Applied Complex Analysis

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Prolegomenon

These are the lecture notes for Amath 567: Applied Complex Analysis. This is the second year these notes are typed up, thus it is guaranteed that these notes are full of mistakes of all kinds, both innocent and unforgivable. Please point out these mistakes to me so they may be corrected for the benefit of your successors. If you think that a different phrasing of something would result in better understanding, please let me know.

These lecture notes are not meant to supplant the textbook used with this course. The main textbook is Ablowitz & Fokas "Complex Variables: introduction and applications" (2nd edition, 2003) (Cambridge Texts in Applied Mathematics).

Contents

Chapter 1

Complex Numbers and elementary functions

1.1 Complex Numbers

Complex numbers are introduced so as to get a number system that is closed under the operation of taking rational powers. Clearly, within the realm of the real numbers, taking rational powers is not always allowed, as $(-1)^{1/2}$, for instance, is not defined. It turns out that to create a number system that is closed under taking rational powers, it is sufficient to deal with the above example. We define

$$i := \sqrt{-1},$$

so that $i^2 = -1$.

A *complex number* is an expression of the form

$$z = x + iy.$$

We call $x = \mathrm{Re}(z) \in \mathbb{R}$ the *real part* of z and $y = \mathrm{Im}(z) \in \mathbb{R}$ the *imaginary part*. The set of all complex numbers is denoted by $\mathbb{C}$ and we may write $\mathbb{C} = \mathbb{R} + i\mathbb{R}$. Thus, a complex number is specified by two real numbers. As such, Gauss realized that a complex number can be geometrically represented by a vector in $\mathbb{R}^2$, as illustrated in Fig. 1.1.

We will see here and throughout the course that this analogy goes quite far and is very useful. Of course, the above implies that we can represent complex numbers using polar coordinates in $\mathbb{R}^2$ just as well. Let

$$x = \rho \cos\theta,$$
$$y = \rho \sin\theta.$$

Then $z = \rho(\cos\theta + i\sin\theta)$ and $\rho = \sqrt{x^2 + y^2} := |z|$, the *modulus* or *absolute value* of z. Further, $\theta := \arg z$, the *argument* of z. We have

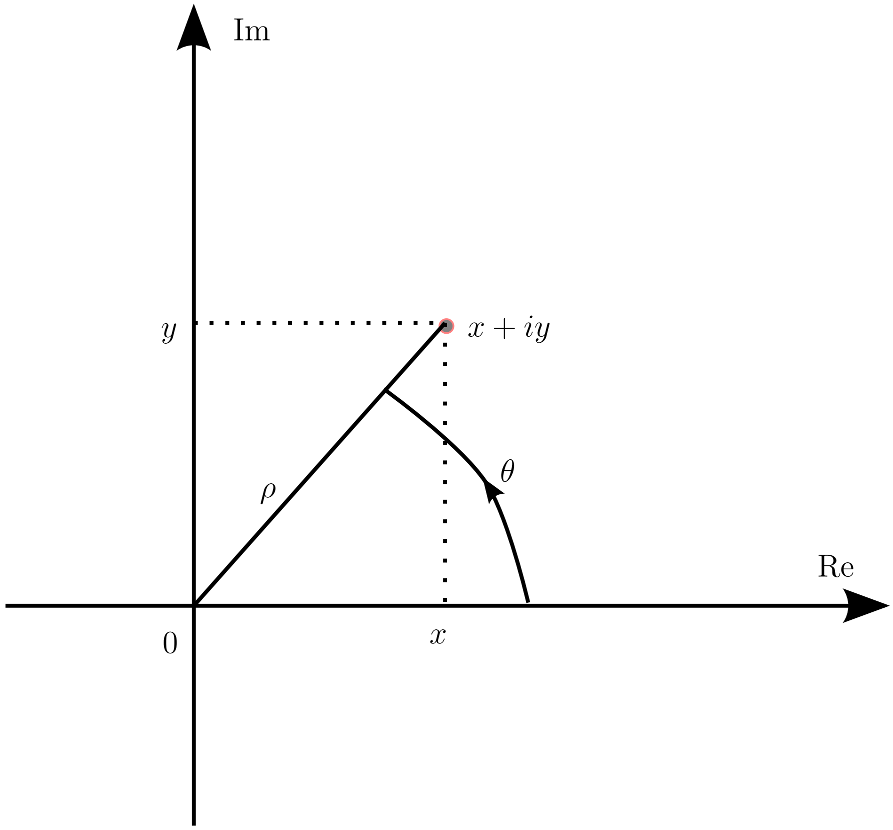

Figure 1.1: The complex plane, as invented by Gauss.

$$
\begin{aligned}
&& \tan\theta &= \frac{y}{x} \\
\Leftrightarrow && \theta &= \alpha\pi + \arctan\left(\frac{y}{x}\right) + 2\pi n,
\end{aligned}
$$

where n is any integer and the quadrant has to be specified, so as to be able to determine whether $\alpha = 0$ or 1.

If $z \in \mathbb{C}$, then the *complex conjugate* $\bar{z}$ is defined as $\bar{z} := x - iy = \rho(\cos\theta - i\sin\theta) = \rho(\cos(-\theta) + i\sin(-\theta))$. These equalities show that the complex conjugate $\bar{z}$ of z is found in the Gaussian complex plane by reflecting z with respect to the real axis.

Note that

$$
\begin{aligned}
z\bar{z} &= (x + iy)(x - iy) \\
&= x^2 + y^2 \\
&= |z|^2.
\end{aligned}
$$

This is useful to determine the real and imaginary part of the division of two complex numbers:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{z_1\overline{z_2}}{z_2\overline{z_2}} \\ &= \frac{z_1\overline{z_2}}{|z_2|^2}.\end{aligned}$$

We will use the triangle inequality quite often.

Theorem 1.1 *For any* $z_1,\ z_2 \in \mathbb{C}$*, we have (i)* $|z_1 + z_2| \leq |z_1| + |z_2|$*, and (ii)* $|z_1 - z_2| \geq ||z_1| - |z_2||$*.*

Proof. We prove (i) first. Consider

$$\begin{aligned}|z_1 + z_2|^2 - (|z_1| + |z_2|)^2 &= \cancel{|z_1|^2} + \cancel{|z_2|^2} + z_1\overline{z_2} + \overline{z_1}z_2 - \cancel{|z_1|^2} - \cancel{|z_2|^2} - 2|z_1||z_2| \\ &= z_1\overline{z_2} + \overline{z_1\overline{z_2}} - 2|z_1||\overline{z_2}| \\ &= 2\left(\mathrm{Re}(z_1\overline{z_2}) - |z_1\overline{z_2}|\right) \\ &\leq 0.\end{aligned}$$

This proves (i). It follows that $|z_1 + z_2| - |z_1| \leq |z_2|$. Let $w_1 := z_1 + z_2$ and $w_2 := z_1$. Then our first inequality becomes $|w_1| - |w_2| \leq |w_1 - w_2|$. Similarly, by switching the indices, we get $|w_2| - |w_1| \leq |w_2 - w_1|$. These two results combined give (ii). ■

The triangle inequality (i) is immediately generalized to

$$\left|\sum_{j=1}^{n} z_j\right| \leq \sum_{j=1}^{n} |z_j|.$$

The geometrical meaning of the triangle inequality (which explains its name) is illustrated in Fig. 1.2: in essence, the inequality states that the length of any side of a triangle is shorter then the sum of the lengths of the other two sides.

At this point, we will run ahead a bit. We define the complex exponential function

$$e^z = e^{x+iy} := e^x(\cos y + i \sin y).$$

For real z (*i.e.*, $y = 0$) this coincides with the real exponential that we know. On the other hand, for imaginary z (*i.e.*, x=0) we have

$$e^{iy} = \cos y + i \sin y,$$

which you may have seen before as Euler's formula, from a course in differential equations, for instance. We will justify the above definition of the complex exponential later. Accepting it for now, we have for $\theta \in \mathbb{R}$:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

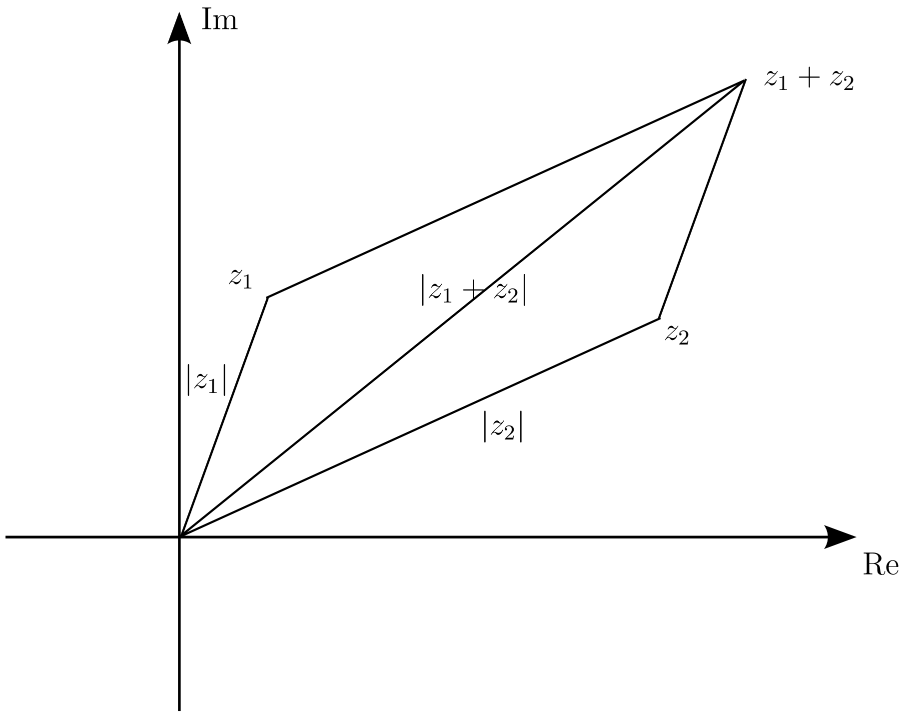

Figure 1.2: Illustrating the triangle inequality.

and

$$\left|e^{i\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

Further

$$\begin{aligned} e^{i(\theta_1+\theta_2)} &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= e^{i\theta_1}e^{i\theta_2}, \end{aligned}$$

as you would want. Using induction, we may prove[1] for any integer m and any real θ that

$$e^{im\theta} = \cos m\theta + i\sin m\theta = (e^{i\theta})^m.$$

Hence, by replacing θ by θ/m $(0 \neq m \in \mathbb{Z})$ we get

$$(e^{i\theta})^{1/m} = e^{i\theta/m}.$$

This gets us to our original goal of being able to take rational powers of complex numbers.

[1] This means that **you** should do this!

Let $z \in \mathbb{C}$. Then for any $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned} z^{1/n} &= \left(\rho e^{i\theta}\right)^{1/n} \\ &= \rho^{1/n}\left[e^{i(\theta+k2\pi)}\right]^{1/n} \\ &= \rho^{1/n} e^{i(\theta+k2\pi)/n}, \end{aligned}$$

where $k = 0, 1, \ldots, n-1$. For integer values of k outside of this range the values obtained merely repeat. It follows from this result that there are (you are not surprised) n distinct roots of the equation $z^n = \alpha$, where n is an integer and $\alpha \neq 0$. Further, these n roots lie on the vertices of a regular polygon of "size" $|\alpha|^{1/n}$ with n sides. Note that the source of the multivaluedness of calculating rational powers is that the argument of a complex number is not uniquely determined. Rather, arguments whose difference is an integer multiple of 2π give rise to the same complex number.

An important special case of the above is that of the roots of unity. These are the solutions of $z^n = 1$. We get

$$z = e^{ik2\pi/n} := \omega^k, \qquad \text{with } \omega := e^{i2\pi/n},$$

and $k = 0, 1, \ldots, n-1$. Thus the roots of unity of order n are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

1.2 Elementary functions of a complex variable

Now that we have gotten arithmetic out of the way, let's see what we will get out of this course. You already know vector calculus. Vector calculus deals with functions of more than one variable. A function of a complex variable takes as input a complex number z and produces as output another complex number w. In other words

$$w = f(z),$$

where $f(\cdot)$ is some prescription specifying how to get w given z. Since $z = x+iy$ is determined by two real numbers x and y, so is w. Hence a function of a complex variable z is a special case of a vector function of two variables. So, what did we gain, if anything?

The fact that a complex function is a vector function that depends only on the combination $x + iy$ and not on x and y individually is the key to the magic you'll encounter in this course! Think of a complex function as a real function that we extend in a very specific way by replacing its variable x by $x + iy$. We will make this more precise and rigorous later on.

Here are some examples of some of the bizarre expressions we will be able to evaluate:

$$\begin{aligned} &\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}, \\ &\sum_{n=1}^{p-1} \frac{1}{\sin^2(n\pi/p)} = \frac{p^2-1}{3}, \qquad 1 < p \in \mathbb{Z}^+ \end{aligned}$$

and many, many more. The consequences and applications of function theory of functions of a complex variable extend much further: fluid mechanics and surface water waves through the use of conformal mappings, asymptotics, integral transforms, new methods (1997) for solving partial differential equations, *etc.*

Before we get there, we need to set up some technical machinery.

Definition 1.2 *An ϵ* neighborhood *of $z_0 \in S \subset \mathbb{C}$ is the set of all $z \in S$ for which $|z-z_0| < \epsilon$, with $\epsilon > 0$.*

Of course, usually we think of ϵ as being small, but that is not necessary.

Definition 1.3 *A point $z_0 \in S$ is called an* interior point *of S if there is an ϵ neighborhood of z_0 entirely contained in S.*

Definition 1.4 *A set S is called* open *if every point of S is an interior point of S.*

Definition 1.5 *A point $z_0 \in S$ is a* boundary point *of S if every neighborhood of z_0 contains at least one point in S and one point not in S.*

Just to avoid giving too many definitions (no peeking ahead!), we use *region* to denote an open set, a closed set, or an open set with some of its boundary points.

Definition 1.6 *A region is called* bounded *if it is entirely contained in a circle of sufficiently large radius.*

Definition 1.7 *A region is* closed *if it contains all of its boundary points.*

Definition 1.8 *A region which is both closed and bounded is called* compact.

Definition 1.9 *A region is called* connected *if any two points α and β in it can be joined by a sequence of line segments $[\alpha z_1]$, $[z_1 z_2]$, $\ldots$, $[z_N \beta]$ (where $[z_j z_k]$ denotes the line segment from z_j to z_k, including the end points).*

Definition 1.10 *A* domain *is a connected open set.*

Given a region $\mathcal{R}$, we denote the boundary of $\mathcal{R}$ by $\partial\mathcal{R}$, and the closure of $\mathcal{R}$ by $\overline{\mathcal{R}} = \mathcal{R} \cup \partial\mathcal{R}$, which should not be confused with the complex conjugate. Hopefully this will be clear from context. I won't make any promises. Knock on wood. Fingers crossed.

Are we definitioned© out yet? Too bad. There's plenty more. Now we define a function of a complex variable.

Definition 1.11 *For each $z \in \mathcal{R}$, assign a unique complex number w. Then $w = f(z)$ is a function of the complex variable z.*

Definition 1.12 *In the above definition of a function of a complex variable, the $\mathcal{R}$ is referred to as the* domain of definition *of f.*

Definition 1.13 *The totality of values w achieved for $z \in \mathcal{R}$ is called the* range *of f.*

After this boatload of definitions, we'll look at a shipload of examples.

Example (Power functions). Let

$$f(z) = z^n, \qquad n = 0, 1, 2, \ldots.$$

Here the domain of definition is $\mathbb{C}$ and the range is also $\mathbb{C}$.

Example (Polynomials). Here

$$f(z) = \sum_{j=0}^{n} a_j z^j, \qquad n = 0, 1, 2, \ldots.$$

The complex numbers a_j, $j = 0, 1, \ldots, n$ are referred to as the coefficients of the polynomial. The domain of definition of $f(z)$ is $\mathbb{C}$ and its range is $\mathbb{C}$ as well. Actually, this last statement is the *fundamental theorem of algebra* in disguise (barely), which we'll come back to later.

Example (Rational functions). Consider

$$R(z) = \frac{P_n(z)}{Q_m(z)},$$

where $P_n(z)$ is a polynomial of degree n and $Q_m(z)$ is a polynomial of degree m. The domain of definition of $R(z)$ is $\mathbb{C}/\{\text{roots of } Q_m(z)\}$. Its range is $\mathbb{C}$.

As a more explicit example consider

$$R = \frac{1}{1+z^2}.$$

Thus its domain of definition is $\mathbb{C}/\{i, -i\}$. In order to find the real and imaginary parts of $w = f(z)$, we let $z = x + iy$, $w = u + iv$. Here and in what follows we will always assume that this suggestive notation indeed implies that x, y, u and v are all real. Thus

$$\begin{aligned} u + iv &= \frac{1}{1+(x+iy)^2} \\ &= \frac{1}{1+x^2-y^2+2ixy} \\ &= \frac{1+x^2-y^2-2ixy}{(1+x^2-y^2)^2+(2xy)^2}, \end{aligned}$$

so that it follows that

$$\begin{aligned} u &= \frac{1+x^2-y^2}{(1+x^2-y^2)^2+4x^2y^2}, \\ v &= \frac{-2xy}{(1+x^2-y^2)^2+4x^2y^2}. \end{aligned}$$

Example (Exponential function). We have already introduced

$$w = e^z = e^x(\cos y + i \sin y).$$

From this definition it is clear that $z \in \mathbb{C}$. It is perhaps less obvious that also $w \in \mathbb{C}/\{0\}$. You should check this. While doing so, you are of course establishing that the domain of definition of the complex logarithm is $\mathbb{C}/\{0\}$. All in due time!

Example (Trigonometric functions). We define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2},$$
$$\sin z := \frac{e^{iz} - e^{-iz}}{2i},$$

and the other trig functions are defined from these per the usual definitions: $\tan z := \sin z / \cos z$, *etc.* Note that the above definitions coincide with the familiar definitions when $z = x \in \mathbb{R}$. All of the familiar identities follow, as expected:

$$\cos^2 z + \sin^2 z = 1,$$

and so on. It is easy to check that $\sin z$ and $\cos z$ are periodic in the real direction with period 2π, while they grow exponentially in the imaginary direction. Thus, geometrically, we could divide $\mathbb{C}$ in vertical strips of width 2π. Then whatever happens in one strip with these functions is repeated in the others by horizontal translation.

Example (Hyperbolic functions). These are perhaps a bit less familiar. For shame! We define

$$\cosh z := \frac{e^z + e^{-z}}{2},$$
$$\sinh z := \frac{e^z - e^{-z}}{2},$$

and the other hyperbolic functions as usual. Note the relationships

$$\sinh z = -i \sin iz,$$
$$\cosh z = \cos iz.$$

Thus, up to constants, hyperbolic functions are trig functions with their arguments rotated by $\pi/2$, since $i = e^{i\pi/2}$. It follows that everything that is true in the complex plane for trigonometric functions is true for hyperbolic functions, but rotated. Thus, hyperbolic functions are periodic in the imaginary direction of the complex plane, with period $2\pi i$. This raises an interesting question: would it be possible to devise a function in $\mathbb{C}$ that is periodic in both the real and the imaginary direction? We'll come back to this.

So far, we have defined the elementary functions by specifying their real and imaginary parts. Another philosophy is to define them using series expansions: a power series expansion around z_0 is an expression of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

Here z_0 and a_n $(n = 0, 1, \ldots)$ are constants. If this series converges for $z \in S$, then it may be used to define a function there. We will say **much** more about power series later, but for now we can say, just like for power series of a real variables, that the series converges with radius of convergence

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

which follows from the ratio test. Then the series converges for $z \in S = \{z \in \mathbb{C} : |z-z_0| < \rho\}$.

We give some elementary and familiar-looking power series:

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\ \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \\ \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}. \end{aligned}$$

For all three series, the radius of convergence $\rho = \infty$.

The extended complex plane

The complex plane as we have treated it so far is not compact, since it is open and unbounded. For many applications it is convenient to compactify the complex plane by adding to it the point at infinity. An ϵ neighborhood of the point at infinity (denoted z_∞ or simply ∞) consists of all points satisfying $|z| > 1/\epsilon$, for small values of ϵ. Another way of doing this is to let $z = 1/t$, after which ∞ is mapped to 0, and all proceeds as before.

Stereographic projection

One popular[2] way of compactifying the complex plane is to use *stereographic projection*. This is illustrated in Fig. 1.3. We put a sphere of radius 1 on top of the complex plane, with its south pole S at the origin. Thus, in three-dimensional coordinates

$$P(0) = S = (0,0,0) \quad \text{and} \quad P(\infty) = N = (0,0,2).$$

Similarly,

$$z = (x, y, 0), \quad \text{and} \quad P(z) := (X, Y, Z).$$

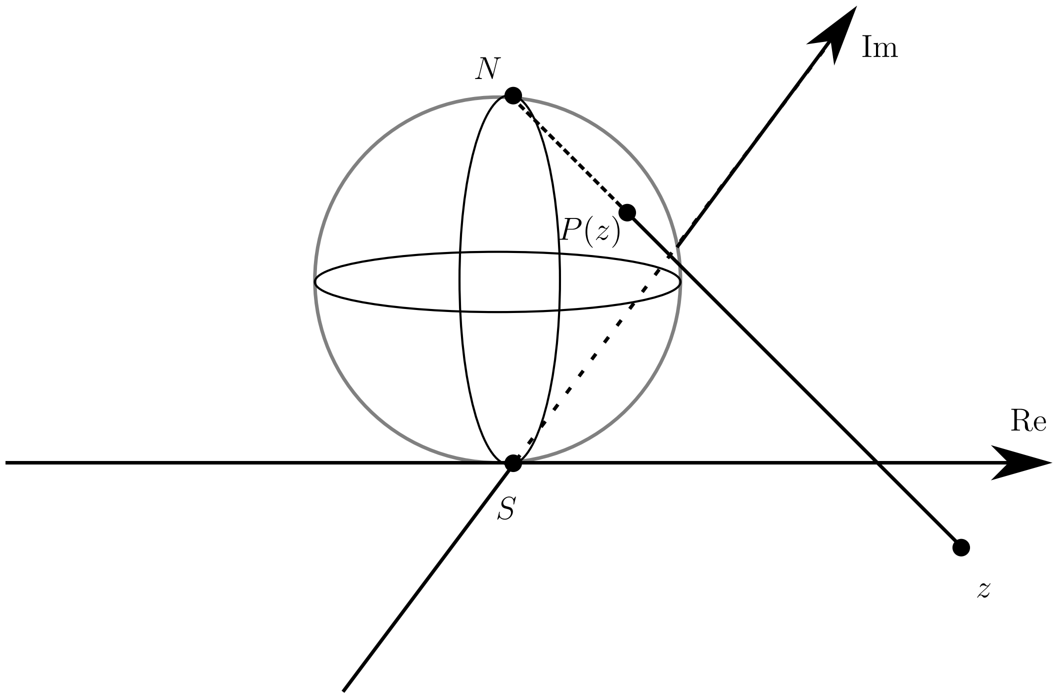

Figure 1.3: Stereographic projection.

We construct the stereographic projection $P(z)$ of z by intersecting the line from z to N with the sphere. This intersection point is the stereographic $P(z)$ of z. Thus, N, $P(z)$ and z lie on a straight line. It follows that the line segment from z to N is proportional to the segment from $P(z)$ to N:

$$\begin{aligned} & s[(x,y,0)-(0,0,2)] = (X,Y,Z)-(0,0,2) \\ \Rightarrow \quad & s(x,y,-2) = (X,Y,Z-2) \\ \Rightarrow \quad & sx = X, \quad sy = Y, \quad -2s = Z-2. \end{aligned}$$

In the above, s is the constant of proportionality. We get $Z = 2-2s$. Expressing that $P(z)$ lies on the sphere, we get

$$\begin{aligned} & X^2+Y^2+(Z-1)^2 = 1 \\ \Rightarrow \quad & s^2(x^2+y^2+4) = 4s \\ \Rightarrow \quad & s = \frac{4}{|z|^2+4}. \end{aligned}$$

[2]As popular as these things get.

It follows that the stereographic projection can be explicitly written as

$$X = \frac{4\mathrm{Re}z}{|z|^2+4},$$
$$Y = \frac{4\mathrm{Im}z}{|z|^2+4},$$
$$Z = \frac{2|z|^2}{|z|^2+4}.$$

Every point on the sphere corresponds to a unique point in $\mathbb{C}$, except the north pole which corresponds to the point at infinity, approached from any direction in the complex plane. The formulae above also show this: given a point on the sphere, $|z|^2$ is determined from the third equation. Once $|z|^2$ is known, the real and imaginary parts of z are obtained using the first and second equation, respectively.

1.3 Limits, continuity and differentiation

Consider $w = f(z)$, defined in a neighborhood of z_0, except maybe at z_0 itself. We say that

$$\lim_{z\to z_0} f(z) = w_0$$

if for all $\epsilon > 0$ sufficiently small, there is a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \epsilon.$$

If z_0 is a boundary point of the domain of definition, we impose that the above definition holds only for paths of approach of z to z_0 that lie entirely in the domain of definition. This is a generalization of the idea of a one-sided limit.

Example. Let's verify that

$$\lim_{z\to -2i} \frac{3z^2 - z + 6zi - 2i}{z+2i} = -6i - 1.$$

We have to show that for a given ϵ, there is a δ such that

$$0 < |z + 2i| < \delta$$

implies

$$\begin{aligned}
& & \left|\frac{3z^2 - z + 6zi - 2i}{z+2i} + 6i + 1\right| &< \epsilon \\
&\Leftrightarrow & \left|\frac{(3z-1)\cancel{(z+2i)}}{\cancel{(z+2i)}} + 6i + 1\right| &< \epsilon \\
&\Leftrightarrow & |3z + 6i| &< \epsilon \\
&\Leftrightarrow & |z + 2i| &< \frac{\epsilon}{3}.
\end{aligned}$$

It follows that if we choose $\delta < \epsilon/3$, the condition is satisfied and we have verified the limit statement.

We extend the limit concept to the point at infinity in the following way: we say that

$$\lim_{z\to\infty} f(z) = w_0,$$

if for sufficiently small $\epsilon > 0$ there is a $\delta > 0$ such that

$$|z| > \frac{1}{\delta} \quad \Rightarrow \quad |f(z) - w_0| < \epsilon.$$

You can verify that the usual algebraic properties hold: the sum of the limits is the limit of the sums provided that all limits involved exist, *etc.*

A function is called *continuous* at z_0 if

$$f(z_0) = \lim_{z\to z_0} f(z).$$

In terms of ϵ and δ, we have that $f(z)$ is continuous at $z = z_0$ if

$$\forall\epsilon > 0 \;\; \exists\delta > 0 : \;\; |z - z_0| < \delta \;\Rightarrow\; |f(z) - f(z_0)| < \epsilon.$$

The same algebraic properties hold that hold for limits hold for continuous functions: the sum of continuous functions is continuous, the product of continuous functions is continuous, *etc.*

It follows immediately from the definition (since all is phrased in terms of absolute values) that if $f(z)$ is continuous at z_0 then so is $\overline{f(z)}$. Thus, if $f(z)$ is continuous at z_0, so are $\overline{f(z)}$, $\text{Re}f(z)$, $\text{Im}f(z)$ and $|f(z)|^2$, since all of these are sums and products of continuous functions in this case.

We say that a function is continuous in a region if it is continuous at every point in the region. Typically, δ depends on which point z_0 in the region is being considered: $\delta = \delta(\epsilon, z_0)$. If δ can be chosen to be independent of z_0, we say $f(z)$ is *uniformly continuous* in the region.

Let $f(z)$ be defined in $\mathcal{R}$, which contains a neighborhood of z_0. Then

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0),$$

is called the derivative of $f(z)$ at z_0 if this limit exists. If so, then also

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

The function $f(z)$ is *differentiable* in a region if its derivative at every point in the region is defined.

Theorem 1.14 *If $f(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0.*

Proof.

$$\begin{aligned}\lim_{z\to z_0}(f(z)-f(z_0)) &= \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)(z-z_0)\\ &= \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)\lim_{z\to z_0}(z-z_0)\\ &= f'(z_0)\cdot 0 = 0.\end{aligned}$$

This proves the theorem. ■

It appears that for many of these proofs, it suffices to repeat the proof of the real case. This is correct, to some extent. However, there are important consequences of being in two dimensions.

Example. Consider $f(z)=\overline{z}$. We know that this function is continuous for every $z_0\in\mathbb{C}$. However,

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}\frac{\cancel{\overline{z}}+\overline{h}-\cancel{\overline{z}}}{h}\\ &= \lim_{h\to 0}\frac{\overline{h}}{h}.\end{aligned}$$

Let $h=\epsilon e^{i\theta}$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}\frac{\overline{h}}{h}\\ \lim_{\epsilon\to 0}\frac{\cancel{\epsilon}e^{-i\theta}}{\cancel{\epsilon}e^{i\theta}}\\ &= e^{-2i\theta}.\end{aligned}$$

Thus the limit is not defined, since the result depends on the direction along which z is approached. It follows that $f(z)=\overline{z}$ is nowhere differentiable. This is reminiscent of the case of a function of two variables like

$$f(x,y)=\frac{x^2}{x^2+y^2}.$$

Here the limit as $(x,y)\to(0,0)$ is undefined. Reminiscent but not quite in the same league: the above example $f(z)=\overline{z}$ is nowhere differentiable!

We will see later that the differentiability of a complex function is what gives

us all the goodies that we wish to benefit from.

It is easy to show that

$$
\begin{aligned}
(f(z)+g(z))' &= f'(z)+g'(z), \\
(f(z)g(z))' &= f'(z)g(z)+f(z)g'(z), \\
[f(g(z))]' &= f'(g(z))g'(z).
\end{aligned}
$$

Of course, we also get the regular differentiation formulae for the elementary functions we defined before:

$$
\begin{aligned}
(z^n)' &= nz^{n-1}, \\
\frac{d}{dz}e^z &= e^z, \\
\frac{d}{dz}\sin z &= \cos z, \\
\frac{d}{dz}\cos z &= -\sin z.
\end{aligned}
$$

1.4 Exercises

1. solve $z^2+z+1=0$ for z, writing the solutions explicitly with real and imaginary parts.

2. Consider the complex-valued expression

   $$f(z) = z^{1/2},$$

   where $z = x+iy$, with x, $y \in \mathbb{R}$. Derive explicit expressions for the real and imaginary part(s) of $f(z)$ in terms of x and y. If you make any choices (*e.g.* for branch cuts), show how they impact your answer. Your answer should not contain any trig functions.

3. Prove the general triangle inequality

   $$\left|\sum_{j=1}^{N} z_j\right| \leq \sum_{j=1}^{N} |z_j|.$$

   What is the condition for equality?

4. Consider the function $E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$, which is defined for all $z \in \mathbb{C}$ (easy check, so easy you don't even have to do it). Show (without using that you've conveniently noticed that $E(z) = e^z$) that $E(z_1+z_2) = E(z_1)E(z_2)$. Can you find other power series with the same property?

5. Examine the function $f(z) = e^{-z^2}$ in the complex plane: where is $f(z)$ real, imaginary, growing, decaying?

6. (**Cardano's formula for the solution of the cubic**) Consider the cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

where a, b and c are given numbers.

- Use the change of variables $x = y - a/3$ to reduce the equation to the form

  $$y^3 + py + q = 0.$$

  Find expressions for p and q.
- Let $y = u + v$. We're replacing one unknown with two, so we get to impose another constraint later. Check that

  $$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

- Now we impose $3uv + p = 0$, so that

  $$u^3v^3 = -p^3/27.$$

  Also, from above, we have

  $$u^3 + v^3 = -q.$$

  Find a quadratic equation satisfied by both u^3 and v^3.
- Solve this quadratic equation, finding expressions for u and v.
- Finally, obtain an expression for x. How many different solutions does your expression give rise to?
- Use your result to solve the cubic $x^3 - 2x^2 + x - 12 = 0$.
- (**Bombelli's equation**) Use your result to solve the cubic $x^3 - 15x - 4 = 0$, writing your result explicitly in terms of real and imaginary parts. The observation $(2 + i)^3 = 2 + 11i$ may be useful.

7. Find all zeroes of $\cosh(z)$, $z \in \mathbb{C}$. What can you conclude about the zeroes of $\cos(z)$, $z \in \mathbb{C}$?

8. Consider $f_\epsilon(z) = 1/(\epsilon^2 + z^2)$, where ϵ is a small positive number, and $z \in \mathbb{C}/\{i\epsilon, -i\epsilon\}$. Plot $|f_\epsilon(z)|$, for various values of ϵ. Discuss the influence the singularities of a function in the complex plane have on its behavior on the real line.

9. Prove that the function

$$\phi(z) = w := i\frac{1 - z}{1 + z}$$

maps the set $\{z \in \mathbb{C} : |z| < 1\}$ one-to-one onto the upper-half plane $\mathrm{Im}(w) > 0$, where Im denotes the imaginary part. The map ϕ is called the Cayley transform.

Chapter 2

Analytic functions and integration

2.1 Analytic functions

The Cauchy-Riemann equations

We have seen that $f(z)$ is differentiable at z if the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

Definition 2.1 *The function $f(z)$ is* analytic *at z_0 if $f(z)$ is differentiable in a neighborhood of z_0. The function $f(z)$ is analytic in an open set S if it is analytic at every point of S.*

Example. e^z is analytic in $\mathbb{C}$. The function $1/z^2$ is analytic in $\mathbb{C}$ except at $z = 0$.

Definition 2.2 *The function $f(z)$ is* entire *if $f(z)$ is analytic in $\mathbb{C}$.*

Thus, the exponential is an entire function. Next, we examine some consequences of analyticity.

Note that the limit is independent of the path taken by $h \to 0$. Let

$$f(z) = u(x, y) + iv(x, y).$$

First we consider $h = \Delta x$, *i.e.*, h is restricted to real values. Then

$$\begin{aligned} f'(z) = & \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} \\ & \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\ & = u_x + iv_x. \end{aligned}$$

Next we consider $h = i\Delta y$, *i.e.*, h is restricted to imaginary values.

$$
\begin{aligned}
f'(z) &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{i\Delta y} \\
&\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(y,y)}{i\Delta y} \\
&= v_y - iu_y.
\end{aligned}
$$

Since these two results are supposed to be equal, we get

$$
\begin{aligned}
& & u_x + iv_x = v_y - iu_y \\
\Leftrightarrow & & u_x = v_y, \quad v_x = -u_y.
\end{aligned}
$$

These are the *Cauchy-Riemann equations.* If, in addition, $u(x, y)$ and $v(x, y)$ are twice differentiable, then

$$
\begin{aligned}
& & u_{xx} = v_{yx} = -u_{yy}, \\
& & v_{xx} = -u_{yx} = -v_{yy}, \\
\Rightarrow & & \nabla^2 u = 0 = \nabla^2 v.
\end{aligned}
$$

A function that satisfies Laplace's equation is called *harmonic*. Thus both the real and imaginary parts of $f(z)$ are harmonic, provided that $f(z)$ is analytic.

Next we show that the Cauchy-Riemann conditions are not only necessary conditions from analyticity, they are also sufficient conditions for analyticity.

