Math 575 Homework 4

Due not very soon

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Problem 1 *Please be working on your next project presentation, scheduled for May 24 and 26. The final paper will then be due on June 9.*

Solution. Project is being worked on! $\square$

Problem 2 *Consider the "all-to-all" coupled system of pulse-coupled phase oscillators on the N-dimensional torus, with coupling strength* $\epsilon > 0$*, from class:*

$$\dot{\theta}_i = \omega + \epsilon z(\theta_i)\frac{1}{N}\sum_{j=1}^{N} g(\theta_j) \tag{1}$$

$i = 1...N$*. Let* $z(\theta_i) = A\sin\theta + B\cos\theta$*, which we noted in class corresponds to Hopf, generalized Hopf, and to saddle-node on a periodic orbit bifurcations, which are the most common co-dimension 1 bifurcations to periodic orbits. Let* $g(\theta) = \sum_{k=1}^{\infty} a_k \sin(k\theta) + b_k \cos(k\theta)$*, a totally general "impulse" function describing the coupling from oscillator j. Beginning with the same coordinate transformation as in class, compute the averaged system*

$$\dot{\psi}_i = \epsilon\frac{1}{N}\sum_{j=1}^{N} f(\psi_j - \psi_i) \tag{2}$$

Recall that the conclusions of the averaging theorem on how the latter equation approximates the first hold here, making the latter equation a useful approximation. [a] Find a general explicit expression for f*, involving the constants* A, B, a_k, b_k *above for appropriate* k*. [b] Building from a previous homework, find a general condition on these constants that guarantees that the averaged system will be a gradient dynamical system. [c] Find a general condition on these constants that guarantees that any solution* $\psi_i \equiv k\ \forall k$ *is a fixed point for the averaged system. These are referred to as synchronized solutions. Compute the Jacobian for these fixed points , and write down the dimension of the stable, unstable, and center manifolds for all possible choices of the constants* A, B, a_k, b_k *.*

Solution.

Consider the "all-to-all" coupled system of pulse-coupled phase oscillators on the N-dimensional torus, with coupling strength $\epsilon > 0$, from class:

$$\dot{\theta}_i = \omega + \epsilon z(\theta_i)\frac{1}{N}\sum_{j=1}^{N} g(\theta_j), \tag{3}$$

$i = 1...N$. Let $z(\theta_i) = A\sin\theta + B\cos\theta$ and

$$g(\theta) = \sum_{k=1}^{\infty} a_k \sin(k\theta) + b_k \cos(k\theta).$$

(a) To find a general explicit expression for f, we take the change of variables and integrate to get the averaged system to be

$$\psi_i' = \epsilon\frac{1}{N}\frac{1}{2\pi}\int_{\psi_i}^{\psi_i+2\pi} z(s)\sum_{j=1}^{N} g(\psi_j - \psi_i + s)\mathrm{d}s$$

$$= \epsilon \frac{1}{N} \sum_{j=1}^{N} \int_{\psi_i}^{\psi_i + 2\pi} z(s) g(\psi_j - \psi_i + s) \mathrm{d}s.$$

Now substituting $g(\theta)$ and $z(s)$ we get

$$\begin{aligned} I &= \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{\psi_i}^{\psi_i+2\pi} (A\sin(s) + B\cos(s))(a_k \sin(k(\psi_j - \psi_i + s)) + b_k \cos(k(\psi_j - \psi_i + s)))\mathrm{d}s \\ &= \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{0}^{2\pi} (A\sin(s) + B\cos(s))(a_k \sin(k(\psi_j - \psi_i + s)) + b_k \cos(k(\psi_j - \psi_i + s)))\mathrm{d}s. \end{aligned}$$

where the second equality is true since the integrand is 2π-periodic. To compute I, we will break up the integral into three sub-integrals. First, consider the integral

$$\begin{aligned} &\int_0^{2\pi} \sin(s)\sin(k(\psi_j - \psi_i + s))\mathrm{d}s \\ &\quad = \frac{1}{2}\int_0^{2\pi} \cos((k-1)s + k\psi_j - k\psi_i) - \cos((k+1)s + k\psi_j - k\psi_i)\mathrm{d}s \\ &\quad = \frac{1}{2(k-1)} \sin((k-1)s + k\psi_j - k\psi_i) - \frac{1}{2(k+1)} \sin((k-1)s + k\psi_j - k\psi_i)\Big|_0^{2\pi} \\ &\quad = \begin{cases} 0 & \text{if } k \neq 1 \\ \pi\cos(\psi_j - \psi_i) & \text{otherwise.} \end{cases} \end{aligned}$$

