AMATH 575
Problem set 4

Working together is welcomed. Please do not refer to previous years' solutions.

I Please be working on your next project presentation, scheduled for May 24 and 26. The final paper will then be due on June 9.

II Consider the "all-to-all" coupled system of pulse-coupled phase oscillators on the N-dimensional torus, with coupling strength $\epsilon > 0$, from class:

$$\dot{\theta}_i = \omega + \epsilon z(\theta_i)\frac{1}{N}\sum_{j=1}^{N} g(\theta_j) \;\; mod \;\; 2\pi \tag{1}$$

$i = 1...N$. Let $z(\theta_i) = A\sin\theta + B\cos\theta$, which we noted in class corresponds to Hopf, generalized Hopf, and to saddle-node on a periodic orbit bifurcations, which are the most common co-dimension 1 bifurcations to periodic orbits. Let $g(\theta) = \sum_{k=1}^{\infty} a_k \sin(k\theta) + b_k \cos(k\theta)$, a totally general "impulse" function describing the coupling from oscillator j. Beginning with the same coordinate transformation as in class, compute the averaged system

$$\dot{\psi}_i = \epsilon\frac{1}{N}\sum_{j=1}^{N} f(\psi_j - \psi_i) \;\; mod \;\; 2\pi \tag{2}$$

Recall that the conclusions of the averaging theorem on how the latter equation approximates the first hold here, making the latter equation a useful approximation. [a] Find a general explicit expression for f, involving the constants A, B, a_k, b_k above for appropriate k. [b] Building from a previous homework, find a general condition on these constants that guarantees that the averaged system will be a gradient dynamical system. [c] Find a general condition on these constants that guarantees that any solution $\psi_i \equiv k \; \forall k$ is a fixed point for the averaged system. These are referred to as synchronized solutions. Compute the Jacobian for these fixed points , and write down the dimension of the stable, unstable, and center manifolds for all possible choices of the constants A, B, a_k, b_k .

III Compute the normal form, up to order two, for a two-dimensional flow with linear part (Jacobian, in real Jordan form)

$$J = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$$

where $\lambda \neq 0$ is an arbitrary real parameter. Note that the case you will study, $\lambda \neq 0$, is for a hyperbolic fixed point that is a saddle or a source. Make sure to cover all of the possible cases; the normal form may differ for different values of λ. As a comment, your result will relate nicely to Sternberg's Theorem (not covered in class): if λ_j are eigenvalues of J, the full flow can be linearized by a diffeomorphism if $\sum_{j=1}^{n} m_j \lambda_j \neq 0$ for all integers m_j.

IV Determine the Takens-Bogdanov normal form to third order.

V In homework 1 we have the Lorenz equations

$$\begin{cases} x' = 10(-x+y) \\ y' = rx - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases} \tag{3}$$

Characterize the bifurcation when $r = 1$.

VI Consider the one-parameter family of one-dimensional maps,

$$x \mapsto x^2 + c, \tag{4}$$

where c is a real-valued parameter.

1. Find the fixed points of this system. For which values of c do they exist? Determine the stability of these fixed points and their dependency on the value of c. Determine if there is a bifurcation, and find the bifurcation point.
2. Focusing on the value $c = -3/4$, compute $f'_{-3/4}(p_-)$, where $f_c(x) = x^2 + c$ and p_- is the smaller of the two fixed points at this value of c. Convince yourself that as c descends through $-3/4$, we see the emergence of an (attracting) 2-cycle. This is a period doubling bifurcation!
3. Solve for the period two points by considering the fixed points of the function $f_c^2(.) = f_c(f_c(.))$, and the domain of c for which the original system has a fixed point.