Math 567 Homework 2
Due October 19 202asd2
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Problem 1 *Evaluate $\oint_C f(z)dz$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:*

a e^{iz}

b e^{z^2}

c $\frac{1}{z-1/2}$

d $\frac{1}{z^2-4}$

e $\frac{1}{2z^2+1}$

f $\sqrt{z-4}$, $0 \leq \arg(z-4) \leq 2\pi$.

Solution.

a Consider when $f(z) = e^{iz}$. First let's show that $f(z)$ is satisfies the Cauchy-Riemann equations. Observe that,

$$f(z) = e^{iz} = e^{ix-y} = e^{-y}(\cos(x) + i\sin(x)) = e^{-y}\cos(x) + ie^{-y}\sin(x) = u(x,y) + iv(x,y)$$

Now let's compute the following partials,

$$\frac{\partial u}{\partial x} = -e^{-y}\sin(x)$$
$$\frac{\partial u}{\partial y} = -e^{-y}\cos(x)$$
$$\frac{\partial v}{\partial x} = e^{-y}\cos(x)$$
$$\frac{\partial v}{\partial y} = -e^{-y}\sin(x)$$

So $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ satisfying the Cauchy-Riemann equations. Furthermore, we can clearly see that the partials of u and v with respect to x and y exist, and are continuous within C. Therefore $f(z)$ is analytic within C and by Cauchy Theorem, $\boxed{\oint_C e^{iz}dz = 0}$.

b Consider when $f(z) = e^{z^2}$. First let's show that $f(z)$ is satisfies the Cauchy-Riemann equations. Observe that,

$$\begin{aligned} f(z) = e^{z^2} &= e^{(x+iy)^2} = e^{x^2+2xyi-y^2} = e^{x^2-y^2}e^{2xyi} \\ &= e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy) = u(x,y) + iv(x,y) \end{aligned}$$

Now let's compute the following partials,

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2e^{x^2-y^2}x\cos(2xy) - 2e^{x^2-y^2}y\sin(2xy) \\ \frac{\partial u}{\partial y} &= -2e^{x^2-y^2}y\cos(2xy) - 2e^{x^2-y^2}x\sin(2xy) \\ \frac{\partial v}{\partial x} &= e^{x^2-y^2} \cdot\ 2x\sin(2xy) + \cos(2xy)\cdot\ 2ye^{x^2-y^2} \\ \frac{\partial v}{\partial y} &= -2e^{x^2-y^2}y\sin(2xy) + 2e^{x^2-y^2}x\cos(2xy) \end{aligned}$$

So $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ satisfying the Cauchy-Riemann equations. Furthermore, we can clearly see that the partials of u and v with respect to x and y exist, and are continuous within C. Therefore $f(z)$ is analytic within C and by Cauchy Theorem, $\boxed{\oint_C e^{z^2}dz = 0}$.

c Consider when $f(z) = \frac{1}{z-1/2}$. Notice that $\oint_C (z-z_0)^n dz$ where $n = -1$ and $z_0 = \frac{1}{2}$. In class we showed that this integral will be $2\pi i$ if $n = -1$ and 0 otherwise when C encloses z_0. Thus we have that $\boxed{\oint_C \frac{1}{z-1/2}dz = 2\pi i}$.

d consider $f(z) = \frac{1}{z^2-4}$. Then we can use the derivative formula to observe,

$$\begin{aligned} \frac{d}{dz}\left(\frac{1}{z^2-4}\right) &= \lim_{\Delta z\to 0} \frac{\frac{1}{(z-\Delta z)^2-z_0} - \frac{1}{z^2-z_0}}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{\frac{z^2-z_0-z^2-2z\Delta z-\Delta z^2+z_0}{((z+\Delta z)^2-z)(z^2-z_0)}}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{-2z-\Delta z}{((z+\Delta z)^2-z)(z^2-z_0)} \\ &= \frac{-2z}{(z^2-z_0)^2} \end{aligned}$$

and thus we see that the derivative is exists and is independent of the path. Since $f(z)$ does not blow up within the unit circle, $f(z)$ is analytic within C. By Cauchy Theorem, $\boxed{\oint_C \frac{1}{z^2-4}dz = 0}$.

