Math 569 Homework 2
Due April 26
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Problem 1 *Consider the wave equation*

$$c^2u_{xx} - u_{tt} = 0,$$

which is a special case of the general quasi-linear equation:

$$au_{xx} + 2bu_{xt} + cu_{tt} = f.$$

Find the slope of each of the two characteristics:

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -z_1 \text{ along } \alpha = \text{ constant characteristic}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -z_2 \text{ along } \beta = \text{ constant characteristic.}$$

Find the expression in terms of x and t for α and β so that the wave equation simplifies to

$$u_{\alpha\beta} = 0.$$

Solution.
Consider the wave equation

$$c^2u_{xx} = u_{tt},$$

which is a special case of the general quasi-linear equation:

$$au_{xx} + 2bu_{xt} + c'u_{tt} = f,$$

where $a = c^2, b = 0, c' = -1,$ and $f = 0$. To classify the PDE, we will use the transformation on the independent variables

$$\begin{cases} \alpha = \phi(x,t) \\ \beta = \psi(x,t), \end{cases}$$

which we require to be locally invertible. This implies that

$$J(x,t) = \det \begin{bmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial t} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial t} \end{bmatrix} \neq 0.$$

Using the chain rule, we can compute the first and second derivates to be

$$\begin{aligned} u_x &= u_\alpha \alpha_x + u_\beta \beta_x \\ u_{xx} &= u_\alpha \alpha_{xx} + u_\beta \beta_{xx} + u_{\alpha\alpha}\alpha_x^2 + 2u_{\alpha\beta}\alpha_x\beta_x + u_{\beta\beta}\beta_x^2, \end{aligned}$$

and

$$\begin{aligned}u_t &= u_\alpha \alpha_t + u_\beta \beta_t \\ u_{tt} &= u_\alpha \alpha_{tt} + u_\beta \beta_{tt} + u_{\alpha\alpha}\alpha_t^2 + 2u_{\alpha\beta}\alpha_t\beta_t + u_{\beta\beta}\beta_t^2.\end{aligned}$$

We now plug the transformation back into the wave equation to get

$$A(\alpha,\beta)u_{\alpha\alpha} + 2B(\alpha,\beta)u_{\alpha\beta} + C(\alpha,\beta)u_{\beta\beta} = F,$$

where F contains all the other contributions, not explicitly written and where

$$\begin{cases} A = a\alpha_x^2 + 2b\alpha_x\alpha_t + c'\alpha_t^2 & = c^2\alpha_x^2 - \alpha_t^2 \\ B = a\alpha_x\beta_x + b(\alpha_x\beta_t + \alpha_t\beta_x) + c'\alpha_t\beta_t & = c^2\alpha_x\beta_x - \alpha_t\beta_t \\ C = a\beta_x^2 + 2b\beta_x\beta_t + c'\beta_t^2 & = c^2\beta_x^2 - \beta_t^2. \end{cases}$$

Now if we pick α and β to satisfy the equation

$$\alpha\varphi_x^2 + 2b\varphi_x\varphi_t + c'\varphi_t^2 = 0,$$

then we achieve that $A = C = 0$ which reduces the PDE to

$$2Bu_{\alpha\beta} = F.$$

To determine α and β, we divide by φ_t^2 to obtain

$$\alpha\left(\frac{\varphi_x}{\varphi_t}\right)^2 + 2b\left(\frac{\varphi_x}{\varphi_t}\right) + c'\varphi_t^2 = 0,$$

and setting $z = \frac{\varphi_x}{\varphi_t}$ we obtain the quadratic equation

$$az^2 + 2bz + c' = c^2z^2 - 1 = 0,$$

and since $b^2 - ac' = c^2 > 0$, the roots will be real and not equal. Furthermore, we can compute the roots to be

$$c^2z^2 - 1 = 0 \implies z_{1,2} = \pm\frac{1}{c}.$$

Since $z_1 \neq z_2$ and $z_1, z_2 \in \mathbb{R}$, we know that

$$\begin{cases} \alpha_x = z_1\alpha_y \\ \alpha_y = z_2\beta_y, \end{cases}$$

and thus we see that $\alpha(x,y) =$ a constant and is one family of characteristics and $\beta(x,y) =$ constant which is a second family of characteristics. The slope of the characteristics in the $x - y$ plane are

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -z_1 = -\frac{1}{c},$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -z_2 = \frac{1}{c}.$$

