Math 586 Homework 1
Due April 10 11pm
By Marvyn Bailly

Problem 1 *In this exercise you will show convergence for a discretization of*

$$\begin{cases} -u''(x) = g(x), \\ u'(0) = \alpha, \\ u(1) = \beta. \end{cases}$$

1. *Consider the* $(m+1) \times (m+1)$ *matrix*

$$A = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Find its Cholesky decomposition.

2. *Show that*

$$A^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} m+1 \\ m \\ \vdots \\ 1 \end{bmatrix}. \tag{1}$$

3. *Now show that*

$$\|A^{-1}\|_1 \leq (m+1)^2, \quad \|A^{-1}\|_\infty \leq (m+1)^2.$$

Solution.

(a) We wish to find the Cholesky decomposition of the $(m+1) \times (m+1)$ matrix

$$A = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Since A is an HPD, there exists a unique Cholesky decomposition L such that $A = LL^*$. We can find L by preforming Gaussian Elimination on A and collecting the row operations in the a matrix L. The first step of Gaussian Elimination looks like

$$A = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix} \xrightarrow{r_1 + r_2 \to r_2} \begin{bmatrix} 1 & -1 & & & \\ 0 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix},$$

where L_1 is given by

$$L_1 = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Continuing this process of adding the ith row to the $i+1$th row gives

$$\begin{bmatrix} 1 & -1 & & & \\ 0 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & & & \\ 0 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & 0 & 1 \end{bmatrix} = U.$$

Thus we have found that

$$(L_m \cdots L_1)A = U.$$

Then we define the Cholesky Factorization as

$$L = (L_m \cdots L_1)^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & & & \\ & & & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -d_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & & & \\ & & & -d_m & 1 \end{bmatrix},$$

where $A = LL^*$.

(b) Next we wish to show that

$$A^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} m+1 \\ m \\ \vdots \\ 1 \end{bmatrix} \implies A \begin{bmatrix} m+1 \\ m \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Observe that

$$\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} m+1 \\ m \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} m+1-m+0 \\ -m-1+2m-m+1+0 \\ \vdots \\ 0+2-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(c) Now we wish to show that

$$\|A^{-1}\|_1 \leq (m+1)^2, \quad \|A^{-1}\|_\infty \leq (m+1)^2.$$

Since A is HPD, A^{-1} is as well. This means that $\|A^{-1}\|_1 = \|A^{-1}\|_\infty$ due to symmetry. Observe that

$$\begin{aligned} \|A^{-1}\|_\infty = \|A^{-1}\|_1 &\leq \|L^{-*}\|_1 \|L^{-1}\|_1 \\ &= \left(1 + \max_{2 \leq +i \leq m} \sum_{j=i+1}^{m} \prod_{l=i}^{j} |d_l|\right)\left(1 + \max_{2 \leq +i \leq m} \sum_{i=1}^{j-1} \prod_{l=i}^{j} |d_l|\right) \\ &\leq (1+m)(1+m) \\ &= (1+m)^2. \end{aligned}$$

□

Problem 2 *Consider the matrix (see (2.54) in LeVeque)*

$$L = h^{-2} \begin{bmatrix} h & -h & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

1. *Compute L^{-1} in terms of A^{-1} and compute bounds for $\|L^{-1}\|_1$ and $\|L^{-1}\|_\infty$.*
2. *Explain why this is not enough to imply convergence for the one-sided approach, (2.53) in LeVeque.*
3. *Use* (1) *to show the method converges in both the grid 1-norm and the ∞-norm.*

Solution.
Consider the matrix

$$L = h^{-2} \begin{bmatrix} h & -h & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

(a) We first wish to compute L^{-1} is terms of A^{-1}. Observe that

$$L = \frac{1}{h^2} \begin{bmatrix} h & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} A = \frac{1}{h^2} BA,$$

and thus we have that

$$L^{-1} = \left(\frac{1}{h^2} AB \right)^{-1} = h^2 A^{-1} B^{-1}.$$

Next we can bound $\|L^{-1}\|_1$ by observing

$$\begin{aligned} \|L^{-1}\|_1 &= h^2 \|A^{-1}B^{-1}\|_1 \\ &= h^2 \|A^{-1}\|_1 \|B^{-1}\|_1 \\ &= h^2 (m+1)^2 \left| \frac{1}{h} \right| \\ &= h(m+1)^2 \\ &= m + 1 = h^{-1}. \end{aligned}$$

Next we wish to bound $\|L^{-1}\|_\infty = h^2\|A^{-1}B^{-1}\|_\infty \leq h^2\|A^{-1}\|_\infty\|B^{-1}\|_\infty$. We know that $\|A^{-1}\|_1 = \|A^{-1}\|_\infty$ and since B is symmetric, we know that B^{-1} is also symmetric and thus $\|B^{-1}\|_1 = \|B^{-1}\|_\infty$. Therefore we have that $\|L^{-1}\|_\infty \leq h^{-1}$.

