Problem 1

Find an equation of the tangent line to the graph of $f(x) = x * \sqrt{2x^2 + 7}$ at $x = 3$.

Solution

Proof. Okay, so we want to find the tangent line to $f(x) = x * \sqrt{2x^2 + 7}$ at $x = 3$. Remember that the tangent line intersects $f(x)$ and has the same slot as $f(x)$ at the point of interest, for us $x = 3$. So there are three steps we have to do

(1) Find the point $(x, y) = (x, f(x))$ when $x = 3$ where the tangent line interests $f(x)$.

(2) Find the slope of $f(x)$ at the $x = 3$.

(3) Write the equation for the tangent line using the above information.

1: Since the tangent line intersections $f(x)$ at $x = 3$, we know that the intersection point is $(3, f(3)) = (3, 3 * \sqrt{2 * 3^2 + 7}) = (3, 15)$. So the point $(3, 15)$ is where the tangent line hits $f(x)$. I graphed it here:

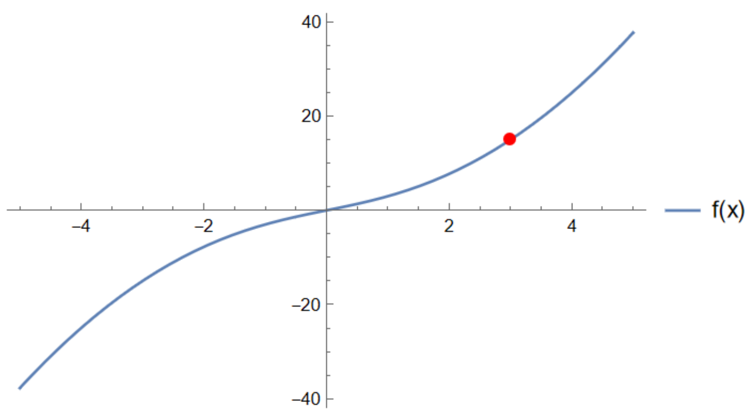


Figure 1

2: To find the slope of $f(x)$ at $x = 3$, remember that the derivative of a function tells us the slope or the rate of change of a function. So let's compute $f'(x)$ and plug in $x = 3$ to find the slope of $f(x)$ at $x = 3$. First we compute the derivative to be

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x * \sqrt{2x^2 + 7}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x * \left(2x^2 + 7\right)^{1/2}\right) \quad \text{let's rewrite the square root}
\end{aligned}$$

now we are going to have to use product rule

$$= \frac{\mathrm{d}}{\mathrm{d}x}(x)\left(2x^2+7\right)^{1/2} + (x)\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^2+7\right)^{1/2}$$

$$= (1)\left(2x^2+7\right)^{1/2} + (x)\left(\frac{1}{2}\cdot(2x^2+7)^{-1/2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2x^2+7)\right) \quad \text{don't forget chain rule}$$

$$= \sqrt{2x^2+7} + \frac{x}{2}(4x)\left(2x^2+7\right)^{-1/2}$$

$$= \sqrt{2x^2+7} + \frac{2x^2}{\sqrt{2x^2+7}}$$

Then we plug in $x = 3$ to find the slope at the point

$$f'(3) = \sqrt{2(3)^2+7} + \frac{2(3)^2}{\sqrt{2(3)^2+7}} = \sqrt{25} + \frac{6}{\sqrt{25}} = 5 + \frac{18}{5} = \frac{43}{5}$$

So the slope at of $f(x)$ at $(3, 15)$ is $\frac{43}{5}$.

3: Now we have everything we need to create the equation of the tangent line. We know that the line must go through $(3, 15)$ and have slope $\frac{43}{5}$, so I will use the point-slope formula which says if a line goes through (a, b) with slope m, then its equation is

$$y - b = m(x - a).$$

So the equation of our line is

$$\begin{aligned} y - 15 &= \frac{43}{5}(x-3) \\ y &= \frac{43}{5}x - \frac{129}{5} + 15 \\ y &= \frac{43}{5}x - \frac{104}{5} \end{aligned}$$

We generally expect you to write the equation in $y = mx + b$ form. I graphed it:

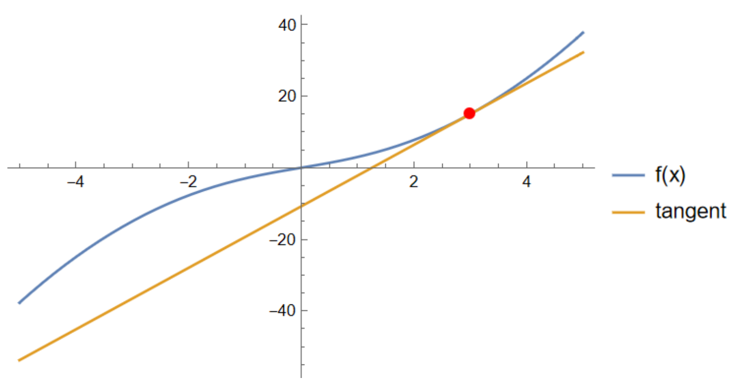


Figure 2

$\square$

Problem 2

A sugar refinery can produce x tons of sugar per week at a weekly cost of $C(x) = .1x^2 + 5x + 2250$ dollars. Find the level of production for which the average cost is at a minimum and show that the average cost equals the marginal cost at that level of product. *Hint: Average cost=*$\frac{C(x)}{x}$

Solution

Proof. Remember, to find a minimum of a function $f(x)$, we have to find the derivative of $f(x)$ and set it equal to zero, $f'(x) = 0$. Solving this equation for x gives us the critical points (these are the points of $f(x)$ that have zero slope, so they are either a min, max, or inflection point) but we need to verify what type of critical point it is. We can use the second derivative test by computing $f''(x)$ and plugging in the critical points. If $f''(x) < 0$, then $f(x)$ is concave down, making the critical point a max. If $f''(x) = 0$, then we are at an inflection point. and If $f''(x) > 0$, then $f(x)$ is concave up, making the critical point a min. I think this image, which I definitely didn't steal from online, does a nice job of showing this:

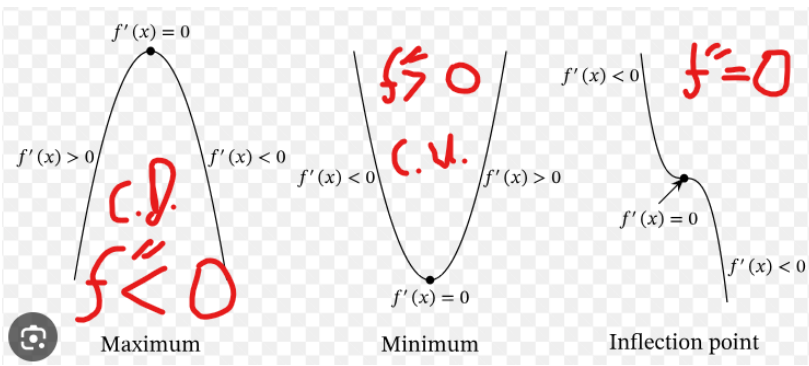


Figure 3: From left to right: The first example has $f''(x) < 0$, meaning concave down and max. The second example has $f''(x) > 0$, meaning concave up and min. The last example has $f''(3x) = 0$, meaning inflection point.

Okay to actually solve the problem. So, we have been tasked with finding the level of production for which the average cost is at a minimum. We know that the sugar refinery can produce x tons of sugar per week at a weekly cost of $C(x) = .1x^2 + 5x + 2250$ dollars. <u>Notice that we don't want to minimize cost but the average cost!</u> From the hint, the average cost is given by

$$AC(x) = \frac{C(x)}{x} = \frac{.1x^2 + 5x + 2250}{x} = .1x^{2-1} + 5x^{1-1} + 2250x^{-1} = .1x + 5 + 2250x^{-1}.$$

Now let's minimize $AC(x)$ by finding the derivative:

$$AC'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(.1x + 5 + 2250x^{-1}\right) = .1 + (-1)2250x^{-2} = .1 - 2250x^{-2}.$$

