Math 569 Homework 4
Due May 10
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Problem 1 *Green's function of the 1-D heat equation in a semi-infinite domain, $G(x,t;\xi,\tau)$, is defined by:*

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)G = \delta(x-\xi)\delta(t-\tau),$$

with $0 < x, \xi < \infty, t, \tau > 0$ subject to zero initial condition: $G = 0$ at $t = 0$. The boundary condition is either (a): $G = 0$ at $x = 0$ and $x \to \infty$, or (b): $\frac{\partial}{\partial x}G = 0$ at $x = 0$ and $x \to \infty$. The solution in a semi-infinite domain can be constructed from the solution in the infinite domain by adding or subtracting another source located at $x = -\xi$, so that the contributions cancel at $x = 0$ for (a), or the contributions are symmetric about $x = 0$. To find the Green's function defined above for boundary condition (a). Then repeat the problem for boundary condition (b).

Solution.
Consider the Green's function of the 1-D heat equation in a semi-infinite domain, $G(x,t;\xi,\tau)$, is defined by:

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)G = \delta(x-\xi)\delta(t-\tau), \tag{1}$$

with $0 < x, \xi < \infty, t, \tau > 0$ subject to zero initial condition: $G = 0$ at $t = 0$.

(a) Suppose that Eq. (1) is subject to the boundary condition $G = 0$ at $x = 0$ and $x \to \infty$. When $t < \tau$, the initial value problem is

$$\begin{cases} \frac{\partial}{\partial t}G - D\frac{\partial^2}{\partial x^2}G = 0, & 0 < x, \xi < \infty, 0 < t < \tau, \\ G = 0 & \text{at } t = 0, \\ G = 0 & \text{at } x = 0, x \to \infty \end{cases}$$

which has the trivial solution

$$G(x,t;\xi,\tau) = 0.$$

When $t > \tau$, then

$$\begin{cases} \frac{\partial}{\partial t}G - D\frac{\partial^2}{\partial x^2}G = 0, & 0 < x, \xi < \infty, t > \tau > 0, \\ G = \delta(x-\xi) & \text{at } t = \tau \text{ (from lecture 12)}, \\ G = 0 & \text{at } x = 0, x \to \infty. \end{cases}$$

Since the problem is in a semi-infinite domain and we require $G = 0$ at $x = 0$, we will use an odd extension of the form

$$g(x) = \begin{cases} \delta(x-\xi), & x > 0, \\ 0, & x = 0, \\ -\delta(-x-\xi), & x < 0. \end{cases}$$

Thus the problem becomes the fundamental problem for the heat equation

$$\begin{cases} \frac{\partial}{\partial t}G - D\frac{\partial^2}{\partial x^2}G = 0, & -\infty < x, \xi < \infty, t > \tau > 0 \\ G = g(x), & t = \tau, \\ G = 0 & \text{at } x = 0, |x| \to \infty, \end{cases}$$

which has a Green's function (using the Drunken Sailor problem)

$$G_1(x,t;\xi,\tau) = \begin{cases} \frac{1}{\sqrt{4\pi D(t-\tau)}} e^{-\frac{(x-\xi)^2}{4D(t-\tau)}}, & t > \tau \\ 0, & t < \tau, \end{cases} = H(t-\tau)e^{-\frac{(x-\xi)^2}{4D(t-\tau)}},$$

defined on the infinite domain and where $H(x)$ is the Heaviside function and we make the assumption that $G_1(x) \to 0$ a $|x| \to \infty$. Thus the Green's function on a semi-infinite domain is given by

$$\begin{aligned} G(x,t;\tau,\xi) &= \int_{-\infty}^{\infty} H(t-\tau)\frac{1}{\sqrt{4\pi D(t-\tau)}} e^{-\frac{(x-y)^2}{4D(t-\tau)}} d(y)\mathrm{d}y, \\ &= -\frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4D(t-\tau)}} \delta(-y-\xi)\mathrm{d}y \\ &\quad + \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4D(t-\tau)}} \delta(y-\xi)\mathrm{d}y \\ &= \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(e^{-\frac{(x+\xi)^2}{4D(t-\tau)}} - e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \right) \end{aligned}$$

where y was used as a dummy variable. Lastly, we verify that G satisfies Eq. (1) at $t = 0$ and the initial and boundary conditions. Observe that G when $t = 0$ yields

$$G(x,0;\xi,\tau) = \frac{H(-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(e^{-\frac{(x+\xi)^2}{4D(t-\tau)}} - e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \right) = 0.$$

