Math 568 Homework 4
Due February 3
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Problem 1 *Consider the weakly nonlinear oscillator:*

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y + \epsilon y^5 = 0,$$

with $y(0) = 0$ *and* $y'(0) = A > 0$ *and with* $0 < \epsilon \ll 1$.

(a) Use a regular perturbation expansion and calculate the first two terms.

(b) Determine at what time the approximation of part (a) fails to hold.

(c) Use a Poincare-Lindstedt expansion and determine the first two terms and frequency corrections.

(d) For $\epsilon = 0.1$*, plot the numerical solution (from MATLAB), the regular expansion solution, and the Poincare-Lindstedt solution for* $0 \leq t \leq 20$.

Solution.
Consider the weakly nonlinear oscillator:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y + \epsilon y^5 = 0, \tag{1}$$

with $y(0) = 0$ and $y'(0) = A > 0$ and with $0 < \epsilon \ll 1$.

(a) First we wish to compute the first two terms of the regular perturbation expansion. First let

$$y(t) = y_0 + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots .$$

Substituting this into Eq. (1) yields

$$(y_0'' + \epsilon y_1'' + \cdots) + (y_0 + \epsilon y_1 + \cdots) + \epsilon(y_0 + \epsilon y_1 + \cdots)^5 = 0.$$

Next we can collect the terms in powers of ϵ which gives

$$\begin{aligned}
&\mathcal{O}(1): && y_0'' + y_0 = 0, && y_0(0) = 0, y_0'(0) = A,\\
&\mathcal{O}(\epsilon): && y_1'' + y_1 + y_0^5 = 0, && y_1(0) = 0, y_1'(0) = 0,\\
&\mathcal{O}(\epsilon^2): && y_2'' + y_2 + 5y_0^4 y_1 = 0, && y_2(0) = 0, y_2'(0) = 0.
\end{aligned}$$

The leading order problem

$$y_0'' + y_0 = 0,$$

with boundary conditions $y_0(0) = 0$ and $y_0'(0) = A$ has the general solution

$$y_0(t) = c_1 \cos(t) + c_2 \sin(t),$$

and applying the first boundary condition gives $c_1 = 0$ which yields the solution

$$y_0(t) = c_2 \sin(t).$$

Enforcing the second boundary conditions gives $c_2 = A$ and thus the final solution is

$$y_0(t) = A \sin(t),$$

which is the first term in the regular perturbation expansion. Substituting this solution into the $\mathcal{O}(\epsilon)$ equation yields

$$y_1'' + y_1 = -A^5 \sin^5(t),$$

with boundary conditions $y_1(0) = 0$ and $y_1'(0) = 0$ which is a second order inhomogeneous differential equation. To solve this differential equation, we need to use the method of variation parameters. First let's solve the homogeneous problem

$$y_1'' + y_1 = 0,$$

which has the general solution

$$y_1(t) = c_1 \cos(t) + c_2 \sin(t).$$

Next, we can find the particular solution by computing

$$y_p(t) = -\bar{y}_1(t) \int \frac{\bar{y}_2(t) f(t)}{W(\bar{y}_1(t), \bar{y}_2(t))} dt + \bar{y}_2(t) \int \frac{\bar{y}_1(t) f(t)}{W(\bar{y}_1(t), \bar{y}_2(t))} dt,$$

where $f(t) = -A^5 \sin^5(t)$, $\bar{y}_1(t) = \cos(t)$ and $\bar{y}_2(t) = \sin(t)$. This yields

$$y_p(t) = \frac{1}{384} A^5 (-80 \sin(t) - 15 \sin(3t) + \sin(5t) + 120t \cos(t)),$$

and thus the general solution is given by

$$y_1 = c_1 \cos(t) + c_2 \sin(t) + \frac{1}{384} A^5 (-80 \sin(t) - 15 \sin(3t) + \sin(5t) + 120t \cos(t)),$$

and enforcing the boundary conditions gives

$$y_1 = \frac{1}{384} A^5 (-80 \sin(t) - 15 \sin(3t) + \sin(5t) + 120t \cos(t)).$$

Thus we have found the first two terms of the regular perturbation expansion to be

$$y(t) = A \sin(t) + \epsilon \left(\frac{1}{384} A^5 (-80 \sin(t) - 15 \sin(3t) + \sin(5t) + 120t \cos(t)) \right) + \mathcal{O}(\epsilon^2).$$

(b) In the second term of the regular perturbation expansion we have a secular growth term that shows the perturbation is the same size as the leading order solution at t $\mathcal{O}(1/\epsilon)$. Thus for t values greater than $\mathcal{O}(1/\epsilon)$, the solution blows up to infinity.

(c) Now we wish to use the Poincare-Lindstedt expansion to remove the secular term. We begin by letting

$$\begin{aligned}&\tau = \omega(\epsilon)t \text{ where } \omega(\epsilon) = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots,\\ &y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots.\end{aligned}$$

Note that by the chain rule $y_{tt} = \omega^2 y_{\tau\tau}$. Plugging these into the Eq. (1) and collecting terms yields

$$\begin{aligned}&O(1): \quad && \omega_0 y_{0_{\tau\tau}} + y_0 = 0, && y_0(0) = 0, w_0 y_{0_\tau}(0) = A,\\ &O(\epsilon): && y_{1_{\tau\tau}} + y_1 = -2\omega_1 y_{0_{\tau\tau}} - y_0^5, && y_1(0) = 0, \omega_1 y_{0_\tau} + \omega_0 y_{1_\tau} = 0.\end{aligned}$$

Let's first solve the leading order term

$$\omega_0 y_{0_{\tau\tau}} + y_0 = 0.$$

Enforcing the Fredholm-Alternative theorem

$$\langle 0, \sin\tau\rangle = 0,$$

shows that the solvability of the leading order always satisfied and thus independent of ω_0. Thus let's let $\omega_0 = 1$. The general solution to the leading order term is given by

$$c_1\cos(\tau) + c_2\sin(\tau) = 0,$$

and applying the boundary conditions $y_0(0) = 0, y_{0_\tau}(0) = A$ gives the leading order term to be

$$y_0 = A\sin(\tau).$$

Plugging the found values into the $\mathcal{O}(\epsilon)$ term gives

$$y_{1_{\tau\tau}} + y_1 = 2\omega_1 A\sin(\tau) - A^5\sin^5(\tau), \quad y_1(0) = 0, y_{1_\tau} = -\omega_1 A.$$

