NA 568 - Winter 2022

Invariant Kalman Filtering

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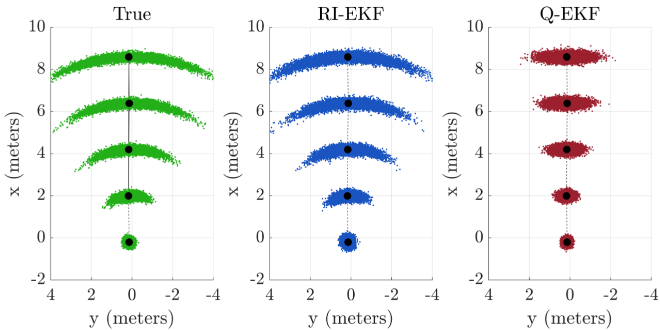
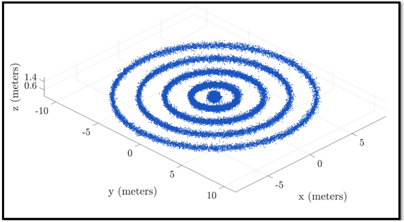
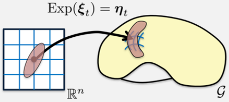
- For a large class of systems defined on matrix Lie groups, the machinery of geometry provides natural coordinates that exploits symmetries of the space.
- The theory of invariant observer design is based on the estimation error being invariant under the action of a matrix Lie group.
- The fundamental result is that by correct parametrization of the error variable, a wide range of nonlinear problems can lead to linear error equations (what's not to like!?).





$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}_t = \mathbf{A}_t \boldsymbol{\xi}_t$$
$$\exp(\boldsymbol{\xi}_t) = \bar{\mathbf{X}}_t \, \mathbf{X}_t^{-1}$$



Consider a deterministic LTI process model $\dot{x} = Ax + Bu$. Let $\bar{x}$ be an estimate of x, i.e., $\dot{\bar{x}} = A\bar{x} + Bu$.

- Define the error $e := x - \bar{x}$.
- Then $\dot{e} = \dot{x} - \dot{\bar{x}} = A(x - \bar{x}) = Ae$ (an autonomous differential equation).
- Given an initial condition $e(0) = e_0$, we can solve for the error at any time $e(t) = \exp(At)e_0$.
- Error propagation is independent of the system trajectory (state estimate).

A Motivating Example: 3D Orientation Propagation

Suppose we wish to estimate the 3D orientation of a rigid body given angular velocity measurements in the body frame, $\omega_t := \mathrm{vec}(\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$. This type of measurement can be easily obtained from a gyroscope.

Figure: Vectornav VN-100 Inertial Measurement Unit.

A Motivating Example: 3D Orientation Propagation

If we let $q_t := \mathrm{vec}(q_x, q_y, q_z)$ be a vector of Euler angles using the $R = R_z R_y R_x$ convention, then the orientation dynamics can be expressed as

$$\frac{d}{dt}\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} 1 & \sin(q_x)\tan(q_y) & \cos(q_x)\tan(q_y) \\ 0 & \cos(q_x) & -\sin(q_x) \\ 0 & \sin(q_x)\sec(q_y) & \cos(q_x)\sec(q_y) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

Remark

$\omega^b = R^\mathsf{T}\dot{R} = R^\mathsf{T}\frac{dR}{dq}\cdot\frac{dq}{dt} = R^\mathsf{T}\sum_{i,j,k=1}^{3}\frac{dRij}{dq_k}\cdot\frac{dq_k}{dt} =: E^{-1}(q)\dot{q}$.
Then we have $\dot{q} = E(q)\omega^b$.

To generate this matrix, see https://github.com/RossHartley/angles.

A Motivating Example: 3D Orientation Propagation

Let $\delta q_t := q_t - \bar{q}_t \in \mathbb{R}^3$ be the error between the true and estimated Euler angles.

The error dynamics can be written as a nonlinear function of the error variable, the inputs, and the state

$$\frac{d}{dt}\delta q_t = g(\delta q_t, \omega_t, q_t).$$

A Motivating Example: 3D Orientation Propagation

To propagate the covariance in an EKF, we need to linearize the error dynamics at the current state estimate, $q_t = \bar{q}_t$ (i.e. zero error).

This leads to a linear error dynamics of the form:

$$\begin{aligned}\frac{d}{dt}\delta q_t &\approx \begin{bmatrix} 0 & (\omega_z \bar{c}_x + \omega_y \bar{s}_x)/\bar{c}_y^2 & \bar{t}_y(\omega_y \bar{c}_x - \omega_z \bar{s}_x) \\ 0 & 0 & \omega_z \bar{c}_x + \omega_y \bar{s}_x \\ 0 & (\bar{s}_y(\omega_z \bar{c}_x + \omega_y \bar{s}_x))/\bar{c}_y^2 & (\omega_y \bar{c}_x - \omega_z \bar{s}_x)/\bar{c}_y \end{bmatrix} \delta q_t \\ &=: A(\omega_t, \bar{q}_t)\delta q_t,\end{aligned}$$

where $\bar{c}_x$, $\bar{s}_x$, and $\bar{t}_x$ are shorthand for $\cos(\bar{q}_x)$, $\sin(\bar{q}_x)$, and $\tan(\bar{q}_x)$.

