Math 573 Homework 4

Due Soon

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Problem 1 *Consider the Modified Vector Derivative NLS equation*

$$B_t + (\|B\|^2 B)_x + \gamma(e_1 \times B_0)\,(e_1 \cdot (B_x \times B_0)) + e_1 \times B_{xx} = 0.$$

This equation describes the transverse propagation of nonlinear Alfvén waves in magnetized plasmas. Here $B = (0, u, v)$, $e_1 = (1, 0, 0)$, $B_0 = (0, B_0, 0)$, *and* γ *is a constant. The boundary conditions are* $B \to B_0$, $B_x \to 0$ *as* $|x| \to \infty$. *By looking for stationary solutions* $B = B(x - Wt)$, *one obtains a system of ordinary differential equations. Integrating once, one obtains a first-order system of differential equations for* u *and* v.

a) *Show that this system is Hamiltonian with canonical Poisson structure, by constructing its Hamiltonian* $H(u, v)$.

b) *Find the value of the Hamiltonian such that the boundary conditions are satisfied. Then* $H(u, v)$ *equated to this constant value defines a curve in the* (u, v)*-plane on which the solution lives. In the equation of this curve, let* $U = u/B_0$, $V = v/B_0$, *and* $W_0 = W/B_0^2$. *Now there are only two parameters in the equation of the curve:* W_0 *and* γ.

c) *With* $\gamma = 1/10$, *plot the curve for* $W_0 = 3$, $W_0 = 2$, $W_0 = 1.1$, $W_0 = 1$, $W_0 = 0.95$, $W_0 = 0.9$. *All of these curves have a singular point at* $(1, 0)$. *This point is an equilibrium point for the Hamiltonian system, corresponding to the constant solution which satisfies the boundary condition. The curves beginning and ending at this equilibrium point correspond to soliton solutions of the Modified Vector Derivative NLS equation. How many soliton solutions are there for the different velocity values you considered? Draw a qualitatively correct picture of the solitons for all these cases.*

Solution. (Worked with Kaitlynn throughout homework and Cade on number 8) Consider the Modified Vector Derivative NLS equation

$$B_t + (\|B\|^2 B)_x + \gamma(e_1 \times B_0)\,(e_1 \cdot (B_x \times B_0)) + e_1 \times B_{xx} = 0,$$

where $B = (0, u, v)^T$, $e_1 = (1, 0, 0)^T$, $B_0 = (0, B_0, 0)^T$ and γ is a constant with boundary conditions $B \to B_0, B_x \to 0$ as $|x| \to \infty$.

a) To reduce the system to a first-order system, consider when $B = B(x - Wt)$. Then the partials are given by

$$\begin{aligned} B_t &= -WB_z(x - Wt) \\ B_x &= B_z(x - Wt) \\ B_{xx} &= B_{zz}(x - Wt). \end{aligned}$$

Plugging these into the NLS equation we get

$$
\begin{aligned}
0 &= B_t + (u^2 B + v^2 B)_z + \gamma(e_1 \times B_0)(e_1 \cdot (B_x \times B_0)) + e_1 \times B_x \\
&= \begin{pmatrix} 0 \\ u_t \\ v_t \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} 0 \\ u^3 + v^2 u \\ vu^2 + v^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \gamma B_0 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} -v_x B_0 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ -v_{xx} \\ u_{xx} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ u_t \\ v_t \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} 0 \\ u^3 + v^2 u \\ vu^2 + v^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -v_x B_0^2 \gamma \end{pmatrix} + \begin{pmatrix} 0 \\ -v_{xx} \\ u_{xx} \end{pmatrix}.
\end{aligned}
$$

Splitting this into two equations and using the partials we computed we get

$$
\begin{cases} -Wu_z + (u^3 + v^2 u)_z - v_{zz} = 0 \\ -Wv_z + (vu^2 + v^3)_z - \gamma B_0^2 v_z + u_{zz} = 0 \end{cases},
$$

and integrating both sides with respect to z gives

$$
\begin{cases} -Wu + (u^3 + v^2 u) - v_z = \alpha \\ -Wv + (vu^2 + v^3) - \gamma B_0^2 v + u_z = \beta \end{cases},
$$

where α and β are integration constants. From our boundary conditions we have that $u \to B_0$, $v \to 0$, $u_z \to 0$ and $v_z \to 0$ as $|x| \to \infty$. Thus $\beta = 0$ and $\alpha = -W\beta_0 + \beta^3$. Since this is an ODE, let

$$
u_z = \frac{\partial \mathcal{H}}{\partial v} \quad v_z = -\frac{\partial \mathcal{H}}{\partial u}
$$

Then our system becomes

$$
\begin{cases} -Wu + (u^3 + v^2 u) + W\beta_0 - \beta^3 = v_z = -\frac{\partial \mathcal{H}}{\partial u} \\ Wv - vu^2 - v^3 + \gamma B_0^2 v = u_z = \frac{\partial \mathcal{H}}{\partial v} \end{cases}.
$$

Integrating both sides gives

$$
\mathcal{H} = \frac{W}{2}U^2 - \frac{1}{4}u^4 - \frac{v^2}{2}u^2 - WB_0 u + B_0^3 u + A(v),
$$

and

$$
\mathcal{H} = \frac{W}{2}v^2 - \frac{u^2}{2}v^2 - \frac{1}{4}v^4 + \frac{1}{2}\gamma B_0^2 v^2 + B(u).
$$

Combining these two equations we find that the Hamiltonian with canonical Poisson structure is given by

$$
H(u,v) = -\frac{1}{4}v^4 + \frac{1}{2}\big(W - u^2 + \gamma B_0^2\big)v^2 - \frac{1}{4}u^4 + \frac{1}{2}Wu^2 - WB_0 u + B_0^3 u.
$$

b) We now wish to find the value of the Hamiltonian such that the boundary conditions are satisfied. Observe that

$$H(B_0,0) = -\frac{1}{4}B_0^4 + \frac{1}{2}WB_0 - WB_0^2 + B_0^4,$$

and since $H(u,v) = H(B_0,0)$ we can rewrite our Hamiltonian by

$$\begin{aligned} &-\frac{1}{4}v^4 + \frac{1}{2}(W - u^2 + \gamma B_0^2)v^2 - \frac{1}{4}u^4 + \frac{1}{2}Wu^2 - WB_0u + B^3u \\ &= -\frac{1}{4}B_0^4 + \frac{1}{2}WB_0^2 - WB_0^2 + B_0^4. \end{aligned}$$

If we let $U = u/B_0, V = v/B_0$, and $W_0 = W/B_0^2$ then $u = B_0U, v = VB_0$, and $W = W_0B_0^2$ then our system becomes

$$\begin{aligned} &-\frac{1}{4}B_0^4V^4 + \frac{1}{2}(W_0B_0^2 - B_0^2U^2 + \gamma B_0^2)V^2B_0^2 - \frac{1}{4}B_0^4U^4 + \frac{1}{2}W_0B_0^4U^2 - W_0B_0^4U + B_0^4U \\ &= -\frac{1}{4}B_0^4 + \frac{1}{2}W_0B_0^4 - W_0B_0^4 + B_0^4. \end{aligned}$$

