

Marvin Abisrror Zarate

**On Selecting  
Heuristic Function Subsets For Domain  
Independent-Planning**

**Brasil**

**Fevereiro, 2016**



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Dissertação apresentada à Universidade Federal de Viçosa, como parte das exigências do Programa de Pós-Graduação em Ciência da Computação, para a obtenção do título de *Magister Scientiae*.

Universidade de Viçosa – UFV

Centro de Ciencias Exactas e Tecnologicas (CCE)

Programa de Pós-Graduação

Orientador: Levi Henrique Santana de Lelis

Co-orientador: Santiago Franco

Brasil

Fevereiro, 2016

Marvin Abisrror Zarate

On Selecting

Heuristic Function Subsets For Domain Independent-Planning/ Marvin Abisrror Zarate.  
– Brasil, Fevereiro, 2016-

50 p. : il. (algumas color.) ; 30 cm.

Orientador: Levi Henrique Santana de Lelis

Tese (Mestrado) – Universidade de Viçosa – UFV

Centro de Ciencias Exactas e Tecnologicas (CCE)

Programa de Pós-Graduação, Fevereiro, 2016.

1. Palavra-chave1. 2. Palavra-chave2. 2. Palavra-chave3. I. Orientador. II. Universidade xxx. III. Faculdade de xxx. IV. Título

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Trabalho aprovado. Brasil, 25 de fevereiro de 2016:

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**Professor**  
Convidado 2

Brasil  
Fevereiro, 2016



*This dissertation is dedicated to my Mother.*





# Acknowledgements

I would like to express my sincere gratitude to my advisor PhD. Levi Henrique Santana de Lelis, for the continuous support and guidance during the dissertation process. His valuable advice, patience and encouragement have been of great importance for this work.

Besides my advisor, I would like to thank to my co—advisor: PhD. Santiago Franco for his insightful feedback, interest and tough questions.

To the professors of the CCE, particularly the Master Degree program with all its members, played an invaluable role in my graduate education.

Last, but not least, I would like to thank my Mother.



*“Não vos amoldeis às estruturas deste mundo,  
mas transformai-vos pela renovação da mente,  
a fim de distinguir qual é a vontade de Deus:  
o que é bom, o que Lhe é agradável, o que é perfeito.  
(Bíblia Sagrada, Romanos 12, 2)*



# Abstract

In this dissertation we present greedy methods for selecting a subset of heuristic functions from a large pool of possibilities with the objective of reducing the running time of search algorithms. Holte et al. (2006) showed that search can be faster if several smaller pattern databases are used instead of one large pattern database. Our methods are able to select good heuristics from a large set of heuristic functions to guide A\* search. If the heuristics are consistent, our method is able to make good subset selections with respect to an approximation of the A\* search tree size. In addition the subset our method selects is also good with respect to the resulting A\* running time. We implemented our method in Fast Downward and showed empirically that it produces heuristics which outperform the state-of-the-art planners in the International Planning Competition benchmarks.

**Key-words:** Heuristics selection; A\*



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# 1 Introduction

State-space search algorithms have been used to solve important real-world problems, such as problems arising in Robotics, domain-independent planning, chemistry, biology, and engineering. For example, in Figure 1 the agent, in the top-right requires to do a plan for the shortest path from the initial location to the location marked with the letter **g** (bottom-left corner); the shaded area represents an obstacle the agent cannot pass. The agent has two options to go to the goal: It can go to the location **b** or location **c**. The agent are going to find the shortest path going through **c**, because there is not obstacle to the goal.

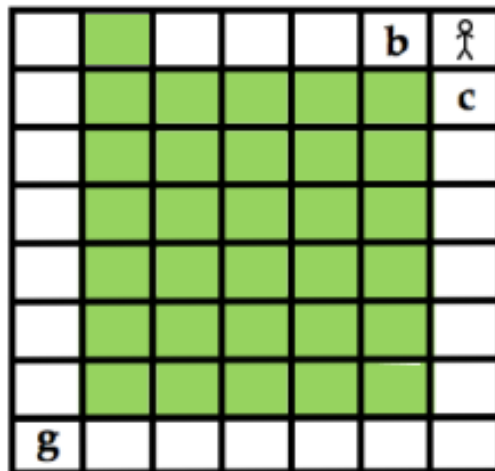


Figure 1 – The agent in the top-right corner requires to do a plan path to the location marked with a letter **g**; the shaded area represents an obstacle the agent cannot pass.

In this dissertation we study methods for selecting a subset of heuristic functions while minimizing the search tree size and the running time of the A\* (HART P. E.; NILSSON; RAPHAEL, 1968) search algorithm while solving state-space search problems.

We are interested in selecting heuristics from a large set of possibilities because Holte et al., (2006) showed that search can be faster if several smaller pattern databases are used instead of one large pattern database. Intuitively, one heuristic can be helpful in a region of the search tree where another heuristic isn't. Then, instead of using one heuristic to find the solution, it would be best to use the most promising subset of heuristics from a possibly large set.

## 1.1 Background work

State-space search algorithms are used to solve some Artificial Intelligence (AI) problems by finding a sequence of actions from the start state to a goal state in the search space. Two well know search algorithms for solving state-space search problems are Depth-First Search (DFS) and Breadth-First Search (BFS). DFS looks for the solution by exploring the subtree rooted node  $n$  before exploring the subtrees rooted at  $n$ 's siblings while looking for a path from start to goal.

Figure 2 shows the ordering in which DFS expands nodes while solving the 8-tile puzzle (explained in detail in Chapter 2), a state-space problem. We can see that DFS expands 31 states before finding a goal state (represented by the bottom right state). BFS looks for the solution by exploring all nodes in a given level before exploring nodes in the next level. In Figure 3 we can see BFS expands 46 states to find the goal. In both Figure 2 and Figure 3 the numbers above of each state represent the order in which the states are visited. DFS and BFS are brute-force search algorithms, as they visit all states encountered during its search before finding a solution. We call brute-force-search tree (BFST) the tree expanded by DFS and BFS.

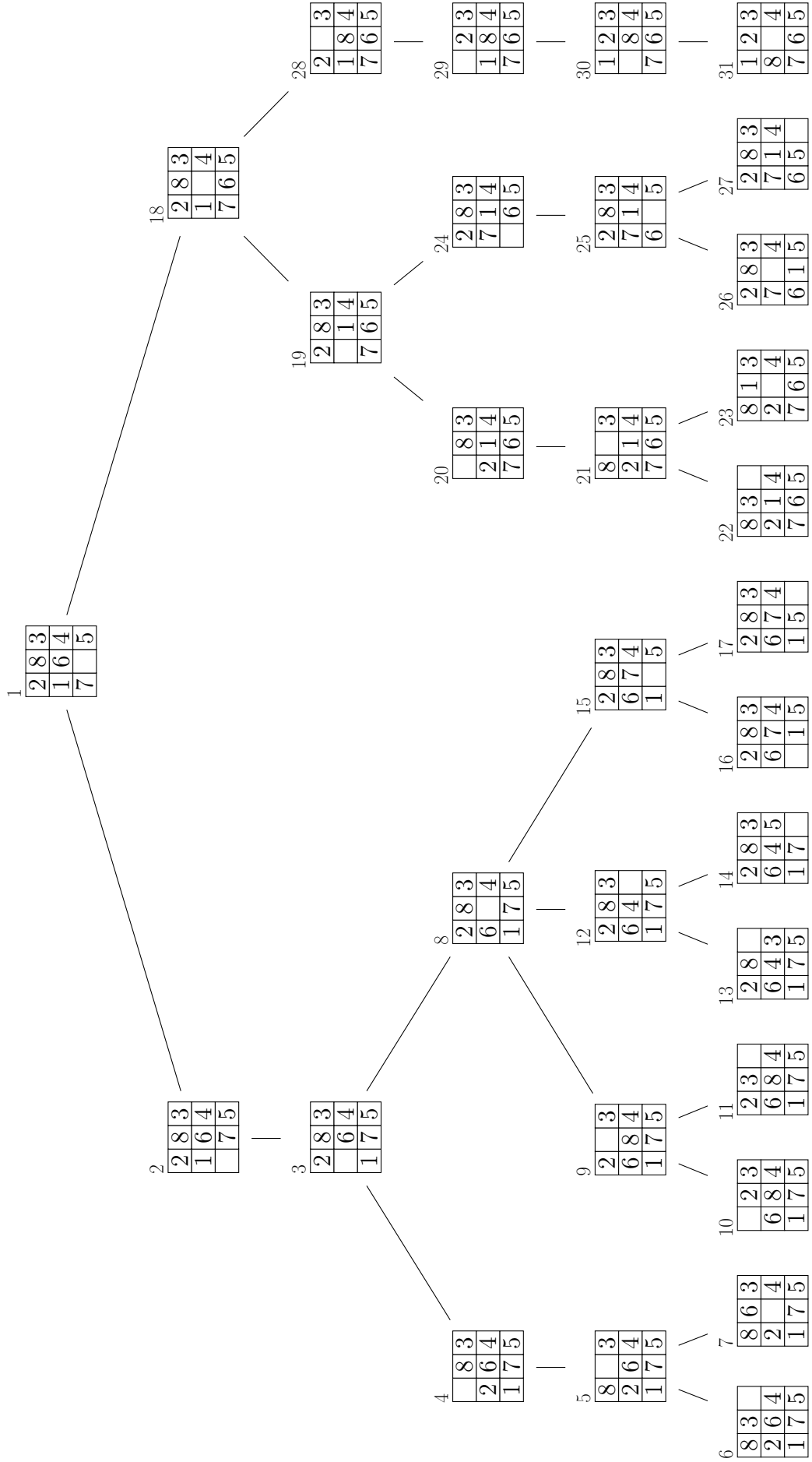


Figure 2 – 8 tile puzzle using DFS. (CHEN, 2011)

