# Math 567 Homework 4
Due October 27
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**Problem 1** *Using residue calculus, calculate*

$$I = \int_{-\infty}^{\infty} \frac{\sin(x)}{\sinh x} dx.$$

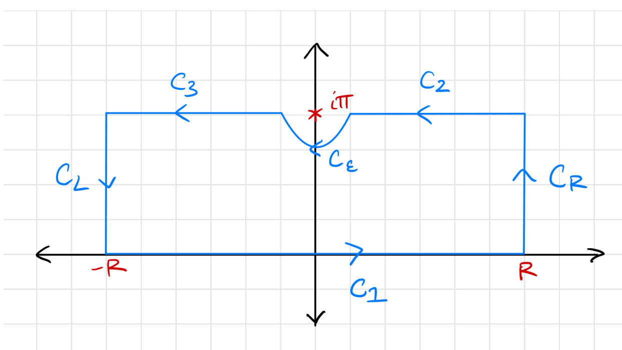
*Solution.*
To evaluate the real integral,

$$I = \int_{-\infty}^{\infty} \frac{\sin(x)}{\sinh x} dx = \fint_{-\infty}^{\infty} \frac{\sin(x)}{\sinh x} dx,$$

we will consider the integral in the complex plane with,

$$\oint_C \frac{\sin(z)}{\sinh z} dz = \left( \int_{C_1} + \int_{C_R} + \int_{C_2} + \int_{C_\epsilon} + \int_{C_3} + \int_{C_L} \right) \frac{\sin(z)}{\sinh z} dz,$$

where $C$ is the contour described in the following figure.




We know that the function is analytic on and inside $C$ without the discontinuity at $z = 0$, we have that

$$\oint_C \frac{\sin(z)}{\sinh z} dz = 0,$$

by Cauchy's Theorem. Next consider $C_R$,

$$\int_{C_R} \frac{\sin(z)}{\sinh z} = \int_0^{\pi} \frac{\sin(R + iy)}{\sinh (R + iy)} i dy = \int_0^{\pi} \frac{e^{iR}e^{-y} - e^{-iR}e^{y}}{e^{R}e^{iy} - e^{-R}e^{-iy}} dy.$$

And since,

$$\lim_{R\to\infty} \left| \frac{e^{iR}e^{-y} - e^{-iR}e^{y}}{e^{R}e^{iy} - e^{-R}e^{-iy}} \right| \leq \lim_{R\to\infty} \frac{\left|e^{iR}\right||e^{-y}| + \left|e^{-iR}\right||e^{y}|}{||e^{R}||e^{iy}| - |e^{-R}||e^{-iy}||}$$

$$
\begin{aligned}
&= \lim_{R\to\infty} \frac{e^{-y}+e^{y}}{|e^{R}-e^{-R}|}\\
&= 0,
\end{aligned}
$$

for $0 \le y \le \pi$. Thus we have that,

$$\lim_{R\to\infty} \int_{C_R} \frac{\sin(z)}{\sinh(z)} dz = 0.$$

Next notice that $C_L$,

$$\int_{C_L} \frac{\sin(z)}{\sinh(z)} dz = \int_{\pi}^{0} \frac{\sin(-R+iy)}{\sinh(-R+iy)} = \int_{\pi}^{0} \frac{e^{iR}e^{-y}-e^{iR}e^{y}}{e^{-R}e^{iy}-e^{R}e^{-iy}} dy.$$

Since

$$
\begin{aligned}
\lim_{R\to\infty} \left| \frac{e^{iR}e^{-y}-e^{iR}e^{y}}{e^{-R}e^{iy}-e^{R}e^{-iy}} \right| &\le \lim_{R\to\infty} \frac{\left|e^{iR}\right||e^{-y}|+\left|e^{iR}\right||e^{y}|}{||e^{-R}||e^{iy}|-|e^{R}||e^{-iy}||}\\
&= \lim_{R\to\infty} \frac{e^{-y}+e^{y}}{|e^{-R}-e^{R}|}\\
&= 0,
\end{aligned}
$$

for $0 \le y \le \pi$ we have shown that,

$$\int_{C_L} \frac{\sin(z)}{\sinh(z)} dz = 0.$$

We have a simple pole at $z = i\pi$,

$$\int_{C_\epsilon} \frac{\sin(z)}{\sinh z} dz = -i\pi \mathrm{Res}(i\pi) = -i\pi \left( \frac{\sin(i\pi)}{\cosh(i\pi)} \right) = -i\pi \left( \frac{i\sinh(\pi)}{\cos(\pi)} \right) = -\pi \sinh(\pi).$$

