Math 568 Homework 5
Due February 19
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Problem 1 *Consider the singular equation*

$$\epsilon y'' + (1+x)^2 y' + y = 0,$$

with $y(0) = y(1) = 1$ *and with* $0 < \epsilon \ll 1$.

(a) *Obtain a uniform approximation which is valid to* $\mathcal{O}(\epsilon)$, *i.e. determine the leading order behavior and first correction.*

(b) *Show that assuming the boundary layer to be at* $x = 1$ *is inconsistent. (hint: use the stretched inner variable* $\xi = (1-x)/\epsilon$.*)*

(c) *Plot the uniform solution for* $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution.
Consider the equation

$$\epsilon y'' + (1+x)^2 y' + y = 0,$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$ which is a singular equation with $b(x) = (1+x)^2$ and $c(x) = 1$. Since $b(x)$ is always positive in the interval, we expect the boundary layer to be on the left.

(a) We first wish to find the uniform approximation which is valid to $\mathcal{O}(\epsilon)$.

Outer Problem: We first consider the outer problem with the expansion

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots,$$

and plugging these into the equation and collecting terms we get

$$\begin{aligned} \mathcal{O}(1): &\quad y_0 + y_{0_x} + 2xy_{0_x} + x^2 y_{0x} = 0, \\ \mathcal{O}(\epsilon): &\quad y_1 + y_{1_x} + 2xy_{1_x} + x^2 y_{1_x} = -y_{0_{xx}}, \end{aligned}$$

with

$$\begin{aligned} y_0(1) &= 1, \\ y_1(1) &= 0. \end{aligned}$$

Since we are considering the outer region, we only apply the right side boundary condition. Then the leading order solution is given by

$$y_0 = e^{-\frac{1}{2}+\frac{1}{1+x}} = y_{\text{out}}.$$

Inner Problem: For the inner problem, we introduce the stretched variable

$$\xi = \frac{x}{\epsilon}.$$

Note that this transformation gives the following chain rules

$$\begin{aligned} y_x &= \frac{1}{\epsilon} y_\xi, \\ y_{xx} &= \frac{1}{\epsilon^2} y_{\xi\xi}. \end{aligned}$$

Plugging this into the equation and multiplying by ϵ yields

$$y_{\xi\xi} + (1 + \xi\epsilon)^2 y_\xi + \epsilon y = 0,$$

with $y(0) = 1$. Now introducing the perturbation expansion

$$y(\xi) = y_0(\xi) + \epsilon y_1(\xi) + \epsilon^2 y_2(\xi) + \cdots,$$

and collecting terms yields the hierarchy of equations in the inner region to be

$$\begin{aligned} \mathcal{O}(1): &\quad y_{0_\xi} + y_{0_{\xi\xi}} = 0, \\ \mathcal{O}(\epsilon): &\quad y_{1_\xi} + y_{1_{\xi\xi}} = -y_0 + 2\xi y_{x_\xi}, \end{aligned}$$

with

$$\begin{aligned} y_0(0) &= 1, \\ y_1(0) &= 0. \end{aligned}$$

Solving the leading order problem gives the solution

$$y_0 = 1 + A - Ae^{-\xi} = y_{\text{in}}.$$

Matching: We finally need to match the inner and outer solutions. To do so, observe that

$$\begin{aligned} \lim_{x\to 0} y_{\text{out}} &= \lim_{\xi\to\infty} y_{\text{in}}, \\ \lim_{x\to 0} e^{-\frac{1}{2}+\frac{1}{1+x}} &= \lim_{\xi\to\infty} 1 + A - Ae^{-\xi}, \\ e^{\frac{1}{2}} &= 1 + A, \\ \implies A &= e^{\frac{1}{2}} - 1. \end{aligned}$$

Thus we have $y_{match} = e^{1/2}$. Therefore the uniform solution to be boundary layer problem is

$$\begin{aligned} y &= y_{\text{in}} + y_{\text{out}} - y_{\text{match}} \\ &= e^{\frac{1}{x+1}-\frac{1}{2}} - \left(\sqrt{e} - 1\right) e^{-\frac{x}{\epsilon}}. \end{aligned}$$