Enforcing the Fredholm-Alternative theorem, applying the trig identify $\sin^5(\tau) = \frac{10\sin(\tau) - 5\sin(3\tau) + \sin(5\tau)}{16}$, and noting that $\sin(\tau)$ is orthogonal to $\sin(3\tau)$ and $\sin(5\tau)$ gives

$$\begin{aligned}&\left\langle 2\omega_1 A\sin(\tau) - A^5\sin^5(\tau), \sin(\tau)\right\rangle = 0\\ \iff &\left\langle 2\omega_1\sin(\tau) - A^4\left(\frac{10\sin(\tau) - 5\sin(3\tau) + \sin(5\tau)}{16}\right), \sin(\tau)\right\rangle = 0\\ \iff &\left\langle 2\omega_1\sin(\tau) - \frac{10A^4}{16}\sin(\tau), \sin(\tau)\right\rangle = 0\end{aligned}$$

$$\iff \left(2\omega_1 - \frac{10A^4}{16}\right)\langle \sin(\tau), \sin(\tau)\rangle = 0.$$

As $\langle \sin(\tau), \sin(\tau)\rangle \neq 0$ we find that

$$\omega_1 = \frac{5}{16}A^4.$$

Plugging this back into the $\mathcal{O}(\epsilon)$ term gives

$$y_{1_{\tau\tau}} + y_1 = \frac{5}{8}A^4\sin(\tau) - A^5\sin^5(\tau), \quad y_1(0) = 0, y_{1_\tau} = -\frac{5}{16}A^5.$$

To solve the ODE, we use variation of parameters to get the homogeneous solution

$$\hat{y}_1 = B\sin(\tau) + C\cos(\tau),$$

and the particular solution

$$y_{1_p} = \frac{1}{384}A^5(-80\sin(\tau) - 15\sin(3\tau) + \sin(5\tau)).$$

Adding the two solution together and applying the boundary conditions $y_1(0) = 0, y_{1_\tau} = -\frac{5}{16}A^5$ gives $\mathcal{O}(\epsilon)$ solution to be

$$y_1 = \frac{1}{384}A^5(-80\sin(\tau) - 15\sin(3\tau) + \sin(5\tau)).$$

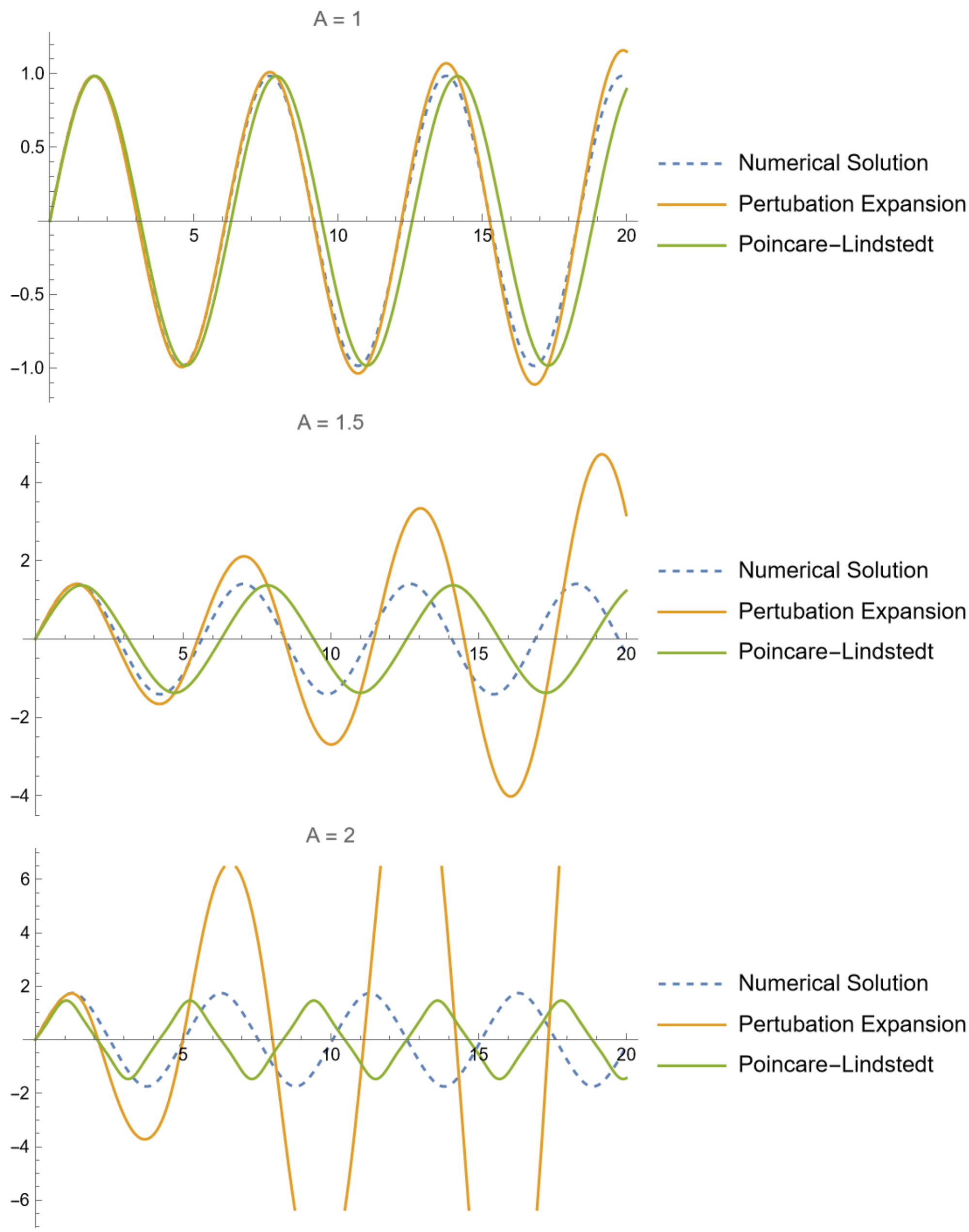
Therefore we have found the first two terms of the Poincare-Lindstedt expansion to be

$$y = A\sin((1+\epsilon\frac{5}{16}A^4)t) + \frac{\epsilon}{384}A^5(-80\sin((1+\epsilon\frac{5}{16}A^4)t) - 15\sin(3(1+\epsilon\frac{5}{16}A^4)t) + \sin(5(1+\epsilon\frac{5}{16}A^4)t)) + \cdots$$

and the first two frequency corrections

$$\omega_0 = 1 \quad \text{and} \quad \omega_1 = \frac{5}{16}A^4.$$

(d) Using Mathematica to plot the numerical solution (using `NDSolve`), the regular expansion solution, and the Poincare-Lindstedt solution for $0 \leq t \leq 20$ with $\epsilon = 0.1$ gives the following plots:



We see that the perturbation expansion is accurate until $\mathcal{O}(1/\epsilon)$ and then the secular term causes the solution to blow up. On the other hand, the Poincare-Lindstedt solution does not have a secular term and does not blow up but differs in frequency to the numerical solution.

$\square$

Problem 2 *Consider Rayleigh's equation*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y + \epsilon \left[-\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^3 \right] = 0,$$

which has only one periodic solution called a "limit cycle" ($0 < \epsilon \ll 1$). Given

$$y(0) = 0,$$

and

$$\frac{\mathrm{d}y(0)}{\mathrm{d}t} = A.$$

(a) *Use a multiple scale expansion to calculate the leading order behavior.*

(b) *Use a Poincare-Lindsted expansion and an expansion of $A = A_0 + \epsilon A_1 + \cdots$ to calculate the leading-order solution and the first non-trivial frequency shift for the limit cycle.*

(c) *For $\epsilon = 0.01, 0.1, 0.2,$ and 0.3, plot the numerical solution and the multiple scale expansion for $0 \le t \le 40$ and for various values of A for your multiple scale solution. Also plot the limit cycle solution calculated from part (b).*

(d) *Calculate the error*

$$E(t) = |y_{numerical}(t) - y_{approximation}(t)|,$$

as a function of time ($0 \le t \le 40$) using $\epsilon = 0.01, 0.1, 0.2,$ and 0.3.

Solution.

(a) Consider Rayleigh's equation

$$y'' + y + \epsilon\left(-y' + \frac{1}{3}y'^3\right) = 0$$

with the initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = \alpha.$$

First we wish to use a multiple scale expansion to calculate the leading order behavior. The scaling takes the form

$$y = y_0(x, t, \tau) + \epsilon y_1(x, t, \tau) + \ldots,$$

where $\tau = \epsilon t$ is the slow variable dependence. Note that the chain rule gives

$$\begin{aligned} y_t &= y_t + \epsilon y_\tau, \\ y_{tt} &= y_{tt} + 2\epsilon y_{t\tau} + \epsilon^2 y_{\tau\tau}. \end{aligned}$$

Plugging these into Rayleigh's equation gives

$$(y_{tt} + 2\epsilon y_{t\tau} + \epsilon^2 y_{\tau\tau}) + y + \epsilon\bigg(-(y_t + \epsilon y_\tau) + \frac{1}{3}(y_t + \epsilon y_\tau)^3\bigg) = 0,$$

with the initial conditions $y(0,0) = 0$ and $y'(0,0) = \alpha$. Next collect terms to get

$$\begin{aligned}
&\mathcal{O}(1): \quad && y_{0_{\tau\tau}} + y_0 = 0 && y_0(0,0) = 0, y_{0_t}(0,0) = \alpha,\\
&\mathcal{O}(\epsilon): \quad && y_{1_{tt}} + y_1 = -2y_{0_{t\tau}} + y_{0_t} - \frac{1}{3}y_{0_t}^3 && y_1(0,0) = 0, y_{1_t}(0,0) = -y_{0_\tau}(0,0).
\end{aligned}$$

The leading order solution produces the general solution

$$y_0(t,\tau) = A(\tau)\cos(t) + B(\tau)\sin(t)$$

where $A(0) = 0$ and $B(0) = \alpha$. Plugging the leading order solution into the $\mathcal{O}(\epsilon)$ equation gives

$$\begin{aligned}
y_{1_{tt}} + y_1 = -2(-\sin(t)A'(\tau) + \cos(t)B'(\tau)) + B(\tau)\cos(t) - A(\tau)\sin(t)\\
- \frac{1}{3}(B(\tau)\cos(t) - A(\tau)\sin(t))^3.
\end{aligned}$$

Using trig identities $\sin^3(t) = \frac{3}{4}\sin(t) - \frac{3}{4}\sin(3t)$ and $\cos^3(t) = \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t)$, we can rewrite the right hand side as

$$\begin{aligned}
y_{1_{tt}} + y_1 =& \bigg(2A' - A - \frac{1}{3}\bigg(-\frac{3}{4}AB^2 - \frac{3}{4}A^3\bigg)\bigg)\sin(t) - \frac{1}{3}\bigg(\frac{1}{4}A^3 - \frac{3}{4}AB^2\bigg)\sin(3t)\\
&+ \bigg(B - 2B' - \frac{1}{3}\bigg(\frac{3}{4}B^3 - \frac{1}{4}A^2B\bigg)\bigg)\cos(t) - \frac{1}{3}\bigg(\frac{1}{4}B^3 - \frac{3}{4}A^2B\bigg)\cos(3t).
\end{aligned}$$

Now we wish to eliminate the terms that cause secular growth which gives us the following system of ODEs

$$2A' - A - \frac{1}{3}\bigg(-\frac{3}{4}AB^2 - \frac{3}{4}A^3\bigg) = 0 \tag{2}$$

$$B - 2B' - \frac{1}{3}\bigg(\frac{3}{4}B^3 - \frac{1}{4}A^2B\bigg) = 0. \tag{3}$$

