Math 584 Homework 3
Due Wednesday Oct 18
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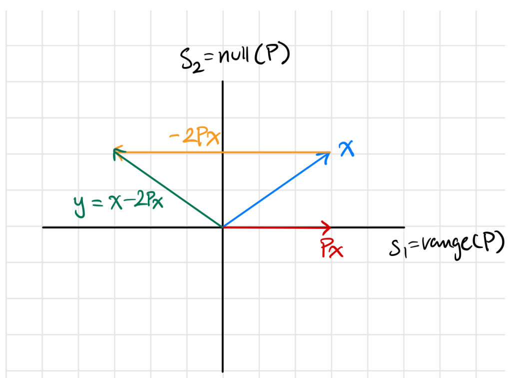
Problem 1 *6.1*

Solution.

Let P be an orthogonal projector. We wish to show that $I - 2P$ is unitary. Since P is a projector we know that $P^2 = P$, and since it is an orthogonal projector we know that $P = P^*$. Now observe that

$$\begin{aligned}(I-2P)^*(I-2P) &= (I-2P^*)(I-2P)\\ &= (I-2P)(I-2P)\\ &= I - I2P - I2P + 4P^2\\ &= I - 4P + 4P^2\\ &= I - 4P + 4P\\ &= I\end{aligned}$$

Thus we have shown that the columns of I are orthogonal.



Looking at the image kindly made by Rohin, we visual how $(I - 2P)x = x - 2Px$ and see that it mirrors over the null space of S_2. We can visually see that the length x remains the same under the transformation. And since the applying the transformation will move the vector back in the opposite direction, $x - 2Px$ is unitary. $\square$

Problem 2 *6.2*

Solution. Let E be the $m \times m$ matrix that extracts "even part" of an vector:

$$Ex = \frac{x + Fx}{2}$$

where F is the $m \times m$ matrix that flips $(x_1, \cdots, x_m)^*$ to $(x_m, \cdots, x_1)^*$. To check if E is an projector observe that Ex can be rewritten as,

$$Ex = \frac{x + Fx}{2} = \frac{1}{2}\left(I + F\right)x.$$

with this simplification see that,

$$\begin{aligned}(E)^2 &= \left(\frac{1}{2}(I+F)\right)^2 \\ &= \frac{1}{4}\left(I+F\right)^2 \\ &= \frac{1}{4}\left(2F + F^2 + 2I\right) \\ &= \frac{1}{4}\left(2F + I + I\right) \\ &= \frac{1}{2}\left(F + I\right) \\ &= E\end{aligned}$$

which implies that $E^2 = E$ verifying that E is a projector. Note that F^2 flips $(x_1, \cdots, x_m)^*$ to $(x_m, \cdots, x_1)^*$ and then flip it back, thus $F^2 = I$. Next let's check to see if E is an orthogonal projector by checking if $E^* = E$. Computing this we see that,

$$E^* = \frac{1}{2}(I+F)^* = \frac{1}{2}(F^* + I)$$

and thus E is orthogonal if F is symmetric, so let's compute the entries of F. We know that F flips $(x_1, \cdots, x_m)^*$ to $(x_m, \cdots, x_1)^*$ and thus F most be of the form,

$$F = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ & \iddots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

This matrix is indeed symmetric and thus E is indeed an orthogonal projector. To find the entries of E, notice that,

$$E = (I + F) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ & \iddots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Thus E is given by

$$E = \frac{1}{2}\begin{cases} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \ddots & & \vdots & & \iddots & 0 & 0 \\ \vdots & \vdots & & 1 & 0 & 1 & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & 1 & 0 & 1 & & \vdots & \vdots \\ 0 & 0 & \iddots & & \vdots & & \ddots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, & \text{when } m \text{ is odd} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & & \vdots & \vdots & & 1 & 0 & 0 \\ \vdots & \vdots & & \ddots & 0 & 0 & \iddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \iddots & 0 & 0 & \ddots & & \vdots & \vdots \\ 0 & 0 & 1 & & \vdots & \vdots & & 1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, & \text{when } m \text{ is even.} \end{cases}$$