e Consider $\frac{1}{2z^2+1}$. We can rewrite this as $\frac{1}{2z^2+1} = \frac{1}{2}\left(\frac{1}{z^2+\frac{1}{2}}\right) = \frac{1}{2}g(z)$. We can see that there are potential zeros at $z_0 = \pm\frac{i}{\sqrt{2}}$ which are contained within C. Next we can use partial fractions to get,

$$g(z) = \left(\frac{1}{z-\frac{i}{\sqrt{2}}}\right)\left(\frac{1}{z+\frac{i}{\sqrt{2}}}\right) = \frac{A}{2-\frac{i}{\sqrt{2}}} + \frac{B}{2+\frac{i}{\sqrt{2}}}$$

$$A = \lim_{z\to\frac{i}{\sqrt{2}}}\left(\frac{1}{z+\frac{i}{\sqrt{2}}}\right) = \frac{\sqrt{2}}{2i}$$

$$B = \lim_{z\to-\frac{i}{\sqrt{2}}}\left(\frac{1}{z-\frac{i}{\sqrt{2}}}\right) = -\frac{\sqrt{2}}{2i}$$

Next we can apply the Cauchy's Integral formula, we know

$$\frac{\sqrt{2}}{4i}\oint_C \frac{1}{z-\frac{i}{\sqrt{2}}}dz = \frac{\sqrt{2}}{4i}(2\pi i) = \frac{\pi\sqrt{2}}{2}$$

and

$$-\frac{\sqrt{2}}{4i}\oint_C \frac{1}{z+\frac{i}{\sqrt{2}}}dz = -\frac{\sqrt{2}}{4i}(2\pi i) = -\frac{\pi\sqrt{2}}{2}$$

Therefore,

$$\oint_C f(z)dz = \oint_C \frac{1}{2}g(z)dz = \frac{1}{2}\left(\frac{\pi\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{2}\right) = 0$$

In conclusion, $\boxed{\oint_C \frac{1}{z^2-4}dz} = 0$.

f Consider $f(z) = \sqrt{z-4}$, $0 \le \arg(z-4) \le 2\pi$. This is in the from $f(z) = (z-z_1)^{\frac{1}{2}}$ where $z_1 = 4$. We have a branch cut that is centered at $z = z_1$ and moves 2π around the complex plane. This restricts $f(z)$ to be a single-valued function within C. Next we have to show that the derivative of $f(z)$ exists within C. Observe that,

$$
\begin{aligned}
\frac{d}{dz}\left(\sqrt{z-4}\right) &= \lim_{\Delta z \to 0} \frac{\sqrt{z+\Delta z-4} - \sqrt{z-4}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\sqrt{z+\Delta z-4} - \sqrt{z-4}}{\Delta z}\left(\frac{\sqrt{z+\Delta z-4} + \sqrt{z-4}}{\sqrt{z+\Delta z-4} + \sqrt{z-4}}\right) \\
&= \lim_{\Delta z \to 0} \frac{(z+\Delta z-4) - (z-4)}{\Delta z\left(\sqrt{z+\Delta z-4} + \sqrt{z-4}\right)} \\
&= \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z\left(\sqrt{z+\Delta z-4} + \sqrt{z-4}\right)} \\
&= \lim_{\Delta z \to 0} \frac{1}{\left(\sqrt{z+\Delta z-4} + \sqrt{z-4}\right)} \\
&= \frac{1}{\sqrt{z-4} + \sqrt{z-4}} \\
&= \frac{1}{2\sqrt{z-4}}
\end{aligned}
$$

And thus we see that the derivative exists within C and is path independent of Δz. Note the branch cut begins at $z = 4$ and moves away from C. Therefore there are no issues within C and $f(z)$ is analytic within C. By Cauchy's Theorem, we have that $\boxed{\oint_C \sqrt{z-4}dz = 0}$ for $0 \leq \arg(z-4) \leq 2\pi$.

