

Marvin Abisrror Zarate

**On Selecting  
Heuristic Function Subsets For Domain  
Independent-Planning.**

**Brasil**

**2016, v-1.9.5**



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Dissertação apresentada à Universidade Federal de Viçosa, como parte das exigências do Programa de Pós-Graduação em Ciência da Computação, para a obtenção do título de *Magister Scientiae*.

Universidade de Viçosa – UFV

Centro de Ciencias Exactas e Tecnologicas (CCE)

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Orientador: Levi Henrique Santana de Lelis

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Brasil  
2016, v-1.9.5



*This dissertation is dedicated to my Mother.*





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*“Não vos amoldeis às estruturas deste mundo,  
mas transformai-vos pela renovação da mente,  
a fim de distinguir qual é a vontade de Deus:  
o que é bom, o que Lhe é agradável, o que é perfeito.  
(Bíblia Sagrada, Romanos 12, 2)*



# Abstract

In this dissertation we present greedy methods for selecting a subset of heuristic functions from a large pool of possibilities with the objective of reducing the running time of search algorithms. Holte et al. (2006) showed that search can be faster if several smaller pattern databases are used instead of one large pattern database. Our methods are able to select good heuristics from a large set of heuristic functions to guide A\* search. If the heuristics are consistent, our method is able to make good subset selections with respect to an approximation of the A\* search tree size. In addition to being consistent, if all heuristics have the same evaluation time, the subset our method selects is also good with respect to the resulting A\* running time. We implemented our method in Fast Downward and showed empirically that it produces heuristics which outperform the state-of-the-art planners in the International Planning Competition benchmarks.

**Key-words:** Heuristics selection; A\*



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# Chapter I

## Introduction



# 1 About the Problem

State-space search algorithms have been used to solve important real-world problems, such as problems arising in Robotics, domain-independent planning, chemistry, biology, and engineering. In this dissertation we study methods for selecting a subset of heuristic functions while minimizing the search tree size and the running time of the A\* search algorithm while solving state-space search problems (Hart P. E. et al., 1968).

We are interested in selecting heuristics from a large set of possibilities because Holte et al., (2006) showed that search can be faster if several smaller pattern databases are used instead of one large pattern database. In fact, we believe that different heuristics can give us valuable information about the solution of the problem. For example, one heuristic can be helpful in a region of the search tree where other heuristics aren't. Then, instead of using one heuristic to find the solution, it would be best to use the most promising subset of heuristics from a possibly large set.

## 1.1 Background work

Search algorithms are used to solve Artificial Intelligence (AI) problems by finding sequence of actions from the start state to the goal state in the search space. Two well know search algorithms are Depth-First Search (DFS) and Breadth-First Search (BFS). DFS looks for the solution by exploring the subtree rooted node  $n$  before exploring the subtrees rooted at  $n$ 's siblings while looking for a path from start to goal. In the Figure 1 we can see DFS expands 31 states until finding a solution to the problem. BFS looks for the solution by exploring the nodes in a given level before exploring the nodes in the next level. In the Figure 2 we can see BFS expands 46 states to find the goal. In both Figure 1 and 2 the numbers above of each state represent the order in which the states are visited. Furthermore, both search algorithms usually visit a large number of states before finding a solution. We call brute-force-search tree (BFST) the tree expanded by DFS and BFS.



Figure 1 – 8 tile puzzle using DFS. (Bernard Chen, 2011)



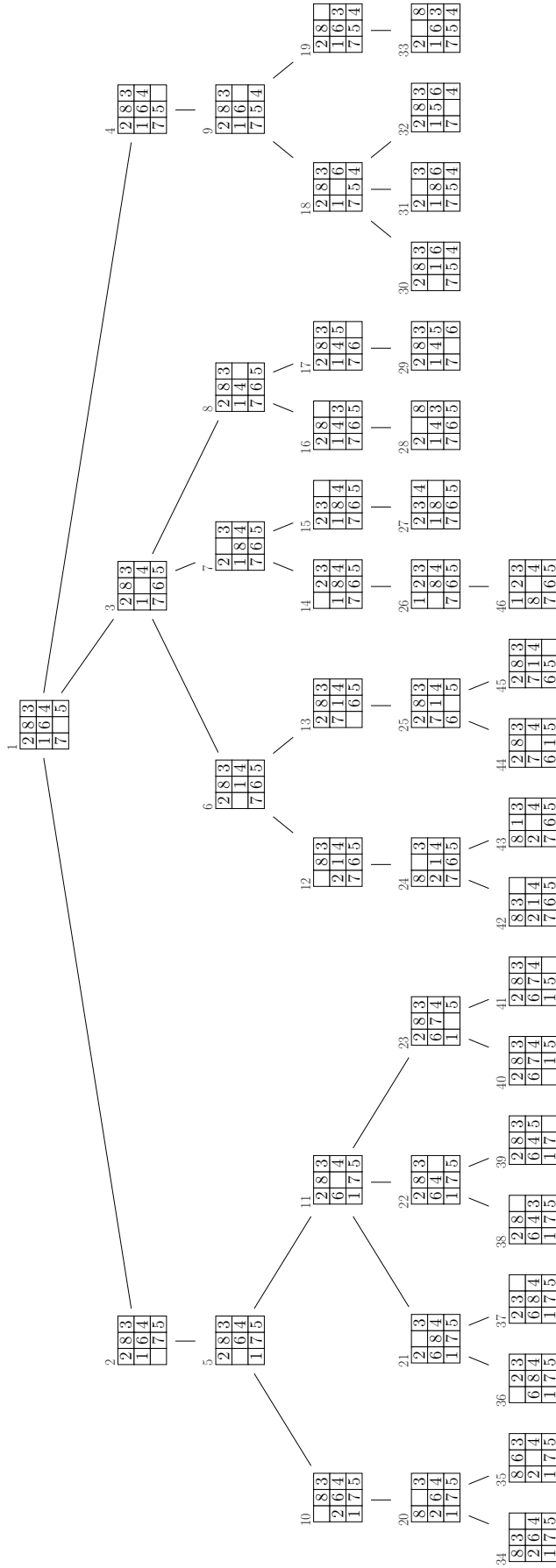


Figure 2 – 8 tile puzzle using BFS. (Bernard Chen, 2011)

There are other types of algorithms called heuristic search algorithms, and the most representative algorithm of this type is A\* (Hart P. E. et al., 1968), which also solves problems optimally. The heuristic is the estimation of the distance for one node in the search tree to get to a goal state. The heuristic search algorithms tend to generate smaller search trees than the BFST, because the heuristic guides the search to more promising parts of the state space. Also, by reducing the search tree size, the heuristic function guidance might also reduce the overall running time of the algorithm.

There are different approaches (Haslum et al., 2007; Edelkamp, 2007; Nissim et al., 2011) for creating heuristics. The way to create heuristics could be using domain knowledge or without domain knowledge. The systems that generate a heuristic are called heuristic generators by Barley et al., (2014). Actually, the approach that have shown successful results in heuristic generation is Pattern Database (PDB) (Culberson and Schaeffer, 1998). The way how PDBs work is the following: The search space of the problem is abstracted into a smaller state space that can be enumerated with exhaustive search. The distance of all abstracted states to the abstracted goal state are stored in a lookup table, which can be used as a heuristic function for the original state space.

Holte et al., (2006) showed that search can be faster if several smaller PDBs are used instead of one large pattern database. In addition Domshlak et al., (2010) and Tolpin et al., (2013) showed that evaluating the heuristic lazily, only when they are essential to a decision to be made in the search process, is worthy in comparison to take the maximum of the set of heuristics.

## 1.2 Aim and Objectives

### 1.2.1 Aim

The objective of this dissertation is to develop meta-reasoning approaches for selecting a heuristic subset from a large set of heuristics with the goal of reducing the running time of the search algorithms employing these functions.

### 1.2.2 Objectives

- Develop an approach to find a subset of heuristics from a large pool of heuristics  $\zeta$  that minimizes the number of nodes expanded by A\* in the process of search.
- Develop an approach to find a subset of heuristics from a large pool of heuristics  $\zeta$  that minimizes the A\* running time.

## 1.3 Scope, Limitations, and Delimitations

We implemented our method in Fast Downward (Helmert, 2006) and we test our methods on the 2011 International Planning Competition (IPC) problem instances.

## 1.4 Justification

Good results have been obtained in domain-independent planning by using heuristic search approach (Blai Bonet and Hector Geffner, 2001). The heuristic function used to guide the A\* search can greatly affect the algorithm's running time.

We use heuristic generators in order to create a large set of heuristics and obtain heuristics to solve problems.

## 1.5 Hypothesis

We test the following hypothesis:

- **H1:** A greedy algorithm is effective for selecting a good subset of heuristics to guide the A\* search.

## 1.6 Contribution of the Dissertation

The main contributions of this dissertation are:

- Provide a meta-reasoning approach for selecting heuristic functions while minimizing the number of nodes generated by A\*.
- Provide a meta-reasoning approach for selecting heuristic functions while minimizing the running time of the A\* search.