When $x = 0$ and $t > 0$ we have

$$G(0,t;\xi,\tau) = \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(e^{-\frac{\xi^2}{4D(t-\tau)}} - e^{-\frac{\xi^2}{4D(t-\tau)}} \right) = 0.$$

And finally

$$\begin{aligned} \lim_{x\to\infty} G(x,t;\xi,\tau) &= \lim_{x\to\infty} \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(e^{-\frac{(x+\xi)^2}{4D(t-\tau)}} - e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \right) \\ &= \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}}(0-0) \\ &= 0. \end{aligned}$$

(b) Suppose that Eq. (1) is subject to the boundary condition $G = 0$ at $x = 0$ and $x \to \infty$. When $t < \tau$, the initial value problem is

$$\begin{cases} \frac{\partial}{\partial t}G - D\frac{\partial^2}{\partial x^2}G = 0, & 0 < x, \xi < \infty, 0 < t < \tau, \\ G = 0 & \text{at } t = 0, \\ G_x = 0 & \text{at } x = 0, x \to \infty \end{cases}$$

which has the trivial solution

$$G(x, t; \xi, \tau) = 0.$$

When $t > \tau$, then

$$\begin{cases} \frac{\partial}{\partial t}G - D\frac{\partial^2}{\partial x^2}G = 0, & 0 < x, \xi < \infty, t > \tau > 0, \\ G = \delta(x - \xi) & \text{at } t = \tau \text{ (from lecture 12)}, \\ G_x = 0 & \text{at } x = 0, x \to \infty. \end{cases}$$

Since the problem is in a semi-infinite domain and we require $G_x = 0$ at $x = 0$, we will use an even extension of the form

$$g(x) = \begin{cases} \delta(x - \xi), & x > 0, \\ 0, & x = 0, \\ \delta(-x - \xi), & x < 0. \end{cases}$$

Thus the problem becomes

$$\begin{cases} \frac{\partial}{\partial t}G - D\frac{\partial^2}{\partial x^2}G = 0, & -\infty < x, \xi < \infty, t > \tau > 0 \\ G = g(x), & t = \tau, \\ G_x = 0 & \text{at } x = 0, |x| \to \infty, \end{cases}$$

which is slightly different from the fundamental heat equation but note that when solving the heat equation using Drunken Sailor, we have the boundary condition that $u \to 0$ as $|x| \to \infty$ and make the additional assumption that $u_x \to 0$ as $|x| \to \infty$, where u is the solution to the heat equation. Thus if we let the boundary condition be that $u_x \to 0$ as $|x| \to \infty$ and make the assumption that $u \to 0$ as $|x| \to \infty$ then the solution will be the same and thus the Green's function is given by

$$G'(x, t; \xi, \tau) = \begin{cases} \frac{1}{\sqrt{4\pi D(t-\tau)}} e^{-\frac{(x-\xi)^2}{4D(t-\tau)}}, & t > \tau \\ 0, & t < \tau, \end{cases} = H(t - \tau)e^{-\frac{(x-\xi)^2}{4D(t-\tau)}},$$

defined on the infinite domain and where $H(x)$ is the Heaviside function. Note that the new assumption is easily verified due to the exponential decay term. Thus the Green's function on a semi-infinite domain is given by

$$G(x, t; \tau, \xi) = \int_{-\infty}^{\infty} H(t - \tau)\frac{1}{\sqrt{4\pi D(t - \tau)}} e^{-\frac{(x-y)^2}{4D(t-\tau)}} d(y)\mathrm{d}y,$$

$$
\begin{aligned}
&= \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4D(t-\tau)}} \delta(-y-\xi)\mathrm{d}y \\
&\quad + \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4D(t-\tau)}} \delta(y-\xi)\mathrm{d}y \\
&= \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(e^{-\frac{(x+\xi)^2}{4D(t-\tau)}} + e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \right)
\end{aligned}
$$

where y was used as a dummy variable. Lastly, we verify that G satisfies Eq. (1) at $t = 0$ and the initial and boundary conditions. Observe that G when $t = 0$ yields

$$
G(x, 0; \xi, \tau) = \frac{H(-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(e^{-\frac{(x+\xi)^2}{4D(t-\tau)}} + e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \right) = 0.
$$

When $x = 0$ and $t > 0$ we have

$$
G_x(0, t; \xi, \tau) = \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(\frac{-1}{4D(t-\tau)} \right) \left((2\xi) e^{-\frac{\xi^2}{4D(t-\tau)}} + (-2\xi) e^{-\frac{\xi^2}{4D(t-\tau)}} \right) = 0.
$$