(b) To see why this bound is not strong enough to imply convergence for the one-sided approach presented in LeVeque, observe the following. Let $E_j = u(x_j) - u_j$, then we have that

$$L[E_j] = \begin{bmatrix} hu''(\xi_0)/2 \\ h^2u''''(\xi_1)/12 \\ \vdots \\ h^2u''''(\xi_m)/12 \end{bmatrix} = \begin{bmatrix} O(h) \\ O(h^2) \\ \vdots \\ O(h^2) \end{bmatrix}$$

Thus we see that

$$\begin{aligned}
\|[E_j]\|_1 &= \left\| L^{-1} \cdot \begin{bmatrix} hu''(\xi_0)/2 \\ h^2u''''(\xi_1)/12 \\ \vdots \\ h^2u''''(\xi_m)/12 \end{bmatrix} \right\|_1 \\
&\leq \|L^{-1}\|_1 \cdot \left\| \begin{bmatrix} hu''(\xi_0)/2 \\ h^2u''''(\xi_1)/12 \\ \vdots \\ h^2u''''(\xi_m)/12 \end{bmatrix} \right\|_1 \\
&\leq \left\| \begin{bmatrix} u''(\xi_0)/2 \\ hu''''(\xi_1)/12 \\ \vdots \\ hu''''(\xi_m)/12 \end{bmatrix} \right\|_1 \\
&\leq \frac{\|u''\|_\infty}{2}.
\end{aligned}$$

Thus we see that the method is not necessarily consistent as $\|[E_j]\|_1$ does not have to go to zero as $h \to \infty$. Therefore, the bound on $\|L\|_1$ is not sufficient to guarantee convergence. A similar result occurs when studying $\|[E_j]\|_\infty$.

(c) To show the method converges in both the grid 1-norm and the ∞-norm, recall that convergence in the ∞-norm implies convergence in the 1-norm. Thus let's find a stricter bound on the ∞-norm that shows the method is consistent and thus convergences. Recall that in part one we found

$$L^{-1} = h^2A^{-1}B^{-1} \implies A^{-1} = \frac{1}{h^2}L^{-1}B,$$

where

$$B = \begin{bmatrix} h & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Then we have that

$$\begin{aligned}
[E_j] &= L^{-1} \\
&= L^{-1} \begin{bmatrix} hu''(\xi_0)/2 \\ h^2 u''''(\xi_1)/12 \\ \vdots \\ h^2 u''''(\xi_m)/12 \end{bmatrix} \\
&= h^2 A^{-1} B^{-1} \begin{bmatrix} hu''(\xi_0)/2 \\ h^2 u''''(\xi_1)/12 \\ \vdots \\ h^2 u''''(\xi_m)/12 \end{bmatrix} \\
&= -h^2 A^{-1} \begin{bmatrix} u''(\xi_0)/2 \\ h^2 u''''(\xi_1)/12 \\ \vdots \\ h^2 u''''(\xi_m)/12 \end{bmatrix} \\
&= -\frac{h^2 u''(\xi_0)}{2} A^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \frac{h^2}{12} A^{-1} \begin{bmatrix} 0 \\ h^2 u''''(\xi_1) \\ \vdots \\ h^2 u''''(\xi_m) \end{bmatrix} \\
&= -\frac{h^2 u''(\xi_0)}{2} \begin{bmatrix} m+1 \\ m \\ \vdots \\ 1 \end{bmatrix} - \frac{h^4}{12} A^{-1} \begin{bmatrix} 0 \\ u''''(\xi_1) \\ \vdots \\ u''''(\xi_m) \end{bmatrix}.
\end{aligned}$$

Then applying the grid ∞-norm yields

$$\begin{aligned}
\|[E_j]\|_\infty &\le \frac{\|u''\|_\infty}{2} h^2 (m+1) + \frac{\|u''''\|_\infty}{12} h^4 (m+1)^2 \\
&= O(h) + O(h^2).
\end{aligned}$$

Thus we have that that $\|[E_j]\|_\infty$ is consistent since $\|[E_j]\|_\infty \to 0$ as $h \to 0$. Thus the method converges under the grid ∞-norm which means that the method also converges under the grid one-norm (as $\|[E_j]\|_1 \le \|[E_j]\|_\infty$).

□

Problem 3 *Consider $u(x) = \cos(k\pi x)\exp(-x^2)$. Determine g, α and β such that*

$$
\begin{cases}
-u''(x) = g(x), \\
u(0) = \alpha, \\
u(1) = \beta.
\end{cases}
$$

1. *Modify the code in* `LinearBVP.ipynb` *to solve this BVP using a second-order accurate method. Plot errors on a log-log scale for $k = 1, 2, 3, 4$.*

2. *Modify the code in* `LinearBVP.ipynb` *to solve this BVP using a fifth-order accurate method. Plot errors on a log-log scale for $k = 1, 2, 3, 4$.*

Solution.

