Math 568 Homework 7
Due March 3
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Problem 1 *Consider the Optical Parametric Oscillator as given in Lecture* 23 *of the notes.*

(a) *Assuming slow time $\tau = \epsilon^2 t$ and slow space $\xi = \epsilon x$, derive the Fisher-Kolmogorov equation for the slow evolution of the instability (the expression after Eq. (518))*

(b) *Derive the Swift-Hohenberg type expression which is governed by Eq. (519) with the scalings detailed in the notes.*

Solution.
Consider the Optical Parametric Oscillator (OPO) that is given by the dimensionless signal (U) and pump (V) of the form

$$
\begin{aligned}
U_t &= \frac{i}{2}U_{xx} + VU^* - (1 + i\Delta_1)U \\
V_t &= \frac{i}{2}\rho V_{xx} + U^2 - (\alpha + i\Delta_2)V + S,
\end{aligned}
$$

where Δ_1 and Δ_2 are the cavity detuning parameters, ρ is the diffraction ration between signal and pump fields, α is the pump-to-signal loss ratio, and S represents the external pumping term. We know that the stable uniform steady-state response of the OPO is given by

$$
\begin{aligned}
U &= 0 \\
V &= \frac{S}{\alpha + i\Delta_2}.
\end{aligned}
$$

Using linearly stability analysis, we find that the critical value of the pumping strength is given by

$$
S_c = (\alpha + i\Delta_2)(1 + i\Delta_1).
$$

To study the behavior near S_c, we use the following slow scales

$$
\begin{aligned}
\tau &= \epsilon^2 t \\
\xi &= \epsilon x,
\end{aligned}
$$

where $\epsilon^2 = |S - S_c| \ll 1$. Now we expand about the steady-state solution by

$$
\begin{aligned}
U &= 0 + \epsilon u(\tau, \xi) \\
V &= \frac{S}{\alpha + i\Delta_2} + \epsilon^2 v(\tau, \xi) \\
S &= S_c + \epsilon^2 C + \mathcal{O}(\epsilon^4),
\end{aligned}
$$

where C is a constant. Note that the chain rules are given by

$$\begin{aligned}U_t &= \epsilon^3 u_\tau\\ V_t &= \epsilon^4 v_\tau\\ U_{xx} &= \epsilon^3 u_{\xi\xi}\\ V_{xx} &= \epsilon^4 v_{\xi\xi}.\end{aligned}$$

Plugging these expansions into the first OPO equation yields

$$\begin{aligned}\epsilon^3 u_\tau &= \frac{i}{2}\epsilon^3 u_{\xi\xi} + \left(\frac{(\alpha+i\Delta_2)(1+i\Delta_1)+\epsilon^2 C}{\alpha+i\Delta_2} + \epsilon^2 v\right)\epsilon u^* - (1+i\Delta_1)\epsilon u + \mathcal{O}(\epsilon^4)\\
\implies \epsilon(1+i\Delta_1)(u-u^*) &= \epsilon^3\left(\frac{i}{2}u_{\xi\xi} - u_\tau + vu^* + \frac{C}{\alpha+i\Delta_2}u^*\right) + \mathcal{O}(\epsilon^4)\\
\implies (1+i\Delta_1)(u-u^*) &= \epsilon^2\left(\frac{i}{2}u_{\xi\xi} - u_\tau + vu^* + \frac{C}{\alpha+i\Delta_2}u^*\right) + \mathcal{O}(\epsilon^4) \qquad (1)\\
\implies u^* &= u - \frac{\epsilon^2}{1+i\Delta_1}\left(\frac{i}{2}u_{\xi\xi} - u_\tau + vu^* + \frac{C}{\alpha+i\Delta_2}u^*\right) + \mathcal{O}(\epsilon^4). \qquad (2)\end{aligned}$$

Similarly, we can plug in the second OPO equation yields

$$\begin{aligned}\epsilon^4 v_\tau &= \frac{i}{2}\rho\epsilon^4 v_{\xi\xi} + \epsilon^2 u - (\alpha+i\Delta_2)\left(\frac{(\alpha+i\Delta_2)(1+i\Delta_1)+\epsilon^2 C}{\alpha+i\Delta_2} + \epsilon^2 v\right)\\
&\quad + (\alpha+i\Delta_2)(1+i\Delta_1) + \epsilon^2 C + \mathcal{O}(\epsilon^4)\\
\implies (\alpha+i\Delta_2)v &= -u^2 + \epsilon^2\left(\frac{i}{2}\rho v_{\xi\xi} - v_\tau\right) + \mathcal{O}(\epsilon^4) \qquad (3)\\
\implies v &= -\frac{u^2}{\alpha+i\Delta_2} + \frac{\epsilon^2}{\alpha+i\Delta_2}\left(\frac{i}{2}\rho v_{\xi\xi} - v_\tau\right) + \mathcal{O}(\epsilon^4). \qquad (4)\end{aligned}$$

Now plugging Equation 4 into itself yields

$$\begin{aligned}v &= -\frac{u^2}{\alpha+i\Delta_2} + \frac{\epsilon^2}{\alpha+i\Delta_2}\left(-\frac{i}{2}\rho\frac{(u^2)_{\xi\xi}}{\alpha+i\Delta_2} + \frac{(u^2)_\tau}{\alpha+i\Delta_2}\right) + \mathcal{O}(\epsilon^4),\\
&= -\frac{u^2}{\alpha+i\Delta_2} + \frac{\epsilon^2}{(\alpha+i\Delta_2)^2}\left((u^2)_\tau - \frac{i}{2}\rho(u^2)_{\xi\xi}\right) + \mathcal{O}(\epsilon^4). \qquad (5)\end{aligned}$$

