).*

Proof:

1. We prove that $\mathcal{T} = \{M \in \mathcal{A} |\ M \in \mathsf{SN}^{\rightarrow} \Rightarrow P(M)\}$ is patriarchal.
   Let $N \in \mathcal{A}$ such that $\rightarrow(N) \subseteq \mathcal{T}$. We want to prove that $N \in \mathcal{T}$:
   Suppose that $N \in \mathsf{SN}^{\rightarrow}$. By Lemma 17 we get that $\forall R \in \rightarrow(N), R \in \mathsf{SN}^{\rightarrow}$. By definition of $\mathcal{T}$ we then get $\forall R \in \rightarrow(N), P(R)$. From the main hypothesis we get $P(N)$. Hence, we have shown $N \in \mathcal{T}$.
   Now by definition of $M \in \mathsf{SN}^{\rightarrow}$, we get $M \in \mathcal{T}$, which can be simplified as $P(M)$ as required.

2. We prove that $\mathcal{T} = \{M \in \mathcal{A} |\ M \in \mathsf{WN}^{\rightarrow} \wedge P(M)\}$ is paternal.
   Let $N \in \mathsf{nf}^{\rightarrow} \subseteq \mathsf{WN}^{\rightarrow}$. By the main hypothesis we get $P(N)$.
   Now let $N \in \rightarrow^{-1}(\mathcal{T})$, that is, there is $R \in \mathcal{T}$ such that $N \rightarrow R$.
   We want to prove that $N \in \mathcal{T}$:
   By definition of $\mathcal{T}$, we have $R \in \mathsf{WN}^{\rightarrow}$, so $N \in \mathsf{WN}^{\rightarrow}$ (because $\mathsf{WN}^{\rightarrow}$ is paternal). We also have $P(R)$, so we can apply the main hypothesis to get $P(N)$. Hence, we have shown $N \in \mathcal{T}$.
   Now by definition of $M \in \mathsf{WN}^{\rightarrow}$, we get $M \in \mathcal{T}$, which can be simplified as $P(M)$ as required.

□

As a first application of the induction principle, we prove the following results:

Lemma 19 *$M \in \mathsf{SN}^{\rightarrow}$ if and only if there is no infinite reduction sequence starting from M (classically, with the countable axiom of choice).*

Proof:

- *only if*: Consider the predicate $P(M)$ "having no infinite reduction sequence starting from M". We prove it by induction in $\mathsf{SN}^{\rightarrow}$. If M starts an infinite reduction sequence, then there is a $N \in \rightarrow(M)$ that also starts an infinite reduction sequence, which contradicts the induction hypothesis.
- *if*: Suppose $M \notin \mathsf{SN}^{\rightarrow}$. There is a patriarchal set $\mathcal{T}$ in which M is not. Hence, there is a $N \in \rightarrow(M)$ that is not in $\mathcal{T}$, and we re-iterate on it, creating an infinite reduction sequence. This uses the countable axiom of choice.

□

Lemma 20

1. *If $\rightarrow_1 \subseteq \rightarrow_2$, then $\mathsf{nf}^{\rightarrow_1} \supseteq \mathsf{nf}^{\rightarrow_2}$, $\mathsf{WN}^{\rightarrow_1} \supseteq \mathsf{WN}^{\rightarrow_2}$, $\mathsf{SN}^{\rightarrow_1} \supseteq \mathsf{SN}^{\rightarrow_2}$, and for all n, $\mathsf{BN}_n^{\rightarrow_1} \supseteq \mathsf{BN}_n^{\rightarrow_2}$.*
2. *$\mathsf{nf}^{\rightarrow} = \mathsf{nf}^{\rightarrow^+}$, $\mathsf{WN}^{\rightarrow} = \mathsf{WN}^{\rightarrow^+}$, $\mathsf{SN}^{\rightarrow} = \mathsf{SN}^{\rightarrow^+}$, and for all n, $\mathsf{BN}_n^{\rightarrow^+} = \mathsf{BN}_n^{\rightarrow}$.*

Proof:

1. By expanding the definitions.
2. For each statement, the right-to-left inclusion is a corollary of point 1.
   For the first statement, it remains to prove that $\mathsf{nf}^{\rightarrow} \subseteq \mathsf{nf}^{\rightarrow^+}$.
   Let $M \in \mathsf{nf}^{\rightarrow}$. By definition, $\rightarrow(M) = \emptyset$, so clearly $\rightarrow^+(M) = \emptyset$ as well.
   For the second statement, it remains to prove that $\mathsf{WN}^{\rightarrow} \subseteq \mathsf{WN}^{\rightarrow^+}$ which we do by induction in $\mathsf{WN}^{\rightarrow}$:
   Assume $M \in \mathsf{WN}^{\rightarrow}$ and the induction hypothesis that either $M \in \mathsf{nf}^{\rightarrow}$ or there is $N \in \rightarrow(M)$ such that $N \in \mathsf{WN}^{\rightarrow^+}$. In the former case, we have $M \in \mathsf{nf}^{\rightarrow} = \mathsf{nf}^{\rightarrow^+}$ and $\mathsf{nf}^{\rightarrow^+} \subseteq \mathsf{WN}^{\rightarrow^+}$. In the latter case, we have $N \in \rightarrow^+(M)$. Because of Lemma 17, $\mathsf{WN}^{\rightarrow^+}$ is stable by $\mathsf{WN}^{\rightarrow^{+^{-1}}}$, and hence $M \in \mathsf{WN}^{\rightarrow^+}$.
   For the third statement, it remains to prove that $\mathsf{SN}^{\rightarrow} \subseteq \mathsf{SN}^{\rightarrow^+}$. We prove the stronger statement that $\forall M \in \mathsf{SN}^{\rightarrow}$, $\rightarrow^*(M) \subseteq \mathsf{SN}^{\rightarrow^+}$ by induction in $\mathsf{SN}^{\rightarrow}$: assume $M \in \mathsf{SN}^{\rightarrow}$ and the induction hypothesis $\forall N \in \rightarrow(M)$, $\rightarrow^*(N) \subseteq \mathsf{SN}^{\rightarrow^+}$. Clearly, $\rightarrow^+(M) \subseteq \mathsf{SN}^{\rightarrow^+}$. Because of

Lemma 17, $\mathsf{SN}^{\rightarrow^+}$ is $\rightarrow^+$-patriarchal, so $M \in \mathsf{SN}^{\rightarrow^+}$, and hence $\rightarrow^*(M) \subseteq \mathsf{SN}^{\rightarrow^+}$.

The statement $\mathsf{BN}_n^{\rightarrow} \subseteq \mathsf{BN}_n^{\rightarrow^+}$ can easily be proved by induction on n.

□

Notice that this result enables us to use a stronger induction principle: in order to prove $\forall M \in \mathsf{SN}^{\rightarrow}, P(M)$, it now suffices to prove

$$\forall M \in \mathsf{SN}^{\rightarrow}, (\forall N \in \rightarrow^+(M), P(N)) \Rightarrow P(M)$$

This induction principle is called the *transitive induction in $\mathsf{SN}^{\rightarrow}$*.

Theorem 21 (Strong normalisation of disjoint union)
Suppose that $(\mathcal{A}_i)_{i \in I}$ is a family of disjoint sets on some index set I, each being equipped with a reduction relation $\rightarrow_i$, and consider the reduction relation $\rightarrow := \bigcup_{i \in I} \rightarrow_i$ on $\bigcup_{i \in I} \mathcal{A}_i$.

We have $\bigcup_{i \in I} \mathsf{SN}^{\rightarrow_i} \subseteq \mathsf{SN}^{\rightarrow}$.

Proof: It suffices to prove that for all $j \in I$, $\mathsf{SN}^{\rightarrow_j} \subseteq \mathsf{SN}^{\rightarrow}$, which we do by induction in $\mathsf{SN}^{\rightarrow_j}$. Assume $M \in \mathsf{SN}^{\rightarrow_j}$ and assume the induction hypothesis $\rightarrow_j(M) \subseteq \mathsf{SN}^{\rightarrow}$. We must prove $M \in \mathsf{SN}^{\rightarrow}$, so it suffices to prove that for all N such that $M \rightarrow N$ we have $N \in \mathsf{SN}^{\rightarrow}$. By definition of the disjoint union, since $M \in \mathcal{A}_i$, all such N are in $\rightarrow_j(M)$ so we can apply the induction hypothesis. □

1.2.2 Termination by simulation

Now that we have established an induction principle on strongly normalising elements, the question arises of how we can prove strong normalisation. In this subsection we re-establish in our framework the well-known technique of simulation, which can be found for instance in [BN98]. The first technique to prove that a reduction relation on the set $\mathcal{A}$ terminates consists in simulating it (in the sense of Definition 6) in another set $\mathcal{B}$ equipped with its own reduction relation known to be terminating.