A Motivating Example: 3D Orientation Propagation

- $$\frac{d}{dt}\delta q_t := A(\omega_t, \bar{q}_t)\delta q_t,$$

- The linear dynamics matrix, $A(\omega_t, \bar{q}_t)$, clearly depends on the estimated angles.

- Therefore, bad estimates will affect the accuracy of the linearization and ultimately the performance and consistency of the filter.

- A process dynamics evolving on the Lie group, for state $X_t \in \mathcal{G}$, is

$$\frac{d}{dt}X_t = f_{u_t}(X_t).$$

- $\bar{X}_t$ denotes an estimate of the state.
- The state estimation error is defined using right or left multiplication of X_t^{-1}.

Left and Right Invariant Error

Definition (Left and Right Invariant Error)

The right- and left-invariant errors between two trajectories X_t and $\bar{X}_t$ are:

$$\eta_t^r = \bar{X}_t X_t^{-1} = (\bar{X}_t L)(X_t L)^{-1} \quad \text{(Right-Invariant)}$$
$$\eta_t^l = X_t^{-1} \bar{X}_t = (L\bar{X}_t)^{-1}(L X_t), \quad \text{(Left-Invariant)}$$

where $L \in \mathcal{G}$ is an arbitrary element of the group.

Theorem (Autonomous Error Dynamics)

A system is group affine if the dynamics, $f_{u_t}(\cdot)$, satisfies:

$$f_{u_t}(X_1 X_2) = f_{u_t}(X_1) X_2 + X_1 f_{u_t}(X_2) - X_1 f_{u_t}(I) X_2$$

for all $t > 0$ and $X_1, X_2 \in \mathcal{G}$. Furthermore, if this condition is satisfied, the right- and left-invariant error dynamics are trajectory independent and satisfy:

$$\frac{d}{dt}\eta_t^r = g_{u_t}(\eta_t^r) \quad \text{where} \quad g_{u_t}(\eta^r) = f_{u_t}(\eta^r) - \eta^r f_{u_t}(I)$$

$$\frac{d}{dt}\eta_t^l = g_{u_t}(\eta_t^l) \quad \text{where} \quad g_{u_t}(\eta^l) = f_{u_t}(\eta^l) - f_{u_t}(I)\eta^l$$

Define A_t to be a $\dim \mathfrak{g} \times \dim \mathfrak{g}$ matrix satisfying

$$g_{u_t}(\exp(\xi)) := (A_t \xi)^\wedge + \mathcal{O}(||\xi||^2).$$

For all $t \geq 0$, let $\xi_t \in \mathbb{R}^{\dim \mathfrak{g}}$ be the solution of the linear differential equation $\frac{d}{dt}\xi_t = A_t \xi_t$.

Theorem (Log-Linear Property of the Error)

Consider the right-invariant error, η_t, between two trajectories (possibly far apart). For arbitrary initial error $\xi_0 \in \mathbb{R}^{\dim \mathfrak{g}}$, if $\eta_0 = \exp(\xi_0)$, then for all $t \geq 0$,

$$\eta_t = \exp(\xi_t);$$

that is, the nonlinear estimation error η_t can be exactly recovered from the time-varying linear differential equation.

Differential Equation of a Curve in Lie Groups

- For a curve $g(t) \in \mathcal{G}$, we have
  $\xi(t) = g(t)^{-1} \cdot \dot{g}(t); \quad i.e., \quad \xi(t) = \left(\ell_{g^{-1}}\right)_* \dot{g}(t).$

- The reasoning behind using $g^{-1}\dot{g}$ rather than just $\dot{g}$ is because $\dot{g} \in T_g\mathcal{G}$ and $g^{-1} : T_g\mathcal{G} \to T_e\mathcal{G} = \mathfrak{g}$ and therefore $g^{-1}\dot{g} \in \mathfrak{g}$.

Suppose we are interested in estimating the 3D orientation of a rigid body given angular velocity measurements in the body frame, $\omega_t := \mathrm{vec}(\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$. This type of measurement can be easily obtained from a gyroscope.