Notice that every term has a B_0^4 so we can cancel them and simplify our equation to get

$$-\frac{1}{4}V^4 + \frac{1}{2}(W_0 - U^2 + \gamma)V^2 - \frac{1}{4}U^4 + \frac{1}{2}W_0U^2 - W_0U + U = \frac{3}{4} - \frac{1}{2}W_0.$$

c) Recall the system we found

$$\begin{cases} -Wu + u^3 + v^2u + WB_0 - B_0^3 = v' \\ Wv - vu^2 - v^3 + \gamma B_0^2 v = u' \end{cases}.$$

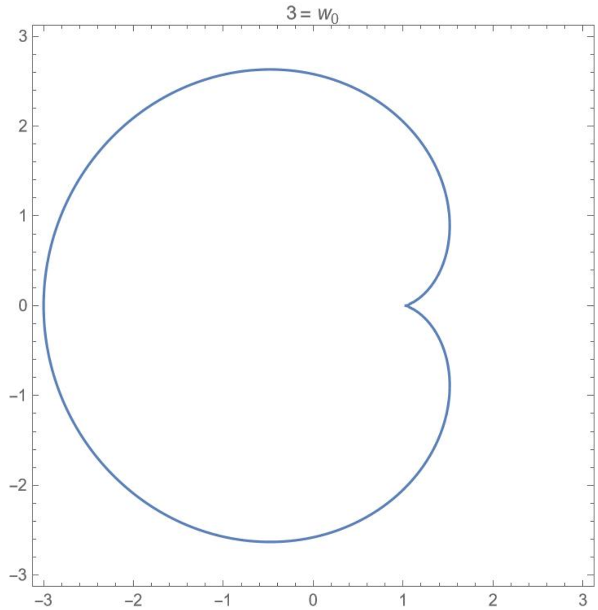
Let $U = u/B_0, V = v/B_0$, and $W_0 = W/B_0^2$. Then our system becomes

$$\begin{cases} B_0^3W_0U + B_0^3U^3 + B_0^3V^2U + B_0^3W_0 - B_0^3 = B_0V' \\ B_0^3V - B_0^3VU^2 - B_0^3V^3 + \gamma B_0^3V = B_0U' \end{cases},$$

and dividing both sides by B_0 and letting $\gamma = \frac{1}{10}$

$$\begin{cases} B_0^2W_0U + B_0^2U^3 + B_0^2V^2U + B_0^2W_0 - B_0^2 = V' \\ B_0^2V - B_0^2VU^2 - B_0^2V^3 + \frac{1}{10}B_0^2V = U' \end{cases}.$$

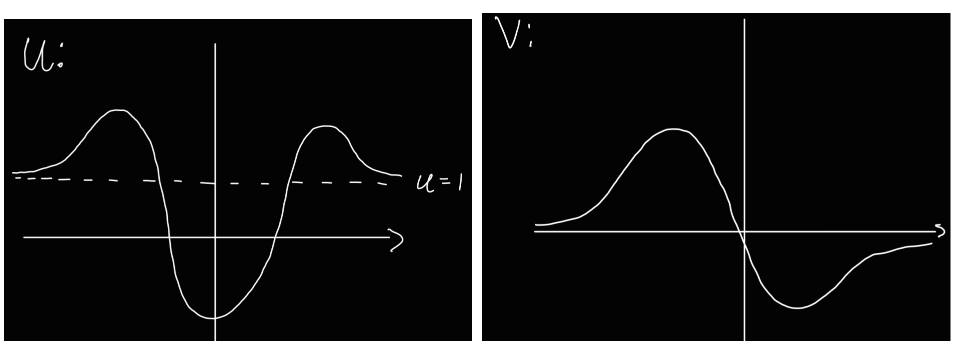
Now consider when $W_0 = 3$.



To find the direction of motion along the line, consider the point $U = -3$ and $V = 0$. Our system becomes

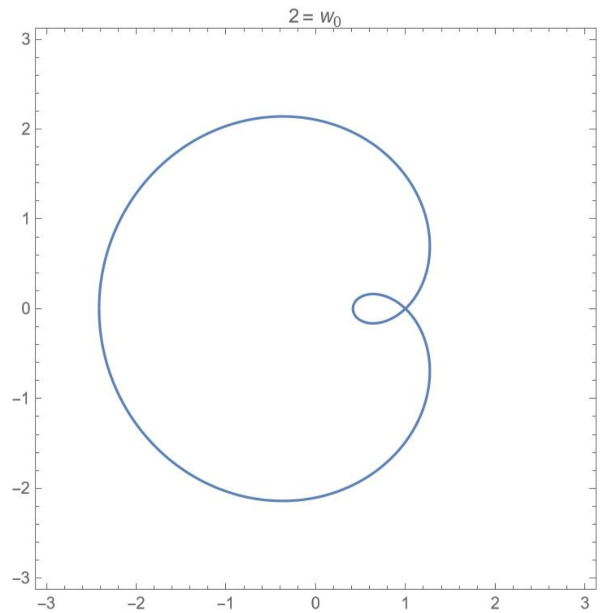
$$\begin{cases} -16B_0^2 = V' < 0 \\ 0 = U' \end{cases},$$

and thus there is counterclockwise motion along the curve. Noting that there is a singularity $(1, 0)$, we have that this is a one soliton case where the soliton for U and V look like



Note that it takes and infinite amount of time for U to reach $U = 1$ and V and infinite amount of time to reach $V = 0$.

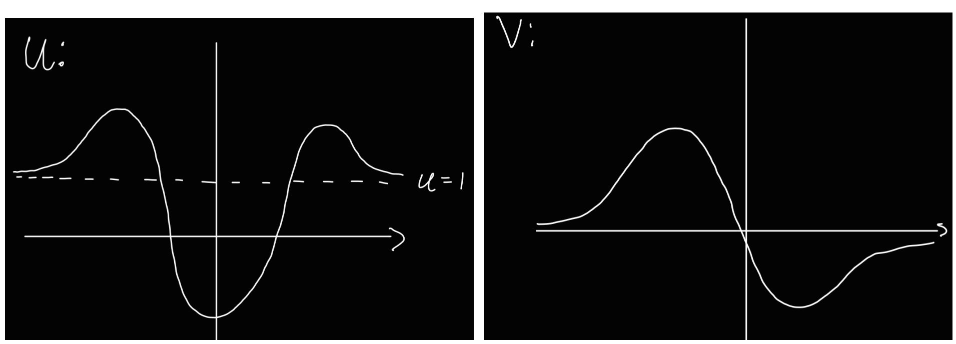
Now consider when $W_0 = 2$.



