Math 569 Homework 3
Due 3 May
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Problem 1 *(a) Solve using Fourier transform in x and Laplace transform in t:*

$$\begin{cases} \frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = \delta(x-\xi)\delta(t-\tau), & -\infty < x < \infty, t > 0, -\infty < \xi < \infty, \tau > 0, \\ u(x,t) \to 0, & \text{as } x \to \pm\infty, t > 0, \\ u(x,0) = 0, & \infty < x < \infty. \end{cases}$$

(b) Solve the same problem as in (a) expect you do not use Laplace transform in t.

Solution.
Consider the equation

$$\begin{cases} \frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = \delta(x-\xi)\delta(t-\tau), & -\infty < x < \infty, t > 0, -\infty < \xi < \infty, \tau > 0, \\ u(x,t) \to 0, & \text{as} x \to \pm\infty, t > 0, \\ u(x,0) = 0, & \infty < x < \infty. \end{cases} \tag{1}$$

(a) We wish to solve 1 using a Fourier transform in x and Laplace transform in t. Let's first take the Fourier transform in x. We first define

$$U(\omega, y) = \mathcal{F}[u(x,y)] = \int_{-\infty}^{\infty} u(x,y)e^{i\omega x}dx.$$

Then using integration by parts twice we find that,

$$\begin{aligned} \mathcal{F}[u_{xx}] &= \int_{-\infty}^{\infty} u_{xx}e^{i\omega x}\mathrm{d}x \\ &= [u_x e^{i\omega x}]_{-\infty}^{\infty} - i\omega\int_{-\infty}^{\infty} u_x e^{i\omega x}\mathrm{d}x \\ &= \cancel{[u_x e^{i\omega x}]_{-\infty}^{\infty}} - i\omega\cancel{[ue^{i\omega x}]_{-\infty}^{\infty}} - \omega^2\int_{-\infty}^{\infty} ue^{i\omega x}\mathrm{d}x \\ &= -\omega^2 U, \end{aligned}$$

where the first term is canceled as $u(x,y) \to 0$ as $x \to \pm\infty$ and the second term is canceled by making the assumption that $u_x \to 0$ as $|x| \to \infty$. We also compute

$$\mathcal{F}[u_t] = \int_{-\infty}^{\infty} u_t e^{i\omega x}\mathrm{d}x = \frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} ue^{i\omega x}\mathrm{d}x = U_t,$$

and

$$\mathcal{F}[\delta(x-\xi)\delta(t-\tau)] = \int_{-\infty}^{\infty} \delta(x-\xi)\delta(t-\tau)e^{i\omega x}\mathrm{d}x$$

$$= \delta(t-\tau)\int_{-\infty}^{\infty}\delta(x-\xi)e^{i\omega x}\mathrm{d}x$$
$$= \delta(t-\tau)e^{i\omega\xi}.$$

Thus we can rewrite equation 1 as

$$U_t + D\omega^2 U = \delta(t-\tau)e^{i\omega\xi}.$$

Next we will take Laplace transform in t. We first define

$$\tilde{U}(\omega,s) = \mathcal{L}[U(\omega,t)].$$

Then we have

$$\begin{aligned}
\mathcal{L}[U(\omega,t)_t] &= \int_0^{\infty} U_t e^{-st}\mathrm{d}t \\
&= \cancel{[Ue^{-st}]_0^{\infty}} + s\int_0^{\infty} Ue^{-st}\mathrm{d}t \\
&= s\tilde{U}(\omega,s),
\end{aligned}$$

if we make the assumption that $u(x,t) \to 0$ as $t \to \infty$. We also have

$$\mathcal{L}[\delta(t-\tau)e^{i\omega\xi}] = \int_0^{\infty}\delta(t-\tau)e^{i\omega\xi}e^{-st}\mathrm{d}t = e^{i\omega\xi}\int_0^{\infty}\delta(t-\tau)e^{-st}\mathrm{d}t = e^{i\omega\xi}e^{-s\tau}.$$

Thus the PDE becomes

$$(s+D\omega^2)\tilde{U}(\omega,s) = e^{i\omega\xi}e^{-s\tau} = e^{i\omega\xi - s\tau},$$

and so

$$\tilde{U}(\omega,s) = \frac{e^{i\omega\xi - s\tau}}{s+D\omega^2}.$$

Now we will compute the inverse Laplace transform

$$U(\omega,t) = \mathcal{L}^{-1}[\tilde{U}(\omega,s)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}\frac{e^{i\omega\xi-s\tau}}{s+D\omega^2}\mathrm{d}s = \frac{e^{i\omega\xi}}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{s(t-\tau)}}{s+D\omega^2}\mathrm{d}s.$$