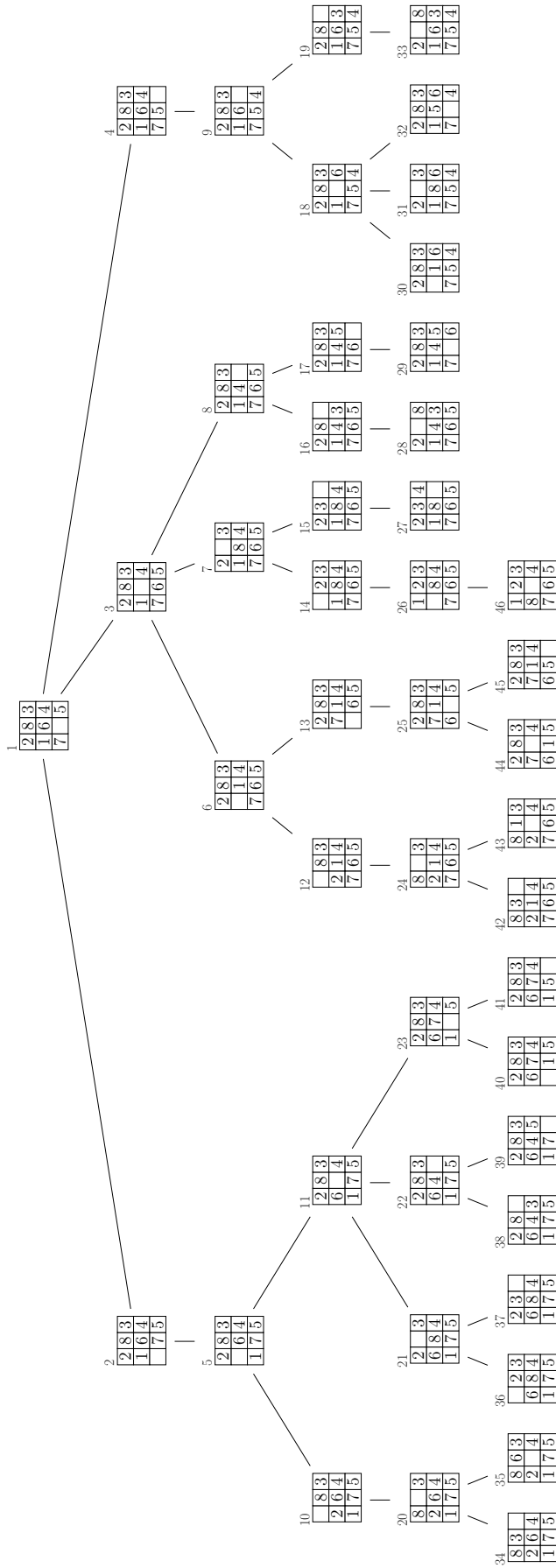


Figure 3 – 8 tile puzzle using BFS. (CHEN, 2011)

There are other types of algorithms called heuristic search algorithms, and the most representative algorithm of this type is A\* (HART P. E.; NILSSON; RAPHAEL, 1968). Heuristic search algorithms use a heuristic function for estimating the distance of a node in the search tree to a goal state. Heuristic search algorithms tend to generate smaller search trees than the BFST because the heuristic guides the search to more promising parts of the state space. Also, by reducing the search tree size, the guidance of the heuristic function might also reduce the overall running time of the algorithm.

There are different approaches (HASLUM et al. 2007; EDELKAMP 2007; NISSIM; HOFFMANN; HELMERT 2011) for creating heuristics. One approach that has shown successful results in heuristic generation is Pattern Database (PDB) (CULBERSON; SCHAEFFER, 1998). The way how PDBs work is as follows: The search space of the problem is abstracted into a smaller state space that can be enumerated with exhaustive search. The distance of all abstracted states to the abstracted goal state are stored in a lookup table, which can be used as a heuristic function for the original state space.

## 1.2 Objectives

The objective of this dissertation is to develop algorithms for selecting a heuristic subset from a large set of heuristics with the goal of solving domain-independent planning problems. Specifically our objectives are the following.

- Develop a method to find a subset of heuristics from a large pool of heuristics  $\zeta$  that minimizes the number of nodes expanded by A\* in the process of search.
- Develop a method to find a subset of heuristics from a large pool of heuristics  $\zeta$  that minimizes the A\* running time.

## 1.3 Scope, Limitations, and Delimitations

We implemented our method in Fast Downward (HELMERT, 2006) and we test our methods on the 2011 International Planning Competition (IPC) problem instances.

## 1.4 Justification

Good results have been obtained in domain-independent planning by using heuristic search approach (BONET; GEFNER, 2001). The heuristic function used to guide the A\* search are known to greatly affect the algorithm's running time. That is why it is important to have methods for selecting a good subset of heuristics to guide the A\* search.

## 1.5 Hypothesis

We test the following hypothesis:

- A greedy algorithm is effective for selecting a good subset of heuristics to guide the A\* search.

## 1.6 Contributions

The main contributions of this dissertation are:

- A meta-reasoning approach for selecting heuristic functions while minimizing the number of nodes generated by A\*.
- A meta-reasoning approach for selecting heuristic functions while minimizing the running time of the A\* search.
- Detailed experiments on domain-independent planning showing the strengths and weaknesses of the proposed approaches. Our experiments also show that one of our proposed approaches is able to outperform all other systems tested.

## 1.7 Organization

This dissertation is organized as follows:

1. In Chapter I, the background of the dissertation is provided, which also includes the objectives and the scope definition.
2. In Chapter II we review the state-of-the-art in selection of heuristic functions.
3. In Chapter III we introduce Greedy Heuristic Selection (GHS).
4. In Chapter IV compare GHS with other planner systems.
5. We conclude in Chapter V.



## 2 Background

The system most similar to the one we present in this dissertation is RIDA\* (BARLEY; FRANCO; RIDDLE, 2014). RIDA\* also selects a subset from a pool of heuristics to guide the A\* search. In RIDA\* this is done by starting with an empty subset and trying subsets of size one before trying subsets of size two and so on. RIDA\* stops after evaluating a fixed number of subsets. While RIDA\* is able to evaluate sets of heuristics with only tens of elements, the method we propose in this dissertation is able to evaluate sets with thousands of elements.

Rayner et al., (2013) present an optimization procedure that is also similar to ours. In contrast with our work, Rayner et al. limited their experiments to a single objective function that sought to maximize the sum of heuristic values in the state space. Moreover, Rayner et al.’s method performs a uniform sampling of the state space to estimate the sum of heuristic values in the state space. Thus, their method is not directly applicable to domain-independent planning. In this dissertation we adapt Rayner et al.’s approach to domain-independent planning by using Stratified Sampling (SS) (CHEN, 1992) to estimate the sum of heuristic values in the state space. Our empirical results show that GHS minimizing an approximation of A\*’s running time is able to substantially outperform Rayner et al.’s approach in domain-independent planning.

Our meta-reasoning requires a prediction of the number of nodes expanded by A\* using any given subset. One of the prediction methods we use is SS. Even though, SS produces good predictions of the Iterative-Deepening A\* (IDA\*) (KORF, 1985) search tree, it does not produce good predictions of A\*’s search tree. This is because SS is unable to detect duplicated nodes during sampling (LELIS; STERN; STURTEVANT, 2014). Although SS does not produce good predictions of the number of nodes generated by A\*, we show empirically that SS allows GHS to make good subset selections.

### 2.1 Planning Task

A *SAS<sup>+</sup> planning task* (BÄCKSTRÖM; NEBEL, 1995) is a 4 tuple  $\nabla = \{V, O, I, G\}$ .  $V$  is a set of *state variables*. Each variable  $v \in V$  is associated with a finite domain of possible  $D_v$ . A state is an assignment of a value to every  $v \in V$ . The set of possible states, denoted  $V$ , is therefore  $D_{v_1} \times \dots \times D_{v_2}$ .  $O$  is a set of operators, where each operator  $o \in O$  is triple  $\{pre_o, post_o, cost_o\}$  specifying the preconditions, postconditions (effects), and non-negative cost of  $o$ .  $pre_o$  and  $post_o$  are assignments of values to subsets of variables,  $V_{pre_o}$  and  $V_{post_o}$ , respectively. Operator  $o$  is applicable to state  $s$  if  $s$  and  $pre_o$  agree on the assignment of values to variables in  $V_{pre_o}$ . The effect of  $o$ , when applied to  $s$ , is to set

the variables in  $V_{post_o}$  to the values specified in  $post_o$  and to set all other variables to the value they have in  $s$ , resulting in a new state, which we call a *child* of  $s$ . We define as  $children(s)$  the set of child nodes of  $s$ .  $G$  is the goal condition, an assignment of values to a subset of variables,  $V_G$ . A state is a goal state if it and  $G$  agree on the assignment of values to the variable in  $V_G$ .  $I$  is the initial state, and the planning task,  $\nabla$ , is to find an optimal (least-cost) sequence of operators leading from  $I$  to a goal state. We denote the optimal solution cost of  $\nabla$  as  $C^*$ .

## 2.2 The 8-tile-puzzle case

The domain 8-tile-puzzle is used to illustrate a few concepts that will be helpful in the other chapters of this dissertation. The 8-tile-puzzle, illustrated in the Figure 4, consists of a board with 8 squares named tiles numbered from 1 to 8 and one empty square. The goal of the game is to order the tiles in some order. For example: From left to right and up to bottom in the following order 1, 2, 3, 4, 5, 6, 7, 8 and empty tile. The goal can be achieved by moving the numbered tiles into the empty tile.

Initial			Goal		
4	1	2	1	2	3
8		3	4	5	6
5	7	6	7	8	

Figure 4 – The left tile-puzzle is one possible initial distribution of tiles and the right tile-puzzle is the goal distribution of tiles. Each one represents a state.

Instead of using DFS or BFS that will analyze all states encountered during search to solve instances of the 8-tile-puzzle, we can obtain heuristics from the domain, which will allow search algorithms such as A\* and IDA\* to find a solution quicker.

## 2.3 Heuristics

There exist many state-space search algorithms, and one of the most important and well known is A\* (HART P. E.; NILSSON; RAPHAEL, 1968). A\* uses the  $f(s) = g(s) + h(s)$  cost function to guide its search to a solution.  $g(s)$  is the cost to go from the start state to state  $s$ , and  $h(s)$  is the estimated cost to go from  $s$  to the goal.

In Figure 5 the optimal distance from the initial state  $I$  to the state  $s$  is 4 and is represented by  $g(s)$ .  $h^*(s) = 3$  represents the optimal distance from  $s$  to the goal state  $G$ , and  $h(s) = 2$  is the estimated cost to go from  $s$  to  $G$ .



Figure 5 – Heuristic Search:  $I$ : Initial state,  $s$ : Some state,  $G$ : Goal state

A heuristic is admissible if  $h(s) \leq h^*(s)$  for all  $s \in V$ . A heuristic is consistent iff  $h(s) \leq c(s, t) + h(t)$  for all states  $s$  and  $t$ , where  $c(s, t)$  is the cost of the cheapest path from  $s$  to  $t$ . For example, the heuristic function provided by a PDB (CULBERSON; SCHAEFFER, 1998) is admissible and consistent.