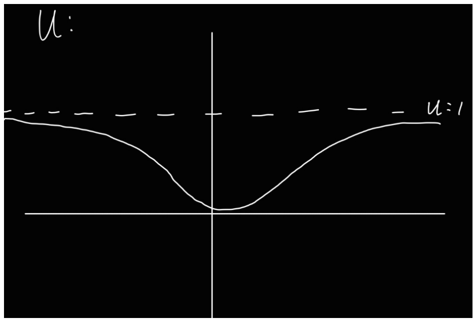
To find the direction of motion along the inner loop, consider the point $U = 0$ and $V = 0$. Our system becomes

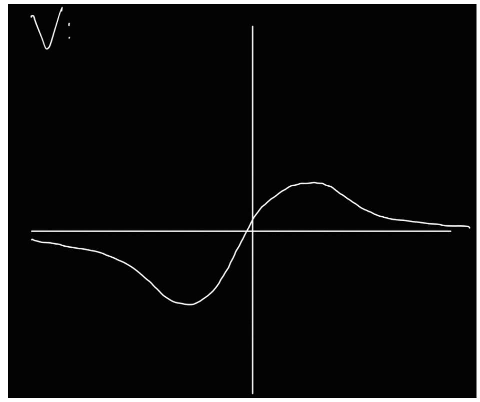
$$\begin{cases} B_0^2 = V' < 0 \\ 0 = U' \end{cases},$$

and thus the inner loop has clockwise motion. similarly we can verify that the outer loop has counterclockwise motion. In this case we have two solitons for U and V. For the outer loop we have similar motion as in the previous case but with the maximums and minimum of U and V closer to the origin. The following plots show the soliton of U and V in the outer loop

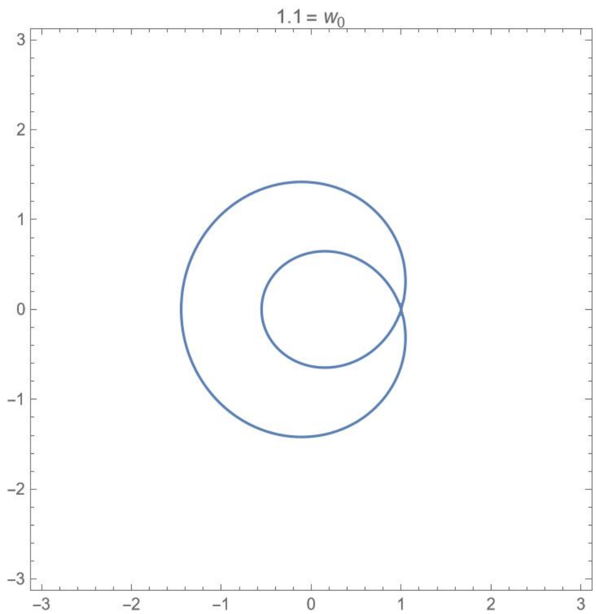


The following plots show the second soliton of U and V from the inner loop

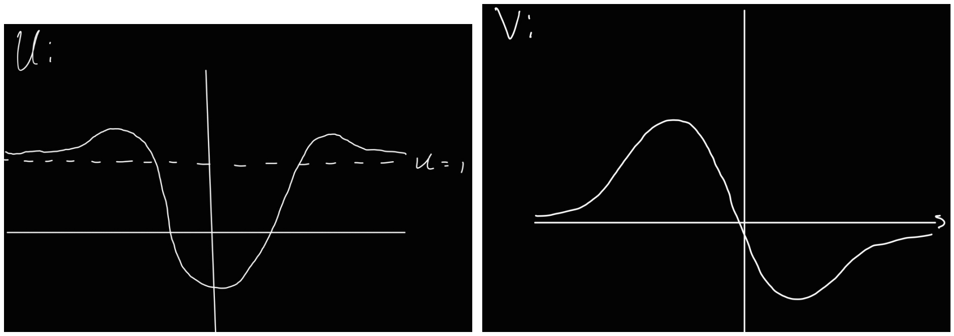




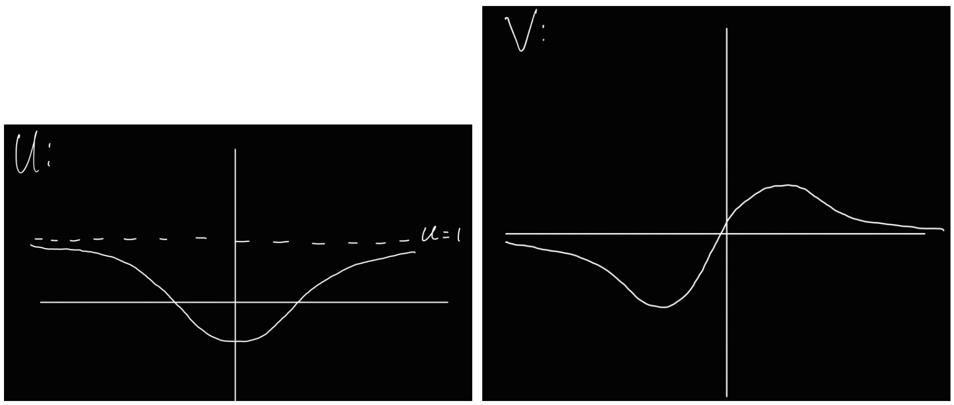
Now consider when $W_0 = 1.1$.



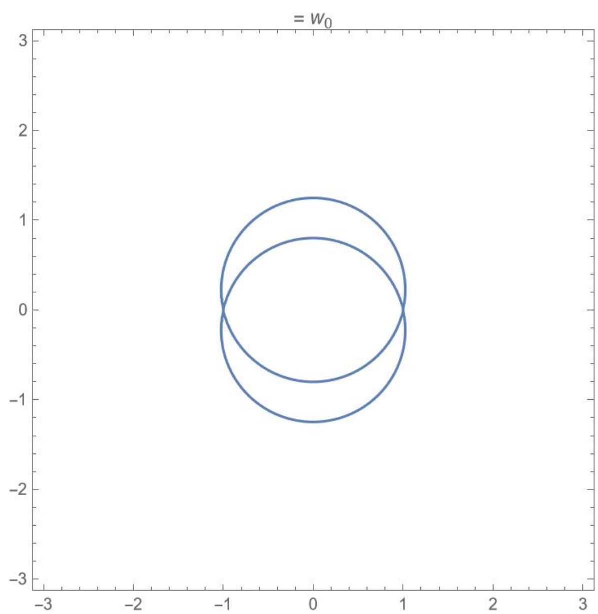
similarly to when $W_0 = 2$, we have the counterclockwise motion on the outer loop and clockwise motion in the inner. We have two solitons for U and V with similar behavior as before. Note that the max and min for the soliton of U and V corresponding to the outer loop is decreasing while the inner loop is increasing. The following figures show the soliton corresponding to the outer loop



The following figures show the soliton corresponding to the inner loop

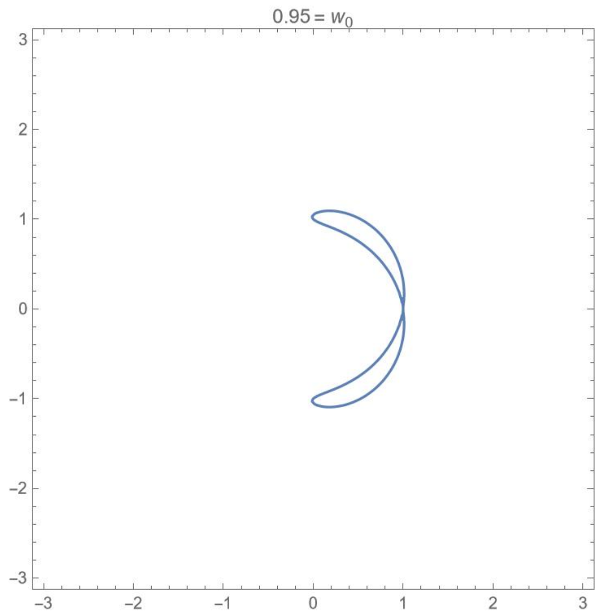


Now consider when $W_0 = 1$.

