Math 584 Homework 2
Due Next Wednesday
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Problem 1 *Exercise 3.2*

Solution. Let $A \in \mathbb{C}^{m\times m}$ and let $x \neq \vec{0}$ be the eigenvector corresponding to the largest eigenvalue of A, λ. Let $\rho(A)$ is the spectral radius of A. Notice that by letting $||\cdot||$ be any norm of $\mathbb{C}^m$ we get,

$$\begin{aligned}
\lambda x &= Ax \\
||\lambda x|| &= ||Ax|| \\
|\lambda|||x|| &= ||Ax|| \\
|\lambda| &= \frac{||Ax||}{||x||} \\
|\lambda| &\leq \sup_{||x||=1} \frac{||Ax||}{||x||} = ||A||
\end{aligned}$$

Therefore $\rho(A) \leq ||A||$. $\square$

Problem 2 *Exercise 3.3*

Solution. Let $x \in \mathbb{C}^m$ and $A \in \mathbb{C}^{m\times n}$

a. We can show that $\|x\|_\infty \leq \|x\|_2$ by considering that, $\|x\|_\infty = \max_{1\leq i\leq m}|x_i| \implies (\|x\|_\infty)^2 = \max_{1\leq i\leq m}|x_i^2|$ but $(\|x\|_\infty)^2 \leq \sum_{j=1}^m (x_j^2)$ so $\|x\|_\infty \leq \left(\sum_{j=1}^m (x_j^2)\right)^{\frac{1}{2}} = \|x\|_2$. An example of equality is when $x = e_1$. Then $\|x\|_\infty = 1 = \|x\|_2$.

b. We can show that $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$ when considering that $\|x\|_2 = \left(\sum_{j=1}^m |x_j|\right)^{\frac{1}{2}} = (|x_1|^2 + |x_2|^2 + \cdots + |x_m|^2)^{\frac{1}{2}} \leq (\max_{1\leq i\leq m}|x_i|^2 + \max_{1\leq i\leq m}|x_i|^2 + \cdots + \max_{1\leq i\leq m}|x_i|^2)^{\frac{1}{2}} = \sqrt{m}\|x\|_\infty$. An example of equality is when $x = (1,1)^T$ then $\|x\|_2 = \sqrt{2} = \sqrt{2}(1) = \|x\|_\infty$.

c. We can show that $\|A\|_\infty \leq \sqrt{n}\|A\|_2$ by recalling that $\|A\|_\infty = \max_{1\leq j\leq m}\|a_i\|_1$. From (a) we have that $|Ax|_\infty \leq \|Ax\|_2$ and from (b) we have that $\frac{1}{\sqrt{n}}\|x\|_2 \leq \|x\|_\infty$. Combining these facts we get

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \frac{\|Ax\|_2}{\frac{1}{n}\|x\|_2}$$

taking the supremum of both sides we get $\|A\|_\infty \leq \sqrt{n}\|A\|_2$. An example of equality is given when $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then the max row sum is 2 and the two norm of A is $\sqrt{2}$, thus we have that $\|A\|_\infty = 2 = \sqrt{2}\|A\|_2 = \sqrt{2}\sqrt{2} = 2$.

d. To show that $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$ recall that (a) states that $\|x\|_\infty \leq \|x\|_2$ and (b) gives that $\|Ax\|_2 \leq \sqrt{m}\|Ax\|_\infty$. Using these two facts we have that

$$\frac{\|Ax\|_2}{\|x\|_2} \leq \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_\infty}.$$

Taking the supremum of both sides we get $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$. An example of equality is given by the 2×2 identity matrix, I_2. Since I_2 is diagonal, we know that $\|I_2\|_2 = \sqrt{1+1} = \sqrt{2}$. And the max row sum of I_2 is 1, thus we have $\sqrt{2} = 1\sqrt{2}$. $\|A\|_2 = \sqrt{2} = \sqrt{2}\|A\|_\infty = \sqrt{2}$

