Natural logs did not break math!
By Marvyn

Hey guys, in our worksheet we had the integral

$$\int \frac{\frac{3}{2}}{x+3}\mathrm{d}x.$$

We found two different ways to compute this integral. The first one is factor out the 3/2 right away and then integrate:

$$\begin{aligned}\int \frac{\frac{3}{2}}{x+3}\mathrm{d}x &= \frac{3}{2}\int \frac{1}{x+3}\mathrm{d}x\\ &\quad \text{let } u = x+3, \text{ then } \mathrm{d}u = \mathrm{d}x\\ &= \frac{3}{2}\int \frac{1}{u}\mathrm{d}u\\ &= \frac{3}{2}\ln|u| + C\\ &= \frac{3}{2}\ln|x+3| + C.\end{aligned}$$

The other method is distributing 3/2 and then integrating:

$$\begin{aligned}\int \frac{\frac{3}{2}}{x+3}\mathrm{d}x &= 3\int \frac{1}{2x+6}\mathrm{d}x\\ &\quad \text{let } u = 2x+6, \text{ then } \frac{1}{2}\mathrm{d}u = \mathrm{d}x\\ &= 3\int \frac{1}{x+3}\cdot\frac{1}{2}\mathrm{d}u\\ &= \frac{3}{2}\ln|u| + C\\ &= \frac{3}{2}\ln|2x+6| + C.\end{aligned}$$

The question is, how are these results equal? They have to be equal somehow since

$$\int \frac{\frac{3}{2}}{x+3}\mathrm{d}x = \int \frac{3}{2x+6}\mathrm{d}x$$

At first I thought maybe there is some property of ln that made

$$\frac{3}{2}\ln|x+3| = \frac{3}{2}\ln|2x+6|,$$

but after graphing these we see that the functions are definitely NOT the same:

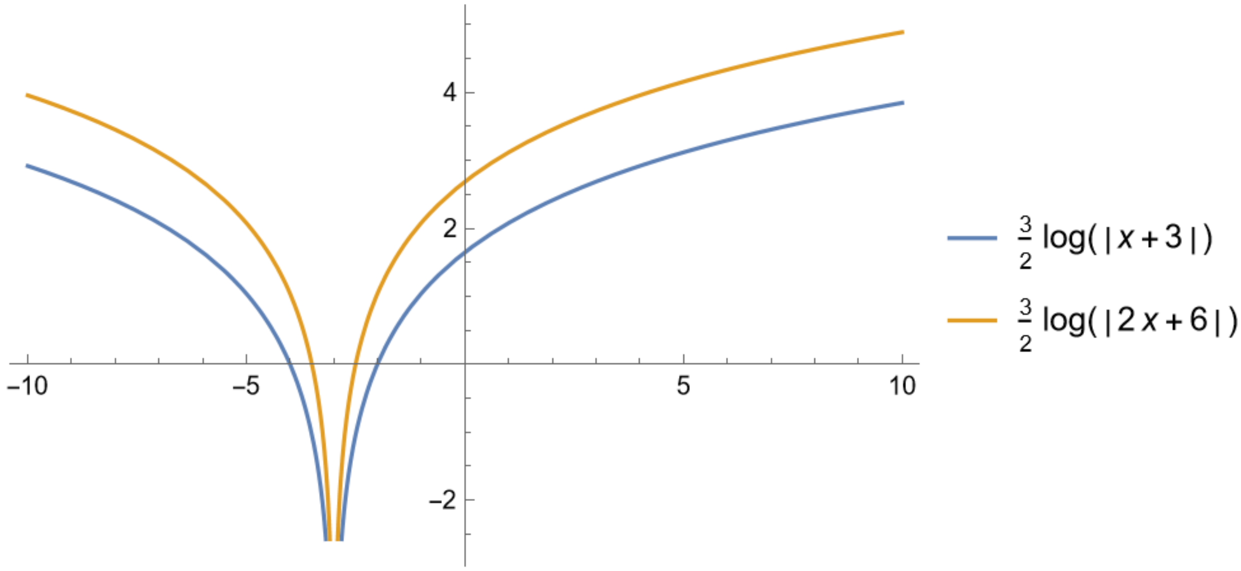

Figure 1

So what is going on? The answer lies in the fact that these are indefinite integrals. There are two ways of thinking about this, one way is that the constant C consumes every constant and the other is by taking the derivative.

Let's first look at the C consumes constants idea. Notice that we can rewrite

$$\begin{aligned}\frac{3}{2}\ln|2x+6|+C &= \frac{3}{2}\ln|2(x+3)|+C\\ &= \frac{3}{2}\ln|2|+\frac{3}{2}\ln|x+3|+C,\end{aligned}$$

and since $\frac{3}{2}\ln|2|$ is just some constant, it basically gets consumed by C. Thus we have

$$\frac{3}{2}\ln|x+3|+C,$$

which matches the other solution!

Another way of thinking about this is by taking the derivative of both terms. For now, let's just drop the absolute values. Taking the derivative of the first result gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{2}\ln(x+3)+C\right)=\frac{3}{2}\cdot\frac{1}{x+3}=\frac{\frac{3}{2}}{x+3}.$$

And taking the derivative of the second result gives (don't forget chain rule!)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{2}\ln(2x+6)+C\right)=\frac{3}{2}\cdot\frac{1}{2x+6}\cdot 2=\frac{3}{2x+6}=\frac{\frac{3}{2}}{x+3}.$$

So even though the results are different, they are indeed the anti-derivatives of the original function.

Long story short, but we did not break math! Indefinite integrals are weird and easy to get confused by. If this was a definite integral (i.e. it had bounds) we would not have had this issue. If you made it this far, thanks for reading and I hope you found this somewhat interesting. C consumes all!