Theorem 2.3 *The function $f(z) = u(x, y) + iv(x, y)$ is analytic at $x + iy$ if and only if u_x, v_x, u_y and v_y are continuous and satisfy the Cauchy-Riemann conditions.*

Proof. We've already proven that the conditions are necessary. Let's prove sufficiency. We have

$$
\begin{aligned}
\Delta u &= u_x \Delta x + u_y \Delta y + \epsilon_1 |\Delta z|, \\
\Delta v &= v_x \Delta x + v_y \Delta y + \epsilon_2 |\Delta z|,
\end{aligned}
$$

where $|\Delta z| := \sqrt{(\Delta x)^2 + (\Delta y)^2}$ and $\lim_{\Delta z \to 0} \epsilon_{1,2} = 0$. Thus

$$
\begin{aligned}
\frac{\Delta f}{\Delta z} &= \frac{\Delta u + i\Delta v}{\Delta z} \\
&= \frac{u_x \Delta x + u_y \Delta y + iv_x \Delta x + iv_y \Delta y + \epsilon_1 |\Delta z| + i\epsilon_2 |\Delta z|}{\Delta z} \\
&= \frac{(u_x + iv_x)\Delta x - v_x \Delta y + iu_x \Delta y + \epsilon_1 |\Delta z| + i\epsilon_2 |\Delta z|}{\Delta z} \\
&= \frac{(u_x + iv_x)(\Delta x + i\Delta y)}{\Delta z} + (\epsilon_1 + i\epsilon_2)\frac{|\Delta z|}{\Delta z},
\end{aligned}
$$

where we have used the Cauchy-Riemann conditions. Let $\Delta z = \epsilon e^{i\theta}$. Then

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = u_x + i v_x + \lim_{\Delta z \to 0} (\epsilon_1 + i\epsilon_2) e^{-i\theta} = f'(z).$$

This concludes the proof. ■

Note that since analyticity requires differentiability in a neighborhood, the Cauchy-Riemann equations only show analyticity at a point if the equality holds in a neighborhood of the point.

There are many consequences of the above theorem. One is that the level curves of $u(x, y)$ and $v(x, y)$ in the (x, y) plane are orthogonal.

Theorem 2.4 *The level curves $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are constants, are orthogonal at all points where $f'(z)$ exists and is not zero.*

Proof. Consider

$$\begin{aligned} 0 \neq |f'(z)|^2 &= u_x^2 + v_x^2 \\ &= u_x^2 + u_y^2 \\ &= |\nabla u|^2, \end{aligned}$$

or

$$\begin{aligned} 0 \neq |f'(z)|^2 &= u_x^2 + v_x^2 \\ &= v_y^2 + v_x^2 \\ &= |\nabla v|^2. \end{aligned}$$

We know that ∇u (∇v) is perpendicular to the level curves $u(x, y) = c_1$ $(v(x, y) = c_2)$. Also, we have that

$$\begin{aligned} \nabla u \cdot \nabla v &= u_x v_x + u_y v_y \\ &= u_x v_x - v_x u_x \\ &= 0. \end{aligned}$$

Thus the gradients of u and v are orthogonal. Rephrased, the normal vectors to the level curves $u = c_1$ and $v = c_2$ are orthogonal. Thus the level curves themselves are orthogonal. This concludes the proof. ■

An example is shown in Fig. 2.1, for $f(z) = z^2$. Note that $f'(z) = 0$ at $z = 0$ and the orthogonality of the level curves breaks down there.

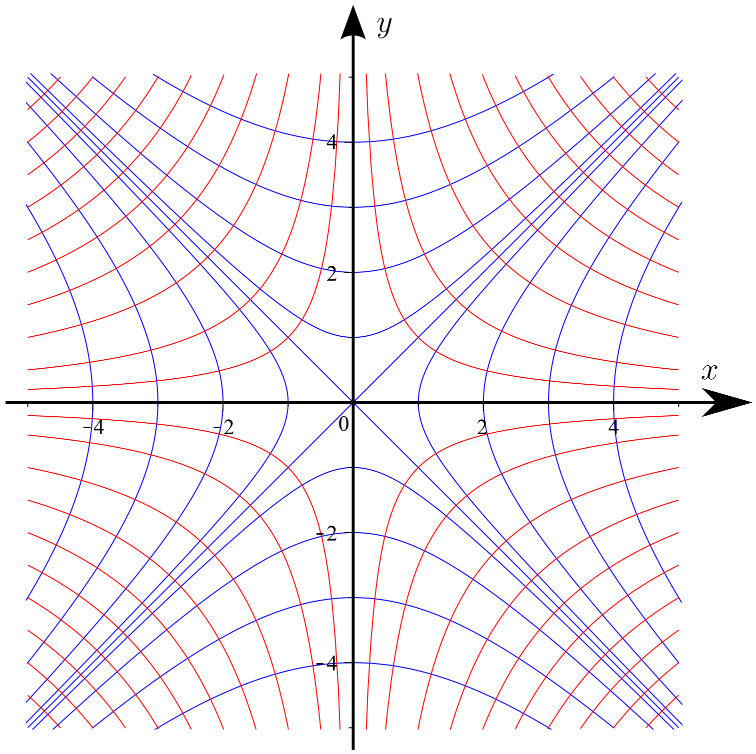

Figure 2.1: The level curves of $u = x^2 - y^2$ (blue) and $v = 2xy$ (red), the real and imaginary parts of $f(z) = z^2$.

For many applications, it is convenient to express the Cauchy-Riemann equations in polar coordinates: $z = re^{i\theta}$. This requires a bit of vector calculus: you find[1] that

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) \\ u_r &= \frac{1}{r}v_\theta, \\ v_r &= -\frac{1}{r}u_\theta, \end{aligned}$$

where, as mentioned above, $x = r\cos\theta$, $y = r\sin\theta$.

Let's do some examples. Apart from checking the analyticity of a given function, the calculations to verify the Cauchy-Riemann equations actually allow us to calculate derivatives.

Example. Let $w = e^z = e^x(\cos y + i\sin y)$. Then

$$\begin{aligned} u &= e^x\cos y, \\ v &= e^x\sin y, \end{aligned}$$

[1] Really. You do. You should check this!

so that

$$\begin{aligned} u_x &= e^x \cos y, \\ v_y &= e^x \cos y, \end{aligned}$$

so that the first of the Cauchy-Riemann equations is satisfied. Similarly,

$$\begin{aligned} u_y &= -e^x \sin y, \\ v_x &= e^x \sin y, \end{aligned}$$

and the second Cauchy-Riemann equation is satisfied. Then

$$f'(z) = u_x + iv_x = e^x(\cos y + i \sin y) = e^z.$$

Note that we can only do this last step after we checked that the Cauchy-Riemann equations are satisfied. Otherwise the derivative might not be defined at all.

Example. Let $w = z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$, where n is a positive integer. Thus

$$\begin{aligned} u &= r^n \cos n\theta, \\ v &= r^n \sin n\theta. \end{aligned}$$

This is the perfect set-up for the polar form of the Cauchy-Riemann equations. We have

$$\begin{aligned} u_r &= nr^{n-1} \cos n\theta, \\ v_r &= nr^{n-1} \sin n\theta, \\ u_\theta &= -r^n n \sin n\theta, \\ v_\theta &= nr^n \cos n\theta, \end{aligned}$$

so that $ru_r = v_\theta$ and $rv_r = -u_\theta$ are immediately satisfied. Thus

$$\begin{aligned} w' &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta}(nr^{n-1} \cos n\theta + inr^{n-1} \sin n\theta) \\ &= e^{-i\theta} nr^{n-1} e^{in\theta} \\ &= nr^{n-1} e^{i(n-1)\theta} \\ &= nz^{n-1}, \end{aligned}$$

verifying the power rule.

Example. Given the real part of an analytic function, we may find its imaginary part and thus the function itself, using the Cauchy-Riemann equations. Consider $u = e^y \cos x$. Let's find $f(z)$, analytic, such that $\mathrm{Re} f = u$.

First we check that u is harmonic:

$$\begin{aligned} u_x &= -e^y \sin x, \\ u_{xx} &= -e^y \cos x, \\ u_{yy} &= e^y \cos x, \end{aligned}$$

so that $\nabla^2 u = u_{xx} + u_{yy} = 0$ and u is harmonic. Next, we find v, the imaginary part of f, using the Cauchy-Riemann equations.

$$\begin{aligned} && v_y &= u_x = -e^y \sin x \\ \Rightarrow && v &= -e^y \sin x + h(x) \\ \Rightarrow && v_x &= -e^y \cos x + h'(x) \\ && &= -u_y = -e^y \cos x \\ \Rightarrow && h'(x) &= 0. \end{aligned}$$

So, up to a constant,

$$\begin{aligned} f(z) &= u + iv \\ &= e^y \cos x + i(-e^y \sin x) \\ &= e^y(\cos x - i \sin x) \\ &= e^y e^{-ix} \\ &= e^{-i(x+iy)} = e^{-iz}. \end{aligned}$$

Example. As you should verify, it is not possible to find an analytic function such that its real part $u = x^2 + y^2$.

2.2 Ideal fluid flow

An ideal fluid is characterized by several properties:

- The fluid has constant density.
- The fluid is not viscous.
- The flow is irrotational.

If these properties are satisfied, then the following equations hold: (1) mass is conserved:

$$\nabla \cdot \boldsymbol{v} = 0,$$

where $\boldsymbol{v}$ is the fluid velocity. It depends on position and time. (2) the irrotationality of the fluid implies that the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ vanishes: $\boldsymbol{\omega} = 0$. From conservation of mass, it follows that the velocity can be written in terms of a vector potential $\boldsymbol{A}$:

$$\boldsymbol{v} = \nabla \times \boldsymbol{A}.$$

On the other hand, the irrotationality leads to the velocity being written as the gradient of a scalar potential:

$$\boldsymbol{v} = \nabla \phi.$$

Taking the divergence of this definition, it follows that

$$\nabla^2 \phi = 0,$$

and the scalar potential ϕ is a harmonic function. This leads us to hope that if we are studying two-dimensional flow, perhaps[2] complex analysis will be a useful tool.

It turns out that it is. For instance, this was used by Nekrasov, Struik, and Levi-Civita (all independent) to prove the existence of periodic solutions in the one-dimensional water-wave problem (*i.e.,* two-dimensional fluid, one-dimensional surface).

With $A_3 = \psi$, we have that $v_1 = \psi_y$, $v_2 = -\psi_x$. Here A_3 is the third component of the vector potential $\boldsymbol{A}$, and ψ is called the stream function. It follows that

$$\begin{aligned} \phi_x &= \psi_y, \\ \phi_y &= -\psi_x. \end{aligned}$$

Thus $\Omega := \phi + i\psi$ satisfies the Cauchy-Riemann equations and is an analytic function of $z = x + iy$ (where the t dependence acts in a parametric role). Further,

$$\begin{aligned} \Omega' &= \phi_x + i\psi_x \\ &= v_1 - iv_2, \end{aligned}$$

and the components v_1 and v_2 of the fluid velocity $\boldsymbol{v}$ can be recovered conveniently from $\overline{\Omega'} = v_1 + iv_2$.

The boundary conditions imposed on the flow depend on the problem considered, of course. Different Ω correspond to different flows.

Example. Consider (v_0 and θ_0 are real constants)

$$\begin{aligned} & & \Omega(z) &= v_0 e^{-i\theta_0} z \\ \Rightarrow & & \Omega' &= v_0 e^{-i\theta_0} \\ \Rightarrow & & \overline{\Omega'} &= v_0 e^{i\theta_0} \\ & & &= (v_0 \cos\theta_0, v_0 \sin\theta_0). \end{aligned}$$

[2]Fingers crossed. Pretty please with sugar on top.

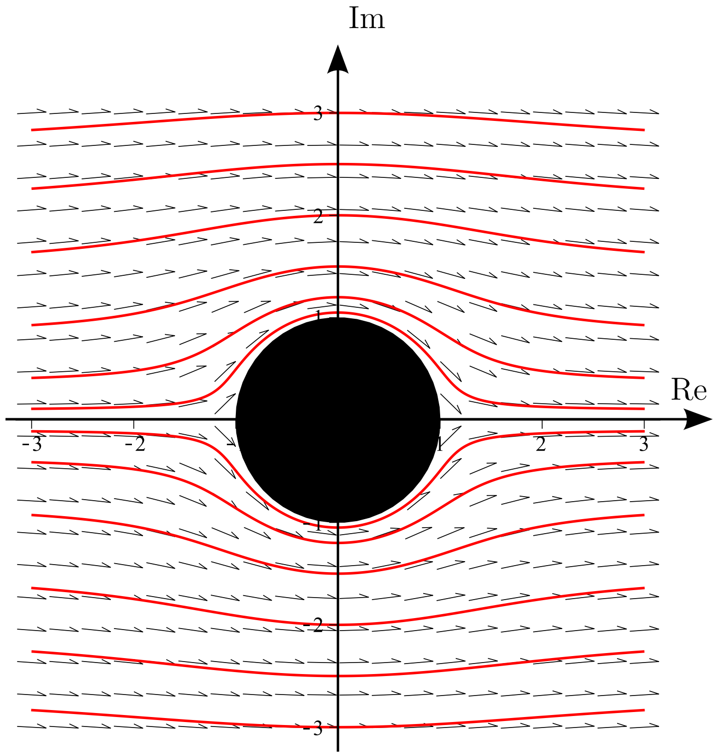

Figure 2.2: The flow around a cyclinder of radius $a = 1$ with $v_0 = 1$.

Thus this complex potential $\Omega(z)$ describes **uniform flow at an angle** θ_0.

Example. Let (v_0 and a are real constants)

$$\begin{aligned}
& & \Omega &= v_0\left(z + \frac{a^2}{z}\right) \\
&\Rightarrow & \Omega' &= v_0\left(1 - \frac{a^2}{z^2}\right) \\
&\Rightarrow & \overline{\Omega'} &= v_0\left(1 - \frac{a^2}{\bar{z}^2}\right) \\
& & &= v_0\left(1 - \frac{a^2}{r^2}e^{2i\theta}\right).
\end{aligned}$$

This complex potential describes the flow around a cylinder, as in Fig. 2.2.

2.3 Multivalued functions

Multivalued functions come about naturally as inverse functions of regular, single-valued functions. Usually, they require much more care.

The simplest example of a multivalued function is the inverse function of z^2. Denote[3]

$$w := z^{1/2}.$$

Let $z = re^{i\theta}$, then

$$w = \sqrt{r}e^{i(\theta+2\pi k)/2}, \quad k = 0, 1.$$

Thus w takes on two values:

$$w_1 = \sqrt{r}e^{i\theta/2} \text{ and } w_2 = \sqrt{r}e^{i(\theta/2+\pi)} = -w_1,$$

much like with real numbers.

For a single-valued function, as z follows a closed curve in the z plane, w follows a closed curve in the w plane. Let's see what happens here. Let z traverse a closed circle with center at the origin of radius ϵ. In other words: $z = \epsilon e^{i\theta}$, where θ starts at 0 and ends at 2π. This is illustrated in Fig. 2.3.

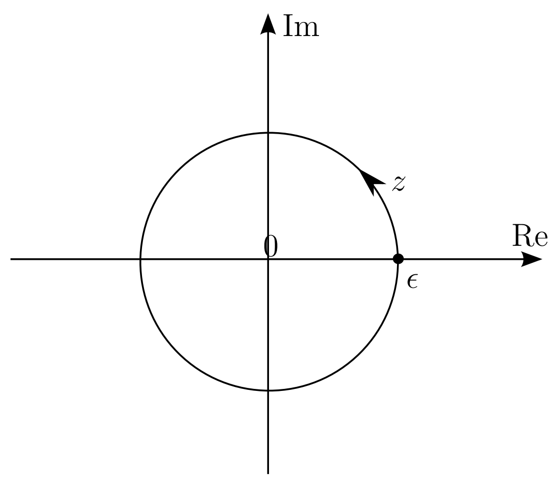

Figure 2.3: The path of z around the circle of radius ϵ.

As z traverses its circle, it follows from the expression for w_1 that it traverses its own circle, but it only gets halfway. The circle has radius $\sqrt{\epsilon}$, of course, but the argument, which starts at 0 only reaches π, as θ reaches 2π. At the same time, w_2 traverses the polar opposite path: it starts at $-\sqrt{\epsilon}$ and ends at ϵ. Both paths are illustrated in Fig. 2.4. Thus neither w_1 or w_2 return to their original starting point. Bummer! This is different from the behavior of an ordinary single-valued function. Suppose that $w = f(z)$, single valued. Let z, starting at z_1, traverse a closed path. In other words, z ends up again at z_1. Thus, as z traverses its closed path, w starts at $f(z_1)$. Since z ends up at z_1, it follows that w will end up at $f(z_1)$ again. Since $f(z)$ is single valued, it follows that w traverses its own closed path in $\mathbb{C}$.

[3]Throughout this course we will try to consistently use fractional powers for multivalued functions and root symbols for single-valued branches of these.

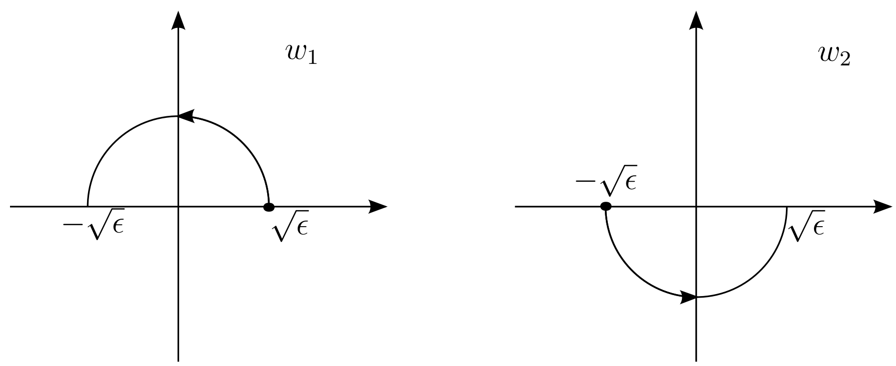

Figure 2.4: The paths of w_1 and w_2 around semicircles of radius $\sqrt{\epsilon}$.

In fact, things are not so different here. Suppose that z traverses some closed curve that does not cross the positive real axis $\arg z = 0$, as depicted in Fig. 2.5. In this case, the phase of z is between θ_1 and θ_2. As a consequence, when we calculate the arguments of, say, w_1, the set of arguments is $[\theta_1/2, \theta_2/2]$. Thus w_1 traverses a closed path. The same is true for w_2, but its path is the polar opposite path, of course. If the z path would have traversed the positive real axis, the w_1 values corresponding to z values above the positive real axis would have small positive argument. On the other hand, the w_1 values for z values just below the positive real axis would have arguments just less than π and w_1 would not traverse a continues path. Thus its path would not be closed. It would experience a discontinuity as z crosses the real axis. All of this is a consequence of choosing $[0, 2\pi)$ as the range for the argument of z. You should examine[4] how all of this changes if we were to choose $\arg z \in [-\pi, \pi)$.

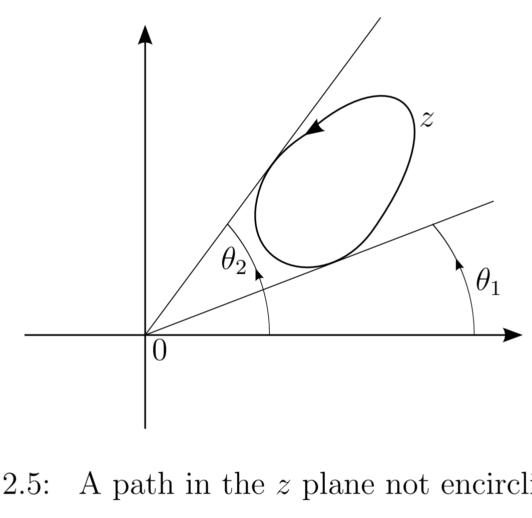

Figure 2.5: A path in the z plane not encircling the origin.

We have established that, apart from at $z = 0$, nothing special happens in the finite complex z-plane. In order to figure out what happens with and around $z = \infty$, we put

[4]You really should. If you only follow up on one footnote in these notes, let it be this one!

everything on the Riemann sphere, with an implied stereographic projection. The z-sphere and the path followed on it are shown in Fig. 2.6. The corresponding paths for w_1 and w_2 are shown in Fig. 2.7.

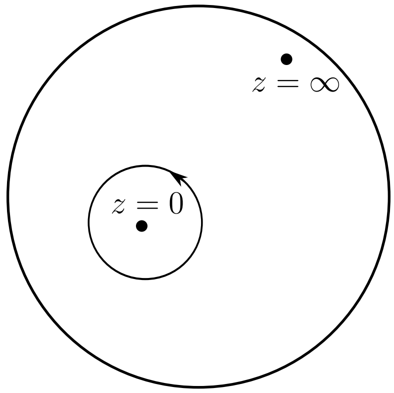

Figure 2.6: The z-sphere and the circular path on it.

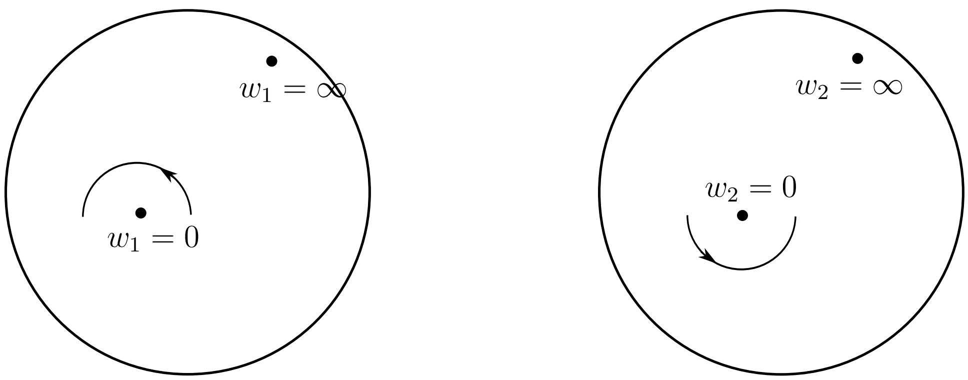

Figure 2.7: The semi-circular paths on the w_1- and w_2-spheres.

Figure 2.6 suggests that a loop around $z = 0$ is also a loop around ∞, but in the opposite direction. This is indeed the case. Let $z = 1/t$. Then $w = 1/t^{1/2}$ and $t = 0$ (which corresponds to $z = \infty$) is a branch point, *i.e.*, w_1 and w_2 do not return to their original values after t traverses its circle. One can also see the change in direction through this substitution, as we have $t = e^{-i\theta}/r$, and as θ increases, the argument of z decreases.

We conclude that the multivalued function

$$w = z^{1/2}$$

has two branch points: $z = 0$ and $z = \infty$. Note that these are the only two z values for which there is only one distinct value of w.

In order to study multivalued functions, we introduce *branch cuts*. Branch cuts are curves connecting branch points[5]. The domain of each branch $w_j,\ j = 1, 2$, is restricted in the sense that z is not allowed to follow a path crossing the branch cut. Almost always branch cuts are chosen to be straight lines. In the example above, the branch cut was chosen implicitly by choosing the argument of z so that $\arg z = [0, 2\pi)$. In other words, arguments cannot be less than 0 or greater than 2π. Thus the positive real axis, connecting the branch points $z = 0$ and $z = \infty$ is a branch cut.

What happens if we were to do the unthinkable? What happens if we cross the branch cut? The complex z-plane with its positive-real-line branch cut is shown in Fig. 2.8. Starting at $z = \epsilon$ and encircling the origin, we know that

$$\begin{aligned} w_1 = \sqrt{\epsilon} &\mapsto w_1 = -\sqrt{\epsilon}, \\ w_2 = -\sqrt{\epsilon} &\mapsto w_2 = \sqrt{\epsilon}, \end{aligned}$$

where each $\mapsto$ implies that z has traversed the circle of radius ϵ once, in the direction of increasing argument. It follows that w_1 and w_2 exchange their roles after z traverses the circle once. If we (hold on to something!) keep on going through the branch cut, z traverses the circle once more, thus:

$$\begin{array}{lllll} w_1 = \sqrt{\epsilon} & \mapsto & w_1 = -\sqrt{\epsilon} & \mapsto & w_1 = \sqrt{\epsilon}, \\ w_2 = -\sqrt{\epsilon} & \mapsto & w_2 = \sqrt{\epsilon} & \mapsto & w_2 = -\sqrt{\epsilon}. \end{array}$$

Riemann didn't like it. He didn't like that for one value of z two values of w were produced. So he set out to fix this problem. This resulted in the construction of what we now call *Riemann surfaces*, in his PhD thesis! For starters, he enlarged the domain: instead of having one z-plane, he uses two: z_1 corresponds to w_1 and z_2 corresponds to w_2. And, so that we can include ∞, let's work with two Riemann spheres: a z_1-sphere and a z_2-sphere. These are shown in Fig. 2.9, together with the branch cuts going from 0 to ∞. Also indicated there are paths going from one side of the branch cut to the other. In other words, following these paths w_1 (w_2) goes to its opposite value by the time the end of the path is reached.

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[5] It can be shown, but this is definitely nontrivial, that the number of branch points, counting multiplicities, is always even.

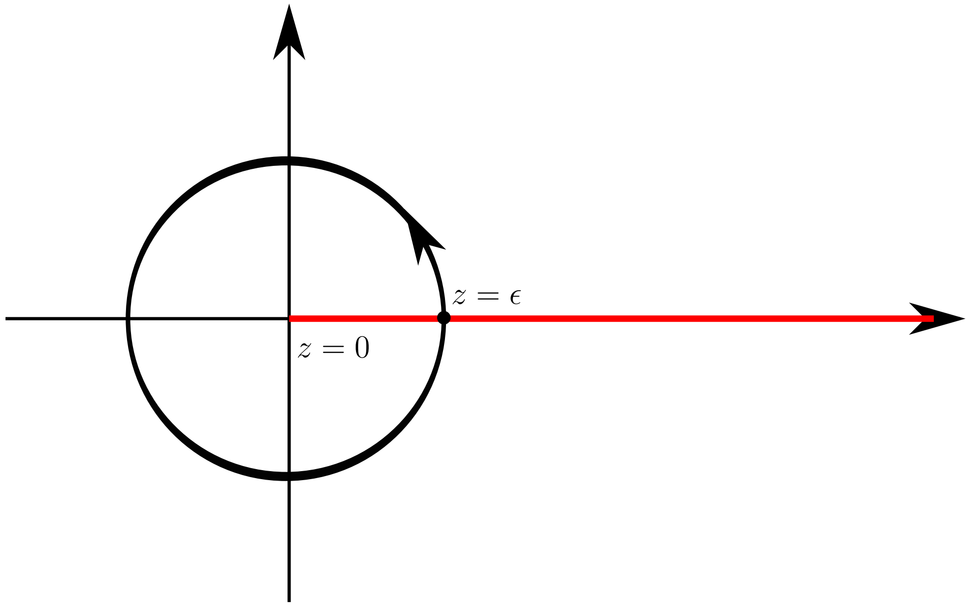

Figure 2.8: The complex z-plane, with the branch cut along the positive real line, indicated in red.

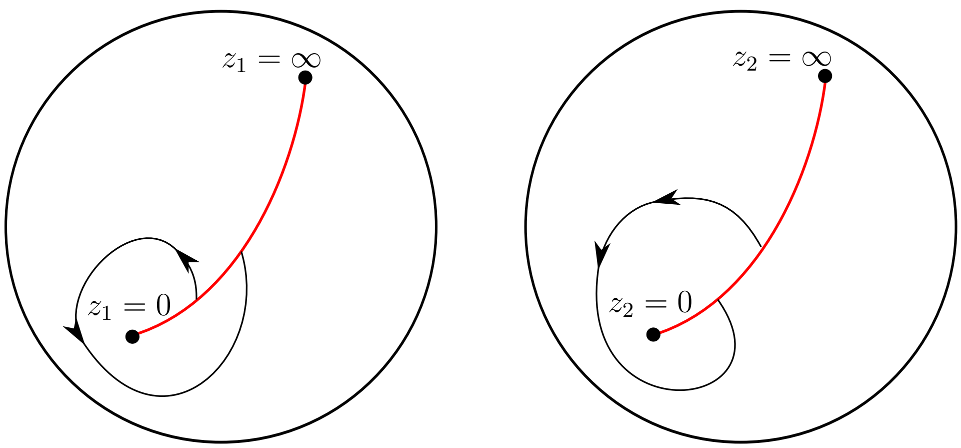

Figure 2.9: The z_1- and z_2-spheres.

From the point of view of the topology of the Riemann spheres for z_1 and z_2, the branch cuts are slits in the spheres, so we open them up, to make this more clear. This is illustrated in Fig. 2.10.

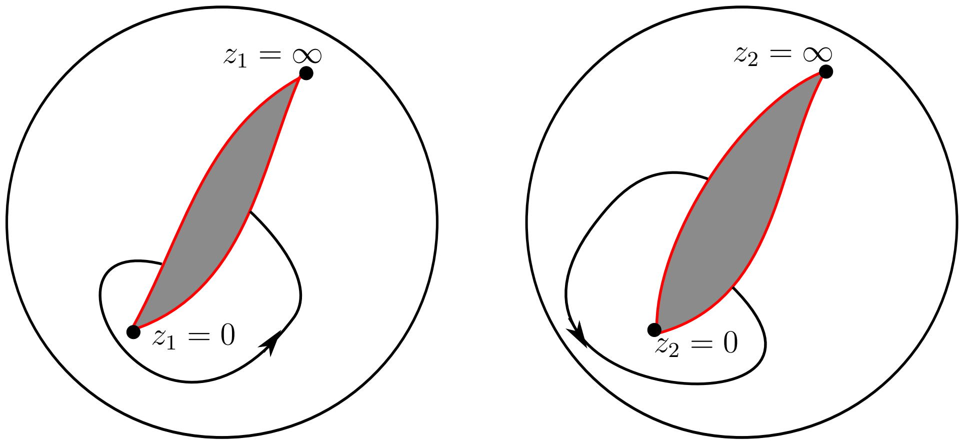

Figure 2.10: The z_1- and z_2-spheres with the branch cuts opened up.

Next, keeping up the topology, we deform the z_1- and z_2-spheres by pulling out the holes, until the spheres look like bags with the opened slits as entries. These holey bags are shown in Fig. 2.11.

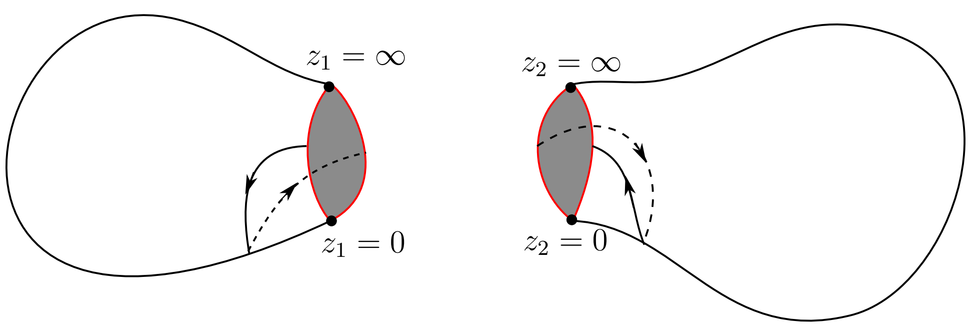

Figure 2.11: The deformed z_1- and z_2-spheres, with the branch cut holes pulled out.

The values on the sides of the branch cuts are the same in the z_1- and z_2-spheres, but they are achieved on opposite sides of the cuts. Thus we can glue the two bags together along the opened cuts since the values obtained there are identical. This gluing process is shown in Fig. 2.12. Note that we are using that $z_1 = 0 = z_2$ and $z_1 = z_\infty = z_2$, our two branch points.

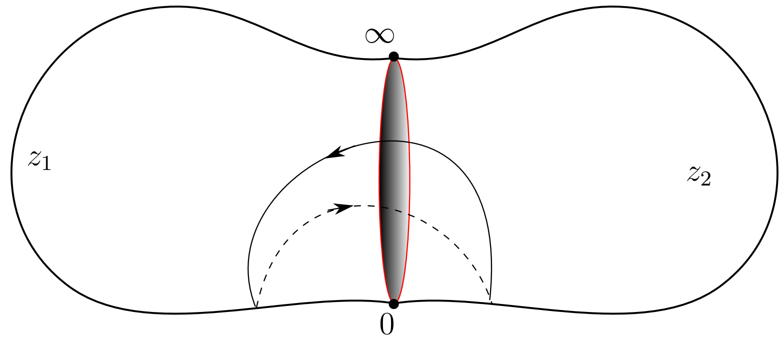

Figure 2.12: The gluing of the two bags along the branch cut.

Figure 2.13 shows the same surface, consisting of the z_1 and z_2 domains, but turned so that the path we have drawn on it is facing us. As a consequence the point at ∞ is at the back of the surface. We see that topologically the surface obtained is still a sphere, but this is not always the case. Since w_1 (w_2) is a single-valued function of z_1 (z_2), it follows that w is a single-valued function of z, where z is chosen from the surface obtained from the z_1 and z_2 domains. In the literature, one often refers to the z surface as a *Riemann surface*. In fact, that is not correct. The w surface that is obtained by mapping the z surface is actually the Riemann surface. Since w is single-valued as a function of z, the Riemann surface is topologically also a sphere. Riemann surfaces are near impossible to visualize for us: they are two-dimensional surfaces in four-dimensional spaces, as are the graphs of complex functions in general.

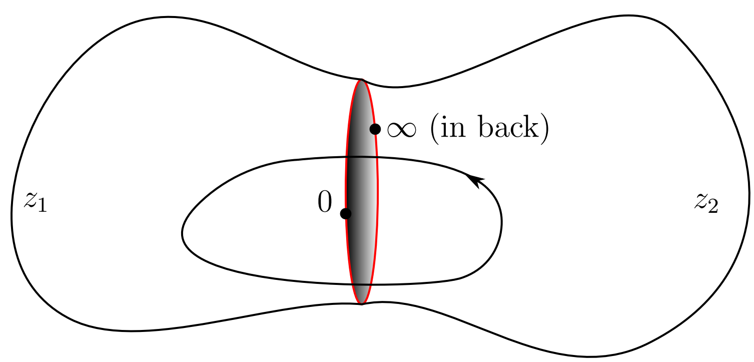

Figure 2.13: A turned version of the surface in Fig. 2.12. The point at infinity is now at the back of this surface.

For simple functions, we can give it a bit of an effort though. Let $z = x+iy$ and $w = u+iv$. Let $z = w^2$. We are interested in the Riemann surface on which w is a single-valued function of z. A representation of this can be obtained by plotting the two-dimensional surface

$$
\begin{aligned}
x &= x(u,v) = u^2 - v^2, \\
y &= y(u,v) = 2uv, \\
v &= v,
\end{aligned}
$$

parameterized by the two real parameters u and v. By coloring the obtained surface according to u, we effectively obtain a two-dimensional object in a four-dimensional space: any point depends on two degrees of freedom (u, v), but lives in four-dimensions: $(x, y, v; u)$ where u is our coloring function. The resulting surface for the square root Riemann surface is shown in Fig. 2.14. The presence of a branch cut along the positive real line seems obvious, but in fact it is not there. It is merely a figment of squeezing down a four-dimensional object without self-intersections into three-dimensional space. The color function, as an independent dimension, shows that no self-intersection is present: as we approach the apparent self-intersection, we are to simply follow the path along which the color changes continuously.

