Cubic Surfaces III

The 27 lines on the blown up projective plane

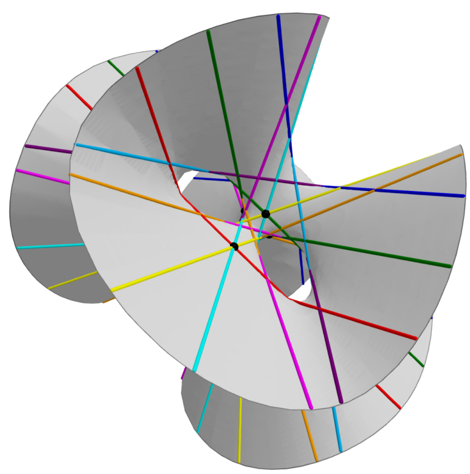
MAT507 Algebraic Geometry I

Marvin Sigg

December 12, 2025

Abstract

After giving some background information, we verify that the induced morphism from the projective plane blown up at six base points in non-degenerate configuration to the projective 3-space embeds the blow-up as a smooth cubic surface. Afterwards we count the number of lines on the surface via basic intersection theory.



Contents

1 Introduction

One of the first results of a highly non-trivial fashion arising from projective geometry is the following:

Theorem 1.1. *Any smooth cubic projective surface contains exactly 27 lines.*

This result seems to be as striking now as it was back when it was demonstrated by Cayley and Salmon in 1849. Since then, a number of proofs have been developed, all using different approaches, language, and machinery. In this talk, we will take a somewhat streamlined classic route, and only occasionally reference the modern language of scheme theory. We will not go the whole way, as this talk is embedded in the middle of a five-part series.

In a nutshell, we will show that the image of blowing up the projective plane at six suitable points is, when embedded in projective 3D-space, a smooth cubic surface. In the main part, we will demonstrate that this surface contains exactly 27 lines. We will use several results from previous talks as prerequisite. Meanwhile, a later talk will demonstrate that any smooth cubic surface can be obtained this way, completing the proof of Theorem 1.1.

This write-up aims to strike a balance between presenting the strategy in a linear fashion, and including an appropriate amount of background explanations and didactic exposé. Throughout, we will be closely following [1].

2 Setting and Preliminaries

Construction 2.1. *The general setup is as follows: Fix six distinct points $p_1, .., p_6 \in \mathbb{P}^2$ called the base points/exceptional points, satisfying the following "generality" condition:*

$$\left.\begin{array}{l}\bullet\ \textit{no three of them lie on a line}\\ \bullet\ \textit{they do not all lie on a conic}\end{array}\right\} (*)$$

We write $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ for the blow-up of $\mathbb{P}^2$ at the base points.

The generality condition $(*)$, as we will see, is necessary to ensure that our blown up surface is not degenerate and in fact a smooth cubic.

In the preceding talk, we have seen the following very useful statement:

Theorem 2.2 (Cubic Surfaces II). *Let X be a projective variety. A morphism $\varphi : X \to \mathbb{P}^n$ maps X isomorphically to its image in $\mathbb{P}^n$ if and only if the linear forms:*

(i) separate points[1]

(ii) separate tangent vectors[2]

[1] i.e., for every pair of distinct points P and Q in X, there is a linear form vanishing at P but not at Q

[2] i.e., for every point P the linear forms vanishing at P span $\mathfrak{m}_P/(\mathfrak{m}_P)^2$, where $\mathfrak{m}_P$ denotes the maximal ideal in the local ring of X at P

This is our key to show that our blow-up is a cubic surface embedded in projective space. We will apply a particularly rephrased variant of Theorem 2.2 suitable for our needs, but for this we need to introduce some new notations.

Notation 2.3. *Let $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the base points $p_1, \ldots, p_6$. Let E_i denote the exceptional divisor over p_i. Let H be the (unique) class of lines in $\mathbb{P}^2$. Consider the complete linear system $|\widetilde{C}|$ defined by*

$$\widetilde{C} = \pi^* 3H - E_1 - \cdots - E_6 \,.$$

Let us unravel this definition by expanding upon implicit assumptions, and recalling and combining definitions we have seen before. Remember that, as a rough picture, one may view $\widetilde{\mathbb{P}^2}$ by taking $\mathbb{P}^2$ and replacing the base points by copies of the projective line $\mathbb{P}^1$. Next, we shall recall basic notions seen in the *Divisors* talk, but already coined onto our setting:

Definition 2.4. A **divisor** D on a variety is a (locally finite) formal linear combination of (irreducible) hypersurfaces V_i, $D = \sum_i' a_i V_i$. A **principal divisor** is a divisor D of the form $D = (f) =$ (zeroes of f) $-$ (poles of f) for some regular function f, given by the valuation.