The mapping from $\mathcal{A}$ to $\mathcal{B}$ is sometimes called the *measure function* or the *weight function*, but Definition 6 generalises the concept to an arbitrary relation between $\mathcal{A}$ and $\mathcal{B}$, not necessarily functional. Similar results are to be found in [Che04], with the notions of *prosimulation*, *insertion*, and *repercussion*. The main point here is that the simulation technique is the typical example where the proof usually starts with "suppose an infinite reduction sequence" and ends with a contradiction. We show how the use of classical logic is completely unnecessary, provided that we use a constructive definition of SN such as ours.

Theorem 22 (Strong normalisation by strong simulation) *Let $\mathcal{R}$ be a relation between $\mathcal{A}$ and $\mathcal{B}$, equipped with the reduction relations $\rightarrow_{\mathcal{A}}$ and $\rightarrow_{\mathcal{B}}$.*

If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, then $\mathcal{R}^{-1}(\mathsf{SN}^{\rightarrow_{\mathcal{B}}}) \subseteq \mathsf{SN}^{\rightarrow_{\mathcal{A}}}$.

Proof: $\mathcal{R}^{-1}(\mathsf{SN}^{\to_\mathcal{B}}) \subseteq \mathsf{SN}^{\to_\mathcal{A}}$ can be reformulated as

$$\forall N \in \mathsf{SN}^{\to_\mathcal{B}}, \forall M \in \mathcal{A}, M\mathcal{R}N \Rightarrow M \in \mathsf{SN}^{\to_\mathcal{A}}$$

which we prove by transitive induction in $\mathsf{SN}^{\to_\mathcal{B}}$. Assume $N \in \mathsf{SN}^{\to_\mathcal{B}}$ and assume the induction hypothesis $\forall N' \in \to_\mathcal{B}^+ (N), \forall M' \in \mathcal{A}, M'\mathcal{R}N' \Rightarrow M' \in \mathsf{SN}^{\to_\mathcal{A}}$. Now let $M \in \mathcal{A}$ such that $M\mathcal{R}N$. We want to prove that $M \in \mathsf{SN}^{\to_\mathcal{A}}$. It suffices to prove that $\forall M' \in \to(M), M' \in \mathsf{SN}^{\to_\mathcal{A}}$. Let M' be such that $M \to_\mathcal{A} M'$. The simulation hypothesis provides $N' \in \to_\mathcal{B}^+ (N)$ such that $M'\mathcal{R}N'$. We apply the induction hypothesis on N', M' and get $M' \in \mathsf{SN}^{\to_\mathcal{A}}$ as required. We illustrate the technique in Fig. 1.5. □

$$\begin{tikzcd}
M \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}"] & N \in \mathsf{SN}^{\to_\mathcal{B}} \arrow[d, "\mathcal{B}+"] \\
M_1 \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}"] & N_1 \arrow[d, "\mathcal{B}+"] \\
M_2 \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}"] & N_2 \arrow[d, "\mathcal{B}+"] \\
{} \arrow[d, dotted] & {} \arrow[d, dotted] \\
M_i \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}"] & \underline{N_i} \\
M_{i+j} \arrow[ur, Rightarrow, "\mathcal{R}"] \arrow[d, dotted, "\mathcal{A}"] & \\
{} \arrow[uur, Rightarrow, "\mathcal{R}"] &
\end{tikzcd}$$

Figure 1.5: Deriving strong normalisation by simulation

1.2.3 Lexicographic termination

The simulation technique can be improved by another standard method. It consists of splitting the reduction relation into two parts, then proving that the first part is strongly simulated by a first auxiliary terminating relation, and then proving that the second part is weakly simulated by it and strongly simulated by a second auxiliary terminating relation. In some sense, the two auxiliary terminating relations act as measures that decrease lexicographically. We express this method in our constructive framework.

Lemma 23 *Given two reduction relations $\to_1$, $\to_2$, suppose that $\mathsf{SN}^{\to_1}$ is stable under $\to_2$. Then $\mathsf{SN}^{\to_1 \cup \to_2} = \mathsf{SN}^{\to_1^* \cdot \to_2} \cap \mathsf{SN}^{\to_1}$*

Proof: The left-to-right inclusion is an application of Theorem 22: $\to_1 \cup \to_2$ strongly simulates both $\to_1^* \cdot \to_2$ and $\to_1$ through Id.