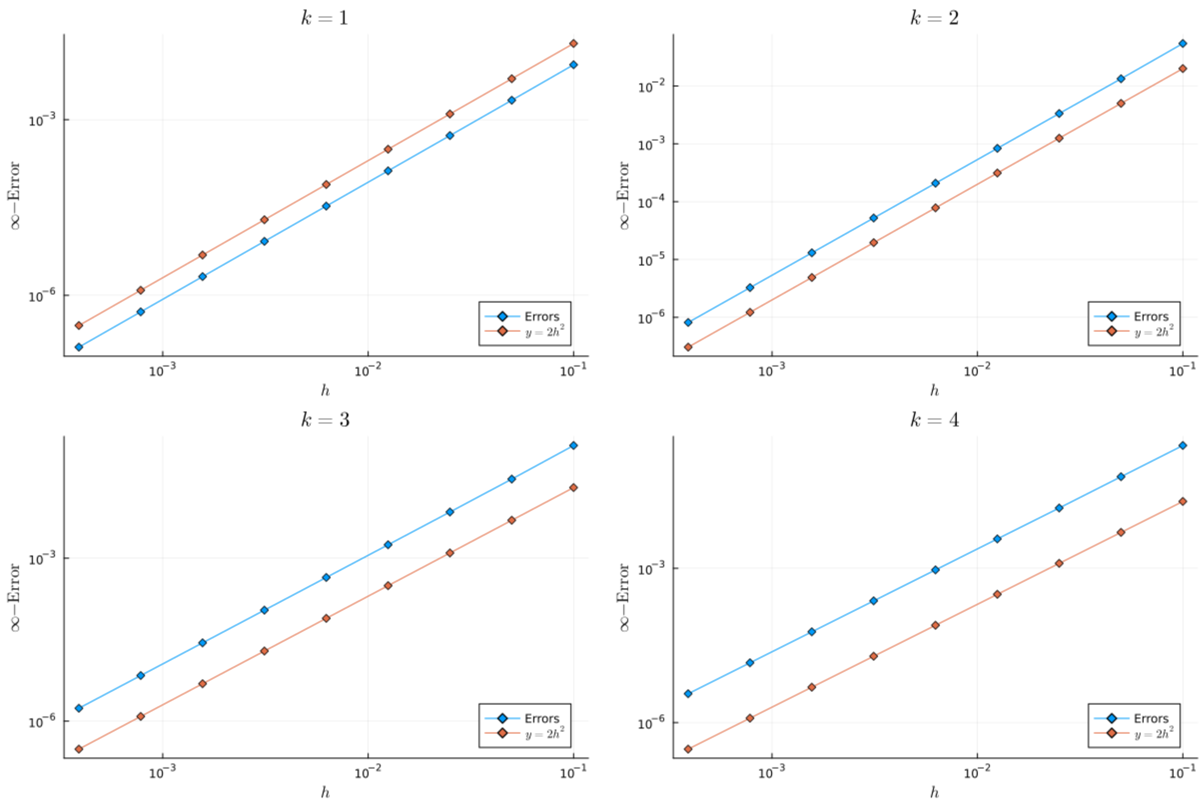

Figure 1: Loglog plots of errors using a second-order accurate method for $k \in \{1, 2, 3, 4\}$ compare to a line of slope h^2.

(a) From Figure 1 we can see that the slope of error for reducing grid sizes for $k \in \{1, 2, 3, 4\}$ is order h^2 as expected.

