Math 573 Homework 5
Due Soonsh
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Problem 1 *Show that*

$$X = \begin{pmatrix} -i\zeta & q \\ \pm q^* & i\zeta \end{pmatrix}, T = \begin{pmatrix} -i\zeta^2 \mp \frac{i}{2}|q|^2 & q\zeta + \frac{i}{2}q_x \\ \pm q^*\zeta \mp \frac{i}{2}q_x^* & i\zeta^2 \pm \frac{i}{2}|q|^2 \end{pmatrix}$$

are Lax Pairs for the Nonlinear Schrödinger equations

$$iq_t = -\frac{1}{2}q_{xx} \pm |q|^2 q.$$

Here the top (bottom) signs of one matrix correspond to the top (bottom) signs of the other. In other words, show that the X, T with the top (bottom) sign are a Lax pair for the Nonlinear Schrödinger equation with the top (bottom) sign.

Solution. (Collaborated with Annie, Kaitlynn, and Cade throughout the homework) We wish to that

$$X = \begin{pmatrix} -i\zeta & q \\ \pm q^* & i\zeta \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} -i\zeta^2 \mp \frac{i}{2}|q|^2 & q\zeta + \frac{i}{2}q_x \\ \pm q^*\zeta \mp \frac{i}{2}q_x^* & i\zeta^2 \pm \frac{i}{2}|q|^2 \end{pmatrix},$$

are Lax Pairs for the Nonlinear Schrödinger equations

$$iq_t = -\frac{1}{2}q_{xx} \pm |q|^2 q.$$

That is, we wish to show that $X_t + XT = T_x + TX$. Using Mathematica we can look at the top and bottom case and using that $|q|^2 = qq^*$ in combination with

$$q_t = -\frac{1}{2i}q_{xx} \pm |q|^2 q \iff (q_t)^* = -\frac{i}{2}q_{xx}^* \pm i|q|^2 q^*,$$

we verify that $X_t + XT - T_x - TX = 0$. Thus X and T are Lax Pairs for the Nonlinear Schrödinger equation.

$\square$

Problem 2 *Let $\psi_n = \psi_n(t)$, $n \in \mathbb{Z}$. Consider the difference equation*

$$\psi_{n+1} = X_n \psi_n,$$

and the differential equation

$$\frac{\partial \psi_n}{\partial t} = T_n \psi_n.$$

What is the compatibility condition of these two equations? Using this result, show that

$$X_n = \begin{pmatrix} z & q_n \\ q_n^* & 1/z, \end{pmatrix} T_n = \begin{pmatrix} iq_n q_{n-1}^* - \frac{i}{2}\left(1/z - z\right)^2 & \frac{i}{z} q_{n-1} - izq_n \\ -izq_{n-1}^* + \frac{i}{z} q_n^* & -iq_n^* q_{n-1} + \frac{i}{2}\left(1/z - z\right)^2 \end{pmatrix}$$

is a Lax Pair for the semi-discrete equation

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - |q_n|^2(q_{n+1} + q_{n-1})$$

Note that this is a discretization of the NLS equation. It is known as the Ablowitz-Ladik lattice. It is an integrable discretization of NLS. For numerical purposes, it is far superior in many ways to the "standard" discretization of NLS:

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - 2|q_n|^2 q_n.$$

Solution.

Let $\psi_n = \psi_n(t)$, $n \in \mathbb{Z}$. Consider the difference equation

$$\psi_{n+1} = X_n \psi_n, \tag{1}$$

and the differential equation

$$\frac{\partial \psi_n}{\partial t} = T_n \psi_n. \tag{2}$$

We wish to find the compatibility condition of these two equations. Firstly observe that 2 gives

$$(\psi_n)_t = T_n \psi_n \implies (\psi_{n+1})_t = T_{n+1} \psi_{n+1}. \tag{3}$$