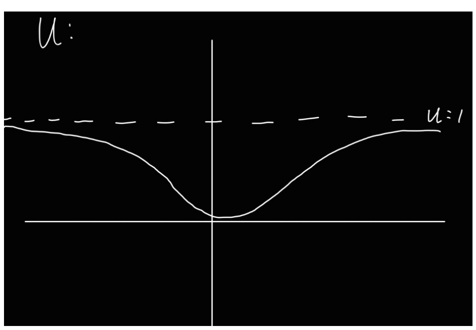


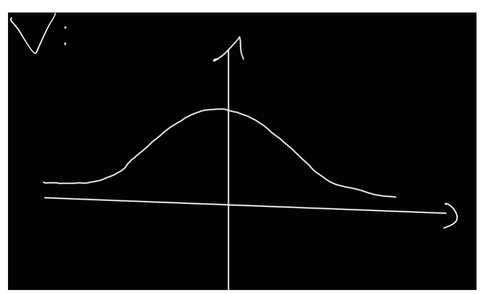
Notice that in this case we have two singularities at ± 1. From the figure above, we see that U has a heteroclinic orbit between -1 and 1. But this means that the boundary condition that $U \to B_0$ as $|x| \to \infty$ is not met and thus this case does not satisfy the system's boundary condition. Therefore there are no solitons in this case for U and V.

Now consider when $W_0 = 0.95$.

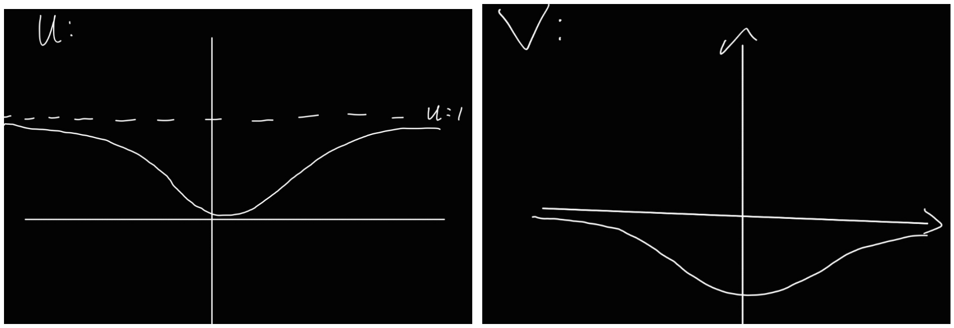


To find the direction of motion, consider the points $U = 0$ and $V = \pm 1$. From our system we get that $V' < 0$ and thus both the upper and lower loop have counterclockwise motion. This is a two soliton case and the following figures show the soliton of U and V for the upper loop

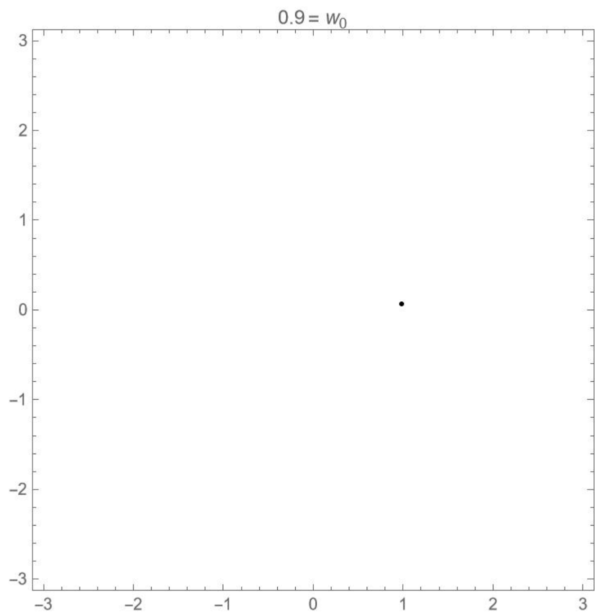




The following figures show the soliton of U and V for the lower loop



Now consider when $W_0 = 0.9$.



Since in this case we just have the singularity at $(0, 1)$, we have no soliton solutions.

$\square$

Problem 2 *Show that the canonical Poisson bracket*

$$\{f,g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right)$$

satisfies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

Solution.
Consider the canonical Poisson bracket

$$\{f,g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right).$$

We wish to show that

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

Observe that,

$$\begin{aligned}
\{\{f,g\},h\} &= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial}{\partial q_i}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right)\frac{\partial h}{\partial p_i} - \frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right)\frac{\partial h}{\partial q_i} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial q_i \partial p_j}\frac{\partial h}{\partial p_i} + \frac{\partial g}{\partial p_j}\frac{\partial f}{\partial q_i \partial q_j}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_i \partial q_j}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial q_j}\frac{\partial f}{\partial q_i \partial p_j}\frac{\partial h}{\partial p_i} \\
&- \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_i \partial p_j}\frac{\partial h}{\partial q_i} - \frac{\partial g}{\partial p_j}\frac{\partial f}{\partial p_i \partial q_j}\frac{\partial h}{\partial g_i} + \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial p_i \partial q_j}\frac{\partial h}{\partial q_i} + \frac{\partial g}{\partial q_j}\frac{\partial f}{\partial p_i \partial p_j}\frac{\partial h}{\partial q_i} \\
\{\{g,h\},f\} &= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial}{\partial q_i}\left(\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q_j} \right)\frac{\partial f}{\partial p_i} - \frac{\partial}{\partial p_i}\left(\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q_j} \right)\frac{\partial f}{\partial q_i} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial g}{\partial q_j}\frac{\partial h}{\partial q_i \partial p_j}\frac{\partial f}{\partial p_i} + \frac{\partial h}{\partial p_j}\frac{\partial g}{\partial q_i \partial q_j}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q_i \partial q_j}\frac{\partial f}{\partial p_i} - \frac{\partial h}{\partial q_j}\frac{\partial g}{\partial q_i \partial p_j}\frac{\partial f}{\partial p_i} \\
&- \frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_i \partial p_j}\frac{\partial f}{\partial q_i} - \frac{\partial h}{\partial p_j}\frac{\partial g}{\partial p_i \partial q_j}\frac{\partial f}{\partial g_i} + \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial p_i \partial q_j}\frac{\partial f}{\partial q_i} + \frac{\partial h}{\partial q_j}\frac{\partial g}{\partial p_i \partial p_j}\frac{\partial f}{\partial q_i} \\
\{\{h,f\},g\} &= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial}{\partial q_i}\left(\frac{\partial h}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q_j} \right)\frac{\partial g}{\partial p_i} - \frac{\partial}{\partial p_i}\left(\frac{\partial h}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q_j} \right)\frac{\partial g}{\partial q_i} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial h}{\partial q_j}\frac{\partial f}{\partial q_i \partial p_j}\frac{\partial g}{\partial p_i} + \frac{\partial f}{\partial p_j}\frac{\partial h}{\partial q_i \partial q_j}\frac{\partial g}{\partial p_i} - \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q_i \partial q_j}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial q_j}\frac{\partial h}{\partial q_i \partial p_j}\frac{\partial g}{\partial p_i} \\
&- \frac{\partial h}{\partial q_j}\frac{\partial f}{\partial p_i \partial p_j}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial p_j}\frac{\partial h}{\partial p_i \partial q_j}\frac{\partial g}{\partial g_i} + \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial p_i \partial q_j}\frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial q_j}\frac{\partial h}{\partial p_i \partial p_j}\frac{\partial g}{\partial q_i}.
\end{aligned}$$