Multiplying the Eq. (2) by A and Eq.(3) by B and adding them together yields

$$\rho_\tau + \frac{1}{4}\rho^2 - \rho = 0,$$

where $\rho = A^2 + B^2$. Noticing that this is Bernoulli equation of the form

$$\rho^{-2}\rho - \rho^{-1} = -\frac{1}{4},$$

gives that

$$\rho = \frac{4}{1 + ce^{-\tau}},$$

where c is an integration constant. Applying the initial conditions $\rho(0) = A^2(0) + B^2(0) = \alpha$ gives

$$c = \frac{4 - \alpha^2}{\alpha^2},$$

and thus

$$\rho = \frac{4\alpha^2}{\alpha^2 + (4 - \alpha^2)e^{-\tau}} = A^2 + B^2.$$

Solving for B^2 and substituting into Eq. (2) gives

$$2A' - A - \frac{1}{3}\left(-\frac{3}{4}A(\rho - A^2) - \frac{3}{4}A^3\right) = 0$$

and applying the boundary condition $A(0) = 0$ shows that $A(\tau) = 0$. Thus $B = \sqrt{\rho}$. Therefore the leading order behavior of the multiple scale expansion is

$$y(t) = \left(\sqrt{\frac{4\alpha^2}{\alpha^2 + (4 - \alpha^2)e^{-\epsilon t}}}\right)\sin(t) + \mathcal{O}(\epsilon).$$

(b) Next we wish to apply a Poincare-Lindstedt expansion

$$\tau = \omega t = (\omega_0 + \epsilon\omega_1 + \cdots)t,$$

and the expansion

$$\alpha = \alpha_0 + \epsilon\alpha_1 + \cdots.$$

Plugging these expansion into Rayleigh's equation yields

$$\omega^2 y_{\tau\tau} + y + \epsilon\left(-\omega y_\tau + \frac{1}{3}\omega^3 y_\tau^3\right) = 0,$$

and collecting the leading order terms gives the following ODE with respect to the given initial conditions

$$\omega_0^2 y_{0_{\tau\tau}} + y_0 = 0, \qquad y_0(0) = 0, \omega_0 y_\tau(0) = \alpha_0.$$

Noting that zero is in the null space and thus the Fredholm-Alternative theorem is satisfied regardless of the ω_0, pick $\omega_0 = 1$. Finding the general solution and applying the initial conditions the leading order solution to be

$$y_0 = \alpha_0 \sin(\tau).$$

To solve for α_0, let's collect the $\mathcal{O}(\epsilon)$ terms

$$\omega_0^2 y_{1_{\tau\tau}} + y_1 = -2\omega_0\omega_1 y_{0_{\tau\tau}} + \omega_0 y_{0_\tau} - \frac{1}{3}w_0^3 y_{0_\tau}^3, \qquad y_1(0) = 0, y_{1_\tau}(0) + \omega_1(y_{0_\tau}) = \alpha_1.$$

Substituting y_0 and ω_0 into the equation we get

$$y_{1_{\tau\tau}} + y_1 = 2\omega_1\alpha_0 \sin(\tau) + \alpha_0\cos(\tau) - \frac{1}{3}\alpha_0^3\cos(\tau)^3.$$

Next we need to enforce the Fredholm-Alternative of the right hand side with both $\sin(\tau)$ and $\cos(\tau)$. Observe that

$$\begin{aligned}
\left\langle 2\omega_1\alpha_0\sin(\tau) + \alpha_0\cos(\tau) - \frac{1}{3}\alpha_0^3\cos(\tau)^3, \sin(\tau)\right\rangle &= 0\\
2\omega_1\alpha_0\langle\sin(\tau),\sin(\tau)\rangle + 0 &= 0\\
\implies 2\omega_1\alpha_0 &= 0,
\end{aligned}$$

and using $\cos^3(t) = \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t)$

$$\begin{aligned}
&\left\langle 2\omega_1\alpha_0\sin(\tau) + \alpha_0\cos(\tau) - \frac{1}{3}\alpha_0^3\cos(\tau)^3, \cos(\tau)\right\rangle\\
0 + \alpha_0\langle\cos(\tau),\cos(\tau)\rangle - \frac{\alpha_0^3}{3}&\left\langle \frac{3}{4}\cos(\tau) + \frac{1}{4}\cos(3\tau), \cos(\tau)\right\rangle = 0\\
&\left(\alpha_0 + \frac{1}{4}\alpha_0^3\right)\langle\cos(\tau),\cos(\tau)\rangle + 0 = 0\\
&\implies \alpha_0 - \frac{1}{4}\alpha^3 = 0.
\end{aligned}$$

Thus we either have $\alpha_0 = 0$ or $\alpha_0 = 2$. Let's pick $\alpha_0 = 2$ and $\omega_1 = 0$ giving us the following ODE subject to the initial conditions

$$y_{1_{\tau\tau}} + y_1 = 2\cos(\tau) - \frac{8}{3}\cos(\tau)^3, \quad y_1(0) = 0, y_{1_\tau}(0) = \alpha_1.$$

Using variation of parameters and applying the initial condition gives the solution to be

$$y_1 = \frac{1}{12}(12\alpha_1\sin(\tau) - \cos(\tau) + \cos(3\tau)).$$

To find the first non-trivial frequency shift, let's collect the $\mathcal{O}(\epsilon^2)$ terms

$$\begin{aligned}
y_{2_{\tau\tau}} + y_2 &= y_{1_\tau} - y_{0_\tau}^2 y_{1_\tau} - 2\omega_2 y_{0_{\tau\tau}}\\
y_{2_{\tau\tau}} + y_2 &= \frac{1}{3}\cos^2(\tau)\left(12\alpha_1\cos(\tau) + \sin(\tau) - 3\sin(3\tau)\right)\\
&\quad + \frac{1}{12}\left(-12\alpha_1\cos(\tau) - \sin(\tau) + 3\sin(3\tau)\right) - 4\omega_2\sin(\tau).
\end{aligned}$$