Next, consider the second integral

$$\begin{aligned} &\int_0^{2\pi} \sin(s)\cos(k(\psi_j - \psi_i + s))\mathrm{d}s \\ &\quad = \frac{1}{2}\int_0^{2\pi} \sin((k+1)s + k\psi_j - k\psi_i) - \sin((k-1)s + k\psi_j - k\psi_i)\mathrm{d}s \\ &\quad = \frac{1}{2(k-1)} \cos((k-1)s + k\psi_j - k\psi_i) - \frac{1}{2(k+1)} \cos((k+1)s + k\psi_j - k\psi_i)\Big|_0^{2\pi} \\ &\quad = \begin{cases} 0 & \text{if } k \neq 1 \\ -\pi\sin(\psi_j - \psi_i) & \text{otherwise.} \end{cases} \end{aligned}$$

Next, consider the third integral

$$\begin{aligned} &\int_0^{2\pi} \cos(s)\sin(k(\psi_j - \psi_i + s))\mathrm{d}s \\ &\quad = \frac{1}{2}\int_0^{2\pi} \sin((k+1)s + k\psi_j - k\psi_i) + \sin((k-1)s + k\psi_j - k\psi_i)\mathrm{d}s \\ &\quad = -\frac{1}{2(k-1)} \cos((k-1)s + k\psi_j - k\psi_i) - \frac{1}{2(k+1)} \cos((k+1)s + k\psi_j - k\psi_i)\Big|_0^{2\pi} \end{aligned}$$

$$= \begin{cases} 0 & \text{if } k \neq 1 \\ \pi \sin(\psi_j - \psi_i) & \text{otherwise.} \end{cases}$$

Finally, consider the fourth integral

$$\int_0^{2\pi} \cos(s)\cos(k(\psi_j - \psi_i + s))\mathrm{d}s$$
$$= \frac{1}{2}\int_0^{2\pi} \cos((k+1)s + k\psi_j - k\psi_i) + \cos((k-1)s + k\psi_j - k\psi_i)\mathrm{d}s$$
$$= \frac{1}{2(k-1)}\sin((k-1)s + k\psi_j - k\psi_i) + \frac{1}{2(k+1)}\sin((k+1)s + k\psi_j - k\psi_i)\Big|_0^{2\pi}$$
$$= \begin{cases} 0 & \text{if } k \neq 1 \\ \pi \cos(\psi_j - \psi_i) & \text{otherwise.} \end{cases}$$

Thus letting $k = 1$ and plugging these back into the original integral we find that

$$I = \frac{1}{2\pi}(Aa_1\pi\cos(\psi_j - \psi_i) - Ab_1\pi\sin(\psi_j - \psi_i) + Ba_1\pi\sin(\psi_j - \psi_i) - Bb_1\pi\cos(\psi_j - \psi_i))$$
$$= \frac{1}{2}(Aa_1 + Bb_1)\cos(\psi_j - \psi_i) + \frac{1}{2}(Ba_1 - Ab_1)\sin(\psi_j - \psi_i).$$

Plugging this back into ψ_i' we find the general expression for f:

$$\psi_i' = \epsilon\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{2}(Aa_1 + Bb_1)\cos(\psi_j - \psi_i) + \frac{1}{2}(Ba_1 - Ab_1)\sin(\psi_j - \psi_i)\right)$$
$$= \epsilon\frac{1}{N}\sum_{j=1}^{N} f(\psi_j - \psi_i).$$

(b) Recall that from Homework 3 we found that a system of this form has a gradient flow if f is an odd function. Thus we let $Aa_1 + Bb_1 = 0$ to enforce f being odd and therefore the system is a gradient flow.