$\square$

Problem 3 *Exercise 4.1*

Solution. We wish to compute the following SVDs

a Consider the matrix

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

Consider Ax, then A stretches values along x_1 by three and flips x_2 before stretching it by 2. Thus A begins by not rotating, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then stretches, $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, and finally rotates to flip x_2, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore we can geometrically construct an SVD to be,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b Consider the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Consider Ax, then A stretches values along x_1 by 2 and x_2 by 3 without rotating. Thus we have that the singular values are $3, 2$. Note that we have to flip the corresponding eigenvectors to match the ordering of the singular values. Therefore we can geometrically construct an SVD to be,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

c Consider the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then Ax maps x_1 to two times x_2 while x_2 is mapped to 0. Thus the singular values are $2, 0$. So A begins by rotating so that x_2 goes to x_1, stretches by the singular values, and then does not rotate. Therefore we can geometrically construct an SVD to be,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

d Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here Ax is mapping x_1 to $x_1 + x_2$ while x_2 goes to zero. This means that we begin by rotating $\pi/4$, then magnifying by $\sqrt{2}$, and then not rotating anymore. Putting these ideas together, we can construct the SVD geometrically to be,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

e Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then Ax scales x_1 and x_2 to $x_1 + x_2$. So we begin by rotating 45 degrees, magnifying by 2, and then finally rotating again to match the basis. Thus we can geometrically construct the SVD to be,

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

$\square$

Problem 4 *Exercise 4.5*

Solution. Consider $A \in \mathbb{R}^{m\times n}$. To show that a real SVD exists, begin by setting $\sigma_1 = \|A\|_2$. Consider $v \in \mathbb{C}^n$ s.t. $\|v\|_2 = 1$ with the form $v = x + iy$ where $x, y \in \mathbb{R}^n$ and let it be the vector that σ_1 greatest magnifies. If either x or y is zero, than the nonzero vector will be the vector with the greatest magnification that is real. If $x \neq 0$ and $y \neq 0$, observe that,

$$\begin{aligned}
\|A\|^2 = \sigma_1^2 &= \frac{\|Av\|^2}{\|v\|^2} \\
&= \frac{A(x+iy)A^*(x+iy)^*}{(x+iy)^*(x+iy)} \\
&= \frac{(A^*x^* + A^*iy^*)(Ax + Aiy)}{(x^* - iy^*)(x+iy)} \\
&= \frac{\|Ax\|^2 + \|Ay\|^2}{\|x\|^2 + \|y\|^2}
\end{aligned}$$

Thus we have that $\sigma_1^2(\|x\|^2 + \|y\|^2) = \|Ax\|^2 + \|Ay\|^2$. By definition we know that,

$$\frac{\|Ax\|^2}{\|x\|^2} \leq \sigma_1^2 \quad \text{and} \quad \frac{\|Ay\|^2}{\|y\|^2} \leq \sigma_1^2$$

which implies that,

$$\|Ax\|^2 \leq \sigma_1^2\|x\|^2 \quad \text{and} \quad \|Ay\|^2 \leq \sigma_1^2\|y\|^2.$$

But since we have shown that $\sigma_1^2(\|x\|^2 + \|y\|^2) = \|Ax\|^2 + \|Ay\|^2$, the inequalities must be equalities. Thus if v is a complex vector, we can choose either the real or complex component as our real vector corresponding to σ_1. Let v_1 be the real vector that is picked. Then $u_1 = Rv_1$ and since σ_1, v_1, and σ_1 are real, so must $u_1 \in \mathbb{R}^m$ with $\|u_1\|_2 = \sigma$. Now consider the extensions of v_1 to an orthonormal basis $\{v_j\}$ of $\mathbb{C}^n$ and u_1 to an orthonormal basis $\{u_j\}$ of $\mathbb{C}^m$. Letting U_1 and V_1 denote the unitary matrices with columns u_j and v_j. Then we have,

$$U_1^* A V_1 = S = \begin{pmatrix} \sigma_1 & w^* \\ 0 & B \end{pmatrix}$$

where B is $(m-1)\times(n-1)$. Thus

$$\left\| \begin{pmatrix} \sigma_1 & w^* \\ 0 & B \end{pmatrix} \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \right\|_2 \geq \sigma_1^2 + w^*w = (\sigma_1^2 + w^*w)^{\frac{1}{2}} \left\| \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \right\|_2$$

we have $\|S\|_2 \geq (\sigma_1^2 + w^*w)^{\frac{1}{2}}$ which means that $w = 0$ since $\|S\|_2 = \|A\|_2 = \sigma_1$. If $m = n = 1$ we have our real SVD. Otherwise we can continue by considering B and taking the first part as our induction hypothesis, we know that B has real SVD $B = U_2\Sigma_2V_2^*$. Which gives that,

$$A = U_1 \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix}^* V_1^*$$

which is also a real SVD of A. This concludes the proof of induction that shows that A will always have a real SVD.