Next observe that taking a t derivative of 1 gives

$$(\psi_{n+1})_t = X_n (\psi_n)_t + (X_n)_t \psi_n, \tag{4}$$

substituting 3 into the LHS and 2 into the RHS of 4 gives

$$T_{n+1} \psi_{n+1} = (X_n T_n + (X_n)_t) \psi_n. \tag{5}$$

Finally plugging 1 into the LHS of 5 gives the compatibility condition to be

$$T_{n+1} X_n = X_n T_n + (X_n)_t. \tag{6}$$

Next we wish to verify that

$$X_n = \begin{pmatrix} z & q_n \\ q_n^* & 1/z, \end{pmatrix} T_n = \begin{pmatrix} iq_n q_{n-1}^* - \frac{i}{2}\left(1/z - z\right)^2 & \frac{i}{z} q_{n-1} - izq_n \\ -izq_{n-1}^* + \frac{i}{z} q_n^* & -iq_n^* q_{n-1} + \frac{i}{2}\left(1/z - z\right)^2 \end{pmatrix}$$

is a Lax Pair for the semi-discrete equation

$$i\frac{\partial q_n}{\partial t} = q_{n+1} - 2q_n + q_{n-1} - |q_n|^2(q_{n+1} + q_{n-1}).$$

Using the same method as described in **Question 1** we can use Mathematica to show that X_n and T_n satisfy the compatibility condition given by 6 for all $n \in \mathbb{Z}$.

$\square$

Problem 3 *For the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ with initial condition $u(x,0) = 0$ for $x \in (-\infty, -L) \cup (L, \infty)$, and $u(x,0) = d$ for $x \in (-L, L)$, with L and d both positive, consider the forward scattering problem.*

- *Find $a(k)$, for all time t.*
- *Knowing that the number of solitons emanating from the initial condition is the number of zeros of $a(k)$ on the positive imaginary axis (i.e., $k = i\kappa$, with $\kappa > 0$), discuss how many solitons correspond to the given initial condition, depending on the value of $2L^2d$. You might want to use Maple, Mathematica or Matlab for this.*
- *What happens for $d < 0$?*
- *In the limit $L \to 0$, but $2dL = \alpha$, $u(x,0) \to \alpha\delta(x)$. What happens to $a(k)$ when you take this limit? Discuss.*

Solution.
Consider the KdV equation $u_t + 6uu_x + u_{xxx}$ with the initial condition $u(x,0) = 0$ for $x \in (-\infty, -L) \cup (L, \infty)$ and $u(x,0) = d$ for $x \in (-L, L)$, with L and d both positive. We wish to apply forward scattering to this problem to study the behavior of the soliton solutions.
Finding a(k): The scattering data for KdV with the above initial conditions is given by

$$\begin{cases} \psi_{xx} + k^2\psi = 0 & x \in (-\infty, -L) \cup (L, \infty) \\ \psi_{xx} + (d + k^2)\psi = 0 & x \in (-L, L), \end{cases}$$

where we desire to have continuity at the boundaries. To find $a(k)$, recall that

$$a(k) = \frac{W(\phi, \varphi)}{2ik},$$

where ϕ and φ are solutions to the spatial Lax pair of KdV such that

$$\begin{aligned} \phi(x,k) &\sim e^{-ikx} \qquad x \to -\infty \\ \varphi(x,k) &\sim e^{ikx} \qquad x \to \infty. \end{aligned}$$

Let's first compute $\phi(x,k)$. For $x < -L$, by definition as $x \to -\infty$

$$\phi \to e^{-ikx}.$$

In addition, $\forall x < -L$ the differential equation is $\psi_{xx} + k^2\psi = 0$ which gives

$$\phi = e^{-ikx}.$$

Similarly for $x > L$,

$$\phi \to e^{ikx}.$$

Thus as $x \to \infty$ we have that

$$\phi = c_1 e^{ikx} + c_2 e^{-ikx},$$

where c_1 and c_2 are arbitrary constants. When $|x| < L$ the differential equation is

$$\phi_{xx} + (d + k^2)\phi = 0,$$

which we can directly solve to get

$$\phi = c_3 e^{i\sqrt{d+k^2}x} + c_4 e^{-i\sqrt{d+k^2}x},$$

where c_3 and c_4 are arbitrary constants. Combining these we get that

$$\phi = \begin{cases} e^{ikx} & x < -L \\ c_3 e^{i\sqrt{d+k^2}x} + c_4 e^{-i\sqrt{d+k^2}x} & |x| < L \\ c_1 e^{ikx} + c_2 e^{-ikx} & x > L \end{cases}.$$

Next we need to impose continuity of ϕ and ϕ_x at the boundaries which will give us our unknown constants. First consider at $x = -L$. That is $\lim_{x \to -L^-} \phi = \lim_{x \to -L^+} \phi$ which expands to

$$\lim_{x \to -L^-} \left(e^{-ikx}\right) = \lim_{x \to -L^+} \left(c_3 e^{i\sqrt{d+k^2}x} + c_4 e^{-i\sqrt{d+k^2}x}\right).$$

