AMSC 660 Homework 2
Due 09/13/23
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Problem 1

Consider the polynomial space $\mathcal{P}_n(x), x \in [-1,1]$. Let $T_k, k = 0, 1, \ldots, n$, be the Chebyshev basis in it. The Chebyshev polynomials are defined via

$$T_k = \cos(k \arccos(x)).$$

(a) Use the trigonometric formula

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

to derive the three-term recurrence relationship for the Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1} = 2xT_k(x) - T_{k-1}(x), \quad k = 1, 2, \ldots.$$

(b) Consider the differentiation map

$$\frac{\mathrm{d}}{\mathrm{d}x} : \mathcal{P}_n \to \mathcal{P}_{n-1}.$$

Write the matrix of the differentiation map with respect to the Chebyshev bases in $\mathcal{P}_n$ and $\mathcal{P}_{n-1}$ for $n = 7$. Hint: you might find helpful properties of Chebyshev polynomials presented in Section 3.3.1 of Gil, Segure, Temme, "Numerical Methods For Special Functions". Chapter 3 of this book is added to Files/Refs on ELMS.

Solution

Proof. Consider the polynomial space $\mathcal{P}_n x, x \in [-1,1]$. Let $T_k, k = 0, 1, \ldots, n$, be the Chebyshev basis in it. The Chebyshev polynomials are defined via

$$T_k = \cos(k\theta),$$

where $\theta = \arccos(x)$.

(a) Applying the trigonometric formula

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right),$$

gives

$$T_{k-1}(x) + T_{k+1}(x) = \cos((k-1)\theta) + \cos((k+1)\theta)$$

$$
\begin{aligned}
&= 2\cos\left(\frac{(k-1)\theta + (k+1)\theta}{2}\right)\cos\left(\frac{(k-1)\theta - (k+1)\theta}{2}\right) \\
&= 2\cos(k\theta)\cos(-\theta) \\
&= 2\cos(\theta)\cos(k\theta) \\
&= 2xT_k(x).
\end{aligned}
$$

Then rearranging the terms gives the three-term recurrence relationship to be

$$T_{k-1}(x) + T_{k+1} = 2xT_k x \implies T_{k+1} = 2xT_k(x) - T_{k-1}(x)$$

(b) Consider the differentiation map

$$\frac{\mathrm{d}}{\mathrm{d}x} : \mathcal{P}_n \to \mathcal{P}_{n-1}.$$

We wish to write the matrix of the differentiation map with respect to the Chebyshev bases in $\mathcal{P}_n$ and $\mathcal{P}_{n-1}$ for $n = 7$.

From the given resources we have the following relation for the Chebyshev polynomial derivatives

$$
\begin{cases}
T_0(x) = T_1'(x), \\
T_1(x) = \frac{1}{4}T_2'(x), \\
T_n(x) = \frac{1}{2}\left(\frac{T_{n+1}'}{n+1} - \frac{T_{n-1}'}{n-1}\right).
\end{cases}
$$

Notice that we can rearrange the terms to find

$$
\begin{aligned}
T_n &= \frac{1}{2}\left(\frac{T_{n+1}'}{n+1} - \frac{T_{n-1}'}{n-1}\right) \\
2T_n &= \frac{T_{n+1}'}{n+1} - \frac{T_{n-1}'}{n-1} \\
\frac{T_{n+1}'}{n+1} &= 2T_n + \frac{T_{n-1}'}{n-1} \\
T_{n+1}' &= (n+1)\left(2T_n + \frac{1}{n-1}T_{n-1}'\right) \\
T_{n+1}' &= 2(n+1)T_n + \frac{n+1}{n-1}T_{n-1}' \\
T_n' &= 2nT_{n-1} + \frac{n}{n-2}T_{n-2}'.
\end{aligned}
$$

We can use this relation to compute the first seven derivatives

$$
\begin{aligned}
T_0' &= 0, \\
T_1' &= T_0, \\
T_2' &= 4T_1,
\end{aligned}
$$

$$
\begin{aligned}
T_3' &= 6T_2 + 3T_1' = 6T_2 + 3T_0,\\
T_4' &= 8T_3 + \frac{4}{2}T_2' = 8T_3 + 8T_1,\\
T_5' &= 10T_4 + \frac{5}{3}T_3' = 10T_4 + 10T_2 + 5T_0,\\
T_6' &= 12T_5 + \frac{6}{4}T_4' = 12T_5 + 12T_3 + 12T_1,\\
T_7' &= 14T_6 + \frac{7}{5}T_5' = 14T_6 + 14T_4 + 14T_2 + 7T_0.
\end{aligned}
$$

Thus the matrix of the differentiation map is given by

$$
A = \begin{pmatrix}
0 & 1 & 0 & 3 & 0 & 5 & 0 & 7\\
0 & 0 & 4 & 0 & 8 & 0 & 12 & 0\\
0 & 0 & 0 & 6 & 0 & 10 & 0 & 14\\
0 & 0 & 0 & 0 & 8 & 0 & 12 & 0\\
0 & 0 & 0 & 0 & 0 & 10 & 0 & 14\\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 14
\end{pmatrix},
$$

where the columns of A correspond to the Chebyshev bases for T_i' for $0 \le i \le 7$, i.e. the A_{00} entry corresponds to the T_0 element of T_0'.

