AMATH 575
Problem set 1

Working together is welcomed. Please do not refer to previous years' solutions.

I Consider the system in the phase plane

$$\begin{aligned}\dot{x} &= f(x)\\ \dot{y} &= g(y)\end{aligned}$$

where $f(x)$ is a continuously differentiable real-valued function of x alone and $g(y)$ is a continuously differentiable real-valued function of y alone (and x and y are both 1-dimensional coordinates defining a plane). Define an oscillatory solution as a trajectory $(x(t), y(t))$ such that $x(t)$ and $y(t)$ are not constant it time and, for any integer N, $x(t+NT) = x(t)$ and $y(t+NT) = y(t)$. Here, T is the period of the oscillation.

(a) Answer YES or NO and give a simple proof or example: Can a system of this form produce an oscillatory solution? (b) Then, repeat this question, but for the discrete time map

$$\begin{aligned}x_{n+1} &= f(x_n)\\ \dot{y}_{n+1} &= g(y_n)\end{aligned}$$

II Consider the 2-D systems below. Find all equilibria and determine where they are lyapunov stable, asymptotically stable, or neither. Here μ is an arbitrary real parameter, so make sure to give answers valid for each relevant range of μ:

- $\dot{x} = 0$, $\dot{y} = \mu x$
- $\dot{x} = 0$, $\dot{y} = \mu y$

III Consider the 2-D system:

$$\begin{aligned}\dot{x} &= -y + \mu(x^2 + y^2)x\\ \dot{y} &= x + \mu(x^2 + y^2)y\end{aligned}$$

where μ is an arbitrary real parameter. Hint: transform to polar coordinates and obtain an exact solution.

a For all possible values of μ, find all fixed points, and determine whether they are lyapunov stable, asymptotically stable, or neither.

b For all possible values of μ, and all possible initial values, determine the maximum duration in both forward and inverse time for which a solution exists.

IV The van der Pol oscillator. Consider:

$$\frac{d^2x}{dt^2} + (x^2 - v)\frac{dx}{dt} + x = 0$$

where v is a parameter that can take any real value.

a Find all fixed points, and the Jacobian evaluated as these fixed point(s).

b State whether the fixed point(s) are lyapunov stable, asymptotically stable, or neither for all possible values of v.

V A general description of a network of N nonlinearly coupled units is given by

$$\frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} w_{ij} g(u_j) \tag{1}$$

Here, u_i is the activity of the i^{th} unit, and the matrix w gives the connection weights among these units; in particular, w_{ij} is the connection weight between unit j and unit i. Finally, $g(\cdot)$ is a monotonically increasing function that describes how the strength of interaction between units depends on their activities.

- For LINEAR interactions: $g(y) = y$, write down a simple bound on the entries w_{ij}, based on the Gershgorin circle theorem, that guarantees that the origin will be a stable equilibrium.
- For NONLINEAR interactions $g(\cdot)$, consider the "energy function"

$$H = -1/2 \sum_{ij} w_{ij} V_i V_j + \sum_i \int_0^{V_i} g^{-1}(V) dV \tag{2}$$

where $V_i = g(u_i)$. [a] If the matrix w is symmetric, show the following bound on the time evolution of the energy:

$$\frac{dH}{dt} \leq 0 \ .$$

Your answer should be valid for a smooth monotonically increasing function $g(\cdot)$. [b] Then, let $\bar{u}$ be an equilibrium for the system. What additional requirements on H would imply that $\bar{u}$ is asympotically stable? [c] Take $g(x) = \tanh(x)$. Construct a simple example of w for which you can find an equilibrium $\bar{u}$ and demonstrate that your function H implies that it is asymptotically stable. A numerical approach is suggested, and rigorous arguments are not needed, though of course if you wish to use analysis instead that is just fine. [d] Finally, does the bound

$$\frac{dH}{dt} \leq 0$$

also hold in general for anti-symmetric w?

VI A specific case of the Lorenz equations is given by

$$\begin{cases} x' = 10(-x + y) \\ y' = rx - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases} \tag{3}$$

a For varying r, find all equilibrium points and discuss their stability.

b Calculate up to second order terms the local invariant manifolds W^u, W^s and W^c for the fixed point at the origin of the Lorenz equations when $r = 1$.