Next, recall the following from the *Blow-Up* talk:

Definition 2.5. Let C be a curve in $\mathbb{P}^2$. The **total transform** of C is its preimage by the blow-up, $\pi^* C := \pi^{-1}(C)$. The **strict transform** of C is the Zariski closure of its preimage sans the base points, $\widetilde{C} := \overline{\pi^{-1}(C \backslash (\bigcup_{i=1}^6 p_i))}$.

Strict transforms of smooth curves in $\mathbb{P}^2$ are smooth curves in $\widetilde{\mathbb{P}^2}$ which do not intersect the exceptional divisors, hence π maps them isomorphically back to the original curve minus the base points.

Definitions 2.4 and 2.5 help us piece together what is going on: We implicitly construct a curve $C \subset \mathbb{P}^2$ by defining its strict transform as a curve (better yet: a divisor!) $\widetilde{C} \subset \widetilde{\mathbb{P}^2}$. Take the total transform of three times the class[3] of the line (that is, a cubic curve), and then subtract the exceptional divisors. The final puzzle piece is the following:

Definition 2.6. A **(complete) linear system** $|D|$ is the family of divisors[4] that are linearly equivalent to some divisor D.

To summarize, $\widetilde{C}$ *is the class of strict transforms of cubics through the base points* $p_1, \ldots, p_6$, while $|\widetilde{C}|$ consists of all divisors on $\widetilde{\mathbb{P}^2}$ in the class of $\widetilde{C}$.

Remark 2.7. *The family of all cubics in three variables (as we are working in $\mathbb{P}^2$) is*

$$k[x_0, x_1, x_2]_3 = span(x_0^3, x_1^3, x_2^3, x_0^2 x_1, x_0^2 x_2, x_1^2 x_0, x_1^2 x_2, x_2^2 x_0, x_0^2 x_1, x_0 x_1 x_2) \,,$$

[3]that is, the equivalence class under linear equivalence: $D \sim D' \iff \exists f : D - D' = (f)$

[4]technically we restrict to *effective* divisors, where we require $a_i > 0$ for all non-zero coefficients a_i.

which naturally forms a 10-dimensional k-vector space. Since we view this as the coordinate ring in the projective plane $\mathbb{P}^2$*, it forms a 9-dimensional linear system (*$3H$*), removing one degree of freedom by scaling. When not stated otherwise, the notion of dimension will refer to linear systems, rather than vector spaces.*

Therefore, Construction 2.1 yields the linear system $|\widetilde{C}|$ *of cubics passing through the six points* $p_1, \dots, p_6$*. That is to say,* $|\widetilde{C}|$ *consists exactly of curves of the form* $\pi^{-1}(C) - E_1 - \dots - E_6$*, where* C *is a cubic curve in* $\mathbb{P}^2$ *containing* $p_1, \dots, p_6$*. The requirement for the cubics to vanish at all ponts yields six independent linear equations, which each reduce the dimension by 1, therefore* $\dim|\widetilde{C}| = 3$.

Choosing a basis of the 4-dimensional vector space, say F_0, F_1, F_2, F_3 (which are linearly independent homogeneous polynomials of degree 3 that vanish at all six base points), induces the rational map

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\ x := [x_0 : x_1 : x_2] &\longmapsto [F_0(x) : F_1(x) : F_2(x) : F_3(x)], \end{aligned}$$

which is not a morphism, since it is not defined at $p_1, \dots, p_6$. Then, as we have seen in *Cubic II*, blowing up $\mathbb{P}^2$ at $p_1, \dots, p_6$ yields the morphism

$$\varphi : \widetilde{\mathbb{P}^2} \longrightarrow \mathbb{P}^3$$

that is central to our endeavour.

3 Proof of the Embedding Claim

The first of the two main tasks is to verify that the morphism φ satisfies the conditions of Theorem 2.2. A complication in this matter is that we need to distinguish between different cases, namely whether the points on $\widetilde{\mathbb{P}^2}$ to be separated are ordinary or exceptional. The good news is that the argument runs fairly similar for separating tangent vectors as well, as they are represented as elements of the blow-up. To this end, we introduce a final bit of notion:

Definition 3.1. We say that a point q is **infinitely near** p if q is a point on the exceptional divisor E of the blow-up $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ of $\mathbb{P}^2$ at p.

We are now ready to state the central claim.