Now we prove the right-to-left inclusion. We first prove the following lemma:

$$\forall M \in \mathsf{SN}^{\to_1}, (\to_1^* \cdot \to_2)(M) \subseteq \mathsf{SN}^{\to_1 \cup \to_2} \Rightarrow M \in \mathsf{SN}^{\to_1 \cup \to_2}$$

We do this by induction in $\mathsf{SN}^{\to_1}$, so not only assume $(\to_1^* \cdot \to_2)(M) \subseteq \mathsf{SN}^{\to_1 \cup \to_2}$, but also assume the induction hypothesis:
$\forall N \in \to_1(M)$, $(\to_1^* \cdot \to_2)(N) \subseteq \mathsf{SN}^{\to_1 \cup \to_2} \Rightarrow N \in \mathsf{SN}^{\to_1 \cup \to_2}$.
We want to prove that $M \in \mathsf{SN}^{\to_1 \cup \to_2}$, so it suffices to prove that both $\forall N \in \to_2(M), N \in \mathsf{SN}^{\to_1 \cup \to_2}$ and $\forall N \in \to_1(M), N \in \mathsf{SN}^{\to_1 \cup \to_2}$. The former case is a particular case of the first hypothesis. The latter case would be provided by the second hypothesis (the induction hypothesis) if only we could prove that $(\to_1^* \cdot \to_2)(N) \subseteq \mathsf{SN}^{\to_1 \cup \to_2}$. But this is true because $(\to_1^* \cdot \to_2)(N) \subseteq (\to_1^* \cdot \to_2)(M)$ and the first hypothesis reapplies.

Now we prove

$$\forall M \in \mathsf{SN}^{\to_1^* \cdot \to_2}, M \in \mathsf{SN}^{\to_1} \Rightarrow M \in \mathsf{SN}^{\to_1 \cup \to_2}$$

We do this by induction in $\mathsf{SN}^{\to_1^* \cdot \to_2}$, so not only assume $M \in \mathsf{SN}^{\to_1}$, but also assume the induction hypothesis $\forall N \in (\to_1^* \cdot \to_2)(M), N \in \mathsf{SN}^{\to_1} \Rightarrow N \in \mathsf{SN}^{\to_1 \cup \to_2}$.
Now we can combine those two hypotheses, because we know that $\mathsf{SN}^{\to_1}$ is stable under $\to_2$: since $M \in \mathsf{SN}^{\to_1}$, we have $(\to_1^* \cdot \to_2)(M) \subseteq \mathsf{SN}^{\to_1}$, so that the induction hypothesis can be simplified in $\forall N \in (\to_1^* \cdot \to_2)(M), N \in \mathsf{SN}^{\to_1 \cup \to_2}$.
This gives us exactly the conditions to apply the above lemma to M. □

Definition 17 (Lexicographic reduction) Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be sets, respectively equipped with the reduction relations $\to_{\mathcal{A}_1}, \ldots, \to_{\mathcal{A}_n}$.

For $1 \leq i \leq n$, let $\to_i$ be the reduction relation on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ defined as follows:

$$(M_1, \ldots, M_n) \to_i (N_1, \ldots, N_n)$$

if $M_i \to_{\mathcal{A}_i} N_i$ and for all $1 \leq j < i$, $M_j = N_j$ and for all $i < j \leq n$, $N_j \in \mathsf{SN}^{\to_{\mathcal{A}_j}}$.
We define the *lexicographic reduction*

$$\to_{\mathsf{lex}} = \bigcup_{1 \leq i \leq n} \to_i$$

We sometimes write $\to_{\mathsf{lexx}}$ for $\to_{\mathsf{lex}}{}^+$, i.e. the transitive closure of $\to_{\mathsf{lex}}$.[9]

Corollary 24 (Lexicographic termination 1)

$$\mathit{SN}^{\to_{A_1}} \times \cdots \times \mathit{SN}^{\to_{A_n}} \subseteq \mathit{SN}^{\to_{lex}}$$

In particular, if $\to_{\mathcal{A}_i}$ is terminating on $\mathcal{A}_i$ for all $1 \leq i \leq n$, then $\to_{lex}$ is terminating on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.

[9] This is the traditional lexicographic order, see e.g. [Ter03].

