Math 567 Homework 7
Due Soon
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Problem 1
construct the bilinear transform

$$w(z) = \frac{az+b}{cz+d}$$

that maps the region between the two circles $|z-\frac{1}{4}| = \frac{1}{4}$ *and* $|z-\frac{1}{2}| = \frac{1}{2}$ *into an infinite strip bounded by the vertical lines* $u = \text{Re}(w) = 0$ *and* $u = \text{Re}(w) = 1$*. To avoid ambiguity, suppose that the outer circle is mapped to* $u = 1$*.*

b *Upon finding the appropriate transformation* w*, carefully show that the image of the inner circle under* w *is the vertical line* $u = 0$*, and similarly for the outer circle.*

Solution.

a We wish to construct the bilinear transform

$$w(z) = \frac{az+b}{cz+d}$$

that maps the region between the two circles $|z-\frac{1}{4}| = \frac{1}{4}$ and $|z-\frac{1}{2}| = \frac{1}{2}$ into an infinite strip bounded by the vertical lines $u = \text{Re}(w) = 0$ and $u = \text{Re}(w) = 1$. To avoid ambiguity, suppose that the outer circle is mapped to $u = 1$. To achieve this mapping, we require $z = \frac{1}{2}$ to map to $w = 0$, thus $z_1 = \frac{1}{2}$. We also need $z = 1$ to map to $w = 1$ thus

$$A\frac{1-\frac{1}{2}}{1} = 1 \implies A = 2.$$

Thus we have that the bilinear transform is given by,

$$w(z) = 2\frac{z-\frac{1}{2}}{z}.$$

b Next we wish to verify the transform by showing that the image of the each circle under w is mapped to their corresponding vertical line. First let's consider the inner circle. Let $z = x+iy$ be on the inner circle,

$$(x-\frac{1}{4})^2 + y^2 = \frac{1}{16}.$$

Then,

$$w(z) = 2\left(\frac{(x+iy)-\frac{1}{2}}{(x+iy)}\right)$$

$$
\begin{aligned}
&= \left(\frac{2(x+iy)-1}{x+iy}\right)\left(\frac{x-iy}{x-iy}\right) \\
&= \frac{2(x^2+y^2)-(x-iy)}{x^2+y^2}.
\end{aligned}
$$

Next observe that

$$
\begin{aligned}
\mathrm{Re}(w) &= \frac{2x^2+2y^2-x}{x^2+y^2} \\
&= 2\left(\frac{x^2+y^2-\frac{1}{2}x}{x^2+y^2}\right) \\
&= \frac{2((x-\frac{1}{4})^2-\frac{1}{16}+y^2)}{x^2+y^2} \\
&= \frac{2(\frac{1}{16}-\frac{1}{16})}{x^2+y^2} \\
&= 0.
\end{aligned}
$$

Thus we see that if $x, y \neq 0$, then $\mathrm{Re}(w(z)) = 0$, otherwise $\mathrm{Re}(w(z)) = z_\infty$ which is the transformation we desired. We can also verify that the inverse transform achieves the desired effect by observing that when $w = ai$

$$
z = \frac{1}{2-w} = \frac{1}{2-ai} = \frac{2}{a^2+4} + \frac{ia}{a^2+4}.
$$

Plugging this into the circle equation with $x = \mathrm{Re}(z) = \frac{2}{a^2+4}$ and $y = \mathrm{Im}(z) = \frac{a}{a^2+4}$ gives

$$
\begin{aligned}
\frac{1}{16} &= \left(\frac{2}{a^2+4} - \frac{1}{4}\right)^2 + \left(\frac{a}{a^2+4}\right)^2 \\
\frac{1}{16} &= -\frac{1}{a^2+4} + \frac{4}{(a^2+4)^2} + \frac{1}{16} + \frac{a^2}{(a^2+4)^2} \\
0 &= -\frac{1}{a^2+4} + \frac{4+a^2}{(a^2+4)^2} \\
0 &= \frac{-a^2-4+a^2+4}{(a^2+4)^2} \\
0 &= 0.
\end{aligned}
$$

Thus the inverse mapping also holds.