□

Problem 2 *We wish to evaluate the integral*

$$\int_0^\infty e^{ix^2} dx$$

Consider the contour

$$I_R = \oint_{C(R)} e^{iz^2} dz,$$

where $C_{(R)}$ is the closed circular sector in the upper half plane with boundary points $0, R$, and $Re^{i\pi/4}$. Show that $I_R = 0$ and that

$$\lim_{R\to\infty} \int_{C_{1(R)}} e^{iz^2} dz = 0,$$

where $C_{1(R)}$ is the line integral along the circular sector from R to $Re^{i\pi/4}$. Hint: Use $\sin(x) \geq \frac{2x}{\pi}$ on $0 \leq x \leq \pi/2$. Then, breaking up the contour $C_{(R)}$ into three component parts, deduce

$$\lim_{R\to\infty} \left(\int_0^R e^{ix^2} dx - e^{i\pi/4} \int_0^R e^{-2r^2} dr \right) = 0$$

and from the well-known result of real integration:

$$\int_0^\infty e^{-2x^2} dx = \frac{\sqrt{\pi}}{2}$$

deduce that $I = e^{i\pi/4}\sqrt{4}/2$.

Solution. Consider

$$\int_0^\infty e^{ix^2} dx$$

and the contour

$$I_R = \oint_{C(R)} e^{iz^2} dz,$$

where $C_{(R)}$ is the closed circular sector in the upper half plane with boundary points $0, R$, and $Re^{i\pi/4}$. We know that e^{iz^2} is entire (from problem 1) and thus analytic within $C_{(R)}$. Then by Cauchy's Theorem $I_R = 0$. Next let's show that

$$\lim_{R\to\infty} \int_{C_{1(R)}} e^{iz^2} dz = 0$$

where $C_{1(R)}$ is the line integral along the circular sector from R to $Re^{i\pi/4}$. Let's also define $C_{2(R)}$ as the line integral from $Re^{i\pi/4}$ to 0 and $C_{3(R)}$ is from 0 to R. Observe that,

$$\begin{aligned}
I_1(R) &= \int_{C_{1(R)}} e^{iz^2} dz \\
&= \int_0^{\pi/4} e^{\left(Re^{i\theta}\right)^2} Rie^{i\theta} d\theta \\
&= Ri \int_0^{\frac{\pi}{4}} e^{iR^2(\cos(\theta)+i\sin(\theta))^2} e^{i\theta} d\theta \\
&= Ri \int_0^{\frac{\pi}{4}} e^{i\left(R^2(\cos(2\theta)+i\sin(2\theta))+\theta\right)} d\theta \\
&= Ri \int_0^{\frac{\pi}{4}} e^{iR^2\cos(2\theta)} e^{-R^2\sin(2\theta)} e^{i\theta} d\theta
\end{aligned}$$

Thus we have,

$$\begin{aligned}
|I_I(R) &= \left| Ri \int_0^{\frac{\pi}{4}} e^{iR^2\cos(2\theta)} e^{-R^2\sin(2\theta)} e^{i\theta} d\theta \right| \\
&\leq R \int_0^{\frac{\pi}{4}} \left| e^{iR^2\cos(2\theta)} e^{-R^2\sin(2\theta)} e^{i\theta} \right| d\theta \\
&= R \int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)} d\theta
\end{aligned}$$

because $\sin(x) \leq \frac{2x}{\pi}$ on $0 \leq x \leq \frac{\pi}{2}$, then $\sin(2x) \leq \frac{4x}{\pi}$ which implies $-R^2\sin(2x) \leq -R^2\frac{4x}{\pi}$ and thus $e^{-R^2\sin(2x)} \leq e^{-\frac{4x}{\pi}R^2}$. Then

$$\begin{aligned}
|I_1(R)| &\leq R \int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)} d\theta \\
&\leq R \int_0^{\frac{\pi}{4}} e^{-\frac{4x}{\pi}R^2} d\theta \\
&= R \left[-\frac{\pi}{4R^2} e^{-R^2\frac{4\theta}{\pi}} \right]_0^{\frac{\pi}{4}} \\
&= -\frac{\pi}{4R}\left(e^{-R^2} - 1\right)
\end{aligned}$$