## 1.7 Organization of the Dissertation

The dissertation is organized as follows:

1. In Chapter I, the background of the dissertation is provided, which also includes the objectives and the scope definition.
2. In Chapter II we review the state-of-the-art in selection of heuristic functions.
3. In Chapter III we introduce Greedy Heuristic Selection (GHS).

4. In Chapter IV compare GHS with other planner systems.
5. We conclude in Chapter V.

In the next chapter, the domain 8-tile-puzzle is used to understand the concepts that will be helpful for the other chapters.

## Chapter II

### Literature Review



## 2 Background

### 2.1 Similar Selection Systems

The system most similar to ours is RIDA\* (Barley et al., 2014). RIDA\* also selects a subset from a pool of heuristics to guide the A\* search. In RIDA\* this is done by starting with an empty subset and trying subsets of size one before trying subsets of size two and so on. RIDA\* stops after evaluating a fixed number of subsets. While RIDA\* is able to evaluate sets of heuristics with only tens of elements. By contrast, the method we propose in this dissertation is able to evaluate sets with thousands of elements.

Rayner et al., (2013) present an optimization procedure that is similar to ours. In contrast with our work, Rayner et al. limited their experiments to a single objective function that sought to maximize the sum of heuristic values in the state space. Moreover, Rayner et al.’s method performs a uniform sampling of the state space to estimate the sum of heuristic values in the state space. Thus, their method is not directly applicable to domain-independent planning. In this dissertation we adapt Rayner et al.’s approach to domain-independent planning by using Stratified Sampling Chen (1992) to estimate the sum of heuristic values in the state space. Our empirical results show that GHS minimizing an approximation of A\*’s running time is able to substantially outperform Rayner et al.’s approach in domain-independent planning.

Our meta-reasoning requires a prediction of the number of nodes expanded by A\* using any given subset. The prediction system we choose is Stratified Sampling (SS system (Lelis et al., 2013)). Even though, SS produce good predictions for Iterative-Deepening A\*, it does not give us good predictions for A\* because it is unable to detect duplicated nodes during search (Lelis et al., 2014). Although SS does not produce good predictions of the number of nodes generated by A\*, we show empirically that SS allows GHS to make good subset selections.

### 2.2 Problem definition

A *SAS<sup>+</sup>planning task* (Bäckström; Nebel, 1995) is a 4 tuple  $\nabla = \{V, O, I, G\}$ .  $V$  is a set of *state variables*. Each variable  $v \in V$  is associated with a finite domain of possible  $D_v$ . A state is an assignment of a value to every  $v \in V$ . The set of possible states, denoted  $V$ , is therefore  $D_{v_1} \times \dots \times D_{v_2}$ .  $O$  is a set of operators, where each operator  $o \in O$  is triple  $\{pre_o, post_o, cost_o\}$  specifying the preconditions, postconditions (effects), and non-negative cost of  $o$ .  $pre_o$  and  $post_o$  are assignments of values to subsets of variables,  $V_{pre_o}$  and  $V_{post_o}$ ,

respectively. Operator  $o$  is applicable to state  $s$  if  $s$  and  $pre_o$  agree on the assignment of values to variables in  $V_{pre_o}$ . The effect of  $o$ , when applied to  $s$ , is to set the variables in  $V_{post_o}$  to the values specified in  $post_o$  and to set all other variables to the value they have in  $s$ .  $G$  is the goal condition, an assignment of values to a subset of variables,  $V_G$ . A state is a goal state if it and  $G$  agree on the assignment of values to the variable in  $V_G$ .  $I$  is the initial state, and the planning task,  $\nabla$ , is to find an optimal (least-cost) sequence of operators leading from  $I$  to a goal state. We denote the optimal solution cost of  $\nabla$  as  $C^*$ .

## 2.3 The 8-tile-puzzle case

The state space problem illustrated in the Figure 3 consists in a board with 8 squares named tiles numbered from 1 to 8 and one square without tile and number named empty tile. The goal of the game is to order the tiles in some order. For example: From left to right and up to bottom in the following order 1, 2, 3, 4, 5, 6, 7, 8 and empty tile. This game is called the 8 Sliding-tile puzzle and its objective is to place the tiles in order by moving the numbered tiles into the empty tile. For this case, the goal would be reached by placing the tiles 1, 2 and 3 in the first row, and 4, 5 and 6 in the following row, and 7, 8 and empty tile in the last row.

Initial			Goal		
4	1	2	1	2	3
8		3	4	5	6
5	7	6	7	8	

Figure 3 – The left tile-puzzle is one possible initial distribution of tiles and the right tile-puzzle is the goal distribution of tiles. Each one represents a state.

Instead of using DFS or BFS that will analyze the states encountered, we can obtain heuristics for the 8-sliding-tile puzzle that will help us solve the problem more quickly.

## 2.4 Heuristics

There exist many state-space algorithms, and one of the most important and well know is A\* (Hart P. E. et al., 1968). A\* uses the  $f(s) = g(s) + h(s)$  cost function to guide its search to a solution.  $g(s)$  is the cost to go from the start state to state  $s$ , and  $h(s)$  is the estimated cost to go from  $s$  to the goal.





Figure 4 – Heuristic Search:  $I$ : Initial state,  $s$ : Some state,  $G$ : Goal state

In the Figure 4 the optimal distance from the initial state  $I$  to the state  $s$  is 4 and is represented by  $g(s)$ . The  $h^*(s) = 3$  represents the optimal distance from  $s$  to the goal state  $G$ , and  $h(s) = 2$  is the estimation distance from  $s$  to  $G$ .

A heuristic is admissible if  $h(s) \leq h^*(s)$  for all  $s \in V$ , where  $h^*(s)$  is the optimal cost of  $s$ . A heuristic is consistent iff  $h(s) \leq c(s, t) + h(t)$  for all states  $s$  and  $t$ , where  $c(s, t)$  is the cost of the cheapest path from  $s$  to  $t$ . For example, the heuristic function provided by a pattern database (PDB) heuristic (Culberson and Schaeffer, 1998) is admissible and consistent.

Given a set of admissible and consistent heuristics  $\zeta = \{h_1, h_2, \dots, h_M\}$ , the heuristic  $h_{max}(s, \zeta) = \max_{h \in \zeta} h(s)$  is also admissible and consistent. When describing our method we assume all heuristics to be consistent. We define  $f_{max}(s, \zeta) = g(s) + h_{max}(s, \zeta)$ , where  $g(s)$  is minimal when A\* using a consistent heuristic expands  $s$ . We call an A\* search tree the tree defined by the states generated by A\* using a consistent heuristic while solving a problem  $\nabla$ .

We now show two heuristics for the 8-puzzle.

### 2.4.1 Out of place (O.P)

This heuristic counts the number of tiles that are out of the goal position. If the tile is not in its goal position, then it counts as one, otherwise it counts as zero.

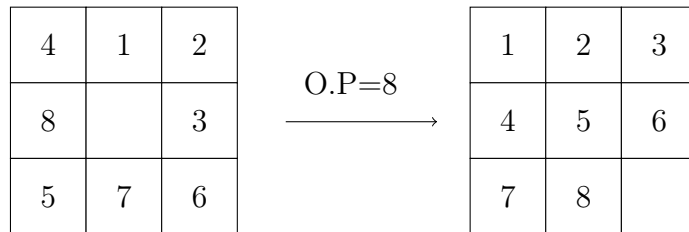


Figure 5 – Out of place heuristic

The tiles numbered with 4, 1, 2, 3, 6, 7, 5, and 8 are out of place, then each tile count as 1 and the sum would be 8, then the heuristic value for this state is 8.

### 2.4.2 Manhatham Distance (M.D)

This heuristic counts the minimum number of operations that should be applied to any tile to place it in its goal position. Let us explain this with an example: Tile 5 is located on the bottom left of the tile puzzle shown on the left-hand side of Figure 6, then the minimum number of moves to get tile 5 to its goal position is 2 (either up and right or right and up), both movements equal to 2.

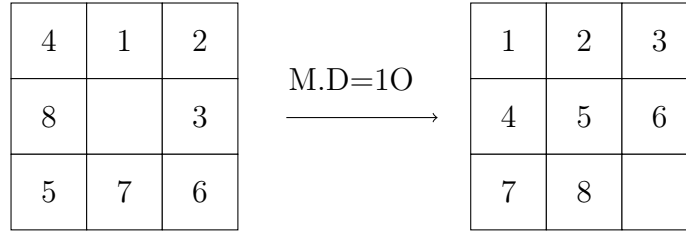


Figure 6 – Manhatham distance heuristic

Tile 4 requires one move to get to its goal position; Tile 1 requires one; Tile 2 requires one; Tile 3 requires one; Tile 6 requires one; Tile 7 requires one; Tile 5 requires two moves; Tile 8 requires two; The sum of required moves is 10.

Out-of-place and Manhattan distance are heuristics that use domain knowledge. There are methods that create heuristics without using domain knowledge. Barley et al., (2014) call them heuristic generators.

## 2.5 Heuristic Generators

Some heuristic generators work by creating abstractions of the original problem space. PDBs (Culberson and Schaeffer, 1998) are an effective way of generating heuristics, PDBs work as follows: The search space of the problem is abstracted into a smaller state space that can be enumerated with exhaustive search. The distance of all abstracted states to the abstracted goal state are stored in a lookup table, which can be used as a heuristic function for the original state space.