Given a set of admissible and consistent heuristics  $\zeta = \{h_1, h_2, \dots, h_M\}$ , the heuristic  $h_{max}(s, \zeta) = \max_{h \in \zeta} h(s)$  is also admissible and consistent. When describing our method we assume all heuristics to be consistent. We define  $f_{max}(s, \zeta) = g(s) + h_{max}(s, \zeta)$ , where  $g(s)$  is minimal when  $A^*$  using a consistent heuristic expands  $s$ . We call an  $A^*$  search tree the tree defined by the states generated by  $A^*$  using a consistent heuristic while solving a problem  $\nabla$ .

We now show two heuristics for the 8-tile-puzzle.

### 2.3.1 Out-of-place Heuristic (OP)

This heuristic counts the number of tiles that are out of the goal position. If the tile is not in its goal position, then it counts as one, otherwise it counts as zero.

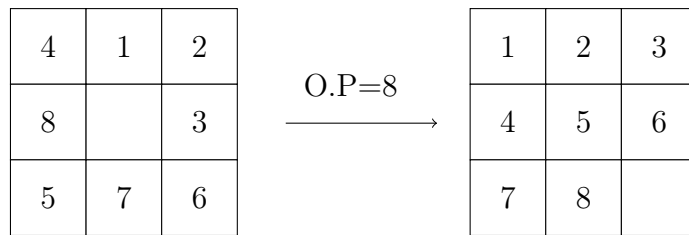


Figure 6 – Out of place heuristic

The tiles numbered with 4, 1, 2, 3, 6, 7, 5, and 8 are out of place, then each tile counts as 1 yielding a total of 8, which is heuristic value for this state.

### 2.3.2 Manhattan Distance Heuristic (MD)

This heuristic counts the minimum number of operations that should be applied to any tile to place it in its goal position. Let us explain this with an example: Tile 5 is located on the bottom left of the state shown on the left-hand side of Figure 7, then the

minimum number of moves to get tile 5 to its goal position is 2 (either up and right or right and up), both movements equal to 2.

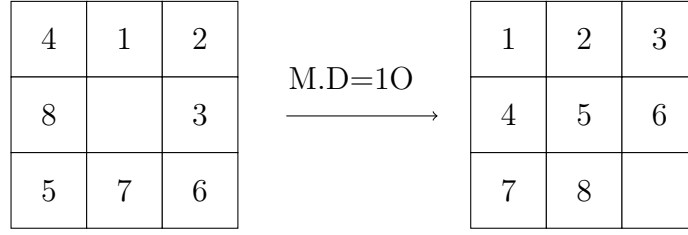


Figure 7 – Manhatham distance heuristic

Tile 4 requires one move to get to its goal position; tile 1 through 7 require one move; tiles 5 and 8 require two moves; the sum of all moves is 10, which is the MD estimate of the cost to go to this particular state.

OP and MD are heuristics that use domain knowledge to estimate the cost to go to different states. Other methods, such as PDBs, do not require domain knowledge to create a heuristic function. In this dissertation we use a set of domain-independent heuristics to guide search.

## 2.4 Using a Set of Heuristics

A\* works using a structure called OPEN, where in each iteration A\* expands the node with the lowest cost function in OPEN. In Figure 8, we have two figures which represent the same graph, the node **a** is the initial state and the node **G** is the goal state. The problem is solved looking for the cost path starting from the initial node and finished when **G**'s node is found. Above of each node we have the heuristic value assigned by the heuristic  $h_1$ . A\* add the node **a** in OPEN and is expanded, generating three nodes **b**, **c** and **d**, the node **a** is removed from OPEN to put it in the solution path and **b**, **c**, **d** are added to OPEN, A\* expands the node with the lowest cost function in OPEN. The cost functions are  $f(b) = 8$ ,  $f(c) = 9$ , and  $f(d) = 12$ , then, the node **b** is chosen. In the next iteration **b** is expanded and the node **e** is generated and added to OPEN their cost function is calculated  $f(e) = 10$ . For the next iteration, **c** is chosen because it has the lowest cost function and expanded and their child **f** is added to OPEN. The node **f** has the lowest cost function in OPEN because their  $f(f) = 9$ . The **f**'s child is **G** and is added to OPEN. **G** is the lowest cost function and A\* ask if this is the goal, if it is, then, A\* stops and return the cost function of the problem which is  $f(G) = 9$ , which is optimal.

Considering that we change  $h_1$  by another heuristic  $h_2$  and for the same problem we will have the Figure 9. The right figure shows that A\* find the optimal cost path for the problem, which is 9. However,  $h_2$  expands more nodes than  $h_1$ . Each time a node is expanded, there is a time of evaluation of the heuristic and time of generation of each

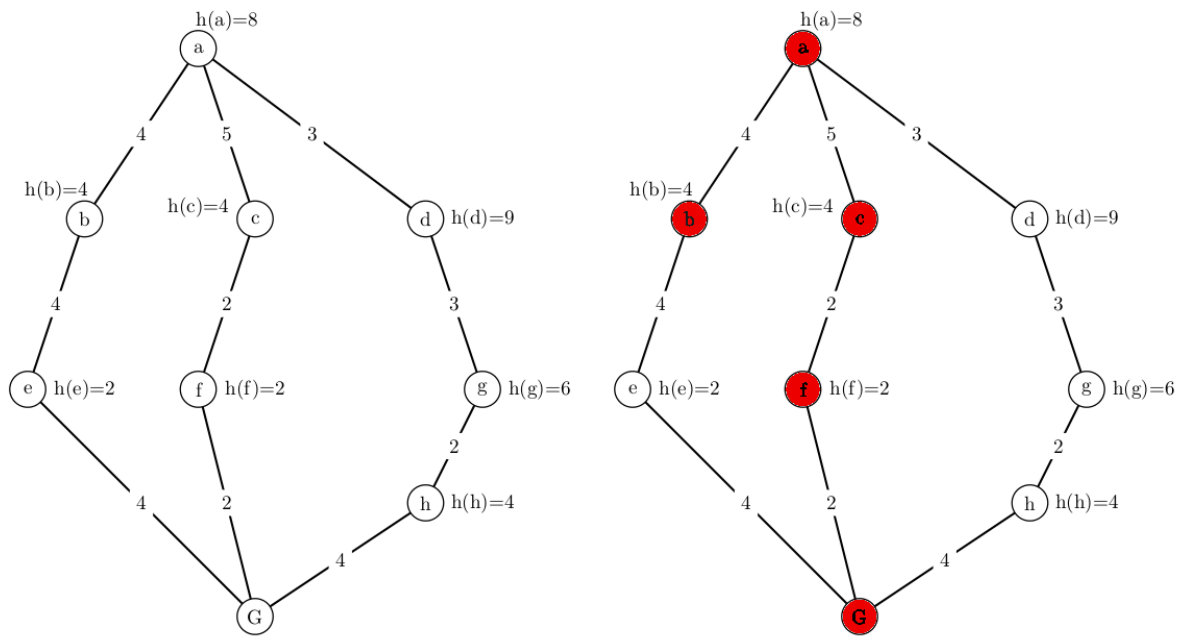


Figure 8 – The left figure represent the graph where above of each node is the heuristic value assigned by the heuristic  $h_1$  and their cost to go from one node to another, the initial state is **a** and the goal state is **G**. The right figure represents the graph with the solution found by A\*.

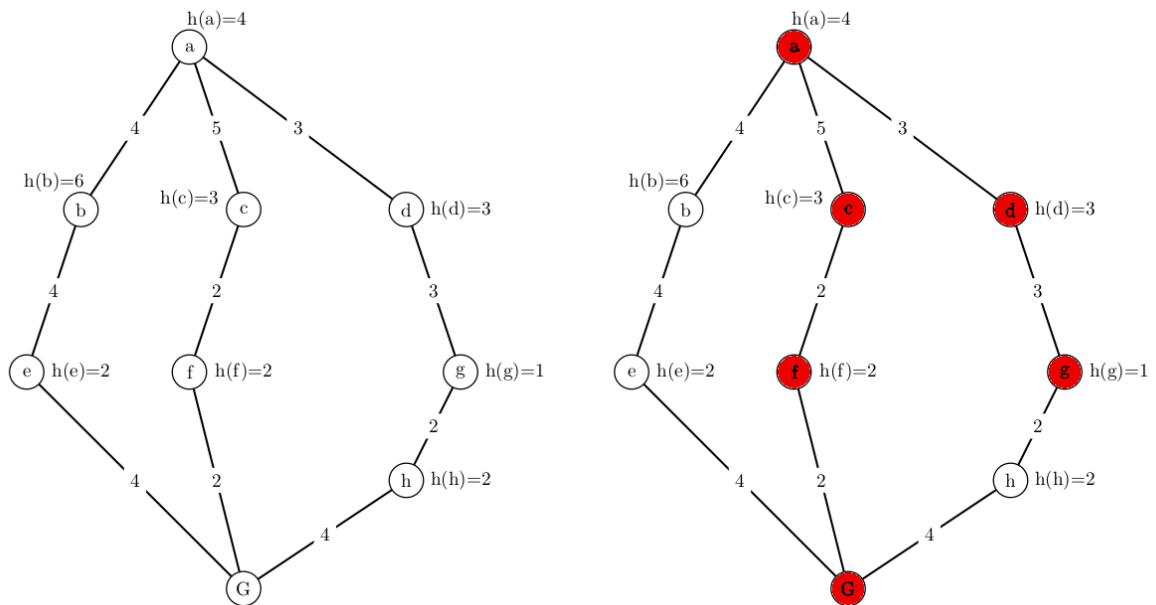


Figure 9 – The left figure represent the graph where above of each node is the heuristic value assigned by the heuristic  $h_2$  and their cost to go from one node to another, the initial state is **a** and the goal state is **G**. The right figure represents the graph with the solution found by A\*.

node involved. If more nodes are expanded in the graph, then more time is required to solve the problem. This mean that A\* with  $h_2$  are going to last more time to find the