Remarks.

- For $w = (z - z_0)^{1/2}$, the branch points are at z_0 and at ∞.
- For $w = (az - z_0)^{1/2}$, the branch points are at z_0/a and ∞. Here a is a complex non-zero constant.

Let's do another example, a bit quicker. Consider

$$w = ((z - z_1)(z - z_2))^{1/2},$$

where z_1 and z_2 are two distinct complex numbers.

1. First we check that z_1 and z_2 are branch points. Near z_1,

   $$z = z_1 + \epsilon e^{i\theta},$$

   where ϵ is a small positive real number. Thus

   $$
   \begin{aligned}
   w &= \left(\epsilon e^{i\theta}(z_1 - z_2 + \epsilon e^{i\theta}\right)^{1/2} \\
   &\sim \sqrt{\epsilon}(z_1 - z_2)^{1/2} e^{i\theta/2},
   \end{aligned}
   $$

   where we have ignored terms of order ϵ. Thus, in the vicinity of z_1, the function behaves like the previous square root example. Thus z_1 is a branch point. Exactly the same argument shows that z_2 is a branch point as well.

2. We check that ∞ is not a branch point. For very large values of z,

   $$w \sim (z^2)^{1/2} = \pm z,$$

   which has two values, but for either choice, w returns to its original value as z traverses a circle of very large radius. Thus, there are two different points at infinity for w, but the behavior near each of those is single valued.

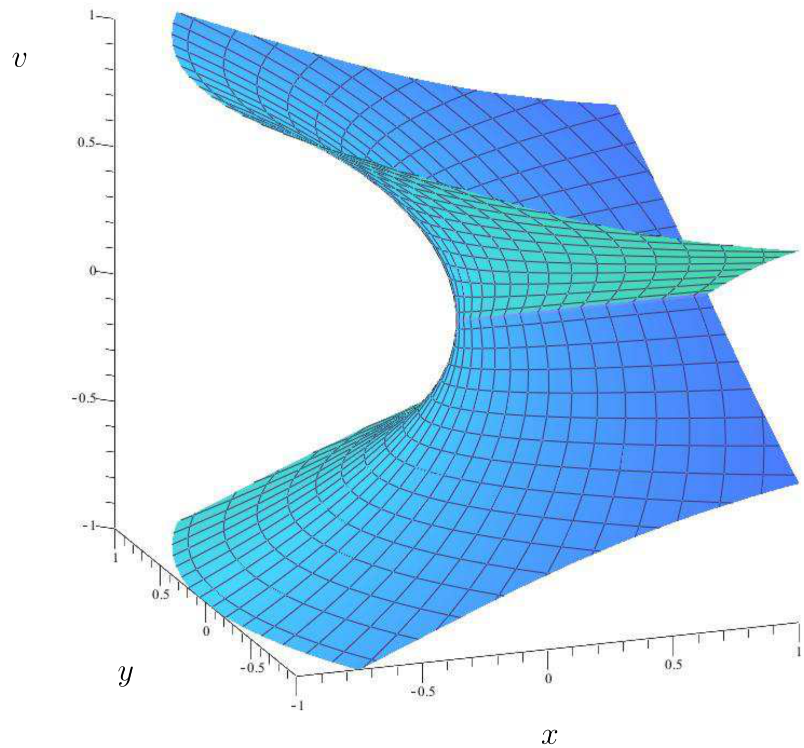

Figure 2.14: A visualization of the Riemann surface for the square root function. Here color is used for the fourth dimension: when we hit the apparent self-intersection and the surface appears not smooth, we simply follow the path along which color changes continuously.

3. Thus we choose a branch cut from z_1 to z_2. The corresponding Riemann spheres are shown in Fig. 2.15, as is its domain on which w is single values.

The logarithm

The logarithmic function is defined as the inverse function of the exponential function:

$$w = \ln z$$

is the inverse function of

$$z = e^w.$$

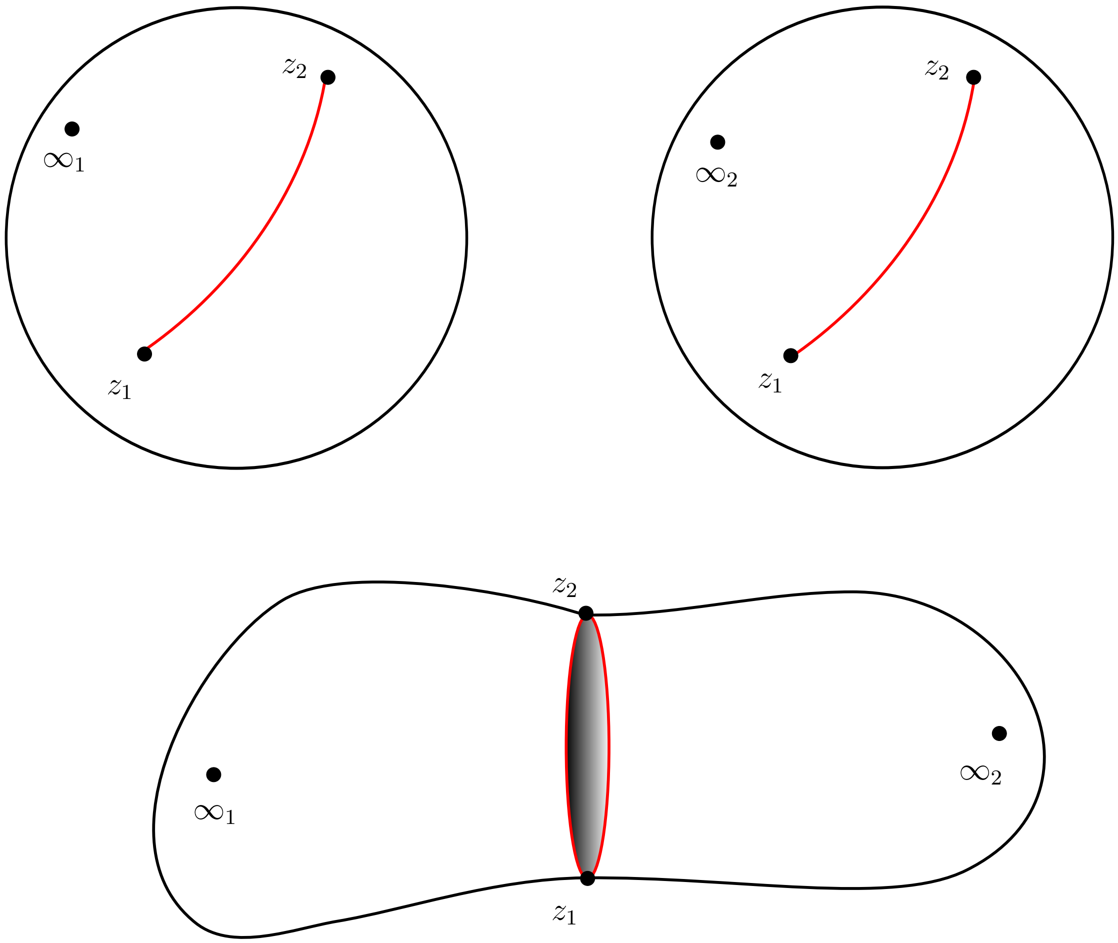

Figure 2.15: The two Riemann spheres required for the Riemann surface construction of $w^2 = (z - z_1)(z - z_2)$ and the domain on which w is single valued.

Let's look at this in more detail. With $z = re^{i\theta}$ and $w = u + iv$, we get

$$re^{i\theta} = e^u e^{iv}$$

$$\Rightarrow \quad \left\{ \begin{array}{rcl} r & = & e^u \\ e^{i\theta} & = & e^{iv} \end{array} \right. ,$$

$$\Rightarrow \quad \left\{ \begin{array}{rcl} u & = & \ln r \\ v & = & \theta + 2\pi n, \quad n \in \mathbb{Z}. \end{array} \right. .$$

In summary,

$$w = \ln |z| + i \arg z,$$

which is a function with an infinite number of values, since the argument is only defined up to multiples of 2π. It would be wise to choose a branch cut! From these calculations and the definition of the argument function, it follows that $z = 0$ and $z = \infty$ are branch points.

Thus we can choose a branch cut to be a straight line connecting these two branch points. Selecting the argument θ to satisfy $\theta \in [0, 2\pi)$, the branch cut becomes the positive real axis. Then $w = \ln z$ for $z \in \mathbb{R}^+$, in agreement with our usual real-valued logarithm, as desired. For $z \in \mathbb{R}^-$ (the negative real line), $\ln z = \ln |z| + i\pi$.

Next we discuss the derivative of the complex logarithm.

The derivative of $\ln z$

We have that

$$\ln z = \ln r + i\theta,$$

where $z = re^{i\theta}$, where any choice of the argument is allowed. First, we check that the Cauchy-Riemann equations are satisfied:

$$\begin{aligned} u_r &= \frac{1}{r} v_\theta && \Rightarrow && \frac{1}{r} = \frac{1}{r}, && \checkmark \\ v_r &= -\frac{1}{r} u_\theta && \Rightarrow && 0 = 0, && \checkmark \end{aligned}$$

and the equations are satisfied for all $z \in \mathbb{C}$, except $z = 0$. It follows that

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) \\ & e^{-i\theta}\left(\frac{1}{r} + 0\right) \\ &= \frac{1}{re^{i\theta}} \\ &= \frac{1}{z}, \end{aligned}$$

which is valid everywhere, except at $z = 0$. Thus, $\ln z$, defined in the complex plane, is analytic everywhere, except at $z = 0$.

Let's examine some other properties of the logarithm that we want to hold: is it true that $\ln z_1 z_2 = \ln z_1 + \ln z_2$? With $z_1 = r_1 e^{i(\theta_1 + 2\pi n_1)}$, $z_2 = r_2 e^{i(\theta_2 + 2\pi n_2)}$, we have

$$\begin{aligned} \ln z_1 z_2 &= \ln r_1 e^{i(\theta_1 + 2\pi n_1)} r_2 e^{i(\theta_2 + 2\pi n_2)} \\ &= \ln r_1 r_2 e^{i(\theta_1 + \theta_2 + 2\pi(n_1 + n_2))} \\ &= \ln r_1 r_2 + i(\theta_1 + \theta_2 + 2\pi(n + n_1 + n_2)) \\ &= \ln r_1 + \ln r_2 + i(\theta_1 + 2\pi n_1) + i(\theta_2 + 2\pi n_2) + i2\pi n \\ &= \ln z_1 + \ln z_2 + i2\pi n. \end{aligned}$$

The extra contribution $i2\pi n$ is added, because the definition of the logarithm allows for any choice of the argument. You see that if we allow this multivaluedness, then this desired logarithm property does not hold. On the other hand if we restrict everything to the logarithm with our favorite branch cut[6], then $n = n_1 = n_2 = 0$ and the property holds.

[6] Any other branch cut choice would do as well.

Some related notes follow. Consider

$$\begin{aligned} e^{\ln z} &= e^{\ln r + i(\theta + 2\pi n)} \\ &= e^{\ln r} e^{i(\theta+2\pi n)} \\ &= re^{i\theta} \\ &= z, \end{aligned}$$

which holds independent of the choice of the argument. On the other hand,

$$\begin{aligned} \ln e^{z} &= \ln e^{x+iy} \\ &= \ln e^{x} + i(y + 2\pi n) \\ &= x + iy + i2\pi n, \end{aligned}$$

and thus $\ln e^z = z$ only if we restrict ourselves to the main branch of the logarithm.

The power function

Using the logarithm, we define the power function. We define

$$z^a := e^{a \ln z}, \quad a \in \mathbb{C}, \quad z \neq 0.$$

Let's see that this definition agrees with our regular one if $a \in \mathbb{Z}$. Then

$$\begin{aligned} z^a &= e^{a \ln z} \\ &= e^{a(\ln r + i(\theta + 2\pi n))} \\ &= e^{a \ln r} e^{ia\theta} \cancelto{1}{e^{i2\pi na}} \\ &= r^a e^{ia\theta}, \end{aligned}$$

which is single valued. In the above, $e^{i2\pi na} = 1$, since a is an integer. On the other hand, if $a = p/q \in \mathbb{Q}$, rational ($p, q \in \mathbb{Z}$), then

$$z^a = r^a e^{ia\theta} e^{i2\pi np/q},$$

and this last factor if a qth root of unity, thus z^a has q different values. It is clear[7] that if a is not rational, then z^a attains an infinity of different values.

Example. Let's examine $f(z) = z^i$. By definition,

$$\begin{aligned} z^i &= e^{i \ln z} \\ &= e^{i(\ln r + i\theta + i2\pi n)} \\ &= e^{i \ln r} e^{-\theta} e^{-2n\pi} \\ &= (\cos \ln r + i \sin \ln r) e^{-\theta} e^{-2n\pi}. \end{aligned}$$

[7]Don't you hate it when instructors write this? I did. I do. Instructors that do this should be shamed on message boards and their Fields medals should be taken away!

This last factor confirms that z^i has an infinite number of sheets, one for each value of $n \in \mathbb{Z}$.

Next, we calculate the derivative of z^a. First, we check that the Cauchy-Riemann equations are satisfied. This requires a bit of work. Let $a = \alpha + i\beta$. Then[8]

$$\begin{aligned} z^a &= e^{a \ln z} \\ &= e^{(\alpha+i\beta)(\ln r + i\theta)} \\ &= e^{\alpha \ln r - \beta\theta} e^{i(\beta \ln r + \alpha\theta)}. \end{aligned}$$

With $z^a = u + iv$, it follows that

$$u = e^{\alpha \ln r - \beta\theta} \cos(\beta \ln r + \alpha\theta),$$

and

$$v = e^{\alpha \ln r - \beta\theta} \sin(\beta \ln r + \alpha\theta).$$

First we verify that $u_r = v_\theta / r$. We have

$$u_r = e^{\alpha \ln r - \beta\theta} \cos(\beta \ln r + \alpha\theta)\frac{\alpha}{r} - e^{\alpha \ln r - \beta\theta} \sin(\beta \ln r + \alpha\theta)\frac{\beta}{r},$$

and

$$v_\theta = -e^{\alpha \ln r - \beta\theta} \sin(\beta \ln r + \alpha\theta)\beta + e^{\alpha \ln r - \beta\theta} \cos(\beta \ln r + \alpha\theta)\alpha,$$

so that the first of the Cauchy-Riemann equations is satisfied. The second one follows just as easily. As before, the presence of $\ln r$ precludes us from concluding anything at $z = 0$. The derivative of z^a is given by

$$\begin{aligned} (z^a)' &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta}\left(e^{\alpha \ln r - \beta\theta} \cos(\beta \ln r + \alpha\theta)\frac{\alpha}{r} - e^{\alpha \ln r - \beta\theta} \sin(\beta \ln r + \alpha\theta)\frac{\beta}{r} + \right. \\ &\qquad \left. i\left(e^{\alpha \ln r - \beta\theta} \sin(\beta \ln r + \alpha\theta)\frac{\alpha}{r} + e^{\alpha \ln r - \beta\theta} \cos(\beta \ln r + \alpha\theta)\frac{\beta}{r}\right)\right) \\ &= e^{-i\theta} e^{\alpha \ln r - \beta\theta}\left(\frac{\alpha}{r} e^{i(\beta \ln r + \alpha\theta)} + i\frac{\beta}{r} e^{i(\beta \ln r + \alpha\theta)}\right) \\ &= e^{-i\theta + \alpha \ln r - \beta\theta + i(\beta \ln r + \alpha\theta)}\frac{a}{r} \\ &= \frac{a}{z} e^{\alpha(\ln r + i\theta) + i\beta(\ln r + i\theta)} \\ &= \frac{a}{z} e^{a(\ln z)} \\ &= a e^{a \ln z} e^{-\ln z} \\ &= a e^{(a-1)\ln z} \\ &= a z^{a-1}, \end{aligned}$$

[8] We won't write out the $2\pi n$ contributions everytime, but you should check that they don't matter for this.

where we had to be careful not assuming any of properties we expect to be true, but don't yet know to be true for the complex case. But, all is well that ends well: the power rule for derivatives holds, as before. Phew!

Inverse trigonometric functions

(And while we're at it, we could do the same for inverse hyperbolic functions.)

As you might expect, the inverse trigonometric functions are even more complicated than the logarithm. Let's look at the inverse of the sin function. Consider

$$\begin{aligned}
& w = \arcsin z \\
\Rightarrow \quad & z = \sin w \\
& \quad = \frac{e^{iw} - e^{-iw}}{2i} \\
\Rightarrow \quad & 2ize^{iw} = e^{2iw} - 1 \\
\Rightarrow \quad & e^{2iw} - 2ize^{iw} - 1 = 0 \\
\Rightarrow \quad & e^{iw} = iz + (1 - z^2)^{1/2} \\
\Rightarrow \quad & iw = \ln(iz + (1 - z^2)^{1/2}) \\
\Rightarrow \quad & w = -i \ln(iz + (1 - z^2)^{1/2}).
\end{aligned}$$

Thus there are two sources of multivaluedness: we have a combination of a ln and of a 1/2 power.

Let's check that $z_1 = 1$ and $z_2 = -1$ are both branch points. Let

$$z_1 = 1 + re^{i\theta},$$

with r small. Then

$$w = -i \ln(i + ire^{i\theta} + (\cancel{1} - \cancel{1} - r^2 e^{2i\theta} - 2re^{i\theta})^{1/2}),$$

which definitely has a discontinuity as θ jumps by 2π. The same argument shows that z_2 is a branch point. Also, $z = \infty$ is a logarithmic branch point. Let $z = 1/t$, then

$$\begin{aligned}
w &= -i \ln\left(\frac{i}{t} + \left(1 - \frac{1}{t^2}\right)^{1/2}\right) \\
&= -i \ln\left(\frac{i}{t} + \frac{(t^2 - 1)^{1/2}}{t}\right) \\
&= -i \ln(i + (t^2 - 1)^{1/2}) + i \ln t,
\end{aligned}$$

which proves that $t = 0$ is a logarithmic branch point, establishing that $z = \infty$ is too. We won't discuss the branch structure of the inverse trig functions more, as we won't need them in this course.

Just for kicks, let's examine the derivative of $w = \arcsin z$:

$$\begin{aligned} w' &= \frac{-i}{iz + (1-z^2)^{1/2}} \left(i + \frac{1}{2}(1-z^2)^{-1/2}(-2z) \right) \\ &= \frac{1 + iz(1-z^2)^{-1/2}}{(1-z^2)^{1/2} + iz} \\ &= \frac{1}{(1-z^2)^{1/2}}, \end{aligned}$$

as expected from the result when working with real functions. This result is even easier to establish using implicit differentiation on $\sin w = z$.

2.4 Complex integration

First, we look at the relatively simple case of a function of a real variable which is complex valued:

$$f(t) = u(t) + iv(t),$$

where both u and v are real-valued functions of the real parameter t. Then

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt,$$

and we have all the usual theorems:

$$\int_a^b f(t)dt = F(b) - F(a),$$

where $F(t)$ is an anti-derivative of $f(t)$: $F'(t) = f(t)$.

Next, we extend this notion of integration to integration along a curve. A *curve* in the complex plane is described by a parameterization

$$z(t) = x(t) + iy(t), \quad t \in [a, b],$$

see Fig. 2.16. We call a curve simple if it does not intersect itself. If $z(a) = z(b)$, we say the curve is closed. An integral along a closed curve is always taken in the positive direction, *i.e.*, the direction which keeps the interior to the left.

Definition 2.5 *A* contour *is a piecewise smooth simple curve.*

Thus contours do not self intersect and their pieces are smooth. Let C be a contour. Then

$$\int_C f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

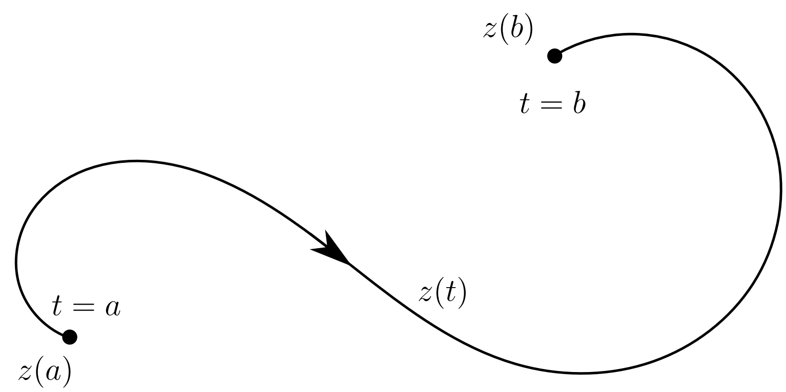

Figure 2.16: A curve in the complex plane, specified by its parameterization.

This definition shows why we want the pieces of a contour to be smooth, so that $z'(t)$ is defined. It is important to note that the value of the integral does not depend on the parameterization of the curve we choose to use. Indeed:

$$\begin{aligned}\int_C f(z)dz &= \int_{t_1}^{t_2} f(z(t))z'(t)dt \\ &= \int_{s_1}^{s_2} f(z(t(s)))z'(t(s))t'(s)ds \\ &= \int_{s_1}^{s_2} f(z(t(s)))[z(t(s))]'ds,\end{aligned}$$

where $t_j = t(s_j)$, $j = 1, 2$. This is the result that the definition of the integral would give if we were to use the s parameter from the beginning. Thus, in the rest of this course, or for that matter, for the rest of your life, you can use whatever parameterization is most convenient.

Some properties:

- $\int_C (af(z) + bg(z))dz = a\int_C f(z)dz + b\int_C g(z)dz$, where $a, b \in \mathbb{C}$, constants,
- $\int_C f(z)dz = -\int_{-C} f(z)dz$, where by $-C$ we denote the contour traversed in the opposite direction, and
- If $C = \sum_{j=1}^N C_j$ (a consecutive traversing of the constitutive paths C_j), then $\int_C f(z)dz = \sum_{j=1}^N \int_{C_j} f(z)dz$.

Theorem 2.6 *Let $F(z)$ be analytic and $f(z) = F'(z)$ is continuous in D. Then for C lying in D with endpoints z_1 and z_2,*

$$\int_C f(z)dz = F(z_2) - F(z_1).$$

Proof.

$$\begin{aligned}
\int_C f(z)dz &= \int_C F'(z)dz \\
&= \int_a^b F'(z(t))z'(t)dt \\
&= \int_a^b \frac{d}{dt}F(z(t))dt \\
&= F(z(b)) - F(z(a)) \\
&= F(z_2) - F(z_1),
\end{aligned}$$

where $z(t)$, $t \in [a, b]$ is a parameterization of C. The assumption that $f(z)$ is continuous guarantees that $\int_C f(z)dz$ is defined. ■

Corollary 2.7 *For a closed curve, $\oint_C f(z)dz = 0$.*

And so it begins. This corollary introduces the symbol $\oint$, which we use to denote the integral around a closed contour. We'll see quite a bit of it.

Corollary 2.8 *Under the conditions of the theorem, the integral of $f(z)$ from z_1 to z_2 is path independent.*

In other words, it does not matter how we get from z_1 to z_2: we may choose two different paths (see Fig. 2.17), we get the same result. Indeed, $C = C_1 - C_2$ is a closed path, thus the integral is zero. It follows that $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

The example below shows that the conditions of the theorem are necessary.

Example. Consider $f(z) = \overline{z}$. We evaluate the integral of $f(z)$ over two different contours, shown in Fig. 2.18.

Along C_1, parameterized by $z = e^{i\theta}$, $\theta \in [0, \pi/2]$, we find

$$\begin{aligned}
\int_{C_1} \overline{z}dz &= \int_0^{\pi/2} e^{-i\theta}e^{i\theta}id\theta \\
&= \frac{i\pi}{2}.
\end{aligned}$$

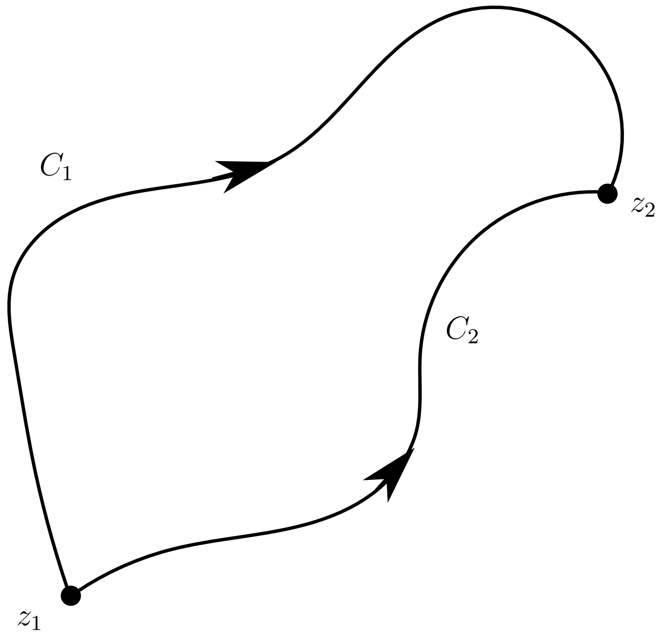

Figure 2.17: Two contours going from z_1 to z_2.

Along C_2, parameterized by $z = 1 - t + it$, $t \in [0, 1]$, we get

$$\begin{aligned}
\int_{C_2} \overline{z} dz &= \int_0^1 (1 - t - it)(-1 + i) dt \\
&= (-1 + i) \int_0^1 (1 - t - it) dt \\
&= (-1 + i) \left[t - \frac{t^2}{2} - i\frac{t^2}{2} \right]_{t=0}^{t=1} \\
&= (-1 + i) \left(1 - \frac{1}{2} - \frac{i}{2} \right) \\
&= i.
\end{aligned}$$

These two results are not equal, and the integral is path dependent.

Example. On the other hand, for the function $f(z) = z$, which does have an analytic anti-derivative, the integral should be path independent. Let's check this directly, without using the theorem, again using the paths from Fig. 2.18.

Along C_1, we get

$$\begin{aligned}
\int_{C_1} z dz &= \int_0^{\pi/2} i e^{2i\theta} d\theta \\
&= \cancel{i} \frac{e^{2i\theta}}{2\cancel{i}} \Bigg|_{\theta=0}^{\theta=\pi/2}
\end{aligned}$$

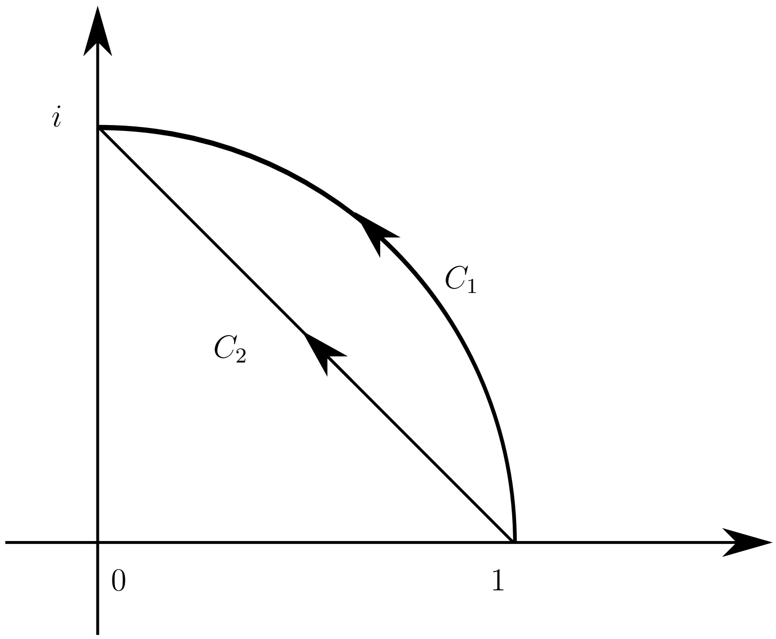

Figure 2.18: Two contours going from $z = 1$ to $z = i$.

$$
\begin{aligned}
&= \frac{1}{2}(-1-1) \\
&= -1.
\end{aligned}
$$

Along C_2, we find

$$
\begin{aligned}
\int_{C_2} z dz &= \int_0^1 (1-t+it)(-1+i)dt \\
&= (-1+i)\left(1-\frac{1}{2}+\frac{i}{2}\right) \\
&= -1.
\end{aligned}
$$

Example. We know that

$$\oint_C \frac{1}{z} dz = 0,$$

provided that C does not encircle the origin. Indeed, $1/z$ has an anti-derivative (the logarithm), which is analytic everywhere, except at the origin. What can we say about this integral if C does encircle the origin?

Such a contour is shown on the left in Fig. 2.19. Instead, we consider the contour

$$\hat{C}_1 = C_1 + L_1 + C_R + L_2,$$

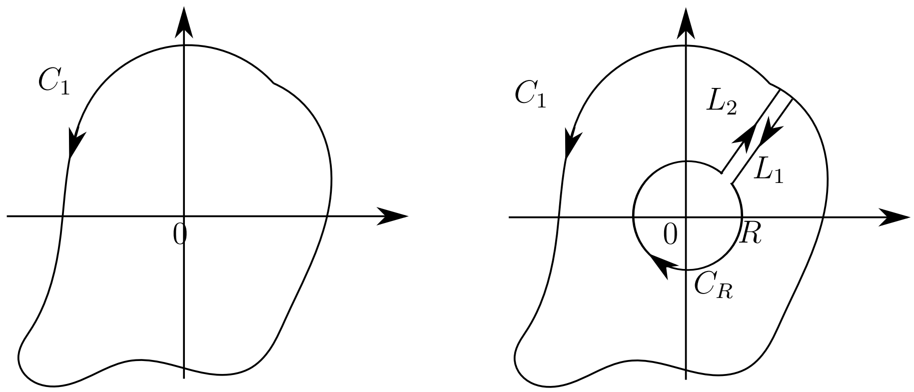

Figure 2.19: The contour deformation for $\int_{C_1} dz/z$.

where these different pieces are indicated on the right in Fig. 2.19. Since $1/z$ has an anti-derivative that is analytic on the inside of $\hat{C}_1$, we have that

$$\oint_{\hat{C}_1} \frac{1}{z} dz = 0 = \oint_{C_1} \frac{1}{z} dz + \int_{L_1} \frac{1}{z} dz + \int_{C_R} \frac{1}{z} dz + \int_{L_2} \frac{1}{z} dz.$$

Here L_1 and L_2 are straight-line segments, lying infinitesimally close together. They are traversed in opposite direction, thus their contributions cancel. It follows that

$$\oint_{C_1} \frac{1}{z} dz = \oint_{-C_R} \frac{1}{z} dz.$$

Since C_R was originally traversed in the clockwise direction (to keep the interior on the left), $-C_R$ is traversed counter clockwise. Thus

$$\begin{aligned}\oint_{C_1} \frac{1}{z} dz &= \oint_{-C_R} \frac{1}{z} dz \\ &= \int_0^{2\pi} \frac{1}{R} e^{-i\theta} R e^{i\theta} i d\theta \\ &= \int_0^{2\pi} i d\theta \\ &= 2\pi i.\end{aligned}$$

In the above, we have used the parameterization of $-C_R$: $z = Re^{i\theta}$, $\theta \in [0, 2\pi]$.

Example. Next, we consider $\oint_C z^n dz$, for integer n. If C does not encircle the origin, the integral is zero, as before. In what follows, we assume C encircles $z = 0$. If $n \geq 0$, the integral is zero, since an analytic anti-derivative exists. We already know the answer for $n = -1$. In general, using the same contour deformation used above (see Fig. 2.19), for $n \neq -1$ and with $R = 1$ (for convenience),

$$\begin{aligned}
\oint_C z^n dz &= \oint_{-C_R} z^n dz \\
&= \int_0^{2\pi} e^{in\theta} e^{i\theta} i d\theta \\
&= i \int_0^{2\pi} e^{i(n+1)\theta} d\theta \\
&= i \left. \frac{e^{i(n+1)\theta}}{i(n+1)} \right|_{\theta=0}^{\theta=2\pi} \\
&= \frac{1}{n+1} \left(e^{2\pi i(n+1)} - 1 \right) \\
&= 0.
\end{aligned}$$

Despite the fact that z^n does not have an analytic anti-derivative inside C for $n < -1$, we get that $\oint_C z^n dz = 0$.

The following theorem gives a bound that we will use quite frequently.

Theorem 2.9 *Let $f(z)$ be continuous on C, then*

$$\left| \int_C f(z) dz \right| \leq ML,$$

where $M \geq |f(z)|$ on C and L is the arclength of C.

Proof.

$$\begin{aligned}
\left| \int_C f(z) dz \right| &= \left| \int_a^b f(z(t)) z'(t) dt \right| \\
&\leq \int_a^b |f(z(t))| |z'(t)| dt \\
&\leq M \int_a^b |z'(t)| dt \\
&= ML,
\end{aligned}$$

since $|z'(t)| = \sqrt{x'^2(t) + y'^2(t)}$. ■

2.5 Cauchy's Theorem and its consequences

Cauchy's Theorem

Next we come to the very important *Cauchy Theorem*, one of the cornerstones of complex analysis. There are different proofs for this. We'll present the simpler one, not the more advanced one (requiring fewer assumptions) due to Goursat. Before we get to Cauchy, we'll deal with Green.

Theorem 2.10 (Green's Theorem) *Let u, v, ∇u and ∇v be continuous throughout $\mathcal{R}$, a simply connected*[9] *region. Then*

$$\oint_C u dx + v dy = \iint_D (v_x - u_y) dx dy,$$

where $D \subset \mathcal{R}$, simply connected and $C = \partial D$ is a closed contour.

We don't prove this theorem here. It is really just a two-dimensional version of the Divergence Theorem. You should have seen the proof of the Divergence Theorem or of Green's Theorem elsewhere already.

Theorem 2.11 (Cauchy). *If $f(z)$ is analytic in R, a simply connected region, then along the simple closed contour $C \subset \mathcal{R}$,*

$$\oint_C f(z) dz = 0.$$

Proof. The proof given here requires $f'(z)$ to be continuous. The Goursat proof allows one (for instance: you) to avoid this assumption. We have

$$\begin{aligned}
\oint_C f(z) dz &= \oint_C (u + iv)(dx + idy) \\
&= \oint_C (udx - vdy) + i \oint_C (vdx + udy) \\
&= -\iint_{\mathcal{R}} (v_x + u_y) dx dy + i \iint_{\mathcal{R}} (u_x - v_y) dx dy \\
&= 0,
\end{aligned}$$

where we used Green's Theorem, followed by the Cauchy-Riemann equations. ■

Corollary 2.12 *An analytic function in a simply connected domain $\mathcal{D}$ has an analytic anti-derivative.*

Proof. Consider the domain $\mathcal{D}$, see Fig. 2.20. Let

$$F(z) := \int_{z_0}^{z} f(s) ds.$$

[9] Recall, this means there are no holes

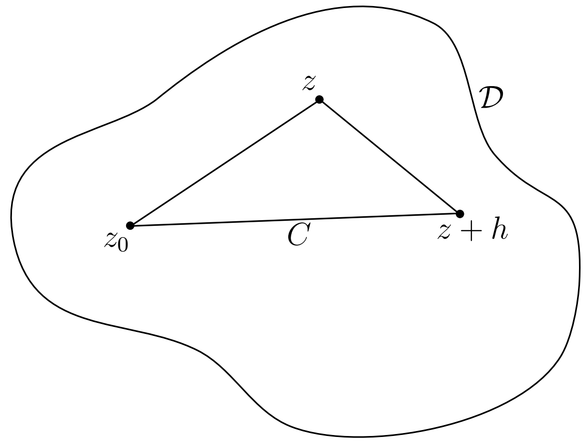

Figure 2.20: The triangular path for the proof of Corollary 2.12.