Now we set the derivative equal to zero and solve for x:

$$\begin{aligned}
AC'(x) = .1 - 2250x^{-2} &= 0 \\
\frac{1}{10} - \frac{2250}{x^2} &= 0 \\
\frac{1}{10} &= \frac{2250}{x^2} \\
\frac{x^2}{10} &= 2250 \\
x^2 &= 2250 * 10 \\
x^2 &= 22500 \\
x &= \pm\sqrt{22500} = \pm 150
\end{aligned}$$

In context of this problem, x are the tons of sugar produced so it doesn't make sense for it to be negative. So we take the positive one and call it a day. But if we wanted to be extra sure, let's find the second derivative:

$$AC''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(.1 - 2250x^{-2}\right) = 0 - (-2)2250x^{-3} = \frac{4500}{x^3}.$$

Then testing our points

$$\begin{aligned}
AC''(150) &= \frac{4500}{150^3} > 0 \\
AC''(-150) &= -\frac{4500}{150^3} < 0
\end{aligned}$$

So $x = 150$ has $AC''(150) > 0$ and is indeed the minimum. Therefore, the optimal level of production is $x = 150$ tons of sugar (don't forget units!) to get the minimum average cost. To do the other part, we need to remember that the marginal cost is the derivative of the cost function, so let's take that derivative:

$$MC(x) = \frac{\mathrm{d}}{\mathrm{d}x}C(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{10}x^2 + 5x + 2250\right) = \frac{2}{10}x + 5 = \frac{1}{5}x + 5.$$

Then we just compute:

$$\begin{aligned}
AC(150) &= \frac{1}{10}(150) + 5 + 2250(150)^{-1} = \frac{150}{10} + 5 + \frac{2250}{150} = 15 + 5 + 15 = 35 \\
MC(150) &= \frac{150}{5} + 5 = \frac{150}{5} + \frac{25}{5} = 35
\end{aligned}$$

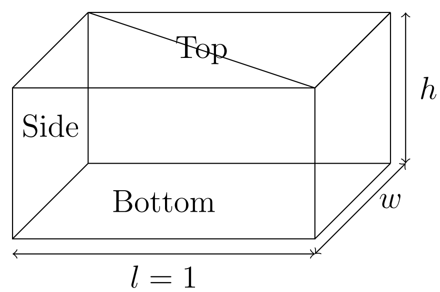
$\square$

Problem 3

A closed rectangular box is to be constructed with one side 1 meter long. The material for the top costs \$20 per square meter, and the material for the sides and bottom costs \$10 per square meter. Find the dimensions of the box with the largest possible volume that can be built at a cost \$240 for the materials. You must use calculus to find your answer. Clearly define your variables, and identify your constraint and objective. You do not need to show that your answer is a maximum.

Solution

Proof. We are tasked with finding the dimensions for the box that gives the largest possible volume at a cost of \$240. Notice that we have an objective, max the cost, that is constrained, volume costs \$240. So this is an optimization problem. To find the objective function, I just look for what we are requiring to max. In this case it is the volume, so the volume will be the objective function. On the other hand, we are constrained to the cost, so the cost function will be the constraint function. Now let's draw what is going on:



Notice that I wrote $l = 1$ since we know that the box will have one side that is 1 meter long. To write the objective volume function, remember that the volume of a box is given by

$$V = lwh = (1)wh = wh \quad \text{since one side is 1 meter long.}$$

To find the equation for the cost, remember that the material for the top costs 20 dollars per square meter (for the area of the top) and the material for the bottom and sides costs 10 dollars. The area of the top and bottom are $l * w = 1 * w = w$. The area of the front and back sides are $lh = 1 * h = h$. The area of the left and right sides are wh. Putting this all together we get:

$$\begin{aligned} C &= 20 * \text{area of top} + 10 * \text{area of bottom} + 10 * 2 * \text{area of front/back} + 10 * 2 * \text{area of left/right} \\ &= 20w + 10w + 10 * 2 * h + 10 * 2 * wh \\ &= 30w + 20h + 20wh. \end{aligned}$$