Thus we have that

$$e^{ikL} = c_3 e^{-i\sqrt{d+k^2}L} + c_4 e^{i\sqrt{d+k^2}L}. \tag{7}$$

And we need $\lim_{x \to -L^-} \phi_x = \lim_{x \to -L^+} \phi_x$ which expands to

$$\lim_{x \to -L^-} \left(-ike^{-ikx}\right) = \lim_{x \to -L^+} \left(i\sqrt{d+k^2} c_3 e^{i\sqrt{d+k^2}x} - i\sqrt{d+k^2} c_4 e^{-i\sqrt{d+k^2}x}\right).$$

Thus we have that

$$-ike^{ikL} = ic_3\sqrt{d+k^2} e^{-i\sqrt{d+k^2}L} - ic_4\sqrt{d+k^2} e^{i\sqrt{d+k^2}L}. \tag{8}$$

Similarly we need to impose continuity of ϕ and ϕ_x at $x = L$. That is $\lim_{x \to L^+} \phi = \lim_{x \to L^-} \phi$ which gives

$$c_1 e^{ikL} + c_2 e^{-ikL} = c_3 e^{i\sqrt{d+k^2}L} + c_4 e^{-i\sqrt{d+k^2}L}. \tag{9}$$

And enforcing $\lim_{x \to L^+} \phi_x = \lim_{x \to L^-} \phi_x$ gives

$$c_1 ike^{ikL} - c_2 ike^{-ikL} = c_3 i\sqrt{d+k^2} e^{i\sqrt{d+k^2}L} - c_6 i\sqrt{d+k^2} e^{-\sqrt{d+k^2}L}. \tag{10}$$

We can now solve the system of equations formed by 7, 8, 9, and 10 for c_1, c_2, c_3 and c_4 using Mathematica to get

$$\phi = \begin{cases} e^{ikx} & x < -L \\ c_3 e^{i\sqrt{d+k^2}x} + c_4 e^{-i\sqrt{d+k^2}x} & |x| < L \\ c_1 e^{ikx} + c_2 e^{-ikx} & x > L \end{cases},$$

where

$$\begin{aligned}
c_1 &= \frac{id \sin\left(2L\sqrt{d+k^2}\right)}{2k\sqrt{d+k^2}} \\
c_2 &= \frac{1}{2}e^{2ikL}\left(2\cos\left(2L\sqrt{d+k^2}\right) - \frac{i\left(d+2k^2\right)\sin\left(2L\sqrt{d+k^2}\right)}{k\sqrt{d+k^2}}\right) \\
c_3 &= \frac{\left(\sqrt{d+k^2}-k\right)e^{iL\left(\sqrt{d+k^2}+k\right)}}{2\sqrt{d+k^2}} \\
c_4 &= \frac{\left(\sqrt{d+k^2}+k\right)e^{-iL\left(\sqrt{d+k^2}-k\right)}}{2\sqrt{d+k^2}}.
\end{aligned}$$

Next let's find φ using a similar process as we did for ϕ. For $\varphi > L$ we have

$$\varphi = e^{ikx},$$

and for $\varphi < -L$

$$\varphi = c_5 e^{ikx} + c_6 e^{-ikx},$$

where c_5 and c_6 are arbitrary constants. When $|\varphi| < L$ we once again get

$$\varphi = c_7 e^{i\sqrt{d+k^2}x} + c_8 e^{-i\sqrt{d+k^2}x},$$

where c_7 and c_8 are arbitrary constants. Thus we have that

$$\varphi = \begin{cases} c_5 e^{ikx} + c_6 e^{-ikx} & x < -L \\ c_7 e^{i\sqrt{d+k^2}x} + c_8 e^{-i\sqrt{d+k^2}x} & |x| < L \\ e^{ikx} & x > L \end{cases}.$$

Imposing continuity $x = L$ for φ gives

$$e^{ikL} = c_7 e^{i\sqrt{d+k^2}L} + c_8 e^{-i\sqrt{d+k^2}L}, \tag{11}$$

and for φ_x gives

$$ike^{ikL} = ic_7\sqrt{d+k^2}e^{i\sqrt{d+k^2}L} - ic_8\sqrt{d+k^2}e^{-i\sqrt{d+k^2}L}. \tag{12}$$