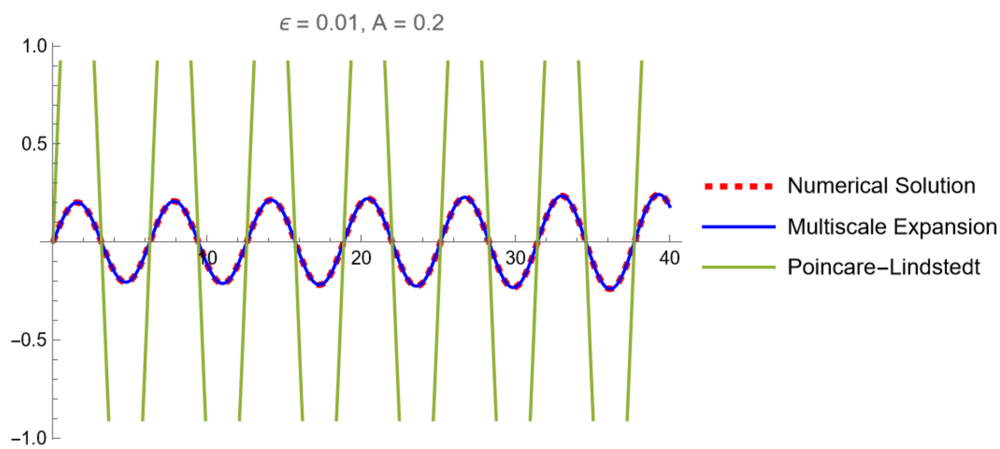
Using Mathematica to enforce the Fredholm-Alternative of the right hand side with both $\sin(\tau)$ and $\cos(\tau)$ and solve the resulting system of two ODEs gives that $\alpha_1 = 0$ and $\omega_2 = -\frac{1}{16}$. Therefore we have the leading order term to be

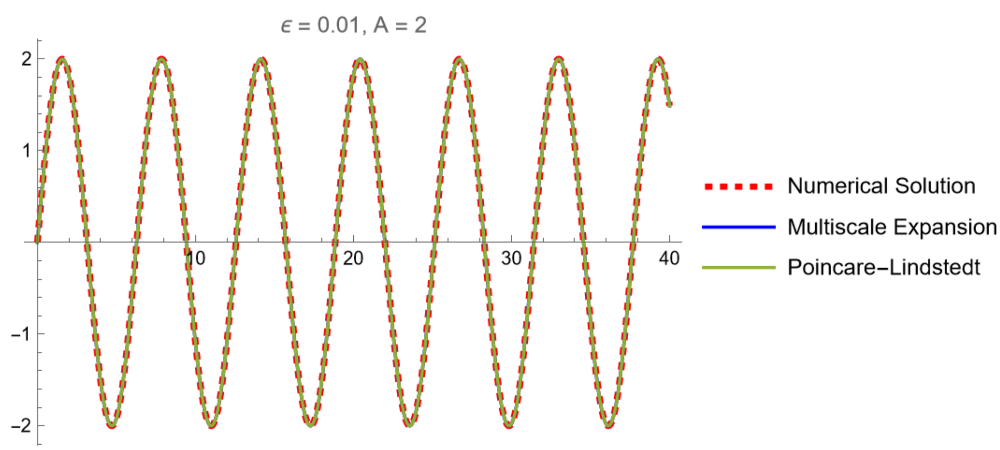
$$y = 2\sin((1 - \frac{\epsilon^2}{16})t) + \cdots$$

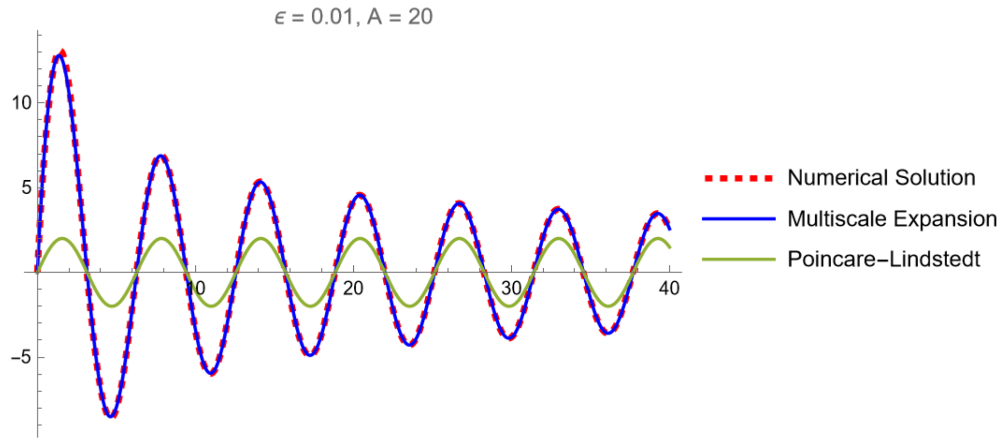
with the first non-trivial frequency shift

$$\omega_2 = -\frac{1}{16}$$

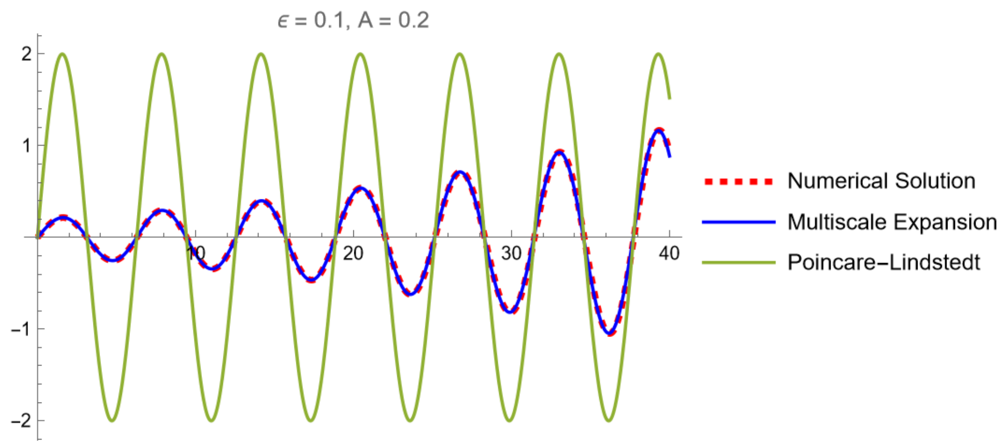
(c) Using Mathematica we can create the following graphs for $\epsilon = 0.01, 0.1, 0.2,$ and 0.3 for $A = 0.2, 2,$ and 20 using the Poincare-Lindstedt expansion, multiscale expansion and Mathematica's numerical solution. We note that the limit cycle is at $A = 2$ so for A values below 2 we see the graph of the multiscale goes up to the solution and vice versa when A is above 2.