Remember that we require the cost to be 240 so we will set this equation equal to 240. Putting what we have together:

$$\text{Objective: } V = wh$$

$$\text{Constraint: } 240 = 30w + 20h + 20wh.$$

We want to find the max of V but it has two variables, so let's solve the constraint function for one of the variables and plug it into V. Looking at the constraint function, solving for h compared to w doesn't look any different so let's just solve for h:

$$\begin{aligned}
240 &= 30w + 20h + 20wh \\
-20h - 20wh &= 30w - 240 \quad \text{move everything with h to one side} \\
h(-20 - 20w) &= 30w - 240 \quad \text{factor out the } h \\
h &= \frac{30w - 240}{-20 - 20w} \quad \text{divide to isolate } h
\end{aligned}$$

and plugging this into V gives

$$V = w \cdot \frac{30w - 240}{-20 - 20w} = \frac{30w^2 - 240w}{-20 - 20w}.$$

Now we will minimize this just like before, first we take the derivative:

$$\begin{aligned}
V' &= \frac{d}{dw}\left(\frac{30w^2 - 240w}{20 - 20w}\right) \quad \text{Quotient rule or Product rule needed} \\
&= \frac{(60w - 240)(20 - 20w) - (30w^2 - 240w)(-20)}{(20 - 20w)^2} \\
&= \frac{-4800 + 6000w - 1200w^2 - 4800w - 600w^2}{400 - 800w + 400w^2} \\
&= \frac{-4800 + 1200w - 1800w^2}{400 - 800w + 400w^2}
\end{aligned}$$

Here I picked to use the Quotient rule rather than the product rule since it will be easier to solve for $V' = 0$. Now we set $V' = 0$ and solve for x:

$$V' = \frac{-4800 + 1200w - 1800w^2}{400 - 800w + 400w^2} = 0,$$

since the denominator will never be zero, we just solve for when the numerator is zero:

$$-4800 + 1200w - 1800w^2 = -600(8 - 2w + 3w^2) = 0,$$

then using the quadratic formula, I find $w = -\frac{4}{3}$ and $w = 2$. Since it doesn't make sense to have a negative width, the maximum must be $w = 2$ meters. Then we can solve for h:

$$h = \frac{30(2) - 240}{-20 - 20(2)} = \frac{-180}{-60} = 3 \text{ meters.}$$

□

Problem 4

The function $h(x) = \sqrt{x^2 - 6x + 10}$ has one relative minimum point for $x \leq 0$. Find the coordinates of the relative minimum. (You do not need to show that the point is a minimum.)

Solution

Proof. We are looking for the minimum that has $x \geq 0$ for $h(x) = \sqrt{x^2 - 6x + 10}$. So let's find $h'(x)$:

$$\begin{aligned}
h'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2 - 6x + 10} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 - 6x + 10\right)^{1/2} \\
&= \frac{1}{2}\left(x^2 - 6x + 10\right)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 - 6x + 10\right) \\
&= \frac{1}{2}\left(x^2 - 6x + 10\right)^{-1/2}(2x - 6) \\
&= \frac{2x - 6}{2}\left(x^2 - 6x + 10\right)^{-1/2} \\
&= \frac{x - 3}{(x^2 - 6x + 10)^{-1/2}}.
\end{aligned}$$

Now we set $h'(x) = 0$ and solve for x:

$$h' = \frac{x - 3}{(x^2 - 6x + 10)^{-1/2}} = 0,$$

so h' is only zero when the numerator is zero:

$$x - 3 = 0 \text{ so } x = 3.$$

The question asks for the coordinates so let's find the $h(3)$:

$$h(3) = \sqrt{3^2 - 3 * 6 + 10} = \sqrt{9 - 18 + 10} = \sqrt{1} = 1.$$

So the coordinates of the relative minimum are $(3, 1)$.
Just for fun, I graphed this and we can see we are right!

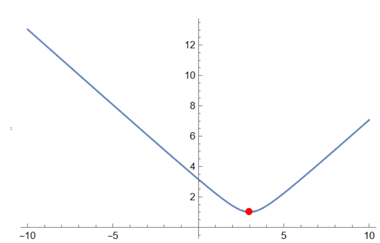


Figure 4

$\square$

Problem 5

Where ist he function $f(x) = -x^3 + 2x^2 - 6x + 3$ decreasing? Justify your answer using calculus.