Then multiplying Equation 5 by u^* yields

$$vu^* = -\frac{1}{\alpha+i\Delta_2}|u|^2 u + \frac{\epsilon^2}{(\alpha+i\Delta_2)^2}\left((u^2)_\tau u^* - \frac{i}{2}\rho(u^2)_{\xi\xi}u^*\right) + \mathcal{O}(\epsilon^4),$$

and plugging in Equation 2 gives

$$vu^* = -\frac{1}{\alpha+i\Delta_2}|u|^2 u + \frac{\epsilon^2}{(\alpha+i\Delta_2)^2}\left((u^2)_\tau u - \frac{i}{2}\rho(u^2)_{\xi\xi}u\right) + \mathcal{O}(\epsilon^4). \qquad (6)$$

Now plugging 6 into 1 yields

$$
\begin{aligned}
R &:= \epsilon^2\left(\frac{i}{2}u_{\xi\xi} - u_\tau - \frac{1}{\alpha + i\Delta_2}|u|^2 u + \frac{C}{\alpha + i\Delta_2}u^*\right) \\
&\quad + \epsilon^4\left(\frac{1}{(\alpha + i\Delta_2)^2}\left((u^2)_\tau u - \frac{i}{2}\rho(u^2)_{\xi\xi}u\right)\right) + \mathcal{O}(\epsilon^6). \qquad (7)
\end{aligned}
$$

Enforcing the Fredholm-Alternative theorem on the RHS of Equation 7 for the leading order, gives the solvability condition

$$
(1 - i\Delta_1)R + (1 + i\Delta_1)R^* = (R + R^*) + i\Delta_1(R^* - R) = 0.
$$

From Equation 2, we have that $u = u^*$ at leading order, thus we have that

$$
\begin{aligned}
R &= \epsilon^2\left(\frac{i}{2}u_{\xi\xi} - u_\tau - \frac{(\alpha - i\Delta_2)u^3}{\alpha^2 + \Delta_2^2} + \frac{(\alpha - i\Delta_2)Cu}{\alpha^2 + \Delta_2^2}\right) + \mathcal{O}(\epsilon^4), \\
R^* &= \epsilon^2\left(-\frac{i}{2}u_{\xi\xi} - u_\tau - \frac{(\alpha + i\Delta_2)u^3}{\alpha^2 + \Delta_2^2} + \frac{(\alpha + i\Delta_2)Cu}{\alpha^2 + \Delta_2^2}\right) + \mathcal{O}(\epsilon^4),
\end{aligned}
$$

then

$$
\begin{aligned}
R + R^* &= \epsilon^2\left(-2u_\tau - \frac{2\alpha u^3}{\alpha^2 + \Delta_2^2} + \frac{2\alpha Cu}{\alpha^2 + \Delta_2^2}\right), \\
R^* - R &= -i\epsilon^2\left(u_{\xi\xi} + \frac{2\Delta_2 u^3}{\alpha^2 + \Delta_2^2} - \frac{2\Delta_2 Cu}{\alpha^2 + \Delta_2^2}\right), \\
\implies i\Delta_1(R^* - R) &= \epsilon^2\Delta_1\left(u_{\xi\xi} + \frac{2\Delta_2 u^3}{\alpha^2 + \Delta_2^2} - \frac{2\Delta_2 Cu}{\alpha^2 + \Delta_2^2}\right).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&0 = (R + R^*) + i\Delta_1(R^* - R) \\
\implies &0 = \epsilon^2\left(-2u_\tau - \frac{2\alpha u^3}{\alpha^2 + \Delta_2^2} + \frac{2\alpha cu}{\alpha^2 + \Delta_2^2} + \Delta_1 u_{\xi\xi} + \frac{2\Delta_1\Delta_2 u^3}{\alpha^2 + \delta_2^2} - \frac{2\Delta_1\Delta_2 Cu}{\alpha^2 - \Delta_1^2}\right) \\
\implies &0 = u^3 - cu + \frac{\Delta_1(\alpha^2 + \Delta_2^2)}{2(\Delta_1\Delta_2 - \alpha)}u_{\xi\xi} - \frac{\alpha^2 + \Delta_2^2}{\Delta_1\Delta_2 - \alpha}u_\tau.
\end{aligned}
$$

Now if we let $u = \left(\frac{\alpha^2+\Delta_2^2}{\Delta_1\Delta_2-\alpha}\right)^{1/2}\varphi$ then we get

$$
\begin{aligned}
&\left(\frac{\alpha^2 + \Delta_2^2}{\Delta_1\Delta_2 - \alpha}\right)^{3/2}\varphi^3 - c\left(\frac{\alpha^2 + \Delta_2^2}{\Delta_1\Delta_2 - \alpha}\right)\varphi + \frac{\Delta_1}{2}\left(\frac{\alpha^2 + \Delta_2^2}{\Delta_1\Delta_2 - \alpha}\right)^{3/2}\varphi_{\xi\xi} - \left(\frac{\alpha^2 + \Delta_2^2}{\Delta_1\Delta_2 - \alpha}\right)^{3/2}\varphi_\tau = 0, \\
\implies &\varphi^3 - \left(\frac{\alpha^2 + \Delta_2^2}{\Delta_1\Delta_2 - \alpha}\right)\varphi + \frac{\Delta_1}{2}\varphi_{\xi\xi} - \varphi_\tau = 0.
\end{aligned}
$$

Finally, if we let $\zeta = \left(\frac{\Delta_1}{2}\right)^{1/2}$ and $\gamma = \frac{|C|(\Delta_1\Delta_2-\alpha)}{\alpha^2+\Delta_2^2}$ then we get the Fisher-Kolmogorov equation

$$\varphi^3 \mp \gamma\varphi + \varphi_{\xi\xi} - \varphi_\tau = 0.$$

□