Using a rotation matrix, $R_t \in \mathrm{SO}(3)$. The dynamics becomes

$$\frac{d}{dt} R_t = R_t \omega_t^\wedge.$$

If we define the error between the true and estimated orientation as $\eta_t := R_t^{\mathsf{T}} \bar{R}_t \in \mathrm{SO}(3)$, then the (left-invariant) error dynamics becomes

$$\begin{aligned} \frac{d}{dt}\eta_t &= R_t^{\mathsf{T}} \frac{d}{dt}\bar{R}_t + \frac{d}{dt}R_t^{\mathsf{T}} \bar{R}_t = R_t^{\mathsf{T}} \bar{R}_t \omega_t^{\wedge} + \left(R_t \omega_t^{\wedge}\right)^{\mathsf{T}} \bar{R}_t \\ &= R_t^{\mathsf{T}} \bar{R}_t \omega_t^{\wedge} - \omega_t^{\wedge} R_t^{\mathsf{T}} \bar{R}_t \\ &= \eta_t \omega_t^{\wedge} - \omega_t^{\wedge} \eta_t \\ &= g(\eta_t, \omega_t). \end{aligned}$$

Using this particular choice of state and error variable yields an autonomous error dynamics function (independent of the state directly).

Example: 3D Orientation Propagation

Now let $\eta_t := \exp(\xi_t)$. Using the first-order approximation for the exponential map, $\exp(\xi_t) \approx I + \xi_t^\wedge$, we have

$$\begin{aligned}
\frac{d}{dt}(\exp(\xi_t)) &= \exp(\xi_t)\omega_t^\wedge - \omega_t^\wedge \exp(\xi_t) \\
\frac{d}{dt}\left(I + \xi_t^\wedge\right) &\approx \left(I + \xi_t^\wedge\right)\omega_t^\wedge - \omega_t^\wedge \left(I + \xi_t^\wedge\right) \\
\frac{d}{dt}\xi_t^\wedge &= \xi_t^\wedge\omega_t^\wedge - \omega_t^\wedge\xi_t^\wedge = (-\omega_t^\wedge\xi_t)^\wedge \\
\frac{d}{dt}\xi_t &= -\omega_t^\wedge\xi_t
\end{aligned}$$

Remark

For all $a,b \in \mathbb{R}^3$, we have $a^\wedge b^\wedge - b^\wedge a^\wedge = [a^\wedge, b^\wedge] = (a \times b)^\wedge$.

Proof.

Let $\eta_0 = \exp(\xi_0)$ be the initial left invariant error. We can show that $\eta_t = R_t^\mathsf{T} \eta_0 R_t$ is the solution to the error dynamics via differentiation.

$$\frac{d}{dt}\eta_t = R_t^\mathsf{T} \eta_0 R_t \omega_t^\wedge - \omega_t^\wedge R_t^\mathsf{T} \eta_0 R_t = \eta_t \omega_t^\wedge - \omega_t^\wedge \eta_t,$$

and

$$\begin{aligned} \eta_t &= R_t^\mathsf{T} \eta_0 R_t \\ \exp(\xi_t) &= R_t^\mathsf{T} \exp(\xi_0) R_t \overset{\text{Adjoint}}{=} \exp(R_t^\mathsf{T} \xi_0) \\ \xi_t &= R_t^\mathsf{T} \xi_0 \end{aligned}$$

By differentiating, we get the log-linear error dynamics.

$$\frac{d}{dt}\xi_t = -\omega_t^\wedge R_t^\mathsf{T} \xi_0 = -\omega_t^\wedge \xi_t$$

□

Associated Noisy System

- A noisy process dynamics evolving on the Lie group, take the following form:

$$\frac{d}{dt}X_t = f_{u_t}(X_t) + X_t w_t^\wedge$$

- $w_t^\wedge \in \mathfrak{g}$ is a continuous white noise whose covariance matrix is denoted by Q_t.

- $\frac{d}{dt}\eta_t^r = g_{u_t}(\eta_t^r) - (\bar{X}_t w_t^\wedge \bar{X}_t^{-1})\eta_t^r = g_{u_t}(\eta_t^r) - (\mathrm{Ad}_{\bar{X}_t} w_t)^\wedge \eta_t^r$

- $\frac{d}{dt}\eta_t^l = g_{u_t}(\eta_t^l) - w_t^\wedge \eta_t^l$

- If observations take a particular form, then the linearized observation model and the innovation will also be autonomous.
- This happens when the measurement, Y_{t_k}, can be written as either

$$Y_{t_k} = X_{t_k} b + V_{t_k} \quad \text{(Left-Invariant Observation)} \quad \text{or}$$
$$Y_{t_k} = X_{t_k}^{-1} b + V_{t_k} \quad \text{(Right-Invariant Observation)}.$$

- b is a constant vector and V_{t_k} is a vector of Gaussian noise.

