CENTRO DE CIENCIAS EXATAS E TECNOLOGICAS - CCE

DEPARTAMENTO DE INFORMATICA

THESIS PROJECT:

ON SELECTING HEURISTIC FUNCTIONS FOR DOMAIN-INDEPENDENT PLANNING.

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Let N be a finite set and F a nonempty collection of subsets N which have the property that $F_1\ \varepsilon\ F\ and\ F_2 \subset \varepsilon\ F$, A real-valued function function z defined on the subsets of N that satisfies $z(S) \leq z(T)$ and $z(S)+z(T) \geq z(S\cup T)+z(S\cap T)$ for all $S,T \subseteq N$ is called nondecreasing and submodular. We consider the problem $max_{S\subseteq N}\{S\varepsilon F, z(S) submodular\ and\ nondecreasing\}$ and several special cases of it.
We analyze greedy and local improvement heuristics, and a linear programming relaxation when *z(S)* is linear. Our results are worst case bounds on the quality of the approximations. For example, when *(N, F)* is described by the intersection of P matroids, we show that a greedy heuristic always produces a solution whose value is at least $\frac{1}{P+1}$ times the optimal value. This bound can be achieved for all postive integers P.

0.1 Introduction

Let N = $\{$*1, ..., n*$\}$ be a finite set and z a real-valued function defined on the subset of N that satisfies

$z(S)+z(T) \geq z(S\cup T)+z(S\cap T)$

for all *S, T* in N. Such a set function is called *submodular*. This paper is the third in a series dealing with approximate methods for maximizing submodular set functions. We additionally assume here that *z(S)* is nondecreasing, i.e., $z(S) \subseteq z(T)$ for all $S \subset T \subseteq N$.

In [2] we studied the uncapacitated location problem

$\max_{S\subseteq N}\{z(S) : z(S) = \sum_{i\varepsilon I} \max_{j\varepsilon S} C_{ij}, |S| \subseteq K\}$

where $C = (C_{ij})$ is a nonnegative matrix with column index set N and row index set I and $z(\emptyset) = 0$, In [7] we generalized the results to the problem.

$$\max_{S\subseteq N}\{z(S) : |S| \subseteq\ K,\ z(S)\ submodular\ and\ nondecrasing\} \qquad (1)$$

Since many combinatorial optimization problems, including the maximum *m-cut* problem [8], a storage allocation problem [1] and the matroid partition problem [3], require an optimal partition or packing, we were motivated to extend our results to the problem.

$$\max_{S_1\subseteq N,\dots,S_m\subseteq N}\{\sum_{i=1}^{m} z_i(S_i) : U_{i=1}^{m} S_i \subseteq N, S_i\cap \S_k = \emptyset, k\neq i,\ z_i(S)\ submodular\ and\ nondecreasing\ i\ = 1,\dots m\} \quad (2)$$

We like to think of Equation 2 as the *"m-box"* model in which putting S_i in box i yields a value of $z_i(S_i)$ and the objective is to maximize the value summed over all boxes.

The *m-box* model can be used to describe a multiproduct version of the uncapacitated location problem. Here each box corresponds to a different product. Assigning the set of locations $S_i \subset N$ to box i means that these locations supply product i. The objective is to maximize $\sum_{i=1}^{m}\sum_{k\varepsilon I}\max_{j\varepsilon S_i} C_{kj}^{i}$

By adding the restricctions $|S_i| \leq 1$ to the *m-box* model we obtain the constraints of an assignment problem. Now by generalizing the objective function to include terms involving pairs of boxed we obtain a model of the quadratic assignment problem. Here the objective is no longer a sum of set functions but multidimensional set function of the form $v(S_i,\dots,S_m)$. We can treat these multidimensional set functions directly by defining a multidimensional version of submodularity, i.e.,

$v(S_1,\dots,S_m)+v(T_1,\dots,T_m) \geq v(S_1\cup T_1,\dots,S_m\cup T_m)+v(S_1\cap T_1,\dots,S_m\cap T_m)$

However, an alternative viewpoint of the box model renders this multidimensional construct unnecessary and provides a more general and unified framework for the extensions of Equation 1 that we consider here.

Let M be the set of boxed, rename the set of elements to be put into the boxes E, let $N = \{(i,j) : i \in M, j \in E\}$ and $N_j = \{(i,j) : i \in M\},\ j \in E$. There is a one-to-one correspondence between packing $(S_1,...S_m)$ of E and subsets $S \subseteq N$ that satisfy $|S \cap N_j| \leq 1, j \in E$. The correspondence is given by $S_i = \{j : (i,j) \in S\}$ and $S = \{(i,j) : j \in S_i, i \in M\}$ Therefore a generalized version of the *m-box* problem Equation 2 is

$$\max_{S \subseteq N}\{z(S) : |S \cap N_j| \leq 1, j \in E.\ z(S)\ \textit{submodular and nondecreasing}\} \tag{3}$$

Now comparing Equation 1 and Equation 2, we see that they differ only in their constraints. However in each case the family F of feasible or independent sets forms a matroid $M = (N,F)$; i.e., $F_1 \in F$ and $F_2 \subset F_1 => F_2 \in F[(N,F)\ \textit{is an independence system}]$ and for all $N' \subseteq N$ every maximal member of $F(N') = \{F : F \in F, F \subseteq N'\}$ has the same cardinality. In Equation 1 M is the matroid in which all subsets of cardinality K or smaller are independent and in 3 M is a partition matroid. Thus a natural generalization of 1 and 3 is

$$\max_{S \subseteq N}\{z(S) : S \in F, M = (N,F)\ \textit{a motroid},\ z(S)\ \textit{submodular and nondecreasing}\} \tag{4}$$

and an obvious generalization of 4 is

$$\max_{S \subseteq N}\{z(S) : S \in \cap_{p=1}^{P} F_p, M_p = (N,F_p)\ \textit{are matroids}\ p = 1,...,P,\ \textit{submodular and nondecreasing.}\} \tag{5}$$

