Math 569 Homework 6
Due May 31
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Problem 1 *Consider the sound waves governed by*

$$\frac{\partial^2}{\partial t^2}\Psi = c^2\nabla^2\Psi, \quad \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

in a circular cylinder of radius a and length L with

$$\Psi = 0 \text{ at } r = a; \ \Psi = 0 \text{ at } z = 0, L.$$

Assume that the sound produced in this tube is symmetric, i.e. no θ dependence. Find the lowest three frequencies. Take $c = 300m/s$, $a = 1cm$, and $L = 0.5m$.

Solution.
Consider the sound waves governed by

$$\frac{\partial^2}{\partial t^2}\Psi = c^2\nabla^2\Psi, \quad \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

in a circular cylinder of radius a and length L with

$$\Psi = 0 \text{ at } r = a; \ \Psi = 0 \text{ at } z = 0, L.$$

We will assume that the sound produced in this tube is symmetric, i.e. there is no θ dependence. Let's use the separation of variables method to solve this PDE so we assume that

$$\Psi(r,\theta,z,t) = T(t)\psi(r,z).$$

Plugging this into our PDE and rearranging terms yields

$$\frac{T''}{c^2T} = \frac{\nabla^2\psi}{\psi} = -\lambda^2,$$

where λ is a positive constant. This breaks down into two problems

$$\frac{T''}{c^2T} = -\lambda^2,$$

and

$$\frac{\nabla^2\psi}{\psi} = -\lambda^2.$$

To solve the first equation, we rearrange the terms to get

$$T'' + c^2\lambda^2T = 0.$$

Solve for T gives

$$T(t) = A\sin(c\lambda t) + B\cos(c\lambda t) = A\sin(\omega t) + B\cos(\omega t).$$

Now we return to the spatial problem. Note that we removed the θ dependence by our assumption and thus we take

$$\psi(r, z) = R(r)Z(z),$$

allowing us to use the separation of variables. This gives

$$\frac{\nabla^2\psi}{\psi} = \frac{1}{rR}\frac{\partial}{\partial r}(rR') + \frac{Z''}{Z} = -\lambda^2,$$

which is only true if

$$\frac{Z''}{Z} = -\lambda^2 - \frac{1}{rR}\frac{\partial}{\partial r}(rR') = -\xi^2,$$

where ξ is a positive constant. Now we can solve for Z using the same method present to solve T to get

$$Z(z) = C\sin(\xi z) + D\cos(\xi z).$$

Now we can apply the boundary condition that $\Psi = 0$ at $z = 0$ which gives that $D = 0$. Next, we apply the boundary condition that $\Psi = 0$ at $z = L$ which gives

$$Z(L) = C\sin(\xi L) = 0,$$

and thus

$$\xi = \frac{n\pi}{L},$$

for $n = 1, 2, \ldots$. Thus we have found that

$$Z_n = \sin\left(\frac{n\pi}{L}z\right),$$

where we let $C = 1$. Now returning to the R equation we have

$$\frac{1}{rR}\frac{\partial}{\partial r}(rR') = \xi^2 - \lambda^2 = -\alpha^2.$$

Then rearranging the terms we get

$$R'' + \frac{1}{r}R' + \alpha^2 R = 0.$$

Now applying the change of variables $s = \alpha r$, we get

$$s^2R''(s) + sR'(s) + s^2R(s) = 0.$$

Noticing that this is a Bessel differential equation we get the solutions

$$R(r) = c_1J_0(\alpha r) + c_2Y_0(\alpha r),$$

where J_0 and Y_0 are Bessel functions of the first and second kind respectively. Now applying the boundary condition $\Psi = 0$ at $r = a$, we see that $c_2 = 0$ since $Y_0 \to -\infty$ as $t \to 0$. Then taking $c_1 = 1$ we get

$$R(r) = J_0(\alpha r).$$

Note that we require $R(a) = 0$ and thus we need the zeros of the Bessel function of the first kind J_0. We will later compute these numerically but for now, we consider the solution to be

$$R_m(r) = J_0(\alpha r).$$

Thus we have the solution

$$\Psi_{n,m}(r, \theta, z, t) = (A \sin(\omega t) + B \cos(\omega t)) \sin\left(\frac{n\pi}{L} z\right) R_m(r),$$

or

$$\Psi = \sum_{n,m=1}^{\infty} (A \sin(\omega t) + B \cos(\omega t)) \sin\left(\frac{n\pi}{L} z\right) R_m(r).$$