Thus for α we have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} = 0, \text{ and } \frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{1}{c},$$

which we can solve using the method of characteristics to find

$$\alpha = x + ct,$$

and similarly for β we find

$$\beta = x - ct.$$

Next let's re-compute the chain rule with α and β to get

$$\begin{aligned} u_x &= u_\alpha + u_\beta \\ u_{xx} &= u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta}, \end{aligned}$$

and

$$\begin{aligned} u_t &= cu_\alpha - cu_\beta \\ u_{tt} &= c^2 u_{\alpha\alpha} - 2c^2 u_{\alpha\beta} + c^2 u_{\beta\beta}. \end{aligned}$$

Plugging this into the wave equation yields

$$\begin{aligned} 0 &= c^2 u_{xx} - u_{tt} \\ &= c^2(u_{\alpha\alpha} + 2u_{\alpha\beta}\alpha_x\beta_x + u_{\beta\beta}) - (c^2 u_{\alpha\alpha} - 2c^2 u_{\alpha\beta} + c^2 u_{\beta\beta}) \\ &= (c^2 - c^2)u_{\alpha\alpha} + (2c^2 + 2c^2)u_{\alpha\beta} + (c^2 - c^2)u_{\beta\beta} \\ &= 4c^2 u_{\alpha\beta} \\ &= 2Bu_{\alpha\beta}, \end{aligned}$$

where $B = 2c^2$ and $A = C = 0$. Thus we have found a transform that reduces the PDE to

$$2Bu_{\alpha\beta} = 0 \iff u_{\alpha\beta} = 0,$$

since $B \neq 0$. Sorry for the roundabout way that I took to get here. $\square$

Problem 2 *Use the Fourier transform method to solve the 2-D Laplace equation in the upper plane for the bounded solution:*

$$\nabla^2 u = 0, \ \ \textit{in } y > 0, -\infty < x < \infty$$

which is subject to $u(x,0) = f(x)$ *where* $f(x)$ *is of compact support;* $u(x,y) \to 0$ *as* $x \to \pm\infty$.

Solution.
Consider the 2-D Laplace equation in the upper plane

$$\nabla^2 u = u_{xx} + u_{yy} = 0,$$

for $y > 0$ and $-\infty < x < \infty$, subject to the initial condition $u(x,0) = f(x)$ where $f(x)$ is of compact support; $u(x,y) \to 0$ as $x \to \pm\infty$. We wish to apply the Fourier transform to solve this PDE. We begin by transforming the PDE and initial condition into the frequency domain. We first define

$$U(\omega, y) = \mathcal{F}[u(x,y)] = \int_{-\infty}^{\infty} u(x,y)e^{i\omega x}dx.$$

Then using integration by parts twice we find that,

$$\begin{aligned}
\mathcal{F}[u_{xx}] &= \int_{-\infty}^{\infty} u_{xx}e^{i\omega x}\mathrm{d}x \\
&= [u_x e^{i\omega x}]_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} u_x e^{i\omega x}\mathrm{d}x \\
&= \cancel{[u_x e^{i\omega x}]_{-\infty}^{\infty}} - i\omega \cancel{[u e^{i\omega x}]_{-\infty}^{\infty}} - \omega^2 \int_{-\infty}^{\infty} u e^{i\omega x}\mathrm{d}x \\
&= -\omega^2 U,
\end{aligned}$$

where the first is canceled as $u(x,y) \to 0$ as $x \to \pm\infty$ and the second term is canceled by making the assumption that $u_x \to 0$ as $|x| \to \infty$. We also compute

$$\mathcal{F}[u_{yy}] = \int_{-\infty}^{\infty} u_{yy}e^{i\omega x}\mathrm{d}x = \frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} u e^{i\omega x}\mathrm{d}x = U_{yy},$$

and transform the initial condition

$$U(\omega, 0) = \mathcal{F}[u(x,0)] = \mathcal{F}[f(x)] = F(\omega),$$

Now we can reformulate the PDE in the frequency domain as the following ODE

$$-\omega^2 U + U_{yy} = 0.$$

which has the general solution

$$U(\omega, y) = c_1(\omega)e^{\omega y} + c_2(\omega)e^{-\omega y}.$$

Notice that if we also assume that $u(x,y) \to 0$ as $y \to \infty$, then we also require $U(\omega,y) \to 0$ as $y \to \infty$. Thus we have two different cases: 1) if $\omega > 0$, then $A(\omega) = 0$ to satisfy the condition, or 2) if $\omega < 0$, then $B(\omega) = 0$ to satisfy the condition. Enforcing this condition we get the solution to be

$$U(\omega,y) = \begin{cases} B(\omega)e^{-\omega y} & \omega > 0 \\ A(\omega)e^{\omega y} & \omega < 0, \end{cases}$$

and enforcing the initial condition we find a particular solution to be

$$U(\omega,y) = \begin{cases} F(\omega)e^{-\omega y} & \omega > 0 \\ F(\omega)e^{\omega y} & \omega < 0. \end{cases}$$

