Math 567 Homework 6
Due November 11 2022
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Problem 1 **(a)** *Let $\hat{f}(s)$ and $\hat{g}(s)$ be the Laplace transforms of one-sided functions $f(t)$ and $g(t)$, respectively. Show that the inverse Laplace transform $\hat{f}(s)\hat{g}(s)$ is;*

$$\int_0^t f(t-\tau)d\tau$$

(b) *Use Laplace transform and the result in (a) to solve the following ordinary differential equation:*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y + 4y = f(t),$$

subject to the initial conditions:

$$y(0) = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 0.$$

Solution.

(a) Let $\hat{f}(s)$ and $\hat{g}(s)$ be defined by

$$\hat{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-sx} f(x)dx$$

$$\hat{g}(s) = \mathcal{L}[g(t)] = \int_0^\infty e^{-s\tau} g(\tau)d\tau.$$

Then we have that

$$\begin{aligned}\hat{f}(s)\hat{g}(s) &= \int_0^\infty e^{-sx} f(x)dx \int_0^\infty e^{-s\tau} g(\tau)d\tau \\ &= \int_0^\infty \int_0^\infty e^{-s(x+\tau)} f(x)g(\tau)dxd\tau,\end{aligned}$$

and if we let $t = x+\tau$, then $x = t-\tau$ and we get that

$$\int_0^\infty \int_\tau^\infty e^{-st} f(t-\tau)g(\tau)dtd\tau.$$

Swapping the order of integration we get

$$\hat{f}(s)\hat{g}(s) = \int_0^\infty \int_0^t e^{-st} f(t-\tau)g(\tau)d\tau dt$$

$$
\begin{aligned}
&= \int_0^\infty e^{-st} \int_0^t f(t-\tau)g(\tau)d\tau dt \\
&= \mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right].
\end{aligned}
$$

Now we can see that when we take the inverse Laplace we get

$$
\mathcal{L}^{-1}\left[\hat{f}(s)\hat{g}(s)\right] = \mathcal{L}^{-1}\left[\mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]\right] = \int_0^t f(t-\tau)g(\tau)d\tau.
$$

(b) Consider the ODE

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2}y + 4y = f(t),
$$

subject to the initial conditions:

$$
y(0) = 0, \ \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 0.
$$

Let's use the Laplace transform on each part of our ODE,

$$
\begin{aligned}
\mathcal{L}y'' &= \int_0^\infty e^{-st} y'' dt \\
&= e^{-st}y'\Big|_0^\infty + s\int_0^\infty e^{-st}y' dt \\
&= s^2\mathcal{L}[y] - sy(0) - y'(0) \\
&= s^2\mathcal{L}[y],
\end{aligned}
$$

where $sy(0) = y'(0) = 0$ due to the initial conditions. We also have that

$$
\mathcal{L}[4y] = 4\mathcal{L}[y] = 4\hat{y}(s) \ \text{ and } \ \mathcal{L}[f(t)] = \hat{f}(s).
$$

Plugging this values into our ODE we get

$$
\begin{aligned}
s^2\mathcal{L}y + 4\mathcal{L}y &= \mathcal{L}f(t) \\
\mathcal{L}y(s^2+4) &= \mathcal{L}f(t) \\
\hat{y}(s) &= \frac{1}{(s^2+4)}\hat{f}(s).
\end{aligned}
$$

Recall that the Laplace transform of $g(t) = \frac{1}{2}\sin(2t)$ is $\mathcal{L}[g(t)] = \hat{g}(s) = \frac{1}{s^2+4}$. Thus we can rewrite the ODE as

$$
\hat{y}(s) = \frac{1}{(s^2+4)}\hat{f}(s) \to \hat{y}(s) = \hat{g}(s)\hat{f}(s).
$$

From part (a) we know how to take inverse Laplace of $\hat{g}(s)\hat{f}(s)$ and thus we have that

$$
\begin{aligned}
y(t) &= \int_0^t g(t-\tau)f(\tau)d\tau \\
&= \int_0^t \frac{1}{2}\sin(2(t-\tau))f(\tau)d\tau.
\end{aligned}
$$