Now let's look at,

$$\left( \int_{C_2} + \int_{C_3} \right) \frac{\sin(z)}{\sinh(z)} dz = \int_{R}^{-R} \frac{\sin(x+i\pi)}{\sinh(x+i\pi)} dx.$$

Observe that,

$$
\begin{aligned}
\frac{\sin(x+i\pi)}{\sinh(x+i\pi)} &= \frac{\sin(x)\cos(i\pi)+\sin(i\pi)\cos(x)}{\sinh(x)\cosh(i\pi)+\sinh(i\pi)\cosh(x)}\\
&= \frac{\sin(x)\cosh(\pi)+i\sinh(\pi)\cos(x)}{\sinh(x)\cos(\pi)+i\sin(\pi)\cosh(x)}\\
&= -\cosh(\pi)\left( \frac{\sin(x)}{\sinh(x)} \right) - i\sinh(\pi)\left( \frac{\cos(x)}{\sinh(x)} \right).
\end{aligned}
$$

We know that $\frac{\cos(x)}{\sinh(x)}$ is an odd function. Using this fact we can rewrite the integral as,

$$\lim_{R\to\infty}\left(\int_{C_2}+\int_{C_3}\right)\frac{\sin(z)}{\sinh(z)}dz = -\cosh(\pi)\lim_{R\to\infty}\int_{R}^{-R}\, dx = \cosh(\pi)I.$$

And since,

$$\lim_{R\to\infty}\int_{C_1}\frac{\sin(z)}{\sinh(z)}dz = \lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(x)}{\sinh x}dx = I.$$

Thus we that,

$$0 = -\pi\sinh(\pi) + I + \cosh(\pi)I,$$

and so we have that,

$$I = \pi\left(\frac{\sinh(\pi)}{1+\cosh(\pi)}\right) = \pi\tanh\left(\frac{\pi}{2}\right).$$

□

**Problem 2** *Use Residue calculus, calculate*

$$I = \int_{-\infty}^{\infty} \frac{1+\cos x}{(x-\pi)^2} dx.$$

*Solution.*
Consider the integral,

$$I = \int_{-\infty}^{\infty} \frac{1+\cos(x)}{(x-\pi)^2} dx = \int_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2} dx,$$

since $\cos(x-\pi) = -\cos(x)$. Then,

$$I = \int_{-\infty}^{\infty} = \frac{1-\cos(x)}{x^2} dx = \fint_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2} dx.$$

Now consider the integral in the complex plane,

$$\begin{aligned} \mathrm{P}\oint_C \frac{1-\cos(x)}{x^2} dx &= \mathrm{Re}\left(\mathrm{P}\int_{-\infty}^{\infty} \frac{1-e^{i}z}{z^2} dz\right) \\ &= \mathrm{Re}\left(\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{i0z}-e^{i}z}{z^2} dz\right), \end{aligned}$$

where $C$ is the closed semicircular sector in the upper half plane from $R$ to $-R$. Now Observe that,

$$\begin{aligned} \lim_{R\to\infty}\left|\frac{1}{z^2}\right| &= \lim_{R\to\infty} \frac{1}{|z^2|} \\ &= \lim_{R\to\infty} \frac{1}{|R^2||e^{i2\theta}|} \\ &= \lim_{R\to\infty} \frac{1}{R^2} \\ &= 0. \end{aligned}$$

So we can apply Jordan's Lemma,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{i0z}-e^{i}z}{z^2} dz = \fint_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2} dx.$$

Now consider that since there is a simpler pole at $z=0$, it must be that,

$$\begin{aligned} \fint_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2} dx &= i\pi \mathrm{Res}(0) \\ &= i\pi\left(\lim_{z\to 0} \frac{1-e^{iz}}{z}\right) \end{aligned}$$

$$
\begin{aligned}
&= i\pi\left(\lim_{z\to 0}\frac{-iz+\mathcal{O}(z^2)}{z}\right)\\
&= i\pi\left(\lim_{z\to 0} -i+\mathcal{O}(z)\right)\\
&= \pi.
\end{aligned}
$$