$\square$

Problem 3 *6.4*

Solution.
Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

a We wish to find an orthogonal projector P_A onto range(A). Let's use the fact that $P_A = A(A^*A)^{-1}A^*$ to compute the orthogonal projector to get,

$$\begin{aligned}
P_A &= A(A^*A)^{-1}A^* \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}^* \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}
\end{aligned}$$

Then we can find the image of the vector $(1, 2, 3)^*$ under P_A by computing,

$$P_A(1, 2, 3)^* = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

b Next we wish to find an orthogonal projector P_B onto the range(B). Let's use the fact that $P_B = B(B^*B)^{-1}B^*$ to compute the orthogonal projector to get,

$$\begin{aligned}
P_B &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}
\end{aligned}$$

Then we can find the image of the vector $(1, 2, 3)^*$ under P_B by computing,

$$P_B(1, 2, 3)^* = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

□

Problem 4 *7.1*

Solution. Consider the matrices,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

a Consider matrix A. First let's find the reduced QR factorization $A = \hat{Q}\hat{R}$ by using the Gram Schmitt algorithm where $a_1 = (1,0,1)^*$ and $a_2 = (0,1,0)^*$. First we can find q_1 by computing,

$$q_1 = \frac{a_1}{\|a_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

which also gives that

$$r_{11} = \sqrt{2}$$

. Next we can compute,

$$\begin{aligned} \tilde{q}_2 &= a_2 - (q_1^* a_2) q_1 \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \left((1/\sqrt{2}, 0, 1/\sqrt{2}) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - (0) \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

so $\tilde{q}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ which means that,

$$q_2 = \frac{\tilde{g}_2}{\sqrt{1}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Along the way we also get that $r_{12} = 0$ and $r_{22} = 1$. Therefore we have that the reduced QR factorization of A is given by,

$$A = \hat{Q}\hat{R} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

To find the full QR factorization we need to compute Q and R. We can easily find R to be,

$$R = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Next we need to find a column that is orthonormal to the columns of A. To find a vector that is orthogonal, consider

$$\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives that $a = -c$ and $b = 0$. And since the vector must also be orthonormal, it is also restricted to be $\sqrt{a^2+b^2+c^2} = \sqrt{a^2+a^2} = \sqrt{2a^2}$ which means that $a = \frac{1}{\sqrt{2}}$. Thus we have found the QR factorization to be,

$$A = QR = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

b Consider matrix B. First let's find the reduced QR factorization $B = \hat{Q}\hat{R}$ by using the Gram Schmitt algorithm where $b_1 = (1,0,1)^*$ and $b_2 = (2,1,0)^*$. First we can find q_1 by computing,

$$q_1 = \frac{b_1}{\|b_1\|} = (1/\sqrt{2}, 0, 1\sqrt{2})^*$$

and this also gives us,

$$r_{11} = \sqrt{2}.$$

Next we can compute,

$$\begin{aligned} \tilde{q}_2 &= b_2 - (q_1^* b_2) q_1 \\ &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 1/\sqrt{2}, 0, 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Thus $\tilde{q}_2 = (1,1,-1)^*$ and we get,

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$$

We also gained along the way that $r_{12} = q_1^* b_2 = \frac{2}{\sqrt{2}}$ and $r_{22} = \sqrt{3}$. Thus we have calculated the reduced QR factorization of B to be,

$$B = \tilde{Q}\tilde{R} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 2/\sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}.$$

To find the full QR factorization, we need to compute Q and R. We can easily find R to be,

$$R = \begin{pmatrix} \sqrt{2} & 2/\sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}.$$

Next we need to find a column that is orthonormal to the columns of B. To find a vector that is orthogonal consider,

$$\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which tells us that $a = -c$ and $b = -2a = 2c$. To restrict the column to be orthonormal we must have $\sqrt{a^2+b^2+c^2} = \sqrt{a^2+(-2a)^2+(-2a)^2} = \sqrt{6a^2} = \sqrt{6}a = 1$ Thus we have that $a = 1/\sqrt{6}$ so $b = -2/\sqrt{6}$ and $c = -1/\sqrt{6}$. Which gives us the full QR factorization to be,

$$B = QR = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 2/\sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}.$$

$\square$

Problem 5 *8.1*

Solution. Consider the Modified Gram-Schmidt algorithm given by,

Algorithm 8.1. Modified Gram–Schmidt

for $i = 1$ **to** n
  $v_i = a_i$
for $i = 1$ **to** n
  $r_{ii} = \|v_i\|$
  $q_i = v_i / r_{ii}$
  for $j = i + 1$ **to** n
    $r_{ij} = q_i^* v_j$
    $v_j = v_j - r_{ij} q_i$

.