$\square$

Problem 5 *Exercise 5.4*

Solution. Let $A \in \mathbb{C}^{m\times m}$ and let its SVD be $A = U\Sigma V^*$. Note that U, V have linearly independent columns and are unitary. Consider the matrix

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

which is a $2m \times 2m$ Hermitian matrix. Consider that $AV = V\Sigma$ and $A^*U = V\Sigma$. This gives us,

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}\begin{pmatrix} V \\ U \end{pmatrix} = \begin{pmatrix} V\Sigma \\ U\Sigma \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}\begin{pmatrix} -V \\ U \end{pmatrix} = \begin{pmatrix} V\Sigma \\ -U\Sigma \end{pmatrix}$$

Putting these together we get,

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}\begin{pmatrix} V & V \\ U & -U \end{pmatrix} = \begin{pmatrix} V\Sigma & V\Sigma \\ U\Sigma & -U\Sigma \end{pmatrix} = \begin{pmatrix} V & V \\ U & -U \end{pmatrix}\begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}$$

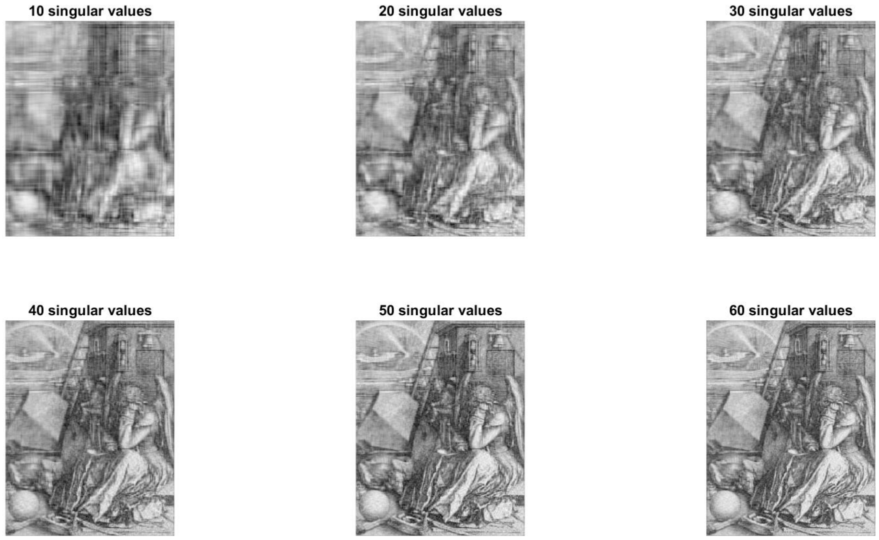
Which implies that

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}\begin{pmatrix} V & -V \\ U & U \end{pmatrix} = \begin{pmatrix} V & -V \\ U & U \end{pmatrix}\begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix}$$

And since V and U are unitary,

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} = \begin{pmatrix} V & -V \\ U & U \end{pmatrix}\begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix}\begin{pmatrix} V & -V \\ U & U \end{pmatrix}^{-1} = X\Lambda X^{-1}$$

Where Σ and $-\Sigma$ are diagonal matrices with eigenvalues as entries. $\square$



Problem 6 *Matlab Fun*

Solution. Using the Matlab code at the end of this problem, I generated the following six images using a different amount of singular values to create a low rank approximation of the image. We can see that at around 30 singular values, the image becomes recognizable. Increasing to 40 singular values gives a sense of most of the details of the image.

```
function hw2()
imagedemo;
%Compute SVD of X
[U,S,V] = svd(X);
for i=1:6
    amount = 10*i;
    X = U(:,1:amount) * S(1:amount,1:amount) * V(:,1:amount)';

    subplot(2,3,i);
    imagesc(X);
    colormap(map);
    axis off;
    axis equal;

    title(amount+" singular values");
end
```

□