(c) To guarantee that any solution $\psi_i = k$ is a fixed point of the averaged system we require

$$0 = \epsilon\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{2}(Aa_1 + Bb_1)\cos(0) + \frac{1}{2}(Ba_1 - Ab_1)\sin(0)\right),$$

and thus

$$Aa_1 + Bb_1 = 0.$$

To determine the stability of the fixed points, let's consider the Jacobian

$$J = J_{i,k} = \begin{cases} \frac{\epsilon}{2N}(Ba_1 - Ab_1)(-(N-1)) & \text{if } i = k, \\ \frac{\epsilon}{2N}(Ba_1 - Ab_1) & \text{if } i \neq k. \end{cases}$$

Next, we can apply Gerschgorin's circle theorem which tells us that all the eigenvalues of J are the same sign or zero depending on the sign of $Ba_1 - Ab_1$. Next, we notice that the Jacobian has a null space of dimension 1 which implies that the center manifold also has dimension 1. We found the dimension of the null space by noting that J is the graph Laplacian of a complete graph with N vertices with uniform weights on the edges $\frac{-\epsilon}{2N}(Ba_1 - Ab_1)$. Thus if $Ba_1 - Ab_1 > 0$, then the stable manifold will have dimension $N - 1$ and the unstable manifold has dimension 0. And if $Ba_1 - Ab_1 < 0$ then the unstable manifold has dimension $N - 1$ and the stable manifold has dimension 0.

$\square$

Problem 3 *Compute the normal form, up to order two, for a two-dimensional flow with linear part (Jacobian, in real Jordan form)*

$$J = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$$

where $\lambda \neq 0$ is an arbitrary real parameter. Note that the case you will study, $\lambda \neq 0$, is for a hyperbolic fixed point that is a saddle or a source. Make sure to cover all of the possible cases; the normal form may differ for different values of λ. As a comment, your result will relate nicely to Sternberg's Theorem (not covered in class): if λ_j are eigenvalues of J, the full flow can be linearized by a diffeomorphism if $\sum_{j=1}^{n} m_j \lambda_j \neq 0$ for all integers m_j.

Solution.
Consider the system

$$w' = G(w),$$

where $w \in \mathbb{R}^2$, $G(w)$ is sufficiently differentiable and has a hyperbolic fixed point at w_0 that is either a saddle or a source. First, we will translate to get the fixed point at the origin by defining

$$v = w - w_0,$$

and then

$$v' = G(v + w_0) = H(v).$$

Next, we will split off the linear part by Taylor expanding and noting that $H(0) = 0$ we get

$$v' = DH(0)v + \hat{H}(v),$$

where $\hat{H}(v)$ is at least quadratic in v and $DH(0)$ is the Jacobian of $H(v)$ evaluated at $v = 0$. Finally, we will transform the linear part by letting

$$v = Tx,$$

where T is the transfomation matrix to bring $DH(0)$ to its real Jordan form. This gives

$$\begin{aligned} Tx' &= DH(0)Tx + \hat{H}(Tx) \\ x' &= T^{-1}DH(0)Tx + T^{-1}\hat{H}(Tx) \\ x' &= Jx + F(x) \\ x' &= Jx + F_2(x) + F_3(x) + \cdots \end{aligned}$$

Where $F_2(x)$ are the quadratic compoments of x and J is the real Jordan form of $DH(0)$ which in known to be of the form

$$J = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix},$$

where $\lambda \neq 0$ since the fixed is hyperbolic. Now we wish to compute the normal form up to the second order and thus we consider

$$x = y + h_2(y),$$

where $h_2(y) = (h_2, h_2)^T$ (apologies for the abuse of notation) is a quadratic function of the components of $y = (y_1, y_2)^T$. Note that the basis for quadratic functions of the components of y is

$$\begin{pmatrix} y_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix}.$$

To see which of these terms are resonant, consider the operator

$$\begin{aligned} L_j^{(2)} h_2 &= J h_2 - D h_2 J y \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} -y_1 \frac{\partial h_1}{\partial y_1} - y_2 \lambda \frac{\partial h_1}{\partial y_2} + h_1 \\ -y_1 \frac{\partial h_2}{\partial y_1} - y_2 \lambda \frac{\partial h_2}{\partial y_2} + \lambda y_2 \end{pmatrix}. \end{aligned}$$