(b) To show that assuming the boundary layer to be at $x = 1$ is inconsistent, consider the stretched variable

$$\xi = \frac{1-x}{\epsilon},$$

which corresponds to a boundary layer at the desired location. Note that the change of variable yields

$$y_x = -\frac{1}{\epsilon} y_\xi,$$
$$y_{xx} = \frac{1}{\epsilon^2} y_{\xi\xi}.$$

Plugging the transformation into the equation and multiplying by ϵ yields

$$y_{\xi\xi} - (2 - \xi\epsilon) y_\xi + \epsilon y = 0,$$

with $y(0) = 1$. Now introducing the perturbation expansion

$$y(\xi) = y_0(\xi) + \epsilon y_1(\xi) + \epsilon^2 y_2(\xi) + \cdots,$$

and collecting terms yields the hierarchy of equations in the inner region to be

$$\begin{aligned} \mathcal{O}(1): \quad & -4y_{0_\xi} + y_{0_{\xi\xi}} = 0, \\ \mathcal{O}(\epsilon): \quad & -4y_{1_\xi} + y_{1_{\xi\xi}} = -y_0 - 4\xi y_{x_\xi}, \end{aligned}$$

with

$$\begin{aligned} y_0(0) &= 1, \\ y_1(0) &= 0. \end{aligned}$$

Solving the leading order problem gives the solution

$$y_0 = \frac{1}{4} B e^{4\xi} - \frac{B}{4} + 1 = y_{\text{in}}.$$

Since the outer solution will remain the same, we will use the same y_{out} to match the solutions

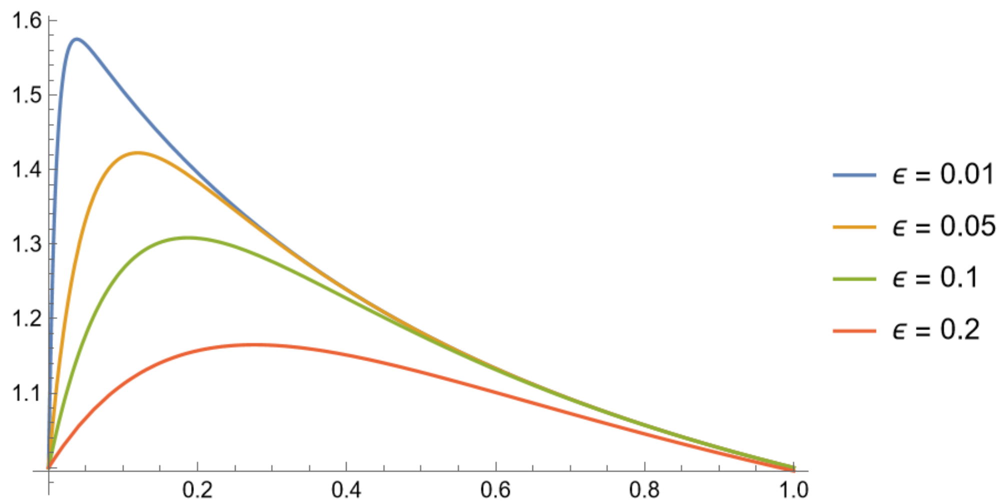
$$\begin{aligned} \lim_{x\to 1} y_{\text{out}} &= \lim_{\xi\to\infty} y_{\text{in}}, \\ \lim_{x\to 1} e^{-\frac{1}{2}+\frac{1}{1+x}} &= \lim_{\xi\to\infty} \frac{1}{4} B e^{4\xi} - \frac{B}{4} + 1, \end{aligned}$$

and thus we choose $B = 0$ to prevent y_{in} from blowing up as $\xi \to \infty$. Then $y_{\text{match}} = 1$ and we get the uniform solution to be

$$y_{\text{unif}} = y_{\text{in}} + y_{\text{out}} - y_{\text{match}} = 1 + e^{-\frac{1}{2}+\frac{1}{1+x}} - 1 = e^{-\frac{1}{2}+\frac{1}{1+x}}.$$

But this solution does not satisfy the boundary condition for the leading order terms at 0 since $y_{\text{unif}}(0) = \sqrt{e} \neq 1$.