Left-Invariant Extended Kalman Filter

- Propagation:

$$\frac{d}{dt}\bar{X}_t = f_{u_t}(\bar{X}_t), \quad t_{k-1} \leq t < t_k$$

$$\frac{d}{dt}\eta^l_t = g_{u_t}(\eta^l_t) - w^\wedge_t \eta^l_t \implies \frac{d}{dt}\xi^l_t = A^l_t \xi^l_t - w_t$$

$$\frac{d}{dt}P^l_t = A^l_t P^l_t + P^l_t {A^l_t}^\mathsf{T} + Q_t$$

- Update: We use $\exp(\xi) \approx I + \xi^{\wedge}$ and neglect the higher order terms.

$$\bar{X}_{t_k}^{+} = \bar{X}_{t_k} \exp\left(L_{t_k}\left(\bar{X}_{t_k}^{-1} Y_{t_k} - b\right)\right)$$

$$X_{t_k}^{-1}\bar{X}_{t_k}^{+} = X_{t_k}^{-1}\bar{X}_{t_k} \exp\left(L_{t_k}\left(\bar{X}_{t_k}^{-1}(X_{t_k} b + V_{t_k}) - b\right)\right)$$

$$\eta_{t_k}^{l+} = \eta_{t_k}^{l} \exp\left(L_{t_k}\left((\eta_{t_k}^{l})^{-1} b - b + \bar{X}_{t_k}^{-1} V_t\right)\right)$$

$$I + \xi_{t_k}^{l+\wedge} = (I + \xi_{t_k}^{l\ \wedge}) \left(I + \left(L_{t_k}\left((I - \xi_{t_k}^{l\ \wedge}) b - b + \bar{X}_{t_k}^{-1} V_t\right)\right)^{\wedge}\right)$$

$$\xi_{t_k}^{l+\wedge} = \xi_{t_k}^{l\ \wedge} + \left(L_{t_k}\left((I - \xi_{t_k}^{l\ \wedge}) b - b + \bar{X}_{t_k}^{-1} V_t\right)\right)^{\wedge}$$

$$\xi_{t_k}^{l+} = \xi_{t_k}^{l} + L_{t_k}\left(-\xi_{t_k}^{l\ \wedge} b + \bar{X}_{t_k}^{-1} V_t\right)$$

- Update: Define the measurement Jacobian, H, such that $H\xi = \xi^{\wedge} b$.

$$\bar{X}^{+}_{t_k} = \bar{X}_{t_k} \exp\left(L_{t_k}\left(\bar{X}^{-1}_{t_k} Y_{t_k} - b\right)\right)$$

$$\xi^{l+}_{t_k} = \xi^{l}_{t_k} - L_{t_k} H \xi^{l}_{t_k} + L_{t_k} \bar{X}^{-1}_{t_k} V_t$$

$$\xi^{l+}_{t_k} = (I - L_{t_k} H)\xi^{l}_{t_k} + L_{t_k} \bar{X}^{-1}_{t_k} V_t$$

$$P^{l+}_{t_k} = (I - L_{t_k} H) P^{l}_{t_k} (I - L_{t_k} H)^{\mathsf{T}} + L_{t_k} \bar{N}_k L^{\mathsf{T}}_{t_k}$$

where

$$\bar{N}_k := \bar{X}^{-1}_{t_k} \mathrm{Cov}[V_k] \bar{X}^{-\mathsf{T}}_{t_k}.$$

- Propagation:

$$\frac{d}{dt}\bar{X}_t = f_{u_t}(\bar{X}_t), \quad t_{k-1} \leq t < t_k$$

$$\frac{d}{dt}P_t^l = A_t^l P_t^l + P_t^l {A_t^l}^\mathsf{T} + Q_t$$

- Update:

$$\bar{X}_{t_k}^+ = \bar{X}_{t_k} \exp\left(L_{t_k}\left(\bar{X}_{t_k}^{-1} Y_{t_k} - b\right)\right)$$

$$P_{t_k}^{l+} = (I - L_{t_k} H) P_{t_k}^l (I - L_{t_k} H)^\mathsf{T} + L_{t_k} \bar{N}_k L_{t_k}^\mathsf{T}$$

where

$$L_{t_k} = P_{t_k}^l H^\mathsf{T} S^{-1}, \quad S = H P_{t_k}^l H^\mathsf{T} + \bar{N}_k$$

- Propagation:

$$\frac{d}{dt}\bar{X}_t = f_{u_t}(\bar{X}_t), \quad t_{k-1} \le t < t_k$$

$$\frac{d}{dt}\eta_t^r = g_{u_t}(\eta_t^r) - (\mathrm{Ad}_{\bar{X}_t} w_t)^\wedge \eta_t^r \implies \frac{d}{dt}\xi_t^r = A_t^r \xi_t^r - \mathrm{Ad}_{\bar{X}_t} w_t$$

$$\frac{d}{dt}P_t^r = A_t^r P_t^r + P_t^r {A_t^r}^\mathsf{T} + \mathrm{Ad}_{\bar{X}_t} Q_t \mathrm{Ad}_{\bar{X}_t}^\mathsf{T}$$