Considering $x = -L$, forcing continuity for φ gives

$$c_5 e^{-ikL} + c_6 e^{ikL} = c_7 e^{-i\sqrt{d+k^2}L} + c_8 e^{i\sqrt{d+k^2}L}, \tag{13}$$

and for φ_x gives

$$ikc_5 e^{-ikL} - ikc_6 e^{ikL} = ic_7\sqrt{d+k^2}e^{-i\sqrt{d+k^2}L} - ic_8\sqrt{d+k^2}e^{i\sqrt{d+k^2}L}. \tag{14}$$

Using Mathematica to solve the system of equations formed by 11, 12, 13, and 14 for c_5, c_6, c_7 and c_8 gives

$$\varphi = \begin{cases} c_5 e^{ikx} + c_6 e^{-ikx} & x < -L \\ c_7 e^{i\sqrt{d+k^2}x} + c_8 e^{-i\sqrt{d+k^2}x} & |x| < L \\ e^{ikx} & x > L \end{cases},$$

where

$$\begin{aligned} c_5 &= \frac{1}{2} e^{2ikL} \left(2\cos\left(2L\sqrt{d+k^2}\right) - \frac{i\left(d+2k^2\right)\sin\left(2L\sqrt{d+k^2}\right)}{k\sqrt{d+k^2}} \right) \\ c_6 &= \frac{id\sin\left(2L\sqrt{d+k^2}\right)}{2k\sqrt{d+k^2}} \\ c_7 &= \frac{\left(\sqrt{d+k^2}+k\right) e^{-iL\left(\sqrt{d+k^2}-k\right)}}{2\sqrt{d+k^2}} \\ c_8 &= \frac{\left(\sqrt{d+k^2}-k\right) e^{iL\left(\sqrt{d+k^2}+k\right)}}{2\sqrt{d+k^2}}. \end{aligned}$$

Now we can use Mathematica to compute

$$a(k) = \frac{W(\phi, \varphi)}{2ik},$$

for each case of ϕ and φ and note that each case is equal to

$$a(k) = -\frac{ie^{2ikL}\left(\frac{\left(d+2k^2\right)\sin\left(2L\sqrt{d+k^2}\right)}{\sqrt{d+k^2}} + 2ik\cos\left(2L\sqrt{d+k^2}\right)\right)}{2k}.$$

Counting Solitons: Recall that the number of solitons corresponds to the number of zeros that $a(k)$ has on the positive imaginary axis. Thus let's plug $k = i\kappa$ for $\kappa \in \mathbb{R}$ s.t.$\kappa > 0$ into $a(k)$ and set it equal to zero yielding

$$a(i\kappa) = -\frac{e^{-2\kappa L}\left(\frac{\left(d-2\kappa^2\right)\sin\left(2L\sqrt{d-\kappa^2}\right)}{\sqrt{d-\kappa^2}} - 2\kappa\cos\left(2L\sqrt{d-\kappa^2}\right)\right)}{2\kappa} = 0.$$

To study when we will have zeros, first observe that if the $\sqrt{d-\kappa^2}$ is complex, then $d < k^2$ and $\sqrt{d-\kappa^2} = i * m$ for some real positive m. Thus $a(i\kappa)$ reduces to

$$a(i\kappa) = -\frac{e^{-2\kappa L}\left(-2\kappa\cosh(2Lm) + \frac{\left(d-2\kappa^2\right)\sinh(2Lm)}{m}\right)}{2\kappa} = 0.$$

Next we can divide out the exponential term and noting that both the denominators are positive, the express will be strictly positive if