Now recalling that

$$-\lambda^2 + \xi^2 = -\alpha^2 \implies \lambda^2 = \xi^2 + \alpha^2,$$

and thus

$$\omega_{n,m} = c\lambda_{n,m} = c\sqrt{\xi_n^2 + \alpha_n^2}.$$

The zeros of $J_0(x)$ are well known and using a good book about Bessel functions we find the first few zeros to be

$$\begin{cases} s = 2.40482558, & m = 1, \\ s = 5.52007811, & m = 2, \\ s = 8.653727913, & m = 3. \end{cases}$$

Since $a = 0.01$ we have

$$\begin{cases} \alpha_1 = 240.482556, \\ \alpha_2 = 552.007811, \\ \alpha_3 = 865.372791. \end{cases}$$

We are searching for the three lowest frequencies and noticing that $\xi_1 = 2\pi, \xi_2 = 4\pi,$ and $\xi_3 = 6\pi$, and thus the three lowest frequencies are given by α_1 and $\xi_{1,2,3}$. Therefore we have found the three lowest frequencies to be

$$\begin{cases} \omega_{1,1} = 300\sqrt{(2\pi)^2 + (240.482556)^2} \approx 72169\text{Hz}, \\ \omega_{2,1} = 300\sqrt{(4\pi)^2 + (240.482556)^2} \approx 72243\text{Hz}, \\ \omega_{3,1} = 300\sqrt{(6\pi)^2 + (240.482556)^2} \approx 72366\text{Hz}, \end{cases}$$

$\square$

Problem 2 *Consider the wave function Ψ for an electron of mass μ in a sphere surrounded by an infinite potential at a radius a from the nucleus, which just means that $\Psi = 0$ at $r = a$.*

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2\mu}\nabla^2\Psi.$$

Find the energy levels for the symmetric case, where Ψ does not depend on θ and ϕ. Your answer should be exact and in terms of the parameters given.

Solution.
Consider the wave function Ψ for an electron of mass μ in a sphere surrounded by an infinite potential at a radius a from the nucleus, which just means that $\Psi = 0$ at $r = a$.

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2\mu}\mathrm{d}^2\Psi.$$

We wish to find the energy levels for the symmetric case and thus Ψ does not depend on θ and ϕ. We will use the separation of variables method to solve this PDE so we assume that

$$\Psi = T(t)\psi.$$

Plugging this back into the PDE and rearranging terms yields

$$-\frac{2i\mu}{\hbar}\frac{T'}{T} = \frac{\nabla^2\psi}{\psi} = -\lambda^2,$$

where λ is a positive constant. This gives the ODE

$$T' + \frac{i\hbar\lambda^2}{2\mu}T = 0,$$

which has the solution

$$T = Ae^{\frac{-i\hbar\lambda^2}{2\mu}t}.$$

Now returning to

$$\frac{\nabla^2\psi}{\psi} = -\lambda^2,$$

where we note that ψ only depends on r since we are in the symmetric case. Thus we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}(()r^2R') = -\lambda^2R.$$

If we preform the change of variable $s = \lambda r$ we get

$$s^2R'' + 2sR' + s^2R = 0.$$

Noticing that this is the spherical Bessel function, we find the solution to be

$$R(s) = Bj_0(s) + Cy_0(s),$$

where j_0 and y_0 are the spherical Bessel functions of the first and second kind respectively. Since the boundary condition requires $\Psi = 0$ at $r = a$ we require $C = 0$ and thus taking $B = 1$ we have

$$R(r) = j_0(\lambda r).$$

Now recalling that

$$j_0 = \frac{\sin(x)}{x},$$

we have

$$R(r) = \frac{\sin(\lambda r)}{\lambda r}.$$

Now enforcing the boundary condition $\Psi = 0$ at $r = a$ we get

$$\frac{\sin(\lambda a)}{\lambda a} = 0 \implies \sin(\lambda a) = 0 \implies \lambda_n = \frac{n\pi}{a},$$

under the assumption that $\lambda \neq 0$ and for $n = 1, 2, \ldots$. From the class notes, we know that the quantized energy of a particle is given by

$$E_n = \frac{\hbar^2}{2\mu}\lambda_n^2.$$

Therefore the energy levels in the symmetric case are

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2\mu a^2}, \quad n = 1, 2, \ldots.$$