- Update: We use $\exp(\xi) \approx I + \xi^\wedge$ and neglect the higher order terms.

$$\bar{X}_{t_k}^{+} = \exp\left(L_{t_k}\left(\bar{X}_{t_k} Y_{t_k} - b\right)\right) \bar{X}_{t_k}$$

$$\eta_{t_k}^{r+} = \exp\left(L_{t_k}\left(\eta_{t_k}^{r} b - b + \bar{X}_{t_k} V_t\right)\right) \eta_{t_k}^{r}$$

$$I + \xi_{t_k}^{r+\wedge} = \left(I + \left(L_{t_k}\left((I + \xi_{t_k}^{r}{}^\wedge) b - b + \bar{X}_{t_k} V_t\right)\right)^\wedge\right)(I + \xi_{t_k}^{r}{}^\wedge)$$

$$\xi_{t_k}^{r+\wedge} = \xi_{t_k}^{r}{}^\wedge + \left(L_{t_k}\left((I + \xi_{t_k}^{r}{}^\wedge) b - b + \bar{X}_{t_k} V_t\right)\right)^\wedge$$

$$\xi_{t_k}^{r+} = \xi_{t_k}^{r} + L_{t_k}\left(\xi_{t_k}^{r}{}^\wedge b + \bar{X}_{t_k} V_t\right)$$

- Update: Define the measurement Jacobian, H, such that $H\xi = -\xi^\wedge b$.

$$\bar{X}_{t_k}^{+} = \exp\left(L_{t_k}\left(\bar{X}_{t_k} Y_{t_k} - b\right)\right)\bar{X}_{t_k}$$

$$\xi_{t_k}^{r+} = \xi_{t_k}^{r} - L_{t_k} H \xi_{t_k}^{r} + L_{t_k}\bar{X}_{t_k} V_t$$

$$\xi_{t_k}^{r+} = (I - L_{t_k} H)\xi_{t_k}^{r} + L_{t_k}\bar{X}_{t_k} V_t$$

$$P_{t_k}^{r+} = (I - L_{t_k} H) P_{t_k}^{r} (I - L_{t_k} H)^\mathsf{T} + L_{t_k}\bar{N}_k L_{t_k}^\mathsf{T}$$

where

$$\bar{N}_k := \bar{X}_{t_k}\mathrm{Cov}[V_k]\bar{X}_{t_k}^\mathsf{T}.$$

Right-Invariant Extended Kalman Filter

- Propagation:

$$\frac{d}{dt}\bar{X}_t = f_{u_t}(\bar{X}_t), \quad t_{k-1} \leq t < t_k$$

$$\frac{d}{dt}P_t^r = A_t^r P_t^r + P_t^r {A_t^r}^\mathsf{T} + \mathrm{Ad}_{\bar{X}_t} Q_t \mathrm{Ad}_{\bar{X}_t}^\mathsf{T}$$

- Update:

$$\bar{X}_{t_k}^+ = \exp\left(L_{t_k}\left(\bar{X}_{t_k} Y_{t_k} - b\right)\right)\bar{X}_{t_k}$$

$$P_{t_k}^{r+} = (I - L_{t_k} H) P_{t_k}^r (I - L_{t_k} H)^\mathsf{T} + L_{t_k} \bar{N}_k L_{t_k}^\mathsf{T}$$

where

$$L_{t_k} = P_{t_k}^r H^\mathsf{T} S^{-1}, \quad S = H P_{t_k}^r H^\mathsf{T} + \bar{N}_k$$

- Consider a robot as a rigid body that operates in 2D or 3D space. Such a robot naturally operates in $\mathrm{SE}(2)$ or $\mathrm{SE}(3)$. An element of the Lie algebra naturally represents the velocity in the body frame and can be measured by sensors attached to the robot.

- The kinematic equation of motion is described by the curve $X(t) \in \mathrm{SE}(2)$ or $\mathrm{SE}(3)$ as

$$\frac{d}{dt} X_t = X_t u_t^{\wedge}, \quad u_t \in \mathfrak{g}.$$

- $u_t = \mathrm{vec}(\omega_t, v_t) \in \mathbb{R}^3$ or $\mathbb{R}^6$ is a vector of angular velocity, ω_t, and linear velocity, v_t.

- Define the deterministic dynamics as $f_{u_t}(X_t) := X_t u_t^\wedge$.