Therefore we have that,

$$
\begin{aligned}
I = \fint_{-\infty}^{\infty}\frac{1-\cos(x)}{x^2}dx &= \mathrm{Re}\left(\fint_{-\infty}^{\infty}\frac{1-e^{iz}}{z^2}\right)\\
&= \mathrm{Re}\left(P\oint_{-\infty}^{\infty}\frac{e^{i0z}-e^{iz}}{z^2}dz\right)\\
&= \pi.
\end{aligned}
$$

$\square$

**Problem 3** *Evaluate the following integral using residue calculus,*

$$I = \int_0^\infty \frac{x^a}{1 + 2x\cos(b) + x^2} dx,$$

*where* $-1 < a < 1, a \neq 0$ *and* $-\pi \leq b \leq \pi, b \neq 0$.

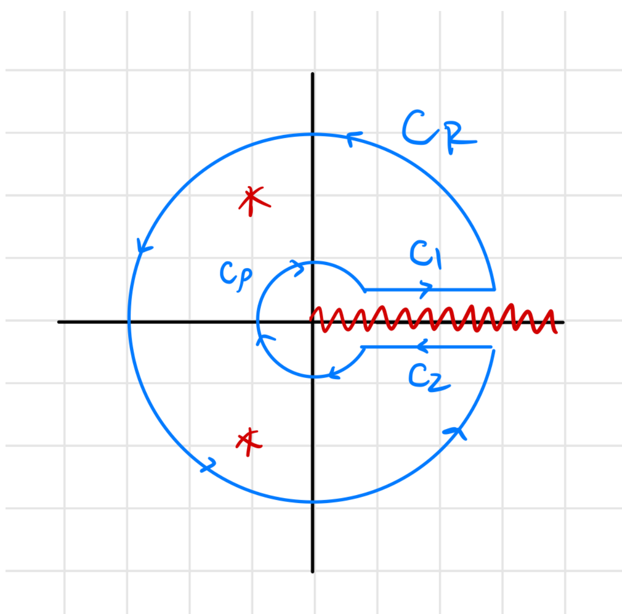
*Solution.*
Consider the integral

$$\begin{aligned} I &= \int_0^\infty \frac{x^a}{1 + 2x\cos(b) + x^2} dx \\ &= \int_0^\infty \frac{x^a}{(x + \cos(b) + i\sin(b))(x + \cos(b) - i\sin(b))} dx \\ &= \int_0^\infty \frac{x^a}{(x + e^{ib})(x + e^{-ib})} dx, \end{aligned}$$

where $-1 < a < 1, a \neq 0$ and $-\pi \leq b \leq \pi, b \neq 0$. Now let's consider this integral in the complex plane

$$\oint_C \frac{z^a}{(z + e^{ib})(z + e^{-ib})} dz = \left( \int_{C_1} + \int_{C_R} + \int_{C_2} + \int_{C_\rho} \right) \frac{z^a}{(z + e^{ib})(z + e^{-ib})} dz,$$

where $0 \leq \arg(z) \leq 2\pi$ is the necessary branch cut to make our $x^a$ single valued and $C$ is the contour described in the figure kindly drawn by Rohin.




Note the two poles at $z = -e^{\pm ib}$ which can be rewritten as

$$-e^{ib} = e^{ib}e^{i\pi} = e^{i(b+\pi)}, \quad \text{and} \quad -e^{-ib} = e^{-ib}e^{\pi} = e^{i(-b+i)},$$

which are within the contour. Since the function is analytic on and within $C$, we can apply Residue theorem

$$\begin{aligned}
\oint_C \frac{z^a}{(z+e^{ib})(z+e^{-ib})}dz &= 2\pi i\big(\mathrm{Res}(-e^{ib}) + \mathrm{Res}(-e^{-ib})\big) \\
&= 2\pi i\left(\lim_{z\to -e^{ib}} \frac{z^a}{z+e^{ib}} + \lim_{z\to -e^{-ib}} \frac{z^a}{z+e^{-ib}}\right) \\
&= 2\pi i\left(\frac{-e^{iab}}{-e^{ib}+e^{-ib}} + \frac{-e^{-iab}}{-e^{-ib}+e^{ib}}\right) \\
&= 2\pi i\left(\frac{e^{iab}-e^{-iab}}{e^{ib}-e^{-ib}}\right) \\
&= 2\pi i\left(\frac{\sin(ab)}{\sin(b)}\right).
\end{aligned}$$

