Lecture 20: Graph SLAM

Objectives:

- Understanding SLAM as:
  * Dynamic Bayes Network
  * Factor Graphs
  * Least squares
- Maximum a-posteriori (MAP) solution.
- Enable you to read papers & understand literature building on these ideas.

Notation and Weighted Norm:

we define $\|r\|_\Sigma^2 := r^T \Sigma^{-1} r$.

Σ (covariance) is symmetric and PSD.

$\Sigma = LL^T$, L is the lower triangular Cholesky factor of Σ.

$$\Sigma^{-1} = (LL^T)^{-1} = L^{-T} L^{-1}.$$

Remark: $(AB)^{-1} = B^{-1}A^{-1}$ and $A^{-T} = (A^T)^{-1} = (A^{-1})^T$.

$$\ast \; \|r\|_\Sigma^2 = r^T \Sigma^{-1} r = r^T L^{-T} L^{-1} r = (L^{-1}r)^T (L^{-1}r)$$
$$= \|L^{-1}r\|^2.$$

Euclidean norm: $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, $x \in \mathbb{R}^n$.

$$x = x_{0:K} = \{x_0, x_1, \cdots, x_K\}$$

$$z = z_{1:K} = \{z_1, z_2, \cdots, z_K\}$$

$$P(x, z) = P(x_{0:K}, z_{1:K-1}, z_K)$$

$$= P(z_K \mid \underbrace{x_{0:K}, z_{1:K-1}}_{\text{Markov assumption}}) P(x_{0:K}, z_{1:K-1})$$

$$= P(z_K \mid x_K) P(x_{0:K-1}, x_K, z_{1:K-1})$$

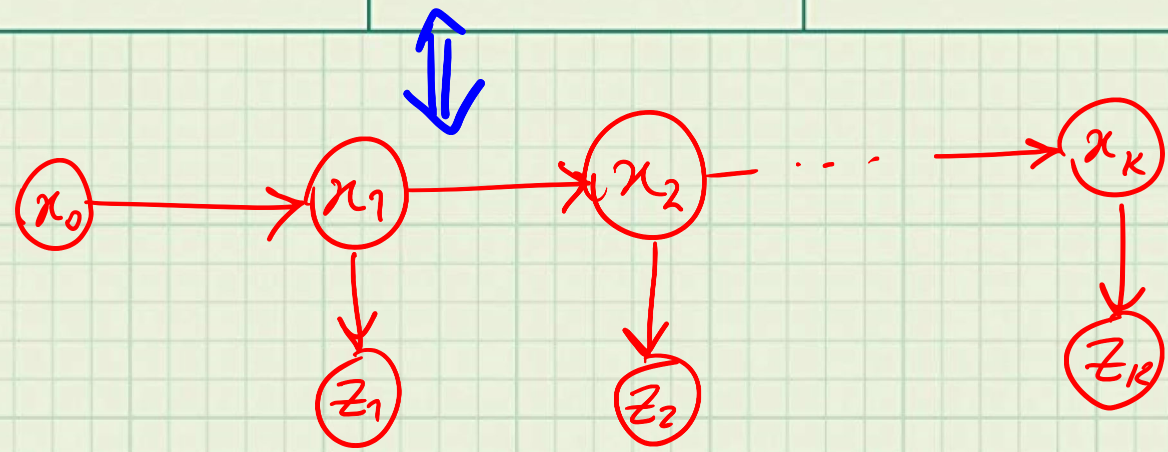
$$= P(z_K \mid x_K) P(x_K \mid \underbrace{x_{0:K-1}, z_{1:K-1}}_{\text{Markov}}) \cdot P(\underbrace{x_{0:K-1}, z_{1:K-1}})$$

