0.1 Study questions

Please note, I completed this while being very rushed so there may be mistakes. Please don't think this covers every topic on the exam. Use this as a study tool and not as a "this is what will show up on the exam." I know as much as you do about the exam.

Problem 1

1. For what values of x does $f(x) = x^4 + 2x^3 - 12x^2 + x$ have a point of inflection?
2. List all the intervals on which $f(x) = 2x^3 - 3x^2 - 12x$ is increasing and those where it is decreasing?

Solution

Proof. (a) Let $f(x) = x^4 + 2x^3 - 12x^2 + x$. Recall that the inflection point can be found by computing $f''(x) = 0$ and solving for x. We have

$$f'(x) = 4x^3 + 2 \cdot 3x^2 - 12 \cdot 2x + 1 = 4x^3 + 6x^2 - 24x + 1,$$

and so

$$f''(x) = 4 \cdot 3x^2 + 6 \cdot 2x - 24 = 12x^2 + 12x - 24.$$

Now we set $f''(x) = 0$ and solve via factoring

$$\begin{aligned} f''(x) = 0 &= 12x^2 + 12x - 24 \\ 0 &= 12(x^2 + x - 2) \\ 0 &= 12((x+1)(x-2)), \quad \text{(two numbers that add to 1 and multiple to } -2), \end{aligned}$$

thus the zeros are $x = -1$ and $x = 2$. This means that the inflection points are $x = -1$ and $x = 2$.

(b) Let $f(x) = 2x^3 - 3x^2 - 12x$. Recall that the $f(x)$ is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. So first let's compute $f'(x)$:

$$f'(x) = 6x^2 - 6x - 12.$$

and setting this equal to zero and factoring just like we did in (a):

$$f'(x) = 0 = 6x^2 - 6x - 12 = 6(x^2 - x - 12) = 6(x+1)(x-2),$$

thus the zeros are $x = -1$ and $x = 2$. We won't have a calculator on the exam, so we need to figure out what the function does when it is less than -1, between $x = -1$

and $x = 2$, and greater than $x = 2$. We can just plug in points in these intervals and see if they are positive or negative:

$$f'(-2) = 6(-2)^2 - 6(-2) - 12 = 12 > 0 \quad (f'(x) > 0)$$
$$f'(0) = 6(0)^2 - 6(0) - 12 = -12 < 0 \quad (f'(x) < 0)$$
$$f(3) = 6(3)^2 - 6(3) - 12 = 24 > 0 \quad (f'(x) > 0)$$

Since $f'(x)$ is positive for $x < -1$ and $x > 2$ and negative for $-1 < x < 2$. This means that $f(x)$ is increasing on $(-\infty, -1) \cup (2, \infty)$ and decreasing on $(-1, 2)$

$\square$

Problem 2

Find the equation of tangent line L to the curve $y = \sqrt{2x-3}$ at the point $x = 6$.

Solution

Proof. Remember the Point-Slope formula: The equation of a line with slope m going through the point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1).$$

We can combine this idea with derivatives, functions, and tangent lines Remember that $f(x_1) = y_1$ gives us the point y_1 and that $f'(x_1) = m$ gives the slope of $f(x)$ at the point x_1. So if it is easier for you, we can rewrite the Point-Slope formula as

$$y - f(x_1) = f'(x_1)(x - x_1),$$

which is the tangent line of $f(x)$ at point x_1. Since the tangent line goes through the point $(x_1, f(x_1))$ with the same slope as $f(x)$ at x_1.

Okay, now let's do the question: let $f(x) = y = \sqrt{2x-3}$ and we want to find tangent line at $x = 6$ so let $x_1 = 6$. Now we can find y_1 to be

$$f(x_1) = y_1 = \sqrt{2(x_1) - 3} = \sqrt{2(6) - 3} = \sqrt{12 - 3} = \sqrt{9} = 3$$

So we have the point $(x_1, f(x_1)) = (x_1, y_1) = (6, 3)$. Now let's find the slope at $x_1 = 6$. First, let's find the derivative using the generalized power rule also known as the Chain Rule to be

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{2x-3} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(2x-3)^{1/2} \quad \text{Rewrite the square root as power} \\
&= \frac{1}{2}(2x-3)^{1/2-1}\frac{\mathrm{d}}{\mathrm{d}x}(2x-3) \\
&= \frac{1}{2}(2x-3)^{-1/2}(2) \\
&= (2x-3)^{-1/2} \\
&= \frac{1}{\sqrt{2x-3}}.
\end{aligned}$$