Now we have that,

$$\oint_{C_R} \frac{z^a}{(z+e^{ib})(z+e^{-ib})}dz = \lim_{\phi\uparrow 2\pi}\int_0^{\phi} \frac{R^a e^{ia\theta} iRe^{i\theta}}{(Re^{i\theta}+e^{ib})(Re^{i\theta}+e^{-ib})}d\theta$$

which we can bound with,

$$\begin{aligned}
\lim_{R\to\infty}\left|\frac{R^a e^{ia\theta} iRe^{i\theta}}{(Re^{i\theta}+e^{ib})(Re^{i\theta}+e^{-ib})}\right| &\le \lim_{R\to\infty} \frac{R^{1+a}}{|R-1||R-1|} \\
&= \lim_{R\to\infty} \frac{R^{a-1}}{\left|1-\frac{1}{R}\right|\left|1-\frac{1}{R}\right|} \\
&= 0.
\end{aligned}$$

Thus we have shown that,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^a}{(z+e^{ib})(z+e^{-ib})}dz = 0.$$

Similarly we can show that,

$$\oint_{C_\rho} \frac{z^a}{(z+e^{ib})(z+e^{-ib})}dz = \lim_{\phi\uparrow 2\pi}\int_{\phi}^{0} \frac{\rho^a e^{ia\theta} i\rho e^{i\theta}}{(\rho e^{i\theta}+e^{ib})(\rho e^{i\theta}+e^{-ib})}d\theta,$$

by parameterizing around $z = \rho e^{i\theta}$. And since

$$\lim_{\rho\to 0}\left|\frac{\rho^a e^{ia\theta} i\rho e^{i\theta}}{(\rho e^{i\theta}+e^{ib})(\rho e^{i\theta}+e^{-ib})}\right| \le \lim_{\rho\to 0} \frac{\rho^{1+a}}{|\rho-1||r-1|} = 0,$$

which gives us that

$$\oint_{C_\rho} \frac{z^a}{(z+e^{ib})(z+e^{-ib})}dz = 0.$$

Next let's consider $C_1$ and us the parameterization $z = r$

$$\lim_{R\to\infty}\int_{C_1}\frac{z^a}{(z+e^{ib})(z+e^{-ib})}dz = \lim_{R\to\infty}\int_0^\infty \frac{r^a}{(r+e^{ib})(r+e^{-ib})}dr = I.$$

Finally let's consider $C_2$ using the parameterization $z = re^{i\phi}$ as $\phi \uparrow 2\pi$

$$\begin{aligned}
\lim_{R\to\infty}\lim_{\phi\uparrow 2\pi}\int_{C_2}\frac{z^a}{(z+e^{ib})(z+e^{-ib})}dz &= \lim_{R\to\infty}\lim_{\phi\uparrow 2\pi}\int_R^0 \frac{r^a e^{i\phi a}e^{i\phi}}{(re^{i\phi}+e^{ib})(re^{i\phi}+e^{-ib})}dr \\
&= \lim_{R\to\infty} -e^{i2\pi a}\int_0^R \frac{r^a}{(r+e^{ib})(r+e^{-ib})} \\
&= -e^{i2\pi a}I.
\end{aligned}$$

Collecting all our integrals we get,

$$-e^{i2\pi a}2\pi i\left(\frac{\sin(ab)}{\sin(b)}\right) = I(1-e^{i2\pi a}),$$

which gives us that,

$$I = \pi\left(\frac{\sin(ab)}{\sin(b)}\right)\left(\frac{-2ie^{i\pi a}}{1-e^{i2\pi a}}\right) = \pi\left(\frac{\sin(ab)}{\sin(b)}\right)\left(\frac{2i}{e^{i\pi a}-e^{-i\pi a}}\right) = \pi\left(\frac{\sin(ab)}{\sin(b)\sin(a\pi)}\right).$$

$\square$