Proof: By induction on n: for $n = 1$, we conclude from $\rightarrow_{\mathcal{A}_1}=\rightarrow_1$.
Then notice that $\rightarrow_{\mathcal{A}_{n+1}}$ strongly simulates $\rightarrow_{n+1}$ through the $(n+1)^{th}$ projection. Hence, by Theorem 22, if $N_{n+1} \in \mathsf{SN}^{\rightarrow_{\mathcal{A}_{n+1}}}$ then $(N_1, \ldots, N_{n+1}) \in \mathsf{SN}^{\rightarrow_{n+1}}$, which we can also formulate as $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n \times \mathsf{SN}^{\rightarrow_{\mathcal{A}_{n+1}}} \subseteq \mathsf{SN}^{\rightarrow_{n+1}}$.
A first consequence of this is $\mathsf{SN}^{\rightarrow_{\mathcal{A}_1}} \times \cdots \times \mathsf{SN}^{\rightarrow_{\mathcal{A}_{n+1}}} \subseteq \mathsf{SN}^{\rightarrow_{n+1}}$ (1). A second one is that $\mathsf{SN}^{\rightarrow_{n+1}}$ is stable under $\rightarrow_1 \cup \ldots \cup \rightarrow_n$ (2). Now notice that $\rightarrow_1 \cup \ldots \cup \rightarrow_n$ strongly simulates $\rightarrow_{n+1}^* \cdot (\rightarrow_1 \cup \ldots \cup \rightarrow_n)$ through the projection that drops the $(n+1)^{th}$ component. We can thus apply Theorem 22 to get $\mathsf{SN}^{\rightarrow_1 \cup \ldots \cup \rightarrow_n} \times \mathcal{A}_{n+1} \subseteq \mathsf{SN}^{\rightarrow_{n+1}^* \cdot (\rightarrow_1 \cup \ldots \cup \rightarrow_n)}$, which, combined with the induction hypothesis, gives $\mathsf{SN}^{\rightarrow_{\mathcal{A}_1}} \times \cdots \times \mathsf{SN}^{\rightarrow_{\mathcal{A}_{n+1}}} \subseteq \mathsf{SN}^{\rightarrow_{n+1}^* \cdot (\rightarrow_1 \cup \ldots \cup \rightarrow_n)}$ (3). From (1), (2), and (3) we can now conclude by using Lemma 23. □

Corollary 25 (Lexicographic termination 2) *Let $\mathcal{A}$ be a set equipped with a reduction relation $\rightarrow$.*

For each natural number n, let $\rightarrow_{\mathsf{lex}n}$ be the lexicographic reduction on $\mathcal{A}^n$.
Consider the reduction relation $\rightarrow_{\mathsf{lex}}= \bigcup_n \rightarrow_{\mathsf{lex}n}$ on the disjoint union $\bigcup_n \mathcal{A}^n$.

$$\bigcup_n (\mathsf{SN}^{\rightarrow})^n \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{lex}}}$$

Proof: It suffices to combine Corollary 24 with Theorem 21. □

Corollary 26 (Lexicographic simulation technique) *Let $\rightarrow_{\mathcal{A}}$ and $\rightarrow'_{\mathcal{A}}$ be two reduction relations on $\mathcal{A}$, and $\rightarrow_{\mathcal{B}}$ be a reduction relation on $\mathcal{B}$. Suppose*

- *$\rightarrow'_{\mathcal{A}}$ is strongly simulated by $\rightarrow_{\mathcal{B}}$ through $\mathcal{R}$*
- *$\rightarrow_{\mathcal{A}}$ is weakly simulated by $\rightarrow_{\mathcal{B}}$ through $\mathcal{R}$*
- *$\mathsf{SN}^{\rightarrow_{\mathcal{A}}} = \mathcal{A}$*

Then $\mathcal{R}^{-1}(\mathsf{SN}^{\rightarrow_{\mathcal{B}}}) \subseteq \mathsf{SN}^{\rightarrow_{\mathcal{A}} \cup \rightarrow'_{\mathcal{A}}}$.
(In other words, if $M\mathcal{R}N$ and $N \in \mathsf{SN}^{\rightarrow_{\mathcal{B}}}$ then $M \in \mathsf{SN}^{\rightarrow_{\mathcal{A}} \cup \rightarrow'_{\mathcal{A}}}$.)