To find the resonant terms, we apply each basis vector to the operator

$$\begin{aligned} L_j^{(2)} \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix} &= \begin{pmatrix} -2y_1 y_2 + y_1^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -y_1^2 \\ 0 \end{pmatrix}, \\ L_j^{(2)} \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix} &= \begin{pmatrix} -y_1 y_2 - \lambda y_1 y_2 + y_1 y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\lambda y_2 y_1 \\ 0 \end{pmatrix}, \\ L_j^{(2)} \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix} &= \begin{pmatrix} -y_2 \lambda 2 y_2 + y_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} (1-2\lambda) y_2^2 \\ 0 \end{pmatrix}, \\ L_j^{(2)} \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix} &= \begin{pmatrix} 0 \\ -y_1 2 y_1 + \lambda y_1^2 \end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda - 2)\lambda_1^2 \end{pmatrix}, \\ L_j^{(2)} \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ -y_1 y_2 - y_2 \lambda y_1 + \lambda y_1 y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -y_1 y_2 \end{pmatrix}, \\ L_j^{(2)} \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix} &= \left(-y_2 \lambda 2 y_2 + \lambda y_2^2\right) = \begin{pmatrix} 0 \\ -\lambda y_2^2 \end{pmatrix}. \end{aligned}$$

Next, we can eliminate the terms in F_2 that are in the range of L_J^2. Recalling that $\lambda \neq 0$, we get the following cases for the normal form of J to be

$$\begin{cases} y' = Jy + F_3 + \cdots & \lambda \neq 1/2 \text{ and } \lambda \neq 2, \\ y' = Jy + a_1 \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix} + \cdots = & \lambda = 1/2, \\ y' = Jy + a_1 \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix} + \cdots & \lambda = 2. \end{cases}$$

Therefore when $\lambda \neq 1/2$ and $\lambda \neq 2$ we can rewrite the system as

$$\begin{cases} y_1 = y_1 + \cdots, \\ y_2 = \lambda y_2 + \cdots. \end{cases}$$

When $\lambda = 1/2$ we can rewrite the system as

$$\begin{cases} y_1 = y_1 + a_1 y_2^2 + \cdots , \\ y_2 = \lambda y_2 + \cdots . \end{cases}$$

And when $\lambda = 2$ we can rewrite the system as

$$\begin{cases} y_1 = y_1 + \cdots , \\ y_2 = \lambda y_2 + a_1 y_1^2 + \cdots . \end{cases}$$

$\square$

Problem 4 *Determine the Takens-Bogdanov normal form to third order.*

Solution.
Suppose that

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The possible third-order terms are

$$H_3 = \text{span}\left\{ \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1^2 y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 y_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_2^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^3 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^2 y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1 y_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2^3 \end{pmatrix} \right\}.$$

We'd like to determine the range of $L_j^{(3)}$. Recall that

$$L_J^{(3)} h_3 = J h_3 - D h_3 J y,$$

where $h_3 = (h_1, h_2)^T \in H_3$ and $y = (y_1, y_2)^T$. In our case we get

$$\begin{aligned} L_J^{(3)} h_3 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} h_2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{\partial h_1}{\partial y_1} y_2 \\ \frac{\partial h_2}{\partial y_1} y_2 \end{pmatrix} \\ &= \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1} y_2 \\ -\frac{\partial h_2}{\partial y_1} y_2 \end{pmatrix}. \end{aligned}$$