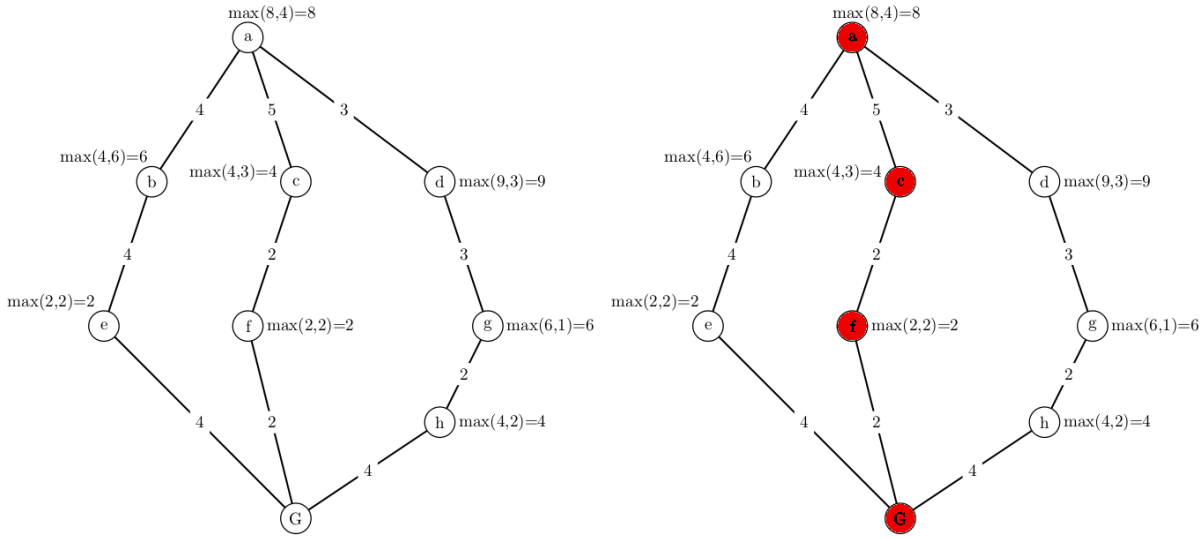


Figure 10 – The left figure represent the graph where above of each node is the maximum heuristic value of  $h_1$  and  $h_2$  and their cost to go from one node to another, the initial state is **a** and the goal state is **G**. The right figure represents the graph with the solution found by A\*.

solution than A\* with  $h_1$ .

One well-know approach to take advantage of a set of heuristics  $\zeta$  is to compute the maximum of all heuristics in  $\zeta$ . For example, given the two heuristics  $h_1$  and  $h_2$ , the maximum of  $h_1$  and  $h_2$  will tend to yield a more informed heuristic than  $h_1$  and  $h_2$  individually. In Figure 10, A\* using the maximum of heuristics values of  $h_1$  and  $h_2$  expands fewer nodes than each heuristic individually. This means that the solution of the problem is found quickly.

One can easily create thousands of heuristics for a problem instance. For example, each different abstraction of a problem domain results in a different PDB heuristic. It is possible to prove that the maximum of all heuristics in  $\zeta$  cannot be less informed than the maximum of any subset of  $\zeta$ . However, if  $\zeta$  is too large, then the resulting heuristic obtained by taking the maximum of all heuristics in  $\zeta$  can be too computationally expensive to be effective to guide search. That is why we select a subset of  $\zeta$  to guide the A\* search. This way we try to select the most informative heuristics in  $\zeta$  while preventing the resulting maximum heuristic to be computationally expensive.

In the next Chapter, we will introduce the meta-reasoning proposed for selecting a subset of  $\zeta$  to guide the A\* search.

### 3 Greedy Heuristic Selection

We present a greedy algorithm selection for solving the heuristic subset selection problem while optimizing different objective functions. We consider the following general optimization problem.

$$\underset{\zeta' \subseteq \zeta}{\text{minimize}} \Psi(\zeta', \nabla) \quad (3.1)$$

Where  $\Psi(\zeta', \nabla)$  is an objective function we want to minimize for a subset of heuristics  $\zeta'$  of  $\zeta$ . We use a hill-climbing algorithm we call Greedy Heuristic Selection (GHS) to solve Equation 3.1 for different functions  $\Psi$ .

---

**Algoritmo 1:** Greedy Heuristic Selection

---

**Input** : problem  $\nabla$ , set of heuristics  $\zeta$   
**Output** : heuristic subset  $\zeta' \subseteq \zeta$

- 1  $\zeta' \leftarrow \emptyset$
- 2 **while**  $\Psi(\zeta', \nabla)$  can be improved **do**
- 3      $h \leftarrow \arg \min_{h \in \zeta} \Psi(\zeta' \cup \{h\}, \nabla)$
- 4      $\zeta' \leftarrow \zeta' \cup \{h\}$
- 5 **return**  $\zeta'$

---

Algorithm 1 shows GHS. GHS receives as input a problem  $\nabla$ , a set of heuristics  $\zeta$ , and it returns a subset  $\zeta' \subseteq \zeta$ . In each iteration GHS greedily selects from  $\zeta$  the heuristics  $h$  which will result in the largest reduction of the value  $\Psi$  (line 3). GHS returns  $\zeta'$  once the objective function cannot be improved. In other words, the algorithm will halt when adding another heuristic does not improve the objective function.

#### 3.1 Minimizing Search Tree Size

The first objective function  $\Psi$  we consider is the number of node generations  $A^*$  performs while solving a given planning problem. The planning problem must be solvable, this means  $C^*$  cannot be infinity. When solving  $\nabla$  using the consistent heuristic function  $h_{max}(\zeta')$  for  $\zeta' \subseteq \zeta$ ,  $A^*$  generates a number of nodes that is bounded above by  $J(\zeta', \nabla)$ , defined as,

$$J(\zeta', \nabla) = |\{\text{children}(s) \in V \mid f_{max}(s, \zeta') \leq C^*\}| \quad (3.2)$$

We write  $J(\zeta')$  or simply  $J$  instead of  $J(\zeta', \nabla)$  whenever  $\zeta'$  and  $\nabla$  are free from the context. What's more, we assume that  $A^*$  expands all nodes  $s$  with  $f(s) \leq C^*$  while solving  $\nabla$ .

### 3.2 Minimizing A\*'s Running Time

Another objective function  $\Psi$  we consider is an approximation of the A\* running time and is defined as follows.

$$T(\zeta', \nabla) = J(\zeta', \nabla) \cdot (t_{h_{max}(\zeta')} + t_{gen}), \quad (3.3)$$

where, for any heuristic function  $h$ , the term  $t_{h_{max}(\zeta')}$  refers to the running time used for computing the  $h_{max}(\zeta')$  of any state  $s$ . We assume that  $t_{h_{max}(\zeta')}$  to be constant in the state space, which is a reasonable assumption for several heuristics such as PDBs.  $t_{gen}$  is the node generation time, which we also assume to be constant.

### 3.3 Estimating Tree Size and Running Time

In practice GHS uses approximations of  $J$  and  $T$  instead of their exact values. This is because computing  $J$  and  $T$  exactly would require solving  $\nabla$ , and this is what we obviously want to avoid.

We use the Culprit Sampler (CS) introduced by Barley et al., (2014) and the Stratified Sampling (SS) algorithm introduced by Chen (1992), for computing  $\hat{J}$  and  $\hat{T}$ . Each method has its strengths and weaknesses, which we explore in Chapter 4, where we make an empirical comparison of GHS with other approaches.

Both CS and SS must be able to quickly estimate the values of  $\hat{J}(\zeta')$  and  $\hat{T}(\zeta')$  for any subset  $\zeta'$  of  $\zeta$  so they can be used in GHS's optimization process.

### 3.4 Culprit Sampler (CS)

CS is an approach introduced by Barley et al., (2014) and works by running a time-bounded A\* search while sampling  $f$ -culprits and  $b$ -culprits. In this dissertation we use CS to estimate the values of  $\hat{J}$  and  $\hat{T}$ .

**Definition 3.4.1.** (*f-culprit*) Let  $\zeta = \{h_1, h_2, \dots, h_M\}$  be a set of heuristics. The  $f$ -culprit of a node  $n$  in an A\* search tree is defined as the tuple  $F(n) = \langle f_1(n), f_2(n), \dots, f_M(n) \rangle$ , where  $f_i(n) = g(n) + h_i(n)$ . For any  $n$ -tuple  $F$ , the counter  $C_F$  denotes the number of nodes  $n$  in the tree with  $F(n) = F$ .

**Definition 3.4.2.** (*b-culprit*) Let  $\zeta = \{h_1, h_2, \dots, h_M\}$  be a set of heuristics and  $b$  a lower bound on the solution cost  $\nabla$ . The  $b$ -culprit of a node  $n$  in an A\* search tree is defined as the tuple  $B(n) = \langle y_1(n), y_2(n), \dots, y_M(n) \rangle$ , where  $y_i(n) = 1$  if  $g(n) + h_i(n) \leq b$  and  $y_i(n) = 0$ , otherwise. For any binary  $n$ -tuple  $B$ , the counter  $C_B$  denotes the number of nodes  $n$  in the tree with  $B(n) = B$ .



CS works by running an  $A^*$  search bounded by a user-specified time limit. Then, CS compresses the information obtained in the  $A^*$  search (i.e., the  $f$ -values of all nodes expanded according to all heuristics  $h$  in  $\zeta$ ) in  $b$ -culprits, which are later used for computing  $\hat{J}$ . The  $f$ -culprits are generated as an intermediate step for computing the  $b$ -culprits, as we explain below. The maximum number of  $f$ -culprits and  $b$ -culprits in an  $A^*$  search tree is equal to the number of nodes in the tree expanded by the time-bounded  $A^*$  search. However, in practice the number of  $f$ -culprits is usually much lower than the number of nodes in the tree. Moreover, in practice, the total number of different  $b$ -culprits tends to be even lower than the total number of  $f$ -culprits. Given a planning problem  $\nabla$  and a set of heuristics  $\zeta$ , CS samples the  $A^*$  search tree as follows.

- 1.- CS runs  $A^*$  using  $h_{min}(s, \zeta) = \min_{h \in \zeta} h(s)$  until reaching a user-specified time limit.  $A^*$  using  $h_{min}$  expands node  $n$  if it were to expand  $n$  while using any of the heuristics in  $\zeta$  individually. For each node  $n$  expanded in this time-bounded search we store  $n$ 's  $f$ -culprit and its counter.
- 2.- Let  $f_{maxmin}$  be the largest  $f$ -value according to  $h_{min}$  encountered in the time-bounded  $A^*$  search described above. We now compute the set  $\mathbb{B}$  of  $b$ -culprits and their counters based on the  $f$ -culprits and on the value of  $f_{maxmin}$ . This is done by iterating over all  $f$ -culprits once.