Solution

Proof. Remember that the derivative of a function tells us about the slope of the function. So if we want to find where $f(x) = -x^3 + 2x^2 - 6x + 3$ decreasing, let's find $f'(x)$ and see for what values $f'(x) < 0$. First let's find the derivative:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(-x^3 + 2x^2 - 6x + 3\right) = -3x^2 + 4x - 6.$$

To find when $f'(x) < 0$, let's find the x-intercepts by setting $f'(x) = 0$ and solving for x:

$$f'(x) = -3x^2 + 4x - 6 = 0$$

which is a quadratic so using the quadratic formula:

$$x = \frac{-4 \pm \sqrt{16 - 4 * (-3)(-6)}}{2(-3)} = \frac{-4 \pm \sqrt{-56}}{2(-3)}$$

But notice that we have $\sqrt{-56}$ which doesn't have any real solutions! So $f'(x)$ never crosses x. This means that the f' is either completely above (always positive) or completely below (always negative) $y = 0$. So let's just test a point like $x = 0$:

$$f'(0) = -6$$

so $f'(0)$ is below the x-axis which shows that $f'(x) < 0$ for all x. Since $f'(x) < 0$, this means that $f(x)$ is always decreasing.
Just for fun plot verifies that it is always decreasing:

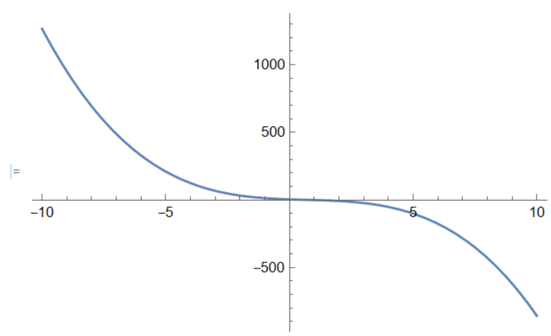


Figure 5

$\square$

Problem 5

Investment A is currently worth \$70,200 and growing at the rate of 13% per year compounded continuously. Investment B is currently worth \$60,000 and is growing at a rate of 14% per year compounded continuously. After how many years will the two investments have the same value?

Solution

Proof. Remember that we have this formula for compound interest:

$$A(t) = P_0 e^{rt},$$

where $A(t)$ is how much money I have at time t, P_0 is the initial amount of money we invested, r is the interest rate, and t is time. So for Investment A, we have the current worth is $70,200$ and has an interest rate of $13\% = .13$. Since we want to know when the two investments are equal from this current time, let's say that the initial amount invested in Investment A is 70200. So we can write the equation for investment A as:

$$A(t) = 70200e^{.13t},$$

Now using the same idea for investment B, we have a current worth is $60,000$ and has investment rate of .14, so we can write its equation:

$$A(t) = 60000e^{.14t}.$$

Now we want to figure out when the two investments are equal, so let's set both the equations equal to each other and solve for t:

$$\begin{aligned}
70200e^{.13t} &= 60000e^{.14t} \\
\frac{70200}{60000} &= \frac{e^{.14}t}{e^{.13t}} \\
\frac{70200}{60000} &= \frac{e^{.14}t}{e^{.13t}} \\
\frac{117}{100} &= e^{.14t-.13t} \\
\frac{117}{100} &= e^{.01t} \\
\ln\left(\frac{117}{100}\right) &= .01t \\
\ln\left(\frac{117}{100}\right) &= t \\
t = 100\ln\left(\frac{117}{100}\right)
\end{aligned}$$

So after $t = 100\ln\left(\frac{117}{100}\right)$ years, the investments will be the same.

$\square$

Problem 6

The velocity of a skydiver at time t seconds is $v(t) = 45 - 45e^{-.2t}$ meters per second. Show that the velocity is always increasing. Then, find the distance traveled by the skydiver during the first 9 seconds.