Then

$$\int_{z_0}^{z+h} f(s)ds + \int_{z+h}^{z} f(s)ds + \int_{z}^{z_0} f(s)ds = 0,$$

by Cauchy's Theorem. Then

$$\begin{aligned} F(z+h) - F(z) &= \int_{z_0}^{z+h} f(s)ds - \int_{z_0}^{z} f(s)ds \\ &= \int_{z}^{z+h} f(s)ds \\ \Rightarrow \qquad \frac{F(z+h)-F(z)}{h} &= \frac{1}{h}\int_{z}^{z+h} f(s)ds \\ \Rightarrow \qquad F'(z) = \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} &= f(z), \end{aligned}$$

by the mean-value theorem. ■

Deformation of contours

Using the same technique used on page 43, we may deform arbitrary contours for integration purposes to sums of small circular contours, each of which encloses one singularity (see Fig. 2.21), so that

$$\oint_C f(z)dz = \sum_{j=1}^{N} \oint_{C_j} f(z)dz,$$

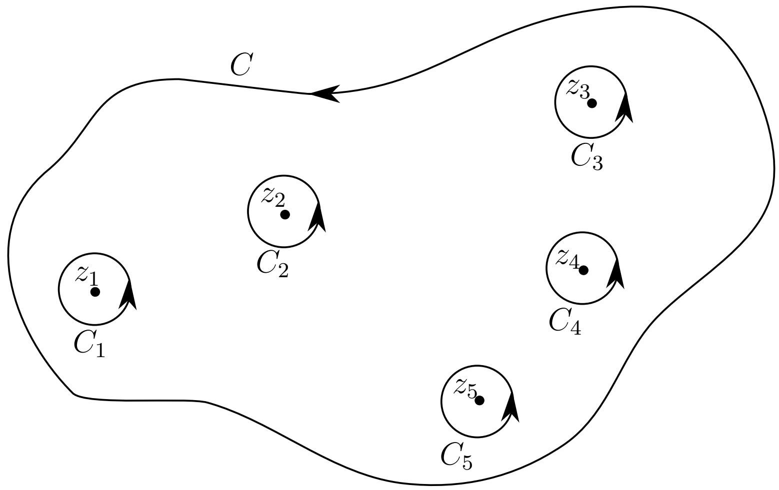

Figure 2.21: Deforming the contour C to small circles C_j enclosing one singularity z_j each.

where N is the number of singularities in C. We have assumed that $f(z)$ is single valued: the introduction of branch cuts requires more careful deformations, as we will see later.

Example. It immediately follows that

$$\oint_C \frac{e^z}{z(z^2-16)}dz = 0,$$

where C is the union of two circles centered at the origin of radius 1 and 3, respectively: $C = \{|z| = 3\} - \{|z| = 1\}$, see Fig. 2.22. Since the integrand is analytic in the annulus, the result follows directly from Cauchy's Theorem.

Example. Let

$$p(z) = A(z-z_1)(z-z_2)\dots(z-z_n),$$

a polynomial of degree n with roots assumed to be distinct. Then[10]

$$\begin{aligned} & \ln p(z) = \ln A + \sum_{k=1}^{n} \ln(z-z_k) \\ \Rightarrow \quad & \frac{p'(z)}{p(z)} = \sum_{k=1}^{n} \frac{1}{z-z_k}. \end{aligned}$$

It follows that

$$\frac{1}{2\pi i}\oint_C \frac{p'(z)}{p(z)}dz = \frac{1}{2\pi i}\sum_{k=1}^{n}\oint_C \frac{1}{z-z_k}dz.$$

[10]We restrict the logarithm to its principal branch, although it does not really matter here (why not?).

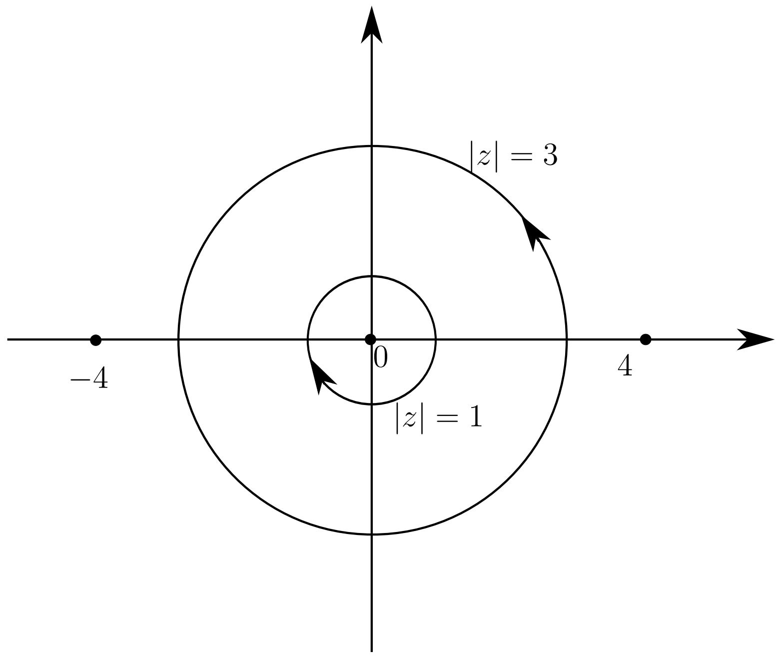

Figure 2.22: The contour $C = \{|z| = 3\} - \{|z| = 1\}$.

We already know that

$$\oint_C \frac{1}{z - z_k} dz = \begin{cases} 2\pi i & \text{if } z_k \text{ inside } C, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\frac{1}{2\pi i} \oint_C \frac{p'(z)}{p(z)} dz = N,$$

where N is the number of roots of $p(z)$ inside C.

Cauchy's Integral Formula and its consequences

We start with *Cauchy's Integral Formula.*

Theorem 2.13 *Let $f(z)$ be analytic inside and on a simple closed contour C, then for all z inside of C,*

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Before we prove this result, let's appreciate what an amazing statement this is: if we know the values of $f(z)$ on a closed curve, we know its values anywhere *inside* of this closed curve. This is definitely not true for functions in $\mathbb{R}^2$!

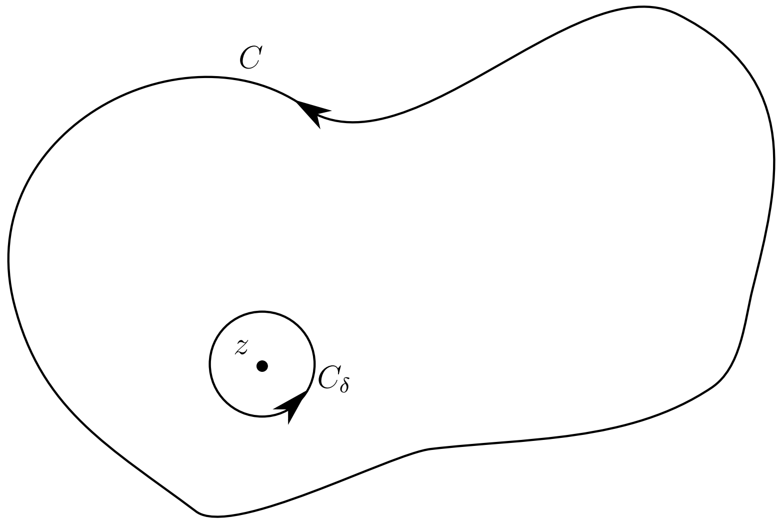

Figure 2.23: The construction for the proof of Cauchy's Formula. The circle C_δ is centered at z and has radius δ.

Proof. (See Fig. 2.23)

$$\begin{aligned}\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta &= \oint_{C_\delta} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \oint_{C_\delta} \frac{f(z) + f(\zeta) - f(z)}{\zeta - z} d\zeta \\ &= f(z) \oint_{C_\delta} \frac{1}{\zeta - z} d\zeta + \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \\ &= 2\pi i f(z) + \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta.\end{aligned}$$

We can bound this last term:

$$\begin{aligned}\left| \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| &\le \oint_{C_\delta} \frac{|f(\zeta) - f(z)|}{|\zeta - z|} |d\zeta| \\ &= \frac{1}{\delta} \oint_{C_\delta} |f(\zeta) - f(z)||d\zeta| \\ &\le \frac{1}{\delta} \oint_{C_\delta} \epsilon |d\zeta| \\ &= \frac{\epsilon}{\delta} 2\pi\delta \\ &= 2\pi\epsilon,\end{aligned}$$

where δ and ϵ are chosen as in the continuity definition for $f(z)$. Thus, as $\epsilon \to 0$,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

which concludes the proof. ■

Example. Consider $f(z) = z$ on C. Then, inside C,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{\zeta}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{\zeta - z + z}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\oint_C \left(1 + \frac{z}{\zeta - z}\right)d\zeta \\ &= \cancelto{0}{\frac{1}{2\pi i}\oint_C d\zeta} + \frac{z}{2\pi i}\oint_C \frac{1}{\zeta - z}d\zeta \\ &= z, \end{aligned}$$

as expected.

Cauchy's Integral Theorem has important consequences.

Theorem 2.14 *Let $f(z)$ be analytic inside of and on a simple closed contour C. Then all derivatives $f^k(z)$, $k = 0, 1, \ldots$ exist inside of C and they are continuous. Further,*

$$f^{(k)}(z) = \frac{k!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}}d\zeta, \quad k = 0, 1, \ldots.$$

This implies that if $f(z)$ is analytic (*i.e.*, its derivative exists), **all** its derivatives exist and are continuous! Quite a strong statement. But then we rock, so this is not so unexpected. Further, it follows from the above formula that all the derivatives are analytic functions themselves, since they have derivatives that exist.

Proof. Let's assume that $f^{(k)}(z)$ is analytic, for some positive integer. Then, from Cauchy's Integral Formula, we have that

$$\begin{aligned} \frac{f^{(k)}(z+h) - f^{(k)}(z)}{h} &= \frac{1}{2\pi i h}\oint_C f^{(k)}(\zeta)\left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right)d\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta - z - h)(\zeta - z)}d\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta - z)^2}\frac{\zeta - z}{\zeta - z - h}d\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta - z)^2}\frac{\zeta - z - h + h}{\zeta - z - h}d\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta - z)^2}\left(1 + \frac{h}{\zeta - z - h}\right)d\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta - z)^2}d\zeta + \frac{h}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta - z)^2(\zeta - z - h)}d\zeta. \end{aligned}$$

For this last term, we have

$$\left|\frac{h}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta-z)^2(\zeta-z-h)}d\zeta\right| \leq \frac{h}{2\pi}ML,$$

where L is the arclength of C and M is the maximum value of the integrand. Since we have assumed that $f^{(k)}(z)$ is analytic, it is bounded on C. Further, denote by 2δ the minimum of $|\zeta - z|$:

$$\min_{\zeta\in C}|\zeta - z| = 2\delta > 0.$$

Further, choose $|h| < \delta$. Then

$$|\zeta - z - h| \geq ||\zeta - z| - |h|| \geq 2\delta - \delta = \delta.$$

Thus

$$M \leq \frac{\max_{\zeta\in C}|f^{(k)}(z)|}{4\delta^3},$$

and the second integral $\to 0$, as $h \to 0$. We conclude that

$$f^{(k+1)}(z) = \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta-z)^2}d\zeta.$$

Thus, if $f^{(k)}(z)$ is analytic, $f^{(k+1)}(z)$ is defined by this formula. Next, we repeat a lot of these arguments: we want to show that $f^{(k+2)}(z)$ exists as well, *i.e.*, $f^{(k+1)}(z)$ is analytic. Let's see how we fare. We have that

$$\begin{aligned}
\frac{f^{(k+1)}(z+h) - f^{(k+1)}(z)}{h} &= \frac{1}{2\pi i h}\oint_C f^{(k)}(\zeta)\left(\frac{1}{(\zeta-z-h)^2} - \frac{1}{(\zeta-z)^2}\right)d\zeta \\
&= \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta-z-h)^2(\zeta-z)^2}(2\zeta-2z-h)d\zeta \\
&= \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta-z)^3}\frac{(2\zeta-2z-h)(\zeta-z)}{(\zeta-z-h)^2}d\zeta \\
&= \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta-z)^3}\frac{(2\zeta-2z-2h+h)(\zeta-z-h+h)}{(\zeta-z-h)^2}d\zeta \\
&= \frac{2}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta-z)^3}\left(1+\frac{h}{2(\zeta-z-h)}\right)\left(1+\frac{h}{\zeta-z-h}\right)d\zeta \\
&= \frac{2}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta-z)^3}\left(1+\frac{3h}{2(\zeta-z-h)}+\frac{h^2}{2(\zeta-z-h)^2}\right)d\zeta.
\end{aligned}$$

The last two terms are shown to $\to 0$ as $h \to 0$, following similar arguments as before. Thus

$$f^{(k+2)}(z) = \frac{2}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{(\zeta-z)^3}d\zeta.$$

Thus, if $f^{(k)}(z)$ is analytic, $f^{(k+1)}(z)$ is analytic, since its derivative is well defined as well. It remains to prove the formula given in the theorem.

From Cauchy's Integral Theorem,

$$f^{(k)}(z) = \frac{1}{2\pi i}\oint_C \frac{f^{(k)}(\zeta)}{\zeta - z}d\zeta.$$

Using the parametrization $C : \zeta = \zeta(t)$, we get

$$f^{(k)}(z) = \frac{1}{2\pi i}\int_a^b \frac{f^{(k)}(\zeta(t))\zeta'(t)}{\zeta(t) - z}dt.$$

Next, we use integration by parts[11]. Let

$$\begin{aligned} u &= \frac{1}{\zeta - z} & \Rightarrow & \quad du = -\frac{1}{(\zeta - z)^2}\zeta'(t)dt \\ dv &= f^{(k)}(\zeta(t))\zeta'(t)dt & \Rightarrow & \quad v = f^{(k-1)}(\zeta(t)). \end{aligned}$$

Thus (the boundary term vanishes, since the curve C is closed)

$$\begin{aligned} f^{(k)}(z) &= \frac{1}{2\pi i}\int_a^b \frac{f^{(k-1)}(\zeta(t))\zeta'(t)}{(\zeta - z)^2}dt \\ &= \frac{1\cdot 2}{2\pi i}\int_a^b \frac{f^{(k-2)}(\zeta(t))\zeta'(t)}{(\zeta - z)^3}dt \\ &\dots \\ &= \frac{k!}{2\pi i}\int_a^b \frac{f(\zeta(t))\zeta'(t)}{(\zeta - z)^{k+1}}dt \\ &= \frac{k!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}}d\zeta, \end{aligned}$$

where we have used repeated integration by parts[12]. This proves the formula from the theorem. ■

Under the above conditions, we can derive some bounds on the derivatives of analytic functions. In

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta,$$

let C be a circle of radius R centered at z. Then $|\zeta - z| = R$. With $|f(\zeta)| \le M$ on C, we have that

$$|f^{(n)}(z)| \le \frac{n!}{2\pi}\frac{M}{R^{n+1}}2\pi R = \frac{n!M}{R^n}.$$

The following theorem is a simple consequence of this, but it is one of the most important in applied complex analysis.

[11] When in doubt, always use integration by parts. I'm not kidding.

[12] Apparently, we were in doubt repeatedly...

Theorem 2.15 (Liouville). *If $f(z)$ is entire and bounded in the z -plane, including at ∞, then $f(z)$ is constant.*

Proof. From the above bounds, we get that on a circle of radius R centered at z that

$$|f'(z)| \leq \frac{M}{R}.$$

Since we can pick the same M valid in all of $\mathbb{C}$, we can allow R to be arbitrarily large. This implies that

$$f'(z) = 0,$$

for any z. Thus

$$f(z) = f(0) + \int_0^z f'(s)ds = f(0),$$

which is a constant. This proves the theorem. ■

Corollary 2.16 Fundamental Theorem of Algebra. *Any polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ ($a_n \neq 0$) of degree $n \geq 1$ (integer, of course) has at least one root.*

Proof. We prove this by contradiction. Assume $p(z) \neq 0$, for all $z \in \mathbb{C}$, then $1/p(z)$ is analytic everywhere and decays to 0 as $z \to \infty$. By Liouville's Theorem, $1/p(z)$ is a constant. This is a contradiction. ■

Of course, having established the existence of one root, we can proceed by long division to establish the existence of n roots, not necessarily all distinct.

Liouville's Theorem is very powerful when used like this[13]. Often we can establish that certain combinations of functions (each of which is not necessarily entire) are entire. Thus this combination is constant, and we have established a relation satisfied by these functions. We will see more examples later on.

Theorem 2.17 (Morera). *If $f(z)$ is continuous in $\mathcal{D}$ and $\oint_C f(z)dz = 0$ for every simple closed contour in D, then $f(z)$ is analytic in $\mathcal{D}$.*

Proof. The proof of Corollary 2.12 really only uses that $\oint_C f(z)dz = 0$, with $f(z)$ continuous. Using the result of the corollary, we have that there exists an analytic function $F(z)$ such that

$$F'(z) = f(z).$$

It follows from Cauchy's Integral Theorem that $f(z) = F'(z)$ is also analytic. ■

Analytic functions satisfy a variety of mean-value theorems. From Cauchy's Integral Formula, with C chosen to be a circle centered at z, so that $\zeta = z + re^{i\theta}$, we have that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

[13] I don't know how else to use it.

Thus, the value at the center of the circle is the average of the values on the circle of radius r. Similarly, by multiplying the above equation by r and integrating over all radii from 0 to R, we get

$$\begin{aligned} f(z)\int_0^R r dr &= \frac{1}{2\pi}\int_0^R \int_0^{2\pi} f(z+re^{i\theta}) r d\theta dr \\ \Rightarrow \qquad f(z) &= \frac{1}{\pi R^2}\iint_A f(z+re^{i\theta})dA, \end{aligned}$$

where the integral is over the entire disk A of radius R. This states that $f(z)$ is the average value of $f(\zeta)$ over the area of the disk. In fact, the mean-value theorem can be modified to where the area one integrates over is not necessarily a disk, but we won't prove that here.

These mean-value results allow us to prove the following theorem.

Theorem 2.18 (Maximum Principle). *(1) If $f(z)$ is analytic in a domain $\mathcal{D}$, then $|f(z)|$ cannot have a maximum in $\mathcal{D}$ unless $f(z)$ is constant. (2) If $f(z)$ is analytic in a bounded region $\mathcal{D}$ and $|f(z)|$ is continuous in $\overline{\mathcal{D}}$, then $|f(z)|$ assumes its maximum on the boundary of the region.*

Proof. Assume that z is an interior point of $\mathcal{D}$ such that $|f(\zeta)| \leq |f(z)|$ for all ζ in the region. Choose a circle of radius R enclosing the disk $\mathcal{D}_0$ lying entirely inside the region. Then

$$\begin{aligned} |f(z)| &\leq \frac{1}{\pi R^2}\iint_{\mathcal{D}_0} |f(\zeta)| dA \\ &\leq \frac{1}{\pi R^2}\iint_{\mathcal{D}_0} |f(z)| dA \\ &= |f(z)|\frac{1}{\pi R^2}\iint_{\mathcal{D}_0} dA \\ &= |f(z)|. \end{aligned}$$

The only way to establish equality at every step if for $|f(\zeta)| = |f(z)|$. If $|f(z)|$ is constant, then so is $f(z)$, as one establishes using the polar coordinate version of the Cauchy-Riemann equations. This proves (1).

To prove (2): since the region is compact, the maximum is attained somewhere. Since it is not attained inside, it must be attained on the boundary. ■

2.6 Exercises

1. Derive the polar-coordinates form of the Cauchy-Riemann equations

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta,$$

where $x = r\cos\theta$ and $y = r\sin\theta$

2. Write $f(z) = R(x, y)\exp(i\theta(x, y))$, where $R(x, y)$ and $\theta(x, y)$ are real-valued functions. Derive equations like the Cauchy-Riemann equations that connect the x and y derivatives of $R(x, y)$ and $\theta(x, y)$ if $f(z)$ is analytic. Use these equations to show that if $R(x, y)$ is constant, than $f(z)$ is constant. Similarly, if $\theta(x, y)$is constant, than so is $f(z)$.

3. Give all values of π^i and i^π (*i.e.*, put them in the form $x + iy$ or $re^{i\theta}$, with x, y, r, θ real)

4. Write out the real and imaginary part of $w = z^z$. Confirm that $w' = z^z(1 + \ln(z))$. What is i^i?

5. Describe[14] the Riemann surface on which the multi-valued function $w(z)$, defined by $w^2 = \prod_{j=1}^{n=3}(z - a_j)$ is single-valued. What happens for $n = 4, 5$? For $n > 5$? You may assume that all the a_j are distinct.

6. (**10 points**) The mean-value theorem for real-valued functions states that if f is a differentiable map of $[a, b] \subset \mathbb{R}$ to $S \subset \mathbb{R}$, then $f(b) - f(a) = (b - a)f'(t)$ for some $t \in [a, b]$. Show that this is typically not true for complex-valued functions. In other words, show that if f is a differentiable map of $[a, b] \subset \mathbb{R}$ to $S \subset \mathbb{C}$, it is not necessarily true that $f(b) - f(a) = (b - a)f'(t)$ for some $t \in [a, b]$.

7. Calculate the average value of $x^2 - y^2 + 4x$ on the closed curve formed by the semi-cirle $y = 3 + \sqrt{25 - x^2}$ and the straight-line segments $L_1 = \{x = -5, -1 \le y \le 3\}$, $L_2 = \{x = 5, -1 \le y \le 3\}$, $L_3 = \{y = -1, -5 \le x \le 5\}$.

8. Let $f(z)$ be an even, single-valued analytic function, except perhaps at $z = 0$. Prove that
$$\oint_C f(z)dz = 0,$$
where C is a simple closed contour encircling $z = 0$.

9. Consider
$$I = \int_{-\infty}^{\infty} \frac{1}{1 + x^2}dx.$$
Evaluate I by considering
$$I_C = \oint_{C_R} \frac{1}{1 + z^2}dz,$$
where C_R is a closer contour consisting of $[-R, R]$ and a semi-circle of radius R in the upper half plane from $z = -R$ to $z = R$. Evaluate the integral using partial fractions and by showing that the contribution from the semi-circle vanishes as $R \to \infty$.

   Having done this, use a similar method to calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$. Can the same method be used to calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^n}$, where $n \ge 2$ is an integer?

[14] Go through the topology steps loosely like we did in Section 2.3.

10. Calculate
$$\oint_C \frac{z^2+1}{z^2-z-2}dz,$$
where C is a circle of radius 12, centered at the origin.

11. The Legendre polynomials $P_n(z)$ $(n = 0, 1, \ldots)$ are defined by
$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left(z^2-1\right)^n.$$
Show that $P_n(z)$ can be represented as
$$P_n(z) = \frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{2^n(t-z)^{n+1}}dt,$$
where C is a simple closed contour encircling z.

12. **Wallis's formula** Let $n \in \mathbb{N}$.
    (a) Parameterize the unit circle $C(0,1)$, to find that
    $$\frac{1}{2\pi i}\oint_{C(0,1)}\left(z+\frac{1}{z}\right)^n\frac{dz}{z} = \frac{2^n}{2\pi}\int_0^{2\pi}\cos^n t dt.$$
    (b) Using the binomial formula on the left-hand side, use the above to find that
    $$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2k} t dt = \frac{(2k)!}{2^{2k}(k!)^2},$$
    for $k \in \mathbb{N}$. Also, find that
    $$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2k+1} t dt = 0,$$
    again for $k \in \mathbb{N}$.

13. Show that
$$I = \int_0^\infty \cos(x^n)dx = \frac{\Gamma(1/n)}{n}\cos\left(\frac{\pi}{2n}\right),$$
and
$$J = \int_0^\infty \sin(x^n)dx = \frac{\Gamma(1/n)}{n}\sin\left(\frac{\pi}{2n}\right),$$
where
$$\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}ds,\ z > 0,$$
the gamma function. Do this by considering $I + iJ$ and a closed contour consisting of a pie segment with straight sides $[0, R]$, $[0, Re^{i\pi/2n}]$ and a circular side arced from R to $Re^{i\pi/2n}$, showing that the contribution from the circular arc vanishes as $R \to \infty$.

Chapter 3

Sequences, series, and singularities

3.1 Complex sequences and series

The sequence of complex functions $\{f_n(z)\}_{n=1}^{\infty}$, defined in a region $\mathcal{R}$, converges to $f(z)$ if

$$\lim_{n\to\infty} f_n(z) = f(z),$$

where $f(z)$ exists and is finite on $\mathcal{R}$, or on a subset of $\mathcal{R}$. Thus, for each z and given $\epsilon > 0$, there is an $N(\epsilon, z)$ such that $\forall n > N(\epsilon, z)$:

$$|f_n(z) - f(z)| < \epsilon.$$

The sequence $\{f_n(z)\}_{n=1}^{\infty}$ is said to converge to $f(z)$ *uniformly*, if $N(\epsilon, z)$ can be chosen to be independent of z.

An infinite *series* can be considered as a special case of a sequence, by considering partial sums

$$S_M(z) = \sum_{n=1}^{M} s_n(z).$$

Then

$$S(z) = \lim_{M\to\infty} S_M(z)$$

is the limit of the series $\sum_{n=1}^{\infty} s_n(z)$. For a series to be convergent, it is necessary that

$$\lim_{n\to\infty} s_n(z) = 0.$$

Example. Consider

$$f_n(z) = \frac{1}{nz} \quad \to \quad f(z) = 0.$$

First, we consider $\mathcal{R}$ to be the annulus $1 < |z| < 2$. Then

$$|f_n(z) - f(z)| = |f_n(z)| = \frac{1}{n|z|} < \epsilon.$$

Thus we need N to be greater than $1/\epsilon|z|$, which is less than $1/\epsilon$. Thus, we can establish that $f_n \to f = 0$ uniformly on the annulus $1 < |z| < 2$.

Next, we consider $\mathcal{R}$ to be the unit disk $|z| < 1$. From the same inequalities as above, it follows a uniform bound is not possible, since $1/|z|$ is unbounded.

Theorem 3.1 *If the elements of the sequence $\{f_n(z)\}_{n=1}^{\infty}$ are continuous on $\mathcal{R}$ and $f_n \to f$ uniformly, then (i) f(z) is continuous, and (ii) for any finite contour C in $\mathcal{R}$*

$$\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz.$$

Before we prove the theorem, we should remark that this theorem tells us that **uniformly convergent series can be integrated term-by-term**. Indeed, suppose the series $\sum_{k=1}^{\infty} s_k(z)$ is uniformly convergent. Then, according to the theorem,

$$\begin{aligned}
& \lim_{n\to\infty} \int_C \sum_{k=1}^{n} s_k(z)dz = \int_C \lim_{n\to\infty} \sum_{k=1}^{n} s_k(z)dz \\
\Leftrightarrow \quad & \lim_{n\to\infty} \sum_{k=1}^{n} \int_C s_k(z)dz = \int_C \sum_{k=1}^{\infty} s_k(z)dz \\
\Leftrightarrow \quad & \sum_{k=1}^{\infty} \int_C s_k(z)dz = \int_C \sum_{k=1}^{\infty} s_k(z)dz,
\end{aligned}$$

which proves the statement.

Proof of the theorem. (i) First we prove the continuity of $f(z)$ at $z_0 \in \mathcal{R}$, arbitrary but fixed. Thus, we seek to establish that for all $\epsilon > 0$, there exists $\delta > 0$, such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < \epsilon$. We have

$$f(z) - f(z_0) = f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0),$$

then

$$|f(z) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|.$$

Since $f_n(z_0) \to f(z_0)$ for $n > N(\epsilon)$, we have that $|f(z) - f_n(z)| \leq \epsilon/3$ and that $|f(z_0) - f_n(z_0)| \leq \epsilon/3$. This fixes $n > N$. Further, with this chosen n, since $f_n(z)$ is continuous at z_0, $|f_n(z) - f_n(z_0)| \leq \epsilon/3$, for $|z - z_0| < \delta$. This is where we determine δ. Thus

$$|f(z) - f(z_0)| \leq \epsilon, \quad \text{for} \quad |z - z_0| \leq \delta.$$

This proves the first part of the theorem.

Next, we show that the integral of $f(z)$ can be obtained as the limit of the integrals. We have

$$\begin{aligned}\left|\int_C f_n(z)dz - \int_C f(z)dz\right| &= \left|\int_C (f_n(z) - f(z))dz\right| \\ &\leq \int_C |f_n(z) - f(z)|dz \\ &\leq \epsilon L,\end{aligned}$$

where L is the arclength of C. This concludes the proof. ■

A useful tool to establish the uniform convergence of a series is the **Weierstrass M-test**.

Theorem 3.2 *Let $|b_j(z)| \leq M_j$ in $\mathcal{R}$, where M_j does not depend on z. If $\sum_{j=1}^{\infty} M_j$ converges, then $S(z) = \sum_{j=1}^{\infty} b_j(z)$ converges uniformly in $\mathcal{R}$.*

Proof.

$$\begin{aligned}|S_n(z) - S_m(z)| &= |\sum_{j=m+1}^{n} b_j(z)| \\ &\leq \sum_{j=m+1}^{n} |b_j(z)| \\ &\leq \sum_{j=m+1}^{n} M_j \\ &\leq \sum_{j=m+1}^{\infty} M_j \\ &\to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

Thus the sequence $S_n(z)$ is a uniform Cauchy sequence and therefore converges uniformly to a limit. This completes the proof. ■

We use the Weierstrass M-test to prove the **Ratio Test.**

Theorem 3.3 (Ratio Test). *If $|b_1(z)|$ is bounded and if*

$$\left|\frac{b_{j+1}(z)}{b_j(z)}\right| \leq M < 1,$$

then $S = \sum_{j=1}^{\infty} b_j(z)$ is uniformly convergent.

Proof. We have that

$$\begin{aligned} b_n(z) &= b_1(z)\frac{b_2(z)}{b_1(z)}\frac{b_3(z)}{b_2(z)}\cdots\frac{b_n(z)}{b_{n-1}(z)} \\ \Rightarrow \qquad |b_n(z)| &\le |b_1(z)|M^{n-1}, \end{aligned}$$

so that

$$|S(z)| \le \sum_{j=1}^{\infty} |b_j(z)| \le |b_1(z)| \sum_{j=1}^{\infty} M^{j-1} = \frac{|b_1|}{1-M} \le \frac{B}{1-M},$$

where B is a bound for $|b_1(z)|$. Thus the series is bounded by a series which is uniformly convergent. By the M-test, so is $S(z)$. ■

3.2 Taylor series

A *power series* around $z = z_0$ is defined by

$$f(z) = \sum_{j=0}^{\infty} b_j (z-z_0)^j.$$

For simplicity, we'll often work with $z_0 = 0$, without loss of generality. Then

$$f(z) = \sum_{j=1}^{\infty} b_j z^j.$$

Theorem 3.4 *If $\sum_{j=0}^{\infty} b_j (z-z_0)^j$ converges for $z_* \neq z_0$, then it converges for all z in $D : \{z : |z-z_0| < |z_* - z_0|\}$. This convergence is uniform in any disk of radius $R < |z_* - z_0|$.*

Proof. For $j \ge J$ and $|z-z_0| < |z_*-z_0|$,

$$\begin{aligned} |b_j(z-z_0)^j| &= \left| b_j \frac{(z-z_0)^j}{(z_*-z_0)^j} (z_*-z_0)^j \right| \\ &\le \left| \frac{z-z_0}{z_*-z_0} \right|^j \\ &\le \frac{R^j}{|z_*-z_0|^j} < 1. \end{aligned}$$

The first inequality follows from the fact that for sufficiently large j, $b_j(z_*-z_0)^j \to 0$, since the series is assumed to converge at z_*. Using $M_j = R^j/|z_*-z_0|^j$ in the Weierstrass M-test for sufficiently large $j > J$, the result follows. ■

Theorem 3.5 (Taylor Series.) *Let $f(z)$ be analytic for $|z - z_0| \leq R$. Then*

$$f(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j,$$

with

$$b_j = \frac{f^{(j)}(z_0)}{j!}, \quad j = 0, 1, 2, \ldots,$$

and the series converges uniformly for $|z - z_0| \leq R_1 < R$.

Proof. From Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where C is a circle of radius R, centered at z_0. Near $z = z_0$, we let $\zeta = z_0 + \xi$. Then the circle of radius R centered at z_0 becomes a circle of radius R centered at 0, still denoted by C[1].

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(z_0 + \xi)}{\xi + z_0 - z} d\xi \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z_0 + \xi)}{\xi} \frac{1}{1 - (z - z_0)/\xi} d\xi. \end{aligned}$$

Note that

$$\left| \frac{z - z_0}{\xi} \right| = \frac{|z - z_0|}{|\xi|} = \frac{|z - z_0|}{R} < 1.$$

Thus

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(z_0 + \xi)}{\xi} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{\xi^n} d\xi \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z_0 + \xi)}{\xi^{n+1}} d\xi \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \frac{2\pi i f^{(n)}(z_0)}{n!} \\ &= \sum_{n=0}^{\infty} b_j (z - z_0)^n, \end{aligned}$$

which is what we had to prove. Thus, provided that we prove that the geometric series used above converges uniformly[2], in $|z - z_0| \leq R_1 < R$, we are done. We'll do this next. ■

Let $x = (z - z_0)/\xi$. Consider

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

[1]I know, I know. This is bad practice. I'm running out of symbols...

[2]So we can integrate the infinite series term-by-term.

We have to show that this series is uniformly convergent for $|x| \leq R_2 < 1$. Let's turn it into a theorem!

Theorem 3.6 *The geometric series $S(x) = \sum_{n=0}^{\infty} x^n \to 1/(1-x)$ uniformly, for $|x| \leq R_2 < 1$.*

Proof. Consider

$$S_N(x) = \sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}.$$

For $|x| \leq R_2 < 1$, we have that

$$\begin{aligned} |S_N(x) - S(x)| &= \left| \frac{x^{N+1}}{1-x} \right| \\ &= \frac{|x|^{N+1}}{|1-x|} \\ &\leq \frac{R_2^{N+1}}{|1-|x||}. \end{aligned}$$

We have that

$$|1 - |x|| = 1 - |x| \geq 1 - R_2 \quad \Rightarrow \quad \frac{1}{|1-|x||} \leq \frac{1}{1-R_2}.$$

Thus

$$|S_N(x) - S(x)| \leq \frac{R_2^{N+1}}{1-R_2}.$$

We wish to show that this is less than ϵ for $N > M(\epsilon)$, independent of x. This is an easy equation to solve for N, establishing the result. ■.

The supremum (or least upper bound) of all R for which a power series converges is called the **radius of convergence.** Since Taylor series converge uniformly within their radius of convergence R, they can be differentiated and integrated term-by-term. The new series resulting from these operations are themselves uniformly convergent for $R_1 < R$.