Now we plug in the point $x_1 = 6$ to find the slope at this point:

$$f'(x_1) = \frac{1}{\sqrt{2x_1 - 3}} = \frac{1}{\sqrt{12-3}} = \frac{1}{3} = m.$$

Then we plug everything into the Point Slope formula:

$$y - f(x_1) = f'(x_1)(x - x_1) \text{ or } y - y_1 = m(x - x_1),$$

becomes

$$\begin{aligned} y - 3 &= \frac{1}{3}(x - 6) \\ y &= \frac{1}{3}x - 2 + 3 \\ y &= \frac{1}{3}x + 1, \end{aligned}$$

which is the tangent line of $f(x)$ at $x = 6$.

$\square$

Problem

We want to make a can out of a material such that it can hold 54π cubic inches of liquid. What should the dimensions be to use the least possible amount of material? *Hint: The volume of a cylinder of height h and radius r is $\pi r^2 h$ and the area of the circular part of the can on the side (excluding the top or bottom) is $2\pi rh$.*

Solution

Proof. Here is my suggested strategy for tackling these types of optimization problems. It is okay if you have a method that works better for yourself, this is just a suggestion.

1. **Draw a picture** of what is going on in the question (in this question, I would draw a can and label the height and radius).

2. **Identify the object function**. One trick is to look at what the question is asking to maximize/minimize (in this question, I see that the question wants to minimize the amount of material, that is the surface area of the can.)

3. **Identify the constraint function**. This is a function that includes both variables and has them equal to something known. It literally constrains our problem. (in this question, I see that the volume of the can is *constrained* to hold 54π cubic inches of liquid.)

4. **Solve the constraint function for one variable and plug it into the objective function**. The point of this is to plug the variable we solved for into the objective function so that it is in terms of only one variable! (Here we can choose to solve for the easier variable to plug into the objective.)

5. **Find the critical points of the objective function.** Now that the object function is in terms of one variable, we can find the critical points:

   - **Take the derivative of the objective function.**
   - **Set the derivative equal to zero and solve for unknown.**

6. **Verify the critical point is the desired max/min.** Depending on what the question is asking for (in our question, we need to min), we can use the second derivative to double-check that the critical points we found are the max or min.

   - **Take the second derivative of the objective function.**
   - **Plug the critical points into the second derivative**. Remember that if the critical point makes the second derivative positive, then we are concave up and the critical point is a min (in our question, this is what we want!). If the critical makes the second derivative negative, then we are concave down and the critical point is a max. (Note that if the critical point makes the second derivative zero, it is neither a max or min.)

7. **Find all the desired dimensions for the problem**. Make sure that you find everything the question is asking for and it wouldn't hurt to include the units (in this question, we need to find both the radius and the height of the can that minimizes the can and the units would be inches.)

Okay, now let's do the problem. First I would draw a picture of the can with the height and radius labeled. The question is asking to minimize the amount of material used, so the objective function will measure the amount of material. Since the can is made from two circles on the top and bottom and the circular part of the can on the side we get the object function to be

$$\text{Sa} = \text{area of two circles} + \text{area of circular part} = 2\pi r^2 + 2\pi rh.$$

Now we need the constraint function. Notice that the volume of the can is literally constrained to be 54π. So let's use the hint to write the volume of the can and set it equal to 54π:

$$V = 54\pi = \pi r^2 h.$$

Now we will solve the constraint function for one variable. We can choose to solve for r or h and looking at the objective function it will be easier to plug in h so let's solve for h:

$$\begin{aligned} 54\pi &= \pi r^2 h \\ 54 &= r^2 h \\ \frac{54}{r^2} &= h \\ 54r^{-2} &= h. \end{aligned}$$

Now we plug it into the objective function and simplify:

$$\begin{aligned} \text{Sa} &= 2\pi r^2 + 2\pi r(54r^{-2}) \\ &= 2\pi r^2 + 108\pi r^{-1}. \end{aligned}$$