Thus we have that

$$
\begin{aligned}
&\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\} \\
&=\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial q_i \partial p_j}\frac{\partial h}{\partial p_i}+\frac{\partial g}{\partial p_j}\frac{\partial f}{\partial q_i \partial q_j}\frac{\partial h}{\partial p_i}-\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_i \partial q_j}\frac{\partial h}{\partial p_i}-\frac{\partial g}{\partial q_j}\frac{\partial f}{\partial q_i \partial p_j}\frac{\partial h}{\partial p_i}\right. \\
&-\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_i \partial p_j}\frac{\partial h}{\partial q_i}-\frac{\partial g}{\partial p_j}\frac{\partial f}{\partial p_i \partial q_j}\frac{\partial h}{\partial g_i}+\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial p_i \partial q_j}\frac{\partial h}{\partial q_i}+\frac{\partial g}{\partial q_j}\frac{\partial f}{\partial p_i \partial p_j}\frac{\partial h}{\partial q_i} \\
&+\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial q_i \partial p_j}\frac{\partial f}{\partial p_i}+\frac{\partial h}{\partial p_j}\frac{\partial g}{\partial q_i \partial q_j}\frac{\partial f}{\partial p_i}-\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q_i \partial q_j}\frac{\partial f}{\partial p_i}-\frac{\partial h}{\partial q_j}\frac{\partial g}{\partial q_i \partial p_j}\frac{\partial f}{\partial p_i} \\
&-\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_i \partial p_j}\frac{\partial f}{\partial q_i}-\frac{\partial h}{\partial p_j}\frac{\partial g}{\partial p_i \partial q_j}\frac{\partial f}{\partial g_i}+\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial p_i \partial q_j}\frac{\partial f}{\partial q_i}+\frac{\partial h}{\partial q_j}\frac{\partial g}{\partial p_i \partial p_j}\frac{\partial f}{\partial q_i} \\
&+\frac{\partial h}{\partial q_j}\frac{\partial f}{\partial q_i \partial p_j}\frac{\partial g}{\partial p_i}+\frac{\partial f}{\partial p_j}\frac{\partial h}{\partial q_i \partial q_j}\frac{\partial g}{\partial p_i}-\frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q_i \partial q_j}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial q_j}\frac{\partial h}{\partial q_i \partial p_j}\frac{\partial g}{\partial p_i} \\
&\left.-\frac{\partial h}{\partial q_j}\frac{\partial f}{\partial p_i \partial p_j}\frac{\partial g}{\partial q_i}-\frac{\partial f}{\partial p_j}\frac{\partial h}{\partial p_i \partial q_j}\frac{\partial g}{\partial g_i}+\frac{\partial h}{\partial p_j}\frac{\partial f}{\partial p_i \partial q_j}\frac{\partial g}{\partial q_i}+\frac{\partial f}{\partial q_j}\frac{\partial h}{\partial p_i \partial p_j}\frac{\partial g}{\partial q_i}\right].
\end{aligned}
$$

Noticing that the bounds on both our summations are the same, we can cancel terms all of these terms to get zero. For example consider the first term

$$\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial q_i \partial p_j}\frac{\partial h}{\partial p_i},$$

which has a corresponding term

$$-\frac{\partial h}{\partial p_j}\frac{\partial g}{\partial p_i \partial q_j}\frac{\partial f}{\partial g_i},$$

and since the $1 \leq i \leq N$ and $1 \leq j \leq N$, these terms will completely cancel. This holds for the rest of the terms and thus we have that

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0.$$

□

Problem 3 *Show that the Sine-Gordon equation*

$$u_{tt} - u_{xx} + \sin(u) = 0$$

is Hamiltonian with canonical Poisson structure and Hamiltonian

$$H = \int \left(\frac{1}{2}p^2 + \frac{1}{2}q_x^2 + 1 - \cos(q) \right) dx,$$

where $q = u$, and $p = u_t$.

Solution.
Consider the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0.$$

Let's assume that

$$H = \frac{1}{2}p^2 + \frac{1}{2}q_x^2 + 1 - \cos(q)dx,$$

is a Hamiltonian of the Sine-Gordon equation where $q = u$ and $p = u_t$. We have that

$$\mathcal{H} = \int (\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + 1 - \cos(u)dx).$$

Observe that

$$\begin{pmatrix} q_t \\ p_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta q} \\ \frac{\delta H}{\delta p} \end{pmatrix} \implies \begin{cases} q_t = \frac{\delta H}{\delta p} \\ p_t = -\frac{\delta H}{\delta q} \end{cases}.$$

Now we can directly compute $\frac{\delta H}{\delta p}$ and $\frac{\delta H}{\delta p}$ to be

$$\frac{\delta H}{\delta p} = \frac{\partial \mathcal{H}}{\partial p} - \partial_x \frac{\partial \mathcal{H}}{\partial p_x} = p - \partial_x(0) = p = q_t = u_t,$$

$$\frac{\delta H}{\delta q} = \sin(q) - \partial_x(q_x) = \sin(q) - q_{xx} = -p_t = -u_{tt}.$$

And thus we have gotten the Sine-Gordon equation since

$$\sin(q) - q_{xx} = -u_{tt} \implies \sin(q) - u_{xx} = -u_{tt} \implies \sin(q) - u_{xx} + u_{tt} = 0.$$

Therefore H is indeed the Hamiltonian and the Sine-Gordon equation is Hamiltonian with canonical Poisson structure. $\square$

Problem 4 *Check explicitly that the conserved quantities* $F_{-1} = \int u dx$*,* $F_0 = \int \frac{1}{2}u^2 dx$*,* $F_1 = \int \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right) dx$*,* $F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2\right) dx$ *are mutually in involution with respect to the Poisson bracket defined by the Poisson structure given by* ∂_x*.*

Solution.
We wish to show that $F_{-1} = \int u dx$, $F_0 = \int \frac{1}{2}u^2 dx$, $F_1 = \int \left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right) dx$, $F_2 = \int \left(\frac{1}{24}u^4 - \frac{1}{2}uu_x^2 + \frac{3}{10}u_{xx}^2\right) dx$ are mutually in involution with respect to the Poisson bracket defined by the Poisson structure given by ∂_x. Note that

$$\{F_i, F_j\} = 0 \implies \{F_j, F_i\} = 0 \text{ as } \{F_i, F_j\} = -\{F_j, F_i\},$$

since the Poisson bracket is antisymmetric. Next observe that $\{F_i, F_{-1}\} = 0$ for all i since

$$\partial_x(\frac{\delta}{\delta u}u) = \partial_x 1 = 0.$$

Thus F_{-1} is mutually an involution with F_0, F_1, and F_2. Next note that

$$\{F_i, F_j\} = 0 \iff \frac{\delta}{\delta u}\left(\frac{\delta F_i}{\delta u}\partial_x \frac{\delta F_j}{\delta u}\right) = 0.$$

