AMSC 660 Homework 12
Due Dec 1
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Problem 1

Consider the KKT system
$$\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} -\mathbf{p} \\ \lambda \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix},$$
where G is a $d \times d$ symmetric positive definite matrix and A is $m \times d$ and has linearly independent rows. Show that the matrix
$$K := \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}$$
is of saddle-point type, i.e., it has d positive eigenvalues and m negative ones.

Solution

Proof. Consider the KKT system
$$\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}\begin{pmatrix} -\mathbf{p} \\ \lambda \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix},$$
where G is a $d \times d$ symmetric positive definite matrix and A is $m \times d$ and has linearly independent rows. Let's denote
$$K := \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix},$$
and consider the decomposition
$$\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} G & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I & X^\top \\ 0 & I \end{pmatrix}.$$
Where we can find G and S by observing
$$\begin{aligned}
\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} G & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I & X^\top \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} G & 0 \\ XG & S \end{pmatrix}\begin{pmatrix} I & X^\top \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} G & GX^T \\ XG & XGX^\top + S \end{pmatrix} \\
&= \begin{pmatrix} G & (XG)^T \\ XG & XGX^\top + S \end{pmatrix},
\end{aligned}$$

and so $X = AG^{-1}$ as $G = G^\top$ since G is SPD and we can solve for S to get

$$S = -XGX^\top = -AG^{-1}G(AG^{-1})^\top = -AG^{-1}A^\top.$$

Thus

$$\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ AG^{-1} & I \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & AG^{-1}A^\top \end{pmatrix} \begin{pmatrix} I & (AG^{-1})^\top \\ 0 & I \end{pmatrix},$$

Since $\begin{pmatrix} I & 0 \\ AG^{-1} & I \end{pmatrix}$ and its transpose are nonsingular, we have that K and $\begin{pmatrix} G & 0 \\ 0 & AG^{-1}A^\top \end{pmatrix}$ are congruent. Then by Sylvester's law of inertia, K and $\begin{pmatrix} G & 0 \\ 0 & AG^{-1}A^\top \end{pmatrix}$ have the same number of positive, zero, and negative eigenvalues. Note that both matrices are size $d+m \times d+m$. Since G is SPD and $d \times d$, it has d real and positive eigenvalues. Next, let (λ, v) be an eigenpair of $AG^{-1}A^\top$, observe that

$$\begin{aligned} -AG^{-1}A^\top v &= \lambda v \\ \Leftrightarrow -v^\top(AG^{-1}A^\top)v &= v^\top \lambda v \\ \Leftrightarrow -(v^\top A)G^{-1}(v^\top A)^\top &= \lambda \|v\|^2 \end{aligned}$$

Since G is SPD, we have that $-(v^\top A)G^{-1}(v^\top A)^\top < 0$ where the strick inequality is since A is of full rank and thus $v^\top A \neq 0, \forall v$. Furthermore, as $\|v\|^2 > 0$, we have that $\lambda < 0$. Thus $AG^{-1}A^\top$ has m negative eigenvalues. Therefore, $\begin{pmatrix} G & 0 \\ 0 & AG^{-1}A^\top \end{pmatrix}$ has d positive eigenvalues and m negative ones which gives that K does as well.

$\square$

Problem 2

Consider an equality-constrained quadratic problem QP

$$\frac{1}{2}x^\top Gx + c^\top x \to \min \text{ subject to}$$
$$Ax = b.$$

The matrix G is symmetric. Assume that A is full rank (i.e. its rows are linearly independent) and $Z^\top GZ$ is positive definite where Z is a basis for the null-space of A, i.e., $AZ = 0$.

(a) Write the KKT system for this case in the matrix form.

(b) Show that the matrix of this system K is invertible. Hint: assume that there is a vector $z := (x, y)^\top$ such that $Kz = 0$. Consider the quadratic form $z^\top Kz$, use logical reasoning and algebra, and arrive at the conclusion that then $z = 0$.

(c) Conclude that there exists a unique vector $(x^*, \lambda^*)^\top$ that solves the KKT system. Note that since we have only equality constraints, the positivity of λ is irrelevant.

Solution

Proof. Consider an equality-constrained quadratic problem QP

$$f(x) = \frac{1}{2}x^\top Gx + c^\top x \to \min \text{ subject to} \tag{1}$$
$$Ax = b, \tag{2}$$

where G is symmetric, A is full rank, and $Z^\top GZ$ is positive definite where Z is a basis for the null-space of A, i.e., $AZ = 0$.