Solution

Proof. To show that the velocity is always increasing, let's look at the derivative of $v(t)$. Remember that the rate of change of a function is given by its derivative, so let's find $v'(t)$ and show that it must always be positive! To find $v'(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(45 - 45e^{-.2t}\big) = -45e^{-.2t}\frac{\mathrm{d}}{\mathrm{d}t}(-.2t) = -45e^{-.2t}(-.2) = 9e^{-.2t}$$

Now since $e^{-t} > 0$, we see that $v'(t) > 0$ for all values of t. Thus the velocity is always increasing since it's derivative is always positive.
Next, we want to find the distance traveled by the skydiver after 9 seconds who is dropping with a velocity of $v(t) = 45 - 45e^{-.2t}$. We can do this by integrating from 0 to 9:

$$\begin{aligned}
\int_0^9 v(t)\mathrm{d}t &= \int_0^9 \big(45 - 45e^{-.2t}\big)\mathrm{d}t \\
&= 45t + \frac{45}{.2}e^{-.2t}\Big|_0^9 \\
&= 45(9) + \frac{45}{.2}e^{-.2(9)} - \left(45(0) + \frac{45}{.2}e^{0}\right) \\
&= 405 + 225e^{-1.8} - 225 \\
&= 180 + 225e^{-1.8}.
\end{aligned}$$

So the skydiver has traveled $180 + 225e^{-1.8}$ meters.

□

Problem 7

(a) Find the total are abounded between the x-axis and the graph of the function $g(x) = x(x-3)(x+3)$. As part of your solution, graph g and shade the area between the function and the x-axis.

(b) Based on your work for part (a) would $\int_{-4}^{3} g(x)\mathrm{d}x$ be greater than or less than 0? Justify your answer with a brief explanation.

Solution

Proof. We wish to find the area bounded between the x-axis and the function $g(x) = x(x-3)(x+3)$. The x-axis can be thought of as the equation $y = 0$. For this problem, we will use that the area from a to b between curves $f(x)$ and $g(x)$ is given by

$$\int_a^b \text{top function} - \text{bot function } \mathrm{d}x.$$

So we will have to find when $g(x)$ is above and below the x-axis. We can do this by finding the intercepts of the functions and testing a point to see which function is above. To find the intercepts, we set the equations equal to each other:

$$g(x) = x(x-3)(x+3) = 0,$$

so there intercepts are $x = 0, x = 3,$ and $x = -3$. So we have the intervals $(-\infty, -3), (-3, 0), (0, 3), (3, \infty)$. We want to find the area bounded between the x-axis and (x) which will be in the interval $[-3, 3]$. Now let's figure out if $g(x)$ is above or below $y = 0$ by testing points:

$$\begin{aligned}
&(-\infty, -3) \text{ let's test pint } x = -4\text{: } g(-4) = -28 < 0 \\
&(-3, 0) \text{ let's test pint } x = -1\text{: } g(-1) = 8 > 0 \\
&(0, 3) \text{ let's test pint } x = 1\text{: } g(1) = -8 < 0 \\
&(3, \infty) \text{ let's test pint } x = 4\text{: } g(4) = 28 > 0
\end{aligned}$$

So if we want to sketch $g(x)$, we know that on $(-\infty, -3)$, $g(x)$ is below $y = 0$, from $(-3, 0)$ it is above $y = 0$ but then goes back below on $(0, 3)$. Finally the $g(x)$ goes above the x-axis on $(3, \infty)$. So I'm not sketching it but it would look something like:

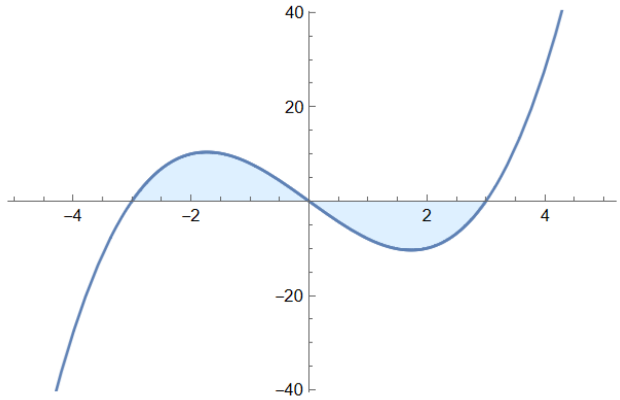


Figure 6

Since in one region $g(x)$ is above and the other $g(x)$ is below, we will use two integrals:

$$\begin{aligned}
A &= \int_{-3}^{0} g(x) - 0\mathrm{d}x + \int_{0}^{-3} 0 - g(x)\mathrm{d}x \\
&= \int_{-3}^{0} x(x-3)(x+3)\mathrm{d}x + \int_{0}^{-3} -x(x-3)(x+3)\mathrm{d}x \\
&= \int_{-3}^{0} x^3 - 9x\mathrm{d}x - \int_{0}^{-3} x^3 - 9x\mathrm{d}x \\
&= \frac{x^4}{4} - \frac{9x^2}{2}\bigg|_{-3}^{0} - \left(\frac{x^4}{4} - \frac{9x^2}{2}\bigg|_{0}^{3}\right) \\
&= 0 - \left(\frac{(-3)^4}{4} - \frac{9(-3)^2}{2}\right) - \left(\frac{(3)^4}{4} - \frac{9(3)^2}{2} - 0\right) \\
&= \frac{81}{4} - \frac{81}{4} \text{ (there might be a mistake in the algebra)} \\
&= 0 \text{ (but this should be zero)}
\end{aligned}$$

We found that the area is 0 which makes sense since all the area on the left side is above the x-axis and the area on the right side is below x-axis. Since its symmetric, the areas cancel each other and we are left with zero.
To figure out if $\int_{-4}^{3} g(x)\mathrm{d}x$ is greater than or less than 0, we are now trying to find the following shaded region:

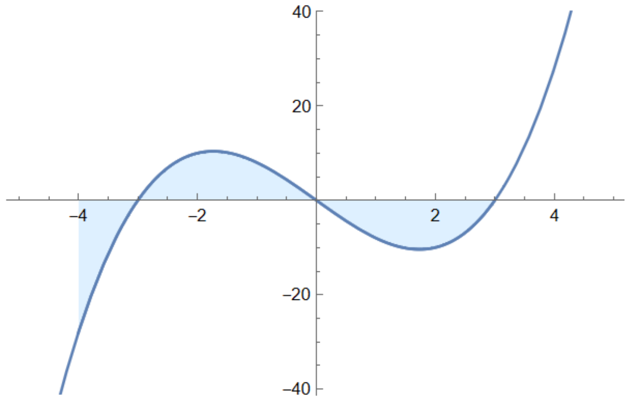


Figure 7

Notice that there is more area under the x-axis and so we expect the integral to be less than zero. Or another way of thinking about it is, the area from -3 to 3 is zero, so we are just left with the negative area from -4 to -3.

$\square$

Problem 8

Let $f(x, y) = 5x^2 e^{xy}$. Find $\frac{\partial f}{\partial x}$.

Solution

Proof. We have $f(x, y) = 5x^2 e^{xy}$ and want to find $\frac{\partial f}{\partial x}$. Remember that when we have a function with more than one variable, we denote differentiation with respect to one of the variables as $\frac{\partial f}{\partial x}$. In other words, $\frac{\partial f}{\partial x}$ means the same thing as $\frac{\mathrm{d}f}{\mathrm{d}x}$ where we treat the other variable y as a constant. Now let's compute $\frac{\partial f}{\partial x}$:

$$\begin{aligned}
\frac{\partial f}{\partial x} f(x, y) &= \frac{\partial f}{\partial x}\left(5x^2 e^{xy}\right) \quad \text{We need product rule here sine both we have } x^2 \text{ times } e^{yx} \\
&= \frac{\partial}{\partial x}\left(5x^2\right)\left(e^{xy}\right) + \left(5x^2\right)\frac{\partial}{\partial x}\left(e^{xy}\right) \\
&= (10x)(e^{xy}) + \left(5x^2\right)\left(e^{xy}\right)\frac{\partial}{\partial x}(xy) \quad \text{we treat } y \text{ as just some constant} \\
&= (10x)(e^{xy}) + \left(5x^2\right)\left(e^{xy}\right)(y) \quad \text{x goes to 1 and we have } 1y = y \\
&= (10x)(e^{xy}) + y\left(5x^2\right)\left(e^{xy}\right) \\
&= (10 + 5yx)(xe^{xy})
\end{aligned}$$

Just in case you want to see in more detail:

$$\frac{\partial}{\partial x} e^{xy} = e^{xy}\frac{\partial}{\partial x}(xy) = e^{xy}(y) = ye^{xy}$$