$$= P(z_K \mid x_K) P(x_K \mid x_{K-1}) P(x_{0:K-1}, z_{1:K-1})$$

$$\vdots$$

$$= P(x_0) \prod_{i=1}^{K} P(x_i \mid x_{i-1}) P(z_i \mid x_i)$$

$$= P(x, z)$$



Dynamic Bayes Network

Ex: $p(x_2 | x_1)$

$p(x_0)$ is prior

$p(x_1 | x_0)$ is the motion model

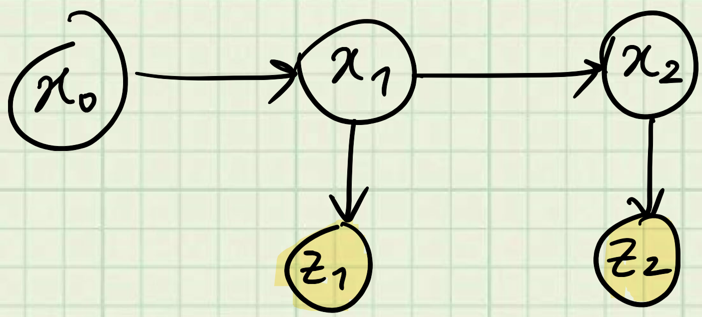
$p(z_1 | x_1)$ is the measurement model

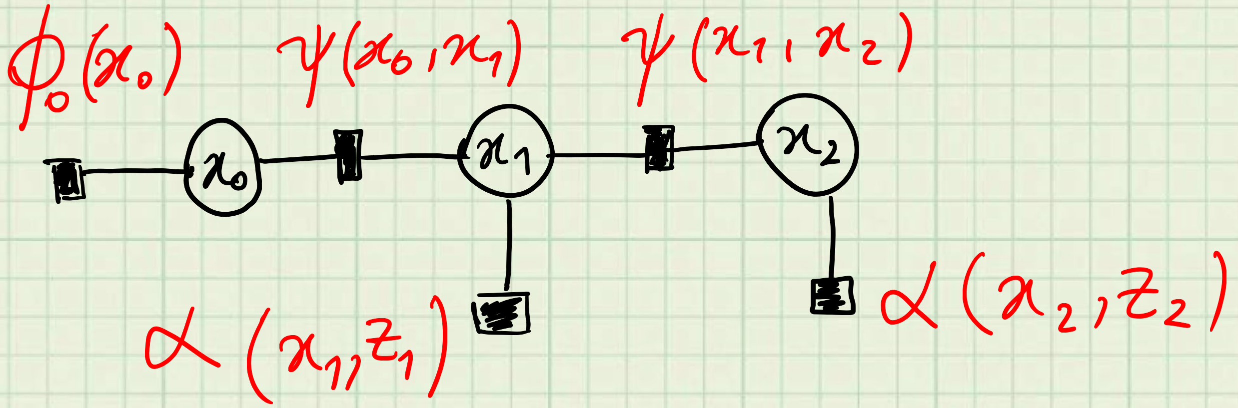
Assumption: Gaussian noise model

$$\begin{cases} x_i = f(x_{i-1}, u_i) + w_i \quad , \quad w_i \sim N(0, \Sigma_{w_i}) \\ p(x_i | x_{i-1}, u_i) \propto \exp\left(-\frac{1}{2} \| f(x_{i-1}, u_i) - x_i \|^2_{\Sigma_{w_i}}\right) \end{cases}$$

$$\begin{cases} z_i = h(x_i) + v_i \quad , \quad v_i \sim N(0, \Sigma_{v_i}) \\ p(z_i | x_i) \propto \exp\left(-\frac{1}{2} \| h(x_i) - z_i \|^2_{\Sigma_{v_i}}\right) \end{cases}$$

Factor Graphs





$$p(x, z) \propto \phi_0(x_0)\, \psi(x_0, x_1)\, \psi(x_1, x_2)\,.$$
$$\alpha(x_1, z_1)\, \alpha(x_2, z_2)$$

$$\phi_0(x_0) \propto p(x_0)$$
$$\psi(x_{i-1}, x_i) \propto p(x_i \mid x_{i-1})$$
$$\alpha(x_i, z_i) \propto p(z_i \mid x_i)$$

* Eech factor is a sensor (measurement) model.