$$2\kappa\cosh(2Lm) - \left(d-2\kappa^2\right)\sinh(2Lm) > 0. \tag{15}$$

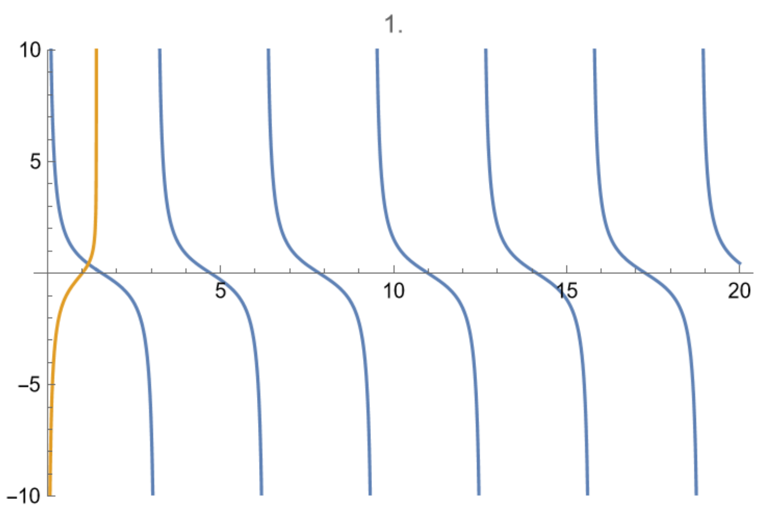
We have assumed that $L, m > 0$ and thus the cosh and sinh terms will be positive. We also have that κ is positive and thus the first term of 15 is positive. Lastly, under the assumption that the square root is complex, $d < k^2 \implies d - 2\kappa^2 < 0$ and thus the last term in 15 also positive. Therefore, 15 is strictly positive and thus has no roots under this assumption. So let's assume that $\sqrt{d-\kappa^2}$ is real and let it $\sqrt{d-\kappa^2} = \frac{s}{2L}$ which implies that $\kappa = \sqrt{d - \left(\frac{s}{2L}\right)^2}$. Then $a(i\kappa)$ becomes

$$a(i\kappa) = \frac{e^{-\sqrt{4dL^2-s^2}}\left(\sin(s)\left(2dL^2 - s^2\right) + s\cos(s)\sqrt{4dL^2-s^2}\right)}{s\sqrt{4dL^2-s^2}},$$

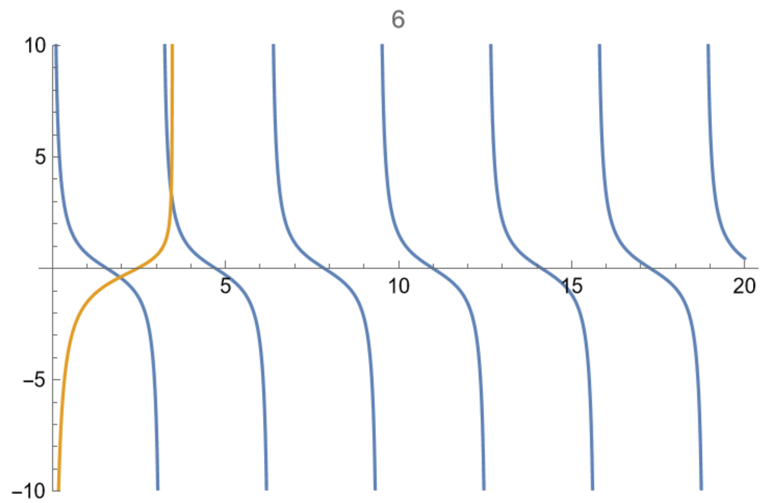
and setting it equal to zero yields

$$\begin{aligned}
0 &= \frac{e^{-\sqrt{4dL^2-s^2}}\left(\sin(s)\left(2dL^2 - s^2\right) + s\cos(s)\sqrt{4dL^2-s^2}\right)}{s\sqrt{4dL^2-s^2}} \\
\iff 0 &= \cos(s) + \frac{2dL^2 - s}{s\sqrt{4dL^2-s}}\sin(s) \\
\iff 0 &= \cot(s) + \frac{p - s^2}{s\sqrt{2p-s^2}} \\
\iff \cot(s) &= \frac{s^2 - p}{s\sqrt{2p-s^2}},
\end{aligned}$$

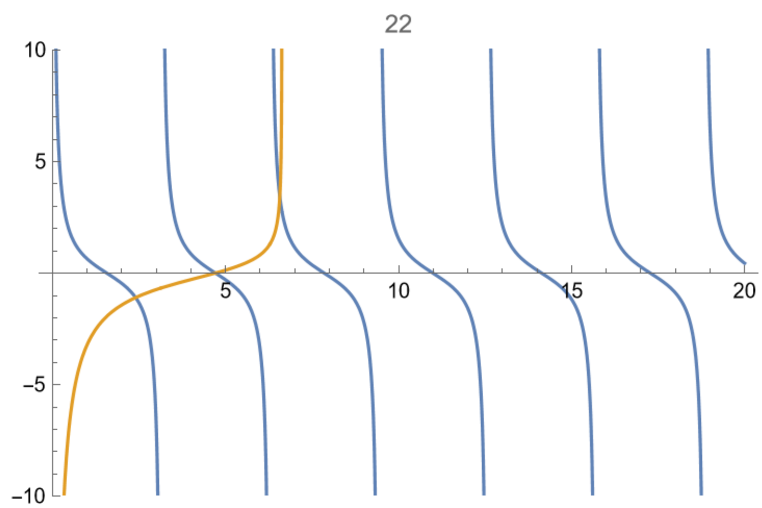
where $p = 2dL^2$. To find the zeros of $a(k)$ let's plot the $\cot(s) = \frac{s^2-p}{s\sqrt{2p-s^2}}$ using Mathematica and observe how p effects the number of solitons. When $p = 1$ we get the following graph



