Numerical Integration

Numerical Introductory Course
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Contents

1. Abstract

1. Introduction

1. Motivation

Integration is an important operation in mathematics. Unfortunately, in real life applications one might find it extremely difficult or even impossible to solve certain integrals in a closed form. Due to the continous improvement in computational power one might address this issue by numercially approximating the integral of interest. In order to do so, several procedures have been developed, each with it's own advantages respectively disadvantages.

2. Literature Review

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3. Theory

In the following section I want to explain some of the most popular methods in numerical integration. These can be distingushed into one and multi-dimensional methods. Furthermore one might distinguesh numerical integration methods further into deterministic and probabilistic methods. But before I start introducing the methods of interest I will do a little recap of the bascis:

3.1 Review: The Riemann Integral

The Riemann Integral is one of the two classic concepts of integrals in analysis. It is named after the German mathematician Bernhard Riemann and it's aim is to calculate the area between the x-axis and a certain limited function $f : [a; b] \to \mathbb{R}$. Loosley speaking, the basic idea behind the concept is to approximate the desired integral by summing up different areas of easier to compute rectangles.

The kind of definition I want to present here is the definition using upper and lower sums introduced by Jean Gaston Darboux:

Let $f : [a; b] \to \mathbb{R}$ be a limited function and $[a; b]$ be an interval. Furthermore, let P be a partition of $[a; b]$ where $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. Then we can define the upper and lower sums accordingly:

$$U(P) = \sum_{k=1}^{n}((x_k - x_{k-1}) \cdot \sup_{x_{k-1}<x<x_k} f(x))$$

$$L(P) = \sum_{k=1}^{n}((x_k - x_{k-1}) \cdot \inf_{x_{k-1}<x<x_k} f(x))$$

Now we can compute the infimum and supremum of the upper and lower sum over all partitions P. Therefore it follows:

$$\sup_P L(P) \leq \inf_P U(P)$$

In case of equality, on says that f in Riemann integrable.

3.2 One-Dimensional Procedures

The one dimensional procedurees elaborated on in this chapter are classified as deterministic methods. Throughout this document the function $f : [-4; 4] \to \mathbb{R}$ with $f(x) = x^2 + 3 \cdot x + 4$ is used for visualizing the procedures introduced.

3.2.1 Midpoint or Rectangular Quadrature

The idea of the Midpoint or Rectangular Quadrature directly derives from the definition of the Riemann Integral. We therefore want to calculate the area between the x-axis and a limited function $f : [a; b] \to \mathbb{R}$. The algorithm works in the way, that we start by partitioning our interval of interest $[a; b]$ into $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. Afterwards we calculate the midpoint $x^{(i)}$ within each subinterval $[x_i; x_{i+1}]$ for $i \in \{0, 1, ..., n-1\}$ and evaluate f for each $x^{(i)}$. For our approximation it then holds that $\int_a^b f(x)dx \approx \sum_{k=0}^{n-1} f(x^{(i)} \cdot (x_{i+1} - x_i)$.

An illustration of the procedure can be found in Figure 1.

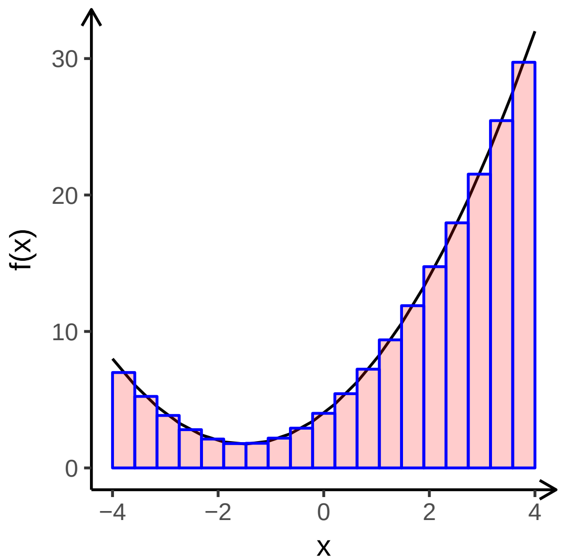

Figure 1: Illustration of the Midpoint or Rectangular Quadrature

3.2.2 Simpson-Rule

The Simposn-Rule is similiar to the Rectangular Quadrature, but instead of rectangles quadratic functions are used in order to calculate the area between the x-axis and our limited function $f : [a; b] \to \mathbb{R}$ more accurately. We again start by partitioning our interval of interest $[a; b]$ into $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ with aquidistant distances which we will in the following denote by Δx. Afterwards we calculate the midpoint $x^{(i)}$ within each subinterval $[x_i; x_{i+1}]$ for $i \in \{0, 1, ..., n-1\}$. Now we use the 3 points $(x_i; f(x_i))$, $\left(x^{(i)}; f(x^{(i)})\right)$ and $(x_{i+1}; f(x_{i+1}))$ within each subinterval to interpolate our quadratic functions $g_i(x) : [x_i; x_{i+1}] \to \mathbb{R}$. For our approximation it then holds that $\int_a^b f(x)dx \approx \frac{\Delta x}{6} \cdot \left(f(x_0) + 2 \cdot \sum_{k=1}^{n-1} f(x_k) + f(x_n) + 4 \cdot \sum_{k=0}^{n-1} f(x^{(k)})\right)$.

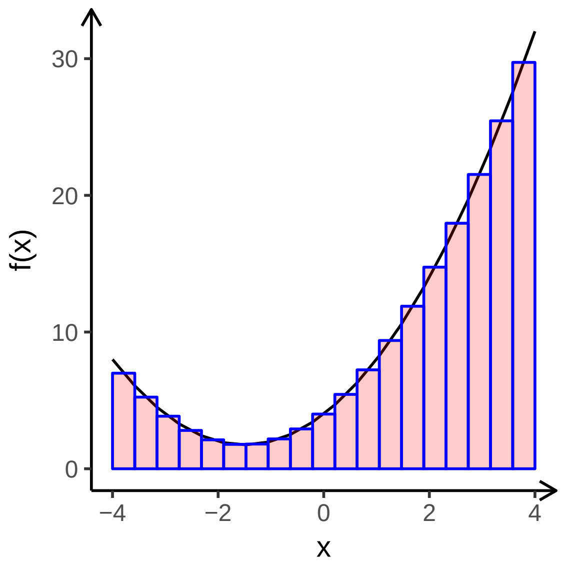

Figure 2: Illustration of the Midpoint or Rectangular Quadrature

3.3 Multi-Dimensional Procedures

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3.4 Integrals over infinite Intervals

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4. Application: Approximation of the Normal Distribution

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5. Conclusion

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6. Bibliography

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