Now we wish to transform the solution out of the frequency space, so let's take the inverse Fourier transform which gives

$$\begin{aligned} u(x,y) &= \mathcal{F}^{-1}[U(\omega,y)] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,y)e^{-i\omega x}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{0} F(\omega)e^{\omega y}e^{-i\omega x}\mathrm{d}\omega + \int_{0}^{\infty} F(\omega)e^{-\omega y}e^{-i\omega x}\mathrm{d}\omega\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-|\omega|y}e^{-i\omega x}\mathrm{d}\omega \\ &= \mathcal{F}^{-1}[F(\omega)e^{-|\omega|y}]. \end{aligned}$$

Recall that the convolution theorem states

$$\mathcal{F}^{-1}[gh] = \mathcal{F}^{-1}[g] * \mathcal{F}^{-1}[h],$$

where $*$ denotes the convolution operation. Applying the theorem to our problem yields

$$\mathcal{F}^{-1}[F(\omega)e^{-|\omega|y}] = \mathcal{F}^{-1}[F(\omega)] * \mathcal{F}^{-1}[e^{-|\omega|y}] = f(x) * \mathcal{F}^{-1}[e^{-|\omega|y}].$$

To find $\mathcal{F}^{-1}[e^{-|\omega|y}]$, observe that

$$\begin{aligned} \mathcal{F}^{-1}[e^{-|\omega|y}] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|y}e^{-i\omega x}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{\omega(y-ix)}\mathrm{d}\omega + \int_{0}^{\infty} e^{-\omega(y+ix)}\mathrm{d}\omega\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{y-ix} + \frac{1}{y+ix}\right) \\ &= \frac{1}{2\pi}\cdot\frac{2y}{x^2+y^2} \\ &= \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}. \end{aligned}$$

Thus we have that

$$u(x,y) = f(x) * \frac{1}{\pi} \cdot \frac{y}{x^2+y^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{y}{(x-t)^2+y^2}\mathrm{d}t.$$

Now let's double check the assumptions we made. As $y \to \infty$,

$$\frac{y}{(x-t)^2+y^2} \to 0,$$

and thus $u(x,y) \to 0$ as $y \to \infty$ as the integrand tends to zero. We also have that

$$u_x(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{y(x-t)}{((x-t)^2+y^2)^2}\mathrm{d}t,$$

and thus $u_x(x,y) \to 0$ as $|x| \to \infty$ as the integrand tends to zero.

$\square$

Problem 3 *Solve the following problem in two ways:*

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u, \quad t > 0;\ 0 < x < \infty,$$

subject to $u(x,0) = 0$, $u(x,t)$ *bounded as* $x \to \infty$ *and* $u(0,t) = T_0$ *a constant,* $t > 0$.

(a) *By the method of similarity transformation. Look for the value of* α *such that the PDE reduces to an ode in* η, $\eta = x/t^\alpha$.

(b) *By an integral transform in t, in this case a Laplace transform (You can use a table of Laplace transform to do the inverse transform).*

Solution.
Consider the problem

$$\begin{aligned} u_t &= u_{xx}, \quad t > 0, 0 < x < \infty \\ u(x,0) &= 0 \\ u(0,t) &= T_0, \end{aligned}$$

where $u(x,t)$ is bounded as $x \to \infty$ and T_0 is a constant.

(a) We wish to solve the problem using the method of similarity transformation. Consider the similarity transformation

$$\eta = \frac{x}{t^\alpha},$$

then we have the following chain rules

$$\frac{\partial u}{\partial t} = \frac{\partial \eta}{\partial t}\frac{\partial u}{\partial \eta} = \frac{-\alpha x}{t^{\alpha+1}}\frac{\partial u}{\partial \eta} = -\frac{\alpha}{t}\eta\frac{\partial u}{\partial \eta},$$

and

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta} = \frac{1}{t^\alpha}\frac{\partial u}{\partial \eta}, \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{1}{t^\alpha}\frac{\partial u}{\partial \eta}\right) = \frac{1}{t^\alpha}\left(\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\right) = \frac{1}{t^{2\alpha}}\frac{\partial^2 u}{\partial \eta^2}. \end{aligned}$$