Thus we can check that the rest are involutions by computing $\frac{\delta}{\delta u}\left(\frac{\delta F_i}{\delta u}\partial_x \frac{\delta F_j}{\delta u}\right)$ and verifying it to equal zero. We used Mathematica and the package `VariationalMethods` to compute the variationally derivatives. We found that $\frac{\delta}{\delta u}\left(\frac{\delta F_0}{\delta u}\partial_x \frac{\delta F_1}{\delta u}\right) = 0$ which implies that $\{F_0, F_1\} = 0$, and similarly we verified that $\{F_0, F_2\} = 0$, and $\{F_1, F_2\} = 0$. Thus we have that F_{-1}, F_0, F_1, and F_2 are mutually involutions with respect to the Poisson bracket.

□

Problem 5 *Find the fourth conserved quantity for the KdV equation $u_t = uu_x + u_{xxx}$, i.e., the conserved quantity which contains $\frac{1}{24}\int u^4 dx$.*

Solution.
We wish to find the fourth conserved quantity for the KdV equation

$$u_t = uu_x + u_{xxx}.$$

Recall that we computed up to $[f] = 6$ to find the third conserved quantity. We also know that there are only conserved quantities for even weights, so let's begin examining $[f] = 8$. When $[f] = 8$, then the possible terms are $u^4, u^2u_{xx}, uu_x^2, u_{xx}^2, u_xu_{xxx}, uu_{4x}$, and u_{6x}. We note that u^2u_{xx} and uu_x^2 are equivalent and same with u_xu_{xxx}, uu_{4x}, and u_{8x}. Thus let's consider

$$f = a_1u^4 + a_2uu_x^2 + a_3u_{xx}^2.$$

We also need for $[g] = 10$. The possible terms for this are $u^5, u^3u_{xx}, u^2u_x^2, u^2u_{4x}, uu_xu_{xxx}, uu_{xx}^2$, $u_x^2u_{xx}, uu_{6x}, u_xu_{5x}, u_{xx}u_{4x}, u_{xxx}^2$, and u_{5x}. This gives that

$$\begin{aligned} g = {} & c_1u^5 + c_2u^3u_{xx} + c_3u^2u_x^2 + c_4u^2u_{4x} + c_5uu_xu_{xxx} + c_6uu_{xx}^2 + c_7u_x^2u_{xx} \\ & + c_8uu_{6x} + c_9u_xu_{5x} + c_{10}u_{xx}u_{4x} + c_{11}u_{xxx}^2 + c_{12}u_{8x}. \end{aligned}$$

Now we wish to compute $f_t + g_x$ so lets compute

$$f_t = \frac{1}{6}u^3u_t + 2a_1uu_xu_{xt} + a_1u_tu_x^2 + 2a_2u_{xx}u_{xxt},$$

and note that we can eliminate the time derivatives in f_t using

$$\begin{aligned} & u_t = uu_x + u_{xxx} \\ \implies & u_{xt} = \partial_x(uu_x + u_{xxx}) \\ \implies & u_{xxt} = \partial_{xx}(uu_x + u_{xxx}). \end{aligned}$$

Plugging these into Mathematica and computing $f_t + g_t$, we can find the following system of equations by looking at the coefficients

$$\begin{aligned} u^4u_x: &\quad -\frac{1}{6} + \frac{30}{6}c_1 = 0 \\ u^3u_{3x}: &\quad -\frac{1}{6} + c_2 = 0 \\ u_x^2u_{5x}: &\quad -a_1 + c_5 + c_7 = 0 \\ u_{3x}u_{4x}: &\quad c_{10} + 2c_{11} \\ u_{2x}u_{5x}: &\quad -2a_2 + c_10 + c_9 \\ u^2u_xu_{xx}: &\quad -2a_1 + 3c_2 + 2c_3 \\ u^2u_{5x}: &\quad c_4 = 0 \end{aligned}$$

$$
\begin{aligned}
u_x u_{xx}^2 :& \quad -6a_2 + c_6 + 2c_7 \\
u_x u_{6x} :& \quad c_8 + c_9 = 0 \\
u u_x^3 :& \quad -3a_1 + 2c_3 = 0 \\
u u_{xx} u_{xxx} :& \quad -2a_2 + c_5 + 2c_6 = 0 \\
u u_x u_{4x} :& \quad -2a_1 + 2c_4 + c_5 = 0 \\
u_{7x} :& \quad c_8 = 0 \\
u_{9x} :& \quad c_{12} = 0.
\end{aligned}
$$

Simplifying this system gives

$$
\begin{aligned}
a_1 &= c_5 + c_7 \\
a_1 &= \frac{1}{4} + c_3 \\
a_1 &= \frac{2}{3} c_3 \\
a_1 &= \frac{1}{2} c_5 \\
a_2 &= \frac{1}{2} c_{10} \\
a_2 &= \frac{1}{6} c_6 + \frac{1}{3} c_7 \\
a_2 &= \frac{1}{2} c_5 + c_6
\end{aligned}
$$

Solving this system gives

$$
a_1 = -\frac{1}{2}, \quad \text{and} \quad a_2 = \frac{3}{10}.
$$

Thus we have found the fourth conserved quantity of the KdV equation to be

$$
\int \left(\frac{1}{24} u^4 - \frac{1}{2} u u_x^2 + \frac{3}{10} u_{xx}^2 \right) dx.
$$

$\square$

Problem 6 Recursion operator *For a Bi-Hamiltonian system with two Poisson structures given by B_0, B_1, one defines a recursion operator $R = B_1 B_0^{-1}$, which takes one element of the hierarchy of equations to the next element. For the KdV equation with $B_0 = \partial_x$ and $B_1 = \partial_{xxx} + \frac{1}{3}(u\partial_x + \partial_x u)$, we get $B_0^{-1} = \partial_x^{-1}$, integration with respect to x. Write down the recursion operator. Apply it to u_x (the zero-th KdV flow) to obtain the first KdV flow. Now apply it to $uu_x + u_{xxx}$ to get (up to rescaling of t_2) the second KdV equation. What is the third KdV equation?*

Solution.

