Math 568 Homework 6
Due February 21
By Marvyn Bailly

Problem 1 *Consider the singular equation:*

$$\epsilon y'' + (1+x)^2 y' + y = 0,$$

with $y(0) = y(1) = 1$ *and with* $0 < \epsilon \ll 1$.

(a) Obtain the leading order uniform solution using the WKB method.

(b) Plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution.
Consider the singular equation:

$$\epsilon y'' + (1+x)^2 y' + y = 0,$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$.

(a) We wish to obtain the leading order uniform solution using the WKB method. We begin by assuming the solution takes the form

$$y(x) = \exp\left(\frac{S_0(x) + \epsilon S_1(x) + \epsilon^2 S_2(x) + \cdots}{\epsilon}\right).$$

Inserting this ansatz into the governing equation, collecting terms and dividing out the expontial yields

$$\begin{aligned} \mathcal{O}(e^{-1}) &: \quad S_{0_x}^2 + (1+x)^2 S_{0_x} = 0, \\ \mathcal{O}(e^{0}) &: \quad S_{0_{xx}} + 2S_{0_x}S_{1_x} + (1+x)^2 S_{1_x} + 1 = 0. \end{aligned}$$

The leading order problem can be rewritten as

$$S_{0_x}\left(S_{0_x} + (1+x)^2\right) = 0,$$

which gives the two solutions $S_{0_x} = 0$ or $S_{0_x} = -(1+x)^2$.

In the case when $S_{0_x} = 0$, then S_0 is a constant. Plugging this result into the $O(1)$ equation gives

$$S_{1_x}(1+x)^2 + 1 = 0,$$

and solving for S_1 gives

$$S_1 = -\int_0^x \frac{1}{(1+\xi)^2}\mathrm{d}\xi = \frac{1}{1+x}.$$

The WKB solution in this case is given by

$$y(x) = \exp\left(\frac{S_0}{\epsilon} + S_1\right) = \exp\left(\frac{S_0}{\epsilon}\right)\exp\left(\frac{1}{1+x}\right) = C_1 e^{\frac{1}{1+x}}.$$

Next consider when $S_{0_x} = -(1+x)^2$. Plugging this into the $O(1)$ equation gives

$$-(x+1)^2 S_{1_x} - 2x = 1,$$

and solving for S_1 gives

$$S_1 = \int_0^x \frac{1}{(1+\xi)^2} \mathrm{d}\xi - \ln[(1+x)^2] = -\frac{1}{1+x} - \ln[(1+x)^2].$$

Note that

$$S_0 = -x - x^2 - \frac{x^3}{3}$$

The WKB solution in this case is given by

$$\begin{aligned} y(x) &= \exp\left(\frac{S_0}{\epsilon} + S_1\right) \\ &= \exp\left(\frac{1}{\epsilon}\left(-x - x^2 - \frac{x^3}{3}\right) - \frac{1}{1+x} - \ln[(1+x)^2]\right) \\ &= \frac{C_2}{(1+x)^2}\exp\left(\frac{1}{\epsilon}\left(-x - x^2 - \frac{x^3}{3}\right) - \frac{1}{1+x}\right). \end{aligned}$$

This gives the solution

$$y(x) = C_1 \exp\left(\frac{1}{1+x}\right) + \frac{C_2}{(1+x)^2}\exp\left(\frac{1}{\epsilon}\left(-x - x^2 - \frac{x^3}{3}\right) - \frac{1}{1+x}\right),$$

and enforcing the initial conditions $y_1(0) = y_1(1) = 1$ we find that

$$C_1 = -\frac{4e^{\frac{7}{3\epsilon}+\frac{1}{2}} - e}{e\left(e - 4e^{\frac{7}{3\epsilon}}\right)},$$

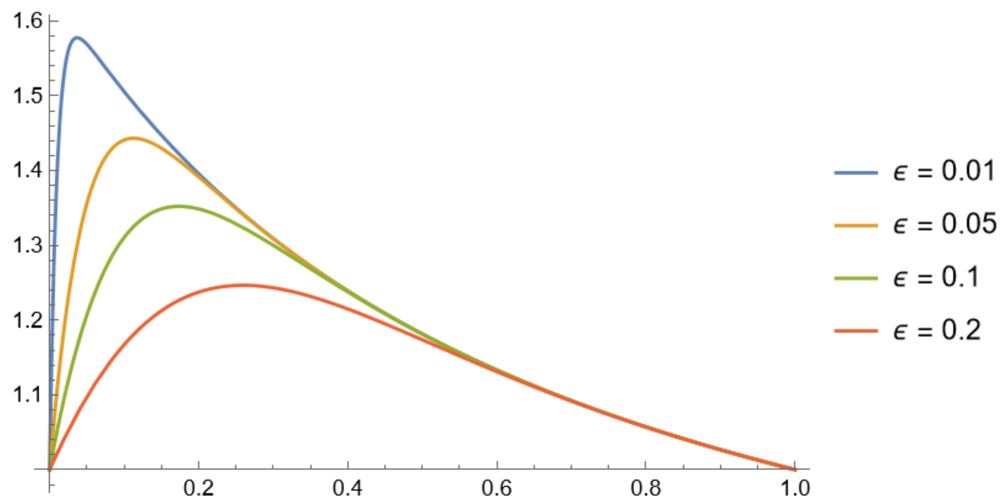
and

$$C_2 = -\frac{4\left(\sqrt{e}-1\right)e^{\frac{7}{3\epsilon}+1}}{4e^{\frac{7}{3\epsilon}} - e}.$$

Thus the uniform solution using the WKB method is given by

$$y(x) = -\frac{4\left(\sqrt{e}-1\right)e^{\frac{-\frac{x^3}{3}-x^2-x}{\epsilon} - \frac{1}{x+1} + \frac{7}{3\epsilon} + 1}}{(x+1)^2\left(4e^{\frac{7}{3\epsilon}} - e\right)} - \frac{e^{\frac{1}{x+1}-1}\left(4e^{\frac{7}{3\epsilon}+\frac{1}{2}} - e\right)}{e - 4e^{\frac{7}{3\epsilon}}}$$

(b) Using Mathematica we can plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$ which gives the following figure.



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