(a) To find the KKT system of this problem, let x_k be the current iterate and consider the update

$$x_{k+1} = x_k + p_k.$$

Plugging this into Equation (1) yields

$$\begin{aligned}
&\frac{1}{2}(x_k + p_k)^\top G(x_k + p_k) + c^\top(x_k + p_k) = \\
&= \frac{1}{2}p_k^\top Gp_k + \left(\frac{1}{2}x_k^\top Gx_k + +c^\top x_k\right) + \frac{1}{2}x_k^\top Gp_k + \frac{1}{2}p_k^\top Gx_k + +c^\top p_k \\
&= \frac{1}{2}p_k^\top Gp_k + (Gx_k + c)^\top p_k + f(x_k).
\end{aligned}$$

Furthermore, notice that $Gx_k + c = \nabla f(x_k)$ and since $f(x_k)$ is independent of p_k, it does not affect the minimizer. Next, we can plug $x_{k+1} = x_k + p_k$ into Equation (2) to get

$$A(x_k + p_k) - b = (Ax_k - b) + Ap_k = Ap_k,$$

since $Ax_k - b = 0$. Therefore, the minimization problem for p_k is of the form

$$\frac{1}{2}p^\top Gp + \nabla f(x_k)^\top p \to \min \text{ subject to}$$
$$Ap = 0.$$

To find the KKT system, we compute the Lagrangian function of the problem to be

$$L(p_k, \lambda) = \frac{1}{2}p_k^\top Gp_k + \nabla f(x_k)^\top p_k - \lambda^\top(Ap_k).$$

Next, we compute the gradients to get

$$\nabla_{p_k} L(p_k, \lambda) = Gp_k + \nabla f(x_k) - A^\top \lambda \text{ and } \nabla_\lambda L(p_k, \lambda) = -Ap_k.$$

We can now rewrite the KKT system in matrix form as

$$\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} -p_k \\ \lambda \end{pmatrix} = \begin{pmatrix} \nabla f(x_k) \\ 0 \end{pmatrix}. \tag{3}$$

(b) Now if we let

$$K = \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix},$$

we wish to show that k is nonsingular. To do so, assume that there exists $z := (x, y)^\top$ such that $Kz = 0$. Then we have that

$$\begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0, \tag{4}$$

giving that $Ax = 0$. Thus we get that

$$0 = \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^\top Gx + (Ax)^\top y + y^\top Ax = x^\top Gx.$$

Now since x is in the null space of A, we can rewrite it as a $x = uZ$ for some vector u. This gives that

$$0 = x^\top Gx = u^\top \big(Z^\top GZ\big)u,$$

and since $Z^\top GZ$ is SPD, we have that $u = 0$ which gives $x = 0$. Furthermore, by plugging $x = 0$ into Equation 4, we find that $A^\top y = 0$. Since A is full row rank, $A^\top y = 0$ if and only if $y = 0$. Therefore $Kz = 0$ if and only if $z = 0$ which shows that K is nonsingular.

(c) Since K is nonsingular, there exists a vector v that satisfies Eq. (3), namely

$$v = \begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} G & A^\top \\ A & 0 \end{pmatrix}^{-1} \begin{pmatrix} \nabla f(x_k) \\ 0 \end{pmatrix}. \tag{5}$$

To show that v is unique, assume there exists another vector u such that $u \neq v$ and u satisfies Eq. (3). Then we have that

$$Kv = \begin{pmatrix} \nabla f(x_k) \\ 0 \end{pmatrix} \text{ and } Ku = \begin{pmatrix} \nabla f(x_k) \\ 0 \end{pmatrix}.$$

But this gives that

$$Kv - Ku = K(v - u) = 0,$$

and since $u \neq v$, then $v - u \neq 0$ which contradicts K being nonsingular. Therefore, there exists a unique vector $(x^*, \lambda^*)^\top$ which satisfies the KKT system Eq. (3).

$\square$

Problem 3

Consider the following quadratic program with inequality constraints:

$$f(x,y) = (x-1)^2 + (y-2.5)^2 \to \min \quad \text{subject to} \tag{6}$$

[1] $$x - 2y + 2 \geq 0 \tag{7}$$

[2] $$-x - 2y + 6 \geq 0 \tag{8}$$

[3] $$-x + 2y + 2 \geq 0 \tag{9}$$

[4] $$x \geq 0 \tag{10}$$

[5] $$y \geq 0 \tag{11}$$

(a) Plot level sets of the objective function and the feasible set.

(b) What is the exact solution to (6) - (11)? Find it analytically with the help of your figure.