Next consider the outer circle. Let $z = x + iy$ be on the circle,

$$
(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}.
$$

Then,

$$\begin{aligned}
w(z) &= 2\left(\frac{(x+iy)-\frac{1}{2}}{(x+iy)}\right) \\
&= \left(\frac{2(x+iy)-1}{x+iy}\right)\left(\frac{x-iy}{x-iy}\right) \\
&= \frac{2(x^2+y^2)-(x-iy)}{x^2+y^2}.
\end{aligned}$$

Next observe that

$$\begin{aligned}
\mathrm{Re}(w) &= \frac{2x^2+2y^2-x}{x^2+y^2} \\
&= \frac{x^2+y^2}{x^2+y^2} + \frac{x^2+y^2-x}{x^2+y^2} \\
&= 1 + \frac{(x-\frac{1}{2})^2-\frac{1}{4}+y^2}{x^2+y^2} \\
&= 1 + \frac{\frac{1}{4}-\frac{1}{4}}{x^2+y^2} \\
&= 1.
\end{aligned}$$

Thus we see that if $x, y \neq 0$, then $\mathrm{Re}(w(z)) = 1$, otherwise $\mathrm{Re}(w(z)) = z_\infty$ which is the transformation we desired. We can also verify that the inverse transform achieves the desired effect by observing that when $w = ai + 1$

$$z = \frac{1}{2-w} = \frac{1}{2-ai-1} = \frac{1}{a^2+1} + \frac{ia}{a^2+1}.$$

Plugging this into the circle equation with $x = \mathrm{Re}(z) = \frac{1}{a^2+1}$ and $y = \mathrm{Im}(z) = \frac{a}{a^2+1}$ gives

$$\begin{aligned}
\frac{1}{4} &= (\frac{1}{a^2+1}-\frac{1}{2})^2 + (\frac{a}{a^2+1})^2 \\
\frac{1}{4} &= -\frac{1}{a^2+1} + \frac{1}{(a^2+1)^2} + \frac{1}{4} + \frac{a^2}{(a^2+9)^2} \\
0 &= \frac{-a^2-9}{(a^2+9)^2} + \frac{9+a^2}{(a^2+9)^2} \\
0 &= 0.
\end{aligned}$$

Thus the inverse mapping also holds.

$\square$

Problem 2 *Use the result of Problem 1 to find the steady state temperature $T(x, y)$ in the region bounded by the two circles, where the inner circle is maintained at $T = 0C$ and the outer circle at $T = 100C$. Assume T satisfies the two-dimensional Laplace equation.*

Solution.

Building off the previous problem consider the steady state temperature $T(x, y)$ in the region bounded by the two circles, where the inner circle is maintained at $T = 0$C and the outer circle at $T = 100$C. Assume T satisfies the two-dimensional Laplace equation. Thus we have that $T_{xx} + T_{yy} = 0$ with $T_{c_1} = 0$ and $T_{c_2} = 100$. Applying the bilinear transform gives that $T_{uu} + T_{vv} = 0$ with $T(0, v) = 0$ and $T(1, v) = 100$. Since T has no v dependence, we have that $T_{vv} = 0$. Thus we find that

$$T_{uu} = 0 \implies T = cu + d,$$

and applying the boundary conditions gives

$$d = 0 \quad \text{and} \quad c = 100.$$

thus

$$T(u, v) = 100u.$$

Now to transform it back

$$\begin{aligned} T(w) &= 100 \cdot \mathrm{Re}(w) \\ &= 100 \cdot \mathrm{Re}\left(\frac{2z - 1}{z}\right) \\ &= 100 \cdot \mathrm{Re}\left(\frac{2(x + iy) - 1}{x + iy}\right) \\ &= 100\left(\frac{2x^2 + 2y^2 - x}{x^2 + y^2}\right). \end{aligned}$$

□