Proof: Clearly, the reduction relation $\rightarrow_{\mathcal{A}}^* \cdot \rightarrow'_{\mathcal{A}}$ is strongly simulated by $\rightarrow_{\mathcal{B}}$ through $\mathcal{R}$, so that by Theorem 22 we get $\mathcal{R}^{-1}(\mathsf{SN}^{\rightarrow_{\mathcal{B}}}) \subseteq \mathsf{SN}^{\rightarrow_{\mathcal{A}}^* \cdot \rightarrow'_{\mathcal{A}}}$.
But $\mathsf{SN}^{\rightarrow_{\mathcal{A}}^* \cdot \rightarrow'_{\mathcal{A}}} = \mathsf{SN}^{\rightarrow_{\mathcal{A}}^* \cdot \rightarrow'_{\mathcal{A}}} \cap \mathsf{SN}^{\rightarrow_{\mathcal{A}}} = \mathsf{SN}^{\rightarrow_{\mathcal{A}} \cup \rightarrow'_{\mathcal{A}}}$, by the Lemma 23 (since $\mathsf{SN}^{\rightarrow_{\mathcal{A}}} = \mathcal{A}$ is obviously stable by $\rightarrow'_{\mathcal{A}}$). □

The intuitive idea behind this corollary is that after a certain number of $\rightarrow_{\mathcal{A}}$-steps and $\rightarrow'_{\mathcal{A}}$-steps, the only reductions in $\mathcal{A}$ that can take place are those that no longer modify the encoding in $\mathcal{B}$, that is, $\rightarrow_{\mathcal{A}}$-steps. Then it suffices to show that $\rightarrow_{\mathcal{A}}$ terminate, so that no infinite reduction sequence can start from M, as illustrated in Fig. 1.6.

$$\begin{tikzcd}
M \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}*"] & N \in \mathsf{SN}^{\to\mathcal{B}} \arrow[d, "\mathcal{B}*"] \\
M_1 \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}'"] & N_1 \arrow[d, "\mathcal{B}+"] \\
M'_1 \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}*"] & N'_1 \arrow[d, "\mathcal{B}*"] \\
M_2 \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}'"] & N_2 \arrow[d, "\mathcal{B}+"] \\
{} \arrow[d, dotted] & {} \arrow[d, dotted] \\
M'_i \arrow[r, Rightarrow, "\mathcal{R}"] \arrow[d, "\mathcal{A}*"] & \underline{N'_i} \\
M_{i+j} \arrow[ur, Rightarrow, "\mathcal{R}"] \arrow[d, dotted, "\mathcal{A}*"] & \\
{} \arrow[uur, Rightarrow, "\mathcal{R}"] &
\end{tikzcd}$$

Figure 1.6: Deriving strong normalisation by lexicographic simulation

1.2.4 Multi-set termination

Now we define the notions of multi-sets their reductions [DM79, BN98]. We constructively prove their termination. Classical proofs of the result can also be found in [Ter03].

Definition 18 (Multi-Sets)

- Given a set $\mathcal{A}$, a *multi-set of* $\mathcal{A}$ is a total function from $\mathcal{A}$ to the natural numbers such that only a finite subset of elements are not mapped to 0.

- Note that for two multi-sets f and g, the function $f + g$ mapping any element M of $\mathcal{A}$ to $f(M) + g(M)$ is still a multi-set of $\mathcal{A}$ and is called the (multi-set) union of f and g. We also define the multi-set $f \setminus g$ as the function mapping each element $M \in \mathcal{A}$ to $\mathsf{max}(f(M) - g(M), 0)$.

- We define the multi-set $\{\!\{N_1, \ldots, N_n\}\!\}$ as $f_1 + \cdots + f_n$, where for all $1 \leq i \leq n$, f_i maps N_i to 1 and every other element to 0.

- We abusively write $M \in f$ if $f(M) \neq 0$.

Definition 19 (Multi-Set reduction relation) Given $\rightarrow$ is a reduction relation on $\mathcal{A}$, we define the multi-set reduction as follows:
if f and g are multi-sets of $\mathcal{A}$, we say that $f \rightarrow_{\mathsf{mul}} g$ if there is a M in $\mathcal{A}$ such that

$$\begin{cases} f(M) = g(M) + 1 \\ \forall N \in \mathcal{A}, f(N) < g(N) \Rightarrow M \rightarrow N \end{cases}$$

We sometimes write $\rightarrow_{\mathsf{mull}}$ for $\rightarrow_{\mathsf{mul}}{}^{+}$, i.e. the transitive closure of $\rightarrow_{\mathsf{mul}}$.[10]

Example 2 (Multi-set reduction) Considering multi-sets of natural numbers, for which the reduction relation is $>$, we have for instance $\{\!\{5,7,3,5,1,3\}\!\} >_{\mathsf{mul}} \{\!\{4,3,1,7,3,5,1,3\}\!\}$. In this case, the element M is 5, and an occurrence has been "replaced" by $4,3,1$, which are all smaller than 5.