We will count the number of operations by type. I'll begin with addition. From the outer loop we have $m - 1$ additions from the $r_i i = \|v_i\|$ term and from the inner loop, we have $m - 1$ additions from the $r_{ij} = q_i^* v_j$ term. Thus there are,

$$\sum_{i=1}^{n} (m-1)(n-i) + \sum_{i=1}^{n} (m-1) = \frac{1}{2}(m-1)n(n-1) + (m-1)(n)$$
$$= \frac{(m-1)n(n+1)}{2}.$$

Next let's count the number of substractions. From the inner loop, there are m substracts from the $v_j = v_j - r_{ij} q_i$ term. Thus there are,

$$\sum_{i=1}^{n} m(n-1) = \frac{mn(n-1)}{2}.$$

Next let's count the multiplications. From the inner loop we have m multiplications from the $r_i i = \|v_i\|$ term, and the inner loop we have, both terms give m multiplications. Thus there are,

$$\sum_{i=1}^{n} 2m(n-1) + \sum_{i=1}^{n} m = mn(n-1) + mn = mn^2.$$

And finally let's count the number of division. From the outer loop we have, m divisions from the $q_i = v_i / r_{ii}$ term. Thus there are mn divisions. Adding all these up we get the total flops to be,

$$\frac{(m-1)n(n+1)}{2} + \frac{mn(n-1)}{2} + mn^2 + mn = \frac{1}{2n((2m-1)n-1)} + mn^2 + mn$$

□

Problem 6 *11.3*

Solution.
I created the MatLab code,

```
function q11()
format long;
m = 50; n = 12;
t = linspace(0,1,m);

A = fliplr(vander(t));
A = A(:,1:12);
b = cos(4*t)';

%Method A - Normal Equation%
R = chol(A'*A);
xa = R\(R'\(A'*b));

%Method D - QR factorization%
[Q, R] = qr(A);
xd = R\(Q'*b);

%Method E - A\b%
xe = A\b;

%Method F - SVD factorization%
[U, S, V] = svd(A,0);
xf = V*(S\(U'*b));

x = [xa, xd, xe, xf]
end
```

to print the least square coefficients of the four methods. The code outputs,

```
Method A              Method D             Method E             Method F
 0.999999996787553    1.000000000996608    1.000000000996607    1.000000000996608
 0.000000350916732   -0.000000422743080   -0.000000422743364   -0.000000422743088
-8.000003028795119   -7.999981235685203   -7.999981235676154   -7.999981235684746
-0.000077893877909   -0.000318763231287   -0.000318763346323   -0.000318763237547
 10.668084035612900   10.669430795858052   10.669430796641096   10.669430795900578
-0.009615585352545   -0.013820287698367   -0.013820290914619   -0.013820287867134
-5.654480747960067   -5.647075628404193   -5.647075619959385   -5.647075627982760
-0.068885958631546   -0.075316022079922   -0.075316036589419   -0.075316022763597
 1.693354534665567    1.693606960559036    1.693606976803618    1.693606961280185
 0.001461547101732    0.006032111063859    0.006032099645104    0.006032110585745
-0.370576739064076   -0.374241704456940   -0.374241699881279   -0.374241704275633
 0.087095866530693    0.088040576259675    0.088040575462356    0.088040576229626
```

which shows us that the results from methods D,E, and F are fairly consistent while method A is unstable. Observe,


```
Method A              Method D             Method E             Method F
 0.999999996787553    1.000000000996608    1.000000000996607    1.000000000996608
 0.000000350916732   -0.000000422743080   -0.000000422743364   -0.000000422743088
-8.000003028795119   -7.999981235685203   -7.999981235676154   -7.999981235684746
-0.000077893877909   -0.000318763231287   -0.000318763346323   -0.000318763237547
 10.668084035612900   10.669430795858052   10.669430796641096   10.669430795900578
-0.009615585352545   -0.013820287698367   -0.013820290914619   -0.013820287867134
-5.654480747960067   -5.647075628404193   -5.647075619959385   -5.647075627982760
-0.068885958631546   -0.075316022079922   -0.075316036589419   -0.075316022763597
 1.693354534665567    1.693606960559036    1.693606976803618    1.693606961280185
 0.001461547101732    0.006032111063859    0.006032099645104    0.006032110585745
-0.370576739064076   -0.374241704456940   -0.374241699881279   -0.374241704275633
 0.087095866530693    0.088040576259675    0.088040575462356    0.088040576229626
```