(c) Using Mathematica we can plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$ which gives the following figure.



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Problem 2 *Consider the singular equation*

$$\epsilon y'' - x^2 y' - y = 0,$$

with $y(0) = y(1) = 1$ *and with* $0 < \epsilon \ll 1$.

(a) *With the method of dominant balance, show that there are three distinguished limits:* $\xi = \epsilon^{1/2}, \xi = \epsilon, \xi = 1$ *(the outer problem). Write down each of the problems in the various distinguished limits.*

(b) *Obtain the leading order uniform approximation (hint: there are boundary layers at* $x = 0$ *and* $x = 1$.*)*

(c) *Plot the uniform solution for* $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution.

Consider the equation

$$\epsilon y'' - x^2 y' - y = 0,$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$.

(a) First we wish to use the method of dominant balance to find the three distinguished limits. Notice that this is a singular problem since ϵ is in front of the highest derivative. Furthermore, $b(x) = -x^2$ and $c(x) = -1$. As $b(x) < 0$ everywhere except at zero, we expect boundary layers at $x = 1$ and $x = 0$ since $b(0) = 0$.

First let's consider the distinguished limit near $x = 0$. Consider the stretched variable

$$\xi = \frac{x}{\delta}.$$

The change of variable gives the chain rule

$$y_x = y_\xi \xi_x = \frac{1}{\delta} y_\xi,$$
$$y_{xx} = \frac{1}{\delta^2} y_{\xi\xi}.$$

Plugging these in and multiplying through by δ^2 yields

$$\epsilon y_{\xi\xi} + \delta^3 \xi^2 y_\xi - \delta^2 y = 0.$$

Dropping the smallest term (the second) gives the dominant balance as

$$\epsilon y_{\xi\xi} - \delta^2 y = 0.$$

To balance this we require

$$\delta = \epsilon^{1/2}.$$

Therefore there is a boundary layer at $x = 0$ which has characteristic width $\mathcal{O}(\epsilon^{1/2})$.

Next let's study the distinguished limit near $x = 1$ with the stretched variable

$$\xi = \frac{1-x}{\delta}.$$

The change of variable gives the chain rule

$$\begin{aligned} y_x &= -\frac{1}{\delta} y_\xi, \\ y_{xx} &= \frac{1}{\delta^2} y_{\xi\xi}. \end{aligned}$$

Plugging these in and multiplying through by δ^2 yields

$$\epsilon y_{\xi\xi} - \delta(\delta\xi - 1)^2 y_\xi - \delta^2 y = 0.$$

Dropping the smallest terms gives the leading order equation

$$\epsilon y_{\xi\xi} + \delta y_\xi = 0.$$

to balance this we require

$$\delta = \epsilon.$$

Therefore there is a boundary layer at $x = 1$ with characteristic width $\mathcal{O}(\epsilon)$.

(b) *Outer Solution* - the outer solution is given by plugging the expansion

$$y = y_0 + \epsilon y_1 + \cdots,$$

into the governing equation yields the leading order equation

$$-x^2 y_{0_x} - y_0 = 0,$$

whose solution is

$$y_0 = Ce^{1/x} = y_{\text{out}}.$$

Inner Solution - There are two boundary layers for the inner problem. Let's first consider the inner solution near $x = 0$ with $\xi = x/\epsilon^{1/2}$ which has a dominant solution

$$y_{\xi\xi} - y = 0.$$

Using the expansion

$$y = y_0 + \epsilon y_1 + \cdots,$$

We get the leading order solution to be

$$y_{0_{\xi\xi}} - y_0 = 0,$$

which has the solution

$$Ae^{-\xi} + Be^{\xi} = 0.$$

For matching, we require our solution to be bounded and thus we set $B = 0$ and applying the boundary condition $y_0(0) = 1$ gives the leading order solution