Consider a Bi-Hamiltonian system for the KdV equation with two Poisson structures given by $B_0 = \partial_x$ and $B_1 = \partial_{xxx} + \frac{1}{3}(u\partial_x + \partial_x u)$ where $R = B_1 B_0^{-1}$ defines the recursion operator. First let's explicitly compute the recursion operator

$$\begin{aligned} R = B_1 B_0^{-1} &= (\partial_{xxx} + \frac{1}{3}u\partial_x + \frac{1}{3}\partial_x u)(\partial_x^{-1}) \\ &= \partial_{xxx}\partial_x^{-1} + \frac{1}{3}u\partial_x\partial_x^{-1} + \frac{1}{3}\partial_x u \partial_x^{-1} \\ &= \partial_{xx} + \frac{1}{3}u + \frac{1}{3}\partial_x u \partial_x^{-1}, \end{aligned}$$

where $B_0^{-1} = \partial_x^{-1} = \int dx$. Now we wish to apply R to u_x to obtain the first KdV flow

$$\begin{aligned} Ru_x &= (\partial_{xx} + \frac{1}{3}u + \frac{1}{3}\partial_x u \partial_x^{-1})u_x \\ &= u_{xxx} + \frac{1}{3}uu_x + \frac{1}{3}\partial_x u \partial_x^{-1} u_x \\ &= u_{xxx} + \frac{1}{3}uu_x + \frac{1}{3}\partial_x u^2 \\ &= u_{xxx} + \frac{1}{3}uu_x + \frac{2}{3}uu_x \\ &= u_{xxx} + uu_x. \end{aligned}$$

Thus $u_{xxx} + uu_x$ is the first KdV flow. Now we wish to apply R to the first KdV flow to find the second KdV equation

$$\begin{aligned} R(u_{xxx} + uu_x) &= (\partial_{xx} + \frac{1}{3}u + \frac{1}{3}\partial_x u \partial_x^{-1})(u_{xxx} + uu_x) \\ &= \partial_x(uu_{xx} + u_x u_x) + \frac{1}{3}u^2 u_x + \frac{1}{3}\partial_x u \partial_x^{-1}\left(\frac{1}{2}u^2\right)_x + u_{5x} + \frac{1}{3}uu_{xxx} + \frac{1}{3}\partial_x(uu_{xx}) \\ &= uu_{xxx} + u_x u_{xx} + 2u_x u_{xx} + \frac{1}{3}u^2 u_x + \frac{1}{2}u^2 u_x + u_{5x} + \frac{1}{3}uu_{xxx} + \frac{1}{3}uu_{xxx} + \frac{1}{3}u_x u_{xx} \\ &= \frac{5}{3}uu_{xxx} + \frac{10}{3}u_x u_{xx} + \frac{5}{6}u^2 u_x + u_{5x}. \end{aligned}$$

Thus we have found the second KdV equation. Now we wish to compute the third KdV equation by applying R to the second KdV equation

$$
\begin{aligned}
&R\left(\frac{5}{3}uu_{xxx}+\frac{10}{3}u_xu_{xx}+\frac{5}{6}u^2u_x+u_{5x}\right)=\left(\partial_{xx}+\frac{1}{3}u+\frac{1}{3}\partial_xu\partial_x^{-1}\right)\left(\frac{5}{3}uu_{xxx}+\frac{10}{3}u_xu_{xx}+\frac{5}{6}u^2u_x+u_{5x}\right)\\
&=\frac{5}{3}\partial_x(uu_{4x}+u_{xxx}u_x)+\frac{5}{9}u^2u_{xxx}+\frac{5}{9}\partial_xu\partial_x^{-1}(uu_{xx}-\frac{1}{2}u_x^2)_x+\frac{10}{3}\partial_x(u_xu_{xxx}+u_{xx}^2)+\frac{10}{9}uu_xu_{xx}\\
&+\frac{10}{9}\partial_xu\partial_x^{-1}\left(\frac{1}{2}u_x^2\right)_x+\frac{5}{6}\partial_x(u^2u_{xx}+2uu_x^2)+\frac{5}{18}u^3u_x+\frac{5}{18}\partial_xu\partial_x^{-1}\left(\frac{1}{3}u^3\right)_x+u_{7x}+\frac{1}{3}uu_{5x}+\frac{1}{3}\partial_x(uu_{4x})\\
&=\frac{5}{3}uu_{5x}+\frac{5}{3}u_xu_{4x}+\frac{5}{3}u_{xx}u_{xxx}+\frac{5}{3}u_xu_{4x}+\frac{5}{9}u^2u_{xxx}+\frac{5}{9}u^2u_{xxx}+\frac{10}{9}uu_xu_{xx}-\frac{10}{18}uu_xu_{xx}-\frac{5}{18}u_x^3\\
&+\frac{10}{3}u_xu_{4x}+\frac{10}{3}u_{xx}u_{xxx}+\frac{20}{3}u_{xx}u_{xxx}+\frac{10}{9}uu_xu_{xx}+\frac{20}{18}uu_xu_{xx}+\frac{10}{18}u_x^3+\frac{5}{6}u^2u_{xxx}+\frac{10}{6}uu_xu_{xx}\\
&+\frac{10}{3}uu_xu_{xx}+\frac{5}{3}u_x^3+\frac{5}{18}u^3u_x+\frac{20}{54}u^3u_x+u_{7x}+\frac{1}{3}uu_{5x}+\frac{1}{3}uu_{5x}+\frac{1}{3}u_xu_{4x}\\
&=u_{7x}+\left(\frac{5}{3}+\frac{1}{3}+\frac{1}{3}\right)uu_{5x}+\left(\frac{5}{3}+\frac{5}{3}+\frac{10}{3}+\frac{1}{3}\right)u_xu_{4x}+\left(\frac{5}{3}+\frac{10}{3}+\frac{20}{3}\right)u_{xx}u_{xxx}+\left(\frac{5}{18}+\frac{20}{54}\right)u^3u_x\\
&+\left(\frac{5}{9}+\frac{5}{9}+\frac{5}{6}\right)u^2u_{xxx}+\left(\frac{10}{9}-\frac{10}{18}+\frac{18}{9}+\frac{20}{18}+\frac{10}{6}+\frac{10}{3}\right)uu_xu_{xx}+\left(\frac{10}{18}+\frac{5}{3}-\frac{5}{18}\right)u_x^3\\
&=u_{7x}+\frac{7}{3}uu_{5x}+7u_xu_{4x}+\frac{35}{3}u_{xx}u_{xxx}+\frac{35}{18}u^2u_{xxx}+\frac{70}{9}uu_xu_{xx}+\frac{35}{18}u_x^3+\frac{35}{54}u^3u_x.
\end{aligned}
$$

Thus we have found the third KdV equation. □

Problem 7 *Consider the function* $U(x) = 2\partial_x^2 \ln\left(1 + e^{kx+\alpha}\right)$. *Show that for a suitable* k, $U(x)$ *is a solution of the first member of the stationary KdV hierarchy (as you've already seen, it is the one-soliton solution):*

$$6uu_x + u_{xxx} + c_0 u_x = 0.$$

(Note: it may be easier to define c_0 *in terms of* k, *instead of the other way around)*
Having accomplished this, let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots)}$. *Determine the dependence of* α *on* t_1, t_2 *and* t_3 *such that* $u(x, t_1, t_2, t_3, \ldots)$ *is simultaneously a solution of the first, second and third KdV equations:*

$$\begin{aligned}
u_{t_1} &= 6uu_x + u_{xxx}, \\
u_{t_2} &= 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x}, \\
u_{t_3} &= 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u_{xx}u_{xxx} + 70u^2u_{xxx} + 42u_xu_{xxxx} + 14uu_{5x} + u_{7x}.
\end{aligned}$$

Based on this, write down a guess for the one-soliton solution that solves the entire KdV hierarchy.