□

Problem 2 *Solve the following Laplace equation*

$$\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = 0,$$

in the upper half place: $-\infty < x < \infty$, $o < y < \infty$, *subject to the boundary conditions:*

$$\phi \to 0 \text{ as } y \to \infty;\ \phi \to 0 \text{ as } x \to \pm\infty;\ \phi(x,0) = \frac{x}{x^2+a^2}.$$

Hint: You can use Fourier transform in x *or Laplace transform in* y*.*

Solution. Consider the Laplace equation

$$\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = 0,$$

in the upper half place $-\infty < x < \infty$, $o < y < \infty$, subject to the boundary conditions:

$$\phi \to 0 \text{ as } y \to \infty;\ \phi \to 0 \text{ as } x \to \pm\infty;\ \phi(x,0) = \frac{x}{x^2+a^2}.$$

To solve the equation, let's assume that ϕ is integrable and take a Fourier transform in x. Let

$$U(\lambda, y) = \mathcal{F}[\phi(x,y)] = \int_{-\infty}^{\infty} e^{i\lambda x}\phi(x,y)dx.$$

Applying this to the Laplace equation we get

$$\mathcal{F}[\phi_{yy}] = \mathcal{F}[\phi_{xx}] = U_{yy}$$

where

$$\mathcal{F}[\phi_{yy}] = \frac{\partial^2}{\partial y^2}\mathcal{F}[U] = U_[yy],$$

and

$$\begin{aligned}
\mathcal{F}[\phi_{xx}] &= \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial^2}{\partial x^2} \\
&= e^{i\lambda x}\frac{\partial}{\partial x}\phi\Big|_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial}{\partial x}\phi dx \\
&= e^{i\lambda x}\frac{\partial}{\partial x}\phi\Big|_{-\infty}^{\infty} - i\lambda e^{i\lambda x}\phi\Big|_{-\infty}^{\infty} + (i\lambda)^2 U \\
&= -\lambda^2 U,
\end{aligned}$$

assuming that $\phi_x \to 0$ as $x \to \pm\infty$ and from the boundary condition $\phi \to 0$ as $x \to \pm\infty$. Thus the Laplace equation becomes,

$$\frac{\partial^2}{\partial y^2}U - \lambda^2 U = 0.$$

Applying the characteristic equation get

$$U'' - \lambda^2 U = 0 \to r^2 - \lambda^2 = 0 \implies r^2 = \lambda^2,$$

and thus $r = \pm\lambda$. Therefore a solution to the equation is given by

$$U(\lambda, y) = A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y}.$$

Now let's transform the boundary condition $\phi(x, 0)$ which $\mathcal{F}[\phi(x, 0)] = U(\lambda, 0)$ and $\phi \to 0$ as $x \to \pm\infty$ implies that $U(\lambda, 0) \to 0$ as $x \to \pm\infty$ and $y \to \infty$. Observe that when $\lambda > 0$, to maintain these conditions, $A(\lambda) = 0$, while when $\lambda < 0$, we have that $B(\lambda) = 0$. Taking the Fourier transform gives

$$\begin{aligned}\mathcal{F}[\phi(x, o)] &= \mathcal{F}\left[\frac{x}{x^2 + a^2}\right] \\ &= \int_{-\infty}^{\infty} e^{i\lambda x}\frac{x}{x^2 + a^2}dx.\end{aligned}$$

Now let

$$I = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{x}{x^2 + a^2}dx,$$

and since $\left|\frac{x^2}{x^2+a^2}\right| \to 0$ as $x \to \infty$, we can apply Jordan's Lemma with the three following cases: $\lambda = 0, \lambda > 0$, and $\lambda < 0$.

Consider when $\lambda = 0$. Then

$$I = \int_{-\infty}^{\infty} \frac{x}{x^2 + a^2}dx = 0,$$

since we have an odd function over symmetric bounds.