$\square$

Problem 2

Let $A = (a_{ij})$ be an $m \times n$ matrix.

(a) Prove that the l_1-norm of A is

$$\|A\|_1 = \max_j \sum_i |a_{ij}|,$$

i.e., the maximal column sum of absolute values. Find the maximizing vector.

(b) Prove that the max-norm or l_∞-norm of A

$$\|A\|_{\max} = \max_i \sum_j |a_{ij}|,$$

i.e., the maximal row sum of absolute values. Find the maximizing vector.

Solution

Proof. Let $A = (a_{ij}) = (a_1|a_2|\cdots|a_n)$ be an $m \times n$ matrix.

(a) Let k be the index such that $\max_j \|a_j\|_1 = \|a_k\|_1$. By definition we have

$$\begin{aligned}
\|A\|_1 &= \max_{\|v\|_1=1} \|Av\|_1 \\
&= \max_{\|v\|_1=1} \left\| \sum_{j=1}^n a_j v_j \right\|_1 \\
&\leq \max_{\|v\|_1=1} \sum_j^n \|a_j v_j\|_1 \\
&= \max_{\|v\|_1=1} \sum_j^n |v_j| \|a_j\|_1 \\
&\leq \max_{\|v\|_1=1} \left(\sum_j^n |v_j| \right) \|a_k\|_1 \\
&= \|a_k\|_1 \\
&= \max_j \|a_j\|_1 \\
&= \max_j \sum_i |a_{ij}|.
\end{aligned}$$

Notice that if we let $v = e_k$, we achieve equality

$$\|A\|_1 = \|Ae_k\|_1 = \|a_k\|_1 = \max_j \sum_i |a_{ij}|.$$

(b) By definition we have

$$\begin{aligned}
\|A\|_\infty &= \max_{\|v\|_\infty=1} \|Av\|_\infty \\
&= \max_{\|v\|_\infty=1} \left\| \sum_j^n a_j v_j \right\|_\infty \\
&\leq \max_{\|v\|_\infty=1} \left\| \sum_j^n \|v\|_\infty a_j \right\|_\infty \\
&= \left\| \sum_j^n a_j \right\|_\infty \\
&= \max_i \sum_j^n |a_{ij}|.
\end{aligned}$$

Notice that if we define the vector v such that $v_i = \mathrm{Sign}(a_{ij})$, then $a_{ij}v_i = |a_{ij}|$ which means

$$\|A\|_\infty = \|Av\|_\infty = \left\| \sum_j^n |a_j| \right\|_\infty = \max_i \sum_j^n |a_{ij}|$$

$\square$

Problem 3

Consider the matrix

$$A = \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix}.$$

(a) Find the Jordan form of A.

(b) Find the 2-norm of A.

Solution

Proof. Consider the matrix

$$A = \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix}.$$

(a) Thus $\lambda = 1$ with algebraic multiplicity 2 are the eigenvalues of A. Next, let's find eigenspace corresponding to $\lambda = 1$ by solving $Av = \lambda v$, where v is the corresponding eigenvector

$$\begin{pmatrix} 1-\lambda & 10 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$
$$\begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0,$$

solving the system gives

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix},$$

and letting $a_1 = 1$ we get the eigenvector $v_1 = (1,0)^T$. Since λ has algebraic multiplicity 2, we can find the second eigenvector in the corresponding eigenspace using the generalized eigenvector of the form

$$(A - I\lambda)v_2 = v_1 \implies \begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 10b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

thus we have that $v_2 = (0, 1/10)^T$. Then to find the Jordan form we compute

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 1/10 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/10 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = J.$$

(b) Recall that

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)} = \sigma_{\max}.$$

Notice that

$$A^*A = \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 10 \\ 10 & 101 \end{pmatrix},$$

whose characteristic polynomial is

$$(1-\lambda)(101-\lambda)-100=\lambda^2-100\lambda+1=0.$$

Solving using the quadratic formula yields

$$\lambda_{1,2}=51\pm 10\sqrt{26}.$$

Thus

$$\|A\|_2=\sqrt{51\pm 10\sqrt{26}}\approx 10.09$$

$\square$