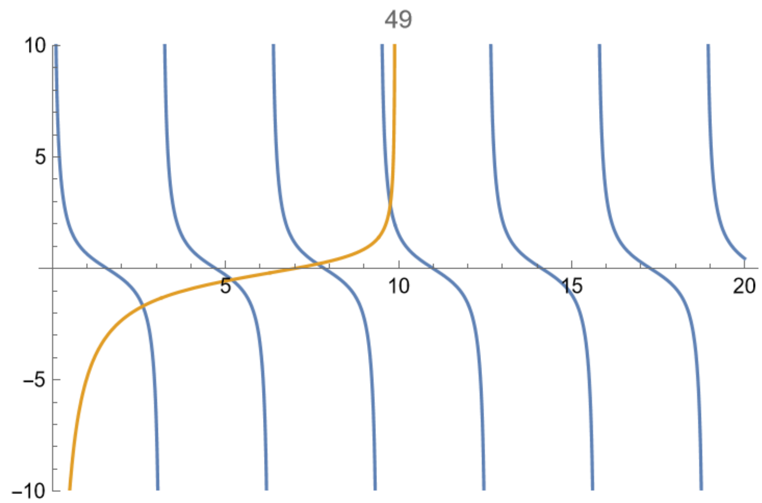
which shows that we have one soliton solution. As we increase p a new soliton emerges around $p = 6$, meaning that there are two solitons.



Continuing to increase p, $a(k)$ gains another soliton around $p = 22$ which gives three solitons.



Noting that the fourth soliton only emerges around $p = 49$, we have that number of solitons corresponding to the given initial condition increases as the value of $p = 2L^2d$ increases but we see that the rate of new emerging solitons decreases as p gets large.



What happens for negative d: In the case when $d < 0$,

$$a(i\kappa) = -\frac{e^{-2\kappa L}\left(\frac{\left(d-2\kappa^2\right)\sin\left(2L\sqrt{d-\kappa^2}\right)}{\sqrt{d-\kappa^2}} - 2\kappa\cos\left(2L\sqrt{d-\kappa^2}\right)\right)}{2\kappa} = 0.$$

will be strictly positive since we have already shown that $\sqrt{d-\kappa^2} \in \mathbb{C} \implies a(i\kappa) > 0$. Thus there are no soliton solutions when $d < 0$.

Limit as L goes to 0: Using Mathematica we can evaluate

$$\lim_{L\to 0} a(k) = 1 - \frac{i\alpha}{2k} = 1 + \frac{\alpha}{2ik} = \frac{\alpha + 2ik}{2ik},$$

under the transformation $2dL = \alpha$. Considering that taking the limit as $L \to 0$ is similar to restricting the initial condition to a delta function $u(x,0) = \alpha\delta(x)$, it is to no surprise that

$$\lim_{L\to 0} a(k) = \frac{\alpha + 2ik}{2ik},$$

based off the work we did in class. $\square$

Problem 4 *This is no problem.*

Solution. The solution is left as an exercise for the reader. □

Problem 5 The Liouville equation. *Consider the horribly nonlinear PDE*

$$u_{xy} = e^u,$$

known as Liouville's equation. Consider the transformation

$$\begin{aligned} v_x &= -u_x + \sqrt{2}e^{(u-v)/2}, \\ v_y &= u_y - \sqrt{2}e^{(u+v)/2}, \end{aligned}$$

where $u(x, y)$ satisfies Liouville's equation above.