(c) Suppose the initial point is $(2,0)$. Initially, constraints 3 and 5 are active, hence start with $\mathcal{W} = \{3,5\}$. Work out all iterations of the active-set method analytically. The arising linear systems should be very easy to solve. For each iteration, you need to write out the set $\mathcal{W}$, the KKT system, its solution, i.e., (p_x, p_y), the vector of Lagrange Multipliers, and the current iterate (x_k, y_k). Plot all iterations on your figure. There should be a total of 5 iterations.

Solution

Proof. Consider the quadratic program with inequality constraints given by (6)-(10).

(a) To plot the level sets of the objective function given in Eq. (6), we notice that these are circles with origin at $(1, 2.5)$. Next, we can rewrite the constraints [1]-[3] to get

$$\begin{aligned} &[1] \quad y \leq \frac{1}{2}x + 1, \\ &[2] \quad y \leq -\frac{1}{2}x + 3, \\ &[3] \quad y \leq \frac{1}{2} - 1. \end{aligned}$$

Plotting these we get Figure 1 where we can see the level sets and the feasible set.

(b) Observing Figure 1, we notice that the true minimum of $f(x,y)$ occurs at $(1, 2.5)$. Thus we can see that the constrained minimum occurs when the level set of $f(x,y)$ intersects constraint [1]. Using that $y = \frac{1}{2}x + 1$, we get

$$f(x) = (x-1)^2 + \left(\frac{1}{2}x - \frac{3}{2}\right)^2.$$

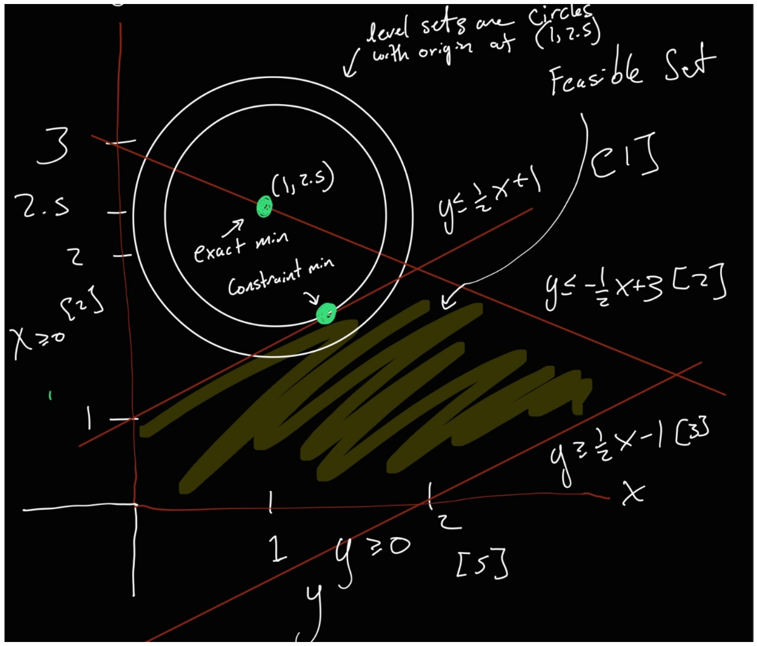

Figure 1: Plot of the quadratic problem (6)-(11). The level sets are shown in white, the constraint functions in red, the feasible set in yellow, and the solutions in green.

Now we can take the derivative and find the minimum to get

$$f'(x) = 2(x-1) + \left(\frac{1}{2}x - \frac{3}{2}\right) = \frac{5}{2}x - \frac{7}{2} = 0 \iff x = \frac{7}{5}.$$

Therefore we find that the solution to (6)-(11) is given by $\left(\frac{7}{5}, \frac{17}{10}\right)$.

(c) To apply the active-step method to this problem, let's rewrite the problem as

$$\begin{aligned} &f(v) = \frac{1}{2}v^\top G v + c^\top v \to \min \text{ subject to} \\ &A_{\mathcal{W}} v = b_{\mathcal{W}}, \end{aligned}$$

where $v = (x, y)^\top$ and $\mathcal{W}$ is the current set of active constraints. Now by expanding

$$f(x, y) = (x-1)^2 + (y-2.5)^2 = x^2 + y^2 - 2x - 5y + 1 + \frac{25}{4},$$

we find that the quadratic term is $x^2 + y^2$ and the linear part is $-2x - 5y$ and thus

$$G = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } c = \begin{pmatrix} -2 \\ -5 \end{pmatrix}.$$

Suppose the initial point is $v_1 = (2, 0)$. Then the active set of constraints is $\mathcal{W} = \{3, 5\}$. Thus we get

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Finally, notice that the gradient is given by

$$\nabla f(x_k, y_k) = (2x - 2, 2y - 5)^\top,$$

so $\nabla f(v_1) = (2, -5)^\top$. Therefore KKT system on the first iteration is given by