Theorem 3.7 (Comparison Test.) *Let $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ be convergent for $|z - z_0| < R$. If for $j \geq N$, $|b_j| \leq |a_j|$, then $\sum_{j=0}^{\infty} b_j (z - z_0)^j$ is also convergent for $|z - z_0| < R$.*

Proof. For $j \geq N$ and $|z - z_0| < R_1 < R$,

$$
\begin{aligned}
|b_j(z-z_0)^j| &\leq |a_j(z-z_0)^j| \\
&= |a_j R_1^j| \left|\frac{z-z_0}{R_1}\right|^j \\
&\leq |a_j R_1^j| \frac{|z-z_0|^j}{R_1^j} \\
&\leq \frac{|z-z_0|^j}{R_1^j} < 1,
\end{aligned}
$$

and convergence follows from the Weierstrass M-test. ■

3.3 Remarks about analyticity

As a summary, we know that the following statements are equivalent in the region $\mathcal{R}$.

1. $f(z)$ is analytic in $\mathcal{R}$,
2. $f'(z)$ exists in $\mathcal{R}$,
3. $f^{(n)}(z)$ exists in $\mathcal{R}$ for all $n \geq 0$,
4. $f(z)$ has a uniformly convergent Taylor series centered at each point of $\mathcal{R}$.

Theorem 3.8 *Let each of $f(z)$ and $g(z)$ be analytic in D. If $f(z) \equiv g(z)$ for $z \in D' \subset D$, or on a curve interior to D, then $f(z) \equiv g(z)$, everywhere in D.*

Proof. For any point $z_0 \in D'$, consider the largest R such that $C : \{|z - z_0| = R\} \subset D$, see Fig. 3.1. Inside C, both $f(z)$ and $g(z)$ may be represented by a Taylor series. By the uniqueness of power series, $f(z) \equiv g(z)$ inside C. Now we have extended D', the domain on which $f(z)$ and $g(z)$ are equal. This argument is repeated to cover all of D. ■

The following theorem is one of the truly exceptional results in complex analysis. It tells us that we can build analytic functions from smaller analytic pieces, as long as those pieces agree on a curve separating where the pieces are defined[3].

[3]Whoa! Really, think about it. Whoa!

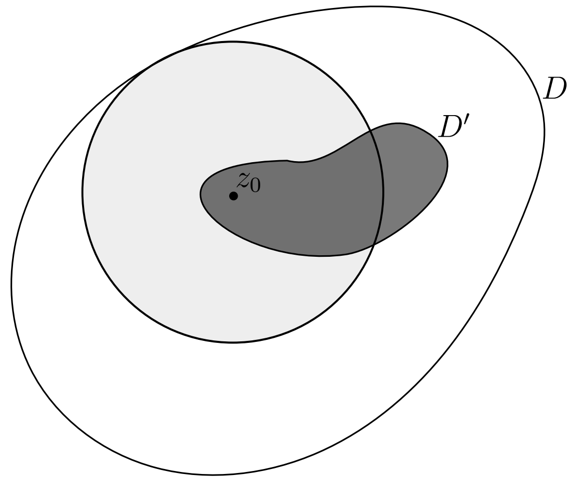

Figure 3.1: The domains D and D'.

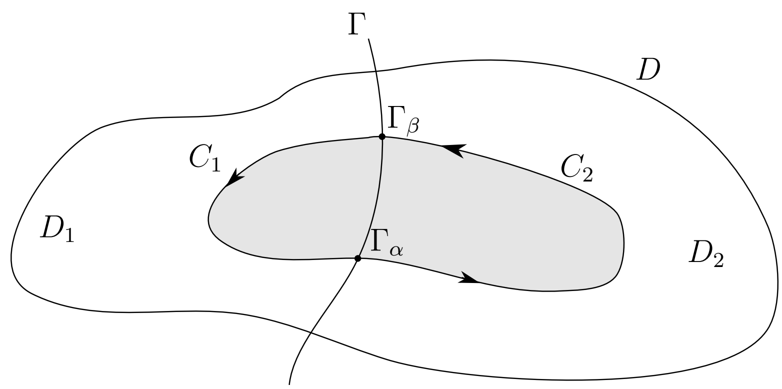

Figure 3.2: The set-up for the analytic continuation theorem.

Theorem 3.9 (Analytic Continuation). *Let D_1 and D_2 be two disjoint domains whose boundaries share a common contour Γ. Let $f(z)$ be analytic in D_1 and continuous on $D_1 \cup \Gamma$. Let $g(z)$ be analytic in D_2 and continuous on $D_2 \cup \Gamma$. Let $f(z) = g(z)$ on Γ. Then*

$$H(z) := \begin{cases} f(z), & z \in D_1 \\ f(z) = g(z), & z \in \Gamma \\ g(z), & z \in D_2 \end{cases},$$

is analytic in $D = D_1 \cup \Gamma \cup D_2$.

The function $g(z)$ is called the analytic continuation of $f(z)$ from D_1 to D_2.

Proof. Consider a simple, closed contour C in D. If $C \subset D_1$ or $C \subset D_2$, then $\oint_C H(z)dz = 0$, by Cauchy's Theorem, applied to $f(z)$ in D_1 or $g(z)$ in D_2. If C crosses Γ, then (see Fig. 3.2)

$$\oint_C H(z)dz = \int_{C_1} f(z)dz + \int_{C_2} g(z)dz + \int_{\Gamma_\alpha}^{\Gamma_\beta} f(z)dz + \int_{\Gamma_\beta}^{\Gamma_\alpha} g(z)dz.$$

The sum of the first and third integral is zero, as is the sum of the second and fourth integral. Thus

$$\oint_C H(z)dz = 0,$$

for all contours C in $D_1 \cup \Gamma \cup D_2$. By Morera's Theorem, $H(z)$ is analytic in D. ■

The following result characterizes the zeros of analytic functions.

Theorem 3.10 *If $f(z) \not\equiv 0$ is analytic in D, then its zeros in D are isolated.*

Proof. About a zero z_0, $f(z)$ has a Taylor expansion of the form

$$f(z) = (z - z_0)^n g(z),$$

where n is the order of the zero and $g(z_0) \neq 0$. Note that $g(z)$ has a Taylor series about z_0. There must be a maximal n, since otherwise the derivatives of all orders of $f(z)$ at z_0 would vanish and the Taylor series would vanish identically, and $f(z)$ with it, identically in a neighborhood of z_0. If it were identically zero on a neighborhood of z_0, by analytic continuation it would be zero on all of D, violating our assumption. ■

Analytic continuation of power series

Consider $f(z)$, analytic in $C_0 = \{z : |z - z_0| < R_0\}$. In C_0, we represent $f(z)$ by its power series:

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

which has radius of convergence R_0. We know that on the boundary of C_0, $f(z)$ has at least one singularity, say at z_s, see Fig. 3.3. Since the power series centered at z_0 is uniformly convergent inside C_0, we may rearrange the series to find a power series centered at z_1, inside

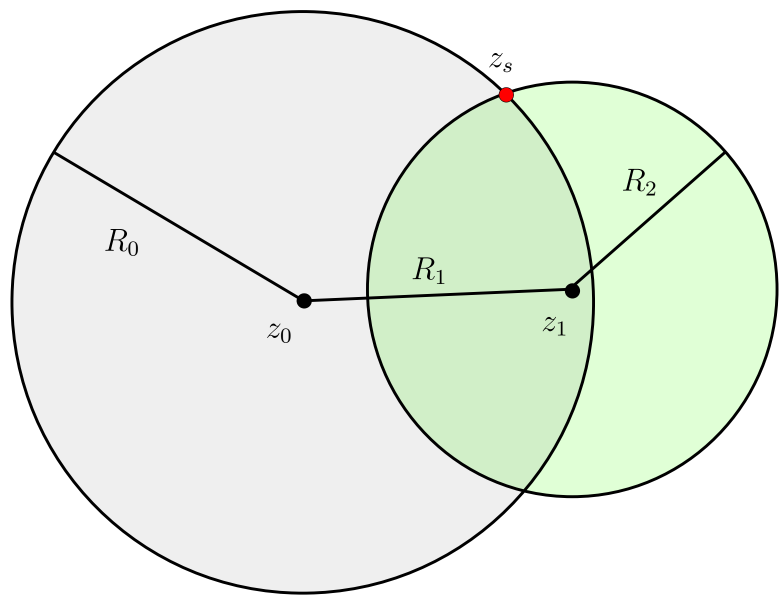

Figure 3.3: Analytic continuation by series rearrangement from one disk of convergence centered at z_0 to one centered at z_1, resulting in an augmented region of analyticity. The singularity is located at z_s.

C_0:

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} b_n (z - z_1 + z_1 - z_0)^n \\
&= \sum_{n=0}^{\infty} b_n \sum_{k=0}^{n} \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_n \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} b_n \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} \\
&= \sum_{k=0}^{\infty} c_k (z - z_1)^k,
\end{aligned}
$$

where

$$
c_k = \sum_{n=k}^{\infty} b_n \binom{n}{k} (z_1 - z_0)^{n-k}.
$$

The new series has its own radius of convergence, say R_2. This is the distance from z_1 to the nearest singularity of $f(z)$, perhaps z_s, perhaps another singularity. If $R_2 > R_0 - |z_1 - z_0|$, then we have extended the domain of analyticity of $f(z)$, as illustrated in Fig. 3.3. This process can be repeated to extend the domain of analyticity of $f(z)$ further. We call this **analytic continuation by series rearrangement**.

3.4 Functions defined by integrals

Functions defined by integrals arise in applications all the time: Fourier transforms, Laplace transforms, *etc.* Consider

$$F(z) = \int_a^b g(z,t)dt, \quad a \le t \le b.$$

Theorem 3.11 *Suppose that for all $t \in (a,b)$, $g(z,t)$ is analytic for $z \in D$, and for all $z \in D$, $g(z,t)$ is a continuous function of t. Then (i) $F(z)$ is analytic in D, and (ii) $F'(z) = \int_a^b g_z(z,t)dt$.*

Proof. Since $g(z,t)$ is analytic in z,

$$g(z,t) = \sum_{j=0}^{\infty} c_j(t)(z-z_0)^j,$$

where z_0 is an arbitrary point in D. Here

$$c_j(t) = \frac{1}{2\pi i}\oint_C \frac{g(\zeta,t)}{(\zeta-z_0)^{j+1}}d\zeta,$$

for $j = 0, 1, \ldots$. We let C be a circle inside D centered at z_0. Since $g(\zeta,t)$ is a continuous function of t, so is $c_j(t)$, Since $g(z,t)$ is analytic in D it is bounded on C, say $|g(\zeta,t)| < M$. Thus

$$|c_j(t)| \le \frac{M}{2\pi}\frac{2\pi R}{R^{j+1}} = \frac{M}{R^j}.$$

We have that

$$g(z,t) = \sum_{j=0}^{\infty} c_j(t)(z-z_0)^j$$

converges uniformly inside C by the Weierstrass M-test and thus we can switch sums and integrals. A similar argument shows that

$$\left|\int_a^b c_j(t)dt\right| \le \int_a^b |c_j(t)|dt \le (b-a)\frac{M}{R^j}.$$

Thus the power series for $F(z)$ can be differentiated term-by-term.

We have

$$
\begin{aligned}
F(z) &= \int_a^b \sum_{j=0}^{\infty} c_j(t)(z-z_0)^j \\
&= \sum_{j=0}^{\infty} (z-z_0)^j \int_a^b c_j(t)dt \\
\Rightarrow \qquad F'(z) &= \sum_{j=1}^{\infty} j(z-z_0)^{j-1} \int_a^b c_j(t)dt \\
&= \int_a^b \sum_{j=1}^{\infty} jc_j(t)(z-z_0)^{j-1}dt \\
&= \int_a^b g_z(z,t)dt.
\end{aligned}
$$

This proves (ii). Since its derivative is defined $F(z)$ is analytic, proving (i). ■

It should be noted that the above proof fails if a or b or both are infinite. In other words, when we consider integrals like the Fourier or Laplace transform, some extra work will be necessary.

3.5 Laurent series

Taylor series allow us to write functions that are analytic at z_0 in terms of certain basis functions, *i.e.*, the positive integer power functions $(z-z_0)^j$, $j = 0, 1, \ldots$. By allowing a larger set of basis functions, we can hope to be able to represent functions that are not analytic at z_0.

Theorem 3.12 (Laurent Series.) *If $f(z)$ is analytic in $R_1 \le |z-z_0| \le R_2$ (but not necessarily at z_0!), then it can be represented by the expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n,$$

in $R_1 < R_a \le |z-z_0| \le R_b < R_2$, and

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n \in \mathbb{Z},$$

where C is a closed contour enclosing z_0 in the annulus, see Fig. 3.4.

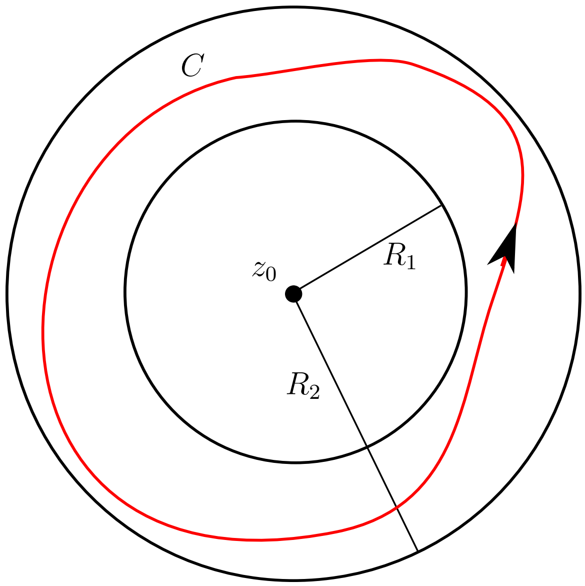

Figure 3.4: The annulus and the contour for the Laurent Series.

Proof. The boundary of the annulus can be written as $C_2 - C_1$, where $C_2 = \{z : |z - z_0| = R_2\}$, $C_1 = \{z : |z - z_0| = R_1\}$. By Cauchy's Formula, for z inside the annulus[4],

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

In the first integral, we let $\zeta - z = \zeta - z_0 + z_0 - z = \zeta - z_0 - (z - z_0)$. Thus

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0 - (z - z_0)} \\ &= \frac{1}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)}, \end{aligned}$$

where

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < 1.$$

Thus

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{j=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^j,$$

[4]But not on its boundary

which is uniformly convergent. For the second integral, we have

$$\begin{aligned}
-\frac{1}{\zeta - z} &= \frac{1}{z-\zeta} \\
&= \frac{1}{z - z_0 - (\zeta - z_0)} \\
&= \frac{1}{(z-z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)} \\
&= \frac{1}{z-z_0} \sum_{j=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^j,
\end{aligned}$$

which is also uniformly convergent. As above, note that here

$$\left|\frac{\zeta - z_0}{z - z_0}\right| < 1.$$

Substitution in Cauchy's Formula and switching integrals and sums, we get

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z_0} d\zeta \sum_{j=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^j + \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{z - z_0} d\zeta \sum_{j=0}^{\infty}\left(\frac{\zeta - z_0}{z - z_0}\right)^j \\
&= \sum_{j=0}^{\infty}(z-z_0)^j \frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta + \sum_{j=0}^{\infty}(z - z_0)^{-(j+1)}\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)^j d\zeta \\
&\stackrel{j \to -1-j'}{=} \sum_{j=0}^{\infty}(z-z_0)^j \frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta + \sum_{j'=-\infty}^{-1}(z - z_0)^{j'}\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{j'+1}} d\zeta \\
&\stackrel{j' \to j}{=} \sum_{j=-\infty}^{\infty}(z-z_0)^j \frac{1}{2\pi i}\oint_{C}\frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta,
\end{aligned}$$

which proves the theorem. ■

The coefficient of the -1 power,

$$c_{-1} = \frac{1}{2\pi i}\oint_C f(\zeta) d\zeta,$$

is referred to as the *residue* of $f(z)$ at z_0. The sum of the negative powers of $f(z)$ is referred to as the *principal part* of the Laurent series of $f(z)$ about z_0.

Special cases

1. If $f(z)$ is analytic in $|z - z_0| < R_2$, then

$$c_j = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta = 0,$$

for $j+1 \leq 0$, *i.e.*, for $j \leq -1$, since in that case the integrand is analytic. Here C is a contour encircling z_0. Thus, for an analytic function, the Laurent series reduces to a Taylor series.

2. Suppose that $f(z)$ is analytic outside of $|z-z_0| = R_1$ (including at ∞). Consider

$$c_j = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta,$$

for $j \geq 0$. By deforming C_1 to a circle of arbitrarily large radius, it is easy to show[5] that c_j is arbitrarily small, and thus zero (for $j \geq 0$). This relies heavily on the fact that $f(\zeta)$ is analytic at ∞ and thus bounded there. Thus for such functions,

$$f(z) = \sum_{j=-\infty}^{0} c_j (z-z_0)^j,$$

and only negative powers and a constant are present in the Laurent series.

Example. Laurent series (and Taylor series for that matter), are often easily obtained by using known series for different functions. For instance, consider

$$f(z) = e^{1/z}.$$

By substitution $z \to 1/z$ in the Taylor series of e^z, we find the Laurent series of $f(z)$:

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}.$$

The proof of the following theorem follows the same methods as before, and we will skip it. Similarly, we easily establish that the Laurent series of a given function is unique.

Theorem 3.13 (Uniform convergence of Laurent Series.) *The Laurent series of a function $f(z)$, analytic in $R_1 \leq |z-z_0| \leq R_2$ is uniformly convergent to $f(z)$ for $R_1 < \rho_1 \leq |z-z_0| \leq \rho_2 < R_2$.*

3.6 Singularities of complex functions

Definition 3.14 *A* **Singular Point** *of a complex function $f(z)$ is a point z_0 at which the function is not analytic.*

It follows that z_0 is a singular point if $f'(z_0)$ is not defined.

[5] Really. You should try it.

Definition 3.15 *The singular point z_0 of $f(z)$ is* **isolated** *if $f(z)$ is analytic in any neighborhood of z_0, excluding z_0.*

We know that near an isolated singular point z_0, the function $f(z)$ can be represented by its Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta,$$

and C is a simple closed contour enclosing z_0. The Laurent series representation of $f(z)$ is our starting point for the classification of isolated singular points.

Removable singularities

If $f(z_0)$ is bounded, then $c_n = 0$, for $n < 0$ and

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is represented by a Taylor series. Thus $f(z)$ is analytic at z_0. In this case the function can be extended to be analytic at z_0.

Example. Consider

$$f(z) = \frac{\sin z}{z}.$$

This function is not defined at $z = 0$ and neither is $f'(z)$. Thus $z = 0$ is an isolated[6] singularity. We have that

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}.$$

We may define

$$\tilde{f}(z) = \begin{cases} f(z), & z \neq 0, \\ 1 = \lim_{z \to 0} f(z), & z = 0. \end{cases}$$

Then $\tilde{f}(z)$ is analytic at 0. It is, in fact, entire. This example shows that the presence of a removable singularity is merely a consequence of choosing an inconvenient representation for $f(z)$.

[6] Since f(z) is defined for all $z \neq 0$.

Poles

If

$$f(z) = \sum_{n=-N}^{\infty} c_n (z - z_0)^n,$$

where N is a positive integer. Then we may write

$$f(z) = \frac{1}{(z - z_0)^N} g(z),$$

where $g(z)$ is analytic at z_0. In this case we say that z_0 is a **pole** of $f(z)$ of **order** N. The coefficient c_{-N} is sometimes[7] called the strength of the pole z_0. Clearly,

$$\lim_{z \to z_0} f(z) = \infty.$$

Example. Consider

$$\begin{aligned} f(z) &= \frac{2\cos z}{z} \\ &= 2\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n)!} \\ &= \frac{2}{z}\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots\right). \end{aligned}$$

Thus $z = 0$ is a pole of $f(z)$ of order 1 and strength 2.

Example. Consider

$$f(z) = \frac{(z-1)^2}{z(z+1)^3} = \frac{1}{z}\frac{(z-1)^2}{(z+1)^3}.$$

This function has a simple pole (*i.e.*, a pole of order 1) at $z = 0$ of strength 1, and a pole of order 3 at $z = -1$ of strength -4.

Example. Consider

$$f(z) = \frac{\ln(z+1)}{z-1} = \frac{1}{z-1}\ln(z+1),$$

which has a branch point at $z = -1$. We will talk about branch points later. There is also a pole of order 1 and strength $\ln 2$ on any of the sheets of $f(z)$.

In the preceding example, the order and the strength of the pole on each sheet is the same. This is not always the case, as the next example shows.

[7] On Tuesdays, between 3-5pm

Example. Consider

$$f(z) = \frac{z^{1/2} - 1}{z - 1}.$$

Let $z = 1 + t$, then

$$\begin{aligned} f(z) &= \frac{(1+t)^{1/2} - 1}{t} \\ &= \frac{1}{t}\left(\pm\left(1 + \frac{t}{2} - \frac{t^2}{8} + \cdots\right) - 1\right) \\ &= \frac{1}{t}\begin{cases} t/2 - t^2/8 + \cdots \\ -2 - t/2 + \cdots \end{cases} \\ &= \begin{cases} 1/2 - t/8 + \cdots \\ -2/t + \cdots \end{cases}. \end{aligned}$$

Thus $z = 1$ is a removable singularity of the + sheet, while it is a first-order pole on the − sheet, of strength -2.

Essential singularities

An essential singularity is any isolated singularity that is neither removable nor a pole. Near an essential singularity, $f(z)$ has a full Laurent expansion, and its principal part has an infinite number of terms. We could regard an essential singularity as a pole of infinite order, but that is not always useful.

Example. Consider

$$f(z) = e^{1/z} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}.$$

Let $z = re^{i\theta}$. Then

$$\begin{aligned} f(z) &= e^{e^{-i\theta}/r} \\ &= e^{\frac{1}{r}(\cos\theta - i\sin\theta)} \\ &= e^{\frac{1}{r}\cos\theta}\left(\cos\left(\frac{\sin\theta}{r}\right) - i\sin\left(\frac{\sin\theta}{r}\right)\right), \end{aligned}$$

so that

$$|f(z)| = e^{\frac{1}{r}\cos\theta}.$$

It follows that if $\cos\theta > 0$, then $|f(z)| \to \infty$, whereas if $\cos\theta < 0$, then $|f(z)| \to 0$. Let $r = \cos\theta/R$, which corresponds to a circle in the complex z plane of radius $1/(2R)$, centered at $1/(2R)$. Indeed, we have

$$x = r\cos\theta = \frac{1}{R}\cos^2\theta = \frac{1}{2R} + \frac{1}{2R}\cos(2\theta),$$

and

$$y = r\sin\theta = \frac{1}{R}\cos\theta\sin\theta = \frac{1}{2R}\sin(2\theta).$$

Then

$$|f(z)| = e^R,$$

and

$$f(z) = e^R\left(\cos(R\tan\theta) - i\sin(R\tan\theta)\right).$$

Thus the argument of $f(z)$ along this circle is $R\tan\theta$. It follows that as $\theta \to \pi/2$, the argument of $f(z)$ takes on any value between 0 and 2π, while its modulus is any positive number, specified by our choice of R. Thus, near $z = 0$, $f(z)$ takes on any value in $\mathbb{C}$! In fact, we can see from this reasoning that $f(z)$ takes on any value in $\mathbb{C}$, infinitely often!

Although we illustrated this last statement only for the one example presented, it is true in general, as proven by Picard. We state Picard's Theorem here[8], although we do not prove it. Rather we prove a simpler result, given in the next theorem[9].

Theorem 3.16 (Picard). *In any neighborhood of an essential singularity, $f(z)$ takes on all values, except possibly one, infinitely often.*

Ah, yes. Except possibly one. In the example above, there is indeed a value which is not taken on by $f(z)$: the value 0 is not attained for any input value of z.

Theorem 3.17 *If $f(z)$ has an essential singularity at $z = z_0$, then for any $w \in \mathbb{C}$, $f(z)$ comes arbitrarily close to w for z in a neighborhood of z_0.*

Proof. Suppose $|f(z) - w| > \epsilon$, for $|z - z_0| < \delta$, where δ is sufficiently small so that $f(z)$ is analytic in $0 < |z - z_0| < \delta$. In this region, we define $g(z)$:

$$g(z) = \frac{1}{f(z) - w},$$

which is analytic there. It is bounded by $1/\epsilon$:

$$|g(z)| = \frac{1}{|f(z) - w|} < \frac{1}{\epsilon},$$

which is also valid as $z \to z_0$. Thus $z = z_0$ is a removable singularity of $g(z)$ and we may write

$$g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

[8] It's too cool not to state it!

[9] This next theorem is not named after someone. Yet. Here's your chance!

By redefining $g(z_0) = c_0$, we have that $g(z)$ is analytic in $|z - z_0| < \delta$. As a consequence, $g(z)$ has only isolated zeroes. We have that

$$f(z) = w + \frac{1}{g(z)}.$$

It follows that $f(z)$ is either analytic in $|z - z_0| < \delta$ (if $g(z)$ has no zeros) or else has pole singularities, with poles at the zeros of $g(z)$. This is a contradiction. ■

Singularities at infinity are examined by the use of the transformation $z = 1/t$, as you might expect.

Example. Consider

$$f(z) = a_n z^n + \cdots + a_n,$$

so that

$$f(1/t) = \frac{a_n}{t^n}\left(1 + \text{polynomial}\right),$$

so that $f(z)$ has a pole of order n at infinity, with strength a_n.

Definition 3.18 Meromorphic functions *are functions whose only singularities in the complex plane are poles.*

Note that meromorphic functions can have essential singularities at infinity. In fact, they often do. Rational functions (*i.e.*, ratios of polynomials) are meromorphic functions. They do not have an essential singularity at infinity. On the other hand, $\sin z/z^2$ is also meromorphic and does have an essential singularity at infinity.

Next we turn to non-isolated singularities.

Branch points

A branch point is a non-isolated singularity: going around the branch point produces a jump discontinuity, no matter how small the radius of the enclosing circle is. When the function is restricted to a single sheet to make it single valued, the branch cut gives rise to a whole line of singularities, and a jump discontinuity exists across it.

Cluster points

A cluster point z_0 is a singular point for which there exists an infinite sequence of isolated singularities approaching z_0. Thus, within any neighborhood of a cluster point, there are an infinite number of isolated singularities.

Example. Consider

$$f(z) = \tan z.$$

This function has isolated singularities at $\pi(n+1/2)$, which limit to ∞ as $n \to \infty$. Or, if you are more comfortable with finite singularities, the function

$$g(s) = f(1/s) = \tan(1/s)$$

has a cluster point at $s = 0$, since the sequence of isolated singularities $\{2/\pi(2n+1)\}$ approaches it as $n \to \infty$.

Boundary jump discontinuities

We have already discussed branch cuts above, which are an example of our final type of singularities. There are more types, but we have to stop at some point.

Example. Consider the function

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{1}{\zeta - z} d\zeta,$$

where C is a simple closed contour. It follows that

$$f(z) = \begin{cases} 1, & z \text{ inside } C, \\ 0, & z \text{ outside } C. \end{cases}$$

We see that the location of the jump discontinuity depends entirely on the location of the contour C, as it does for branch cuts.

3.7 Infinite products

An *infinite product* is denoted by

$$P = \prod_{k=1}^{\infty} (1 + a_k).$$

The product converges if (i) the sequence

$$\{P_n = \prod_{k=1}^{n} (1 + a_k)\}$$

converges to a finite limit, and (ii) for N sufficiently large,

$$\lim_{n\to\infty} \prod_{k=N}^{n} (1 + a_k) \neq 0.$$

The reason for this second condition is that the product is related to the series

$$S = \sum_{k=1}^{\infty} \ln(1 + a_k),$$

which would not make sense if $P = 0$. For this same reason, we assume in what follows that $a_k \neq -1$.

Definition 3.19 *If $\prod_{k=1}^{\infty}(1 + |a_k|)$ converges, then P converges* **absolutely**.

Definition 3.20 *If $\prod_{k=1}^{\infty}(1 + |a_k|)$ diverges, but P converges, then we say that the convergence is* **conditional**.

Note that $a_n \to 0$ is a necessary condition for convergence, since

$$P_n = (1 + a_n)P_{n-1} \quad \Rightarrow \quad a_n = \frac{P_n}{P_{n-1}} - 1 \;\to\; 0.$$

Theorem 3.21 *If $S = \sum_{k=1}^{\infty} \ln(1 + a_k)$ converges, then so does $P = \prod_{k=1}^{\infty}(1 + a_k)$, where we restrict* ln *to its principal branch.*

Proof. Let $S_n = \sum_{k=1}^{n} \ln(1 + a_k)$. Then

$$e^{S_n} = P_n,$$

and as $n \to \infty$, we have that $e^{S_n} \to e^S = P$. ■

Theorem 3.22 (Weierstrass M test). *Let $a_k(z)$ be analytic in a domain D for all k. Suppose that for all $z \in D$ and $k \geq N$ either (a) $|\ln(1 + a_k(z))| \leq M_k$, or (b) $|a_k(z)| \leq M_k$, and $\sum_{k=1}^{\infty} M_k = M < \infty$, where each M_k is constant. Then $P(z) = \prod_{k=1}^{\infty}(1 + a_k(z))$ is uniformly convergent to an analytic function $P(z)$ in D. Furthermore, $P(z)$ is zero only when a finite number of the factors $1 + a_k(z)$ are zero in D.*

Proof. For $n \geq N$, define

$$P_n(z) = \prod_{k=N}^{n}(1 + a_n(z)),$$

and

$$S_n(z) = \sum_{k=N}^{n} \ln(1 + a_k(z)).$$

Then

$$\begin{aligned}
|S_n(z)| &\leq \sum_{k=N}^{n} |\ln(1+a_k)| \\
&\leq \sum_{k=N}^{n} M_k \\
&\leq \sum_{k=1}^{\infty} M_k \\
&< \infty,
\end{aligned}$$

for $z \in D$. It follows that, for $n > m$,

$$\begin{aligned}
|S_n(z) - S_m(z)| &= \left| \sum_{k=m+1}^{n} \ln(1+a_k(z)) \right| \\
&\leq \sum_{k=m+1}^{n} M_k \\
&\leq \sum_{k=m+1}^{\infty} M_k \\
&\leq \epsilon_m,
\end{aligned}$$

and $\epsilon_m \to 0$ as $m \to \infty$. It follows that $\{S_n(z)\}$ is a uniformly convergent Cauchy sequence. The sequence is uniformly convergent since ϵ_m does not depend on z. It follows that $S_n(z)$ converges uniformly to a limit.

Next, consider

$$\begin{aligned}
|P_n(z) - P_m(z)| &= \left| e^{S_n(z)} - e^{S_m(z)} \right| \\
&= \left| e^{S_m(z)} \right| \left| e^{S_n(z) - S_m(z)} - 1 \right| \\
&\leq e^{|S_m(z)|} \left(e^{|S_n(z) - S_m(z)|} - 1 \right) \\
&\leq e^{M} \left(e^{\epsilon_m} - 1 \right) \\
&\to 0, \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Thus $\{P_n(z)\}$ is a uniformly convergent Cauchy sequence. We conclude that $P_n(z)$ converges uniformly to a limit $\tilde{P}(z)$.

In the above, we used the following bounds:

$$|e^w| = \left| \sum_{n=0}^{\infty} \frac{w^n}{n!} \right| \leq \sum_{n=1}^{\infty} \frac{|w|^n}{n!} = e^{|w|},$$

and

$$|e^w - 1| = \left| \sum_{n=1}^{\infty} \frac{w^n}{n!} \right| \leq \sum_{n=1}^{\infty} \frac{|w|^n}{n!} = e^{|w|} - 1.$$

Returning to the main proof, we have that $P_n(z) \to \tilde{P}(z)$, uniformly as $n \to \infty$. Thus, $\tilde{P}(z)$ is the uniform limit of analytic functions. It follows immediately from Morera's Theorem that $\tilde{P}(z)$ is analytic. Further,

$$P(z) = \prod_{k=1}^{N-1}(1 + a_k(z))\tilde{P}(z),$$

which is also analytic.

We proceed to show that $P(z)$ is zero only if a finite number of the factors $1 + a_k(z)$ is zero. Indeed,

$$\begin{aligned}
|P_n(z)| &= \left|e^{S_n(z)}\right| \\
&= \frac{1}{|e^{-S_n(z)}|} \\
&\geq \frac{1}{e^{|S_n(z)|}} \\
&\geq \frac{1}{e^M} \\
&= e^{-M}.
\end{aligned}$$

It follows that $\tilde{P}(z)$ is not zero, from which the assertion follows. ■

The theorem works under two sets of assumptions. We only used one in our proof. Now we show that the first assumption in the theorem follows from the second one. Thus, in practice, we can verify whichever assumption is easiest to verify.

Consider

$$\begin{aligned}
& & \ln(1+w) &= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{w^k}{k}, \qquad \text{for } |w| < 1, \\
\Rightarrow & & |\ln(1+w)| &\leq \sum_{k=1}^{\infty}\frac{|w|^k}{k} \\
& & &= |w|\sum_{k=1}^{\infty}\frac{|w|^{k-1}}{k} \\
& & &= |w|\left(1 + \frac{|w|}{2} + \frac{|w|^2}{3} + \cdots\right).
\end{aligned}$$

Provided $|w| < 1/2$, we have

$$\begin{aligned}
|\ln(1+w)| &\leq |w|\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) \\
&= \frac{|w|}{1 - 1/2} \\
&= 2|w|.
\end{aligned}$$

Thus, for $|a_k(z)| < 1/2$, we have that $|\ln(1 + a_k)| \leq 2|a_k|$ and the first assumption follows from the second one.

Let us consider some examples[10].

Example. Consider the infinite product

$$F(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Thus $F(z)$ represents an entire function with zeros at $\pm k$, *i.e.*, all integers except zero. Here $a_k = -z^2/k^2$. Inside any circle $|z| < R$ we have

$$|a_k| = \frac{|z|^2}{k^2} < \frac{R^2}{k^2},$$

and $F(z)$ is analytic since

$$\sum_{k=1}^{\infty} |a_k| < R^2 \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} R^2,$$

which is finite. Since this holds for all R, $F(z)$ is entire.

Example. Consider

$$G(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{z/k}.$$

Using the same argument as above, it is clear that the product would not converge without the exponential factor. With the factor we have

$$\begin{aligned} \left(1 - \frac{z}{k}\right) e^{z/k} &= \left(1 - \frac{z}{k}\right) \left(1 + \frac{z}{k} + \frac{z^2}{2k^2} + \cdots\right) \\ &= \left(1 - \cancel{\frac{z}{k}} - \frac{z^2}{k^2} + \cancel{\frac{z}{k}} + \frac{z^2}{2k^2} - \cdots\right) \\ &= \left(1 - \frac{z^2}{2k^2} + \cdots\right), \end{aligned}$$

and the exponential has the effect of eliminating the first-order contributions, which were the cause of the divergence. We will see below that $G(z)$ as defined with the exponential does indeed give rise to a convergent product.

We have

$$\begin{aligned} \ln\left(\left(1 - \frac{z}{k}\right) e^{z/k}\right) &= \ln\left(1 - \frac{z}{k}\right) + \frac{z}{k} \\ &= -\left(\cancel{\frac{z}{k}} + \frac{z^2}{2k^2} + \frac{z^3}{3k^3} + \cdots\right) + \cancel{\frac{z}{k}} \\ &= -\frac{z^2}{k^2}\left(\frac{1}{2} + \frac{z}{3k} + \frac{z^2}{4k^2} + \cdots\right). \end{aligned}$$

[10] It's about time!