In what follows we always assume that $\mathcal{A}$ is a set with a reduction relation $\rightarrow$.

Lemma 27 *If $f_1, \ldots, f_n, g$ are multi-sets of $\mathcal{A}$ and $f_1 + \cdots + f_n \rightarrow_{\mathsf{mul}} g$ then there is $1 \leq i \leq n$ and a multi-set f'_i such that $f_i \rightarrow_{\mathsf{mul}} f'_i$ and $f_1 + \cdots + f_{i-1} + f'_i + f_{i+1} + \cdots + f_n = g$.*

Proof: We know that there is a M in $\mathcal{A}$ such that

$$\begin{cases} f_1(M) + \cdots + f_n(M) = g(M) + 1 \\ \forall N \in \mathcal{A}, f_1(N) + \cdots + f_n(N) < g(N) \Rightarrow M \rightarrow N \end{cases}$$

An easy lexicographic induction on two natural numbers p and q shows that if $p + q > 0$ then $p > 0$ or $q > 0$. By induction on the natural number n, we extend this result: if $p_1 + \cdots + p_n > 0$ then $\exists i, p_i > 0$. We apply this result on $f_1(M) + \cdots + f_n(M)$ and get some $f_i(M) > 0$. Obviously there is a unique f'_i such that $f_1 + \cdots + f_{i-1} + f'_i + f_{i+1} + \cdots + f_n = g$, and we also get $f_i \rightarrow_{\mathsf{mul}} f'_i$. $\square$

Definition 20 (Sets of multi-sets) Given two sets $\mathcal{N}$ and $\mathcal{N}'$ of multi-sets, we define $\mathcal{N} + \mathcal{N}'$ as $\{f + g \mid f \in \mathcal{N}, g \in \mathcal{N}'\}$.

We define for every M in $\mathcal{A}$ its *relative multi-sets* as all the multi-sets f of $\mathcal{A}$ such that if $N \in f$ then $M \rightarrow^* N$. We denote the set of relative multi-sets as $\mathcal{M}_M$.

Remark 28 Notice that for any $M \in \mathcal{A}$, $\mathcal{M}_M$ is stable under $\rightarrow_{\mathsf{mul}}$.

Lemma 29 *For all $M_1, \ldots, M_n$ in $\mathcal{A}$, if $\mathcal{M}_{M_1} \cup \ldots \cup \mathcal{M}_{M_n} \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$ then $\mathcal{M}_{M_1} + \cdots + \mathcal{M}_{M_n} \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$.*

Proof: Let $\mathcal{W}$ be the relation between $\mathcal{M}_{M_1} + \cdots + \mathcal{M}_{M_n}$ and $\mathcal{M}_{M_1} \times \cdots \times \mathcal{M}_{M_n}$ defined as: $f_1 + \cdots + f_n \mathcal{W} (f_1, \ldots, f_n)$ for all $f_1, \ldots, f_n$ in $\mathcal{M}_{M_1} \times \cdots \times \mathcal{M}_{M_n}$.

[10] This is the traditional multi-set order, see e.g. [Ter03].

We consider as a reduction relation on $\mathcal{M}_{M_1} \times \cdots \times \mathcal{M}_{M_n}$ the lexicographic composition of $\rightarrow_{\mathsf{mul}}$. We denote this reduction relation as $\rightarrow_{\mathsf{mullex}}$. By Corollary 24, we know that $\mathcal{M}_{M_1} \times \cdots \times \mathcal{M}_{M_n} \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mullex}}}$.
Hence, $\mathcal{W}^{-1}(\mathsf{SN}^{\rightarrow_{\mathsf{mullex}}}) = \mathcal{M}_{M_1} + \cdots + \mathcal{M}_{M_n}$.

Now we prove that $\mathcal{M}_{M_1} + \cdots + \mathcal{M}_{M_n}$ is stable by $\rightarrow_{\mathsf{mul}}$ and that $\rightarrow_{\mathsf{mullex}}$ strongly simulates $\rightarrow_{\mathsf{mul}}$ through $\mathcal{W}$. Suppose $f_1 + \cdots + f_n \rightarrow_{\mathsf{mul}} g$. By Lemma 27 we get a multi-set f'_i such that $f_1 + \cdots + f_{i-1} + f'_i + f_{i+1} + \cdots + f_n = g$ and $f_i \rightarrow_{\mathsf{mul}} f'_i$.
Hence, $f'_i \in \mathcal{M}_{M_i}$, so that $(f_1, \ldots, f_{i-1}, f'_i, f_{i+1}, \cdots, f_n) \in \mathcal{M}_{M_1} \times \cdots \times \mathcal{M}_{M_n}$ and even $(f_1, \cdots, f_n) \rightarrow_{\mathsf{mullex}} (f_1, \ldots, f_{i-1}, f'_i, f_{i+1}, \cdots, f_n)$.