Consider when $\lambda > 0$ consider the contour that is a half circle in the upper plane going from $-R$ to R. Then by Jordan's Lemma

$$I = \lim_{R\to\infty} \oint_C e^{i\lambda z}\frac{z}{z^2 + a^2}dz.$$

We have two simple poles at $\pm ai$ of which only $z = ai$ is within the contour. Let's compute the residue at this point

$$\begin{aligned}\text{Res}(ai) &= \lim_{z\to ai}(z - ai)\left(e^{i\lambda z}\frac{z}{z^2 + a^2}\right) \\ &= \lim_{z\to ai} e^{i\lambda z}\frac{z}{z^2 + ai}\end{aligned}$$

$$
\begin{aligned}
&= e^{i\lambda ai}\frac{ai}{2ai} \\
&= \frac{1}{2}e^{-\lambda a}.
\end{aligned}
$$

Thus by Residue Theorem we have

$$
I = 2\pi i\left(\frac{1}{2}e^{-\lambda a}\right) = \pi i e^{-\lambda a}.
$$

Consider when $\lambda < 0$ consider the contour that is a half circle in the lower plane going from $-R$ to R. Then by Jordan's Lemma

$$
\begin{aligned}
I &= \lim_{R\to\infty}\int_{-\infty}^{\infty} e^{i\lambda z}\frac{z}{z^2+a^2}dz \\
&= \lim_{R\to\infty}\int_{R}^{-R} e^{i\lambda z}\frac{z}{z^2+a^2}dz \\
&= -\lim_{R\to\infty}\oint_C e^{i\lambda z}\frac{z}{z^2+a^2}dz
\end{aligned}
$$

We have two simple poles at $\pm ai$ of which only $z = -ai$ is within the contour. Let's compute the residue at this point

$$
\begin{aligned}
\mathrm{Res}(-ai) &= \lim_{z\to -ai}(z+ai)\left(e^{i\lambda z}\frac{z}{z^2+a^2}\right) \\
&= \lim_{z\to -ai} e^{i\lambda z}\frac{z}{z^2-ai} \\
&= e^{i\lambda(-ai)}\frac{-ai}{-2ai} \\
&= \frac{1}{2}e^{\lambda a}.
\end{aligned}
$$

Thus by Residue Theorem we have

$$
I = -2\pi i\left(\frac{1}{2}e^{\lambda a}\right) = -\pi i e^{\lambda a}.
$$

Now collecting these we get that

$$
\begin{aligned}
U(\lambda, 0) = \mathcal{F}[\phi(x,0)] = \mathcal{F}\left[\frac{x}{x^2+a^2}\right] &\\
&= \int_{-\infty}^{\infty} e^{i\lambda x}\frac{x}{x^2+a^2}dx \\
&= \mathrm{sgn}(\lambda)\pi i e^{-|\lambda| a}.
\end{aligned}
$$

Plugging this back into our transformed Laplace equation gives

$$U(\lambda, y) = \operatorname{sgn}(\lambda)\pi i e^{-|\lambda|a} e^{-|\lambda|y}.$$

Now we lets take the inverse

$$\begin{aligned}
\phi(x, y) &= \mathcal{F}^{-1} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{sgn}(\lambda)\pi i e^{-|\lambda|a} e^{-|\lambda|y} d\lambda \\
&= \lim_{\delta\to 0}\left(\frac{i}{2}\int_{\delta}^{\infty} e^{-i\lambda x}e^{-\lambda a}e^{-\lambda y}d\lambda - \frac{i}{2}\int_{-\infty}^{-\delta} e^{-i\lambda x}e^{\lambda a}e^{\lambda y}d\lambda\right) \\
&= \lim_{\delta\to 0}\left(\frac{i}{2}\int_{\delta}^{\infty} e^{-\lambda(ix+a+y)}d\lambda - \frac{i}{2}\int_{-\infty}^{-\delta} e^{\lambda(-ix+a+y)}d\lambda\right) \\
&= \lim_{\delta\to 0}\left(\frac{i}{2}\cdot\frac{-1}{ix+a+y}e^{-\lambda(ix+a+y)}\Big|_{\delta}^{\infty}\right) - \lim_{\delta\to 0}\left(\frac{i}{2}\cdot\frac{1}{-ix+a+y}e^{\lambda(-ix+a+y)}\Big|_{-\infty}^{-\delta}\right) \\
&= \frac{i}{2}\left(\frac{1}{ix+a+y} - \frac{1}{-ix+a+y}\right) \\
&= \frac{i}{2}\left(\frac{-2ix}{x^2+y^2+a^2+2ay}\right) \\
&= \frac{x}{x^2+(a+y)^2}.
\end{aligned}$$

And indeed we see that ϕ is integrable and $\phi_x \to 0$ as $x \to \pm\infty$ as desired. Thus we have found our solution.