* Graphical modeling is an easy way to describle conditional independence and write the joint distribution.

* MAP Estimation.

$$x^* = \underset{x}{\arg\max}\ p(x|z) = \arg\max p(x,z)$$

$$= \underset{x}{\arg\min} -\log p(x,z)$$

Remark: $p(x|z) = \dfrac{p(x,z)}{p(z)}$.

* General framework for Sensor Fusion

$$x^* = \arg\min_x -\log p(x, z)$$

$$p(x,z) = p(x_0) \prod_{i=1}^{M} p(x_i | x_{i-1}, u_i) \prod_{j=1}^{K} p(z_j | x_{ji})$$

$$-\log p(x_0) \propto \frac{1}{2} \| \hat{x}_0 - x_0 \|^2_{\Sigma_0}$$

$$-\log p(x_i | x_{i-1}, u_i) \propto \frac{1}{2} \| f(x_{i-1}, u_i) - x_i \|^2_{\Sigma_{w_i}}$$

$$-\log p(z_j | x_{ji}) \propto \frac{1}{2} \| h(x_{ji}) - z_j \|^2_{\Sigma_{v_j}}$$

$$x^* = \arg\min_x -\log p(x, z)$$

$$= \arg\min_x \frac{1}{2} \| \hat{x}_0 - x_0 \|^2_{\Sigma_0} + \frac{1}{2} \| f(x_{i-1}, u_i) - x_i \|^2_{\Sigma_{w_i}}$$

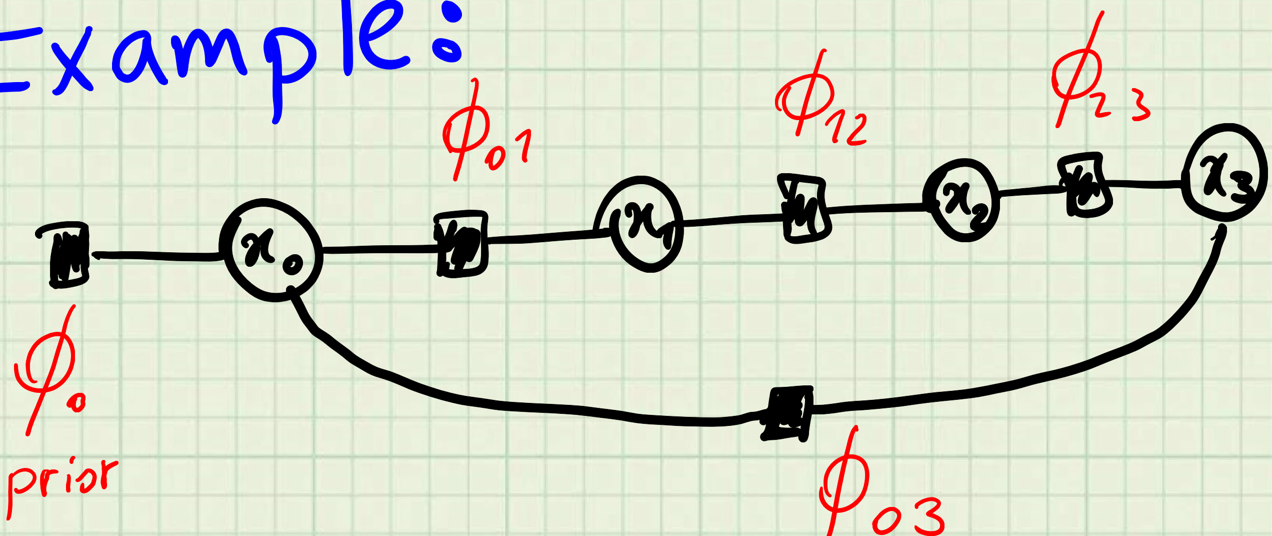
$$+ \frac{1}{2} \| h(x_{ji}) - z_j \|^2_{\Sigma_{v_j}}$$

$x = x_{0:K}$ (state)

$$p(x_0) \propto \exp\left(-\frac{1}{2} \| \hat{x}_0 - x_0 \|^2_{\Sigma_0}\right)$$

Remark: $\hat{x}_0$ is the mean prior.
In many problems, e.g., HW7, we can set $\hat{x}_0 = 0$.