□

Problem 9

Let $f(x,y) = x^2 - 2xy + 3y^2 - 16y + 22$. Identify the point, (x,y) where f has a possible relative extremum. Then, use the second derivative test to determine, if possible, the nature of $f(x,y)$ at these points. If the second derivative test is inconclusive, state so. *Hint:* $D(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)$

Solution

Proof. We have the function $f(x,y) = x^2-2xy+3y^2-16y+22$, which has two variables. The process for finding and determining the possible extremum for $f(x,y)$ is similar to finding them for $f(x)$ but with a few more steps:

1. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$
2. Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ using each other for (x,y) points. These are the critical points (just like $f'(x) = 0$ give the critical points of $f(x)$).
3. Compute $D(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)$ and plug the critical values into them. This is the analog of the second derivative test for $f(x,y)$. Then we can use the following to determine the type of critical value:
   (a) If $D(a,b) > 0$ and $\frac{\partial^2 f(a,b)}{\partial x^2} > 0$, then (a,b) is a local minimum.
   (b) If $D(a,b) > 0$ and $\frac{\partial^2 f(a,b)}{\partial x^2} < 0$, then (a,b) is a local maximum.
   (c) If $D(a,b) < 0$, then $f(a,b)$ is saddle point.
   (d) If $D(a,b) = 0$, then inconclusive.

Now let's do it:

1. Let's compute everything. Remember that the partials are derivatives where you treat the other variable as a constant:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\big(x^2 - 2xy + 3y^2 - 16y + 22\big) = 2x - 2y \quad \text{Treat } y \text{ as just some constant}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(x^2 - 2xy + 3y^2 - 16y + 22\big) = -2x + 6y - 16 \quad \text{Treat } x \text{ as just some constant}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x - 2y) = 2 \quad \text{This is taking the second derivative of } x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-2x + 6y - 16) = 6 \quad \text{This is taking the second derivative of } y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-2x + 6y - 16) = -2$$

take the derivative with respect to x of the derivative with respect to y.

2. Next, let's find the critical values by solving the system of equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Observe that

$$\frac{\partial f}{\partial x} = 2x - 2y = 0 \text{ means } 2x = 2y \text{ means } x = y$$

Using that $x = y$ we can solve

$$\begin{aligned} \frac{\partial f}{\partial y} &= -2x + 6y - 16 = 0 \\ -2x + 6x - 16 &= 0 \\ 4x &= 16 \\ x &= 4 \end{aligned}$$

and since $x = y$, that means $y = 4$ as well. So we have found the critical point to be at $(4, 4)$.

3. To classify the critical point $(4, 4)$. First let's find $D(x, y)$ using the parts we found in step (1):

$$D(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right) = 2 * 6 - (-2)^2 = 12 - 4 = 8.$$

This means $D(x, y) > 0$ for all values and most importantly $D(4, 4) > 0$. Furthermore, we have $\frac{\partial^2 f}{\partial x^2} = 2 > 0$. This tells us that $(4, 4)$ is a local minimum of $f(x, y)$.

Just-for-fun-plot shows that we are right!

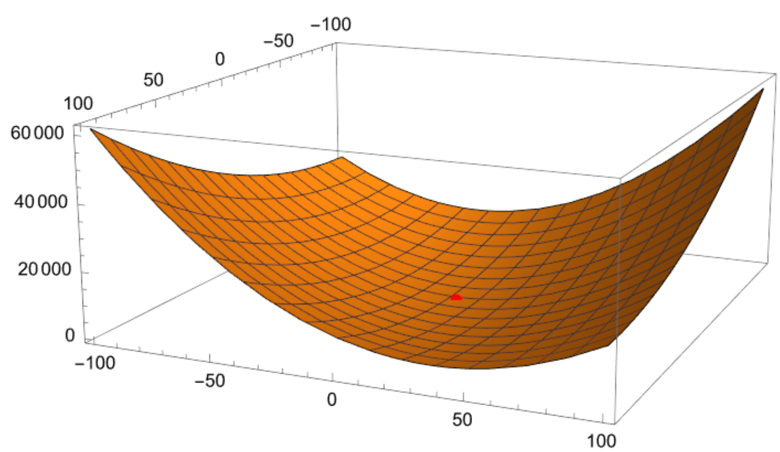


Figure 8