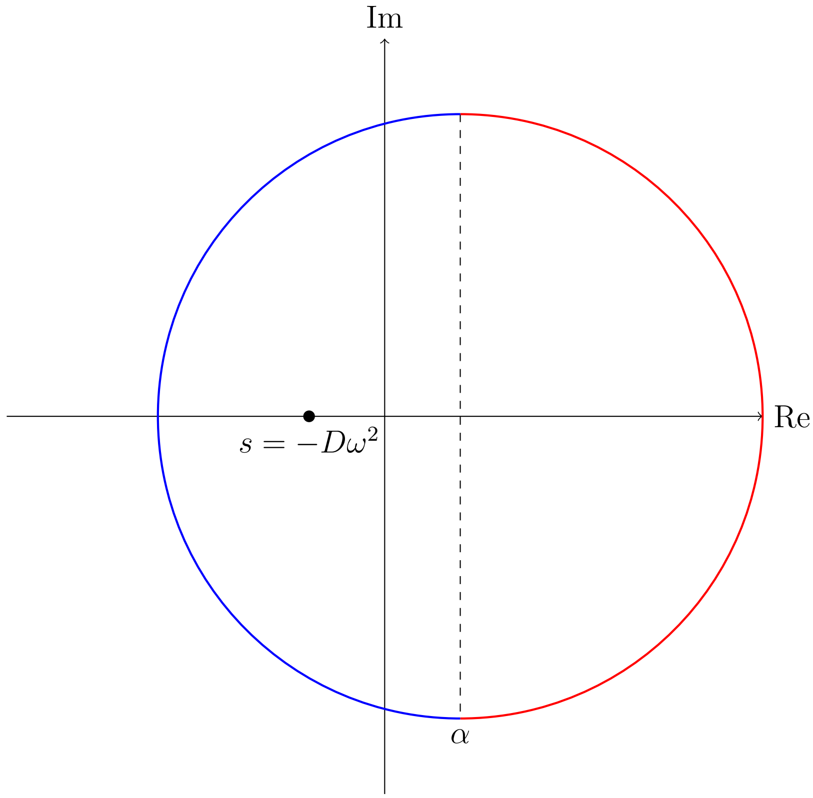

Figure 1: Bromwich Contour

Note that there exists a simple pole at $s = -D\omega^2$. To compute the integral, we first consider when $t < \tau$ which gives exponential decay when $s > 0$. Thus we use a Bromwich contour, C^+ on the right side centered at α (up along the dotted line and down along the red arc). Then we have that

$$\int_{C^+} \frac{e^{s(t-\tau)}}{s + D\omega^2} \mathrm{d}s = 0,$$

by Jordan's lemma since the integrand is analytic for $s > 0$. When $t > \tau$, we have exponential decay when $s < 0$ and thus we will use a Bromwich contour, C^- on the left side centered at α (down along the dotted line and up along the blue arc). We note that C^- now contains the simple pole. Then using Jordan's lemma we have

$$\begin{aligned} \int_{C^-} \frac{e^{s(t-\tau)}}{s + D\omega^2} \mathrm{d}s &= 2\pi i \sum \mathrm{Res}\left(\frac{e^{s(t-\tau)}}{s + D\omega^2} \right) \\ &= 2\pi i \frac{e^{(-D\omega^2)(t-\tau)}}{(s + D\omega^2)'} \\ &= 2\pi i e^{-D\omega^2(t-\tau)}. \end{aligned}$$

Thus we have found

$$U(\omega,t) = \begin{cases} 0, & t < \tau \\ e^{i\omega\xi}e^{-D\omega^2(t-\tau)}, & t > \tau \end{cases},$$

and if we let $H(x)$ denote the Heaviside function we can write this as

$$U(\omega,t) = H(t-\tau)e^{i\omega\xi - D\omega^2(t-\tau)},$$

which is the solution in the frequency domain. Finally we have to take the inverse Fourier transform

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1}[U(\omega,t)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\omega}e^{i\xi\omega}H(t-\tau)e^{i\omega\xi - D\omega^2(t-\tau)}\mathrm{d}\omega \\ &= \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(x-\xi)+D\omega^2(t-\tau)}\mathrm{d}\omega \\ &= \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty} e^{-D(t-\tau)\left(\omega^2 + \frac{i\omega(x-\xi)}{D(t-\tau)}\right)}\mathrm{d}\omega \\ &= \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty} e^{-D(t-\tau)\left(\left(\omega+\frac{i(x-\xi)}{2D(t-\tau)}\right)^2 + \left(\frac{x-\xi}{2D(t-\tau)}\right)^2\right)}\mathrm{d}\omega \\ &= \frac{H(t-\tau)}{2\pi}e^{-\left(\frac{x-\xi}{2D(t-\tau)}\right)^2}\int_{-\infty}^{\infty} e^{-D(t-\tau)\left(\omega+\frac{i(x-\xi)}{2D(t-\tau)}\right)^2}\mathrm{d}\omega. \end{aligned}$$