Note that the solution to the original PDE is

$$\Psi = \sum_{n=1}^{\infty} e^{-\frac{iE_n}{\hbar}t} j_0\left(\frac{n\pi}{a}r\right).$$

$\square$

Problem 3 *Consider the Legendre's equation:*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right)+n(n+1)y(x)=0,\quad -1\le x\le 1,$$

with the initial condition that $y(\pm 1)$ *are bounded. The solutions are the Legendre polynomials,* $P_n(x)$*, which are given by the Rodrigue's formula:*

$$P_n(x)=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

Compute the first four coefficients in the Legendre expansion (similar to Fourier sine or cosine series expansion):

$$f(x)=\sum_{n=0}^{\infty}a_nP_n(x),\ \textit{where}\ a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)\mathrm{d}x,$$

for

$$f(x)=\begin{cases}0 \textit{ for } -1<x<0,\\ x \textit{ for } 0<x<1.\end{cases}$$

Plot the approximation of the sum consisting of one, two, three and four terms along with the original function $f(x)$*.*

Solution.

We wish to compute the first four coefficients in the Legendre expansion of the form

$$f(x)=\sum_{n=0}^{\infty}a_nP_n(x),$$

where

$$a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)\mathrm{d}x,$$

and

$$P_n(x)=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n,$$

for

$$f(x)=\begin{cases}0 \text{ for } -1<x<0,\\ x \text{ for } 0<x<1.\end{cases}$$

The first coefficient in the expansion is

$$\begin{aligned}a_0&=a_0P_0(x)\\ &=\frac{1}{2}\int_{-1}^{1}f(x)(1)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-1}^{0} 0\mathrm{d}x + \frac{1}{2}\int_{0}^{1} x\mathrm{d}x \\
&= \frac{1}{4}.
\end{aligned}
$$

Thus the first term in the Legendre expansion is

$$f_0(x) = \frac{1}{4}.$$

The second coefficient in the expansion is

$$
\begin{aligned}
a_1 &= a_1 P_1(x) \\
&= \frac{3}{2}\int_{-1}^{1} f(x)\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\mathrm{d}x \\
&= \frac{3}{2}\int_{0}^{1} (x)\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\right)\mathrm{d}x \\
&= \frac{3}{2}\left(\int_{0}^{1} x^2\mathrm{d}x\right) \\
&= \frac{3}{2}\left(\frac{1}{3}\right) \\
&= \frac{1}{2}
\end{aligned}
$$

Thus the second term in the Legendre expansion is

$$f_1(x) = \frac{1}{2}x.$$

The third coefficient in the expansion is

$$
\begin{aligned}
a_2 &= a_2 P_2(x) \\
&= \frac{5}{2}\int_{0}^{1} x\left(\frac{1}{2}(3x^2-1)\right)\mathrm{d}x \\
&= \frac{5}{4}\int_{0}^{1} 3x^3 - x\mathrm{d}x \\
&= \frac{5}{4}\left(\frac{3}{4}-\frac{1}{2}\right) \\
&= \frac{5}{16}.
\end{aligned}
$$

Thus the third term in the Legendre expansion is

$$f_2(x) = \frac{5}{16}\left(\frac{1}{2}(3x^2-1)\right) = \frac{5}{32}\big(3x^2-1\big).$$

The fourth coefficient in the expansion is

$$\begin{aligned}
a_3 &= a_3 P_3(x) \\
&= \frac{7}{2}\int_0^1 x\left(\frac{1}{2}(5x^3 - 3x)\right)\mathrm{d}x \\
&= \frac{7}{4}\int_0^1 5x^4 - 3x^2\mathrm{d}x \\
&= \frac{7}{4}(1-1) \\
&= 0.
\end{aligned}$$