## 2.6 Using a Set of Heuristics

One well-know approach to take advantage of heuristics is to compute the maximum of all heuristics  $\zeta$ . For example, given two heuristics  $h1$  and  $h2$ , the maximum of  $h1$  and  $h2$  will tend to yield a more informed heuristic than  $h1$  and  $h2$  individually.

The heuristic generators can create hundreds or even thousands of heuristics and we need to figure it out how to take advantage of this large set of heuristics. In fact, there exist different ways to take advantage of those heuristics, however we need to take into

account that if we want to use all the heuristics created by the heuristic generator, the resulting heuristic would be too expensive to guide search. This is because, the main problem involved would be the time required to evaluate all the heuristics for each node in the search tree.

## 2.7 Heuristic Subset

Let's suppose we have to run our meta-reasoning considering that we have a fixed amount of memory  $M$ . Then, one question is raised: How many heuristics our system should consider? Holte et al., (2006) observed that maximizing over  $N$  pattern databases of size  $M/N$ , for  $N < M$ , can produce a search tree size smaller than the search tree size generated by using a single pattern database of size  $M$ . Thus we will use heuristic generators to create a large number of heuristics to fill the amount  $M$  of memory available.

Heuristic generator systems can create a large number of heuristics. Let us suppose  $|\zeta| = 1000$  heuristics were created and that our meta-reasoning for example selects  $N = 50$  heuristics. This would imply count the following number of combinations one can get from solving the combinatorial problem:

$$\binom{1000}{50} \approx 10^{85} \text{possibilities}$$

In the next Chapter, we will introduce the meta-reasoning proposed for selecting heuristics.



## Chapter III

### Approach Proposal



## 3 Meta-Reasoning for selection

### 3.1 Greedy Heuristic Selection (GHS)

We present a greedy algorithm selection for solving the heuristic subset selection problem while optimizing different objective functions. We consider the following general optimization problem.

$$\underset{\zeta' \subseteq \zeta}{\text{minimize}} \Psi(\zeta', \nabla) \quad (3.1)$$

Where  $\Psi(\zeta', \nabla)$  is an objective function we want to minimize for a subset of heuristics  $\zeta'$  of  $\zeta$ . We use an algorithm based on the local search we call Greedy Heuristic Selection (GHS) to solve Equation 3.1 for different functions  $\Psi$ .

---

**Algoritmo 1:** Greedy Heuristic Selection

---

**Input** : problem  $\nabla$ , set of heuristics  $\zeta$

**Output** : heuristic subset  $\zeta' \subseteq \zeta$

```

1  $\zeta' \leftarrow \emptyset$ 
2 while  $\Psi(\zeta', \nabla)$  can be improved do
3    $h \leftarrow \arg \min_{h \in \zeta} \Psi(\zeta' \cup \{h\}, \nabla)$ 
4    $\zeta' \leftarrow \zeta' \cup \{h\}$ 
5 return  $\zeta'$ 
```

---

Algorithm 1 shows GHS. GHS receives as input a problem  $\nabla$ , a set of heuristics  $\zeta$ , and it returns a subset  $\zeta' \subseteq \zeta$ . In each iteration GHS greedily selects from  $\zeta$  the heuristics  $h$  which will result in the largest reduction of the value  $\Psi$  (line 3). GHS returns  $\zeta'$  once the objective function cannot be improved. In other words, the algorithm will halt when adding another heuristic does not improve the objective function.

### 3.2 Minimizing Search Tree Size

The first objective function  $\Psi$  we consider accounts for the number of node generations  $A^*$  performs while solving a given planning problem. The planning problem must be solvable, this means  $C^*$  cannot be infinity. When solving  $\nabla$  using the consistent heuristic function  $h_{max}(\zeta')$  for  $\zeta' \subseteq \zeta$ ,  $A^*$  generates a number of nodes that is bounded above by  $J(\zeta', \nabla)$ , defined as,

$$J(\zeta', \nabla) = |\{children(s) \in V | f_{max}(s, \zeta') \leq C^*\}| \quad (3.2)$$

$$J(\zeta', \nabla) = |\{children(s) \in V | h_{max}(s, \zeta') \leq C^* - g(s)\}| \quad (3.3)$$

We write  $J(\zeta')$  or simply  $J$  instead of  $J(\zeta', \nabla)$  whenever  $\zeta'$  and  $\nabla$  are free from the context. What's more, we assume that  $A^*$  expands all nodes  $s$  with  $f(s) \leq C^*$  while solving  $\nabla$ , as shown in Equation 3.2.

### 3.3 Minimizing $A^*$ 's Running Time

Another objective function  $\Psi$  we consider is an approximation of the  $A^*$  running time and is defined as follows.

$$T(\zeta', \nabla) = J(\zeta', \nabla) \cdot (t_{h_{max}(\zeta')} + t_{gen}), \quad (3.4)$$

where, for any heuristic function  $h$ , the term  $t_{h_{max}(\zeta')}$  refers to the running time used for computing the  $h_{max}$  of any state  $s$ . We assume that  $t_{h_{max}(\zeta')}$  to be constant in the state space, which is a reasonable assumption for several heuristics such as PDBs.  $t_{gen}$  is the node generation time, which we also assume to be constant.

### 3.4 Estimating Tree Size and Running Time

In practice GHS uses approximations of  $J$  and  $T$  instead of their exact values. This is because computing  $J$  and  $T$  exactly would require solving  $\nabla$ , and this is what we obviously want to avoid.

We use the Culprit Sampler (CS) introduced by Barley et al., (2014) and the Stratified Sampling (SS) algorithm introduced by Chen (1992), for computing  $\hat{J}$  and  $\hat{T}$ . Each method has its strengths and weaknesses, which we explore in Chapter 4, where we make an empirical comparison of GHS with other approaches.

Both CS and SS must be able to quickly estimate the values of  $\hat{J}(\zeta')$  and  $\hat{T}(\zeta')$  for any subset  $\zeta'$  of  $\zeta$  so they can be used in GHS's optimization process.

### 3.5 Culprit Sampler (CS)

CS is an approach introduced by Barley et al., (2014) and works by running a time-bounded  $A^*$  search while sampling  $f$ -culprits and  $b$ -culprits. In this dissertation we use CS to estimate the values of  $\hat{J}$  and  $\hat{T}$ .

**Definition 3.5.1.** (*f-culprit*) Let  $\zeta = \{h_1, h_2, \dots, h_M\}$  be a set of heuristics. The  $f$ -culprit of a node  $n$  in an  $A^*$  search tree is defined as the tuple  $F(n) = \langle f_1(n), f_2(n), \dots, f_M(n) \rangle$ ,



where  $f_i(n) = g(n) + h_i(n)$ . For any  $n$ -tuple  $F$ , the counter  $C_F$  denotes the number of nodes  $n$  in the tree with  $F(n) = F$ .

**Definition 3.5.2.** (*b-culprit*) Let  $\zeta = \{h_1, h_2, \dots, h_M\}$  be a set of heuristics and  $b$  a lower bound on the solution cost  $\nabla$ . The *b-culprit* of a node  $n$  in an  $A^*$  search tree is defined as the tuple  $B(n) = \langle y_1(n), y_2(n), \dots, y_M(n) \rangle$ , where  $y_i(n) = 1$  if  $g(n) + h_i(n) \leq b$  and  $y_i(n) = 0$ , otherwise. For any binary  $n$ -tuple  $B$ , the counter  $C_B$  denotes the number of nodes  $n$  in the tree with  $B(n) = B$ .

CS works by running an  $A^*$  search bounded by a user-specified time limit. Then, CS compresses the information obtained in the  $A^*$  search (i.e., the  $f$ -values of all nodes expanded according to all heuristics  $h$  in  $\zeta$ ) in *b-culprits*, which are later used for computing  $\hat{J}$ . The *f-culprits* are generated as an intermediate step for computing the *b-culprits*, as we explain below. The maximum number of *f-culprits* and *b-culprits* in an  $A^*$  search tree is equal to the number of nodes in the tree expanded by the time-bounded  $A^*$  search. However, in practice the number of *f-culprits* is usually much lower than the number of nodes in the tree. Moreover, in practice, the total number of different *b-culprits* tends to be even lower than the total number of *f-culprits*. Given a planning problem  $\nabla$  and a set of heuristics  $\zeta$ , CS samples the  $A^*$  search tree as follows.

- 1.- CS runs  $A^*$  using  $h_{\min}(s, \zeta) = \min_{h \in \zeta} h(s)$  until reaching a user-specified time limit.  $A^*$  using  $h_{\min}$  expands node  $n$  if it were to expand  $n$  while using any of the heuristics in  $\zeta$  individually. For each node  $n$  expanded in this time-bounded search we store  $n$ 's *f-culprit* and its counter.
- 2.- Let  $f_{\max\min}$  be the largest *f-value* according to  $h_{\min}$  encountered in the time-bounded  $A^*$  search described above. We now compute the set  $\mathbb{B}$  of *b-culprits* and their counters based on the *f-culprits* and on the value of  $f_{\max\min}$ . This is done by iterating over all *f-culprits* once.