- This process satisfies the group affine property:

$$
\begin{aligned}
&f_{u_t}(X_1 X_2) = X_1 X_2 u_t^\wedge \\
&f_{u_t}(X_1) X_2 + X_1 f_{u_t}(X_2) - X_1 f_{u_t}(I) X_2 = \\
&\qquad\qquad X_1 u_t^\wedge X_2 + X_1 X_2 u_t^\wedge - X_1 u_t^\wedge X_2 = X_1 X_2 u_t^\wedge.
\end{aligned}
$$

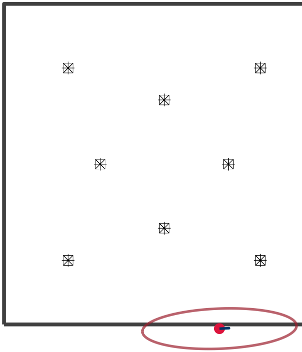
- The right-invariant error dynamics is

$$
\frac{d}{dt}\eta_t^r = g_{u_t}(\eta^r) = f_{u_t}(\eta^r) - \eta^r f_{u_t}(I) = \eta^r u_t^\wedge - \eta^r u_t^\wedge = 0.
$$

$$
\implies \frac{d}{dt}\xi_t^r = 0, \quad \text{and } A_t^r = 0.
$$

Example: Right-Invariant EKF Localization

A robot is operating in a known 2D map with point landmarks. The robot has 3 DOF and the state space is $\mathrm{SE}(2)$. The sensor provides relative 2D position of nearby landmarks. We use the velocity motion model and landmarks measurement model within a Right-Invariant EKF to localize the robot.



Example: Right-Invariant EKF Localization

The robot pose at any time-step is $X_k = \begin{bmatrix} R_k & p_k \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(2)$.

For the prediction step, we discretize the velocity motion model:

$$\bar{X}_{k+1} = \bar{X}_k \exp(u_k^\wedge) \quad \text{motion model}$$

$$\Phi = \exp(A_t^r \Delta t) = I \quad \text{transition matrix}$$

$$P_{k+1} = \Phi P_k \Phi^\mathsf{T} + \mathrm{Ad}_{\bar{X}_k} Q_d \mathrm{Ad}_{\bar{X}_k}^\mathsf{T} \quad \text{covariance propagation}$$

$$P_{k+1} = P_k + \mathrm{Ad}_{\bar{X}_k} Q_d \mathrm{Ad}_{\bar{X}_k}^\mathsf{T}$$

$$Q_d \approx \Phi Q_t \Phi^\mathsf{T} \Delta t = Q_t \Delta t \quad \text{discrete noise covariance}$$

Example: Right-Invariant EKF Localization

The global map of landmarks, $m \in \mathbb{R}^2$, is given. The relative landmark measurement model corresponds to the right-invariant observation form:

$$Y_k = \bar{X}_k^{-1} b + V_k$$

$$\begin{bmatrix} y_k^1 \\ y_k^2 \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{R}_k^{\mathsf{T}} & -\bar{R}_k^{\mathsf{T}} p_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ 1 \end{bmatrix} + \begin{bmatrix} v_k \\ 0 \end{bmatrix}$$

Example: Right-Invariant EKF Localization

Using the Right-Invariant EKF equations, we proceed to find H.

$$H\xi_k^r = -\xi_k^{r\wedge} b = -\begin{bmatrix} 0 & -\xi_k^\omega & \xi_k^{v_1} \\ \xi_k^\omega & 0 & \xi_k^{v_2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m^1 \\ m^2 \\ 1 \end{bmatrix}$$

$$= -\begin{bmatrix} -\xi_k^\omega m^2 + \xi_k^{v_1} \\ \xi_k^\omega m^1 + \xi_k^{v_2} \\ 0 \end{bmatrix} = \begin{bmatrix} m^2 & -1 & 0 \\ -m^1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_k^\omega \\ \xi_k^{v_1} \\ \xi_k^{v_2} \end{bmatrix}$$

$$H = \begin{bmatrix} m^2 & -1 & 0 \\ -m^1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Remark

Notice that since the map is known, H is constant. The last row corresponds to the homogeneous coordinates and can be removed during the implementation.

Now let's look into the observability of the linearized system. Using $\Phi = I$ and $H = \begin{bmatrix} m^2 & -1 & 0 \\ -m^1 & 0 & -1 \end{bmatrix}$, the discrete-time observability matrix is

$$\mathcal{O} = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} m^2 & -1 & 0 \\ -m^1 & 0 & -1 \\ m^2 & -1 & 0 \\ -m^1 & 0 & -1 \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

The first column is a linear combination of the second and third columns and, therefore, the first dimension (orientation) is unobservable.

Remark

Using only one landmark the robot heading angle is not observable. How many landmarks make the robot pose fully observable?