$$y_0(\xi) = e^{-\xi} = y_{\text{in (left)}}.$$

Next let's consider the inner solution near $x = 1$ with $\xi = (1 - x)/\epsilon$ which has a dominant solution of

$$y_{\xi\xi} - y_\xi = 0.$$

Using the expansion

$$y = y_0 + \epsilon y_1 + \cdots ,$$

We get the leading order solution to be

$$y_{0_{\xi\xi}} + y_{0_\xi} = 0,$$

which has the solution

$$y_0 = Ae^{-\xi} + (1 - A) = y_{\text{in (right)}}.$$

Matching - Now we must match our solutions on the left and right. Let's first consider $x = 0$ which gives us

$$\begin{aligned} \lim_{x\to 0} u_{\text{out}} &= \lim_{x\to\infty} u_{\text{in(left)}} \\ \lim_{x\to 0} Ce^{1/x} &= \lim_{x\to\infty} e^{-x/\epsilon^{1/2}}, \end{aligned}$$

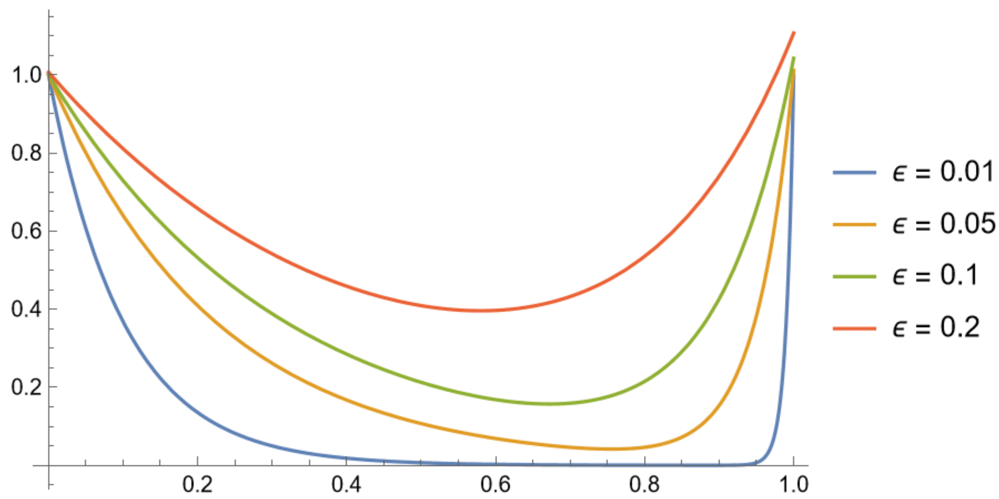
and to bound the solution of the outer, we require $C = 0$. Thus $u_{\text{out}} = 0$. Next we match the solution near $x = 1$ and thus

$$\begin{aligned} \lim_{x\to 1} u_{\text{out}} &= \lim_{x\to\infty} u_{\text{in(right)}} \\ 0 &= \lim_{x\to\infty} Ae^{-\xi} + (1 - A), \end{aligned}$$

which is only true if $A = 1$. The uniform solution is of the form

$$\begin{aligned} u &= u_{\text{out}} + u_{\text{in (left)}} + u_{\text{in (right)}} - u_{\text{match (left)}} - u_{\text{match (left)}} \\ u &= 0 + e^{-\frac{x}{\epsilon^{1/2}}} + e^{-\frac{1-x}{\epsilon}} - 0 \\ u &= \exp\left(-\frac{x}{\epsilon^{1/2}}\right) + \exp\left(-\frac{1-x}{\epsilon}\right). \end{aligned}$$

(c) Using Mathematica we can plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$ which gives the following figure.


1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
ϵ = 0.01
ϵ = 0.05
ϵ = 0.1
ϵ = 0.2

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