$\square$

Problem 3 *Use Fourier transform to solve the following wave equation:*

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u, \quad -\infty < x < \infty, 0 < t < \infty,$$

subject to the initial condition: $u(x,0) = 0$, $\frac{\partial}{\partial t}u = \delta(x)$ *at* $t = 0$ *and boundary conditions:* $u(x,t) \to 0$ *as* $x \to \pm\infty$.

Solution.

Consider the wave equation given by

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u, \quad -\infty < x < \infty, 0 < t < \infty,$$

subject to the initial condition: $u(x,0) = 0$, $\frac{\partial}{\partial t}u = \delta(x)$ at $t = 0$ and boundary conditions: $u(x,t) \to 0$ as $x \to \pm\infty$. Let's take the Fourier transform in x to simplify this problem. Let

$$U(\lambda,t) = \int_{-\infty}^{\infty} e^{i\lambda x}u(x,t)dx = \mathcal{F}[u(x,t),]$$

under the assumption that $u(x,t)$ is integrable. Then the Fourier transform of the wave equation is

$$\mathcal{F}[u_{tt}] = c^2\mathcal{F}[u_{xx}],$$

where

$$\mathcal{F}[u_{tt}] = \frac{\partial^2}{\partial t^2}U,$$

and

$$\begin{aligned}
\mathcal{F}[u_{xx}] &= \int_{-\infty}^{\infty} e^{i\lambda x}u_{xx}dx \\
&= u_x e^{i\lambda x}\big|_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}u_x dx \\
&= u_x e^{i\lambda x}\big|_{-\infty}^{\infty} - i\lambda\Big(ue^{i\lambda x}\big| - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}u dx\Big) \\
&= u_x e^{i\lambda x}\big|_{-\infty}^{\infty} - i\lambda\, ue^{i\lambda x}\big|_{-\infty}^{\infty} - \lambda^2 U \\
&= -\lambda^2 U,
\end{aligned}$$

assuming that $u_x \to 0$ as $x \to \pm\infty$ and from the boundary condition $u \to 0$ as $x \to \pm\infty$. Thus the wave equation becomes

$$\frac{\partial^2}{\partial t^2}U = -\lambda^2 U,$$

which we know there is a solution of the form,

$$U(\lambda,t) = A(\lambda)\cos(c\lambda t) + B(\lambda)\sin(c\lambda t).$$

Next let's Fourier transform the initial conditions,

$$\mathcal{F}[u(x,0)] = \mathcal{F}[0] = 0 = U(\lambda, 0),$$

and

$$\mathcal{F}[u_t(x,0)] = \mathcal{F}[\delta(x)] = U_t(\lambda, 0) = 1.$$

Applying these to our solution we get

$$U_t(\lambda, t) = -c\lambda A(\lambda)\sin(c\lambda t) + c\lambda B(\lambda)\cos(c\lambda t),$$

which gives

$$U_t(\lambda, 0) = 1 = c\lambda B(\lambda) \implies B(\lambda) = \frac{1}{c\lambda}.$$

And in combination of the other boundary condition we get $A(\lambda) = 0$ and thus we have

$$U(\lambda, t) = \frac{1}{c\lambda}\sin(c\lambda t).$$

Next let's take the inverse transform

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[U(\lambda, t)] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} U(\lambda, t) d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{c\lambda}\sin(c\lambda t) d\lambda. \end{aligned}$$

We know how to evaluate this integral from Homework 5 question 2,

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\left(\frac{1}{2}(\operatorname{sgn}(x+ct) - \operatorname{sgn}(x-ct))\right) \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(y) dy. \end{aligned}$$

This $u(x,t)$ is indeed integrable and $u_x \to 0$ as $x \to \pm\infty$ since for sufficiently large $|x|$, we have that $u = 0$. Thus we have our solution. $\square$