Example: Right-Invariant EKF Localization

To resolve the observability problem, we use two landmarks, $m_1, m_2 \in \mathbb{R}^2$, in each correction step. The stacked right-invariant observation model becomes:

$$\begin{bmatrix} Y_{1,k} \\ Y_{2,k} \end{bmatrix} = \begin{bmatrix} \bar{X}_k^{-1} & 0 \\ 0 & \bar{X}_k^{-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} V_k \\ V_k \end{bmatrix},$$

and the stacked measurement Jacobian, H, becomes

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} m_1^2 & -1 & 0 \\ -m_1^1 & 0 & -1 \\ m_2^2 & -1 & 0 \\ -m_2^1 & 0 & -1 \end{bmatrix}$$

Remark

It is easy to verify that the observability Gramian has rank 3 which makes the pose observable.

Example: Right-Invariant EKF Localization

See `riekf_localization_se2.m` for details and code.

- The inertial Measurement Units (IMUs) are ubiquitous and are available in most of the modern robotic systems. For an exaple of an IMU sensor see: `https://www.vectornav.com/products/vn-100`

- The IMU measurements, angular velocity $\tilde{\omega}_t$ and linear acceleration $\tilde{a}_t$ in the body frame.

- They are modeled as $\tilde{\omega}_t = \omega_t + w_t^g$, $w_t^g \sim \mathcal{GP}\left(0_{3,1}, \Sigma^g \, \delta(t-t')\right)$ and $\tilde{a}_t = a_t + w_t^a$, $w_t^a \sim \mathcal{GP}\left(0_{3,1}, \Sigma^a \, \delta(t-t')\right)$, where $\mathcal{GP}$ denotes a Gaussian process and $\delta(t-t')$ denotes the Dirac delta function.

- The IMU dynamics can be written as:

$$\begin{aligned}
\dot{R}_t &= R_t(\tilde{\omega}_t - w_t^g)^\wedge \\
\dot{v}_t &= R_t(\tilde{a}_t - w_t^a) + g \\
\dot{p}_t &= v_t,
\end{aligned}$$

  where g is the gravity vector.

- This model in deterministic form satisfies the group affine property.

- Assuming a zero-order hold on the incoming IMU measurements between times t_k and t_{k+1}, we have:

$$\begin{aligned} R_{k+1} &= R_k \, \Gamma_0(\bar{\omega}_k \Delta t) = R_k \exp(\bar{\omega}_k \Delta t) \\ v_{k+1} &= v_k + R_k \Gamma_1(\bar{\omega}_k \Delta t) \bar{a}_k \Delta t + g \Delta t \\ p_{k+1} &= p_k + v_k \Delta t + R_k \Gamma_2(\bar{\omega}_k \Delta t) \bar{a}_k \Delta t^2 + \frac{1}{2} g \Delta t^2, \end{aligned}$$

  where $\bar{\omega}_t := \tilde{\omega}_t - \bar{b}_t^g$ and $\bar{a}_t := \tilde{a}_t - \bar{b}_t^a$ are the "bias-corrected" inputs.

- These discrete dynamics are an exact integration of the continuous-time system under the assumption that the IMU measurements are constant over Δt.

- $\Gamma_0(\phi) = I + \frac{\sin(\|\phi\|)}{\|\phi\|}(\phi^\wedge) + \frac{1-\cos(\|\phi\|)}{\|\phi\|^2}(\phi^\wedge)^2$
- $\Gamma_1(\phi) = I + \frac{1-\cos(\|\phi\|)}{\|\phi\|^2}(\phi^\wedge) + \frac{\|\phi\|-\sin(\|\phi\|)}{\|\phi\|^3}(\phi^\wedge)^2$
- $\Gamma_2(\phi) = \frac{1}{2}I + \frac{\|\phi\|-\sin(\|\phi\|)}{\|\phi\|^3}(\phi^\wedge) + \frac{\|\phi\|^2+2\cos(\|\phi\|)-2}{2\|\phi\|^4}(\phi^\wedge)^2$
- $\Gamma_m(\phi) := \left(\sum_{n=0}^{\infty} \frac{1}{(n+m)!}(\phi^\wedge)^n\right)$

Remark

$\Gamma_0(\phi)$ *is simply the exponential map of* $\mathrm{SO}(3)$. $\Gamma_1(\phi)$ *is also known as the left Jacobian of* $\mathrm{SO}(3)$.