And then we have,

$$\lim_{R\to\infty} -\frac{\pi}{4R}\left(e^{-R^2} - 1\right) = 0$$

Therefore,

$$
\begin{aligned}
&0 \le \lim_{R\to\infty} |I_1(R)| \le \lim_{R\to\infty} -\frac{\pi}{4R}\left(e^{-R^2}-1\right) = 0 \\
&\implies \lim_{R\to\infty} |I_1(R)| = 0
\end{aligned}
$$

Noting that we can break the contour into three components as,

$$I_R = I_1(R) + I_2(R) + I_3(R) = I_2(R) + I_3(R) = 0$$

then $\lim_{r\to\infty}(I_2(R) + I_3(R)) = 0$ which gives that $\lim_{r\to\infty} I_3(R) = -\lim_{r\to\infty} I_2(R)$. Therefore,

$$
\begin{aligned}
\int_0^\infty e^{ix^2}dx &= \lim_{R\to\infty}\int_0^R e^{ix^2}dx \\
&= \lim_{R\to\infty} I_3(R) \\
&= -\lim_{R\to\infty} I_2(R) \\
&= -\lim_{R\to\infty}\int_R^0 e^{i(r^2e^{i\pi/4})}e^{i\pi/4}dr \\
&= e^{i\pi/4}\lim_{R\to\infty}\int_0^R e^{-r^2}dr \\
&= \frac{e^{i\pi/4}\sqrt{\pi}}{2}
\end{aligned}
$$

And so we have shown that $I = \frac{e^{i\pi/4}\sqrt{\pi}}{2}$. □

Problem 3 *Consider the integral*

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}.$$

Show how to evaluate this integral by considering

$$\oint_{C(R)} \frac{dz}{z^2+1},$$

where $C(R)$ is the closed semicircle in the upper half plane with endpoints at $(-R,0)$ and $(R,0)$ plus the x axis. Hint: use

$$\frac{1}{z^2+1} = \frac{-1}{2i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right)$$

and show that the integral along the open semicircle in upper half plane vanishes as $R \to \infty$. Verify your answer by usual integration in real variables.

Solution. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

and $C(R)$ is the closed semicircle in the upper half plane with endpoints at $(-R,0)$ and $(R,0)$ plus the x axis. Let C_1 be the line from $-R$ to R and C_2 the upper semicricle from R to $-R$. Note that there is one singularity within $C(R)$ at $z = i$. Next we can decompose the fraction to get,

$$\begin{aligned}
\oint \frac{1}{z^2+1}dz &= -\frac{1}{2i}\oint_{C_R} \frac{1}{z^2+1}dz \\
&= -\frac{1}{2i}\oint_{C_R} \frac{1}{z+i} - \frac{1}{z-i}dz \\
&= -\frac{1}{2i}\oint_{C_R} \frac{1}{z+i}dz + \frac{1}{2i}\oint_{C_R} \frac{1}{z-i}dz \\
&= 0 + \frac{1}{2i}(2\pi i) \\
&= \pi
\end{aligned}$$

where the $\oint_{C_R} \frac{1}{z-i}dz = 2\pi i$ since the singularity is contained within $C(R)$. Therefore we have,

$$\begin{aligned}
\pi = \oint_{C(R)} \frac{dz}{z^2+1} &= \oint_{C(1)} \frac{dz}{z^2+1} + \oint_{C(2)} \frac{dz}{z^2+1} \\
\implies \oint_{C(1)} \frac{dz}{z^2+1} &= \pi - \oint_{C(2)} \frac{dz}{z^2+1}
\end{aligned}$$

We can apply a transformation to get,

$$\oint_{C(2)} \frac{dz}{z^2+1} = \int_0^{\pi} \frac{Rie^{i\theta}}{R^2e^{i2\theta}+1}d\theta$$