The process described above is performed only once GHS's execution. The value of  $\hat{J}(\zeta', \nabla)$  for any subset  $\zeta'$  of  $\zeta$  is then computed by iterating over all  $b$ -culprits  $\mathbf{B}$  and summing up the relevant values of  $C_B$ . The relevant values of  $C_B$  represent the number of nodes  $A^*$  would expand in a search bounded by  $b$  if using  $h_{max}(\zeta')$ . This computation can be written as follows.

$$\hat{J}(\zeta', \nabla) = \sum_{\mathbf{B} \in \mathbb{B}} W(\mathbf{B}) \quad (3.4)$$

Where  $W(\mathbf{B})$  is 0 if there is a heuristic in  $\zeta'$  whose  $y$ -value in  $\mathbf{B}$  is zero (i.e., there is a heuristic in  $\zeta'$  that prunes all nodes compressed into  $\mathbf{B}$ ), and  $C_B$  otherwise. If the time-bounded  $A^*$  search with  $h_{min}$  expands all nodes  $n$  with  $f(n) \leq C^*$ , then  $\hat{J} = J$ . In practice, however, our estimate  $\hat{J}$  will tend to be much lower than  $J$ .

The value of  $\hat{T}$  is computed by multiplying  $\hat{J}$  by the sum of the evaluation time of each heuristic in  $\zeta'$ . The evaluation time of the heuristics in  $\zeta'$  is measured in a separate process, before executing CS, by sampling a small number of nodes from  $\nabla$ 's start state.

### 3.5 Stratified Sampling (SS)

Chen (1992), presented a method for estimating the search tree size of backtracking search algorithms by using a stratification of the search tree to guide its sampling. We define Chen's stratification as a type system.

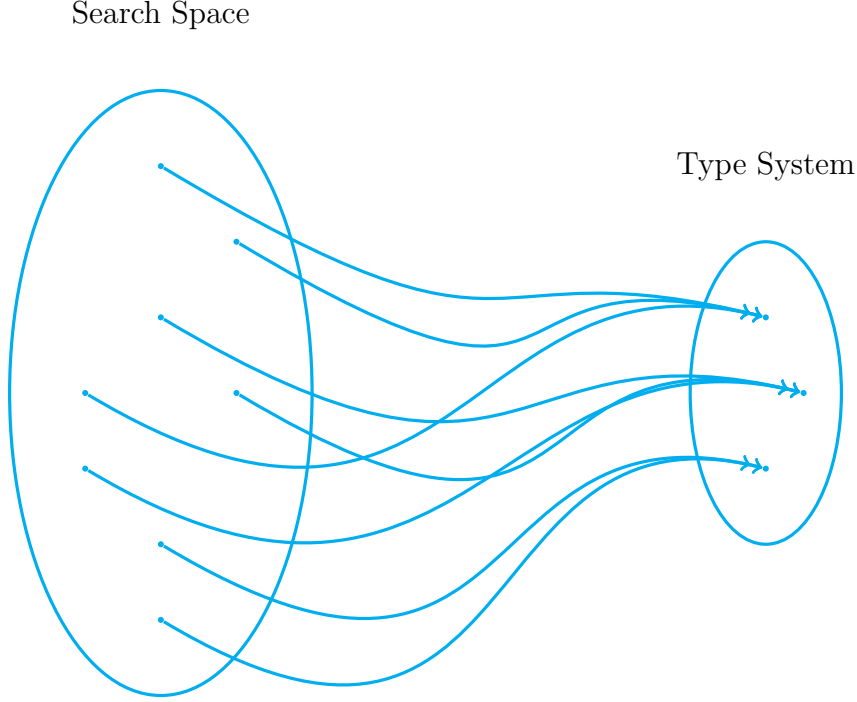


Figure 11 – Type system and the search space representation.

#### 3.5.1 Type System

A type system can be defined accounting for any property of nodes in the search tree. A common definition is the one that assigns two nodes to the same type if they have the same  $f$ -value (LELIS; ZILLES; HOLTE, 2013). A type system is effectively compressing the state space, as depicted in Figure 11, by mapping a set of states in the original state space to a single type.

**Definition 3.5.1.** *Type System* Let  $S = (N, E)$  be a search tree, where  $N$  is its set of nodes and for each  $n \in N$ ,  $\{n' | (n, n') \in E\}$  is  $n$ 's set of child nodes.  $TS = \{t_1, \dots, t_k\}$  is a type system for  $S$  if it is a disjoint partitioning of  $N$ . If  $n \in N$  and  $t \in TS$  with  $n \in t$ , we write  $TS(n) = t$ .

SS is able to approximate functions of the form  $\varphi = \sum_{n \in S} z(n)$ , where  $z$  is a function assigning a numerical value to a node, consequently  $\varphi$  is a numerical property of

$S$ . For example, if  $z(n) = |\text{children}(n)|$  for all  $n \in S$ , then  $\varphi$  is the size of the tree. Instead of summing all the  $z$ -values from the tree, **SS** considers subtrees rooted at nodes of the same type will have equal values of  $\varphi$  and only one of each type, chosen at random, is expanded. Given a search tree  $S$  and a type system  $TS$ , **SS** predicts  $\varphi$  in the following way: First, the search tree is sampled, returning the set of *representative-weight* pairs, with one such pair for every unique type seen during sampling. Second, in the pair  $\langle s, w \rangle$  in  $A$  for type  $t \in TS$ ,  $n$  is the unique node of type  $t$  that was expanded during search and  $w$  is an estimate of the number of nodes of type  $t$  in the tree. Finally,  $\varphi$  is then approximated by  $\hat{\varphi}$ , defined as,  $\hat{\varphi} = \sum_{\langle s, w \rangle \in A} w \times z(n)$ .

By making  $z(n) = |\text{children}|$  for all  $n \in S$  **SS** produces an estimate  $\hat{J}$  of  $J$ . Similarly to our approach with **CS**, we obtain  $\hat{T}$  by multiplying  $\hat{J}$  by the sum of heuristic evaluation time and generation time.

---

**Algoritmo 2:** SS, a single probe

---

**Input** : root  $n^*$  of a tree and a type system  $TS$ , upper bound  $d$ , heuristic function  $h$

**Output** : an array of sets  $A$ , where  $A[i]$  is the set of (node, weight) pairs  $\langle s, w \rangle$  for the nodes  $n$  expanded at level  $i$ .

```

1  $A[0] \leftarrow \{\langle s^*, 1 \rangle\}$ 
2  $i \leftarrow 0$ 
3 while  $A[i]$  is not empty do
4   for each element  $\langle s, w \rangle$  in  $A[i]$  do
5     for each child  $\hat{n}$  of  $n$  do
6       if  $g(\hat{n}) + h(\hat{n}) \leq d$  then
7         if  $A[i+1]$  contains an element  $\langle n', w' \rangle$  with  $TS(n') = TS(\hat{n})$  then
8            $w' \leftarrow w' + w$ 
9           with probability  $w/w'$ , replace  $\langle n', w' \rangle$  in  $A[i+1]$  by  $\langle \hat{n}, w' \rangle$ 
10        else
11          insert new element  $\langle \hat{n}, w \rangle$  in  $A[i+1]$ 
12    $i \leftarrow i + 1$ 

```

---

In **SS** the types are required to be partially ordered: a node's type must be strictly greater than the type of its parent. This can be guaranteed by adding the depth of a node to the type system and then sorting the types lexicographically. That is why in our implementation of **SS** types at one level are treated separately from types at another level by the division of  $A$  into groups  $A[i]$ , where  $A[i]$  is the set of *representative-weight* pairs for the types encountered at level  $i$ . If the same type occurs on different levels the occurrences will be treated as if they were different types – the depth of search is implicitly included into all of our type systems.

Algorithm 2 shows **SS** in detail. Representative nodes from  $A[i]$  are expanded to

get representative nodes for  $A[i + 1]$  as follows.  $A[0]$  is initialized to contain only the root of the search tree to be probed, with weight 1 (Line 1). In each iteration (Lines 4 through 11), all nodes in  $A[i]$  are expanded. The children of each node in  $A[i]$  are considered for inclusion in  $A[i + 1]$  if their  $f$ -value do not exceed an upper bound  $d$  provided as input to **SS**. If a child  $\hat{n}$  has a type  $t$  that is already represented in  $A[i + 1]$  by another node  $n'$ , then a *merge* action on  $\hat{n}$  and  $n'$  is performed. In a merge action we increase the weight in the corresponding *representative-weight* pair of type  $t$  by the weight  $w(n)$  of  $\hat{n}$ 's parent  $n$  (from level  $i$ ) since there were  $w(n)$  nodes at level  $i$  that are assumed to have children of type  $t$  at level  $i + 1$ .  $\hat{n}$  will replace  $n'$  according to the probability shown in Line 9. Chen (1992), proved that this probability reduces the variance of the estimation. Once all the states in  $A[i]$  are expanded, we move to the next iteration.

One run of the **SS** algorithm is called a *probe*. Chen (1992), proved that the expected value of  $\hat{\varphi}$  converges to  $\varphi$  in the limit as the number of probes goes to infinity. As Lelis et al., (2014) showed, **SS** is not able to detect duplicated nodes in its sampling process. As a result, since  $A^*$  does not expand duplicates, **SS** usually overestimates the actual number of nodes  $A^*$  expands. Thus, in the limit, as the number of probes grows large, **SS**'s prediction converges to a number which is likely to overestimate the  $A^*$  search tree size. We test empirically whether **SS** is able to allow **GHS** to make good subset selects despite being unable to detect duplicated nodes during sampling.

Similarly to **CS**, we also define a time-limit to run **SS**. We use **SS** with an iterative-deepening approach in order to ensure an estimate of  $\hat{J}$  and  $\hat{T}$  before reaching the time limit. We set the upper bound  $d$  to the heuristic value of the start state and, after performing  $p$  probes, if there is still time, we increase  $d$  to twice its previous value. The values of  $\hat{J}$  and  $\hat{T}$  is given by the prediction produced for the last  $d$ -value in which **SS** was able to sample from.

**SS** must also be able to estimate the values of  $\hat{J}(\zeta')$  and  $\hat{T}(\zeta')$  for any subset  $\zeta'$  of  $\zeta$ . This is achieved by using **SS** to estimate  $b$ -culprits (see Definition 3.4.2) instead of the search tree size directly. Similarly to **CS**, **SS** used  $h_{min}$  of the heuristics in  $\zeta$  to decide when to prune a node (see Line 6 of Algorithm 2) while sampling. This ensures that **SS** expands a node  $n$  if  $A^*$  employing at least one of the heuristics in  $\zeta$  would expand  $n$  according to bound  $d$ . The  $C_B$  counter of each  $b$ -culprit  $B$  encountered during **SS**'s probe is given by,

$$C_B = \sum_{\langle n, w \rangle \in A \wedge B(n)=B} w \quad (3.5)$$

We recall that to compute  $B(n)$  for node  $n$  one needs to define a bound  $b$ . Here we use the bound  $d$  used by **SS**. The average value of  $C_B$  across  $p$  probes is used to predict the search tree size for a given subset  $\zeta'$ . As explained for **CS**, this can be done by traversing

all  $b$ -culprits once.