For $|z/k| < 1/2$, we get that

$$\begin{aligned}
\left|\ln\left(\left(1-\frac{z}{k}\right)e^{z/k}\right)\right| &\leq \frac{|z|^2}{k^2}\left(\frac{1}{2}+\frac{1}{3}\frac{1}{2}+\frac{1}{4}\frac{1}{2^2}+\cdots\right)\\
&\leq \frac{z^2}{k^2}\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)\\
&= \frac{|z|^2}{2k^2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)\\
&= \frac{|z|^2}{2k^2}\frac{1}{1-1/2}\\
&= \frac{|z|^2}{k^2}.
\end{aligned}$$

It follows that for $|z| < R$ and $k > 2R$,

$$\left|\ln\left(\left(1-\frac{z}{k}\right)e^{z/k}\right)\right| \leq \frac{|z|^2}{k^2} < \frac{R^2}{k^2},$$

and it follows from the Weierstrass M-test that $G(z)$ is analytic for $|z| < R$, which is valid for any R, thus $G(z)$ is entire.

Note that if $F(z)$ and $G(z)$ are entire and have the same zeros (including multiplicities), then there is an analytic function $h(z)$ such that

$$F(z) = e^{h(z)}G(z).$$

This follows from a simple argument: consider $F(z)/G(z)$. This function is entire, since all its singularities (at the zeros of $G(z)$) are removable. Further, $F(z)/G(z)$ has no zeros. This implies that it has a logarithm which is analytic everywhere, concluding the argument.

Is it possible to construct an entire function with prescribed zeros at prescribed points, or, more general, a meromorphic function which has prescribed zeros and poles? The answers to these questions lead to **Weierstrass representations** and **Mittag-Leffler expansions.**

3.8 Representations of meromorphic functions

Suppose that $f(z)$ is meromorphic, with poles of order N_j at $z = z_j$, $j = 1, \cdots, N$. Then in a neighborhood of z_j,

$$f(z) = p_j(z) + \sum_{n=0}^{\infty} b_{nj}(z-z_j)^n,$$

where $p_j(z)$ is the principal part of $f(z)$ about z_j:

$$p_j(z) = \sum_{n=1}^{N_j} \frac{a_{nj}}{(z-z_j)^n}.$$

Consider

$$g(z) = f(z) - \sum_{j=1}^{N} p_j(z).$$

It follows that $g(z)$ is entire, since the sum effectively subtracts off all of the singular behavior of $f(z)$. Thus

$$g(z) = \sum_{k=0}^{\infty} c_k z^k,$$

a Taylor series with infinite radius of convergence. Thus

$$f(z) = \sum_{j=1}^{N} p_j(z) + \sum_{k=0}^{\infty} c_k z^k,$$

which we may justifiably call the **partial fraction decomposition** of $f(z)$. This construction is valid provided the number of poles is finite. If $f(z)$ is rational, the Taylor series for $g(z)$ has a finite number of terms, and the representation above is the classical partial fraction expansion.

If the number of poles of $f(z)$ is infinite, the series obtained may or may not converge. Consider $f(z) = \lim_{n\to\infty} f_n(z)$, where

$$\begin{aligned} f_n(z) &= \sum_{k=-n, k\neq 0}^{n} \frac{1}{z-k} \\ &= \sum_{k=1}^{n} \left(\frac{1}{z-k} + \frac{1}{z+k} \right) \\ &= 2z \sum_{k=1}^{n} \frac{1}{z^2 - k^2}. \end{aligned}$$

Due to the $1/k^2$ behavior, $f(z)$ is well defined, except of course at the integers. On the other hand, if we use

$$f_n(z) = \sum_{k=1}^{n} \frac{1}{z-k},$$

then $f(z)$ is not defined, since the series obtained diverges for all values of z.

Mittag-Leffler expansions

If $f(z)$ has only simple poles, then

$$\begin{aligned} p_j(z) &= \frac{a_j}{z - z_j} \\ &= -\frac{a_j}{z_j} \frac{1}{1 - z/z_j}, \end{aligned}$$

where we have assumed that $z_j \neq 0$. It follows that for $|z/z_j| < 1$, there exists a sufficiently large m such that

$$g_j(z) = -\frac{a_j}{z_j}\left(1 + \frac{z}{z_j} + \frac{z^2}{z_j^2} + \cdots + \frac{z^{m-1}}{z_j^{m-1}}\right)$$

approximates $p_j(z)$ arbitrarily close.

Let

$$L(w, m) := \frac{1}{w-1} + 1 + w + w^2 + \cdots + w^{m-1} = \frac{w^m}{w-1}.$$

We need this function in the statement of the following theorem.

Theorem 3.23 (Mittag-Leffler, simple pole case). *Let $\{z_k\}$ and $\{a_k\}$ be sequences with all z_k distinct, and $|z_k| \to \infty$ as $k \to \infty$. Let m be an integer such that*

$$\sum_{k=1}^{\infty} \frac{|a_k|}{|z_k|^{m+1}} < \infty.$$

Consider the function

$$f(z) = p_0(z) + \sum_{j=1}^{\infty} \left(\frac{a_j}{z_j}\right) L\left(\frac{z}{z_j}, m\right) + h(z),$$

where $h(z)$ is entire and $p_0(z)$ is the principal part of $f(z)$ about $z = 0$. Then $f(z)$ is a meromorphic function whose only singularities are simple poles at z_j (which may include 0) with residues a_j.

Proof. (i) $f(z)$ has a pole at $z = z_j$. Further $f(z)$ has no singularities other than the elements of $\{z_k\}$. (ii) By construction, the residue of $f(z)$ at z_j is a_j. It remains to show that the infinite series in the definition of $f(z)$ is convergent in $\mathbb{C}$. We have that

$$|w - 1| \geq ||w| - 1| = 1 - |w| \geq 1/2,$$

for $|w| < 1/2$. It follows that then

$$|L(w, m)| \leq 2|w|^m.$$

We get

$$\begin{aligned}
\left|\sum_{j=1}^{\infty} \left(\frac{a_j}{z_j}\right) L\left(\frac{z}{z_j}, m\right)\right| &\leq \sum_{j=1}^{\infty} \left|\frac{a_j}{z_j}\right| \left|L\left(\frac{z}{z_j}, m\right)\right| \\
&\leq 2\sum_{j=1}^{\infty} \left|\frac{a_j}{z_j}\right| \left|\frac{z}{z_j}\right|^m \\
&= 2|z|^m \sum_{j=1}^{\infty} \frac{|a_j|}{|z_j|^{m+1}},
\end{aligned}$$

which converges uniformly for $|z/z_j| < 1/2$. Let $|z| < R$ and take j large enough so that $|z_j| > 2R$. Then $|z/z_j| < 1/2$. This holds for R arbitrarily large. It follows that the series in the definition of $f(z)$ is uniformly convergent in $\mathbb{C}$. This concludes the proof. ∎

Example. Consider

$$f(z) = \pi \cot \pi z.$$

This function has simple poles at $z_j = j$, for $j \in \mathbb{Z}$. Indeed, near $z = 0$[11], we have that

$$\begin{aligned} f(z) &= \pi \frac{\cos \pi z}{\sin \pi z} \\ &\sim \pi \frac{\cos \pi z}{\pi z} \\ &\sim \frac{1}{z} + \cdots . \end{aligned}$$

Thus all poles are simple, and their residues are all equal to 1. Thus

$$p_j(z) = \frac{1}{z - j}.$$

We know that

$$\sum_{j=-\infty}^{\infty}{}' \frac{|a_j|}{|z_j|^{m+1}} = \sum_{j=-\infty}^{\infty}{}' \frac{1}{j^2} < \infty,$$

where $m = 1$ is the smallest value that guarantees convergence. The prime on the sum indicates that the $j = 0$ term is omitted. Thus

$$L\left(\frac{z}{z_j}, 1\right) = \frac{1}{\frac{z}{z_j} - 1} + 1 = \frac{j}{z - j} + 1.$$

It follows from the theorem that

$$\begin{aligned} f(z) &= \frac{1}{z} + \sum_{j=-\infty}^{\infty}{}' \frac{1}{j}\left(\frac{j}{z-j} + 1\right) + h(z) \\ &= \frac{1}{z} + \sum_{j=-\infty}^{\infty}{}' \left(\frac{1}{z-j} + \frac{1}{j}\right) + h(z) \\ &= \frac{1}{z} + 2z \sum_{j=1}^{\infty} \frac{1}{z^2 - j^2} + h(z). \end{aligned}$$

Since the sum is uniformly (and absolutely) convergent, we may rearrange its terms, as we did to group the j and $-j$ terms. We find that

$$\pi \cot \pi z = \sum_{j=-\infty}^{\infty} \frac{z}{z^2 - j^2} + h(z).$$

[11] Nope. Not zero to the eleventh. Actually a footnote. You should convince yourself that the behavior at the other poles is identical.

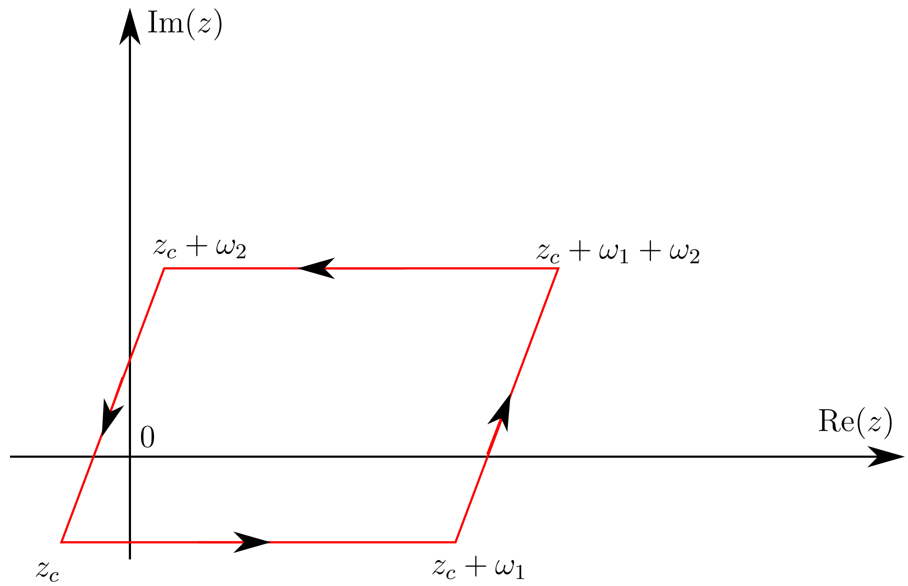

Figure 3.5: A tile in the complex plane in which we wish to define an elliptic function.

You will show on the homework that $h(z) \equiv 0$.

Theorem 3.24 (Mittag-Leffler, general case). *Let $f(z)$ be a meromorphic function in $\mathbb{C}$ with poles at the elements of $\{z_j\}$ and corresponding principal parts $\{p_j(z)\}$. Then there exist polynomials $\{g_j(x)\}$ such that*

$$f(z) = p_0(z) + \sum_{j=1}^{\infty} (p_j(z) - g_j(z)) + h(z)$$

holds, and this series converges uniformly on every bounded set in $\mathbb{C}$ not containing $\{z_j\}$.

We do not prove this theorem here. Rather, we present an example in the form of the **Weierstrass elliptic function** $\wp(z)$.

Example. Suppose we want to construct a function that is periodic in two linearly independent directions in the complex plane. In other words, this function generalizes the trigonometric or the hyperbolic functions, which are periodic in one direction in the complex plane. A figure to guide us is drawn in Fig. 3.5. In the red parallelogram, we wish to define a function $\wp(z)$ which is periodic (*i.e.*, its values on opposing edges of the parallelogram are equal) so that we can use the parallelogram to tile $\mathbb{C}$. Here the two complex numbers ω_1 and ω_2 are linearly independent: their ratio is not real.

1. Can we accomplish the task with a function that is analytic in the parallelogram? We cannot. If we could, we could tile the entire complex plane with this analytic function.

By Liouville's Theorem, the function would have to be constant, and thus boring. We want a non-trivial solution to our problem.

2. Given that an analytic function will not do the job, how about a function with a single simple pole in the parallelogram? It turns out this does not work either. Suppose this was possible. Integrate this function along the edges of the parallelogram. Since the contributions on opposing edges are equal and the edges are traversed in opposite direction, the contour integral is zero. On the other hand, we know that the value of the integral is $2\pi i$ times the residue of the function at the pole. In other words, this residue is zero, eliminating the pole, bringing us back to the analytic case.

3. It follows that we need a function with either two simple poles or else a double pole. Jacobi followed the first path, resulting in the invention of the elliptic functions that are named after him. Weierstrass followed the second approach. He defined

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n=-\infty}^{\infty}{}' \left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right).$$

This function has a double pole at the origin and at its bi-periodic counterparts. By construction the function is periodic with independent periods ω_1 and ω_2.

Weierstrass factorizations

Next we discuss how to represent functions with prescribed zeros and multiplicities for these zeros.

If $f(z)$ is entire, then $f'(z)/f(z)$ is meromorphic with simple poles $\{z_k\}$ and residues $\{a_k\}$. Here $\{z_k\}$ are the zeros of $f(z)$ and $\{a_k\}$ are their corresponding multiplicities. Using the simple-pole case of the Mittag-Leffler Theorem, we get

$$\frac{f'(z)}{f(z)} = \frac{a_0}{z} + \sum_{j=1}^{\infty} \left(\frac{-a_j}{z_j} \frac{1}{1 - z/z_j} + \frac{a_j}{z_j} \sum_{k=0}^{m-1} \left(\frac{z}{z_j} \right)^k \right) + h(z),$$

where $h(z)$ is entire and

$$\sum_{j=1}^{\infty} \frac{|a_j|}{|z_j|^{m+1}} < \infty.$$

Then, integrating term-by-term:

$$\begin{aligned} \ln f(z) &= a_0 \ln z + \sum_{j=1}^{\infty} \left(a_j \ln(1 - z/z_j) + a_j \sum_{k=0}^{m-1} \frac{1}{k+1} \left(\frac{z}{z_j} \right)^{k+1} \right) + \int h(z) dz \\ &= a_0 \ln z + \sum_{j=1}^{\infty} \left(a_j \ln(1 - z/z_j) + a_j \sum_{k=1}^{m} \frac{1}{k} \left(\frac{z}{z_j} \right)^k \right) + \int h(z) dz. \end{aligned}$$

This leads to the following theorem.

Theorem 3.25 (Weierstrass factorization theorem). *The function*

$$f(z) = z^{a_0} \prod_{j=1}^{\infty} \left(\left(1 - \frac{z}{z_j}\right) e^{\sum_{k=1}^{m} \frac{1}{k} \left(\frac{z}{z_j}\right)^k} \right)^{a_j} g(z),$$

where $g(z)$ is an analytic function without zeros, is an entire function with zeros $\{z_j\}$ and corresponding multiplicities $\{a_j\}$.

Example. Consider the function

$$f(z) = \sin \pi z.$$

This function has zeros of multiplicity one at the integers. We already know that

$$\pi \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}.$$

Integrating term-by-term[12] we get

$$\begin{aligned} & & \ln \sin \pi z &= \ln z + \sum_{j=1}^{\infty} \left(\ln(z^2 - j^2) - A_j\right) + A_0 \\ &\Rightarrow & \ln \frac{\sin \pi z}{z} &= \sum_{j=1}^{\infty} \left(\ln(z^2 - j^2) - A_j\right) + A_0. \end{aligned}$$

We have introduced the constant A_j, $j = 0, 1, \ldots$. You might think it suffices to introduce a single integration constant, but it is convenient to introduce the constant inside the sum as well, to ensure uniform convergence of the product we are about to find. Evaluating the above result at $z = 0$, we get

$$\ln \pi = \sum_{j=1}^{\infty} \left(\ln(-j^2) - A_j\right) + A_0,$$

and it is convenient to choose $A_0 = \ln \pi$ and $A_j = \ln(-j^2)$, $j = 1, 2, \ldots$. We get

$$\begin{aligned} & & \ln \sin \pi z &= \ln \pi z + \sum_{j=1}^{\infty} \ln \frac{z^2 - j^2}{-j^2} \\ & & &= \ln \pi z + \sum_{j=1}^{\infty} \ln \left(1 - \frac{z^2}{j^2}\right) \\ &\Rightarrow & \sin \pi z &= \pi z \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right), \end{aligned}$$

quite an amazing result, if you ask me.

[12] Recall that the sum is uniformly convergent.

3.9 Exercises

1. About $\sin z$ and $\sinh z$...

   (a) Write down Taylor series for $\sin z$ and $\sinh z$, valid around $z = 0$. What are the radii of convergence of these series?

   (b) Show that for all $z \in \mathbb{C}$
   $$|\sin z| \le \sinh(|z|).$$

   (c) Are there $z \in \mathbb{C}$ for which $|\sin z| = \sinh |z|$?

2. **Bernoulli numbers:** Consider the function
   $$f(z) = \frac{z}{e^z - 1}.$$

   (a) Show that $f(z)$ has a removable singularity at $z = 0$. Assume from now on that the definition of $f(z)$ has been extended to remove the singularity.

   (b) Suppose you were to find a Taylor series for $f(z)$, centered at $z = 0$. What would be its radius of convergence?

   (c) Find the Taylor series in the form
   $$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$
   The numbers B_n are known as the Bernoulli numbers.

   (d) Find a recursion formula for the Bernoulli numbers, and use it to find B_0, ..., B_{12}.

   (e) Show that $B_{2n+1} = 0$ for $n \ge 1$.

   (f) Use your result to find a Taylor series for $z \coth z$, in terms of the Bernoulli numbers. Where is this series valid? Using this result, find a Laurent series for $\cot z$. Where is this series valid?

3. One representation of the **Riemann zeta function** $\zeta(z)$ is
   $$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$
   where the principal branch of the logarithm is used to define each term.

   (a) For $z = x \in \mathbb{R}$, where is this series defined? Your answer should be in the form "the series converges for $x > x_0$". (so, the question is: what's x_0).

   (b) Show that the series is uniformly convergent for $\mathrm{Re}(z) \ge \delta$, where δ is any real number so that $\delta > x_0$. Thus, the Riemann zeta function as defined above is analytic for $\mathrm{Re}(z) > x_0$.

(c) What is the derivative of the zeta function in $\text{Re}(z) > x_0$?

Riemann used the zeta function to study the distribution of the prime numbers among the natural numbers. Probably the most important open problem in mathematics today is to prove the Riemann hypothesis, which states that the zeta function has infinitely many nonreal roots, and that all such roots satisfy $\text{Re}(z) = 1/2$. Of course, a different representation of the function has to be used to analytically continue the zeta function to other areas of the complex plane then the one we've been dealing with here.

4. **Analytic continuation:**

   (a) Consider
   $$F(z) = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n.$$
   Where is this function analytic?

   (b) Use the above representation to induce a Taylor representation of $F(z)$ centered at $z = -1/2$. Call this representation $G(z)$. Your final result should be of the form
   $$G(z) = \sum_{m=0}^{\infty} c_m \left(z + \frac{1}{2}\right)^m.$$
   Where is this series valid? If you can answer this question without using that both $F(z)$ and $G(z)$ are representations of $1/(1-z)$, you'll get to pat yourself on the back.

5. **More analytic continuation:** let's put some of the steps above in a numerical light. Consider
$$F(z) = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n.$$
and
$$G(z) = \sum_{m=0}^{\infty} c_m \left(z + \frac{1}{2}\right)^m,$$
found above. As above, we will forget and not use that both are representations of $1/(1-z)$. As you found above, the radius of convergence of $F(z)$ is 1, while the radius of convergence of $G(z)$ is $3/2$. You'll need a computer to do the problem below.

   (a) Find how many terms you need to approximate the value of $F(z)$ at $z = 0.9$ to within 10^{-6}, *i.e.*, find N such that $|\sum_{n=0}^{N} z^n - 10| \leq 10^{-6}$.

   (b) Use this value of N to compute $G(0.9)$: with this value of N you can compute approximate values for c_m by truncating the infinite series that defines c_m. Use these values to find $G(0.9)$.

(c) Use this value of N to compute $G(-1.1)$. Note that $z = -1.1$ is well inside the disk of convergence. Compare this with $1/(1-z)$, evaluated at $z = -1.1$, which is approximately 0.47619.

(d) Explain why the value you found s#&ks.

The moral is: analytic continuation is not so easy to do numerically.

6. Classify the singularities (both finite and at infinity) of the functions given below as one of removable singularity, pole, essential singularity, branch point, cluster point. In case of a pole, also specify the order. Also state whether the singularity is isolated or not.

   a) $f(z) = \dfrac{z^p - 1}{z - 1}$, $p \in \mathbb{R}$ (discuss all cases, depending on the value of p)

   b) $f(z) = \cosh(z^{1/2})$

7. We have defined the Weierstrass $\wp$-function as

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k=-\infty}^{\infty} \left(\frac{1}{(z - j\omega_1 - k\omega_2)^2} - \frac{1}{(j\omega_1 + k\omega_2)^2} \right),$$

where $(j,k) = (0,0)$ is excluded from the double sum. Also, you may assume that ω_1 is a positive real number, and that ω_2 is on the positive imaginary axis. All considerations below are meant for the entire complex plane, except the poles of $\wp(z)$.

(a) Show that $\wp(z + M\omega_1 + N\omega_2) = \wp(z)$, for any two integers M, N. In other words, $\wp(z)$ is a doubly-periodic function: it has two independent periods in the complex plane. Doubly periodic functions are called *elliptic* functions.

(b) Establish that $\wp(z)$ is an even function: $\wp(-z) = \wp(z)$.

(c) Find Laurent expansions for $\wp(z)$ and $\wp'(z)$ in a neighborhood of the origin in the form

$$\wp(z) = \frac{1}{z^2} + \alpha_0 + \alpha_2 z^2 + \alpha_4 z^4 + \ldots,$$

and

$$\wp'(z) = -\frac{2}{z^3} + \beta_1 z + \beta_3 z^3 + \ldots.$$

Give expressions for the coefficients introduced above.

(d) Show that $\wp(z)$ satisfies the differential equation

$$(\wp')^2 = a\wp^3 + b\wp^2 + c\wp + d,$$

for suitable choices of a, b, c, d. Find these constants. You may need to invoke Liouville's theorem to obtain this final result. It turns out that the function $\wp(z)$ is determined by the coefficients c and d, implying that it is possible to recover ω_1 and ω_2 from the knowledge of c and d.

8. The Korteweg-deVries (KdV) equation arises whenever long waves of moderate amplitude in dispersive media are considered. For instance, it describes waves in shallow water, and ion-acoustic waves in plasmas. The equation is given by

$$u_t = 6uu_x + u_{xxx},$$

where indices denote partial differentiation.

(a) By looking for solutions $u(x,t) = U(x)$, derive a first-order ordinary differential equation for $U(x)$. Introduce integration constants as required.

(b) Let $U = U_0 \wp(x - x_0)$. Determine U_0 so that $u = U(x)$ solves the KdV equation.

Chapter 4

The Residue Theorem and its applications

4.1 Cauchy's Residue Theorem

Consider a function $f(z)$ with an isolated singularity at $z = z_0$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

with

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is a simple closed contour encircling $z = z_0$. For $n = -1$, we have

$$\oint_C f(z) dz = 2\pi i c_{-1} = 2\pi i \operatorname*{Res}_{z=z_0} f(z_0),$$

which we may regard as a generalization of the Cauchy Theorem for nonanalytic functions.

Theorem 4.1 (Residue Theorem). *Let C be a simple closed contour inside and on which $f(z)$ is analytic, except at a finite number of isolated singular points z_1, z_2, ..., z_N in C. Then*

$$\oint_C f(z) dz = 2\pi i \sum_{n=1}^{N} \operatorname*{\mathit{Res}}_{z=z_n} f(z).$$

Proof. A cartoon of the set-up for this theorem is presented in Fig. 4.1. We have that

$$\begin{aligned}
\oint_C f(z) dz &= \sum_{n=1}^{N} \oint_{C_n} f(z) dz \\
&= 2\pi i \sum_{n=1}^{N} \operatorname*{Res}_{z=z_n} f(z),
\end{aligned}$$

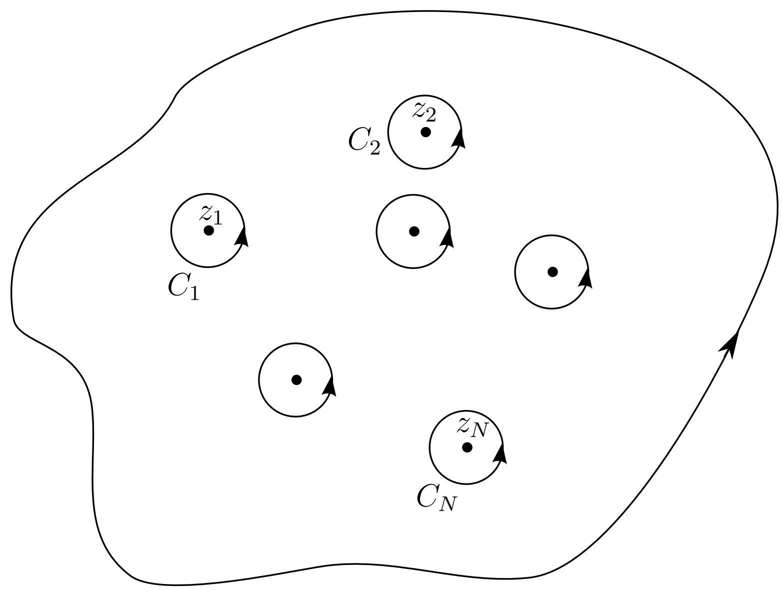

Figure 4.1: The set-up for the Residue Theorem.

by our previous result. This concludes the proof. ■

The applicability of this result cannot be overstated[1]. We will spend this entire chapter illustrating its multitude of uses.

Example. We have that

$$\oint_C z^n dz = 2\pi i \delta_{n,-1},$$

where C is a simple closed contour around $z = 0$ and δ_{jk} denotes the Krönecker delta: $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk=0}$ if $j \neq k$.

Example.

$$\begin{aligned} I &= \frac{1}{2\pi i} \oint_{|z|=1} z e^{1/z} dz \\ &= \operatorname*{Res}_{z=0} z e^{1/z} \\ &= \operatorname*{Res}_{z=0} z \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots \right) \\ &= \operatorname*{Res}_{z=0} \left(z + 1 + \frac{1}{2z} + \cdots \right) \\ &= \frac{1}{2}. \end{aligned}$$

[1]But I'll try. The Residue Theorem is VERY, VERY, VERY useful. Did that work?

Example. Consider

$$I = \oint_{|z|=2} \frac{z+2}{z(z+1)} dz = 2\pi i \left(\operatorname*{Res}_{z=0} \frac{z+2}{z(z+1)} + \operatorname*{Res}_{z=-1} \frac{z+2}{z(z+1)} \right).$$

Thus we need to calculate

$$\alpha_1 = \operatorname*{Res}_{z=0} \frac{z+2}{z(z+1)} \quad \text{and} \quad \alpha_2 = \operatorname*{Res}_{z=0} \frac{z+2}{z(z+1)}.$$

Since

$$\frac{z+2}{z(z+1)} = \frac{1}{z}\,\frac{z+2}{z+1},$$

we conclude that $\alpha_1 = (z+2)/(z+1)|_{z=0} = 2$. Similarly,

$$\frac{z+2}{z(z+1)} = \frac{1}{z+1}\,\frac{z+2}{z}.$$

Thus $\alpha_2 = (z+2)/z|_{z=-1} = -1$. We get

$$I = 2\pi i(\alpha_1 + \alpha_2) = 2\pi i.$$

The following theorem provides a convenient way to calculate residues.

Theorem 4.2 *If $z = z_k$ is a pole of order N of $f(z)$ then*

$$\operatorname*{\mathit{Res}}_{z=z_k} f(z) = \frac{1}{(N-1)!} \lim_{z\to z_k} \frac{d^{N-1}}{dz^{N-1}} (z-z_k)^N f(z).$$

Proof. The function $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{(z-z_k)^N},$$

where $\phi(z)$ is analytic in a neighborhood of z_k. Thus

$$(z-z_k)^N f(z) = \phi(z) = \sum_{n=0}^{\infty} b_n (z-z_k)^n.$$

We need the coefficient of $(z-z_k)^{N-1}$. This will give us the coefficient of $(z-z_k)^{-1}$ in $f(z)$. Thus

$$b_{N-1} = \frac{1}{(N-1)!} \phi^{(N-1)}(z_k),$$

from which the conclusion follows. ■

Example. Consider

$$I = \frac{1}{2\pi i}\oint_{|z|=2} \frac{3z+1}{z(z-1)^3}dz$$
$$= \operatorname*{Res}_{z=0} \frac{3z+1}{z(z-1)^3} + \operatorname*{Res}_{z=1} \frac{3z+1}{z(z-1)^3}.$$

We have

$$\operatorname*{Res}_{z=0} \frac{3z+1}{z(z-1)^3} = \lim_{z\to 0} \frac{3z+1}{(z-1)^3} = \frac{1}{(-1)^3} = -1,$$

and

$$\begin{aligned}
\operatorname*{Res}_{z=1} \frac{3z+1}{z(z-1)^3} &= \lim_{z\to 1} \frac{1}{2}\frac{d^2}{dz^2}\frac{3z+1}{z} \\
&= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(\cancel{3}+\frac{1}{z}\right) \\
&= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left(\frac{-1}{z^2}\right) \\
&= \frac{1}{2}\lim_{z\to 1}\left(\frac{2}{z^3}\right) \\
&= 1.
\end{aligned}$$

Thus

$$I = -1 + 1 = 0.$$

The same conclusion could be obtained by using the behavior at infinity: we have

$$I = \frac{1}{2\pi i}\oint_{|z|=2} \frac{3z+1}{z(z-1)^3}dz.$$

Thus

$$\begin{aligned}
f(z)dz &= \frac{3z+1}{z(z-1)^3}dz \\
&= \frac{(3+t)}{(1-t)^3}t^3\left(\frac{-dt}{t^2}\right) \\
&= -\frac{t(3+t)}{(1-t)^3}dt,
\end{aligned}$$

which has zero coefficient of t^{-1}. Thus, interpreting the contour integral as going around ∞ (in the opposite direction), we find that $I = 0$.

In general, the residue at infinity of $f(z)$ is defined as the coefficient of $f(1/t)/t^2$. The minus sign originating from the differential $dz = -dt/t^2$ takes care of the switch in orientation of the contour when we switch what we mean by inside and outside.

Example. Consider

$$f(z) = \frac{P_n(z)}{Q_m(z)},$$

where $P_n(z)$ and $Q_m(z)$ are polynomials of degree n and m, respectively. We have

$$\frac{1}{2\pi i}\oint_C \frac{P_n(z)}{Q_m(z)}dz = \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k} \frac{P_n(z)}{Q_m(z)},$$

where C encircles all m roots of $Q_m(z)$. Alternatively,

$$\frac{1}{2\pi i}\oint_C \frac{P_n(z)}{Q_m(z)}dz = \operatorname*{Res}_{z=\infty} \frac{P_n(z)}{Q_m(z)},$$

which requires the calculation of only a single residue. Using this approach it is not even necessary to know where the zeros of $Q_m(z)$ are!

4.2 The winding number

Let C be a closed curve (not necessarily simple), encircling z_0. We define the *winding number* of C around z_0 as

$$\begin{aligned}
\omega(z_0) &= \frac{1}{2\pi i}\oint_C \frac{1}{z-z_0}dz \\
&= \frac{1}{2\pi i}\oint_C d\ln(z-z_0) \\
&= \frac{1}{2\pi i}\oint_C d(\cancel{\ln|z-z_0|} + i\arg(z-z_0)) \\
&= \frac{1}{2\pi}\oint_C d\arg(z-z_0) \\
&= \frac{1}{2\pi}\Delta\arg(z-z_0),
\end{aligned}$$

where $\Delta(z-z_0)$ measures the increase in the argument of z along the curve. Since the curve is closed, this will be an integer multiple of 2π, and the winding number is an integer. Thus the winding number $\omega(z_0)$ measures the number of times C goes around z_0 in a counterclockwise sense.

The winding number allows us to generalize the Cauchy Residue Theorem to non-simple closed curves.

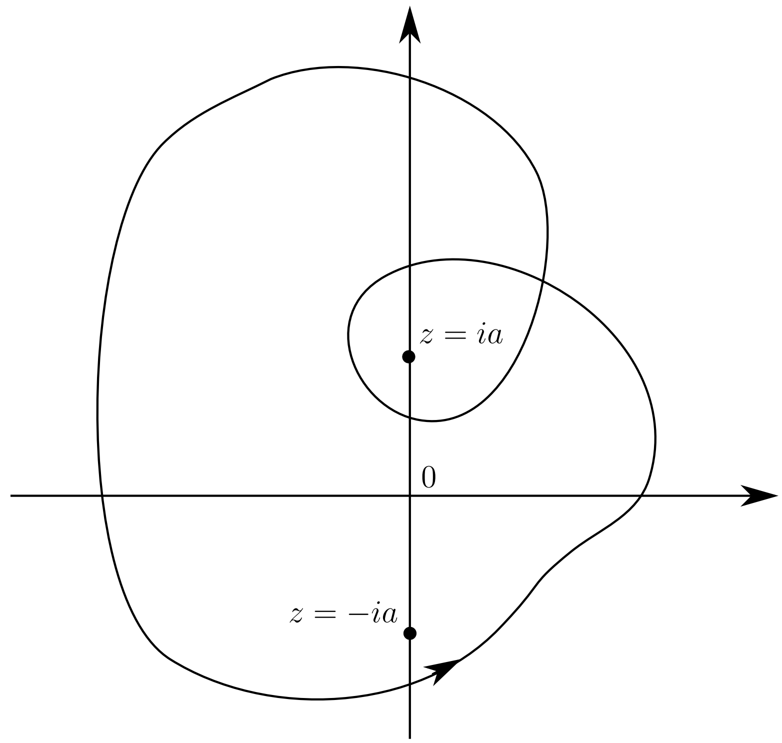

Figure 4.2: The contour C, encircling $z = ia$ twice and $z = -ia$ once.

Theorem 4.3 *For a closed piecewise smooth curve C,*

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{n} \omega(z_j) \underset{z=z_j}{Res} f(z),$$

where $\{z_j\}$ is the set of singularities of $f(z)$ enclosed by C.

Example. Consider

$$I = \oint_C \frac{dz}{z^2 + a^2},$$

where C is shown in Fig. 4.2. Thus

$$I = 2\pi i \left(2 \operatorname*{Res}_{z=ia} \frac{1}{z^2 + a^2} + \operatorname*{Res}_{z=-ia} \frac{1}{z^2 + a^2} \right).$$

4.3 Evaluation of integrals and sums using contour integration

One of the main uses of the Residue Theorem is to evaluate certain real integrals, and to establish certain identities. Before we show this, we need to talk about the Cauchy principal value a bit.

The Cauchy principal value

Suppose that $f(x)$ is singular at $x = x_0 \in [a, b]$. We define the principal value integral of $f(x)$ over the interval $[a, b]$ to be

$$I_p = \lim_{\epsilon \to 0} \left(\int_a^{x_0 - \epsilon} + \int_{x_0+\epsilon}^b \right) f(x)dx := ⨍_a^b f(x)dx,$$

provided this limit exists[2]. In other words, the singularity at x_0 is approached symmetrically from the left and from the right.