Thus the fourth term in the Legendre expansion is

$$f_3(x) = 0,$$

so let's continue to the fifth term. The fifth coefficient in the expansion is

$$\begin{aligned}
a_4 &= a_4 P_4(x) \\
&= \frac{9}{2}\int_0^1 x\left(\frac{1}{8}\left(35x^4 - 30x^2 + 3\right)\right)\mathrm{d}x \\
&= \frac{9}{16}\int_0^1 35x^5 - 30x^3 + 3x\mathrm{d}x \\
&= \frac{9}{16}\left(\frac{35}{6} - \frac{30}{4} + \frac{3}{2}\right) \\
&= -\frac{3}{32}.
\end{aligned}$$

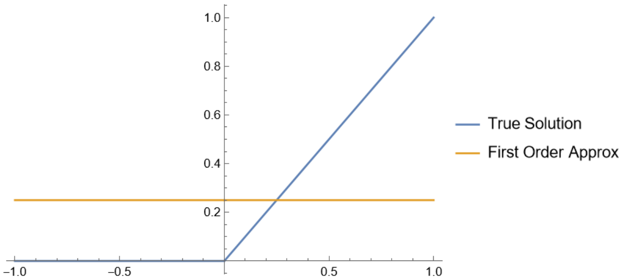
Thus the fifth term in the Legendre expansion is

$$f_4(x) = \frac{-3}{256}\left(35x^4 - 30x^2 + 3\right),$$

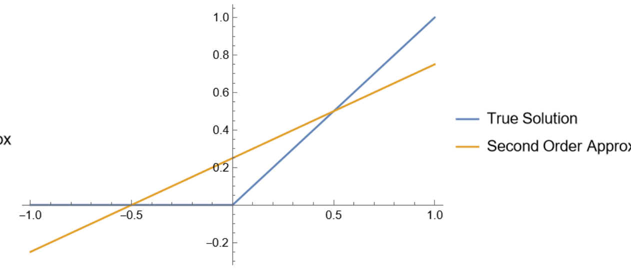
Therefore the Legendre expansion with four terms is given by

$$f(x) = \frac{1}{4} + \frac{1}{2}x + \frac{5}{32}\left(3x^2 - 1\right) - \frac{3}{256}\left(35x^4 - 30x^2 + 3\right) + \cdots.$$

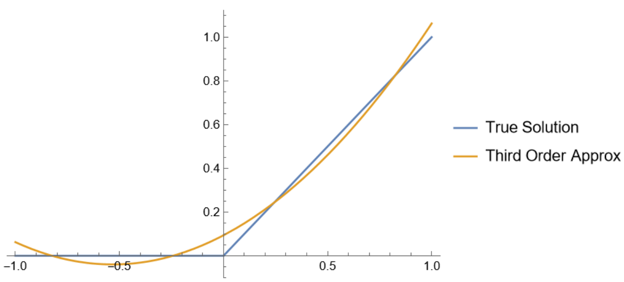
Plotting the true solution along with each order approximation we get the figures seen in Figure 1.



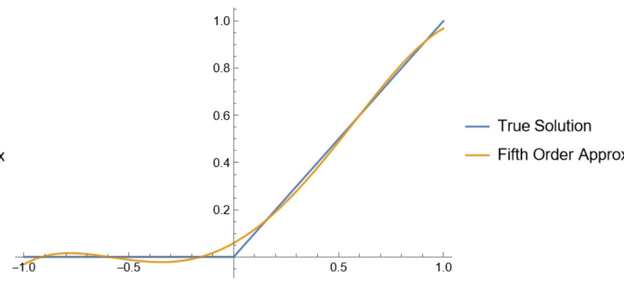
(a) Approximation with one term.



(b) Approximation with two terms.



(c) Approximation with three terms.



(d) Approximation with four terms.

Figure 1: True solution to Legendre's equation seen in blue while different order Legendre expansion approximations in orange.

$\square$