To find the resonant terms, we apply the operator to each basis element of H_3 to get

$$\begin{aligned} L_J^{(3)} \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix} &= \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1} y_2 \\ -\frac{\partial h_2}{\partial y_1} y_2 \end{pmatrix} = \begin{pmatrix} -3 y_1^2 y_2 \\ 0 \end{pmatrix}, \\ L_J^{(3)} \begin{pmatrix} y_1^2 y_2 \\ 0 \end{pmatrix} &= \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1} y_2 \\ -\frac{\partial h_2}{\partial y_1} y_2 \end{pmatrix} = \begin{pmatrix} -2 y_1 y_2^2 \\ 0 \end{pmatrix}, \\ L_J^{(3)} \begin{pmatrix} y_1 y_2^2 \\ 0 \end{pmatrix} &= \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1} y_2 \\ -\frac{\partial h_2}{\partial y_1} y_2 \end{pmatrix} = \begin{pmatrix} -y_2^3 \\ 0 \end{pmatrix}, \\ L_J^{(3)} \begin{pmatrix} y_2^3 \\ 0 \end{pmatrix} &= \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1} y_2 \\ -\frac{\partial h_2}{\partial y_1} y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ L_J^{(3)} \begin{pmatrix} 0 \\ y_1^3 \end{pmatrix} &= \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1} y_2 \\ -\frac{\partial h_2}{\partial y_1} y_2 \end{pmatrix} = \begin{pmatrix} y_1^3 \\ 3 y_1^2 y_2 \end{pmatrix} = \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ y_1^2 y_2 \end{pmatrix}, \\ L_J^{(3)} \begin{pmatrix} 0 \\ y_1^2 y_2 \end{pmatrix} &= \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1} y_2 \\ -\frac{\partial h_2}{\partial y_1} y_2 \end{pmatrix} = \begin{pmatrix} y_1^2 y_2 \\ -2 y_1 y_2^2 \end{pmatrix} = \begin{pmatrix} y_1^2 y_2 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ y_1 y_2^2 \end{pmatrix}, \end{aligned}$$

$$L_J^{(3)}\begin{pmatrix} 0 \\ y_1y_2^2 \end{pmatrix} = \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1}y_2 \\ -\frac{\partial h_2}{\partial y_1}y_2 \end{pmatrix} = \begin{pmatrix} y_1y_2^2 \\ -y_2^3 \end{pmatrix} = \begin{pmatrix} y_1y_2^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ y_2^3 \end{pmatrix},$$

$$L_J^{(3)}\begin{pmatrix} 0 \\ y_2^3 \end{pmatrix} = \begin{pmatrix} h_2 - \frac{\partial h_1}{\partial y_1}y_2 \\ -\frac{\partial h_2}{\partial y_1}y_2 \end{pmatrix} = \begin{pmatrix} y_2^3 \\ 0 \end{pmatrix}.$$

Thus we have found that

$$L_J^{(3)}(H_3) = \text{span}\left\{\begin{pmatrix} y_1^2y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1y_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_2^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2^3 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1y_2^2 \end{pmatrix}, \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ y_1^2y_2 \end{pmatrix}\right\}.$$

Then clearly $(0, y_1^3)^T$ is not in $L_J^{(3)}(H_3)$ and we have the choice of picking either $(y_1^3, 0)^T$ or $(0, y_1^2y_2)^T$ for our second resonant term. Let's pick

$$\begin{pmatrix} 0 \\ y_1^3 \end{pmatrix}, \begin{pmatrix} y_1^3 \\ 0 \end{pmatrix},$$

to be our resonant terms and therefore the Takens-Bogdanov normal form up to third order is

$$\begin{cases} y_1' = y_2 + a_1y_1^2 + a_3y_1^3 + F_4 + \cdots, \\ y_2' = a_2y_1^2 + a_4y_1^3 + F_4 + \cdots. \end{cases}$$

□

Problem 5 *In homework 1 we have the Lorenz equations*

$$\begin{cases} x' = 10(-x+y) \\ y' = rx - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases} \tag{4}$$

Characterize the bifurcation when $r = 1$.