Example. Consider $f(x) = 1/x$, on $[-1, 1]$. We know that the integral of $f(x)$ on $[-1, 1]$ is not defined. On the other hand,

$$⨍_{-1}^1 \frac{1}{x} dx = 0.$$

Indeed,

$$\begin{aligned}
⨍_{-1}^1 \frac{1}{x} dx &= \lim_{\epsilon \to 0} \left(\int_{-1}^{-\epsilon} + \int_\epsilon^1 \right) \frac{1}{x} dx \\
&= \lim_{\epsilon \to 0} \left(\ln(|x|)|_{x=-1}^{-\epsilon} + \ln(|x|)|_\epsilon^1 \right) \\
&= \lim_{\epsilon \to 0} \left(\ln \frac{\epsilon}{1} + \ln \frac{1}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} (\ln 1) \\
&= \lim_{\epsilon \to 0} 0 \\
&= 0.
\end{aligned}$$

Graphically, it is easy to see what is going on: the function $1/x$ is odd around $x = 0$ and the contributions from the left and the right (for finite ϵ, these contributions are finite) cancel exactly.

If we integrate over all of $\mathbb{R}$, we can also define a principal value:

$$I_p = ⨍_{-\infty}^{\infty} f(x)dx := \lim_{R \to \infty} \int_{-R}^R f(x)dx.$$

[2]Sometimes the notation of an integral sign in combination with the letters P and V is used.

Example. Consider $f(x) = x$. Then

$$\begin{aligned}
⨍_{-\infty}^{\infty} f(x)dx &= \lim_{R\to\infty} \int_{-R}^{R} x dx \\
&= \lim_{R\to\infty} \left.\frac{x^2}{2}\right|_{-R}^{R} \\
&= \lim_{R\to\infty} \frac{1}{2}(R^2 - R^2) \\
&= 0.
\end{aligned}$$

We see that it is not even necessary for $f(x) \to 0$ as $x \to \pm\infty$ for the principal value integral on $\mathbb{R}$ to be defined!

It is clear from the above that principal value integrals are often defined when improper integrals are not. On the other hand, when an improper integral is defined, any limiting approach to the singularity will result in the same value, since otherwise the limit would not be properly defined. Thus, **a well-defined improper integral is equal to its principal value**, which may be more convenient to calculate.

The calculation of rational integrals

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{N(x)}{D(x)} dx,$$

where both $N(x)$ and $D(x)$ are polynomials, and $D(x) \neq 0$ for $x \in \mathbb{R}$. We require that the degree of $D(x)$ is at least two greater than the degree of $N(x)$. In this case the improper integral converges.

Using the Residue Theorem, we have that

$$\int_{-R}^{R} \frac{N(x)}{D(x)} dx + \int_{C_R} \frac{N(z)}{D(z)} dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} \frac{N(z)}{D(z)},$$

where these integration paths are shown in Fig. 4.3. Further, R is chosen sufficiently large so that all zeros n of $D(z)$ in the upper half plane are contained in the contour. Taking the limit as $R \to \infty$, we have that

$$I = 2\pi i \operatorname*{Res}_{z=z_k} \frac{N(z)}{D(z)} - \int_{C_\infty} \frac{N(z)}{D(z)} dz.$$

We have defined the limit using the principal value integral, but this is allowed since the improper integral converges by the assumptions above. We will show below that the last term is zero, so that

$$I = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} \frac{N(z)}{D(z)}.$$

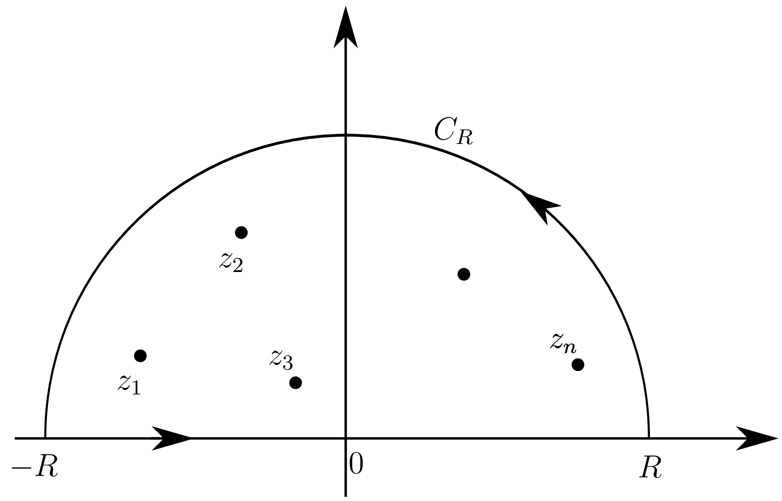

Figure 4.3: The contour to integrate $N(x)/D(x)$.

Thus the improper integral has been reduced to the calculation of a finite number of easily found residues.

Next, we show that

$$\int_{C_\infty} \frac{N(z)}{D(z)} dz = \lim_{R\to\infty} \int_{C_R} \frac{N(z)}{D(z)} dz = 0.$$

We have, assuming a worst case scenario where the degree of the denominator is only two greater than the degree of the numerator,

$$\begin{aligned} & \frac{N(z)}{D(z)} = \frac{a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0}{b_{m+2}z^{m+2} + \cdots + b_0} \\ \Rightarrow \qquad & \left|\frac{N(z)}{D(z)}\right| \le \frac{|a_m||z|^m + \cdots + |a_0|}{|b_{m+2}||z|^{m+2} - |b_{m+1}||z|^{m+1} - \cdots - |b_0|} \\ & \qquad = \frac{|a_m|R^m + \cdots + |a_0|}{|b_{m+2}|R^{m+2} - \cdots - |b_0|}. \end{aligned}$$

It follows that

$$\left|\int_{C_R} \frac{N(z)}{D(z)} dz\right| \le \frac{|a_m|R^m + \cdots + |a_0|}{|b_{m+2}|R^{m+2} - \cdots - |b_0|} R \int_0^\pi d\theta \to 0,$$

which is what we had to show.

Example. Consider

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = 2\pi i \left(\operatorname*{Res}_{z=e^{i\pi/4}} \frac{1}{1+z^4} + \operatorname*{Res}_{z=e^{3i\pi/4}} \frac{1}{1+z^4} \right).$$

It remains to calculate the two residues. We have

$$\begin{aligned}
\operatorname*{Res}_{z=e^{i\pi/4}} \frac{1}{1+z^4} &= \lim_{z\to e^{i\pi/4}} \frac{z-e^{i\pi/4}}{1+z^4} \\
&= \lim_{z\to e^{i\pi/4}} \frac{1}{4z^3} \\
&= \frac{1}{4e^{3i\pi/4}} \\
&= \frac{1}{4}e^{-3i\pi/4}.
\end{aligned}$$

Similarly,

$$\operatorname*{Res}_{z=e^{3i\pi/4}} \frac{1}{1+z^4} = \frac{1}{4}e^{-i\pi/4}.$$

It follows that

$$I = \frac{2\pi i}{4}(e^{-3i\pi/4} + e^{-i\pi/4}) = \frac{\sqrt{2}\pi}{2}.$$

Fourier-type integrals

Next we discuss integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)e^{ikx}dx,$$

where $k \in \mathbb{R}$. Without loss of generality we may assume that $k > 0$. By taking real and imaginary parts, we can consider integrals like

$$\int_{-\infty}^{\infty} f(x)\cos kx dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin kx dx.$$

Such integrals arise naturally when working with Fourier transforms.

Lemma 4.4 (Jordan.) *If on C_R (a circular arc of radius R in the upper half plane from $z = R$ to $z = -R$) $f(z) \to 0$ uniformly as $R \to \infty$, then*

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{ikz}dz = 0, \quad \textit{for } k > 0.$$

Proof. Consider

$$\begin{aligned}
I &= \left| \int_{C_R} e^{ikz} f(z) dz \right| \\
&\leq R \int_0^{\pi} e^{-ky} |f(z)| d\theta \\
&\leq R \sup_{z \in C_R} |f(z)| \int_0^{\pi} e^{-kR\sin\theta} d\theta \\
&= 2R \sup_{z \in C_R} |f(z)| \int_0^{\pi/2} e^{-kR\sin\theta} d\theta \\
&\leq 2R \sup_{z \in C_R} |f(z)| \int_0^{\pi/2} e^{-2kR\theta/\pi} d\theta \\
&= 2R \sup_{z \in C_R} |f(z)| \, e^{-2kR\theta/\pi} \left(\frac{-\pi}{2kR} \right) \Bigg|_{z=0}^{z=\pi/2} \\
&= \sup_{z \in C_R} |f(z)| \frac{\pi}{k} \left(1 - e^{-kR} \right).
\end{aligned}$$

This approaches zero as $R \to \infty$ since $\sup_{z \in C_R} |f(z)| \to 0$ as $R \to \infty$. In the above we have used that $\sin\theta$ is even around $\theta = \pi/2$, and that $\sin\theta \geq 2\theta/\pi$ for $\theta \in [0, \pi/2]$. This finishes the proof. ■

Jordan's Lemma makes the strong, albeit not totally unexpected, statement that an integral with decaying exponential in the integrand limits to zero, under rather mild assumptions on the kernel function. Note that Jordan's Lemma also holds for circular arcs that are smaller than half circles, but the version presented above is what we need below. We are now in a position to evaluate many Fourier-type integrals.

Example. Consider the two real integrals

$$I_1 = \int_{-\infty}^{\infty} \frac{\cos kx}{(x+b)^2 + a^2} dx,$$

and

$$I_2 = \int_{-\infty}^{\infty} \frac{\sin kx}{(x+b)^2 + a^2} dx,$$

where we assume, without loss of generality, that $a > 0$, $b > 0$, and $k > 0$. We can kill two birds with one stone:

$$I_1 + iI_2 = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(x+b)^2 + a^2} dx.$$

Since $k > 0$, the exponential decays in the upper half plane, leading to the contour drawn in Fig. 4.4. The denominator has two singularities, but only one, $z = -b + ia$, is enclosed

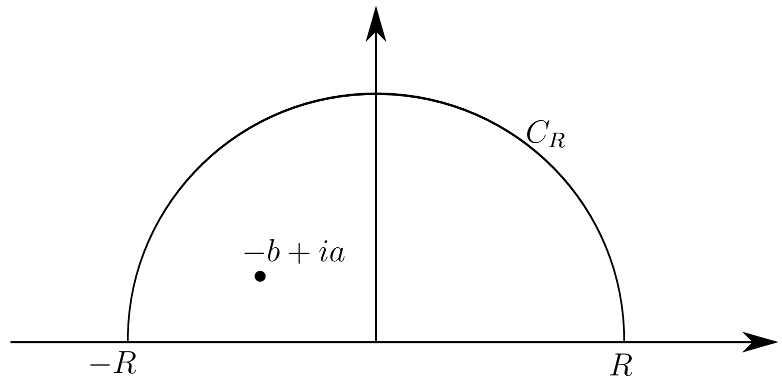

Figure 4.4: The contour integral encircling $z = -b + ia$.

by the contour. Note that

$$\lim_{z\to\infty} \frac{1}{(z+b)^2 + a^2} = 0,$$

uniformly along C_R, thus Jordan's Lemma is applicable. It follows that

$$\begin{aligned}
I_1 + iI_2 &= \int_{-\infty}^{\infty} \frac{e^{ikx}}{(x+b)^2 + a^2} dx + \lim_{R\to\infty} \int_{C_R} \frac{e^{ikz}}{(z+b)^2 + a^2} dz \\
&= \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{e^{ikx}}{(x+b)^2 + a^2} dx + \int_{C_R} \frac{e^{ikz}}{(z+b)^2 + a^2} dz \right) \\
&= 2\pi i \operatorname*{Res}_{z=-b+ia} \frac{e^{ikz}}{(z+b)^2 + a^2} \\
&= 2\pi i \lim_{z\to -b+ia} \frac{e^{ikz}(z+b-ia)}{(z+b)^2 + a^2} \\
&= 2\pi i e^{-ak-ikb} \lim_{z\to -b+ia} \frac{1}{2(z+b)} \\
&= 2\pi i e^{-ak-ikb} \frac{1}{2ia} \\
&= \frac{\pi}{a} e^{-ak-ikb},
\end{aligned}$$

so that

$$I_1 = \frac{\pi}{a} e^{-ka} \cos kb, \quad I_2 = -\frac{\pi}{a} e^{-ka} \sin kb.$$

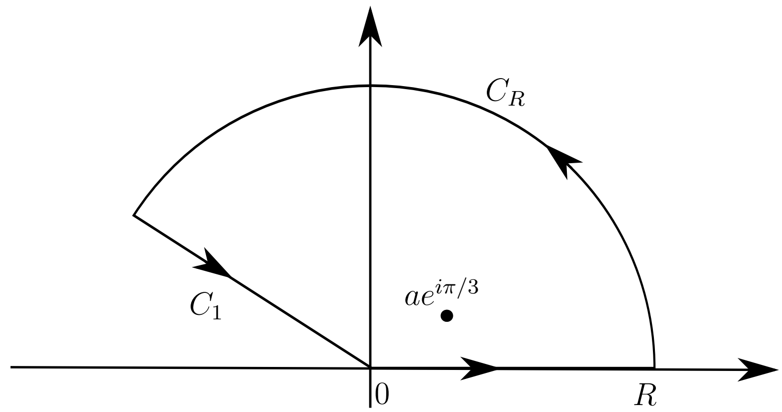

Figure 4.5: The contour for the evaluation of $\int_0^\infty dx/(x^3+a^3)$.

The art of choosing the contour

A trickier example is presented by the following integral. Note that this is not a Fourier-type integral.

Example. Consider

$$I = \int_0^\infty \frac{dx}{x^3+a^3}, \quad a > 0.$$

We consider the contour drawn in Fig. 4.5. Here C_1 is the ray from the origin to $Re^{2\pi i/3}$. We have that, for sufficiently large R,

$$\begin{aligned}
& \int_0^R \frac{1}{x^3+a^3}dx + \int_{C_R} \frac{1}{z^3+a^3}dz + \int_{C_1} \frac{1}{z^3+a^3}dz = 2\pi i \operatorname*{Res}_{z=ae^{i\pi/3}} \frac{1}{z^3+a^3} \\
\Rightarrow \quad & I + \lim_{R\to\infty} \int_{C_R} \frac{1}{z^3+a^3}dz + \lim_{R\to\infty} \int_{C_1} \frac{1}{z^3+a^3}dz = 2\pi i \lim_{z\to ae^{i\pi/3}} \frac{z-ae^{i\pi/3}}{z^3+a^3} \\
\Rightarrow \quad & I + 0 + e^{2\pi i/3} \int_\infty^0 \frac{1}{t^3+a^3}dt = 2\pi i \lim_{z\to ae^{\pi i/3}} \frac{1}{3z^2} \\
\Rightarrow \quad & I - e^{2\pi i/3} I = \frac{2\pi i}{3a^2} e^{-2\pi i/3} \\
\Rightarrow \quad & I = \frac{2\pi i}{3a^2} \frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}} \\
& = \frac{2\pi}{3\sqrt{3}a^2},
\end{aligned}$$

where the last step requires a bit of algebra. The contribution along C_R vanishes as $R \to \infty$, following the same argument as on page 103. To evaluate the integral along C_1, we used the parameterization $z = te^{2\pi i/3}$.

It is clear from this example that the art of using contour integration for the evaluation of integrals is in the choice of the contour. This is why a solid understanding of the behavior in the complex plane of the functions appearing in the integrand is of crucial importance[3].

Evaluating sums

Contour integration is useful as well for the evaluation of finite or infinite sums. You will look at many interesting infinite sums on the homework. Here's a finite sum. Consider

$$I(p) := \sum_{n=1}^{p-1} \frac{1}{\sin^2 \frac{n\pi}{p}}.$$

Can we evaluate this sum in closed form? On a personal note, this sum came up in my graduate research. It was the first time I used complex analysis to derive a new result. I am getting all nostalgic thinking about it. I was so much older then. . . .

In order to evaluate $I(p)$ using the residue theorem, we wish to construct a function whose sum of residues over a suitable contour will give us the sum. Perhaps a direct evaluation of the contour integral will give us a closed form expression for the sum.

The function that does the trick is

$$f(z) = \frac{p}{\sin^2(z)\tan(pz)}.$$

Indeed, $f(z)$ has triple poles at $z = 0$, $z = \pi$, but, most importantly for us, $f(z)$ has simple poles at $z = n\pi/p$, for $n \in \mathbb{Z}$. Furthermore, $f(z)$ is periodic with period π. Thus, we are led to use the contour given in Fig. 4.6. It avoids the poles at $z = 0$ and $z = \pi$, by taking small radius-ϵ semi-circle detours around them, see also the next section. Otherwise, it follows straight-line segments along the imaginary axis, and above and below $z = \pi$. Lastly, there are horizontal segments, which will not contribute, as their contribution will vanish as they are pushed to $i\infty$[4] and $-i\infty$.

From the Residue Theorem,

$$\frac{1}{2\pi i}\oint_C f(z)dz = \sum_{n=1}^{p-1} \operatorname*{Res}_{z=n\pi/p} f(z).$$

It remains to calculate the residues on the right-hand side, and the integral on the left-hand

[3] A better understanding of the behavior of these functions can distinguish the complex plane Rembrandt from the paint-by-the-numbers amateur.

[4] This means ∞ in the vertical direction

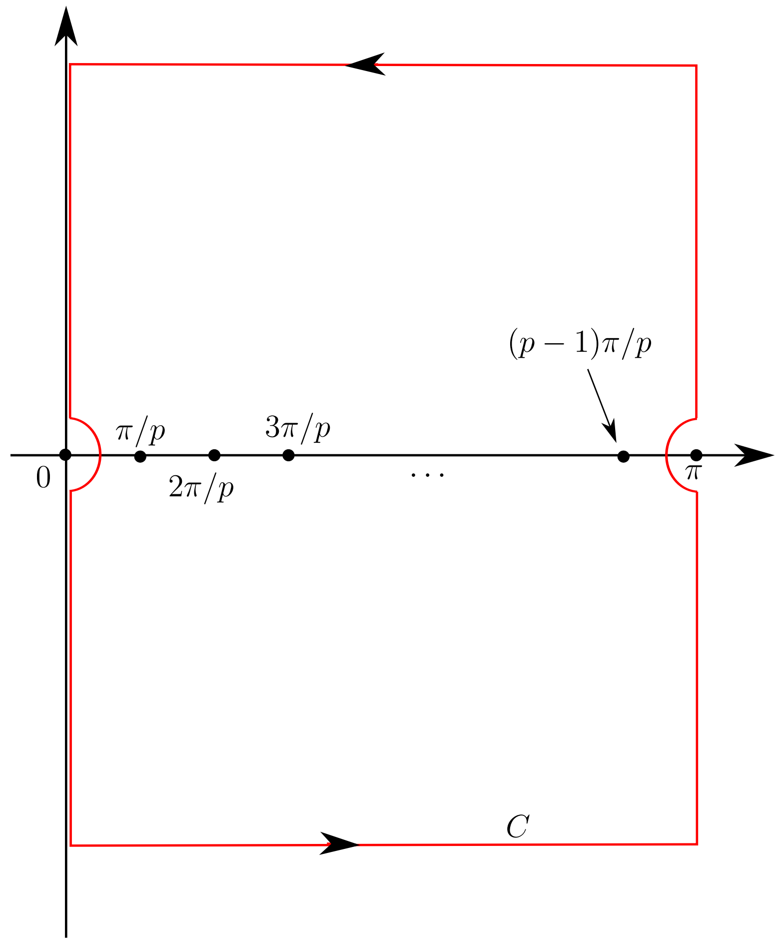

Figure 4.6: The contour for the evaluation of $I(p)$.

side. In other words, it remains to compute everything. We begin with

$$\begin{aligned}
\operatorname*{Res}_{z=n\pi/p} \frac{p}{\sin^2 z \tan(pz)} &= p \lim_{z\to n\pi/p} \frac{z-n\pi/p}{\sin^2 z \tan(pz)} \\
&= \frac{p}{\sin^2(n\pi/p)} \lim_{z\to n\pi/p} \frac{z-n\pi/p}{\tan(pz)} \\
&= \frac{p}{\sin^2(n\pi/p)} \lim_{z\to n\pi/p} \frac{1}{p\sec^2(pz)} \\
&= \frac{1}{\sin^2(n\pi/p)}.
\end{aligned}$$

Thus the right-hand side is exactly $I(p)$.

Next, we tackle the integral. First, note that the contributions from the vertical parts will cancel: the integrand is periodic with period π, thus its values are identical on the left and right vertical parts. But these parts are traversed in opposite direction. Second, we show that the contribution of the horizontal parts vanishes as they are pushed to infinity. On the upper horizontal part C_1, we let $z = x+iR$, and we will consider the limit as $R \to \infty$. Using

$$|\sin(x+iy)| \geq \sinh y \quad \text{and} \quad |\cos(x+iy)| \leq \cosh y,$$

where we have assumed that $x, y \in \mathbb{R}$, we have that

$$\begin{aligned}
\left|\int_{C_1} f(z)dz\right| &= p\left|\int_{C_1} \frac{\cos(pz)}{\sin^2 z \sin(pz)} dz\right| \\
&\leq p \int_0^\pi \frac{|\cos p(x+iR)|}{|\sin^2(x+iR)||\sin p(iR+x)|} dx \\
&\leq p \int_0^\pi \frac{\cosh(pR)}{\sinh^2 R \sinh(pR)} dx \\
&= p\pi \frac{\cosh(pR)}{\sinh^2 R \sinh(pR)} \\
&\to 0, \quad \text{as } R \to \infty.
\end{aligned}$$

It remains to consider the contributions from the semi-circles. Note that by the periodicity of the integrand, we can move the semi-circle around $z = \pi$ to one to the left of $z = 0$. Now these two semi-circles around zero can make some beautiful music together, resulting in a residue contribution around $z = 0$! Since the origin is encircled the wrong way, we get

$$\sum_{n=1}^{p-1} \frac{1}{\sin^2 \frac{n\pi}{p}} = -\operatorname*{Res}_{z=0} f(z).$$

We have

$$\begin{aligned}
\operatorname*{Res}_{z=0} \frac{p}{\sin^2 z \tan(pz)} &= p \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2} \frac{z^3}{\sin^2 z \tan(pz)} \\
&= \frac{p^2-1}{3},
\end{aligned}$$

since $z = 0$ is a triple pole of $f(z)$. I leave you the joy of computing the limit and verifying the last step[5]. Thus we have established that

$$\sum_{n=1}^{p-1} \frac{1}{\sin^2 \frac{n\pi}{p}} = \frac{p^2-1}{3}.$$

You should think of whether you know of any way to establish this without using complex function theory!

[5] If you are reading this on a Monday, you could use Maple or Mathematica or Sage to calculate the limit.

4.4 Indented contours, integrals with branch points

In this section we consider integrals whose integrands may have singularities on the path of integration. Judicious use of the principal value will be adamant!

Example. Consider

$$I(a) = \int_{-\infty}^{\infty} \frac{\sin(ax)}{x}.$$

For now, we will assume that $a > 0$. This integral is not absolutely convergent. Rather it is conditionally convergent, so we may proceed. Note that

$$I(a) = \int_{-\infty}^{\infty} \frac{e^{iax} - e^{-iax}}{2ix}.$$

We might be inclined to split up the integral, wanting to use Jordan's Lemma in the upper half plane for the first integral and in the lower half plane for the second integral. Not a good idea: each integral separately is not defined, because of the singularity at the origin. This singularity is removable in $I(a)$, but not in each integral of the above formula. We proceed as follows instead. Since $I(a)$ is defined, it is equal to its principal value:

$$I(a) = \fint_{-\infty}^{\infty} \frac{\sin(ax)}{x} dx,$$

where at this point the principal value is in place to address our approach to $\pm\infty$. At this point, the singularity at 0 is removable and does not pose a problem. Next, we write

$$I(a) = \fint_{-\infty}^{\infty} \frac{\mathrm{Im}(e^{iax})}{x} dx.$$

We wish to take the imaginary part out of the integral, which would usually be allowed. If we do this here, we get

$$I(a) = \mathrm{Im} \fint_{-\infty}^{\infty} \frac{e^{iax}}{x} dx,$$

and we have added $\fint_{-\infty}^{\infty} \frac{\cos(ax)}{x} dx$ to the integral. If this was not a principal value integral, this would be an illegal operation, since the improper integral would not converge and we would, in effect, be adding ∞. However, it is a principal value integral, and since $\cos(ax)/x$ is an odd function, we are adding zero! That is allowed. Thus, we can proceed to evaluate

$$J(a) := \fint_{-\infty}^{\infty} \frac{e^{iax}}{x} dx,$$

obtaining $I(a)$ from the imaginary part of $J(a)$.

Since the exponential (with $a > 0$) is decaying in the upper half plane, we choose a contour that closes using C_R there, see Fig. 4.7. Also, since 0 is a singular point (no longer removable), we choose a path of integration that avoids it, passing above it on a small circle of

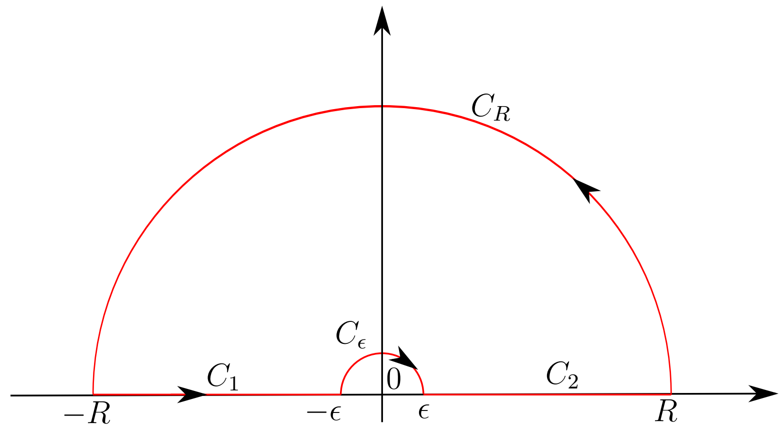

Figure 4.7: The contour for the evaluation of the sinc integral.

radius ϵ instead. You should verify that you get the same answer if you choose a small circle that passes below 0 instead. We now evaluate our integrand along the contour of Fig. 4.7. Since there are no singularities inside the contour[6], we have from Cauchy's Theorem (or the Residue Theorem without singularities inside)

$$0 = \left(\int_{C_1} + \int_{C_\epsilon} + \int_{C_2} + \int_{C_R} \right) \frac{e^{iaz}}{z} dz.$$

The limit as $R \to \infty$ and $\epsilon \to 0$ of the first and the third term equals $J(a)$. Further, the condition for Jordan's Lemma is satisfied ($i.e.$, $|1/z| \to 0$ uniformly along C_R as $R \to \infty$), thus as $R \to \infty$, the last term vanishes. Thus

$$J(a) = -\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iaz}}{z} dz.$$

Below, we will derive a result that allows for the evaluation of this right-hand side integral, but it is just as easy to calculate it directly. Parameterizing the integrand with $z = \epsilon e^{i\theta}$, we have

$$\begin{aligned}
\int_{C_\epsilon} \frac{e^{iaz}}{z} dz &= \int_\pi^0 \frac{e^{ia\epsilon e^{i\theta}}}{\cancel{\epsilon e^{i\theta}}} \cancel{\epsilon e^{i\theta}} i d\theta \\
&= -i \int_0^\pi \left(1 + ia\epsilon e^{i\theta} + \mathcal{O}(\epsilon^2)\right) d\theta \\
&= -i\pi + \mathcal{O}(\epsilon).
\end{aligned}$$

Here $\mathcal{O}(\epsilon^n)$ indicates contributions that are of order ϵ^n, $i.e.$, the limit of these terms divided by ϵ^n is finite (possibly 0). It follows that

$$J(a) = i\pi \quad \Rightarrow \quad I(a) = \pi.$$

[6]This would be different if you worked with a small semi-circle that passes below 0.

If $a < 0$, we would have had to close the contour using a C_R in the lower half plane. This would have resulted in an extra minus sign, as you should verify. It is also easy to see directly. Suppose $b = -a < 0$. Then

$$I(b) = I(-a) = \int_{-\infty}^{\infty} \frac{\sin(-ax)}{x} dx = -\int_{-\infty}^{\infty} \frac{\sin(ax)}{x} dx = -I(a) = -\pi.$$

Thus, for any real a,

$$I(a) = \operatorname{sgn}(a)\pi,$$

where sgn denotes the sign function: $\operatorname{sgn}(a) = 1$ if $a > 0$, $\operatorname{sgn}(a) = -1$ if $a < 0$. We will leave it to the philosophers to decide what $\operatorname{sgn}(0)$ is.

The above example uses a so-called *indented* contour: a contour with small detours to avoid singularities on the contour. The choice of the indentations is not arbitrary: above or below the singularity might result in a residue contribution. Most importantly, we have to keep the eye on the prize: in the above example, we wanted the limit of the contributions along C_1 and C_2. The rest was a means to an end.

The following two theorems are quite useful.

Theorem 4.5 *If on a small arc C_ϵ centered at z_0 of angle ϕ and radius ϵ (see Fig. 4.8), $(z - z_0)f(z) \to 0$ as $\epsilon \to 0$, then*

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) dz = 0.$$

Proof. Parameterizing C_ϵ using $z = z_0 + \epsilon e^{i\theta}$, we have

$$\begin{aligned}
\left| \int_{C_\epsilon} f(z) dz \right| &= \left| \int_{\phi_1}^{\phi_1 + \phi} f(z_0 + \epsilon e^{i\theta}) \epsilon i e^{i\theta} d\theta \right| \\
&\leq \int_{\phi_1}^{\phi_1 + \phi} \left| f(z_0 + \epsilon e^{i\theta}) \right| \epsilon d\theta \\
&= \int_{\phi_1}^{\phi_1 + \phi} \left| \epsilon e^{i\theta} f(z_0 + \epsilon e^{i\theta}) \right| d\theta \\
&\leq K_\epsilon \phi \\
&\to 0,
\end{aligned}$$

as $\epsilon \to 0$. Here K_ϵ is defined using that

$$\epsilon |f(z_0 + \epsilon e^{i\theta})| < K_\epsilon,$$

independent of the angle on C_ϵ, as guaranteed by the assumption of the theorem. This concludes the proof. ■

Thus, if $f(z)$ behaves less bad than $1/(z - z_0)$ near z_0, the integral along an ϵ circle segment of $f(z)$ vanishes and the contribution of such integrals may be ignored.

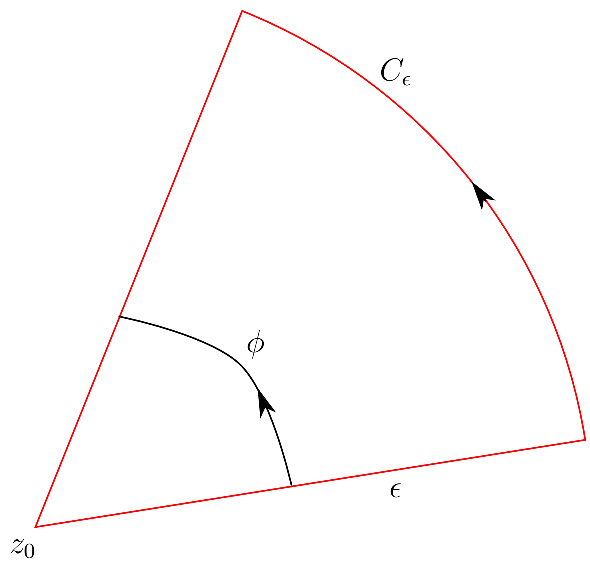

Figure 4.8: A small arc C_ϵ centered at z_0 of angle ϕ and radius ϵ.

Theorem 4.6 *If $f(z)$ has a simple pole of strength (or residue) c_{-1}, then*

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)dz = i\phi c_{-1}.$$

Proof. The proof is a direct calculation. With $z = z_0 + \epsilon e^{i\theta}$ and $z_1 = z_0 + \epsilon e^{i\phi_1}$ and $z_2 = z_0 + \epsilon e^{i(\phi_1+\phi)}$ we have

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{z_1}^{z_2} f(z)dz &= \lim_{\epsilon\to 0}\int_{z_1}^{z_2}\left(\frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots\right)dz \\
&= \lim_{\epsilon\to 0}\int_{\phi_1}^{\phi_1+\phi}\left(\frac{c_{-1}}{\epsilon e^{i\theta}} + c_0 + c_1\epsilon e^{i\theta} + \cdots\right)\epsilon i e^{i\theta}d\theta \\
&= \lim_{\epsilon\to 0}\int_{\phi_1}^{\phi_1+\phi}\left(ic_{-1} + c_0 i\epsilon e^{i\theta} + \mathcal{O}(\epsilon^2)\right)d\theta \\
&= ic_{-1}\phi,
\end{aligned}$$

which concludes the proof. ■

Note that, unlike for the Residue Theorem, more singular contributions would result in an infinite contribution to the integral. The following theorem is proven using a direct calculation, just like the two previous ones. We skip the proof here. It provides a limit result for integrals along arcs of large radii if no exponentials are present and Jordan's Lemma cannot be used.

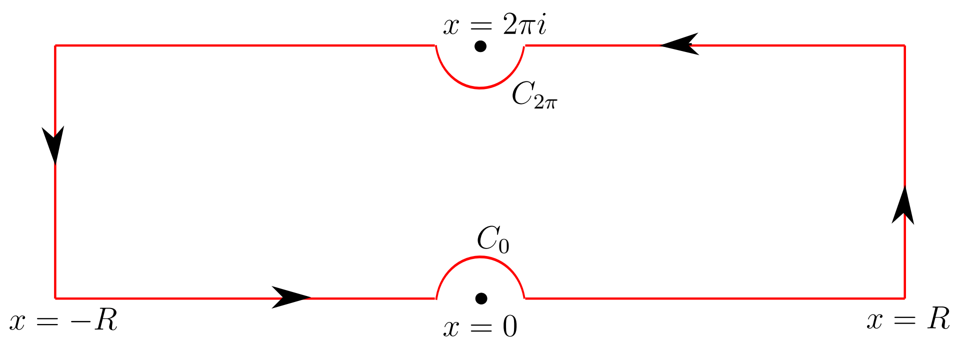

Figure 4.9: The contour to calculate $J(p)$.

Theorem 4.7 *If on a circular arc C_R centered at 0 of radius R, $zf(z) \to 0$ uniformly as $R \to \infty$, then*

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

Let us examine another example using indented contours.

Example. Consider

$$I(p,q) := \int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1-e^x} dx,$$

where $p, q \in (0,1)$. The denominator is zero at $x = 0$, but so is the numerator and the singularity there is removable. It follows that

$$I(p,q) = \fint_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1-e^x} dx = \fint_{-\infty}^{\infty} \frac{e^{px}}{1-e^x} dx - \fint_{-\infty}^{\infty} \frac{e^{qx}}{1-e^x} dx = J(p) - J(q),$$

where

$$J(p) := \fint_{-\infty}^{\infty} \frac{e^{px}}{1-e^x} dx, \quad p \in (0,1).$$

Note that $J(p)$ is well defined. Thus the equality $I(p,q) = J(p) - J(q)$ makes sense.

