AMATH 586 SPRING 2023
HOMEWORK 1 — DUE APRIL 7 ON GRADESCOPE BY 11PM

Be sure to do a `git pull` to update your local version of the `amath-586-2023` repository.
Homeworks must be typeset and uploaded to `Gradescope` for submission.
Code should be uploaded to `GitHub`

Problem 1: In this exercise you will show convergence for a discretization of

$$\begin{cases} -u''(x) = g(x), \\ u'(0) = \alpha, \\ u(1) = \beta. \end{cases}$$

(a) Consider the $(m+1) \times (m+1)$ matrix

$$A = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Find its Cholesky decomposition.

(b) Show that

$$A^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} m+1 \\ m \\ \vdots \\ 1 \end{bmatrix}. \tag{1}$$

(c) Now show that

$$\|A^{-1}\|_1 \leq (m+1)^2, \quad \|A^{-1}\|_\infty \leq (m+1)^2.$$

Problem 2: Consider the matrix (see (2.54) in LeVeque)

$$L = h^{-2} \begin{bmatrix} h & -h & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

(a) Compute L^{-1} in terms of A^{-1} and compute bounds for $\|L^{-1}\|_1$ and $\|L^{-1}\|_\infty$.
(b) Explain why this is not enough to imply convergence for the one-sided approach, (2.53) in LeVeque.
(c) Use (1) to show the method converges in both the grid 1-norm and the ∞-norm.

Problem 3: Consider $u(x) = \cos(k\pi x)\exp(-x^2)$. Determine g, α and β such that

$$\begin{cases} -u''(x) = g(x), \\ u(0) = \alpha, \\ u(1) = \beta. \end{cases}$$

(a) Modify the code in `LinearBVP.ipynb` to solve this BVP using a second-order accurate method. Plot errors on a log-log scale for $k = 1, 2, 3, 4$.

(b) Modify the code in `LinearBVP.ipynb` to solve this BVP using a fifth-order accurate method. Plot errors on a log-log scale for $k = 1, 2, 3, 4$.

Problem 4: Modify the code in `NonlinearBVP.ipynb` to solve

$$\begin{cases} w'(x) - \epsilon w'''(x) = 0, \\ w(0) = 0, \\ w(L) = 0, \\ w'(L) = 1. \end{cases}$$

Demonstrate the convergence rate by comparing the computed solution to the true solution for $\epsilon = 0.1, 0.01$.