- a *Find an equation satisfied by $v(x, y)$: $v_{xy} = \dots$. Your right-hand side cannot have any u's. Those should all be eliminated.*
- b *Write down the general solution for $v(x, y)$ from the equation you obtained.*
- c *Use this solution for v in your Bäcklund transformation and solve for u, obtaining the general solution of the Liouville equation!*

Solution. Consider the Liouville's equation

$$u_{xy} = e^u,$$

and consider the transformation

$$\begin{aligned} v_x &= -u_x + \sqrt{2}e^{(u-v)/2}, \\ v_y &= u_y - \sqrt{2}e^{(u+v)/2}, \end{aligned}$$

where $u(x, y)$ satisfies Liouville's equation above.

a First we wish to find an equation v_{xy} that satisfies $v(x, y)$. First observe that we can rewrite v_x as

$$v_x = -u_x + \sqrt{2}e^{(u-v)/2} \implies u_x = -v_x + \sqrt{2}e^{(u-v)/2},$$

and taking a y derivative of this gives

$$u_{xy} = -v_{xy} + \frac{\sqrt{2}}{2}e^{(u-v)/2}(u_y - v_y). \tag{16}$$

Similarly we can rewrite v_y as

$$v_y = u_y - \sqrt{2}e^{(u+v)/2} \implies u_y = v_y + \sqrt{2}e^{(u+v)/2},$$

and taking a x derivative of this gives

$$u_{yx} = v_{yx} + \frac{\sqrt{2}}{2}e^{(u+v)/2}(u_x + v_x). \tag{17}$$

Setting 16 and 17 equal under the assumption that the mixed derivatives are equal gives,

$$
\begin{aligned}
-v_{xy} + \frac{\sqrt{2}}{2}e^{(u-v)/2}(u_y - v_y) &= v_{yx} + \frac{\sqrt{2}}{2}e^{(u+v)/2}(u_x + v_x) \\
-2v_{xy} &= -\frac{\sqrt{2}}{2}e^{(u-v)/2}(u_y - v_y) + \frac{\sqrt{2}}{2}e^{(u+v)/2}(u_x + v_x) \\
4v_{xy} &= \sqrt{2}e^{(u-v)/2}(u_y - v_y) - \sqrt{2}e^{(u+v)/2}(u_x + v_x). \qquad (18)
\end{aligned}
$$

Recalling that our transformation gives

$$
\begin{aligned}
v_x = -u_x + \sqrt{2}e^{(u-v)/2} &\implies \sqrt{2}e^{(u-v)/2} = v_x + u_x \\
v_y = u_y - \sqrt{2}e^{(u+v)/2} &\implies \sqrt{2}e^{(u+v)/2} = u_y - v_y,
\end{aligned}
$$

we are able to rewrite 18 as

$$4v_{xy} = (v_x + u_x)(u_y - v_y) - (u_y - v_y)(u_x + v_x) = 0.$$

Therefore we have found that

$$v_{xy} = 0.$$

b Since $v_{xy} = 0$ we have that a general solution for $v(x, y)$ is of the form

$$v(x, y) = f(x) + g(y),$$

where $f(x)$ and $g(y)$ are arbitrary functions of x and y respectively.

c Finally let's plug the general form v back into the Bäcklung transformation and solve for u to get a general solution to the Liouville equation. Plugging the general solution of v into the transformation gives

$$
\begin{cases}
v_x &= -u_x + \sqrt{2}e^{(u-f(x)-g(y))/2} \\
v_y &= u_y - \sqrt{2}e^{(u+f(x)+g(y))/2}.
\end{cases}
$$

To get an integration factor, let's rewrite the system as following

$$
\begin{cases}
e^{-(u+f(x))/2}(u + f(x))_x &= \sqrt{2}e^{-g(y)/2}e^{-f(x)} \\
e^{-(u-g(y))/2}(u - g(y))_y &= \sqrt{2}e^{f(x)/2}e^{g(y)}
\end{cases},
$$

and after integrating we have

$$
\begin{cases}
-2e^{-(u+f(x))/2} &= \sqrt{2}e^{-g(y)/2}\left(\int e^{-f(x)}dx + c_1(y)\right) \\
-2e^{-(u-g(y))/2} &= \sqrt{2}e^{f(x)/2}\left(\int e^{g(y)}dy + c_2(x)\right),
\end{cases}
$$

where c_1 and c_2 are integration constants. Next we can take the ln of both sides and rewrite the system as following

$$\begin{cases} \frac{-u-f(x)}{2} & = \ln\left(\frac{-1}{\sqrt{2}} e^{-g(y)/2}\left(\int e^{-f(x)}dx + c_1(y)\right)\right) \\ \frac{-u+g(y)}{2} & = \ln\left(\frac{-1}{\sqrt{2}} e^{f(x)/2}\left(\int e^{g(y)}dy + c_2(x)\right)\right) \end{cases}$$

$$\implies \begin{cases} u & = -2\ln\left(\frac{1}{\sqrt{2}}\right) + g(y) - 2\ln\left(-\int e^{-f(x)}dx + c_1(y)\right) - f(x) \\ u & = -2\ln\left(\frac{1}{\sqrt{2}}\right) - f(x) - 2\ln\left(-\int e^{g(y)}dy + c_2(x)\right) + g(y) \end{cases}$$

$$\implies \begin{cases} u & = \ln(2) + g(y) - 2\ln\left(-\int e^{-f(x)}dx + c_1(y)\right) - f(x) \\ u & = \ln(2) - f(x) - 2\ln\left(-\int e^{g(y)}dy + c_2(x)\right) + g(y) \end{cases}.$$