By Theorem 22 we then get $\mathcal{W}^{-1}(\mathsf{SN}^{\rightarrow_{\mathsf{mullex}}}) \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$, which concludes the proof because $\mathcal{W}^{-1}(\mathsf{SN}^{\rightarrow_{\mathsf{mullex}}}) = \mathcal{M}_{M_1} + \cdots + \mathcal{M}_{M_n}$. □

Lemma 30 $\forall M \in \mathit{SN}^{\rightarrow}, \mathcal{M}_M \subseteq \mathit{SN}^{\rightarrow_{\mathit{mul}}}$

Proof: By transitive induction in $\mathsf{SN}^{\rightarrow}$. Assume that $M \in \mathsf{SN}^{\rightarrow}$ and assume the induction hypothesis $\forall N \in \rightarrow^+(M), \mathcal{M}_N \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$.

Let us split the reduction relation $\rightarrow_{\mathsf{mul}}$: if $f \rightarrow_{\mathsf{mul}} g$, let $f \rightarrow_{\mathsf{mul1}} g$ if $f(M) = g(M)$ and let $f \rightarrow_{\mathsf{mul2}} g$ if $f(M) > g(M)$. Clearly, if $f \rightarrow_{\mathsf{mul}} g$ then either $f \rightarrow_{\mathsf{mul1}} g$ or $f \rightarrow_{\mathsf{mul1}} g$. This is an intuitionistic implication since the equality of two natural numbers can be decided.

Now we prove that $\rightarrow_{\mathsf{mul1}}$ is terminating on $\mathcal{M}_M$.

Let $\mathcal{W}'$ be the following relation (actually, a function) between $\mathcal{M}_M$ to itself: for all f and g in $\mathcal{M}_M$, $f\mathcal{W}g$ if $g(M) = 0$ and for all $N \neq M$, $f(N) = g(N)$.

For a given $f \in \mathcal{M}_M$, let $N_1, \ldots, N_n$ be the elements of $\mathcal{A}$ that are not mapped to 0 by f and that are different from M. Since $f \in \mathcal{M}_M$, for all $1 \leq i \leq n$ we know $M \rightarrow^+ N_i$, and we also know that $\mathcal{W}'(f) \in \mathcal{M}_{N_1} + \cdots + \mathcal{M}_{N_n}$. Hence, we apply the induction hypothesis and Lemma 29 to get $\mathcal{M}_{N_1} + \cdots + \mathcal{M}_{N_n} \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$.
Hence, $\mathcal{W}'(f) \in \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$.

Now notice that $\rightarrow_{\mathsf{mul}}$ strongly simulates $\rightarrow_{\mathsf{mul1}}$ through $\mathcal{W}'$, so by Theorem 22, $f \in \mathsf{SN}^{\rightarrow_{\mathsf{mul1}}}$.

Now that we know that $\rightarrow'_{\mathsf{mul}}$ is terminating on $\mathcal{M}_M$, we notice that the decreasing order on natural numbers strongly simulates $\rightarrow_{\mathsf{mul2}}$ and weakly simulates $\rightarrow_{\mathsf{mul1}}$ through the function that maps every $f \in \mathcal{M}_M$ to the natural number $f(M)$.

Hence, we can apply Corollary 26 to get $\mathcal{M}_M \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$. □

Corollary 31 (Multi-Set termination) *Let f be a multi-set of $\mathcal{A}$.*
If for every $M \in f$, $M \in \mathit{SN}^{\rightarrow}$, then $f \in \mathit{SN}^{\rightarrow_{\mathit{mul}}}$.

Proof: Let $M_1, \ldots, M_n$ be the elements of $\mathcal{A}$ that are not mapped to 0 by f. Clearly, $f \in \mathcal{M}_{M_1} + \cdots + \mathcal{M}_{M_n}$. By Lemma 30, $\mathcal{M}_{M_1} \cup \ldots \mathcal{M}_{M_n} \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$, and by Lemma 29, $\mathcal{M}_{M_1} + \cdots + \mathcal{M}_{M_n} \subseteq \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$, so $f \in \mathsf{SN}^{\rightarrow_{\mathsf{mul}}}$. □