The process described above is performed only once GHS's execution. The value of  $\hat{J}(\zeta', \nabla)$  for any subset  $\zeta'$  of  $\zeta$  is then computed by iterating over all *b-culprits*  $\mathbf{B}$  and summing up the relevant values of  $C_B$ . The relevant values of  $C_B$  represent the number of nodes  $A^*$  would expand in a search bounded by  $b$  if using  $h_{\max}(\zeta')$ . This computation can be written as follows.

$$\hat{J}(\zeta', \nabla) = \sum_{\mathbf{B} \in \mathbb{B}} W(\mathbf{B}) \quad (3.5)$$

Where  $W(\mathbf{B})$  is 0 if there is a heuristic in  $\zeta'$  whose  $y$ -value in  $\mathbf{B}$  is zero (i.e., there is a heuristic in  $\zeta'$  that prunes all nodes compressed into  $\mathbf{B}$ ), and  $C_B$  otherwise. If the

time-bounded A\* search with  $h_{min}$  expands all nodes  $n$  with  $f(n) \leq C^*$ , then  $\hat{J} = J$ . In practice, however, our estimate  $\hat{J}$  will tend to be much lower than  $J$ .

The value of  $\hat{T}$  is computed by multiplying  $\hat{J}$  by the sum of the evaluation time of each heuristic in  $\zeta'$ . The evaluation time of the heuristics in  $\zeta'$  is measured in a separate process, before executing CS, by sampling a small number of nodes from  $\nabla$ 's start state.

## 3.6 Stratified Sampling (SS)

Chen, (1992) presented a method for estimating the search tree size of backtracking search algorithms by using a stratification of the search tree to guide its sampling. We define Chen's stratification as a type system.

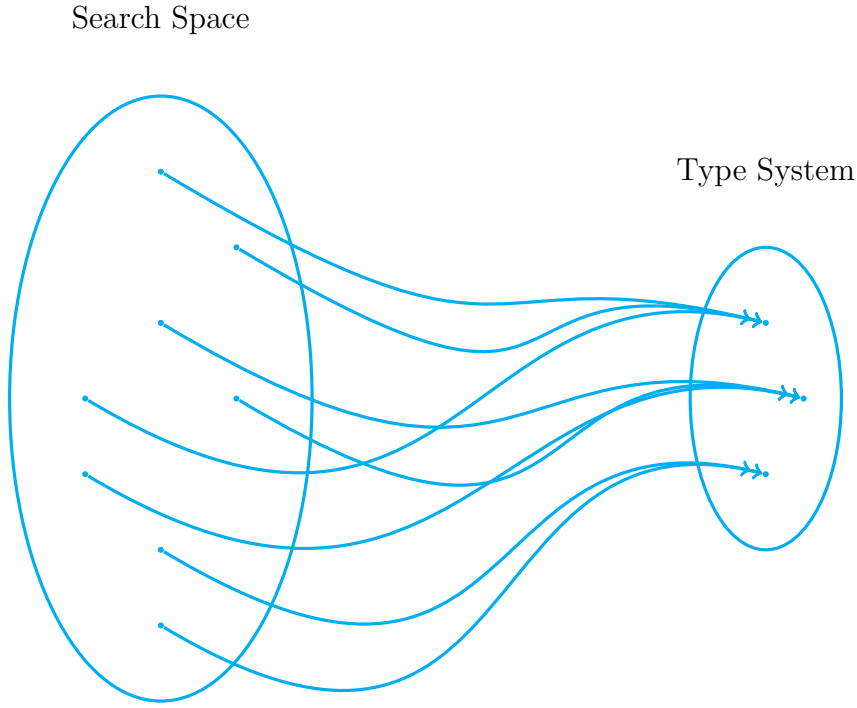


Figure 7 – Type system and the search space representation.

### 3.6.1 Type System

The type system is calculated based on any property of nodes in the search tree. A common definition is the one that assigns two nodes to the same type if they have the same  $f$ -value (Lelis et al., 2013). In the Figure 7 we can see that the type system is a partition of the state space.

**Definition 3.6.1.** *Type System* Let  $S = (N, E)$  be a search tree, where  $N$  is its set of nodes and for each  $n \in N$ ,  $\{n' | (n, n') \in E\}$  is  $n$ 's set of child nodes.  $TS = \{t_1, \dots, t_k\}$  is a type system for  $S$  if it is a disjoint partitioning of  $N$ . If  $n \in N$  and  $t \in TS$  with  $n \in t$ , we write  $TS(n) = t$ .

According to Lelis et al., (2013), SS is able to approximate functions of the form  $\varphi = \sum_{n \in S} z(n)$ , where  $z$  is a function assigning a numerical value to a node, consequently  $\varphi$  is a numerical property of  $S$ . For example, if  $z(n) = 1$  for all  $n \in S$ , then  $\varphi$  is the size of the tree. Instead of summing all the  $z$ -values from the tree, SS considers subtrees rooted at nodes of the same type will have equal values of  $\varphi$  and only one of each type, chosen at random, is expanded. With input a search tree  $S$  and a type system  $TS$ , SS predicts  $\varphi$  in the following way: First, the search tree is sampled, returning the set of *representative-weight* pairs, with one such pair for every unique type seen during sampling. Second, in the pair  $\langle s, w \rangle$  in  $A$  for type  $t \in TS$ ,  $n$  is the unique node of type  $t$  that was expanded during search and  $w$  is an estimate of the number of nodes of type  $t$  in the tree. Finally,  $\varphi$  is then approximated by  $\hat{\varphi}$ , defined as,  $\hat{\varphi} = \sum_{\langle s, w \rangle \in A} w \times z(n)$ .

By making  $z(n) = |\text{children}|$  for all  $n \in S$  SS produces an estimate  $\hat{J}$  of  $J$ . Similarly to our approach with CS, we obtain  $\hat{T}$  by multiplying  $\hat{J}$  by the sum of heuristic evaluation time and generation time.

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**Algoritmo 2:** SS, a single probe

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**Input** : root  $n^*$  of a tree and a type system  $TS$ , upper bound  $d$ , heuristic function  $h$

**Output** : an array of sets  $A$ , where  $A[i]$  is the set of (node, weight) pairs  $\langle s, w \rangle$  for the nodes  $n$  expanded at level  $i$ .

```

1  $A[0] \leftarrow \{\langle s^*, 1 \rangle\}$ 
2  $i \leftarrow 0$ 
3 while  $A[i]$  is not empty do
4   for each element  $\langle s, w \rangle$  in  $A[i]$  do
5     for each child  $\hat{n}$  of  $n$  do
6       if  $g(\hat{n}) + h(\hat{n}) \leq d$  then
7         if  $A[i+1]$  contains an element  $\langle n', w' \rangle$  with  $TS(n') = TS(\hat{n})$  then
8            $w' \leftarrow w' + w$ 
9           with probability  $w/w'$ , replace  $\langle n', w' \rangle$  in  $A[i+1]$  by  $\langle \hat{n}, w' \rangle$ 
10        else
11          insert new element  $\langle \hat{n}, w \rangle$  in  $A[i+1]$ 
12    $i \leftarrow i + 1$ 

```

---

In SS the types are required to be partially ordered: a node's type must be strictly greater than the type of its parent. This can be guaranteed by adding the depth of a node to the type system and then sorting the types lexicographically. That is why in

our implementation of **SS** types at one level are treated separately from types at another level by the division of  $A$  into groups  $A[i]$ , where  $A[i]$  is the set of *representative-weight* pairs for the types encountered at level  $i$ . If the same type occurs on different levels the occurrences will be treated as if they were different types – the depth of search is implicitly included into all of our type systems.

Algorithm 2 shows **SS** in detail. Representative nodes from  $A[i]$  are expanded to get representative nodes for  $A[i + 1]$  as follows.  $A[0]$  is initialized to contain only the root of the search tree to be probed, with weight 1 (Line 1). In each iteration (Lines 4 through 11), all nodes in  $A[i]$  are expanded. The children of each node in  $A[i]$  are considered for inclusion in  $A[i + 1]$  if their  $f$ -value do not exceed an upper bound  $d$  provided as input to **SS**. If a child  $\hat{n}$  has a type  $t$  that is already represented in  $A[i + 1]$  by another node  $n'$ , then a *merge* action on  $\hat{n}$  and  $n'$  is performed. In a merge action we increase the weight in the corresponding *representative-weight* pair of type  $t$  by the weight  $w(n)$  of  $\hat{n}$ 's parent  $n$  (from level  $i$ ) since there were  $w(n)$  nodes at level  $i$  that are assumed to have children of type  $t$  at level  $i + 1$ .  $\hat{n}$  will replace  $n'$  according to the probability shown in Line 9. Chen, (1992) proved that this probability reduces the variance of the estimation. Once all the states in  $A[i]$  are expanded, we move to the next iteration.