Computing the modulus we get the following upper bound,

$$\begin{aligned}
\left|\oint_{C(2)} \frac{dz}{z^2+1}\right| &= \left|\frac{Rie^{i\theta}}{R^2e^{i2\theta}+1}d\theta\right| \\
&\leq \int_0^{\pi} \frac{R}{\left|R^2e^{i2\theta}+1\right|}d\theta \\
&= \frac{1}{R}\int_0^{\pi} \frac{1}{\left|e^{i2\pi\theta}+\frac{1}{R^2}\right|}
\end{aligned}$$

Noticing that $\left|e^{i2\pi\theta}+\frac{1}{R^2}\right| \geq \left|\left(\left|e^{i2\pi\theta}\right| - \left|-\frac{1}{R^2}\right|\right)\right| = \left|1-\frac{1}{R^2}\right|$ we can continue to simplify as

$$\begin{aligned}
\frac{1}{R}\int_0^{\pi} \frac{1}{\left|e^{i2\pi\theta}+\frac{1}{R^2}\right|} &\leq \frac{1}{R}\int_0^{\pi} \frac{1}{\left|1-\frac{1}{R^2}\right|} \\
&= \frac{1}{R}\int_0^{\pi} \frac{1}{1-\frac{1}{R^2}} \\
&= \frac{1}{R(1-\frac{1}{R^2})}\int_0^{\pi} d\theta \\
&= \frac{\pi}{R(1-\frac{1}{R^2})}
\end{aligned}$$

Thus,

$$\begin{aligned}
0 \geq \lim_{R\to\infty}\left|\oint_{C(2)} \frac{dz}{z^2+1}\right| &\leq \lim_{R\to\infty} \frac{\pi}{R(1-\frac{1}{R^2})} = 0 \\
\implies \lim_{R\to\infty}\oint_{C(2)} \frac{dz}{z^2+1} &= 0
\end{aligned}$$

Therefore we have that,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{R\to\infty}\oint_{C(R)} \frac{dz}{z^2+1} = \pi - 0 = \pi.$$

and by directly evaluating we can verify that

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^2+1} &= \lim_{R\to\infty}\left[\arctan(x)\right]_{-R}^{R} \\
&= \lim_{R\to\infty}(\arctan(R) - \arctan(R)) \\
&= \frac{\pi}{2} + \frac{\pi}{2} \\
&= \pi
\end{aligned}$$

□

Problem 4 *Let*

$$f(z) = e^{\frac{t}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n.$$

Show from the definition of Laurent series and using properties of integration that

$$\begin{aligned} Jn(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} d\theta \\ &= \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta - t\sin\theta) d\theta. \end{aligned}$$

The functions $J_n(t)$ are called Bessel functions, which are well-known special functions in mathematics and physics.

Solution.
Let's consider

$$f(z) = e^{\frac{t}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n.$$

where by definition

$$\begin{aligned} Jn(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} d\theta \\ &= \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta - t\sin\theta) d\theta. \end{aligned}$$

Notice that,

$$\begin{aligned} J_n(t) &= \frac{1}{2\pi i} \oint \frac{e^{\frac{t}{2}(z-\frac{1}{z})}}{e^{i\theta(n+1)}} ie^{i\theta} d\theta \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta}-e^{-i\theta})}}{e^{i\theta(n+1)}} ie^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(2i\sin\theta)}}{e^{i\theta n}} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ti\sin\theta - i\theta n} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} d\theta \end{aligned}$$

and since,

$$e^{-i(n\theta - t\sin\theta)} = \cos(n\theta - t\sin(\theta)) - i\sin(n\theta - t\sin(\theta)).$$

Therefore we have

$$\begin{aligned}
J_n(t) &= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi}\cos(n\theta - t\sin(\theta))d\theta - i\int_{-\pi}^{\pi}\sin(n\theta - t\sin(\theta))d\theta\right] \\
&= \frac{1}{2\pi}\left[2\int_{0}^{\pi}\cos(n\theta - t\sin(\theta))d\theta\right] \\
&= \frac{1}{\pi}\int_{0}^{\pi}\cos(n\theta - t\sin(\theta))d\theta
\end{aligned}$$

□