Homework 2

The following problems, whenever coming from the textbook, refer to the book "Numerical Analysis" (second edition) by Walter Gautschi.

Theoretical problems:

T1. Problem 4 on page 201 of [Gautschi].

T2. Find a two-point Gauss quadrature to approximate

$$\int_0^1 \frac{e^x}{\sqrt{x}}\,\mathrm{d}x.$$

T3. Suppose that the interval $[a, b]$ is divided into equal subintervals of length h each such that $n = (b-a)/h$ is even. Denote by R_1 the result of applying the composite trapezoidal method with step size $2h$ and by R_2 the result of applying the same method with step size h. Show that one application of Richardson extrapolation, reading

$$S = \frac{4R_2 - R_1}{3}$$

yields the composite Simpson method.

Coding problems (attach the code you used to generate the results):

C1. One can approximate the derivative of a function $f(x)$ by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}. \tag{1}$$

In class, we derived that

$$\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| = O(h).$$

Let $f(x) = \sin(x)$.

(a) Use (1) to approximate $f'(x)$ at $x_0 = 1.2$. Take `h = 10.^i; i = 0:-1:-16` and plot the absolute error $\left| f'(x_0) - \frac{f(x_0+h)-f(x_0)}{h} \right|$ vs. h using `loglog`. Does the plot behave as you expect? Explain.

(b) By the trig identity $\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$, one has

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{2\cos(x_0 + \frac{h}{2})\sin(\frac{h}{2})}{h}.$$

Use this formula to approximate $f'(x)$ at $x_0 = 1.2$. Make the same plot as in (a). Does the plot behave as you expect? Explain.

C2. Suppose we know

$$\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x = \pi.$$

Use the composite midpoint rule, composite trapezoidal rule, and composite Simpson's rule to approximate the above integral. Plot the error of each method w.r.t. `h = 10.^i; i = -1:-1:-8`.