And finally

$$
\begin{aligned}
\lim_{x\to\infty} G_x(x, t; \xi, \tau) &= \lim_{x\to\infty} \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(\frac{-1}{4D(t-\tau)} \right) \left(2(x+\xi) e^{-\frac{(x+\xi)^2}{4D(t-\tau)}} - 2(x-\xi) e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} \right) \\
&= \frac{H(t-\tau)}{\sqrt{4\pi D(t-\tau)}} \left(\frac{-1}{4D(t-\tau)} \right) (0 - 0) \\
&= 0.
\end{aligned}
$$

$\square$

Problem 2 *Find the Greens function for the wave equation in two-dimensions governed by*

$$\begin{cases} \frac{\partial^2}{\partial t^2}G - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)G = \delta(t)\delta(x)\delta(y), \\ G \to 0 \ \textit{as}\ r \to \infty, \\ G = 0 \ \textit{for}\ t < 0, \end{cases}$$

where $r^2 = x^2 + y^2$*. The solution is*

$$G = \frac{1}{2\pi}\frac{H(t-\tau)}{\sqrt{t^2 - r^2}},$$

where H *is the Heaviside function.*

(a) *Derive this solution using Fourier transform in* x *and* y*. Hint: In the inverse transform, use polar coordinates to get*

$$G = \frac{1}{2\pi}\int_0^\infty J_0(kr)\sin kt\mathrm{d}k.$$

Then use integral tables.

(b) *Derive this solution using Laplace transform in* t*. Hint: First show that the Laplace transform of* G *is*

$$\tilde{G} = \frac{1}{2\pi}K_0(sr),$$

where K *is the modified Bessel function of the second kind. Then use Laplace transform tables.*

Solution.

Consider the 2-D wave equation governed by

$$\begin{cases} \frac{\partial^2}{\partial t^2}G - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)G = \delta(t)\delta(x)\delta(y), \\ G \to 0 \text{ as } r \to \infty, \\ G = 0 \text{ for } t < 0, \end{cases} \tag{2}$$

where $r^2 = x^2 + y^2$.

(a) We first wish to solve Eq. (2) using 2D Fourier transform in x and y. Observe that

$$\mathcal{F}\left[\frac{\partial^2}{\partial t^2}G\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial t^2}Ge^{i(k_1x+k_2y)}\mathrm{d}x\mathrm{d}y = \frac{\partial^2}{\partial t^2}\mathcal{F}[G],$$

and using integration by parts twice yields

$$\int_{-\infty}^{\infty} G_{xx}e^{ik_1x+ik_2y}\mathrm{d}x = e^{ik_2y}\left([G_xe^{ik_1x}]_{-\infty}^{\infty} - ik_1\int_{-\infty}^{\infty}G_xe^{ik_1x}\mathrm{d}x\right)$$

$$
\begin{aligned}
&= e^{ik_2 y}\left(\cancel{[G_x e^{ik_1 x}]_{-\infty}^{\infty}} - ik_1\cancel{[Ge^{ik_1 x}]_{-\infty}^{\infty}} - k_1^2\int_{-\infty}^{\infty} Ge^{ik_1 x}\mathrm{d}x\right)\\
&= -k_1^2\int_{-\infty}^{\infty} Ge^{ik_1 x + ik_2 y}\mathrm{d}x,
\end{aligned}
$$

where we assume that $G_x \to 0$ as $|x| \to \infty$ and similarly find that

$$
\int_{-\infty}^{\infty} G_{yy}e^{ik_1 x + ik_2 y}\mathrm{d}y = -k_2^2\int_{-\infty}^{\infty} Ge^{ik_1 x + ik_2 y}\mathrm{d}y,
$$

where we assume that $G_y \to 0$ as $|x| \to \infty$. Thus we have that

$$
\mathcal{F}[G_{xx} + G_{yy}] = -(k_1^2 + k_2^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ge^{ik_1 x + ik_2 y}\mathrm{d}y\mathrm{d}x = -k^2\mathcal{F}[G]
$$

where $k = \sqrt{k_1^2 + k_2^2}$. We also have that

$$
\mathcal{F}[\delta(t)\delta(x)\delta(y)] = \delta(t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x)\delta(y)e^{i(k_1 x + k_2 y)}\mathrm{d}y\mathrm{d}x = \delta(t).
$$

Thus Eq. (2) becomes

$$
\frac{\partial^2}{\partial t^2}\mathcal{F}[G] + k^2\mathcal{F}[G] = \delta(t).
$$