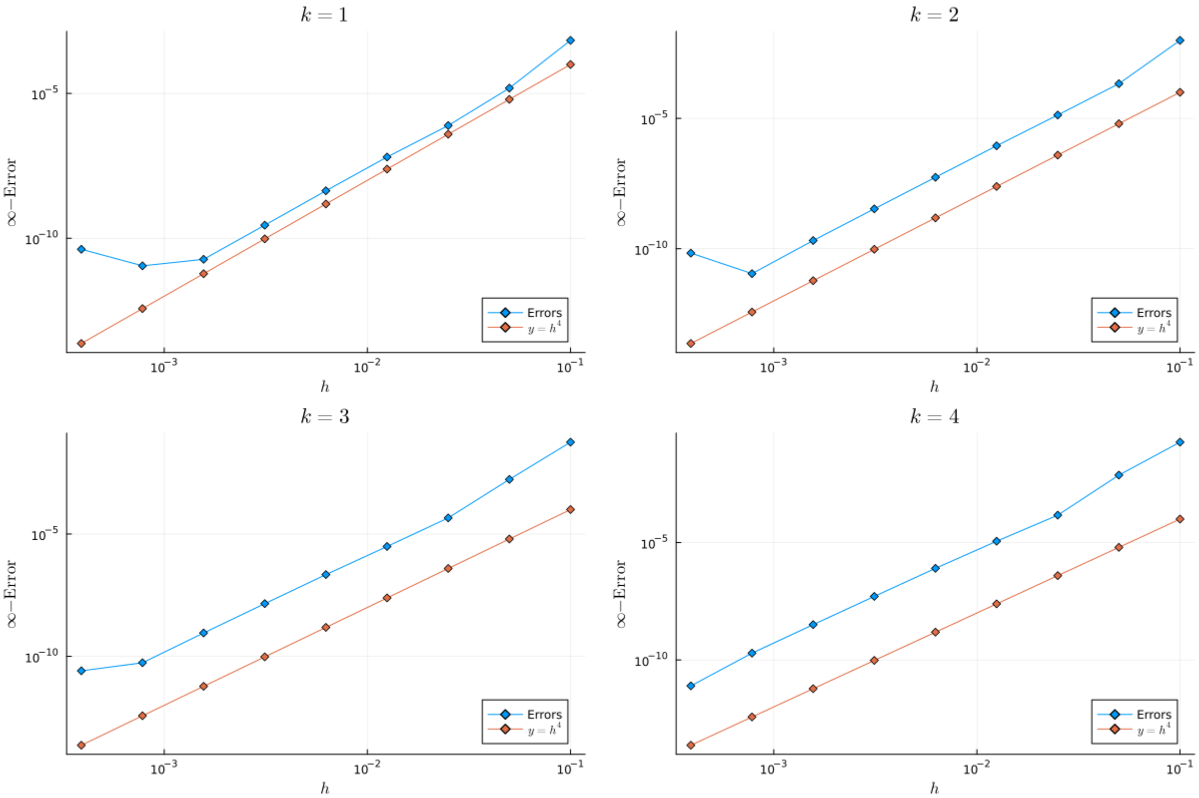

Figure 2: Loglog plots of errors using a fourth-order accurate method for $k \in \{1, 2, 3, 4\}$ compared to a line of slope h^4.

(b)

From Figure 2 we can see that the slope of error for reducing grid sizes for $k \in \{1, 2, 3, 4\}$ is order h^4 as expected.

$\square$

Problem 4 *Modify the code in* `NonlinearBVP.ipynb` *to solve*

$$\begin{cases} w'(x) - \epsilon w'''(x) = 0, \\ w(0) = 0, \\ w(L) = 0, \\ w'(L) = 1. \end{cases}$$

Demonstrate the convergence rate by comparing the computed solution to the true solution for $\epsilon = 0.1, 0.01$*.*

Solution.

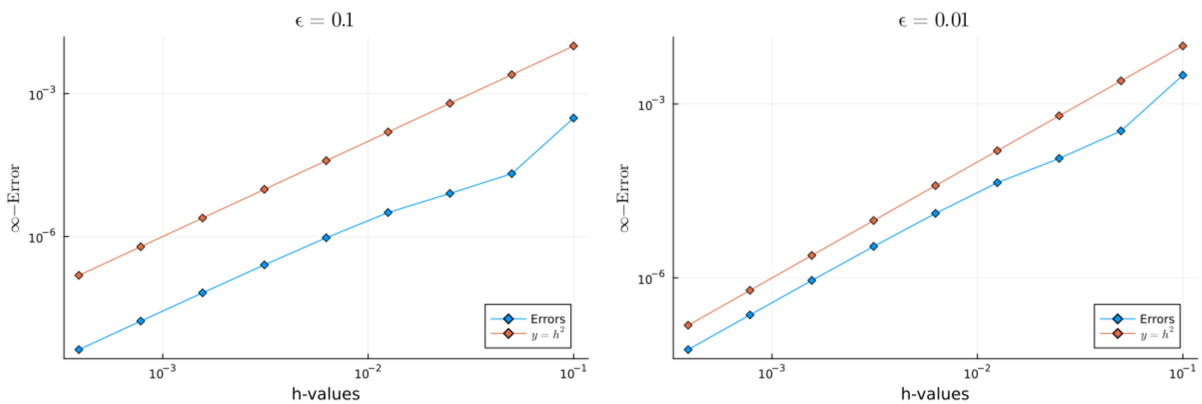

Figure 3: Loglog plots of errors using a second-order accurate method for $\epsilon \in \{0.1, 0.01\}$ compared to a line of slope h^2.

From Figure 3 we see that the constructed method for solving the nonlinear BVP is second-order accurate for $\epsilon \in \{0.1, 0.01\}$ when comparing the slope of the error across decreasing step sizes to $y = h^2$. Note, we used $L = 1$ in this example.

$\square$