Example:



$$\phi_0(x_0) \propto \exp\left(-\frac{1}{2}\|x_0 - \hat{x}_0\|^2_{\Sigma_0}\right)$$

$$\propto p(x_0)$$

$$\phi_{ij} \propto \exp\left(-\frac{1}{2}\|x_j - x_i - u_{ij}\|^2_{W_{ij}}\right)$$

$i<j$

$$\propto p(x_j \mid x_i, u_{ij})$$

$$x_j = f(x_i, u_{ij}) + w_{ij} = x_i + u_{ij} + w_{ij}, \quad w_{ij} \sim N(0, W_{ij})$$

$$r_{ij} := x_j - f(x_i, u_{ij}) = x_j - x_i - u_{ij}$$

MAP problem:

$$-\log p(x) = \frac{1}{2}\|x_0 - \hat{x}_0\|^2_{\Sigma_0} + \frac{1}{2}\sum_{\substack{ij \\ i<j}} \|x_j - x_i - u_{ij}\|^2_{W_{ij}}$$

$$= \frac{1}{2}\|L_0^{-1}(x_0 - \hat{x}_0)\|^2 + \frac{1}{2}\sum_{ij}\|L_{ij}^{-1}(x_j - x_i - u_{ij})\|^2$$

$$=: \frac{1}{2}\|r_0\|^2 + \frac{1}{2}\sum_{ij}\|r_{ij}\|^2$$

$$= \frac{1}{2}\sum_K \|r_K\|^2 \quad , \quad K = \{0, 01, 12, 23, 03\}$$

$$= \frac{1}{2}\sum_K r_K^T r_K$$

$$x^* = \underset{x}{\operatorname{argmin}} \frac{1}{2}\sum_K r_K^T r_K$$

Next step: Linearize!

$$r_0 = L_0^{-1}(x_0 - \hat{x}_0)$$

$$\frac{\partial r_0}{\partial x_0} = L_0^{-1}$$

$$r_{ij} = L_{ij}^{-1}(x_j - x_i - u_{ij}) = \begin{bmatrix} -L_{ij}^{-1} & L_{ij}^{-1} \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} - L_{ij}^{-1} u_{ij}$$

$$\frac{\partial r_{ij}}{\partial x_i} = -L_{ij}^{-1}$$

$$\frac{\partial r_{ij}}{\partial x_j} = L_{ij}^{-1}$$

$$Ax = b$$

$$\begin{array}{c} \begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} L_0^{-1} & 0 & 0 & 0 \\ -L_{01}^{-1} & L_{01}^{-1} & 0 & 0 \\ 0 & -L_{12}^{-1} & L_{12}^{-1} & 0 \\ 0 & 0 & -L_{23}^{-1} & L_{23}^{-1} \\ -L_{03}^{-1} & 0 & 0 & L_{03}^{-1} \end{bmatrix}_{5\times4} \end{array} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}_{4\times1} = \begin{bmatrix} L_0^{-1}\hat{x}_0 \\ L_{01}^{-1} u_{01} \\ L_{12}^{-1} u_{12} \\ L_{23}^{-1} u_{23} \\ L_{03}^{-1} u_{03} \end{bmatrix}_{5\times1}$$

Sparse QR factorization

$$A = QR$$

$$Ax = QRx = b$$

$$Q^T Q R x = Q^T b$$

$$Rx = Q^T b$$

solve via back subtitution.

Remark: In GTSAM, we build the graph, given data. The library solves the problem.