Math 584 Homework 1
Due Wednesday
By Marvyn Bailly

Problem 1 *Exercise 1.1*

Solution.
Let $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$.

a. We can write the actions as the following matrices,

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$
$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b. Let $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

and $C = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then,

$$ABC = \begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

If we let $B = \begin{pmatrix} 1 & b2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 1 & 2 & 3 \\ 5 & 6 & 7 & 8 \end{pmatrix}$ then following the given steps B results in,

$$B = \begin{pmatrix} -3.5 & -3 & -3 \\ 6 & 7 & 7 \\ -5.5 & -6 & -6 \\ 0 & 0 & 0 \end{pmatrix}.$$

This can be verified using the following MatLab code,

```
        a

a =

     1    -1     0     0
     0     1     0     0
     0    -1     1     0
     0    -1     0     1

b

b =

     1     0     1     0
     0     1     0     0
     0     0     1     0
     0     0     0     1

c

c =

    1.0000         0         0         0
         0    1.0000         0         0
         0         0    0.5000         0
         0         0         0    1.0000

d

d =

     2     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1

e

e =

     0     0     0     1
```

```
     0     1     0     0
     0     0     1     0
     1     0     0     0

f

f =

     1     0     0     0
     0     1     0     0
     0     0     1     1
     0     0     0     0

g

g =

     0     0     0
     1     0     0
     0     1     0
     0     0     1

B

B =

     1     2     3     4
     5     6     7     8
     9     1     2     3
     5     6     7     8

a*b*c*B*d*e*f*g

ans =

   -3.5000   -3.0000   -3.0000
    6.0000    7.0000    7.0000
   -5.5000   -6.0000   -6.0000
         0         0         0

exit
```

□

Problem 2 *Exercise 2.1*

Solution. Let A be an $m \times m$ triangular and unitary matrix (square since unitary). Since A is nonsingular the determinant of A is nonzero. As A is triangular, the determinant is the product of the elements along the main diagonal. Thus $a_{ij} \neq 0$ for all $i = j$. Without loss of generality, assume A is upper triangular. First lets show that A^{-1} is also upper triangular. Let $A^{-1} = B$ where b_{ij} are the elements of B. For the sake of contradiction, assume that B is not upper triangular. Than there exists some $b_{ij} \neq 0$ for $i > j$. Let b_{il} be the first nonzero element in its row. Let $AB = C = c_{ij}$ and notice that,

$$c_{il} = b_{i1}a_{1k} + \cdots + b_{il}a_{ll} + \cdots + b_{im}a_{ml} = 0 + b_{il}a_{ll} + \cdots + b_{im}a_{ml}.$$

But A is upper triangular, so $a_{ij} = 0$ for all $i > j$. Thus all the terms after $b_{il}a_{ll}$ vanish leaving us with $c_{il} = b_{il}a_{ll} \neq 0$. Since $i > l$, we have a nonzero element off the main diagonal on C which is a contradiction since $AB = AA^{-1} = I$. Thus A^{-1} is upper triangular. Note that a similar argument holds when A is lower triangular. Since A is unitary, $A^* = A^{-1}$ which gives that A^* is also upper triangular. But A^* is the conjugate transpose of A and thus must be lower triangular. This implies that A^* is diagonal, meaning that $\forall i, j$ such that $i \neq j$, $\bar{a}_{ij} = 0$. Therefore, A is also diagonal. The same follows when A is lower triangular.
□

Problem 3 *Exercise 2.2*

Solution. Let x_i be a set of n orthogonal vectors. We wish to show that

$$||\sum_{i=1}^{n} x_i||^2 = \sum_{i=1}^{n} ||x_i||^2.$$

a. First lets prove the case $n = 2$ through explicit computation of $||x_1 + x_2||^2$. We can see that,

$$\begin{aligned}
||\sum_{i=1} nx_i||^2 &= ||x_1 + x_2||^2 \\
&= \sqrt{(x_1 + x_2)^*(x_1 + x_2)}^2 \\
&= (x_1 + x_2)^*(x_1 + x_2) \\
&= x_1^*x_1 + x_1^*x_2 + x_2^*x_1 + x_2^*x_2 \\
&= x_1^*x_1 + x_2^*x_2 \\
&= ||x_1||^2 + ||x_2||^2 \\
&= \sum_{i=1}^{n} ||x_i||^2
\end{aligned}$$

where $x_1^*x_2 = x_2^*x_1 = 0$ since the vectors are orthogonal.