Note that any independence system can be described as the intersection of P matroids for suitably large P.
Finally, a different generalization of problem Equation 3 is

$$\max_{S \subseteq N}\{z(S) : N = \cup_{j=1}^{n} N_j, Nj \cap N_k = \emptyset, j \neq k, S \cap N_j \in F^j,\ \textit{submodular and nondecreasing}\} \tag{6}$$

where $(N_j, F^j)\ j = 1,...n$ are independence systems, each the intersection of P or fewer matroids. Note that combining the disjoint independence system gives a problem over N of the form 5 involving the intersection of P matroids. Alternatively we can view 6 as a generalization of 5, where 5 is obtained from 6 by taking n = 1.

We now summarize our results. In section 2 we consider a greedy heuristic for problem 5 with the constraint $|S| = 1$ to obtain a set S' and then iteratively builds a nested sequence of sets $\{S'\}$, *t = 2,3,...,* where $\{S'\} \in \cap_{i=1}^{P} F_i$ and $|S'| = t.S^{t+1}$ is determinated by adding to S^t (if possible) a $j^\star$ such that

$z(S^t \cup \{j^\star\}) = \max\{z(S^t \cup \{j\}) : S^t \cup \{j\} \in \cap_{i=1}^{P} F_i, j \notin S^t\}$
We obtain the tight bound

$$\frac{\textit{value of greedy approximation}}{\textit{value of optimal solution}} \geq \frac{1}{P+1}$$

We also show that without regard to P, if K is the cardinality of a largest independent set, and *k+1* the cardinality of a smallest dependent (not independent) set,

$$\frac{\textit{value of greedy approximation}}{\textit{value of optimal solution}} \geq 1 - (\frac{K-1}{K})^k$$

Problem 1. is the special case of this model with $k = K$. In Section 3 we assume that $z(S)$ is linear, in which case Equation 5 can be represented as an intenger program. We study the linear programming relaxation of this integer program, which is obtained by suppressing the integrality restrictions. Our result is

$$\frac{\textit{value of greedy approximation}}{\textit{value of linear programming solution}} \geq \frac{1}{P},$$

which is a bound on the duality gap and also implies the bound obtained by JenKyns 5 and Korte and Hausmann [6] on the ratio of the greedy and integer solutions.

In Section [4] we examine problem Equation 6 and show that the greedy heuristic can be simplified and the bound of $\dfrac{1}{P+1}$ maintained. Also, for example Equation 3 when $z(S)$ has a certain symmetry with respect to the boxed, the bound of $\frac{1}{2}$ can be improved to $\frac{m}{2m-1}$, where m is the number of boxes.

In Section 5 we examine a local improvement heuristic for model 5. We show that when *P = 1*

$$\frac{value\ of\ local\ improvement\ approximation}{value\ of\ optimal\ solution} \geq \frac{1}{2}$$

but that the heuristic is arbitrarily bad when $P \geq 2$

We close this section by giving two other equivalent definitions of submodularity that are proved in [7]. Although this paper can be read independently, we strongly recomend the prior reading of [7].

Let $\rho_j(S) = z(S \cup \{j\}) - z(S)$

Proposition 1.1 *Each of the following statements is equivalent and defines a nondecreasing submodular set function.*

1. $z(S) + z(T) \geq z(S \cup T) + z(S \cap T), \forall S, T \subseteq N$
   $z(S) \leq z(T), \forall S \subseteq T \subseteq N$
2. $\rho_j(S) \geq \rho_j(T) \geq 0, \forall S \subset T \subset N$ *and* $j \in N - T$
3. $z(T) \leq z(S) + \sum_{j \in T-S} \rho_j(S),\ \forall S, T \subseteq N$

Finally, we assume throughout the paper that $z \neq z(\emptyset)$ and therefore exclude the trivial possibility of Ø being an optimal solution.

0.2 The greedy heuristic

We first describe the greedy heuristic and then obtain two worst case bounds for problem Equation 5

The greedy heuristic for nondecreasing set functions on independence system (N, F)

Initialization. Let $S^0 = \emptyset, N^{\emptyset} = N$ and set $t = 1$.

Iteration t

Step 0 If $N^{t-1} = \emptyset$, stop with S^{t-1} the greedy solution.

Step 1 Select $i(t) \in N^{t-1}$ for which $\rho_{i(t)}(S^{t-1}) = \max_{i \in N^{t-1}} \rho_i(S^{t-1})$, with ties settled arbitrarily.

Step 2a If $S^{t-1} \cup \{i(t)\} \notin F,$ *set* $N^{t-1} = N^{t-1} - \{i(t)\}$ and return to Step 0.

Step 2b If $S^{t-1} \cup \{i(t)\} \in F,$ *set* $\rho_{t-1} = \rho_{i(t)}(S^{t-1}), S^t = S^{t-1} \cup \{i(t)\}$ and $N^t = N^{t-1} - \{i(t)\}$

Step 3 Set $t \rightarrow t+1$ and continue.

Let U^t be the set of elements considered in the first $t+1$ iterations of the greedy heuristic before the addition of a $(t+1)$st element $\mathscr{F} = \cap_{p=1}^{P} \mathscr{F}_p$, where $\mathscr{M}_p = (N, \mathscr{F}_p)$ are matroids $p = 1, \ldots P$. Define $r_p(S)$, called the rank of S in matroid p, and define $sp^p(S) = \{j \in N : r_p(S \cup \{j\}) = r_p(S)\}$