Claim 3.2. *The linear system* $|\widetilde{C}|$ *embeds* $\widetilde{\mathbb{P}^2}$ *as a cubic surface in* $\mathbb{P}^3$*. In particular, we claim that*

1. $\widetilde{C} \cdot \widetilde{C} = 3$ *(the degree of the closure of the image is 3)*
2. $\dim|\widetilde{C}| = 3$

3. φ separates distinct points $P, Q \in \widetilde{\mathbb{P}^2}$, for[5]

3a) $P, Q \in \mathbb{P}^2 \backslash \{p_1, \ldots, p_6\}$

3b) P infinitely near p_1, $Q \in \mathbb{P}^2 \backslash \{p_1, \ldots, p_6\}$

3c) P infinitely near p_1, Q infinitely near p_2

3d) P, Q both infinitely near p_1

4. φ separates tangent vectors at $P \in \widetilde{\mathbb{P}^2}$, for

4a) P infinitely near $Q \in \mathbb{P}^2 \backslash \{p_1, \ldots, p_6\}$

4b) P infinitely near Q which is infinitely near p_1

Note that P *and* Q *get separated if and only if* $p_1, \ldots, p_6, P, Q$ *determine a 1-dimensional system.* Further recall the following lemma from the previous talk:

Lemma 3.3 (Cubic Surfaces II)**.** *Consider points on the projective plane $\mathbb{P}^2$.*

(i) Seven distinct points $q_1, .., q_7$ define a 2-dimensional linear system of cubic curves if and only if no five of them lie on a line.

(ii) Six distinct points $q_1, .., q_6$ with a tangent direction at q_1, specified by line l through q_1, define a 2-dimensional linear system of cubic curves if and only if no five of $q_1, .., q_6$ lie on a line and no three of $q_2, .., q_6$ lie on l.

This result, together with the preceding statement, is key to our proof. For Claim 3, in each case we show that $p_1, .., p_6, P, Q$ determine a 1-dimensional linear system, that is to say, they impose an "independence condition".

Definition 3.4. Let r be a natural number smaller than 9. We say points $q_1, \ldots, q_r \in \mathbb{P}^2$ impose the **independence condition** if the linear system they determine has dimension $9-r$. Conversely, we say that $q_1, \ldots, q_r$ *fail* the condition if the linear system they determine has dimension bigger than $9-r$.

Note that by $(*)$ the six base points impose the independence condition, so the linear system they determine has dimension 3. This takes care of Claim 2.

We proceed by showing that the seven distinct points $p_2, .., p_6, P, Q$ determine a 2-dimensional system, in which there exists a member that does not contain p_1, after appropriate reordering. In the following, the statement is shown for Claim 3a). The remaining subclauses 3b) - 4b) are dealt with in a very similar manner, by noting that the independence condition and Lemma 3.3 are applicable even if at most two points are infinitely near other points.

[5] Without loss of generality, we can reorder the base points as needed.

Proof. Claim 3a) Let P, Q be (ordinary) points in $\mathbb{P}^2$. By $(*)$, at most two of the points $p_2, \ldots, p_6$ are collinear. Therefore, even if P and Q were collinear with some p_i, $i \neq 1$, at most four of the points $p_2, \ldots, p_6, P, Q$ are collinear. So by Lemma 3.3, $p_2, \ldots, p_6, P, Q$ determine a 2-dimensional linear system (i.e. they satisfy the independence condition).
Now, let C be the unique conic through $p_4, \ldots, p_6, P, Q$, if it exists. If it does not, let C be a union of two lines[6] passing through $p_4, \ldots, p_6, P, Q$. At least one of p_1, p_2, p_3 is not on C, say p_1 (otherwise, either all 6 p_i's lie on a conic, or three lie on the same line, violating $(*)$ in both cases). Define L to be the line connecting p_2 and p_3; L does not contain p_1 by $(*)$. Then is $C \cup L$ a member of the 2-dimensional system which does not contain p_1. □

To conclude, any two points on the blow-up, together with the six base points satisfying $(*)$, impose the independence condition. *Therefore, φ embeds $\widetilde{\mathbb{P}^2}$ as a cubic surface S into $\mathbb{P}^3$.*

4 Counting the Lines

After constructing the cubic surface S, it is now a fairly simple matter to explicitly produce the lines sitting on S. However, the recalling of another notion is needed.

Definition 4.1. The **intersection number** $D \cdot D'$ of two divisors D, D' is, roughly speaking, a measure of how many times they geometrically intersect.

Note that this mapping is bilinear. We state the following auxiliary remark without proof:

Remark 4.2. *Let D, D' be any divisors on $\mathbb{P}^2$. The following identities are useful to speed up the computation:*

- $V_1 \cdot V_2 = \deg(V_1)deg(V_2)$ *for any two distinct ordinary subvarieties, by Bézout's Theorem*[7]*. In particular if L, L' are distinct lines and C, C' distinct conics in $\mathbb{P}^2$, then $L \cdot L' = 1$, $L \cdot C = 2$, and $C \cdot C' = 4$. If $3H$ the generic class of cubics as above, then $C \cdot 3H = 6$.*
- $\pi^* D \cdot \pi^* D' = D \cdot D'$.
- *If D is a non-exceptional divisor, it has **self-intersection** $D \cdot D = \deg(D)^2$. Of course, any subvariety intersects with itself infinitely many times. The rough idea of self-intersection is to count how many times D intersects with a "slightly perturbed" version of itself; at this stage we can invoke Bézout's Theorem.*
- $E_i \cdot E_j = -\delta_{ij}$ *since distinct exceptional divisors do not intersect. Exceptional divisors have "self-intersection -1" for reasons beyond the scope of this write-up. Furthermore, exceptional divisors do not intersect with generic non-excpetional divisors as well.*

Now we are equipped and ready for the final strike.