Solution.
Consider the function $U(x) = 2\partial_x^2 \ln\left(1 + e^{kx+\alpha}\right)$. We wish to show that for a suitable k, $U(x)$ is a solution to the first member of the stationary KdV hierarchy

$$6uu_x + u_{xxx} + c_0 u_x = 0.$$

Using Mathematica, we plugged $U(x)$ into the first KdV equation to get

$$0 = -\frac{2e^{kx+\alpha}\left(-1 + e^{kx+\alpha}\right)k^3(k^2 + c_0)}{(1 + e^{kx+\alpha})^3},$$

which holds when $k^2 = -c_0$.
Next let $u(x, t1, t2, t3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots)}$. We wish to determine the dependence of α on t_1, t_2, and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution to the first, second, and third KdV equations. Using Mathematica, we plugged

$$u(x, t1, t2, t3) = 2\partial_x^2 \ln\left(1 + e^{kx+\alpha(t_1,t_2,t_3)}\right), \tag{1}$$

into the first KdV equation

$$0 = -u_{t_1} + 6uu_x + u_{xxx},$$

which gives

$$0 = 4k^2 \text{csch}(kx + \alpha)^3 \sinh\left(\frac{1}{2}(kx + \alpha)\right)^4 (-k^3 + \alpha_{t_1}).$$

To satisfy this equality, we require $\alpha_{t_1} = k^3$.
Next we plugged (1) into the second KdV equation

$$0 = -u_{t_2} + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x}$$

which gives

$$0 = 4k^2 \operatorname{csch}(kx+\alpha)^3 \sinh\left(\frac{1}{2}(kx+\alpha)\right)^4 (-k^5 + \alpha_{t_2}).$$

To satisfy this equality, we require $\alpha_{t_2} = k^5$.
Finally we plugged (1) into the third KdV equation

$$0 = -u_{t_3} + 140u^3 u_x + 70u_x^3 + 280uu_x u_{xx} + 70u_{xx}u_{xxx} + 70u^2 u_{xxx} + 42u_x u_{xxxx} + 14uu_{5x} + u_{7x},$$

which gives

$$0 = 4k^2 \operatorname{csch}(kx+\alpha)^3 \sinh\left(\frac{1}{2}(kx+\alpha)\right)^4 (-k^5 + \alpha_{t_3}).$$

To satisfy this equality, we require $\alpha_{t_3} = k^7$. Thus we see a pattern in the α dependence on $t_1, t_2, \ldots$ to be

$$\alpha_{t_n} = k^{1+2n}.$$

Therefore, I'd guess that a one-soliton solution that solves the entire KdV hierarchy is given by

$$u(x, t_1, t_2, t_3, \ldots) = 2\partial_x^2 \ln\left(1 + e^{kx+\alpha}\right),$$

where

$$\alpha = k^3 t_1 + k^5 t_2 + \cdots + k^{1+2n} t_n + \cdots .$$

$\square$

Problem 8 Warning: maple/mathematica-intensive. *Consider the function*

$$U(x) = 2\partial_x^2 \ln\left(1 + e^{k_1 x+\alpha} + e^{k2_x+\beta} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x + k_2 x+\alpha+\beta}\right).$$

Show that for a suitable k_1, k_2, $U(x)$ is a solution of the second member of the stationary KdV hierarchy:

$$30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x + u_{xxx}) + c_0u_x = 0.$$

(Note: it may be easier to define c_1, c_0 in terms of k_1 and k_2 instead of the other way around) Having accomplished this, let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots),\beta=\beta(t_1,t_2,t_3,\ldots)}$. Determine the dependence of α and β on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations, given above.

Based on this, write down a guess for the two-soliton solution of the entire KdV hierarchy.

Solution.

Consider the function

$$U(x) = 2\partial_x^2 \ln\left(1 + e^{k_1 x+\alpha} + e^{k2_x+\beta} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x + k_2 x+\alpha+\beta}\right).$$

We first wish to show that for a suitable k_1 and k_2, $U(x)$ is a solution of the second member of the stationary KdV hierarchy

$$30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x} + c_1(6uu_x + u_{xxx}) + c_0u_x = 0.$$

Using Mathematica, we plugged $U(x)$ into the second KdV equation resulting in am extremely long expression involving many fractions. To simplify the problem, I used Mathematica's `Together` function to get a common denominator. Since we are looking for when this fraction will be zero, I used Mathematica's `Numerator` to get the numerator of the fraction. Finally I arbitrarily picked two terms from the numerator and used Mathematica's `Solve` to solve the two by two system created by the coefficients of the two terms for c_0 and c_1. If the solve function was unable to find a solution, I picked two other terms. Once I got a solutions, I plugged them back into the original equation to verify that $U(x)$ with the new values. If the new values didn't zero $U(x)$, I picked two new terms and repeated the process. Using this method I found that $U(x)$ satisfies the second KdV equation when $c_0 = k_1^2k_2^2$ and $c_1 = -k_1^2 - k_2^2$.

Next let $u(x, t_1, t_2, t_3, \ldots) = U(x)|_{\alpha=\alpha(t_1,t_2,t_3,\ldots),\beta=\beta(t_1,t_2,t_3,\ldots)}$. We wish to determine the dependence of α and β on t_1, t_2 and t_3 such that $u(x, t_1, t_2, t_3, \ldots)$ is simultaneously a solution of the first, second and third KdV equations. Using the same method described above, I plugged

$$u(x, t1, t2, t3) = 2\partial_x^2 \ln\left(1 + e^{k_1 x+\alpha(t_1,t_2,t_3)} + e^{k2_x+\beta(t_1,t_2,t_3)} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x + k_2 x+\alpha(t_1,t_2,t_3)+\beta(t_1,t_2,t_3)}\right) \tag{2}$$

into the first KdV equation and found that

$$\alpha_{t_1} = k_1^3 \ \text{ and } \ \beta_{t_1} = k_2^3.$$

Next I plugged (2) into the second KdV equation and found that

$$\alpha_{t_2} = k_1^5 \ \text{ and } \ \beta_{t_2} = k_2^5.$$

Finally I plugged (2) into the third KdV equation to get

$$\alpha_{t_3} = k_1^7 \ \text{ and } \ \beta_{t_3} = k_2^7.$$

Once again we see a pattern in the α and β dependence on $t_1, t_2, \ldots$ to be

$$\alpha = k_1^{2n+1} t_n \ \text{ and } \ \beta = k_2^{2n+1} t_n.$$

Thus I guess that a two-soliton solution for the entire KdV hierarchy is

$$u(x, t1, t2, \ldots) = 2\partial_x^2 \ln \left(1 + e^{k_1 x + \alpha} + e^{k 2_x + \beta} + \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 e^{k_1 x + k_2 x + \alpha + \beta} \right),$$

where

$$\alpha = k_1^3 t_1 + k_1^5 t_2 + \cdots + k_1^{2n+1} t_n + \cdots, \ \text{ and } \ \beta = k_2^3 t_1 + k_2^5 t_2 + \cdots + k_2^{2n+1} t_n + \cdots.$$

$\square$