### 3.6 An Example of an SS Execution

In Figure 12 we see an example of an SS run. Nodes have different types if they have different colors or are in different levels of the search tree.

In level 1 we have the root node, and its  $w$  is set to one. In level 2 three nodes are generated by the root node. The nodes in level 2 have the following types: red and blue, and initially each node receives the same  $w$  from their parent (in this case  $w = 1$ ). Since SS assumes that nodes of the same type root subtrees of the same size, then only one node of each type needs to be expanded. In level 2 there are two nodes with the red type. In that way, we choose randomly one of them to expand. Let us suppose we choose the right red node. Then, we have to update the number of nodes with the red type by summing their  $w$ -values. Since both nodes of the red type have  $w = 1$ , then we sum the  $w$ -values results in  $w = 2$ .

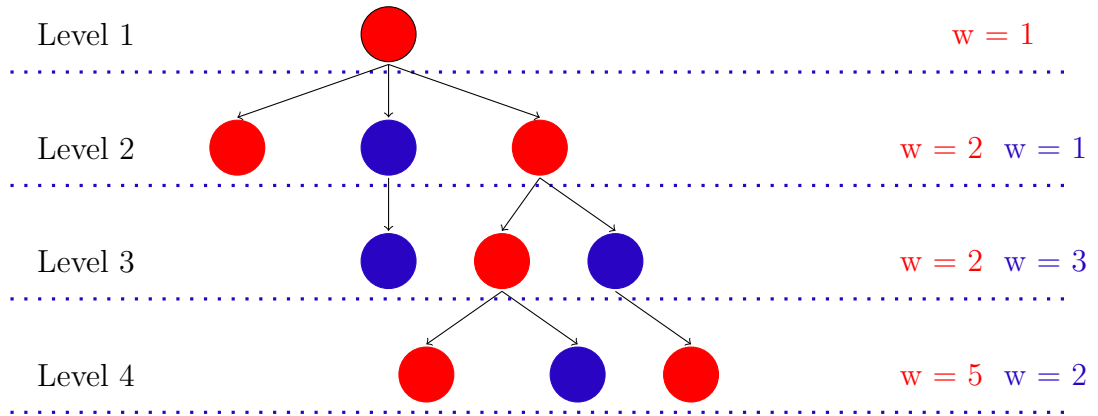


Figure 12 – Search tree using type system

Let us now suppose that nodes in level 2 are expanded. The blue node generates one node of the blue type and the red node generates two nodes with the following types: red and blue. To compute the  $w$ -values of the nodes at level 3 we do the following. The  $w$ -value of nodes of the red type is computed by summing up the  $w$ -values of all parents that generated a node of the red type at level 3. In this case only one node at level 2 generated a node of the red type at level 3. Since this node has a  $w$ -value of 2, then SS estimates that there is 2 nodes of the red type at level 3 (red type's  $w$ -value). Two nodes at level 2 generate a blue node at level 3, thus we have to sum their  $w$ -values, resulting in  $w = 2 + 1 = 3$  for blue nodes.

Finally, the number of nodes expanded in the search tree is obtained summing all  $w$ . In summary, SS estimates that the size of the search tree sampled is  $15 + 1 = 16$ .

If we run the algorithm again maybe in level 2 instead of choosing the right red node we could choose the left red node and the nodes generated in level 3 would be different. As a consequence, the predicted number of nodes generated by SS could be higher or lower than the current estimate. In other words, each probe of the algorithm could result in a different value of the estimated tree size. In order to obtain accurate estimates of the search tree size one usually has to perform several probes of SS and average the results.

## 4 Empirical Evaluation

This chapter is organized as follows. First we show that **SS** is able to produce very accurate estimates of the number of nodes generated by IDA\*, a search algorithm that does not detect duplicated nodes during search, while solving domain-independent planning problems. Next, we show that **SS** produces very poor estimates of the number of nodes generated by A\* on the same problem domains. Although **SS** is unable to produce good predictions for A\*, we show empirically that the **SS**'s predictions allow one to select, from a pool of heuristics, the one that will result in the smallest A\* search tree size.

After testing the effectiveness of **SS**, we test whether the heuristic subsets selected by **GHS** (using both **SS** and **CS**) are near optimal. That is, we are interested in knowing how  $J(\zeta', \nabla)$  compares with  $J(\zeta, \nabla)$ , which is provably optimal.

Finally, we compare the effectiveness of **GHS** with other state-of-the-art planners.

### 4.1 SS for Predicting the IDA\* Search Tree Size

**SS** was shown to produce good predictions of the IDA\* search tree size ([LELIS; ZILLES; HOLTE, 2013](#)). IDA\* is a heuristic search algorithm that uses the same cost function used by A\* but it searches the state space in a depth-first manner. Moreover, in contrast with A\*, IDA\* does not detect duplicated nodes.

In this section we show that **SS** is able to produce good predictions for IDA\* also in domain-independent planning, and we will show that it produces very inaccurate predictions for A\*. In this experiment **SS** estimates the search tree size generated by IDA\* using a consistent heuristic. **SS** estimates the size of the search tree up to some defined  $f$ -layer in the tree.

We first run IDA\* in domain-independent planning benchmark problems with a 30-minute time limit. Then we run **SS** limited by different  $f$ -layers encountered in the IDA\* searches. We experiment with the following number of probes: 1, 10, 100, 1000 and 5000.

We use the *relative unsigned error* ([LELIS; ZILLES; HOLTE, 2012](#)), which we call relative-error, for short, to measure prediction accuracy. The relative-error is shown in Equation 4.1 below.

$$\frac{\sum_{s \in PI} \frac{Pred(s,d) - R(s,d)}{R(s,d)}}{|PI|} \quad (4.1)$$

Table 1 – Precision of SS against IDA\* for 1, 10, 100, 1000 and 5000 probes using *hmax* heuristic.

Domain	hmax													
	IDA*	time	relative-error							time				n
			1	10	100	1000	5000	1	10	100	1000	5000		
Barman	8835990.00	6016.38	0.60	0.45	0.20	0.07	0.04	0.06	0.32	3.21	32.57	214.59	20	
Elevators	1012570.00	4987.57	0.84	0.42	0.23	0.13	0.10	1.40	9.85	96.37	994.33	4425.93	20	
Floortile	30522300.00	3919.72	2.02	0.62	0.40	0.14	0.11	0.01	0.07	0.69	6.93	36.60	2	
Nomystery	6565740.00	3256.86	0.53	0.26	0.07	0.03	0.01	0.07	0.38	3.63	36.35	181.03	20	
Openstacks	80108.50	4017.19	0.03	0.03	0.03	0.03	0.03	94.79	774.86	1067.84	10929.00	11174.30	20	
Parcprinter	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.04	0.35	3.48	17.29	20	
Parking	374925.00	5607.50	0.17	0.04	0.01	0.00	0.00	1.79	11.36	114.28	1196.83	5835.03	20	
Pegsol	68763.70	5.00	0.17	0.04	0.02	0.01	0.00	0.01	0.04	0.37	3.69	17.88	20	
Scanalyzer	8449890.00	4920.58	0.43	0.25	18.63	0.02	0.01	3.13	28.79	273.74	3033.06	10254.00	20	
Sokoban	3118530.00	3932.69	0.41	0.26	0.11	0.05	0.04	0.31	2.00	21.42	222.47	1056.61	20	
Tidybot	444473.00	5632.08	300.86	1072.40	5.88	0.01	0.01	4.40	26.48	238.76	2747.10	11925.40	20	
Transport	2622880.00	2253.51	0.63	0.54	0.24	0.15	0.11	0.09	0.61	5.89	59.37	290.31	20	
Visitall	71032400.00	3704.78	0.12	0.04	0.01	0.00	0.00	0.00	0.05	0.56	5.77	28.07	20	
Woodworking	5139070.00	4944.76	1.28	0.69	0.27	0.17	0.07	0.15	1.33	13.21	130.82	664.08	20	

- $PI$  is the set of all instances of a problem domain.
- $Pred(s, d)$  is the estimated search tree size generated by IDA\* for start state  $s$  and cost bound  $d$ .
- $R(s, d)$  is the real number of nodes expanded by IDA\* for start state  $s$  and cost bound  $d$ .

A relative error of 0.0 means that SS was able to produce perfect predictions.

Table 1 shows the relative-error for different number of probes. The heuristic used in this experiment is *hmax*. The column  $n$  shows the number of problems that were used to compute the relative-error, the average number of nodes generated by IDA\*, and the average running time of IDA\* and SS. The table of results also shows the average number of nodes generated by IDA\*.

The relative-error tends to reduce as one increases the number of probes. For example, in Barman, the relative-error goes from 0.60 for 1 probe to 0.45 for 10 probes, 0.20 for 100 probes, 0.07 for 1000 probes, and 0.04 for 5000 probes. In the case of running time, while the number of probes increase, SS needs to spend more time approximating the size of the search tree. That is why the overall running time of SS increases as one increases the number of probes. For example, in Barman, the time goes from 0.06 seconds for 1 probe to 0.32 seconds for 10 probes, 3.21 seconds for 100 probes, 32.57 seconds for 1000 probes and 214.59 seconds for 5000 probes. There are domains such as: Parcprinter, Parking, Pegsol and Visitall that have perfect score using 5000 probes. In the case of Tidybot, the relative-error using 1 probe is smaller than using 10 probes, which happens due to the stochastic nature of SS.

Each domain of the 2011 IPC benchmark contains 20 instances. The results of Floortile in Table 1 were computed using only 2 instances ( $n = 2$ ). This is because IDA\*



was able to fully complete an iteration for its initial cost bound for only two instances (opt-p01-001 and opt-p03-006). In summary, we showed that for 2011 IPC domains, SS is able to produce good predictions of the search tree expanded by IDA\*.