[6] if $p_4, \ldots, p_6, P, Q$ fail to define a conic, three of them are collinear.
[7] see the talk next week.

Theorem 4.3. *The cubic surface $S \cong \widetilde{\mathbb{P}^2}$ in $\mathbb{P}^3$ contains exactly 27 lines.*

Proof. We are exhibiting varies divisors in $\widetilde{\mathbb{P}^2}$ and showing that they intersect with the class of cubics running through the base points, from which the embedding arises, exactly once. This shows that, embedded in $\mathbb{P}^3$, those divisors have degree 1, making them a line sitting on S.

- *6 lines from the exceptional divisors:* Fix an exceptional divisor E_i. Then

$$\begin{aligned} E_i \cdot \widetilde{C} &= E_i \cdot \Big(\pi^* 3H - \sum_{k=1}^{6} E_k\Big) \\ &= -E_i \cdot E_i = 1\,, \end{aligned}$$

  with negative self-intersection and all other terms vanishing, by Remark 4.2. There are 6 exceptional divisors.

- *15 lines from lines through base points:* Let F_{ij} be the proper transform in $\widetilde{\mathbb{P}^2}$ of the line L_{ij} in $\mathbb{P}^2$ running through the i-th and j-th base point. Then

$$\begin{aligned} F_{ij} \cdot \widetilde{C} &= \big(\pi^* L_{ij} - E_i - E_j\big) \cdot \Big(\pi^* 3H - \sum_{k=1}^{6} E_k\Big) \\ &= L_{ij} \cdot 3H + E_i \cdot E_i + E_j \cdot E_j = 3 - 2 = 1\,, \end{aligned}$$

  with the pull-back property mentioned in Remark 4.2. There are $\binom{6}{2} = 15$ such distinct lines L_{ij} in $\mathbb{P}^2$, by $(*)$.

- *6 lines from conics through five base points:* Let G_i be the proper transform in $\widetilde{\mathbb{P}^2}$ of the conic C_i in $\mathbb{P}^2$ running through all base points with the exception of p_i (which exists by $(*)$). Then

$$\begin{aligned} G_i \cdot \widetilde{C} &= \Big(\pi^* C_i - \sum_{j \neq i}^{6} E_j\Big) \cdot \Big(\pi^* 3H - \sum_{k=1}^{6} E_k\Big) \\ &= C_i \cdot 3H + \sum_{k \neq i} E_k = 6 - 5 = 1\,. \end{aligned}$$

  There are $\binom{6}{5} = 6$ such conics in $\mathbb{P}^2$.

Finally, we show that the number $6 + 15 + 6 = 27$ is not just a lower bound, but the exact number of lines on S. Let L be any line not arising from an exceptional divisor. Then must the locus $\pi(L) \subset \mathbb{P}^2$ be a line or a conic[8]. We can decompose L as follows:

$$L = \pi^* \pi(L) - \sum_{p_i \in \pi(L)} E_i\,.$$

[8] by the so-called genus formula.

By assumption, L intersects $\widetilde{C}$ exactly once, so

$$\begin{aligned} 1 \overset{!}{=} L \cdot \widetilde{C} &= \Big(\pi^*\pi(L) - \sum_{p_i \in \pi(L)} E_i\Big) \cdot \Big(\pi^* 3H - \sum_{k=1}^{6} E_k\Big) \\ &= 3H \cdot \pi(L) + \sum_{p_i \in \pi(L)} E_i \cdot E_i \end{aligned}$$

There are two cases to consider. If $\pi(L)$ is a line, then

$$1 \overset{!}{=} 3 + \sum_{p_i \in \pi(L)} (-1)\,,$$

so $\pi(L)$ must contain exactly two base points. If $\pi(L)$ is a line, then

$$1 \overset{!}{=} 6 + \sum_{p_i \in \pi(L)} (-1)\,,$$

so $\pi(L)$ must contain exactly five base points. Hence L must be a line of the form F_{ij} or G_i, so there are no other lines. □

We conclude: *There are exactly 27 lines on the cubic surface which arose from the blow-up! There are 6 E_i's, 15 F_{ij}'s, and 6 G_i's.*

In the next two talks, we will see that every cubic surface in $\mathbb{P}^3$ can be constructed via blow-up, and discuss further properties of the lines on the cubic.

References

[1] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry.* 1st ed. John Wiley & Sons, Inc., 1994.