When $t \neq 0$, we have $\frac{\partial^2}{\partial t^2}\mathcal{F}[G] + k^2\mathcal{F}[G] = 0$ which has the solution

$$
\mathcal{F}[G] = c_1\sin(kt) + c_2\cos(kt).
$$

To solve for the unknowns, recall that $G = 0$ for $t < 0$ which implies that $\mathcal{F}[G] = 0$ for $t < 0$. Thus

$$
\lim_{t\to 0}\mathcal{F}[G] = c_2 = 0.
$$

We can also find the matching condition at $t = 0$ by integrating across $t = 0$ as follows

$$
\int_{0^-}^{0^+}\frac{\partial^2}{\partial t^2}\mathcal{F}[G] + k^2\mathcal{F}[G]\mathrm{d}t = \int_{0^-}^{0^+}\delta(t)\mathrm{d}t,
$$

where $\int_{0^-}^{0^+}\mathcal{F}[G]\mathrm{d}t = 0$ since G defined to be finite and $\int_{0^-}^{0^+}\delta(t)\mathrm{d}t = 1$. Thus we have that

$$
\int_{0^-}^{0^+}\frac{\partial^2}{\partial t^2}\mathcal{F}[G]\mathrm{d}t = \frac{\partial}{\partial t}\mathcal{F}[G]|_{t=0^+} - \cancel{\frac{\partial}{\partial t}\mathcal{F}[G]|_{t=0^-}} = \frac{\partial}{\partial t}\mathcal{F}[G]|_{t=0^+} = 1,
$$

where we canceled the term since $\mathcal{F}[G] = 0$ for $t < 0$. Thus we can solve for c_1

$$
\frac{\partial}{\partial t}\mathcal{F}[G]|_{t=0^+} = 1 \iff c_2 k\cos(0^+) = 1 \iff c_2 = \frac{1}{k}.
$$

Therefore we have found

$$\mathcal{F}[G] = \frac{1}{k}\sin(kt).$$

Now to compute G we apply the 2D Fourier inverse transform

$$G = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{k}\sin(kt)e^{-i(k_1x+k_2y)}\mathrm{d}k_1\mathrm{d}k_2.$$

Converting to polar coordinates gives

$$G = \frac{1}{4\pi^2}\int_0^{\infty}\int_{-\pi}^{\pi}\sin(kt)e^{-ih\cos(\theta)x-ih\sin(\theta)y}\mathrm{d}\theta\mathrm{d}h,$$

then if we let $\vec{r} = (x,y)^T$ and $\vec{h} = (h\cos(\theta), h\sin(\theta))^T$ and taking the norm yields

$$\langle\vec{r},\vec{h}\rangle = h\cos(\theta)x + h\sin(\theta)y = rk\cos(\theta),$$

where $|\vec{r}| = r$ and $|\vec{h}| = h$. Thus we can rewrite the integral as

$$\begin{aligned}
G &= \frac{1}{4\pi^2}\int_0^{\infty}\int_{-\pi}^{\pi}\sin(kt)e^{irh\cos(\theta)}\mathrm{d}\theta\mathrm{d}h \\
&= \frac{1}{2\pi}\int_0^{\infty}\sin(kt)\left(\frac{1}{\pi}\int_0^{\pi}e^{i(-rh\cos(\theta))}\mathrm{d}\theta\right)\mathrm{d}h \\
&= \frac{1}{2\pi}\int_0^{\infty}\sin(kt)\left(\frac{1}{\pi}\int_0^{\pi}\cos(-rh\sin(\theta))\right)\mathrm{d}\theta\mathrm{d}h \\
&= \frac{1}{2\pi}\int_0^{\infty}\sin(kt)\left(\frac{1}{\pi}\int_0^{\pi}\cos(rh\sin(\theta))\mathrm{d}\theta\right)\mathrm{d}h \\
&= \frac{1}{2\pi}\int_0^{\infty}\sin(kt)J_0(hr)\mathrm{d}h,
\end{aligned}$$

where $J_0(hr)$ is the Bessel function of first kind with $n = 0$. Consulting a table of integrals we find that

$$G = \frac{1}{2\pi}\frac{H(t-\tau)}{\sqrt{t^2-\tau^2}}.$$

Note that since the derivative of the Heaviside function is the delta function, our assumptions are easily verified and the boundary conditions are also satisfied.