.

where the highlighted lines show possible rounding error.This leads me to believe that the normal equations (method A) exhibit unstable behavior. □

Problem 7 *Least Squares of Sport Teams*

Solution. Consider the system of equations,

$$\begin{aligned} r_1 - r_2 &= 4, \\ r_3 - r_1 &= 9, \\ r_1 - r_4 &= 6, \\ r_3 - r_4 &= 3, \\ r_2 - r_4 &= 7. \end{aligned}$$

a If we rewrite the system of equations as a matrix we get,

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 6 \\ 3 \\ 7 \end{pmatrix}.$$

Now if we assume that $(r_1, r_2, r_3, r_4)^*$ is a solution for the system, observe $(r_1 + c, r_2 + c, r_3 + c, r_4 + c)^*$ is also a solution for any constant c since,

$$\begin{aligned} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} r_1 + c \\ r_2 + c \\ r_3 + c \\ r_4 + c \end{pmatrix} &= \begin{pmatrix} 1 \cdot (a+c) + (-1)(b+c) + 0 \cdot (c+c) + 0 \cdot (d+c) \\ (-1)(a+c) + 0 \cdot (b+c) + 1 \cdot (c+c) + 0 \cdot (d+c) \\ 1 \cdot (a+c) + 0 \cdot (b+c) + 0 \cdot (c+c) + (-1)(d+c) \\ 0 \cdot (a+c) + 0 \cdot (b+c) + 1 \cdot (c+c) + (-1)(d+c) \\ 0 \cdot (a+c) + 1 \cdot (b+c) + 0 \cdot (c+c) + (-1)(d+c) \end{pmatrix} \\ &= \begin{pmatrix} a - b \\ c - a \\ a - d \\ c - d \\ b - d \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 9 \\ 6 \\ 3 \\ 7 \end{pmatrix} \end{aligned}$$

as desired. Thus we append a sixth equation $r_1 + r_2 + r_3 + r_4 = 20$ to make the solution unique where 20 is the total number of ranking points.

b Notice that by adding a sixth equation, our new system of equations in matrix form will be,

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 6 \\ 3 \\ 7 \\ 20 \end{pmatrix}.$$

We can see that the sixth equation will be exactly satisfied by observing,

$$\begin{aligned} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &= \begin{pmatrix} 1\cdot a + (-1)\, b + 0\cdot c + 0\cdot d \\ (-1)\, a + 0\cdot b + 1\cdot c + 0\cdot d \\ 1\cdot a + 0\cdot b + 0\cdot c + (-1)\, d \\ 0\cdot a + 0\cdot b + 1\cdot c + (-1)\, d \\ 0\cdot a + 1\cdot b + 0\cdot c + (-1)\, d \\ 1\cdot a + 1\cdot b + 1\cdot c + 1\cdot d \end{pmatrix} \\ &= \begin{pmatrix} a-b \\ c-a \\ a-d \\ c-d \\ b-d \\ a+b+c+d \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 9 \\ 6 \\ 3 \\ 7 \\ 20 \end{pmatrix} \end{aligned}$$

and thus we see that we can set the total number of ranking points to any number and the system will satisfy it.

c I created the Matlab code,

```
A = [1,-1,0,0;-1,0,1,0;1,0,0,-1;0,0,1,-1;0,1,0,-1;1,1,1,1];
b = [4;9;6;3;7;20];
[Q, R] = qr(A);
results = R\(Q'*b)
```

that outputs,

```
results =
    5.2500
```

```
4.6250
9.1250
1.0000
```

using QR factorization to solve the least squares problem we have been discussing. based of this result, we know T_3 is in the lead with T_1 in second and T_2 in third.

$\square$