Recall that the definite integral of a Gaussian function is given by

$$\int_{-\infty}^{\infty} e^{-a(x+b)^2} = \sqrt{\frac{\pi}{a}},$$

and thus we can solve to the integral to be

$$\int_{-\infty}^{\infty} e^{-D(t-\tau)\left(\omega+\frac{i(x-\xi)}{2D(t-\tau)}\right)^2}\mathrm{d}\omega = \sqrt{\frac{\pi}{D(t-\tau)}}.$$

Therefore we have found the solution to be

$$u(x,t) = \frac{H(t-\tau)}{2\pi}e^{-\left(\frac{x-\xi}{2D(t-\tau)}\right)^2}\sqrt{\frac{\pi}{D(t-\tau)}} = \frac{H(t-\tau)e^{-\left(\frac{x-\xi}{2D(t-\tau)}\right)^2}}{\sqrt{4\pi D(t-\tau)}},$$

where the Heaviside function covers the case when $t < \tau$. Finally we can check the assumptions we made. Clearly we have that $u_x \to 0$ as $x \to \pm\infty$ do to the exponential term and $u \to 0$ as $t \to \infty$ since the exponential terms goes to 1 and the denominator grows large.

(b) Now we wish to solve equation 1 expect not using a Laplace transform. Recall that equation 1 in the frequency domain is given by

$$U_t + D\omega^2 U = \delta(t-\tau)e^{i\omega\xi}.$$

Note that $\delta(t-\tau) = 0$ for $t \neq \tau$ by definition and thus we can break up the ODEs into

$$\begin{cases} U_t + D\omega^2 U = 0, & t < \tau, \\ U_t + D\omega^2 U = 0, & t > \tau, \end{cases}$$

which have the general solutions

$$U(\omega, t) = \begin{cases} C(\omega)e^{-D\omega^2 t}, & t < \tau, \\ D(\omega)e^{-D\omega^2 t}, & t > \tau, \end{cases}$$

subject to the initial condition $u(x, 0) = 0$ which gives $U(\omega, 0) = 0$. Since we assumed that $\tau > 0$, then the initial condition only applies to the case when $t < \tau$ which gives that $C(\omega) = 0$ and thus

$$U(\omega, t) = \begin{cases} 0, & t < \tau, \\ D(\omega)e^{-D\omega^2 t}, & t > \tau. \end{cases}$$

To solve for $D(\omega)$, we will find a matching condition across $t = \tau$ by integrating across $t = \tau$ by taking τ^+ and τ^- to be on either side of $t = \tau$. Observe that

$$\int_{\tau^-}^{\tau^+} U_t + D\omega^2 U \mathrm{d}t = \int_{\tau^-}^{\tau^+} \delta(t-\tau)e^{i\omega\xi}\mathrm{d}t = e^{i\omega\xi},$$

and

$$\begin{aligned} \int_{\tau^-}^{\tau^+} U_t + D\omega^2 U \mathrm{d}t &= \int_{\tau^-}^{\tau^+} U_t \mathrm{d}t + D\omega^2 \cancel{\int_{\tau^-}^{\tau^+} U \mathrm{d}t} \\ &= U(\omega, \tau^+) - \cancel{U(\omega, \tau^-)} \\ &= U(\omega, \tau^+) = D(\omega)e^{-D\omega^2 \tau}, \end{aligned}$$

were the first cancellation was due to U being finite and the second since $U(\omega, t) = 0$ when $t < \tau$. Thus we have that

$$D(\omega)e^{-D\omega^2\tau} = e^{i\omega\xi} \implies D(\omega) = e^{i\omega\xi + D\omega^2\tau},$$

which gives that

$$U(\omega, t) = \begin{cases} 0, & t < \tau \\ e^{i\omega\xi - D\omega^2(t-\tau)} \end{cases} = H(t-\tau)e^{i\omega\xi - D\omega^2(t-\tau)},$$

where H denotes the Heaviside function. This is the same solution in frequency space we found prior and thus applying the inverse Fourier transform yields

$$u(x,t) = \mathcal{F}^{-1}[u(\omega,t)] = \frac{H(t-\tau)e^{-\left(\frac{x-\xi}{2D(t-\tau)}\right)^2}}{\sqrt{4\pi D(t-\tau)}}.$$

$\square$