## 4.2 SS for Predicting the A\* Search Tree Size

Table 2 shows the relative-error of SS while predicting the number of nodes generated by A\*. In this experiment we tested different heuristic functions: iPDB (HASLUM et al., 2007), LM-Cut (POMMERENING; HELMERT, 2013) and M&S (NISSIM; HOFFMANN; HELMERT, 2011). Column  $n$  shows the number of instances solved by A\* using the three heuristics. For this experiment we only use the instances that are independently solved by A\* using each of the three heuristics within a 30-minute time limit. The columns with A\* represents the average of number of nodes expanded by A\*. The column with relative-error represents the relative-error for the solved instances. We used 500 probes in this experiment.

Table 2 – Poor prediction of SS against A\* using ipdb, LM-Cut and M&S with 500 probes

Domain	ipdb		LM-Cut		M&S		n
	A*	relative-error	A*	relative-error	A*	relative-error	
Barman	1.72e+07	8.68e+31	7.45e+06	2.21e+30	6.67e+06	1.26e+36	4
Floortile	1.40e+07	1.74e+18	702435	4.68e+14	4.46e+06	1.90e+12	4
Nomystery	40169.7	6.71e+32	267100	6.14e+19	8236	1.20e+20	9
Openstacks	570099	0.61884	570099	0.677425	569984	0.672143	4
Parcprinter	1157	2.56e+22	1363.67	2.33e+21	766.333	6.36e+20	3
Pegsol	841693	2901.39	398221	6859.86	933430	779.017	16
Scanalyzer	337894	3.94e+33	334747	7.58e+31	337833	2.42e+31	3
Sokoban	376755	1.04e+07	45374	2.74e+06	739775	5.60e+08	9
Transport	1.89e+06	2.91e+38	1.49e+06	1.15e+25	1.73e+06	1.50e+29	2
Visitall	253710	1.69e+46	253195	1.69e+46	253521	1.71e+46	8
Woodworking	3.21e+06	2.53e+18	3.20e+06	2.76e+18	3.21e+06	2.48e+18	3

As can be observed in Table 2, SS usually overestimates the number of nodes expanded by A\* by several orders of magnitude, for example, in Barman SS’s relative-error is  $1.26 \times 10^{36}$  (M&S),  $8.68 \times 10^{31}$  (iPDB), and  $2.21 \times 10^{30}$  (LM-Cut). The exception in Table 2 is Openstacks, where SS produces accurate predictions. This is probably because the A\* search tree does not have a large number of duplicated nodes in this domain. That is, the A\* search tree is similar to the IDA\* search tree for Openstack problems.

## 4.3 Approximation Analysis for SS and A\*

Although SS is not able to produce good predictions of the A\* search tree size, here we show empirically that SS allows one to make good heuristic subset selections. In this experiment we use the following heuristics: iPDB, LM-Cut, and 10 independently generated GA-PDBs (EDELKAMP, 2007).

Our goal with this experiment is to show that in order to select heuristics, the predictions produced by **SS** do not have to be accurate in absolute terms, but they have to be accurate in relative terms. That is, if  $A^*$  using heuristic  $h_1$  expands fewer nodes than when using  $h_2$ , then **SS**'s predictions could be arbitrarily inaccurate for both  $h_1$  and  $h_2$ , as long as the prediction estimates that  $A^*$  expands fewer node when using  $h_1$ .

In order to investigate whether **SS** is able to allow one to select the best heuristic in a set of size two:  $\{h_1, h_2\}$ , we construct a graph where the y-axis is the ratio  $J(h_1)/J(h_2)$  (the actual number of nodes generated by  $A^*$  using  $h_1$  to the number of nodes generated by  $A^*$  using  $h_2$ ); and the x-axis is the ratio  $\hat{J}(h_1)/\hat{J}(h_2)$  (the predicted search tree size of  $h_1$  to the predicted search tree size of  $h_2$ ). The plot is divided in four quadrants, as shown in Figure 13. If a data point falls in quadrants II and III it means that **SS** is able to correctly select the heuristic that will yield the smallest search tree size. If the point falls in quadrants I or IV it means that **SS** selects the heuristic that will yield the largest tree size.

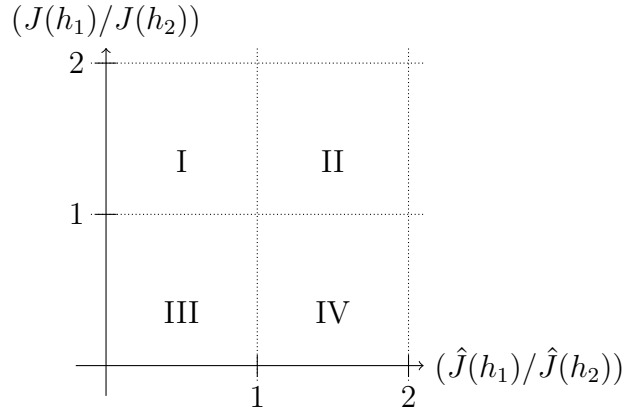


Figure 13 – Cartesian Plane with domain  $\langle 0, 2 \rangle$  and range  $\langle 0, 2 \rangle$ . Note that the ratio values could be larger than 2, and we assume that any ratio larger than 2 is automatically set to 2 in this experiment.

A data point falls in each quadrant (I, II, III, and IV) if the corresponding boolean expression is true.

$$\text{I} - J(h_1) > J(h_2) \wedge \hat{J}(h_2) \geq \hat{J}(h_1)$$

$$\text{II} - \left( J(h_1) > J(h_2) \wedge \hat{J}(h_1) > \hat{J}(h_2) \right) \vee J(h_1) = J(h_2).$$

$$\text{III} - J(h_2) > J(h_1) \wedge \hat{J}(h_2) > \hat{J}(h_1).$$

$$\text{IV} - J(h_2) > J(h_1) \wedge \hat{J}(h_1) \geq \hat{J}(h_2).$$

We assign a data point to quadrant II if  $J(h_1) = J(h_2)$  because in this situation  $A^*$  generates the same number of nodes when using  $h_1$  or  $h_2$ .

Domain	II and III (%)
Elevators	78.57
Floortile	96.08
Nomystery	71.82
Parcprinter	70.50
Pegsol	96.83
Scanalyzer	100.00
Sokoban	89.31
Tidybot	100.00
Transport	51.78
Visitall	98.05
Woodworking	100.00

Table 3 – Percentage of choices **SS** made correctly.

For each problem instance of the IPC 2011 benchmark we generate 12 heuristics: iPDB, LM-Cut, and 10 independently generated **GA-PDBs**, and we generate a data point  $(\hat{J}(h_1)/\hat{J}(h_2)$  and  $J(h_1)/J(h_2)$ ) for each possible pairwise combination of heuristics. In Table 3 we present the percentage of data points falling in quadrants II and III. This percentage is above 50% for all domains, meaning that **SS** is able to correctly select the heuristic that yield smaller search trees in at least half of the binary decisions tested (i.e., should one choose  $h_1$  or  $h_2$ ?). This experiment indicates that although **SS** produces inaccurate estimates of the A\* search tree size, **SS** is capable of selecting the heuristic that will allow A\* to generate fewer nodes in the majority of the cases tested.

## 4.4 Empirical Evaluation of $\hat{J}$ and $\hat{T}$

In the previous experiment we tested whether **SS** allows one to correctly select the heuristic, from a set of size two, that yield the smallest A\* search tree size. Next, we test whether the approximations of  $\hat{J}$  provided by **CS** and **SS** allow **GHS** to make good subset selections from larger sets, with thousands of heuristics. This test is made by comparing  $J(\zeta')$  with  $J(\zeta)$ , which is provably minimal. We restrict our experiment to  $J$  because, in contrast with  $J$ , there is no easy way to find the minimum of  $T$  for a subset in general.

We collect values of  $J(\zeta)$  and  $J(\zeta')$  as follows. For each problem instance  $\nabla$  in our test set we generate a set of PDB heuristics using the **GA-PDB** algorithm as described by Barley et al., (2014)—we call each PDB generated by this method a **GA-PDB**. The number of **GA-PDBs** generated is limited in this experiment by 1,200 seconds and 1GB of memory. Also, all **GA-PDBs** we generate have 2 million entries each. The **GA-PDBs** are generated for our  $\zeta$  set. **GHS** then selects a subset  $\zeta'$  of  $\zeta$ . Finally, we use  $h_{max}(\zeta')$  and  $h_{max}(\zeta)$  to independently try to solve  $\nabla$ . We call the system which uses A\* with  $h_{max}(\zeta)$  the **Max** approach. For **GHS** we allow 600 seconds for selecting  $\zeta'$  and for running A\* with  $h_{max}(\zeta')$ , and for **Max** we allow 600 seconds for running A\* with  $h_{max}(\zeta)$ . Since we use 1,200 seconds

Table 4 – Ratios of the number of nodes generated using  $h_{max}(\zeta')$  to the number of nodes generated using  $h_{max}(\zeta)$ .

Domain	SS		CS		$ \zeta $	$n$
	Ratio	$ \zeta' $	Ratio	$ \zeta' $		
Barman	1.11	17.70	1.50	30.25	5,168.50	20
Elevators	11.50	2.00	1.04	21.00	168.00	1
Floortile	1.02	43.07	1.02	42.36	151.29	14
Openstacks	1.00	1.00	1.00	1.00	390.69	13
Parking	1.00	5.53	1.01	7.26	21.74	19
Parcprinter	3.62	1.00	2.21	13.00	1,189.00	1
Pegsol	1.00	31.00	1.00	57.00	90.00	2
Scanalyzer	1.23	30.57	1.57	19.43	72.86	7
Sokoban	1.32	7.00	1.01	24.00	341.00	1
Tidybot	1.00	2.35	1.00	8.59	3,400.18	17
Transport	1.00	14.70	1.03	14.30	171.70	10
Visitall	1.03	99.33	1.19	48.67	256.33	3
Woodworking	32.43	3.00	199.66	5.00	1,289.00	5

to generate the heuristics, both **Max** and **GHS** are allowed 1,800 seconds in total for solving each problem. In this experiment we test both **CS** and **SS**.

In this experiment we refer to the approach that runs A\* guided by a heuristic subset selected by **GHS** using **CS** as **GHS+CS**. Similarly, we write **GHS+SS** when **SS** is used as predictor to make the heuristic subset selection.

Table 4 shows the average ratios of  $J(\zeta')$  to  $J(\zeta)$  for both **SS** and **CS** in different problem domains. The value of  $J$ , for a given problem instance, is computed as the number of nodes generated up to the largest  $f$ -layer which is fully expanded by all approaches tested (**Max**, **GHS+SS** and **GHS+CS**). We only present results for instances that are not solved during **GHS**'s **CS** sampling process. Column “ $n$ ” shows the number of instances used to compute the averages of each row. We also show the average number of **GA-PDBs** generated ( $|\zeta|$ ) and the average number of **GA-PDBs** selected by **GHS** ( $|\zeta'|$ ). This experiment shows that for most of the problems **GHS**, using **CS** or **SS**, is selecting good subsets of  $\zeta$ . For example, in Tidybot **GHS** selects only a few **GA-PDBs** out of thousands when using either **SS** or **CS**.