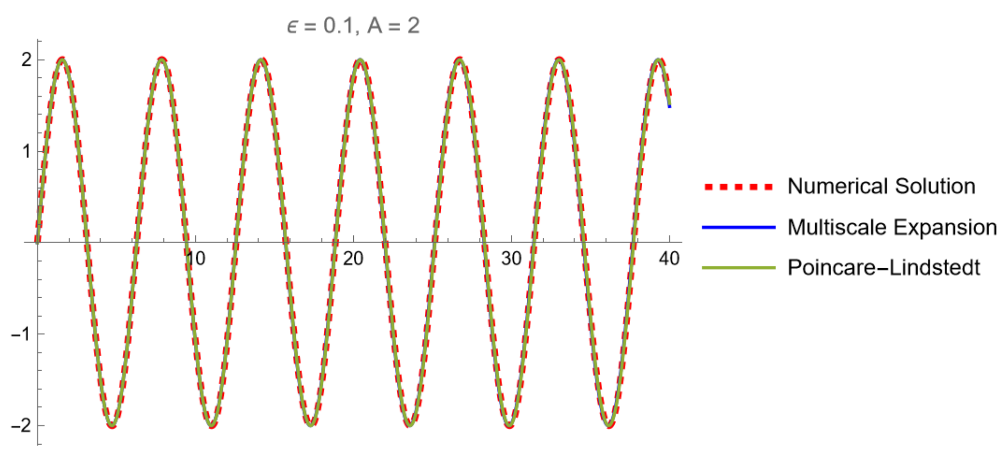




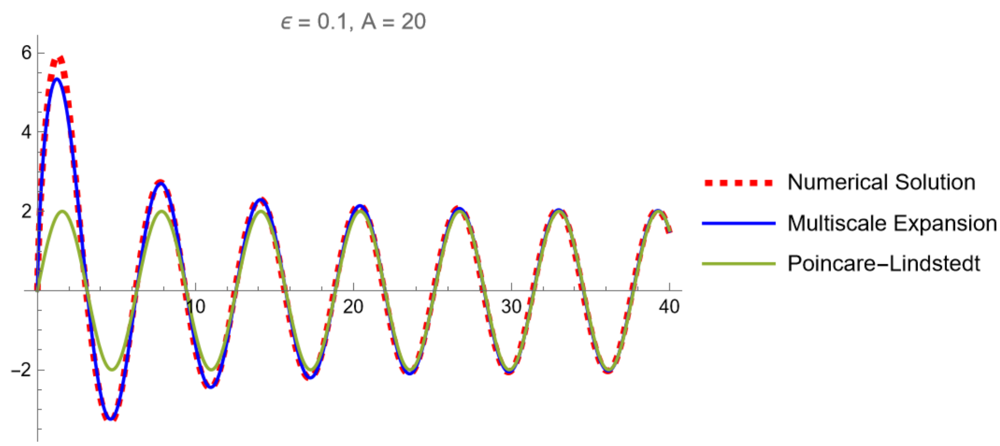
ϵ = 0.01, A = 20
10
5
–5
10
20
30
40
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt

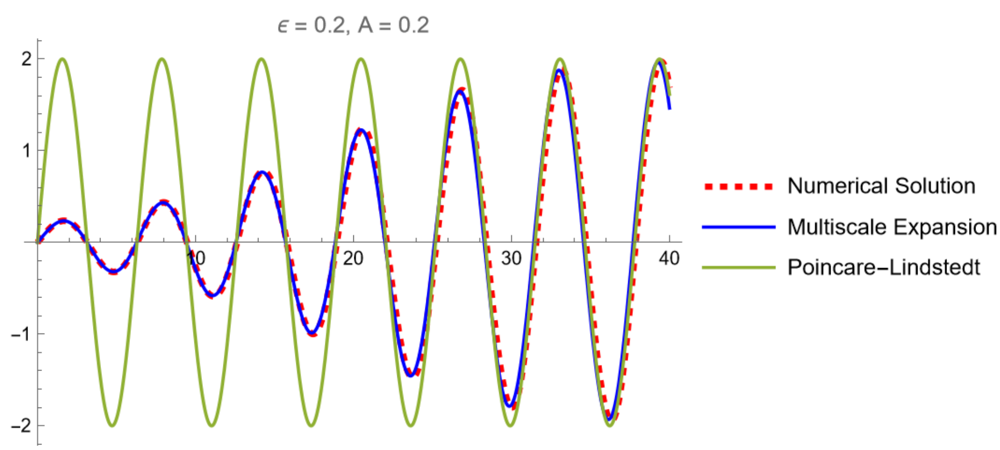


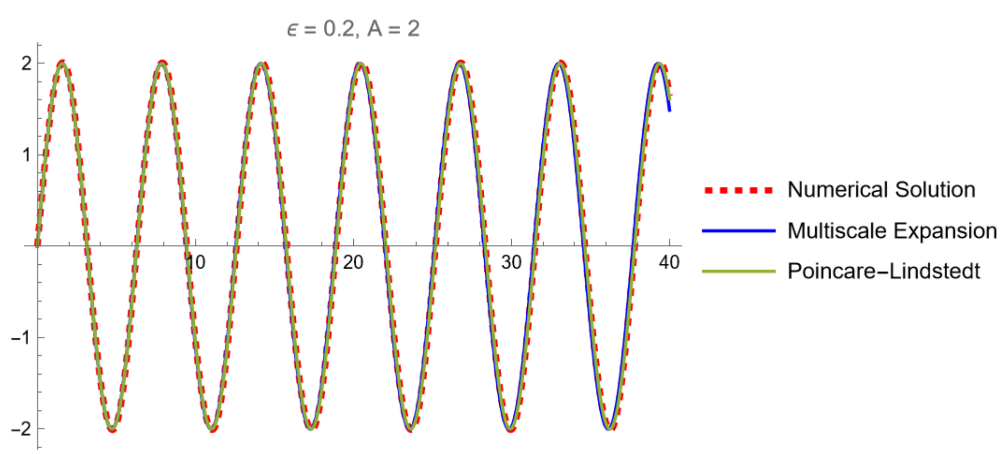
ϵ = 0.1, A = 0.2
2
1
–1
–2
10
20
30
40
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt

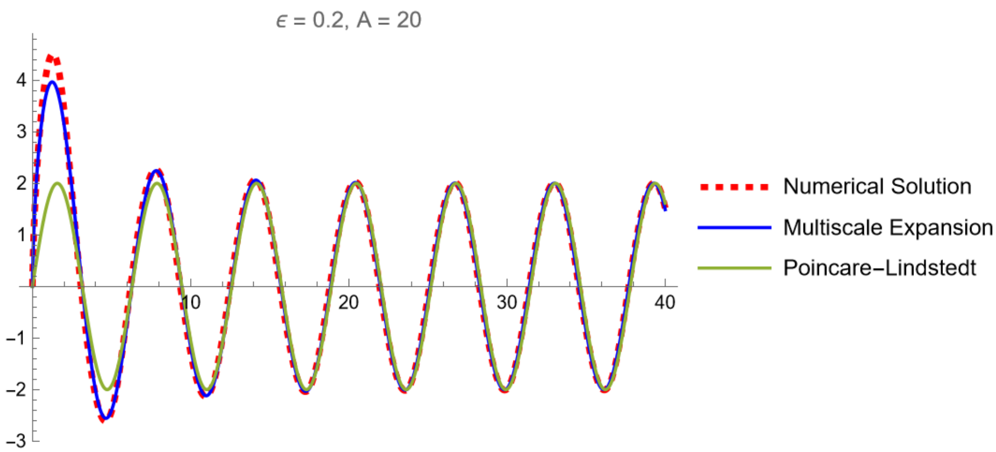


ϵ = 0.1, A = 2
2
1
–1
–2
10
20
30
40
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt


ϵ = 0.1, A = 20
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt


ϵ = 0.2, A = 0.2
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt


ϵ = 0.2, A = 2
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt


ϵ = 0.2, A = 20
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt
4
3
2
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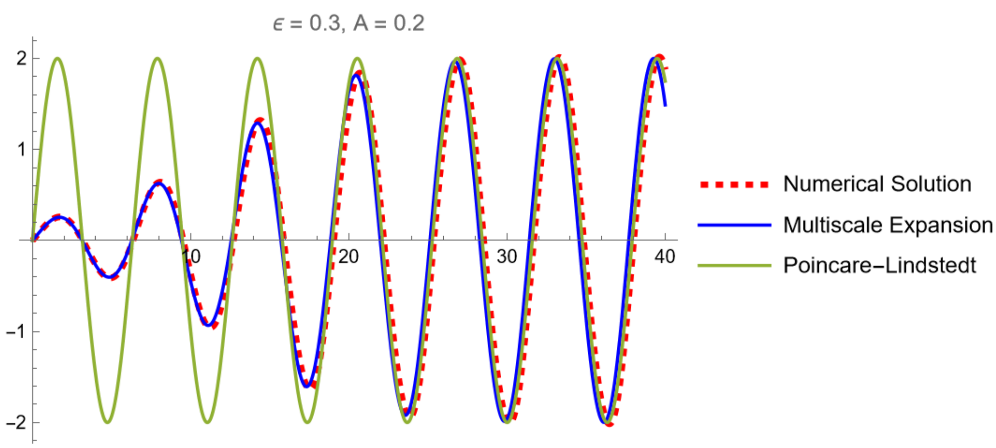
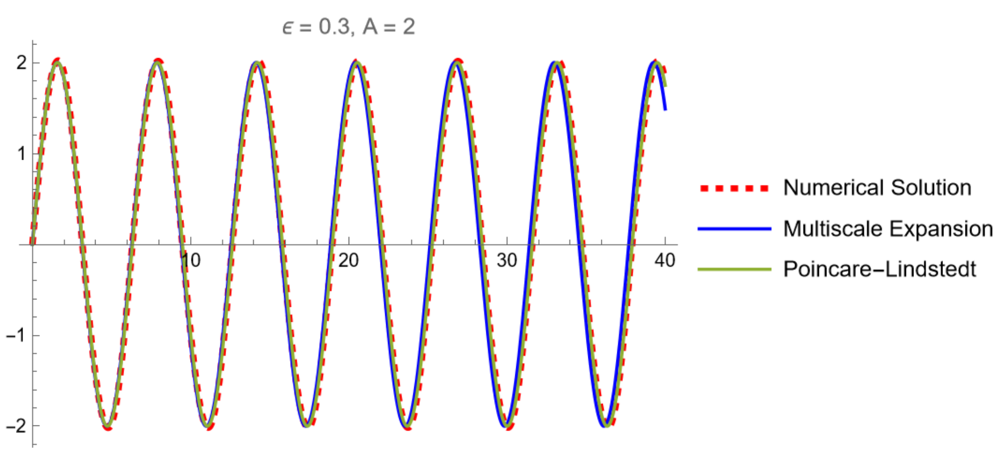
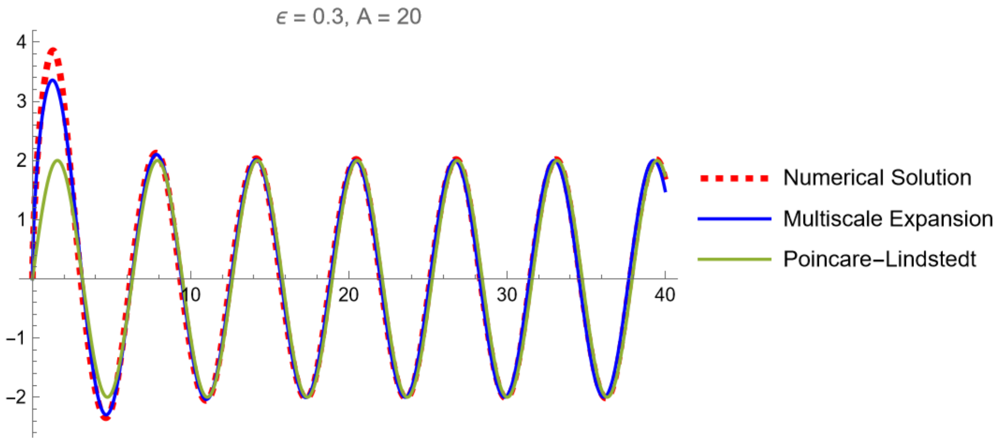
-1
-2
-3
10
20
30
40


ϵ = 0.3, A = 0.2
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt
2
1
-1
-2
10
20
30
40