Solution.
Consider the Lorenz equations

$$\begin{cases} x' = 10(-x+y) \\ y' = rx - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases} \tag{5}$$

From homework 1 we found that there are three fixed points at

$$\overline{(x,y,z)} \in \left\{ (0,0,0), \left(-2\sqrt{\frac{2}{3}}\sqrt{r-1}, -2\sqrt{\frac{2}{3}}\sqrt{r-1}, r-1 \right), \right.$$
$$\left. \left(2\sqrt{\frac{2}{3}}\sqrt{r-1}, 2\sqrt{\frac{2}{3}}\sqrt{r-1}, r-1 \right) \right\},$$

and thus when $r < 1$ there is only one fixed point which we found to be stable. When $r > 1$, there are three fixed points. Of these three fixed points, the fixed point at the origin we found to be unstable while the other two fixed points are stable for $1 < r < \frac{470}{19}$. These conditions correspond to a pitchfork bifurcation and thus the bifurcation at $r = 1$ is a pitchfork.

$\square$

Problem 6 *Consider the one-parameter family of one-dimensional maps,*

$$x \mapsto x^2 + c, \tag{6}$$

where c is a real-valued parameter.

1. *Find the fixed points of this system. For which values of c do they exist? Determine the stability of these fixed points and their dependency on the value of c. Determine if there is a bifurcation, and find the bifurcation point.*

2. *Focusing on the value $c = -3/4$, compute $f'_{-3/4}(p_-)$, where $f_c(x) = x^2 + c$ and p_- is the smaller of the two fixed points at this value of c. Convince yourself that as c descends through $-3/4$, we see the emergence of an (attracting) 2-cycle. This is a period doubling bifurcation!*

3. *Solve for the period two points by considering the fixed points of the function $f_c^2(.) = f_c(f_c(.))$, and the domain of c for which the original system has a fixed point.*

Solution.

Consider the one-parameter family of one-dimensional maps,

$$x \mapsto x^2 + c,$$

where c is a real-valued parameter.

(a) To find the fixed points of the system,

$$\bar{x} = \bar{x}^2 + c \implies \bar{x} = \frac{1 \pm \sqrt{1-4c}}{2}.$$

Thus when $c = \frac{1}{4}$ there is one fixed point, when $c < \frac{1}{4}$ there are two fixed points, and otherwise there are no fixed points. Next, we wish to determine the stability of the fixed points so consider that

$$x' = 2x.$$

Then when $c = \frac{1}{4}$ then $\bar{x} = \frac{1}{2}$ and

$$x' = 2\left(\frac{1}{2}\right) = 1,$$

and thus the stability is inconclusive. When $c < \frac{1}{4}$ we have

$$x' = 2\left(\frac{1 \pm \sqrt{1-4c}}{2}\right) = 1 \pm \sqrt{1-4c},$$

so the fixed point at $1 + \sqrt{1-4c}$ is unstable since $|x'| > 1$. On the other hand, the fixed point at $1 - \sqrt{1-4c}$ is stable on $-3/4 < c < 1/4$ since $|x'| < 1$ and is unstable

$c < -3/4$ since $|x'| > 1$. Since the stability changes at $c = -3/4$, there is a bifurcation at

$$(c, x) = \left(-3/4, \frac{1 - \sqrt{1 - 4c}}{2}\right)$$

.

(b) Now we let $c = -3/4$, $f_c(x) = x^2 + c$, and $p_- = \frac{1-\sqrt{1-4c}}{2}$. Then

$$f_c'(p_-) = 2x = 1 - \sqrt{1 - 4(-3/4)} = -1,$$

and thus the bifurcation at $c - 3/4$ is a period-doubling bifurcation.

(c) Now let's find the period two orbits by computing

$$x = f_c^2(x) = x^4 + 2cx^2 + c^2 + c,$$

and recalling that $0 = x^2 - x + c$ we find

$$x^4 + 2cx^2 - x + c^2 + c = (x^2 - x + c)(x^2 + x + c + 1) = 0,$$

using long division. Thus there the period two points are at

$$x = \frac{-1 \pm \sqrt{-4c - 3}}{2},$$

which is defined for $c < -3/4$ but we also note that the original system has a fixed point only if $c \leq 1/4$.

$\square$