Since the denominator is periodic in the imaginary direction with period 2π, we use the contour depicted in Fig. 4.9. Here C_0 and $C_{2\pi}$ are semi-circles centered at $x = 0$ and $x = 2\pi i$ respectively, both of radius ϵ, as indicated. From Cauchy's Theorem, we have

$$\left(\int_{-R}^{-\epsilon} + \int_{C_0} + \int_{\epsilon}^{R} + \int_{R}^{R+2\pi i} + \int_{R+2\pi i}^{\epsilon+2\pi i} + \int_{C_{2\pi}} + \int_{-\epsilon+2\pi i}^{-R+2\pi i} + \int_{-R+2\pi i}^{-R} \right) \frac{e^{px}}{1-e^x} dx = 0.$$

In the limit $R \to \infty$ and $\epsilon \to 0$, the sum of the first and third integral gives $J(p)$. It remains to determine the other integrals in these limits. Let us start with the fourth integral. Using the parameterization $x = R + 2\pi i t$, we have that

$$\begin{aligned}
\left|\int_R^{R+2\pi i}\frac{e^{px}}{1-e^x}dx\right| &= \left|\int_0^1\frac{e^{p(R+2\pi it)}}{1-e^{R+2\pi it}}2\pi i dt\right| \\
&\le 2\pi\int_0^1\frac{e^{pR}}{e^R-1}dt \\
&= 2\pi\frac{e^{pR}}{e^R-1} \\
&\to 0, \quad \text{as} \quad R\to\infty,
\end{aligned}$$

where we have used that $p\in(0,1)$. This demonstrates that we may omit the contributions from the fourth integral. The last integral undergoes the same fate. Using the parameterization $x=2\pi i+t$ on the top horizontal segments, we have that

$$\begin{aligned}
& J(p) - ⨍_{-\infty}^{\infty}\frac{e^{p(2\pi i+t)}}{1-e^{2\pi i+t}}dt + \lim_{\epsilon\to 0}\left(\int_{C_0}+\int_{C_{2\pi}}\right)\frac{e^px}{1-e^x}dx = 0 \\
\Rightarrow \quad & J(p) - e^{2\pi ip}J(p) + (-\pi)i\operatorname*{Res}_{x=0}\frac{e^{px}}{1-e^x} + (-\pi)i\operatorname*{Res}_{x=2\pi i}\frac{e^{px}}{1-e^x} = 0,
\end{aligned}$$

so that

$$\begin{aligned}
\Rightarrow \quad J(p)(1-e^{2\pi ip}) &= i\pi\lim_{x\to 0}\frac{e^{px}x}{1-e^x} + i\pi\lim_{x\to 2\pi i}\frac{e^{px}(x-2\pi i)}{1-e^x} \\
&= i\pi\lim_{x\to 0}\frac{1}{-e^x} + i\pi e^{2\pi ip}\lim_{x\to 2\pi i}\frac{1}{-e^x} \\
&= -i\pi - i\pi e^{2\pi ip} \\
\Rightarrow \quad J(p) &= \frac{-i\pi - i\pi e^{2\pi ip}}{1-e^{2\pi ip}} \\
&= \pi\cot\pi p,
\end{aligned}$$

from which

$$I(p,q) = \int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^x}dx = \pi(\cot\pi p - \cot\pi q).$$

4.5 The Argument Principle

The *Argument Principle* is a straightforward, yet powerful, consequence of the Residue Theorem.

Theorem 4.8 (Argument Principle.) *Let $f(z)$ be meromorphic inside and analytic on C, a simple closed contour. Further, assume none of the zeros or poles of $f(z)$ are on C. Then*

$$\frac{1}{2\pi i}\oint_C\frac{f'(z)}{f(z)}dz = N - P,$$

where N (P) is the number of zeros (poles) of $f(z)$ that lie inside C, counting multiplicities.

Note that the integral can be written as $\frac{1}{2\pi}\Delta_C \arg f(z)$, the change in the argument of $f(z)$ over C divided by 2π.

Proof. Since the zeros and poles of a meromorphic function are isolated, we may write

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\sum_{k=1}^{N+P}\oint_{C_k}\frac{f'(z)}{f(z)}dz,$$

where C_k encircles only one pole or zero of $f(z)$, possibly of higher multiplicity. Next, we determine what the contribution of each pole or zero is.

Suppose z_k is a pole or zero of $f(z)$. Then in a small neighborhood of z_k we may write

$$f(z) = (z - z_k)^n \phi(z),$$

where n is positive for a zero, and negative for a pole. Further, $|n|$ is its multiplicity. We may assume that the analytic function $\phi(z) \neq 0$ in a neighborhood of z_k. Of course, $\phi(z_k) \neq 0$. Then

$$f'(z) = n(z - z_k)^{n-1}\phi(z) + (z - z_k)^n \phi'(z),$$

so that

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_k} + \frac{\phi'(z)}{\phi(z)}.$$

Since the second term is analytic in a neighborhood of z_k, we have that

$$\frac{1}{2\pi i}\oint_{C_k}\frac{f'(z)}{f(z)}dz = n,$$

which is a positive contribution for a zero, and a negative contribution for a pole. This proves the theorem. ■

Example. We can use the argument principle to see how many roots of $f(z) = z^6 + z + 1$ lie in the first quadrant. To this end we use the contour in Fig. 4.10. Here R is large.

- We start at $z = 0$, following the real axis to $z = R$. Here $f(z)$ is real and positive, thus its argument is zero.

- Next, we traverse the circular arc. Along this arc, $z = Re^{i\theta}$, where θ starts at zero, goes to $\pi/2$. It follows that

$$\begin{aligned} f(z) &= z^6 + z + 1 \\ &= R^6 e^{6i\theta} + Re^{i\theta} + 1 \\ &= R^6 e^{6i\theta}\left(1 + R^{-5}e^{-5i\theta} + R^{-6}e^{-6i\theta}\right). \end{aligned}$$

  Thus for $R \gg 1$, the argument of $f(z)$ goes from 0 to 3π, as the arc is traversed.

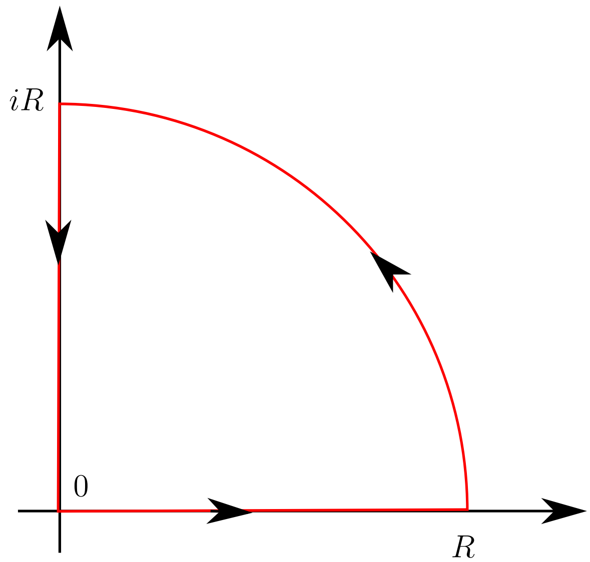

Figure 4.10: A large circular arc contour around the first quadrant.

- Last, we return to the origin along the imaginary axis. Here $z = it$, where t starts at R, decreasing to 0. We have that

$$\begin{aligned} f(z) &= z^6 + z + 1 \\ &= i^6 t^6 + it + 1 \\ &= -t^6 + it + 1 \end{aligned}$$

  Thus the argument of $f(z)$ satisfies $\tan \arg f(z) = t/(1 - t^6)$. For large t, this corresponds to a tangent that is close to zero but negative, in agreement with our previous result of an argument that is near to but slightly less than 3π. As t decreases, the tangent passes through ∞ once, when $t = 1$, at which point the argument has decreased to $5\pi/2$. It decreases further to 2π as $t \to 0$.

In summary, the change in the argument of $f(z)$ is 2π, allowing for the conclusion that there is one root of $f(z)$ in the first quadrant. As z traverses the contour of Fig. 4.10, $f(z)$ traverses the path of Fig. 4.11, winding around the origin once.

The following theorem is a consequence of the Argument Principle.

Theorem 4.9 (Rouché). *Let $f(z)$ and $g(z)$ be analytic in and on C, a simple, closed contour. If $|f(z)| > |g(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.*

Proof. Since $|f(z)| > |g(z)| \geq 0$ on C, it follows that $f(z) \neq 0$ on C. Define

$$w = \frac{f(z) + g(z)}{f(z)}.$$

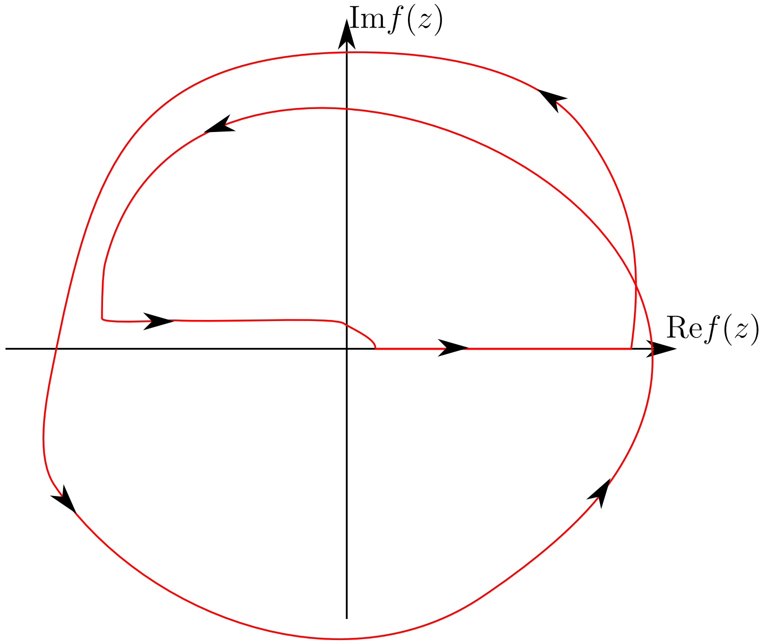

Figure 4.11: A cartoon of the path followed by $f(z)$ in the complex plane. Note that this path winds around the origin once.

Then $\frac{1}{2\pi i}\oint_C \frac{w'}{w} dz$ is well defined. But

$$w(z) - 1 = \frac{f(z) + g(z) - f(z)}{f(z)} = \frac{g(z)}{f(z)},$$

and $|w(z) - 1| < 1$ on C. Thus, as z traverses C, $w(z)$ always lies on the right side of the origin, never touching it. It follows that the winding number of $w(z)$ around the origin is zero. Thus the number of zeros and the number of poles of $w(z)$ are equal. This proves the theorem. ■

Example. For the second time, we prove the Fundamental Theorem of Algebra as a simple consequence of a theorem in complex analysis. Consider

$$f(z) = z^n,$$

and

$$g(z) = a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0.$$

For $|z| > 1$, $|f(z)| = |z|^n$ and

$$|g(z)| \leq (|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|)|z|^{n-1}.$$

Further, $|f(z)| > |g(z)|$, for $|z|$ sufficiently large. It follows from Rouché's Theorem that $f(z) + g(z)$ (an arbitrary polynomial of degree n) has the same number of zeros as $f(z)$, which is n, counting multiplicities. It follows from our proof that all these roots are inside the circle $|z| < R$, with $R > \max(1, |a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|)$.

Example. We show that all the roots of $p(z) = z^8 - 4z^3 + 10$ lie in $1 \leq |z| \leq 2$. First, using $C_1 : |z| = 1$, we choose $f(z) = 10$, $g(z) = z^8 - 4z^3$. Then $|f| = 10$ and $|g(z)| \leq |z|^8 + 4|z|^3 \leq 5$. Thus $|f| > |g|$ on C_1. It follows that $p(z)$ has no zeros inside C_1, since $f(z) = 10$ is never zero there.

Next, using $C_2 : |z| = 2$, we choose $f(z) = z^8$ and $g(z) = -4z^3 + 10$. Then $|f(z)| = 2^8 = 256$. Further, $|g(z)| \leq 4 \times 2^3 + 10 = 42$. As before, this implies that $p(z)$ has as many zeros inside C_2 as does $f(z)$, namely 8. Since these zeros are not inside C_1, they are in the annulus, which we had to show.

4.6 Fourier and Laplace transforms

We won't prove the following theorem, but it is a pretty important one. You can find the proof in the book. It uses series expansions and uniform convergence.

Theorem 4.10 *Consider*

$$F(z) = \int_a^b g(z,t)dt,$$

where (i) $g(z,t)$ is analytic in $z \in D$, and (ii) $g(z,t)$ is a continuous function of t for all $z \in D$. Then (A) $F(z)$ is analytic in D, and (B) we have that

$$F'(z) = \int_a^b \frac{\partial g}{\partial z} dt.$$

The theorem does not immediately apply to the case of the Fourier or Laplace transform, since in those cases both or one of a or b are infinity, and extra uniform convergence requirements are necessary. Be that as it may, the *Fourier transform* of a function $f(x)$ is defined as

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx.$$

From $\hat{f}(k)$, we may recover $f(x)$ from the *inverse Fourier transform*:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)dk.$$

Note that this second integral is defined using a principal value! It is not trivial to see that these two transformations are indeed inverses of each other. This requires either a serious amount of distribution theory, or else an unhealthy amount of advanced calculus and more uniform convergence than is good for you[7]. In any case, in order for these formulas to result in well-defined expressions, it suffices that either

$$f(x) \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty,$$

or,

$$f(x) \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

The validity of the Fourier transform and its inverse imply that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} e^{-iky} f(y) dy.$$

What can be said about the analyticity of the Fourier transform $\hat{f}(k)$ a function of k? This is the topic of the Paley-Wiener Theorems. In general, nothing can be said about the analyticity of the Fourier transform, since the integral has a growing exponential kernel if k is not on the real axis: indeed, since x takes on both positive and negative values in the integral, the exponential grows in both the upper- and lower-half plane. More conditions need to be imposed on $f(x)$ to make any statements. For instance, if one considers $f(x)$ to be non-zero only for $x > 0$, the integral reduces to

$$\hat{f}(k) = \int_{0}^{\infty} e^{-ikx} f(x) dx.$$

In this case, the exponential is decaying for $\text{Im}k < 0$, and the Fourier transform may be analytically extended off the real k axis into the lower-half plane. All Paley-Wiener Theorems work this way: extra conditions are required on $f(x)$, after which the analyticity of $\hat{f}(k)$ in some domain in $\mathbb{C}$ follows. Popular conditions are those of compact support, exponential decay to zero as $|x| \to \infty$, *etc.*

We can use Fourier transforms as a road to Laplace transforms. Suppose that $f(x)$ has support only for positive x. In other words $f(x) = 0$ for $x < 0$. Consider $e^{-cx} f(x)$, for some $c > 0$. This allows for the consideration of $f(x)$ that are exponentially growing: we can kill this growth with a sufficiently large choice of c so that the product $e^{-cx} f(x) \to 0$ as $x \to \infty$. Substituting this into the above formula expressing that the Fourier transform and its inverse are indeed inverse of each other, we get

$$\begin{aligned} e^{-cx} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk \int_{0}^{\infty} e^{-iky} e^{-cy} f(y) dy \\ \Rightarrow \qquad f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(ik+c)x} dk \int_{0}^{\infty} e^{-(ik+c)y} f(y) dy. \end{aligned}$$

Let $s = c + ik$ or $k = (s-c)/i$, then

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx} ds \int_{0}^{\infty} e^{-sy} f(y) dy.$$

[7] I know many people who say uniform convergence is good for you. But always in moderation, of course!

This leads us to define the *Laplace transform* of $f(x)$ as

$$\mathcal{L}[f(x)](s) = L(s) := \int_0^\infty e^{-sy} f(y) dy,$$

and its inverse

$$\mathcal{L}^{-1}[L(s)](x) = \frac{1}{2\pi i} \fint_{c-i\infty}^{c+i\infty} e^{sx} L(s) ds := \frac{1}{2\pi i} \lim_{R\to\infty} \int_{c-iR}^{c+iR} e^{sx} L(s) ds = f(x).$$

The contour in this principal value integral is referred to as a *Bromwich contour*[8].

Most likely, you have already seen the Laplace transform in a course on differential equations. If so, you probably used tables to do Laplace transform inversion, as opposed to the inversion formula given above. This is infinitely more satisfying!

Example. Consider $f(x) = e^{ax}$. It is a trivial calculation to check that

$$L(s) = \frac{1}{s-a}.$$

Next, we use the Bromwich formula to invert this result. We wish to use the Residue Theorem, requiring us to close the contour. We close it using a semi-circle in the left-half plane: this will give rise to a damped exponential, and a zero contribution, from Jordan's Lemma. This is illustrated in Fig. 4.12. The choice of c does not really matter, as long as it is sufficiently large so that all singularities of the integrand lie to the left of the Bromwich contour.

We get that

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \fint_{c-i\infty}^{c+i\infty} L(s) e^{sx} ds \\ &= \operatorname*{Res}_{s=a} L(s) e^{sx} \\ &= \operatorname*{Res}_{s=a} \frac{1}{s-a} e^{sx} \\ &= e^{ax}, \end{aligned}$$

as expected.

Thus, instead of doing lots of partial fraction expansions, to reduce expressions to the simple ones that appear in tables, we simply calculate some residues. The next example is

[8]Thomas John l'Anson Bromwich (1875–1929) was an English mathematician, and a Fellow of the Royal Society. He was senior Wrangler at Cambridge and had a mustache.

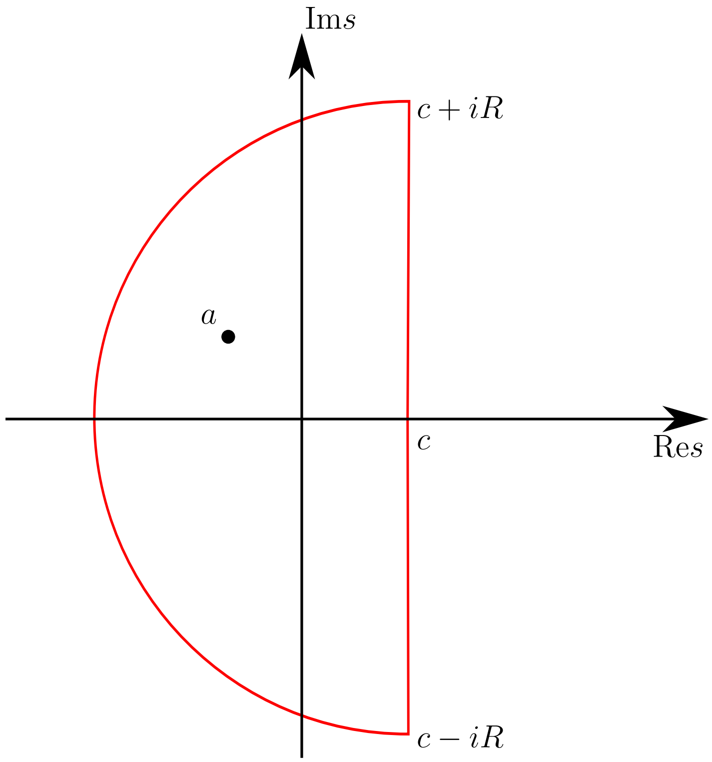

Figure 4.12: The Bromwich contour for $L(s) = 1/(s-a)$.

more involved.

Example. Consider $f(x) = x^\alpha$, with $\alpha \in (-1, 0)$. Then, using the substitution $z = sx$,

$$\begin{aligned} L(s) &= \int_0^\infty x^\alpha e^{-sx} dx \\ &= \frac{1}{s}\int_0^\infty \frac{z^\alpha}{s^\alpha} e^{-z} dz \\ &= \frac{1}{s^{\alpha+1}} \int_0^\infty z^\alpha e^{-z} dz \\ &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}. \end{aligned}$$

Here $\Gamma(y)$ denotes the Gamma function:

$$\Gamma(y) := \int_0^\infty z^{y-1} e^{-z} dz.$$

Can we calculate the inverse transform?

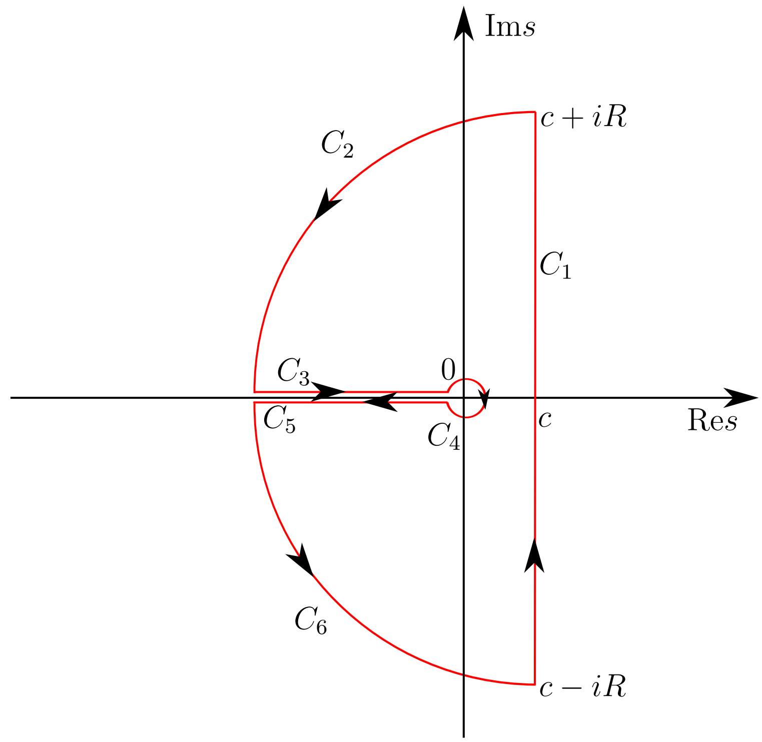

Figure 4.13: The keyhole contour for the inverse Laplace transform of x^α.

The inverse transform is given by

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\left[\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right](x) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\alpha+1)s^{-\alpha-1}e^{sx}ds. \end{aligned}$$

We use the keyhole contour shown in Fig. 4.13. As usual for an inverse Laplace transform, we close the Bromwich contour with a semi-circle on the left. However, due to the presence of a branch point at $s = 0$, we have a branch cut along the negative real axis. The keyhole contour goes around the branch cut, using two large quarter-circle arc segments of radius R and a small circle of radius ϵ to go around $s = 0$. We have that

$$\frac{1}{2\pi i}\left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5} + \int_{C_6}\right)\Gamma(\alpha+1)s^{-\alpha-1}e^{sx}ds = 0,$$

since there are no singularities inside the closed keyhole contour. From Jordan's Lemma, we have that the integrals over the large quarter circles approach zero as $R \to \infty$. Thus

$$y(x) + \frac{1}{2\pi i} \lim_{R\to\infty,\epsilon\to 0} \int_{C_3} \Gamma(\alpha+1) s^{-\alpha-1} e^{sx} ds +$$
$$\frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_{C_4} \Gamma(\alpha+1) s^{-\alpha-1} e^{sx} ds + \frac{1}{2\pi i} \lim_{R\to\infty,\epsilon\to 0} \int_{C_5} \Gamma(\alpha+1) s^{-\alpha-1} e^{sx} ds = 0.$$

We investigate these different parts individually. Let us begin with the integral over the small circle around the branch point at the origin. Using the parameterization $s = \epsilon e^{i\theta}$

$$\begin{aligned}
\frac{1}{2\pi i} \int_{C_4} \Gamma(\alpha+1) s^{-\alpha-1} e^{sx} ds &= \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\pi}^{-\pi} \epsilon^{-\alpha-1} e^{-i(\alpha+1)\theta} e^{x\epsilon e^{i\theta}} \epsilon e^{i\theta} i d\theta. \\
&= -\frac{\Gamma(\alpha+1)}{2\pi} \int_{-\pi}^{\pi} \epsilon^{-\alpha} e^{-i\alpha\theta} (1 + \mathcal{O}(\epsilon)) d\theta \\
&\to 0,
\end{aligned}$$

since $\alpha < 0$. Thus there is no contribution from the branch point at $s = 0$.

Next we consider the contribution from integrating along the upper part of the branch cut. We use the parameterization $s = te^{i\pi}$. We get

$$\begin{aligned}
\frac{1}{2\pi i} \int_{C_3} \Gamma(\alpha+1) s^{-\alpha-1} e^{sx} ds &= \frac{\Gamma(\alpha+1)}{2\pi i} \int_{R}^{\epsilon} (te^{i\pi})^{-\alpha-1} e^{xte^{i\pi}} e^{i\pi} dt \\
&= -\frac{\Gamma(\alpha+1)}{2\pi i} \int_{\epsilon}^{R} t^{-\alpha-1} e^{-i\pi(\alpha+1)} e^{-xt} e^{i\pi} dt \\
&= -\frac{\Gamma(\alpha+1)}{2\pi i} \int_{\epsilon}^{R} t^{-\alpha-1} e^{-i\pi\alpha} e^{-xt} dt \\
&= -\frac{\Gamma(\alpha+1)}{2\pi i} e^{-i\pi\alpha} \int_{\epsilon}^{R} t^{-\alpha-1} e^{-xt} dt \\
&\to -\frac{\Gamma(\alpha+1)}{2\pi i} e^{-i\pi\alpha} \mathcal{L}\left[t^{-\alpha-1}\right](x) \\
&= -\frac{\Gamma(\alpha+1)}{2\pi i} e^{-i\pi\alpha} \frac{\Gamma(-\alpha)}{x^{-\alpha}} \\
&= -\frac{\Gamma(\alpha+1)\Gamma(-\alpha) e^{-i\pi\alpha}}{2\pi i} x^{\alpha}.
\end{aligned}$$

The calculation for the integration along C_5, the bottom of the branch cut is similar, using the parameterization $s = te^{-i\pi}$. We get

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{C_5}\Gamma(\alpha+1)s^{-\alpha-1}e^{sx}ds &= \frac{\Gamma(\alpha+1)}{2\pi i}\int_\epsilon^R (te^{-i\pi})^{-\alpha-1}e^{xte^{-i\pi}}e^{-i\pi}dt \\
&= \frac{\Gamma(\alpha+1)}{2\pi i}\int_\epsilon^R t^{-\alpha-1}e^{i\pi(\alpha+1)}e^{-xt}e^{-i\pi}dt \\
&= \frac{\Gamma(\alpha+1)}{2\pi i}\int_\epsilon^R t^{-\alpha-1}e^{i\pi\alpha}e^{-xt}dt \\
&= \frac{\Gamma(\alpha+1)}{2\pi i}e^{i\pi\alpha}\int_\epsilon^R t^{-\alpha-1}e^{-xt}dt \\
&\to \frac{\Gamma(\alpha+1)}{2\pi i}e^{i\pi\alpha}\mathcal{L}\left[t^{-\alpha-1}\right](x) \\
&= \frac{\Gamma(\alpha+1)}{2\pi i}e^{i\pi\alpha}\frac{\Gamma(-\alpha)}{x^{-\alpha}} \\
&= \frac{\Gamma(\alpha+1)\Gamma(-\alpha)e^{i\pi\alpha}}{2\pi i}x^\alpha.
\end{aligned}
$$

Combining all these results, we obtain

$$
y(x) = -\Gamma(\alpha+1)\Gamma(-\alpha)\frac{\sin\pi\alpha}{\pi}x^\alpha.
$$

We could use this as a rather indirect and long-winded proof that

$$
-\Gamma(\alpha+1)\Gamma(-\alpha)\frac{\sin\pi\alpha}{\pi} = 1,
$$

but more satisfyingly, we can prove this directly, using the product expansions of the Γ function and of sin. Indeed, using

$$
\Gamma(z) = \frac{1}{ze^{\gamma z}}\frac{1}{\prod_{n=1}^\infty (1+z/n)\,e^{-z/n}},
$$

and

$$
\sin\pi z = \pi z\prod_{n=-\infty, n\neq 0}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}
$$

it is a matter of some algebraic manipulations to show that

$$
\Gamma(z+1)\Gamma(-z) = \frac{-\pi}{\sin\pi z},
$$

as desired.

4.7 Exercises

1. Calculate
$$\int_0^\infty \frac{\sin^2(x)}{x^2} dx,$$
using contour integration.

2. Calculate
$$\int_0^\infty \frac{\sin(x)}{\sinh(x)} dx,$$
using contour integration.

3. **Summing series by residues, part 1**

   (a) Show that
   $$\text{Res}_{z=k} f(z)\cot(\pi z) = \frac{1}{\pi} f(k),$$
   provided $f(z)$ is analytic at $z = k$ ($k \in \mathbb{Z}$).

   (b) Let Γ_N be a square contour, with corners at $(N + 1/2)(\pm 1 \pm i)$, $N \in \mathbb{Z}^+$. Show that
   $$|\cot(\pi z)| \leq 2,$$
   for z on Γ_N.

   (c) Suppose $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials, so that the degree of $q(z)$ is at least two more than the degree of $p(z)$. Show that
   $$\lim_{N\to\infty} \left| \oint_{\Gamma_N} \frac{p(z)}{q(z)} \cot(\pi z) dz \right| = 0.$$

   (d) Suppose, in addition, that $q(z)$ has no roots at the integers. Show that
   $$\sum_{k=-\infty}^{\infty} \frac{p(k)}{q(k)} = -\pi \sum_j \text{Res}_{z=z_j} f(z)\cot(\pi z),$$
   where the z_j's are the roots of $q(z)$. Notice that the sum on the right-hand side has a finite number of terms.

4. Use the result of the previous problem to evaluate the following sums:

   (a) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1}$,

   (b) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^3+a^3}$, $a \notin \mathbb{Z}$,

(c) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^4+1}$,

(d) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{(4k^2-1)^2}$.

5. **Summing series by residues, part 2** You can use results from Problem 3, if useful. The notation used here is like that in Problem 3.

   Suppose that $f(z) = p(z)/q(z)$ is a rational function with degree $q(z) \geq 2+$ degree $p(z)$. Suppose that $q(z)$ has no roots at the integers, except possibly at 0. Show that

$$\sum_{k=-\infty}^{\infty}{}' \frac{p(k)}{q(k)} = -\pi \sum_{j} \mathrm{Res}_{z=z_j} f(z) \cot(\pi z),$$

   where the z_j's are the roots of $q(z)$ (including 0). Notice that the sum on the right-hand side has a finite number of terms. Here the prime on the left-hand sum indicates that the $k = 0$ term is skipped.

6. Use the result of the previous problem to evaluate the following sums:

   (a) $\displaystyle\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^2}$,

   (b) $\displaystyle\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^3}$,

   (c) $\displaystyle\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^4}$.

7. On page 91, we found the Laurent series expansion of $\coth(z)$ centered at $z = 0$ in terms of the Bernoulli numbers B_n:

$$\coth z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n-1}.$$

   (a) Write down the Laurent series centered at $z = 0$ for $\cot z$.

   (b) Use this result to calculate

$$\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^n}, \quad n \in \mathbb{N}_0.$$

8. Here's another way to evaluate
$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$
due to Euler. We've seen that
$$\frac{\sin \pi z}{\pi z} = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$
   (a) Equate the coefficients of z^2 on both sides, to recover the desired sum. This is easy.
   (b) Equate the coefficients of z^4 on both sides, to find $\sum_{k=1}^{\infty} \frac{1}{k^4}$. This is harder.

   By equating coefficients of higher powers of z, one can recover other identities too. This gets harder as you go on.

9. Calculate
$$I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x} dx.$$

10. Consider the piecewise defined function $f(x) = e^{-x}$ for $x > 0$, $f(x) = 0$ for $x < 0$ and $f(0) = 1/2$. (a) Calculate the Fourier transform of this function. (b) Calculate the inverse transform of your result, and note where you need the inverse transform to be defined as a principal value integral to recover your original $f(x)$.

11. Using the definition of the inverse Laplace transform, calculate the inverse Laplace transforms of (assume $a > 0$)
   (a) $F(s) = 1/(s - a)$
   (b) $F(s) = 1/(s - a)^2$
   (c) $F(s) = s/(s^2 + a^2)$

12. Show that there is exactly one root of $e^z - 4z - 1$ inside of $C_1 : |z| = 1$.

13. Let $f : z \to w = f(z)$ be an analytic function on the closed disk $D(z_0, R)$ of radius R centered at z_0. Denote the boundary of $D(z_0, R)$ by $C(z_0, R)$. Assume that $f : D(z_0, R) \to f(D(z_0, R))$ is one-to-one and onto.
   (a) Show that the inverse function to f is
$$g(w) = \frac{1}{2\pi i} \oint_{C(z_0,R)} \frac{t f'(t)}{f(t) - w} dt,$$
   for $w \in f(D(z_0, R))$, not including the boundary.

(b) Use this result to calculate the Taylor series of the inverse function of $w = ze^z$ around $(z_0, w_0) = (0, 0)$. What is the radius of convergence of this series? Does this radius of convergence correspond with what you expect from real analysis?

14. Show that

$$\int_0^\infty \frac{1}{x^N+1}dx = \frac{\pi}{N\sin(\pi/N)},$$

where $N \geq 2$ is a positive integer.

15. Consider

$$I = \frac{1}{2\pi i}\oint_C \phi(z)\frac{f'(z)}{f(z)}dt,$$

where C is a simple closed contour, $\phi(z)$ is analytic inside and on C, and $f(z)$ is meromorphic inside and on C without zeros or poles on C. Use the same reasoning as in the proof of the Argument Principle to show that

$$I = \sum_{j=1}^M \alpha_j\phi(v_j) - \sum_{j=1}^N \beta_j\phi(w_j),$$

where v_j, $j = 1, \ldots, M$ are the zeros of $f(z)$ in C with respective multiplicities α_j. Similarly, w_j, $j = 1, \ldots, N$ are the poles of $f(z)$ in C with respective multiplicities β_j.

16. Consider an elliptic function $f(z)$. In other words, there exist complex numbers ω_1, ω_2, ω_1/ω_2 not real, such that $f(z + m\omega_1 + n\omega_2) = f(z)$, for all integers m and n. Use the result of Problem 15 with $\phi(z) = z$ to show that the difference between the sum of the location of the zeros (with multiplicities) and the sum of the location of the poles (with multiplicities) is of the form $M\omega_1 + N\omega_2$, for integers M and N.

17. Consider the Weierstrass $\wp$ function

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n=-\infty}^{\infty}{}' \left(\frac{1}{(z-m\omega_1-n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2}\right),$$

whose base parallelogram S has corners $(\pm\omega_1/2, \pm\omega_2/2)$, with ω_1/ω_2 not real. For a given u and v in $\mathbb{C}$ so that $\wp(u) \neq \wp(v)$, define A and B to be the solutions of the linear system

$$\begin{cases} A\wp(u) + B &= \wp'(u), \\ A\wp(v) + B &= \wp'(v). \end{cases}$$

(a) Consider $f(\xi) = \wp'(\xi) - A\wp(\xi) - B$. How many poles (including multiplicities) does this function have in a S? As a consequence, how many zeros does $f(\xi)$ have in S?

(b) Having argued there are three zeros of $f(\xi)$ in S, you already know two of them: $f(u) = 0 = f(v)$. Use the result of Problem (16) to determine that the third root in S is $w = -u - v + M\omega_1 + N\omega_2$, for some choice of M and N.

(c) Now you have three values of ξ such that $f(\xi) = 0$. By eliminating A and B between the three expressions $f(u) = 0$, $f(v) = 0$, $f(w) = 0$, show that

$$\det \begin{pmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{pmatrix} = 0.$$

This is an addition theorem for the Weierstrass $\wp$ function!