One run of the **SS** algorithm is called a *probe*. Chen, (1992) proved that the expected value of  $\hat{\varphi}$  converges to  $\varphi$  in the limit as the number of probes goes to infinity. As Lelis et al., (2014) showed, **SS** is not able to detect duplicated nodes in its sampling process. As a result, since  $A^*$  does not expanded duplicates, **SS** usually overestimates the actual number of nodes  $A^*$  expands. Thus, in the limit, as the number of probes grows large, **SS**'s prediction converges to a number which is likely to overestimate the  $A^*$  search tree size. We test empirically whether **SS** is able to allow **GHS** to make good subset selects despite being unable to detect duplicated nodes during sampling.

Similarly to **CS**, we also define a time-limit to run **SS**. We use **SS** with an iterative-deepening approach in order to ensure an estimate of  $\hat{J}$  and  $\hat{T}$  before reaching the time limit. We set the upper bound  $d$  to the heuristic value of the start state and, after performing  $p$  probes, if there is still time, we increase  $d$  to twice its previous value. The values of  $\hat{J}$  and  $\hat{T}$  is given by the prediction produced for the last  $d$ -value in which **SS** was able to sample from.

**SS** must also be able to estimate the values of  $\hat{J}(\zeta')$  and  $\hat{T}(\zeta')$  for any subset  $\zeta'$  of  $\zeta$ . This is achieved by using **SS** to estimate  $b$ -culprits (see Definition 3.5.2) instead of the search tree size directly. Similarly to **CS**, **SS** used  $h_{min}$  of the heuristics in  $\zeta$  to decide when to prune a node (see Line 6 of Algorithm 2) while sampling. This ensures that **SS** expands a node  $n$  if  $A^*$  employing at least one of the heuristics in  $\zeta$  would expand  $n$  according to bound  $d$ . The  $C_B$  counter of each  $b$ -culprit  $B$  encountered during **SS**'s probe

is given by,

$$C_B = \sum_{\langle n, w \rangle \in A \wedge B(n)=B} w \quad (3.6)$$

We recall that to compute  $B(n)$  for node  $n$  one needs to define a bound  $b$ . Here we use the bound  $d$  used by SS. The average value of  $C_B$  across  $p$  probes is used to predict the search tree size for a given subset  $\zeta'$ . As explained for CS, this can be done by traversing all  $b$ -culprits once.

### 3.7 SS step by step

In the Figure 8, we can see an example of an SS run. Thus, nodes have different types if they have different colors or are in different levels of the search tree.

In the level 1, we have the root node, and their  $w$  is initialized with one. In the level 2 three nodes are generated by the root node. The nodes in level 2 have the following types: red and blue, and each node receive the same  $w$  from their parent. Since we assume that nodes of the same type root subtrees of the same size, then only one node of each type needs to be expanded. In level 2 there are two nodes with the red type. In that way, we choose randomly one of them to expand. Let us suppose we choose the right red node. Then, we have to update the number of nodes with the red type by summing their  $w$ -values, since both nodes of the red type have  $w = 1$ , then we sum the  $w$  resulting in  $w = 2$ .

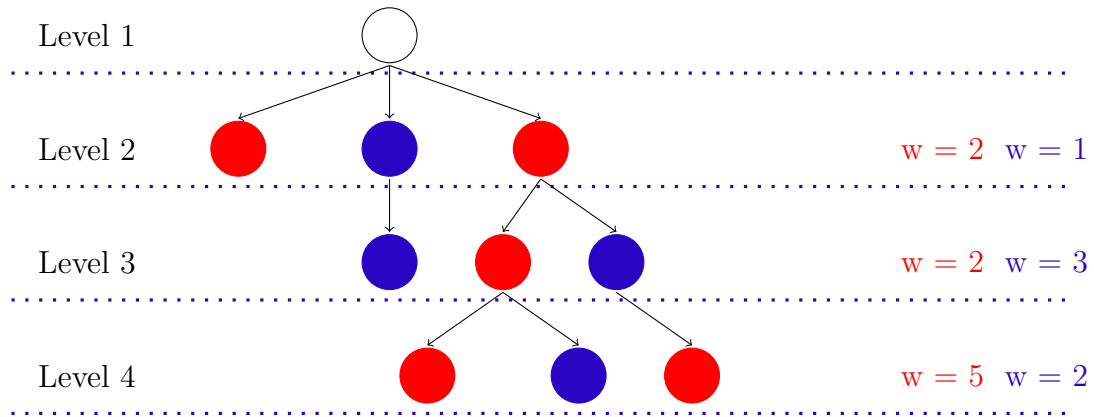


Figure 8 – Search tree using type system

Let's suppose that nodes in level 2 are expanded. The blue node generates one node of the blue type and the red node generates two nodes with the following types: red and blue. To compute the  $w$ -values of the nodes at level 3 we do the following. The  $w$ -value of nodes of the red type is computed by summing up the  $w$ -values of all parents that generated a node of the red type at level 3. In this case only one node at level 2

generated a node of the red type at level 3. Since this node has a  $w$ -value of 2, then **SS** estimates that there is 2 nodes of the red type at level 3 (red type's  $w$ -value). Two nodes at level 2 generate a blue node at level 3, thus we have to sum their  $w$ -values, resulting in  $w = 2 + 1 = 3$  for blue nodes.

Finally, the number of nodes expanded in the search tree is obtained summing all  $w$  plus one (the root node). In summary, the number of nodes expanded in the search tree would be  $15 + 1 = 16$ .

If we run the algorithm again maybe in level 2 instead of choosing the right red node we could choose the left red node and the nodes generated in level 3 would be different. As a consequence, the number of nodes generated could be higher or lower than we have now. In other words, in each probe of the algorithm could result in a different value of the estimated tree size. In order to obtain accurate estimates of the search tree size one has to usually perform several probes of **SS** and average the results.

### 3.8 Comparison between **SS** and IDA\*

**SS** was shown to produce good predictions of the IDA\* search tree size (Lelis et al., 2013). Iterative-Deepening A\* (IDA\*) is a heuristic search algorithm that uses cost function for looking the solution in the search space and in each iteration uses **DFS** to prune the branch until reaching the cost threshold (Korf, Reid and Edelkamp, 2001).

**SS** is able to produce good predictions for IDA\* also in domain-independent planning, and we will show that it produces very inaccurate predictions for A\*

In this experiment **SS** estimates the search tree size generated by IDA\* using a consistent heuristic. **SS** estimates the size of the search tree up to some defined  $f$ -layer in the tree.

We first run IDA\* in domain-independent planning benchmark problems within 30 minutes time limit. Then we run **SS** limited by different  $f$ -layers encountered in the IDA\* searches. We experiment with the following number of probes: 1, 10, 100, 1000 and 5000.

We use the *relative unsigned error* (Lelis et al., 2012), which we call relative-error, for short, to measure prediction accuracy. The relative-error is shown in Equation 3.7 below.

$$\frac{\sum_{s \in PI} \frac{Pred(s,d) - R(s,d)}{R(s,d)}}{|PI|} \quad (3.7)$$

- $PI$  represent all the instances for a domain.

Table 1 – Comparison between SS and IDA\* for 1, 10, 100, 1000 and 5000 probes using *hmax* heuristic.

Domain	<i>hmax</i>												
	IDA*	time	relative-error					time					n
			1	10	100	1000	5000	1	10	100	1000	5000	
Barman	8835990.00	6016.38	0.60	0.45	0.20	0.07	0.04	0.06	0.32	3.21	32.57	214.59	20
Elevators	1012570.00	4987.57	0.84	0.42	0.23	0.13	0.10	1.40	9.85	96.37	994.33	4425.93	20
Floortile	30522300.00	3919.72	2.02	0.62	0.40	0.14	0.11	0.01	0.07	0.69	6.93	36.60	2
Nomystery	6565740.00	3256.86	0.53	0.26	0.07	0.03	0.01	0.07	0.38	3.63	36.35	181.03	20
Openstacks	80108.50	4017.19	0.03	0.03	0.03	0.03	0.03	94.79	774.86	1067.84	10929.00	11174.30	20
Parcprinter	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.04	0.35	3.48	17.29	20
Parking	374925.00	5607.50	0.17	0.04	0.01	0.00	0.00	1.79	11.36	114.28	1196.83	5835.03	20
Pegsol	68763.70	5.00	0.17	0.04	0.02	0.01	0.00	0.01	0.04	0.37	3.69	17.88	20
Scanalyzer	8449890.00	4920.58	0.43	0.25	18.63	0.02	0.01	3.13	28.79	273.74	3033.06	10254.00	20
Sokoban	3118530.00	3932.69	0.41	0.26	0.11	0.05	0.04	0.31	2.00	21.42	222.47	1056.61	20
Tidybot	444473.00	5632.08	300.86	1072.40	5.88	0.01	0.01	4.40	26.48	238.76	2747.10	11925.40	20
Transport	2622880.00	2253.51	0.63	0.54	0.24	0.15	0.11	0.09	0.61	5.89	59.37	290.31	20
Visitall	71032400.00	3704.78	0.12	0.04	0.01	0.00	0.00	0.00	0.05	0.56	5.77	28.07	20
Woodworking	5139070.00	4944.76	1.28	0.69	0.27	0.17	0.07	0.15	1.33	13.21	130.82	664.08	20

- $Pred(s, d)$  is the estimation of the search tree size expanded by IDA\* for start state  $s$  and cost bound  $d$ .
- $R(s, d)$  is the real number of nodes expanded by IDA\* for start state  $s$  and cost bound  $d$ .