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 2 & 1 \\ -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -p_{1_x} \\ -p_{1_y} \\ \lambda_{1_x} \\ \lambda_{1_y} \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 0 \\ 0 \end{pmatrix}.$$

Now let's solve the KKT system for p_1 and λ_1. Observe that

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 2 & 1 \\ -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -p_{1_x} \\ -p_{1_y} \\ \lambda_{1_x} \\ \lambda_{1_y} \end{pmatrix} = \begin{pmatrix} -2p_{1_x} - \lambda_{1_x} \\ 2\lambda_{1_x} + \lambda_{1_y} - 2p_{1_y} \\ p_{1_x} - 2p_{1_y} \\ -p_{1_y} \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 0 \\ 0 \end{pmatrix}$$

and solving the linear system of equations yields

$$p_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \lambda_1 = \begin{pmatrix} -2 \\ -1 \end{pmatrix},$$

since both components of λ_1 are negative, we remove both of the constraints and set $v_2 = v_1 + p_1$.

On the second iteration, the iterate is $v_2 = (2, 0)$ and the active set is $\mathcal{W} = \emptyset$. We compute that $\nabla f(v_2) = (2, -5)^\top$. Therefore the KKT system on the second iteration is given by

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -p_{2_x} \\ -p_{2_y} \end{pmatrix} = \begin{pmatrix} -2p_{2_x} \\ -2p_{2_y} \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$

Solving this system gives

$$p_2 = \begin{pmatrix} -1 \\ 5/2 \end{pmatrix}.$$

Now notice that $v_2 + p_2$ is out of the feasible set, so let's find the step size given by

$$\alpha_2 = \min\left\{1, \min_{i \notin \mathcal{W}, a_i^\top p_2} \frac{b_i - a_i^\top p_2}{a_i p_2}\right\} = \min\left\{1, \frac{b_1 - a_1^\top p_2}{a_1 p_2}\right\} = \min\left\{1, \frac{2}{3}\right\} = \frac{2}{3},$$

and thus we update $v_3 = v_2 + \alpha_2 p_2$.

On the third iteration, the iterate is $v_3 = (4/3, 5/3)$ and so the first constraint is active meaning that $\mathcal{W} = \{1\}$. Then $A = (1, -2)^\top$ and $\nabla f(v_3) = (2/3, -5/3)$. Therefore the KKT system is given by

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} p_{3_x} \\ p_{3_y} \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -5/3 \\ 0 \end{pmatrix}$$

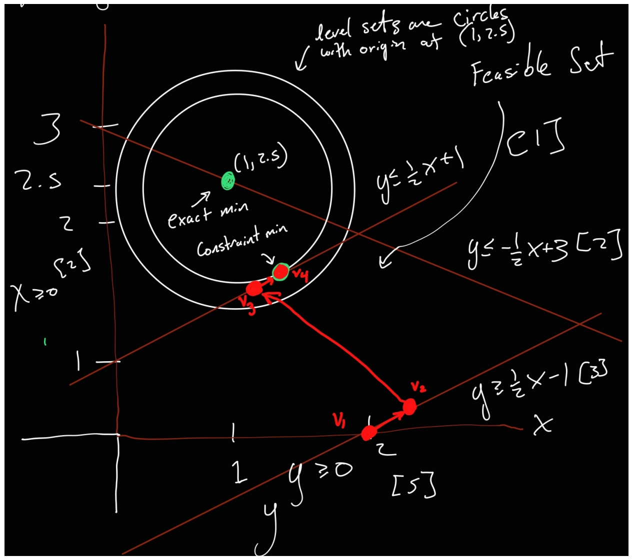

Figure 2: The iterates of the KKT active-step method can be seen in bright red.

Solving this system gives

$$p_3 = \begin{pmatrix} 1/15 \\ 1/30 \end{pmatrix} \text{ and } \lambda_3 = 4/5,$$

and thus we update $v_4 = v_3 + p_3$.

On the fourth iteration, the iterate $v_4 = (7/5, 17/10)$ and so the first constraint us active meaning that $\mathcal{W} = \{1\}$. Then $A = (1, -2)^\top$ and $\nabla f(v_4) = (4/5, -8/5)$. Therefore the KKT system is given by

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -2 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} p_{4_x} \\ p_{4_y} \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 4/5 \\ -8/5 \\ 0 \end{pmatrix}$$

Solving this system gives

$$p_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \lambda_4 = 4/5.$$

Since $p_4 = \vec{0}$ and $\lambda_4 > 0$, we have found the solution to be $(7/5, 17/10)$ which corresponds with the solution, we found in part (b). The iterates can be seen in Figure 2.

$\square$