Table: Summary of World-centric State Estimator

State Definition	Deterministic Nonlinear Dynamics
$X_t := \begin{bmatrix} R_{WB} & {}_W v_B & {}_W p_{WB} \\ 0_{1,3} & 1 & 0 \\ 0_{1,3} & 0 & 1 \end{bmatrix}$	$f_{u_t}(\bar{X}_t) = \begin{bmatrix} \bar{R}_t \tilde{\omega}_t^\wedge & \bar{R}_t \tilde{a}_t + g & \bar{v}_t \\ 0_{1,3} & 0 & 0 \\ 0_{1,3} & 0 & 0 \end{bmatrix}$
Log-Linear Right-Invariant Dynamics	**Log-Linear Left-Invariant Dynamics**
$A_t^r = \begin{bmatrix} 0 & 0 & 0 \\ g^\wedge & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$ $\bar{Q}_t^r = \mathrm{Ad}_{\bar{X}_t} \mathrm{Cov}[w_t] \mathrm{Ad}_{\bar{X}_t}^\mathsf{T}$	$A_t^l = \begin{bmatrix} -\tilde{\omega}_t^\wedge & 0 & 0 \\ -\tilde{a}_t^\wedge & -\tilde{\omega}_t^\wedge & 0 \\ 0 & I & -\tilde{\omega}_t^\wedge \end{bmatrix}$ $\bar{Q}_t^l = \mathrm{Cov}[w_t]$

A robot is equipped with an IMU and the Global Positioning System (GPS) sensors. We use a Left-Invariant EKF to estimate its pose (R_k, p_k), and velocity, v_k, in the world frame. The state is modeled using $\mathrm{SE}_2(3)$ such that

$$X_k = \begin{bmatrix} R_k & v_k & p_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SE}_2(3).$$

For the prediction step, we use the discretized IMU model, and we update the predictions using GPS measurements.

- The (world-centric) left-invariant error dynamics matrix only depends on the IMU inputs that are assumed to be constant between times t_k and t_{k+1}. The state transition matrix can be simply computed from the matrix exponential.

$$\Phi^l(t_{k+1}, t_k) = \exp(A_t^l \Delta t).$$

- This state transition matrix also has an analytical solution of the form:

$$\Phi^l(t_{k+1}, t_k) = \begin{bmatrix} \Phi_{11}^l & 0 & 0 \\ \Phi_{21}^l & \Phi_{22}^l & 0 \\ \Phi_{31}^l & \Phi_{32}^l & \Phi_{33}^l \end{bmatrix}.$$

- The individual terms are

$$
\begin{aligned}
\Phi_{11}^{l} &= \Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t) \\
\Phi_{21}^{l} &= -\Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t)(\Gamma_1(\bar{\omega}_t \Delta t)\bar{a}_t)^{\wedge} \Delta t \\
\Phi_{31}^{l} &= -\Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t)(\Gamma_2(\bar{\omega}_t \Delta t)\bar{a}_t)^{\wedge} \Delta t^2 \\
\Phi_{22}^{l} &= \Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t) \\
\Phi_{32}^{l} &= \Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t) \Delta t \\
\Phi_{33}^{l} &= \Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t)
\end{aligned}
$$

Example: IMU-GPS Left-Invariant EKF

The GPS measurement model corresponds to the left-invariant observation form:

$$Y_k = \bar{X}_k b + V_k$$

$$\begin{bmatrix} y_k \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{R}_k & \bar{v}_k & \bar{p}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_k \\ 0 \\ 0 \end{bmatrix}$$

Using the Left-Invariant EKF equations, we proceed to find H.

$$
\begin{aligned}
H\xi_k^r = \xi_k^{r\wedge} b &= \begin{bmatrix} \xi_k^{\omega\wedge} & \xi_k^v & \xi_k^p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \xi^p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ 0_{1,3} & 0_{1,3} & 0_{1,3} \\ 0_{1,3} & 0_{1,3} & 0_{1,3} \end{bmatrix} \begin{bmatrix} \xi_k^\omega \\ \xi_k^v \\ \xi_k^p \end{bmatrix},
\end{aligned}
$$

and in its reduced form is

$$
H = \begin{bmatrix} 0 & 0 & I \end{bmatrix}.
$$

Switching Between Left and Right-Invariant Errors

We can switch between the left and right error forms through the use of the adjoint map.

$$\begin{aligned} \eta_t^r &= \bar{X}_t X_t^{-1} = \bar{X}_t \eta_t^l \bar{X}_t^{-1} \\ \exp(\xi_t^r) &= \bar{X}_t \exp(\xi_t^l) \bar{X}_t^{-1} = \exp(\mathrm{Ad}_{\bar{X}_t} \xi_t^l) \\ \xi_t^r &= \mathrm{Ad}_{\bar{X}_t} \xi_t^l \end{aligned}$$

Remark

This transformation is exact, which means that we can easily switch between the covariance of the left and right invariant errors using

$$P_t^r = \mathrm{Ad}_{\bar{X}_t} \, P_t^l \, \mathrm{Ad}_{\bar{X}_t}^{\mathsf{T}}.$$

Invariant EKF Summary

- Use it if the process dynamics naturally evolves on a Lie group;
- Works well in practice despite the fact by the addition of noise and calibration parameters the theoretical result is lost;
- Excellent consistency and no spurious correlation among state and parameters.
- Highly efficient and suitable for high-frequency state estimation tasks.

References and Readings

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