A relative error of 0.0 means that SS was able to produce perfect predictions.

The Table 1 shows the relative-error, the heuristic used in this experiment is *hmax*. The column n shows the number of problems that were used to compute the relative-error, the average number of nodes expanded by IDA\*, and the average running time of the algorithms. The table of results also shows the average number of nodes expanded by A\* and IDA\*'s average running time. The relative-error tends to reduce as one increases the number of probes. For example, in Barman, the relative-error goes from 0.60 for 1 probe to 0.45 for 10 probes, 0.20 for 100 probes, 0.07 for 1000 probes and 0.04 for 5000 probes. In the case of running time, while the number of probes increase, SS need to spend more time approximating the size of the search tree. That is why the overall running time of SS increases as one increases the number of probes. For example, in Barman, the time goes from 0.06 *seconds* for 1 probe to 0.32 *seconds* for 10 probes, 3.21 *seconds* for 100 probes, 32.57 *seconds* for 1000 probes and 214.59 *seconds* for 5000 probes. There are domains such as: Parcprinter, Parking, Pegsol and Visitall that have perfect score using 5000 probes. In the case of Tidybot, the relative-error using 1 probe is smaller than using 10 probes, which happens due to the stochastic nature of SS.

The 2011 IPC domains contains 20 instances per domain. Floortile only have 2 instances, it means that when running IDA\* for all the instances of Floortile only two instances (opt-p01-001.pddl and opt-p03-006.pddl) have found the solution. In summary, we showed that for 2011 IPC domains, SS is able to produce good predictions of the search tree expanded by IDA\* when the number of probes goes to infinity.

### 3.9 Comparison between SS and A\*

The Table 2 shows that SS is not a good estimator for A\*. This is because SS does not account for duplicate nodes and A\* does. As a result SS usually overestimates by a several orders of magnitude the A\* search tree size.

Table 2 – Poor prediction of SS against A\* using ipdb, LM-Cut and M&S with 500 probes

Domain	ipdb		LM-Cut		M&S		n
	A*	SS-error	A*	SS-error	A*	SS-error	
Barman	1.72e+07	8.68e+31	7.45e+06	2.21e+30	6.67e+06	1.26e+36	4
Floortile	1.40e+07	1.74e+18	702435	4.68e+14	4.46e+06	1.90e+12	4
Nomystery	40169.7	6.71e+32	267100	6.14e+19	8236	1.20e+20	9
Openstacks	570099	0.61884	570099	0.677425	569984	0.672143	4
Parcprinter	1157	2.56e+22	1363.67	2.33e+21	766.333	6.36e+20	3
Pegsol	841693	2901.39	398221	6859.86	933430	779.017	16
Scanalyzer	337894	3.94e+33	334747	7.58e+31	337833	2.42e+31	3
Sokoban	376755	1.04e+07	45374	2.74e+06	739775	5.60e+08	9
Transport	1.89e+06	2.91e+38	1.49e+06	1.15e+25	1.73e+06	1.50e+29	2
Visitall	253710	1.69e+46	253195	1.69e+46	253521	1.71e+46	8
Woodworking	3.21e+06	2.53e+18	3.20e+06	2.76e+18	3.21e+06	2.48e+18	3

Three heuristics were used: iPDB (Haslum et al., 2007), LM-Cut (Pommerening and Helmert, 2013) and M&S (Nissim et al., 2011). The last column *n* shows the number of instances solved by A\* using the three heuristics. For this experiment we use only the instances that are solved by the three heuristics at the same time. The columns with A\* represents the average of number of nodes expanded by A\* for the instances solved using a specific heuristic. The column with SS-error represents the relative-error for the solved instances.

SS usually overestimates the number of nodes expanded by A\* by several orders of magnitude, for example, in Barman with M&S, A\* expands on average  $6.67 \times 10^6$  which is less nodes than iPDB— $1.72 \times 10^7$  and LM-Cut— $7.45 \times 10^6$ . However, using M&S, SS-error is  $1.26 \times 10^{36}$ , iPDB— $8.68 \times 10^{31}$  and LM-Cut— $2.21 \times 10^{30}$ , which indicates that the number of nodes expanded by SS in average is in the order of magnitude from 30 to 40. In Visitall, SS-error shows that SS overestimates A\* highly, which represent a very bad prediction of SS. In Openstack: Using the three heuristics, A\* expands almost the same number of nodes for the 4 instances solved, and SS-error shows a score near to the perfect and the reason is that SS expands less nodes than A\*.

### 3.10 Approximation Analysis for SS and A\*

Although SS is not able to produce good predictions of the A\* search tree size, here we show empirically that SS allows one to make good heuristic selections. Again we use iPDB, LM-Cut, and GA-PDBs (Edelkamp, 2007) in this experiment.



In order to select heuristics the predictions produced by **SS** do not have to be accurate in absolute terms, but they have to be accurate in relative terms. That is, if  $A^*$  using heuristic  $h_1$  expands fewer nodes than when using  $h_2$ , then **SS**'s predictions could be arbitrarily inaccurate for both  $h_1$  and  $h_2$  as long as its prediction estimated that  $h_1$  is better than  $h_2$  then one is able to select  $h_1$  over  $h_2$ .

In order to investigate whether **SS** is able to allow one to select the best heuristic in a set of size two:  $\{h_1 \text{ and } h_2\}$ , we construct a graph where the y-axis is the value of  $J(h_1)/J(h_2)$  (the actual number of nodes generated by  $A^*$  using  $h_1$  over the number of nodes generated by  $A^*$  using  $h_2$ ); and the x-axis is the value of  $\hat{J}(h_1)/\hat{J}(h_2)$  (the predicted search tree size of  $h_1$  over the predicted search tree size of  $h_2$ ).

We limit our plot to the value of 2, and any fraction that is greater than 2 is set to 2. The plot is divided in four quadrants, as shown in Figure 9. If a point falls in quadrants II and III it means that **SS** is able to correctly select the heuristic that will make  $A^*$  expand the fewer number of nodes. If the point falls in quadrants I or IV it means that **SS** was unable to select the heuristic that expands fewer nodes.

Points that fall in each of the regions:

- I  $J(h_1) > J(h_2)$  for  $A^*$ ,  $\hat{J}(h_2) > \hat{J}(h_1)$  according to **SS**.
- II  $J(h_1) > J(h_2)$  for  $A^*$  and **SS** agrees.
- III  $J(h_2) > J(h_1)$  for  $A^*$  and **SS** agrees.
- IV  $J(h_2) > J(h_1)$  for  $A^*$ ,  $\hat{J}(h_1) > \hat{J}(h_2)$  according to **SS**.

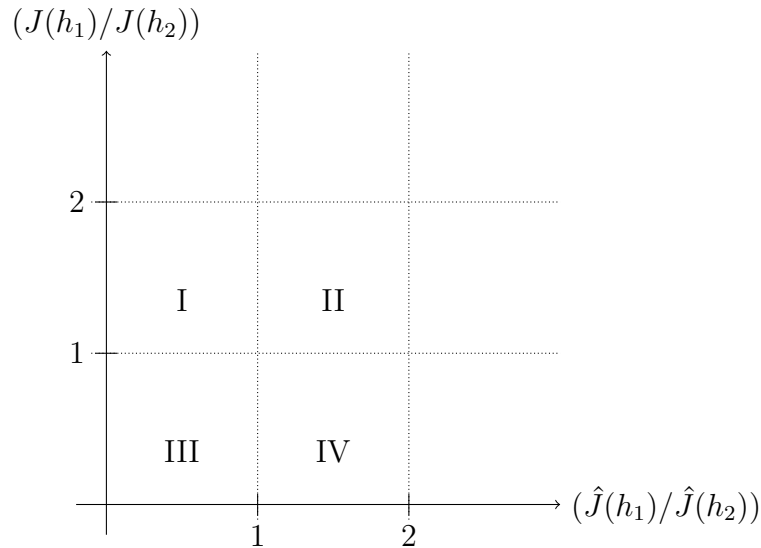


Figure 9 – Cartesian Plane with domain  $\langle 0, 2 \rangle$  and range  $\langle 0, 2 \rangle$

In Table 3 we present a single number for each domain representing the percentage of choices **SS** make correctly. From all the domains all are above of the 50% which means that at least the half of the points represent good relation of heuristics. With this experiment we prove that **SS** is not as bad as we thought it would be when we want to make selection.

Domain	II and III (%)
Elevators	78.57
Floortile	96.08
Nomystery	71.82
Parcprinter	70.50
Pegsol	96.83
Scanalyzer	100.00
Sokoban	89.31
Tidybot	100.00
Transport	51.78
Visitall	98.05
Woodworking	100.00

Table 3 – Percentage of choices **SS** made correctly.

## Chapter IV

### Empirical Evaluation



## 4 Empirical Evaluation

The practical effectiveness of **GHS** depends on its ability of finding good approximations  $\hat{J}$  and  $\hat{T}$ . In order to verify its practical effectiveness, we have implemented **GHS** in Fast Downward (Helmert, 2006) and tested the A\* performance using subsets of heuristics selected by **GHS** while minimizing different objective functions.