Plugging these into our equation

$$-\frac{\alpha}{t}\eta\frac{\partial u}{\partial \eta} = \frac{1}{t^{2\alpha}}\frac{\partial^2 u}{\partial \eta^2},$$

noticing that if we let $\alpha = 1/2$, the t variable is eliminated. Thus

$$\eta = \frac{x}{\sqrt{t}},$$

and our problem has become an ODE of the form

$$
\begin{aligned}
-\frac{\eta}{2}\frac{\partial u}{\partial \eta} &= \frac{\partial^2 u}{\partial \eta^2}, \\
u(0) &= T_0,
\end{aligned}
$$

and $u(\eta) \to 0$ as $\eta \to \infty$. To solve the ODE, let $v = u'$ which yields

$$\frac{v'}{v} = \frac{-1}{2},$$

and solving using separation of variables gives the solution to be

$$v(\eta) = c_1 e^{-\eta^2/4}.$$

Thus we have found

$$u(\eta) = \int_0^\eta c_1 e^{-\eta^2/4} \mathrm{d}\eta = c_1 \mathrm{erf}\Big(\frac{\eta}{2}\Big) + c_2.$$

Now we can enforce that $u(0) = T_0$ and $u(\eta) \to 0$ as $\eta \to \infty$. Recalling that $\mathrm{erf}(0) = 0$ and $\mathrm{erf}(\eta) \to 1$ as $\eta \to \infty$. The first condition gives $c_2 = T_0$ and the second condition gives $c_1 = -c_2 = -T_0$. Thus

$$u(\eta) = -T_0 \mathrm{erf}\Big(\frac{\eta}{2}\Big) + T_0 = \mathrm{erfc}\Big(\frac{\eta}{2}\Big),$$

and undoing our transform gives the solution to be

$$u(x,t) = T_0 \mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

(b) Next we wish to solve the problem using a Laplace transform. Let $\tilde{u}(x,s)$ be the Laplace transform of the solution

$$\tilde{u}(x,s) = \mathcal{L}[u(x,t)] = \int_0^\infty e^{-st} u(x,t) \mathrm{d}t.$$

Then we have

$$\mathcal{L}[u_{xx}] = \int_0^\infty e^{-st} \frac{\partial^2}{\partial x^2} u(x,t) \mathrm{d}t = \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} u(x,t) \mathrm{d}t = \tilde{u}_{xx}(x,s),$$

and using integration by parts

$$
\begin{aligned}
\mathcal{L}[u_t] &= \int_0^\infty \frac{\partial}{\partial t} u e^{-st} \mathrm{d}t \\
&= \cancel{[u e^{-st}]_0^\infty} + s \int_0^\infty u e^{-st} \mathrm{d}t
\end{aligned}
$$

$$= s\tilde{u}(x,s),$$

if we make the assumption that $ue^{-st} \to 0$ as $t \to \infty$. Plugging these into our PDE we have

$$s\tilde{u} = \tilde{u}_{xx},$$

which is an ODE where $s > 0$. We now Laplace transform the initial condition

$$\tilde{u}(0,s) = \int_0^\infty u(0,t)e^{-st}\mathrm{d}t = \int_0^\infty T_0 e^{-st}\mathrm{d}t = \frac{T_0}{s}.$$

Now the general solution to the ODE is

$$\tilde{u}(x,s) = c_1(s)e^{\sqrt{s}x} + c_2(s)e^{-\sqrt{s}x}.$$

To find a particular solution, we make the assumption that $u(x,t) \to 0$ as $x \to \infty$ which implies that $c_1(s) = 0$. Since $\tilde{u}(0,s) = \frac{T_0}{s}$, we have that $c_2 = \frac{T_0}{s}$. Thus the solution becomes

$$\tilde{u}(x,s) = \frac{T_0}{s}e^{-\sqrt{s}x}.$$

Using a table of Laplace transforms (Mathematica) we find the inverse transform of the solution

$$u(x,t) = \mathcal{L}^{-1}[\tilde{u}(x,s)] = T_0\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right),$$

which is the solution we found in part (a). Note that since $\frac{x}{2\sqrt{t}} \to 0$ as $t \to \infty$ and the $\mathrm{erfc}(0) = 1$, we can verify that

$$\lim_{t\to\infty} ue^{-st} = \lim_{t\to\infty} T_0 e^{-st} = 0,$$

since $s > 0$. We also have that $u(x,t) \to 0$ as $x \to \infty$ since $\mathrm{erfc}x \to 0$ as $x \to \infty$.

$\square$