The exceptions in Table 4 are the ratios for Elevators, Parcprinter, and Woodworking. In all three domains **GHS+SS** failed to make good heuristic selections because **SS** was not able to sample “deep enough” into the problem’s search tree. For instance, for the Elevators instance **SS** was able to perform only 245 probes with the initial  $f$ -boundary within the time limit of 300 seconds. **GHS+CS** is able to make better selections for this instance because it is able to sample deeper into the search tree. For the Parcprinter instance as well as the Woodworking instances neither **SS** nor **CS** are able to sample deep enough to make good heuristic selections. These results suggest that **SS** could benefit from

Table 5 – Coverage of **SS**, **CS** and **Max** on the 2011 IPC benchmarks. For GHS using only **GA-PDBs** heuristics.

Domain	SS	CS	Max
Barman	8	7	4
Elevators	19	19	19
Floortile	10	10	9
Nomystery	20	20	20
Openstacks	17	17	11
Parcprinter	17	15	14
Parking	1	1	1
Pegsol	19	19	19
Scanalyzer	10	10	10
Sokoban	20	20	20
Tidybot	14	13	11
Transport	14	14	14
Visitall	18	18	18
Woodworking	12	11	12
Total	199	194	182

an adaptive approach for choosing **SS**’s number of probes. For example, for the Elevators instance **SS** could have performed better by sampling deeper into the tree with fewer probes. We intend to investigate this direction in future works.

In total, out of the 280 instances of the IPC 2011 benchmark set, **GHS+SS** solved 199 problem instances in this experiment, while **GHS+CS** only solved 194 problem instances. The numbers of instances solved are shown in Table 5.

## 4.5 Comparison with Other Planning Systems

The objective of the next experiment is to test the quality of the subset of heuristics **GHS** selects while optimizing different objective functions. Our evaluation metric is coverage, i.e., number of problems solved within a 1,800 second time limit. We note that the 1,800-second limit includes the time to generate  $\zeta$ , select  $\zeta'$ , and run  $A^*$  using  $h_{max}(\zeta')$ . The  $\zeta$  set of heuristics is composed of a number of different **GA-PDBs**, a PDB heuristic produced by the **iPDB** method and the **LM-Cut** heuristic. The generation of **GA-PDBs** is limited by 600 seconds and 1GB of memory. We use one fourth of 600 seconds to generate **GA-PDBs** with each of the following number of entries:  $\{2 \cdot 10^3, 2 \cdot 10^4, 2 \cdot 10^5, 2 \cdot 10^6\}$ . Our approach allows one to generate up to thousands of **GA-PDBs**. For every problem instance, we use exactly the same  $\zeta$  set for **Max** and all **GHS** approaches.

## 4.6 Systems Tested

GHS is tested while minimizing the A\* search tree size (**Size**) and the A\* running time (**Time**). We also use GHS to maximize the sum of heuristic values in the state space (**Sum**), as suggested by Rayner et al., (2013). Rayner et al., (2013) assumed that one could uniformly sample states in the state space in order to estimate the sum of the heuristic values for a given heuristic subset. Since we are not aware of any method to uniformly sample the state space of domain-independent problems, we adapted the Rayner et al.’s method by using SS to estimate the sum of heuristic values in the search tree rooted at  $\nabla$ ’s start state. We write **Size+SS** to refer to the approach that used A\* guided by a heuristic selected by GHS while minimizing an estimate of the search tree size provided by SS. We follow the same pattern to name the other possible combinations of objective functions and prediction algorithms (e.g., **Time+CS**).

In addition to experimenting with all combinations of prediction algorithms (**CS** and **SS**) and objective functions (**Time**, **Size**), we also experiment with an approach that minimizes both the search tree size and the running time as follows. First we create a pool of heuristics  $\zeta$  composed solely of **GA-PDB** heuristics, then we apply GHS while minimizing tree size and using **SS** as predictor. We call the selection of a subset of **GA-PDBs** as the *first selection*. Once the first selection is made, we test all possible combinations of the resulting  $h_{max}(\zeta')$  added to **iPDB** and **LM-Cut** heuristics while minimizing the running time as estimated by **CS**—we call this step the *second selection*. We call the overall approach **Hybrid**.

We compare the coverage of the GHS approaches with several other state-of-the-art planners. Namely, we experiment with **RIDA\*** (BARLEY; FRANCO; RIDDLE, 2014), two variants of **StoneSoup** (**StSp1** and **StSp2**) (HELMERT; RÖGER; KARPAS, 2011), two versions of **Symba** (**SY1** and **SY2**) (TORRALBA, 2015), and A\* being independently guided by the maximum of all heuristics in  $\zeta$  (**Max**), **iPDB**, **LM-cut** and **Merge & Shrink** (**M&S**) (NISSIM; HOFFMANN; HELMERT, 2011). The results are presented in Table 6.

Table 6 – Coverage of different planning systems on the 2011 IPC benchmarks.

Domains	Hybrid	CS		SS		Sum	Max	RIDA*	SY1	SY2	StSp1	StSp2	iPDB	LM-Cut	M&S
		Time	Size	Time	Size										
Barman	7	5	4	4	4	4	4	4	10	11	4	4	4	4	4
Elevators	19	19	19	19	19	19	19	19	20	20	18	18	18	17	12
Floortile	15	14	14	14	14	14	14	14	14	14	14	14	14	8	10
Nomystery	20	20	20	19	19	20	20	20	16	16	20	20	14	19	18
Openstacks	17	17	15	17	15	15	11	15	20	20	17	17	15	17	17
Parcprinter	18	18	18	16	15	19	18	18	17	17	18	18	17	16	16
Parking	7	7	2	7	2	2	2	7	2	1	5	5	2	7	7
Pegsol	18	18	19	19	19	19	19	19	19	20	19	19	17	20	19
Scanalyzer	13	14	12	11	14	14	14	14	9	9	14	14	12	10	11
Sokoban	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20
Tidybot	17	16	16	16	16	16	15	17	15	17	16	16	16	14	9
Transport	14	13	10	11	13	11	9	10	10	11	7	8	6	8	7
Visitall	18	18	18	15	18	18	18	18	12	12	16	16	10	16	16
Woodworking	16	15	15	12	16	16	16	15	20	20	15	15	15	9	9
Total	219	214	202	200	204	207	199	210	204	208	203	204	180	185	175

## 4.7 Discussion of the Results

The system that solves the largest number of instances is **Hybrid** (219 in total). Its combination of the strengths of both **SS** and **CS** has proven particularly effective on the Barman domain where **Hybrid**’s first selection contains good subsets of GA-PDBs and its second selection recognizes when it must not add the **iPDB** and **LM-Cut** heuristics to the first selection. As a result, **Hybrid** solves more problems on this domain than any other **GHS** approach.

**Time+CS** also performs well—it solves 214 problems. **Hybrid** and **Time+CS** are superior to all other approaches tested. When minimizing  $\hat{J}$  or maximizing **Sum**, **GHS** tends to add accurate heuristics to the selected subset, irrespective of their evaluation time. Thus, **GHS** frequently selects **LM-Cut** which is often the heuristic that most reduces the search tree size and most increases the sum of heuristic values. However, **LM-Cut** is computationally expensive, and often the search is faster if **LM-Cut** is not in  $\zeta'$ . Both **Hybrid** and **Time+CS** often recognize when **LM-Cut** should not be in  $\zeta'$  because they account for the heuristics’ evaluation time.

Interestingly, while **Time+CS** solves 214 instances, **Time+SS** solves only 200. We conjecture that this is due to **SS** not detecting duplicate nodes during sampling and thus substantially overestimating  $A^*$ ’s running time. As a result, similarly to the **Size** and **Sum** approaches, **Time+SS** often mistakenly adds the accurate but expensive **LM-Cut** heuristic in cases where  $A^*$  would be faster without **LM-Cut**.

**RIDA\*** is the most similar system to **GHS**; it selects a subset of heuristics by using an evaluation method similar to **CS**. Starting with an empty subset, it evaluates all subsets of size  $i$  before evaluating subsets of size  $i + 1$ . This limits **RIDA\*** to considering only tens of heuristics in its pool. Specifically, **RIDA\*** uses 42 GA-PDBs, **iPDB**, and **LM-Cut** in its pool. By contrast, **GHS** may consider thousands of heuristics.

Selecting from large sets of heuristics can be helpful, even if most of the heuristics in the set are redundant with each other—as is the case with the GA-PDBs. The process of generating GA-PDBs is stochastic, thus one increases the chances of generating a helpful heuristic by generating a large number of them. **GHS** is an effective method for selecting a small set of informative heuristics from a large set of mostly uninformative ones. This is illustrated in Table 6 on the Transport domain. Compared to systems which use multiple heuristics (**StSp** 1 and 2, and **RIDA\***), **Hybrid** solves the largest number of Transport instances, which is due to the selection of a few key GA-PDBs.

The best **GHS** approach, **Hybrid**, substantially outperforms **Max**; **Hybrid** solves 20 more instances than **Max**. Finally, **Hybrid** and **Time+CS** substantially outperform all other approaches tested, with **RIDA\*** being the closest competitor with 210 instances solved.





## 5 Concluding Remarks

This dissertation showed that the problem of finding the subset of a set of heuristics  $\zeta$  for a given problem task is solved using models of A\* search tree size and models of the A\* running time. We introduced the **GHS** algorithm which selects heuristics from  $\zeta$  one at a time and is able to produce a good subset  $\zeta'$ , with respect to the search tree size. In addition to minimizing the search tree size and the running time, we also experimented with an objective function that accounts for the sum of heuristic values in the state-space, as suggested by Rayner et al., (2013).

Since we cannot compute the values of the objective functions exactly, **GHS** effectiveness depends on the quality of the approximations we can obtain. We tested two prediction algorithms, **CS** and **SS**, for estimating the values of the objective functions and showed empirically that both **CS** and **SS** allow **GHS** to make good subset selections with respect to the search tree size and running time.

Finally, experiments on optimal domain-independent problems showed that **GHS** minimizing approximations of the A\* running time outperformed all the other approaches tested, which demonstrates the effectiveness of our method for the heuristic subset selection problem.



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