(b) Next, we wish to derive the solution of Eq. (2) using a Laplace transform in t. Observe that

$$\begin{aligned}
\mathcal{L}[G_{tt}] &= \int_0^{\infty} G_{tt}e^{st}\mathrm{d}t \\
&= [G_te^{st}]_0^{\infty} - [Ge^{st}]_0^{\infty} + s^2\int_0^{\infty} Ge^{st}\mathrm{d}t
\end{aligned}$$

$$= s^2\mathcal{L}[G],$$

under the assumptions that G and $G_t \to 0$ for $t = 0$ and $t \to \infty$. We also have that

$$\mathcal{L}[\nabla^2 G] = \nabla^2 \int_0^\infty G e^{st}\mathrm{d}t = \nabla\mathcal{L}[G],$$

and

$$\mathcal{L}[\delta(t)\delta(x)\delta(y)] = \delta(x)\delta(y)\int_0^\infty \delta(t)e^{st}\mathrm{d}t = \delta(x)\delta(y).$$

Thus we can rewrite Eq. (2) as

$$\begin{cases} s^2\tilde{G}(x,y,s) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{G}(x,y,s) = \delta(x)\delta(y), \\ \tilde{G}(x,y,s) \to 0 \text{ as } r \to \infty, \\ \tilde{G}(x,y,s) = 0 \text{ for } t < 0, \end{cases}$$

where $\mathcal{L}[G] = \tilde{G}$. Converting this equation into polar coordinates and assuming that it does not depend on the angle θ yields

$$s^2\tilde{G}(r) - \tilde{G}_{rr}(r) - \frac{1}{r}\tilde{G}_r(r) = \delta(r),$$

where $r = x^2 + y^2$. When $r \neq 0$ and multiplying through by $-r$ gives

$$r\tilde{G}_{rr} + \tilde{G}_r - s^2 r\tilde{G} = 0.$$

Now letting $g(r) = \tilde{G}(r/s)$ such that $g_r(r) = \frac{1}{s}\tilde{G}_r$ and $g_{rr}(r) = \frac{1}{s^2}\tilde{G}_{rr}$, our equation becomes

$$\frac{r}{s}s^2 g_{rr} + s g_r - s^2\frac{r}{s}g = 0,$$

and dividing through by s gives

$$r g_{rr} + g_r - r g = 0.$$

Notice that this is in the form of a Modified Bessel equation which has the solution

$$g(r) = c_1 I_0(r) + c_2 Y_0(r) \implies \tilde{G}(r) = c_1 I_0(rs) + c_2 Y_0(rs),$$

where I_0 and Y_0 denote the modified Bessel functions of the first and second kind of order 0. Now we impose the boundary condition that $\tilde{G} \to 0$ as $r \to \infty$ and since $K_0(rs) \to 0$ and $I_0 \to \infty$ as $r \to \infty$ we have that $c_1 = 0$ and thus

$$\tilde{G} = c_2 K_0(rs)$$

Now we can enforce the matching condition

$$\int_0^{2\pi}\int_0^{\epsilon} s^2\tilde{G} - \frac{1}{r}\tilde{G}_r - \tilde{G}_{rr}\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\int_0^{\epsilon}\delta(r)\mathrm{d}r\mathrm{d}\theta$$

$$\implies \int_0^{2\pi}\int_0^{\epsilon} s^2\tilde{G}\mathrm{d}r\mathrm{d}\theta - \int_0^{2\pi}\int_0^{\epsilon}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\tilde{G}\right)\mathrm{d}r\mathrm{d}\theta = 1.$$

Recall that $\lim_{r\to 0} K_0(rs) = 0$ and thus the first integral drops out leaving us with

$$2\pi r c_2 \frac{\partial}{\partial r}K_0(rs)|_0^{\epsilon} = 2\pi r c_2\left(\frac{1}{rs}s\right) = 1,$$

where we used the fact that $K_0(rs) \approx -\log(rs)$ as $r \to 0$, since $K_0(x) = -\log(x/2)I_0(x)$ as $|x| \to 0$ and since $I_0(x) \to 1$ as $|x| \to 0$. Thus we have found that $c_2 = \frac{1}{2\pi}$ giving

$$\tilde{G} = \frac{1}{2\pi}K_0(rs),$$

and consulting a table of integrals we find that the inverse Laplace transform gives

$$G = \frac{1}{2\pi}\frac{H(t-\tau)}{\sqrt{t^2-r^2}}.$$

We note that the assumptions we made are satisfied since $G = 0$ when $t = 0$ and as $t \to \infty$. Similarly taking the derivative of G results in a term containing a delta function and thus $G_t = 0$ when $t = 0$ and $t \to \infty$.

Reference I consulted DLMF for Bessel function properties and integrals throughout this problem. $\square$