We run two sets of experiments. In the first set we verify whether the approximations  $\hat{J}$  and  $\hat{T}$  provided by **CS** and **SS** allow **GHS** selects good subset selections for A\* search tree size and running time. In the second set of experiments we test the effectiveness of **GHS** by measuring the total number of problem instances solved by A\* using a heuristic subset selected by **GHS**.

**GHS** is executed up to adding another heuristic does not improve the objective function. In each iteration **GHS** greedily selects from  $\zeta$  the heuristic  $h$  which will result in the largest reduction of the value of the objective function  $\Psi$ . We can't control the size of the resulting subset because **GHS** stops if adding another heuristic  $h$  to the  $\zeta'$  does not improve the objective function and returns the current subset  $\zeta'$ .

We ran our experiments on the 2011 International Planning Competition (IPC) instances. We used the 2011 instances instead of the 2014 instances because the former do not have problems with conditional effects, which are currently not handled by **PDB** heuristics. All experiments are run on 2.67 GHz machines with 4GB, and are limited to 1,800 seconds of running time.

Table 4 – Ratios of the number of nodes generated using  $h_{max}(\zeta')$  to the number of nodes generated using  $h_{max}(\zeta)$

Domain	SS		CS		$ \zeta $	n
	Ratio	$ \zeta' $	Ratio	$ \zeta' $		
Barman	1.11	17.70	1.50	30.25	5168.50	20
Elevators	11.50	2.00	1.03	21.00	168.00	1
Floortile	1.02	43.07	1.01	42.35	151.28	14
Openstacks	1.00	1.00	1.00	1.00	390.69	13
Parking	1.00	5.52	1.01	7.26	21.73	19
Pegsol	1.00	31.00	1.00	57.00	90.00	2
Scanalyzer	1.22	30.57	1.56	19.42	72.85	7
Tidybot	1.00	2.35	1.00	8.58	3400.17	17
Transport	1.00	14.70	1.02	14.30	171.7	10
Visitall	1.02	99.33	1.18	48.66	256.33	3
Woodworking	32.42	3.00	199.65	5.00	1289.00	5

## 4.1 Empirical Evaluation of $\hat{J}$ and $\hat{T}$

We test whether the approximation  $\hat{J}$  provided by **CS** and **SS** allows **GHS** to make good subset selections. This test is made by comparing  $J(\zeta')$  with  $J(\zeta)$ , which is minimal. The condition  $J(\zeta') \leq \alpha \cdot J(\zeta)$ , is sufficient to show that  $J(\zeta')$  is within  $\alpha$  times good with respect to all subsets of any size, for some constant  $\alpha$ .

In contrast with objective function  $J$ , there is no easy way to find the minimum of  $T$  for a subset in general. We experiment then with the special case in which all heuristics in  $\zeta$  have the same evaluation time. This way we are able to test whether the estimates  $\hat{T}$  are allowing **GHS** to make good subset selections while minimizing the A\* running time. This is because by only selecting heuristics which have the same evaluation time, if **GHS** is making good subset selections with respect to  $J$ , then **GHS**, also must be making good subset selections with respect to  $T$ .

We collect values of  $J(\zeta)$  and  $J(\zeta')$  as follows. For each problem instance  $\nabla$  in our test set we generate a set of PDB heuristics using the **GA-PDB** algorithm Edelkamp, (2007) as described by Barley et al., (2014) – we call each PDB generated by this method a **GA-PDB**. We chose to use **GA-PDBs** in this experiment because they all have nearly the same evaluation time and will allow us to verify whether **GHS** is making good selections when minimizing  $J$  and  $T$ , as explained above. The number of **GA-PDBs** generated is limited in this experiment by 1,200 seconds and 1GB of memory. Also, all **GA-PDBs** we generate have 2 millions entries each. The **GA-PDBs** generated form our  $\zeta$  set. **GHS** then selects a subset  $\zeta'$  of  $\zeta$ . Finally, we use  $h_{max}(\zeta')$  and  $h_{max}(\zeta)$  to independently try to solve  $\nabla$ . We call the system which uses A\* with  $h_{max}(\zeta)$  the **Max** approach. For **GHS** we allow 600 seconds for selecting  $\zeta'$  and for running A\* with  $h_{max}(\zeta')$ , and for **Max** we allow 600 seconds for running A\* with  $h_{max}(\zeta)$ . Since we used 1,200 seconds to generate the heuristics, both **Max** and **GHS** were allowed 1,800 seconds in total for solving each problem. In this experiment we test both **CS** and **SS**.

In this experiment we refer to the approach that runs A\* guided by a heuristic subset selected by **GHS** using **CS** as **GHS+CS**. Similarly, we write **GHS+SS** when **SS** is used as predictor to make the heuristic subset selection.

Table 4 shows the average ratios of  $J(\zeta')$  to  $J(\zeta)$  for both **SS** and **CS** in different problem domains. The value of  $J$ , for a given problem instance, is computed as the number of nodes expanded up to the largest  $f$ -layer which is fully expanded by all approaches tested (**Max**, **GHS** using **SS** and **GHS** using **CS**). We only present results for instances that are not solved during **GHS**'s **CS** sampling process. The column “ $n$ ” shows the number of instances used to compute the averages of each row. We also show the average number of **GA-PDBs** generated ( $|\zeta|$ ) and the average number of **GA-PDBs** selected by **GHS** ( $|\zeta'|$ ). This experiment shows that for most of the problems **GHS**, using **CS** or **SS**, is selecting good

Table 5 – Coverage of **SS**, **CS** and **Max** on the 2011 IPC benchmarks. For **GHS** using only **GA-PDBs** heuristics.

Domain	SS	CS	Max
Barman	8	7	4
Elevators	19	19	19
Floortile	10	10	9
Nomystery	20	20	20
Openstacks	17	17	11
Parcprinter	17	15	14
Parking	1	1	1
Pegsol	19	19	19
Scanalyzer	10	10	10
Sokoban	20	20	20
Tidybot	14	13	11
Transport	14	14	14
Visitall	18	18	18
Woodworking	12	11	12
Total	199	194	182

subset of  $\zeta$  for  $A^*$  search tree size and running time. For example, in Tidybot **GHS** selects only a few **GA-PDBs** out of thousands when using either **SS** or **CS**.

The exceptions in Table 4 are the ratios for Elevators, Scanalyzer and Woodworking. In Elevators **SS** has an average ratio of 11.50 and **CS** of 1.03. By looking at the ratios of **SS** for individual instances of Scanalyzer (results now show in Table 4), we noticed that **SS** is able to make good selections for all but 3 of the 7 instances considered in this experiment. Since we do not know a priori what is the instance’s optimal solution cost, **SS** samples nodes with  $f$ -values much larger than the instance’s optimal solution cost. We believe that, in this particular instance of Scanalyzer, by sampling a portion of the state space that is not expanded during the actual  $A^*$ , **SS** is biasing the subset selection to select heuristics that do not contribute to reducing the actual  $A^*$  search tree size.

The **SS**’s ability of sampling deep into the search space is not always harmful. For example, **SS** allows **GHS** to make good selections for instances of the Woodworking domain. By contrast, **CS**’s systematic approach to sampling only allow a shallow sample of the  $A^*$  search tree. As a result, **GHS** makes a limited selection of heuristics to guide  $A^*$  search. While **GHS** using **SS** selects an average of 3 heuristics in Woodworking instances, **GHS** using **CS** selects only an average of 5 heuristics. This difference on sampling strategies reflects on the number of problems solved by  $A^*$ . While **GHS+SS** solves 12 instances of the Woodworking domain, **GHS+CS** solves only 11. In total, out of the 280 instances of the IPC 2011 benchmark set, **GHS+SS** solves 199 problem instances in this experiment, while **GHS+CS** only solves 194 problem instances. (The numbers of instances solved are shown in Table 5).

## 4.2 Comparison with Other Planning Systems

The objective of this second set of experiments is to test the quality of the subset of heuristics **GHS** selects while optimizing different objective functions. Our evaluation metric is coverage, i.e., number of problems solved within a 1,800 second time limit. We note that the 1,800-second limit includes the time to generate  $\zeta$ , select  $\zeta'$ , and run A\* using  $h_{max}(\zeta')$ . The  $\zeta$  set of heuristics is composed of a number of different **GA-PDBs**, a PDB heuristic produced by the **iPDB** method Haslum et al., (2007) and the **LM-Cut** heuristic (Pommerening and Helmert, 2013). The generation of **GA-PDBs** is limited by 600 seconds and 1GB of memory. We use one fourth of 600 seconds to generate **GA-PDBs** with each of the following number of entries:  $\{2 \cdot 10^3, 2 \cdot 10^4, 2 \cdot 10^5, 2 \cdot 10^6\}$ . Our approach allows one to generate up to thousands of **GA-PDBs**. For every problem instance, we use exactly the same  $\zeta$  set for **Max** and all **GHS** approaches.