Now we take the derivative:

$$\text{Sa}' = 4\pi r - 108\pi r^{-2},$$

and set it equal to zero and solve for r:

$$\begin{aligned} \text{Sa}' = 0 &= 4\pi r - 108\pi r^{-2} \\ 108\pi r^{-2} &= 4\pi r \\ 108r^{-2} &= 4r \\ 27r^{-2} &= r \\ 27 &= rr^2 \end{aligned}$$

$$27 = r^3,$$

and taking the cubed root of both sides we get $r = 3$ inches. Now let's verify that this is indeed the min so let's compute the second derivative:

$$\mathrm{Sa}'' = 4\pi + 216\pi r^{-3},$$

then if we plug in $r = 3$ we get

$$\mathrm{Sa}''(3) = 4\pi + 216\pi(3)^{-3} = 4\pi\frac{216\pi}{27} > 0.$$

Since the second derivative is positive, we are concave up and thus have a min as we want! The last thing to do is find h using r. We can return to the constraint function to find:

$$h = 54(3)^{-2} = 6,$$

so $h = 6$ inches. We have found that the can should have the dimensions $r = 3$ inches and $h = 6$ inches which will make the can have a volume of 54π inches cubed while minimizing the amount of material used! Pretty neat I think!

$\square$

Problem

We want to build a house that has four walls and one flat top (no floor). I want the house to be twice as wide as it is deep. The front part of the house will be made out of a material that costs \$16 per square foot. The sides and back of the house will be made out of a material that costs \$8 per square foot. The top of the house will be made out of a material that costs \$24 per square foot. The budget for building the house is \$14,400. Find the dimensions to max the volume of the house.

This question is a little harder than the similar question on the practice exam since it doesn't walk you through the steps as the practice exam does. I'm going to go a little faster through this one for the sake of getting this to you before it is late so please use this to double-check your process on the practice exam.

Solution

Proof. I would begin by drawing the house. The object function will be the volume of the house we is given by

$$V = W \cdot h \cdot d,$$

where W is how wide the house is, h is how tall the house is, and d is the depth of the house. This has three variables! We want it to have two. The important thing to notice here is that the house is going to be twice as wide as it is deep. In math, we can write this as the width will be $W = 2w$ and the depth will be $d = w$ which will make the width twice as long as the depth. This makes the objective function:

$$V = 2w \cdot h \cdot w = 2hw^2.$$

The cost of building the house is constrained $14,400$ so the cost of the sides, front, back, and top must be $14,400$. Using the picture you drew out, we can find that the area of the front of the building is $2wh$, the area of the side is hw, and since there are two, $2hw$; the area of the back is $2wh$, and the area of the top is $2w \cdot w = 2w^2$ (I use here that the width is twice the depth, draw the photo and label carefully). Now multiplying by the cost of each area we get the constraint to be

$$\text{Cost} = 14400 = 16(2wh) + 8(2wh) + 8(2wh) + 24(2w^2) = 64wh + 48w^2.$$

Now we want to solve this for one variable and looking at the objective function, lets do h:

$$\begin{aligned} 14400 &= 64wh + 48w^2 \\ 14400 - 48w^2 &= 64wh \\ \frac{14400}{64w} - \frac{48}{64w}w^2 &= h \\ h &= 225w^{-1} - \frac{3}{4}w \end{aligned}$$

Now plugging into the objective function

$$
\begin{aligned}
V &= 2(225w^{-1} - \frac{3}{4}w)w^2 \\
V &= (450w^{-1} - \frac{3}{2}w)w^2 \\
V &= 450w - \frac{3}{2}w^3.
\end{aligned}
$$

Next, take derivative

$$V' = 450 - \frac{9}{2}w^2,$$

and set it equal to zero and solve:

$$
\begin{aligned}
0 &= 450 - \frac{9}{2}w^2 \\
\frac{9}{2}w^2 &= 450 \\
w^2 &= 450\frac{2}{9} \\
w^2 &= \frac{900}{9} \\
w^2 &= 100 \\
w &= \pm\sqrt{100} \\
w &= \pm 10.
\end{aligned}
$$

Now we have $w = \pm 10$ so let's check which is the max using the second derivative. First, compute the second derivative:

$$V'' = -\frac{18}{2}w = -9w.$$

We want the max, so we want to be concave down which means $V'' < 0$. This happens when $w = 10$ as

$$V''(10) = -90 < 0,$$

so $w = 10$ feet. So the depth is $d = w = 10$ feet, the width is $W = 2w = 20$ feet, and we still have to find that height using the constraint function.

$$h = 255(10)^{-1} - \frac{3}{4}10 = 18 \text{ feet}.$$

I hope this helps and best of luck tomorrow! I believe in you guys! Sorry this came out so late, I've been pretty packed with my own homework :(

$\square$