b. From Part A, we have shown the base case. If we assume $||\sum_{i=1}^{n} x_i||^2 = \sum_{i=1}^{n} ||x_i||^2$ for the n case, then we have,

$$\begin{aligned}
||\sum_{i=1}^{n+1} x_i||^2 &= ||\sum_{i=1}^{n} x_i + x_{n+1}||^2 \\
&= ||\sum_{i=1}^{n} x_i||^2 + ||x_{n+1}||^2 \\
&= \sum_{i=1}^{n} ||x_i||^2 + ||x_{n+1}||^2 \quad \text{(by induction)} \\
&= \sum_{i=1}^{n+1} ||x_i||^2
\end{aligned}$$

and thus we have shown the general case using induction.

□

Problem 4 *Exercise 2.3*

Solution. Let $A \in \mathbb{C}^{m \times m}$ be Hermitian. An eigenvector of A is nonzero vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

a. First we want to show that all eigenvalues of A are real, i.e. $\lambda = \lambda^*$. Consider,

$$\begin{aligned}
\lambda||x||^2 &= \lambda(x^* x) \\
&= x^*(\lambda x) \\
&= x^*(Ax) \\
&= x^* A^* x \quad (\text{since} A = A^*) \\
&= (Ax)^* x \\
&= (\lambda x)^* x \\
&= \lambda^*(x^* x) \\
&= \lambda^* ||x||^2
\end{aligned}$$

and since $||x||^2 = ||x||^2$, we have shown that $\lambda = \lambda^*$. Thus λ must be real and so all the eigenvalues of A are real.

b. Let $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ where $\lambda_1 \neq \lambda_2$. We want to show that x and y are orthogonal, i.e. $x^* y = 0$. Consider that,

$$\begin{aligned}
\lambda_1 y^* x &= y^*(\lambda_1 x) \\
&= y^* A x \\
&= (y^* A^*) x \\
&= (\lambda_2 y)^* x \\
&= y^* \lambda_2 x
\end{aligned}$$

So we have that $\lambda_1 y^* x = y^* \lambda_2 x \implies (\lambda_1 - \lambda_2) y^* x = 0$. Since $\lambda_1 \neq \lambda_2$, we have shown that $y^* x = 0$ and thus x, y are orthogonal.

□

Problem 5 *Exercise 2.6*

Solution. Let $u, v \in \mathbb{C}^m$ such that u, v are nonzero. Consider the matrix $A = I + uv^*$. Suppose that A is singular. Then $Ax = 0$ for some $x \in C^m$ such that $x \neq 0$. This allows us to see that x is some scalar multiple of u by observing,

$$\begin{aligned} xA &= 0 \\ x(I + uv^*) &= 0 \\ x + u(v^* x) &= 0 \\ x &= -u(v^* x) \\ x &= \alpha u \end{aligned}$$

for some scalar α. Therefore, $\alpha u + u(v^* \alpha u) = \alpha u(1 + uv^*) = 0$ which implies that $uv^* = -1$. Thus A will be singular when $uv^* = -1$ and we have that $\text{null}(A) = \{\alpha u : \alpha \in \mathbb{R}\}$. Next suppose that A is nonsingular. Then we wish to show that $A^{-1} = I + \alpha uv^*$. If we let $A^{-1} = [a_1, \ldots, a_m]$ where a_i are vectors, then

$$AA^{-1} = (I + uv^*)[a_1, \ldots, a_m] = [a_1 + uv^* a_1, \ldots, a_m + uv^* a_m]$$

Since $AA^{-1} = I$ by definition, we know that $a_i + u(v^* a_i) = e_i$. If we let $v^* a_i = c_i$ for some scalar c_i, we have $a_u + uc_i = e_i$ which gives $a_i = e_i - uc_i$ for $1 \leq i \leq m$. Let $c = c_i$. Then,

$$\begin{aligned} &I = AA^{-1} = (I + uv^*)(I - uc^*) = I - uc^* + uv^* - uv^* uc^* \\ \implies &0 = uv^* - uc^*(1 + uv^*) \\ \implies &uc^*(1 + uv^*) = uv^* \\ \implies &c^* = \frac{v^*}{1 + uv^*} \end{aligned}$$

Therefore we have that $A^{-1} = I - \frac{uv^*}{1+uv^*}$. By letting $\alpha = -\frac{1}{1+uv^*}$, then indeed $A^{-1} = I + \alpha uv^*$.
□