## 4.3 Systems Tested

**GHS** is tested while minimizing the A\* search tree size (**Size**) and the A\* running time (**Time**). We also use **GHS** to maximize the sum of heuristic values in the state space (**Sum**), as suggested by Rayner et al., (2013). Rayner et al., (2013) assumed that one could uniformly sample states in the state space in order to estimate the sum of the heuristic values for a given heuristic subset. Since we are not aware of any method to uniformly sample the state space of domain-independent problems, we adapted the Rayner et al., (2013)’s method by using **SS** to estimate the sum of heuristic values in the search tree rooted at  $\nabla$ ’s start state. We write **Size+SS** to refer to the approach that used A\* guided by a heuristic selected by **GHS** while minimizing an estimate of the search tree size provided by **SS**. We follow the same pattern to name the other possible combinations of objective functions and prediction algorithms (e.g., **Time+CS**).

In addition to experimenting with all combinations of prediction algorithms (**CS** and **SS**) and objective functions (**Time**, **Size**), we also experiment with an approach that minimizes both the search tree size and the running time as follows. First we create a pool of heuristics  $\zeta$  composed solely of **GA-PDB** heuristics, then we apply **GHS** while minimizing tree size and using **SS** as predictor. We call the selection of a subset of **GA-PDBs** as the *first selection*. Once the first selection is made, we test all possible combinations of the resulting  $h_{max}(\zeta')$  added to **iPDB** and **LM-Cut** heuristics while minimizing the running time as estimated by **CS**—we call this step the *second selection*. We call the overall approach **Hybrid**.

The intuition behind **Hybrid** is that we apply **GHS** with its strongest settings. **GHS** makes good selections respect to  $J$  and  $T$  when selecting from a pool of heuristics with the same evaluation time. After such a selection is made, we reduce the size of the pool of



heuristics from possible thousands to only three (the maximum of a subset of the initial **GA-PDBs**, **iPDB**, and **LM-Cut**). With only three heuristics we are able to choose the exact combination that minimizes the  $A^*$  running time the most. The reason we chose to use **SS** instead of **CS** for the first selection in **Hybrid** is that the former is able to make better subset selections in this setting, as suggested by the results discussed in the previous Chapter 3. Finally, as we show below, **CS** is more effective if one is interested in minimizing the  $A^*$  running time while selecting from a pool of heuristic with different evaluation times. That is why we use **CS** as predictor for the second selection in **Hybrid**.

We compare the coverage of the **GHS** approaches with several other state-of-the-art planners. Namely, we experiment with **RIDA\*** Barley et al., (2014), two variants of **StoneSoup** (**StSp1** and **StSp2**) as described by Nissim et al., (2011), two versions of **Symba** (**SY1** and **SY2**) (Torralba, 2015), and  $A^*$  being independently guided by the maximum of all heuristics in  $\zeta$  (**Max**), **iPDB**, **LM-cut** and **Merge & Shrink(M&S)** Nissim et al., (2011). The results are presented in Table 6.

## 4.4 Discussion of the Results

The system that solves the largest number of instances is **Hybrid**— it solves 219 problems on average. As explained above, we combine in **Hybrid** the strengths of both **SS** and **CS** in a single system. **GHS** uses **SS** to select heuristics from a pool of heuristics with similar evaluation time, and only then **CS** is used for selecting heuristics with different evaluation times. This strategy has proven particularly effective on the Barman domain where **Hybrid**’s first selection is able to select good subsets of **GA-PDBs** and its second selection is able to recognize that it must not include the **iPDB** and **LM-Cut** heuristics to the subset selected by its first selection. As a result, **Hybrid** solves more problems on this domain than any other **GHS** approach.

**Time+CS** also performed well in our experiments—the approach solves 216 problems on average. Clearly **Hybrid** and **Time+CS** are far superior to all other approaches tested. For example, **Size+SS** and **Sum** solves only 206 and 207 problems, respectively. While minimizing the search tree size or maximizing the sum of heuristic values, **GHS** will tend to add accurate heuristics to the selected subset, independently of their evaluation time. As a result, if not minimizing the running time, **GHS** often adds the **LM-Cut** heuristic to  $\zeta'$  as **LM-Cut** is often the heuristic that is able to reduce the most the search tree size and to increase the most the sum of heuristic values. However, **LM-Cut** is very computationally expensive, and in various cases the search is faster if **LM-Cut** is not in  $\zeta'$ . Both **Hybrid** and **Time+CS** are able to recognize when **LM-Cut** should not be included in  $\zeta'$  because they account for the heuristics’ evaluation time.

Note that the difference on the number of problems solved by **Time+CS** and

Table 6 – Coverage of different planning systems on the 2011 IPC benchmarks. For the **GHS** and **Max** approaches we also present the average number of heuristics **GHS** selects ( $|\zeta'|$ ).

Domains	Hybrid	CS		SS		Sum	Max	RIDA*	SY1	SY2	StSp1	StSp2	iPDB	LM-Cut	M&S
		Time	Size	Time	Size										
Barman	7	5	4	4	4	4	4	4	10	11	4	4	4	4	4
Elevators	19	19	19	19	19	19	19	19	20	20	18	18	18	17	12
Floortile	15	14	14	14	14	14	14	14	14	14	14	14	14	8	10
Nomystery	20	20	20	19	19	20	20	20	16	16	20	20	14	19	18
Openstacks	17	17	15	17	15	15	11	15	20	20	17	17	15	17	17
Parcprinter	18	18	18	16	15	19	18	18	17	17	18	18	17	16	16
Parking	7	7	2	7	2	2	2	7	2	1	5	5	2	7	7
Pegsol	18	18	19	19	19	19	19	19	19	20	19	19	17	20	19
Scanalyzer	13	14	12	11	14	14	14	14	9	9	14	14	12	10	11
Sokoban	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20
Tidybot	17	16	16	16	16	16	15	17	15	17	16	16	16	14	9
Transport	14	13	10	11	13	11	9	10	10	11	7	8	6	8	7
Visitall	18	18	18	15	18	18	18	18	12	12	16	16	10	16	16
Woodworking	16	15	15	12	16	16	16	15	20	20	15	15	15	9	9
Total	219	214	202	200	204	207	199	210	204	208	203	204	180	185	175

**Time+SS**: While the former solves 214 instances, the latter solved only 200. We conjecture that this happens because **SS** is not able to detect duplicated nodes during sampling. As a result, **SS** often overestimates by several orders of magnitude the actual  $A^*$ 's running time. Similarly to the **Size** and **Sum** approaches, due to **SS**'s overestimations, **Time+SS** often mistakenly adds the accurate but expensive **LM-Cut** heuristic in cases where the  $A^*$  search would be faster without **LM-Cut**'s guidance.

**RIDA\*** is the most similar system to **GHS**, as it also selects a subset of heuristics from a pool of heuristics by using an evaluation method similar to **CS**. **RIDA\*** uses a systematic approach for selecting a subset of heuristics. Namely, it starts with an empty subset and evaluates all subsets of size  $i$  before evaluating subsets of size  $i + 1$ . This procedure allows **RIDA\*** to consider only tens of heuristics in their pool. By contrast, **GHS** is able to consider thousands of heuristics while making its selection.

The ability to handle large set of heuristics can be helpful, even if most of the heuristics in the set are redundant with each other—as is the case with the **GA-PDBs**. The process of generating **GA-PDBs** is stochastic, thus one increases the chances of generating helpful heuristic by generating a large number of them. **GHS** is an effective method for selecting a small set of informative heuristics from a large set of mostly uninformative ones. This is illustrated in Table 6 on the **Transport** domain. Compared to systems which use multiple heuristics (**StSp1** and 2, and **RIDA\***), **Time+CS** solves the largest number of **Transport** instances, which is due to the selection of a few key **GA-PDBs**.

The best **GHS** approach, **Hybrid**, substantially outperforms the number of instances solved by **Max**; **Hybrid** solves on average more than 20 instances than **Max**. Finally, **Hybrid** and **Time+CS** substantially outperforms all other approaches tested, with **RIDA\*** being

the closest competitor with 210 instances solved.



Chapter V

Conclusion



## 5 Concluding Remarks

This dissertation showed that the problem of finding the optimal subset of a set of heuristics  $\zeta$  for a given problem task is solved using the models of  $A^*$  search tree size and, under mild assumptions, with respect to the  $A^*$  running time. Thus, the **GHS** algorithm which selects heuristics from  $\zeta$  one at a time is able to produce a good subset  $\zeta'$ , with respect to the search tree size. Furthermore, if all heuristics in  $\zeta$  have the same evaluation time, then we have a good subset selection with respect to running time. In addition to minimizing the search tree size and the running time, we also experimented with an objective function that accounts for the sum of heuristic values in the state-space, as suggested by Rayner et al., (2013).

Since we cannot compute the values of the objective functions exactly, **GHS** effectiveness depends on the quality of the approximations we can obtain. We tested two prediction algorithms, **CS** and **SS**, for estimating the values of the objective functions and showed empirically that both **CS** and **SS** allow **GHS** to make good subset selections with respect to the search tree size and running time.

Finally, experiments on optimal domain-independent problems showed that **GHS** minimizing approximations of the  $A^*$  running time outperformed all the other approaches tested, which demonstrates the effectiveness of our method for the heuristic subset selection problem.





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