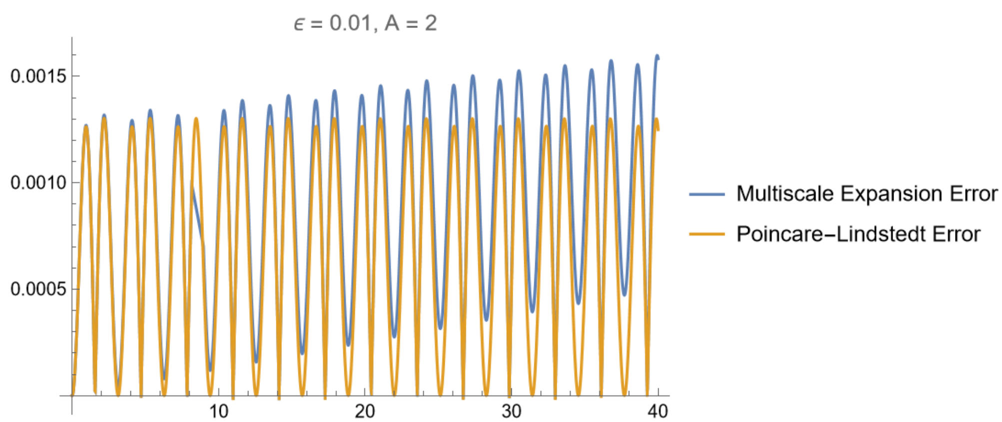

ϵ = 0.3, A = 2
Numerical Solution
Multiscale Expansion
Poincare–Lindstedt
2
1
-1
-2
10
20
30
40

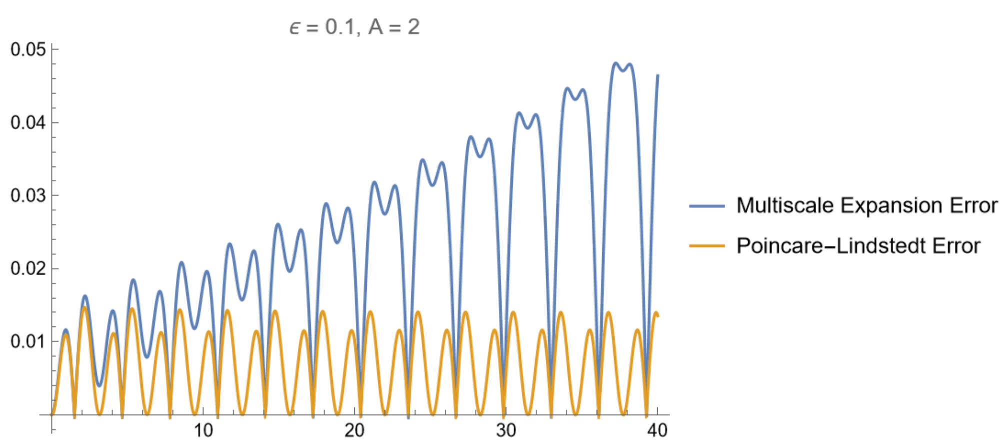


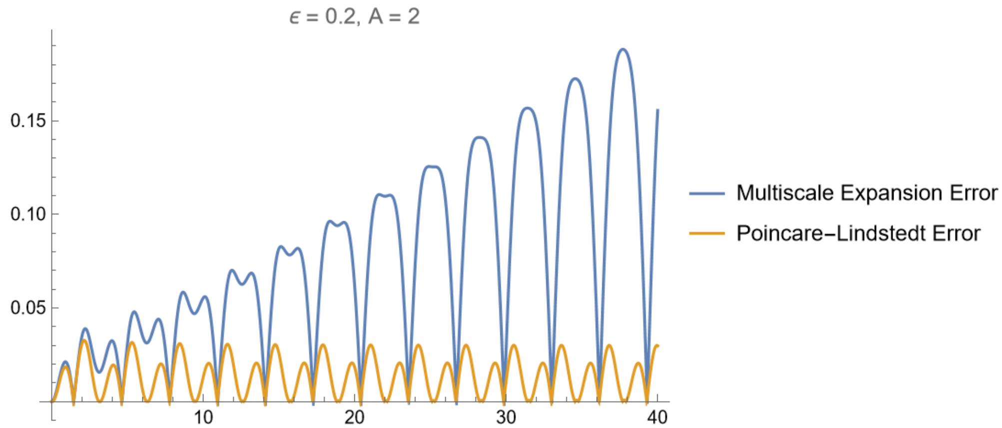
(d) Next we wish to calculate the error of each approximation using

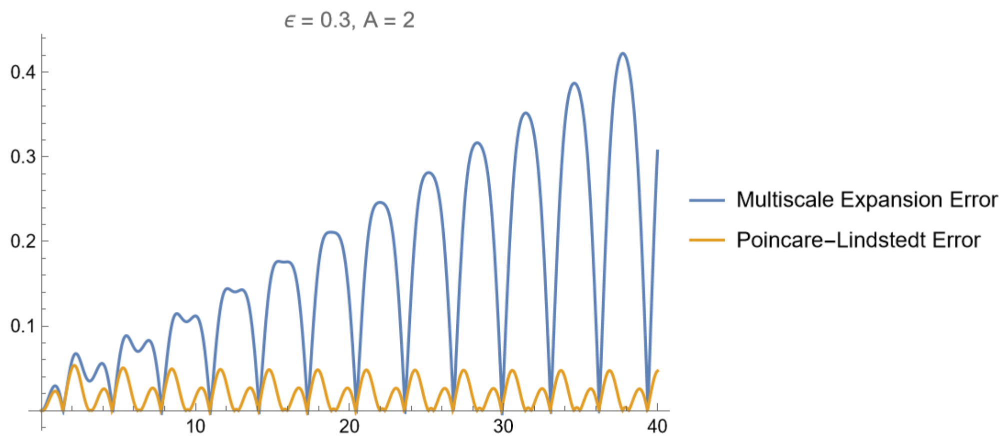
$$E(t) = |y_{\text{numerical}}(t) - y_{\text{approximation}}(t)|,$$

and Mathematica's numerical solution.






ϵ = 0.2, A = 2
0.15
0.10
0.05
10
20
30
40
Multiscale Expansion Error
Poincare–Lindstedt Error


ϵ = 0.3, A = 2
0.4
0.3
0.2
0.1
10
20
30
40
Multiscale Expansion Error
Poincare–Lindstedt Error

□