Combining these equations gives the general solution of the Liouville equation to be

$$u = \ln(2) + g(y) - f(x) - 2\ln\left(-\int e^{-f(x)}dx - \int e^{g(y)}dy\right).$$

$\square$

Problem 6 The sine-Gordon equation. *Consider the sine-Gordon equation*

$$u_{xt} = \sin u,$$

also horribly nonlinear.

a *Show that the transformation*

$$v_x = u_x + 2\sin\frac{u+v}{2},$$
$$v_t = -u_t - 2\sin\frac{u-v}{2},$$

is an auto-Bäcklund transformation *for the sine-Gordon equation. In other words, v satisfies the same equation as u.*

b *Let $u(x,t)$ be the simplest solution of the sine-Gordon equation. With this $u(x,y)$ solve the auto-Bäcklund transformation for $v(x,t)$, to find a more complicated solution of the sine-Gordon equation. Congratulations! You just found the one-soliton solution of the sine-Gordon equation.*

Solution.

Consider the sine-Gordon equation

$$u_{xt} = \sin u.$$

a We wish to show that the transformation

$$v_x = u_x + 2\sin\frac{u+v}{2}, \tag{19}$$
$$v_t = -u_t - 2\sin\frac{u-v}{2}, \tag{20}$$

is an *auto-Bäcklund transformation* for the sine-Gordon equation. In other words, v satisfies the same equation as u. First let's take a t derivative of 19 to get

$$v_{xt} = u_{xt} + \cos\left(\frac{u+v}{2}\right)(u_t + v_t)$$

and plugging in the sine-Gordon equation gives

$$v_{xt} = \sin u + \cos\left(\frac{u+v}{2}\right)(u_t + v_t).$$

Solving 20 for u_t and plugging it into the previous equation gives

$$v_{xt} = \sin u + \cos\left(\frac{u+v}{2}\right)(-2\sin\left(\frac{u-v}{2}\right) - v_t + v_t)$$

$$
\begin{aligned}
&= \sin(u) - 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \\
&= \sin(u) - 2\left(\frac{\sin\left(\frac{u+v+u-v}{2}\right) - \sin\left(\frac{u+v-u+v}{2}\right)}{2}\right) \\
&= \sin(u) - \sin(u) + \sin(v) \\
&= \sin(v).
\end{aligned}
$$

Thus we have That

$$v_{xt} = \sin(v),$$

which verifies that v satisfies the same equation as u and thus the transformation given by 19 and 20 is an auto-Bäcklung transformation for the sine-Gordon equation.

b Next we wish to find the one-soliton solution of the sine-Gordon equation. Let's begin by letting $u(x,t) = 0$ which is the simplest solution to the sine-Gordon equation. Then the auto-Bäcklung transformation becomes

$$v_x = 2\sin\left(\frac{v}{2}\right) \tag{21}$$

$$v_t = -2\sin\left(\frac{-v}{2}\right). \tag{22}$$

Observe that solving 21 for v gives

$$\int \frac{v_x}{2\sin\left(\frac{v}{2}\right)}dx = \int 1dt.$$

Using Mathematica to evaluate this integral gives

$$\ln\left(\tan\left(\frac{v}{4}\right)\right) = t + c_1(x) \implies v = 4\tan^{-1}(e^{t+c_1(x)}),$$

where c_1 is an integration constant. Noticing that 22 can be rewritten in the form of 21 since

$$v_t = -2\sin\left(\frac{-v}{2}\right) = 2\sin\left(\frac{v}{2}\right) = v_x,$$

gives that solving 22 for v yields

$$v = 4\tan^{-1}(e^{t+c_2(t)}),$$

where c_2 is an integration constant. Furthermore, since $v_x = v_t$, $c_1 =$ and $c_2 = t$. Therefore

$$v(x,t) = 4\tan^{-1}\left(e^{x+t}\right),$$

which is the one-soliton solution of the sine-Gordon equation.

□