Math 567 Homework 2
Due October 19 2022
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Problem 1 *AF: 4.2.1: c and d*

Solution.

c Consider the integral,

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} \quad a^2, b^2 > 0,$$

and $a, b > 0$ with out loss of generality. Since the function is even we can get,

$$\frac{1}{2}\int_\infty^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}.$$

Let's consider the integral in the complex plane,

$$I_R = I_{C_1} + I_{C_2} = \oint_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)}.$$

where C_R is the contour made from C_1 which runs along the real line from $-R$ to R and C_2 which is the upper semi circle from R to $-R$. Consider the two cases, when $a \neq b$ and when $a = b$. First let's consider when $a \neq b$, since a^2 and b^2 are greater than positive, the only singularities within this contour are $z_1 = ia$ and $z_2 = ib$. By the residue theorem, we know that

$$I_R = \oint_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i \sum \text{of the residues.}$$

Next we need to find the residues. First let's find the residue at z_1 which can be found by,

$$\begin{aligned} a_{-1} &= \lim_{z\to ia} \frac{z-ia}{(z^2+a^2)(z^2+b^2)} \\ &= \lim_{z\to ia} \frac{z-ia}{(z-ia)(z+ia)(z^2+b^2)} \\ &= \lim_{z\to ia} \frac{1}{(z+ia)(z^2+b^2)} \\ &= \frac{1}{(ia+ia)((ia)^2+b^2)} \\ &= \frac{1}{(2ia)(-a^2+b^2)} \end{aligned}$$

$$= \frac{1}{(2ia)(b^2 - a^2)}$$

Next let's find the residue at z_2,

$$\begin{aligned}
a_{-1} &= \lim_{z \to ib} \frac{z - ib}{(z^2 + a^2)(z^2 + b^2)} \\
&= \lim_{z \to ib} \frac{z - ib}{(z^2 + a^2)(z - ib)(z + ib)} \\
&= \lim_{z \to ib} \frac{1}{(z + ib)(z^2 + a^2)} \\
&= \frac{1}{(ib + ib)((ib)^2 + a^2)} \\
&= \frac{1}{(2ib)(-b^2 + a^2)} \\
&= \frac{1}{(2ib)(a^2 - b^2)}.
\end{aligned}$$

Thus from residue theorem,

$$\begin{aligned}
I_R &= 2\pi i \left[\frac{1}{(2ia)(b^2 - a^2)} + \frac{1}{(2ib)(a^2 - b^2)} \right] \\
&= 2\pi \left[\frac{1}{(2a)(b^2 - a^2)} - \frac{1}{(2b)(b^2 - a^2)} \right] \\
&= \pi \left[\frac{1}{(a)(b^2 - a^2)} - \frac{1}{(b)(b^2 - a^2)} \right] \\
&= \pi \left[\frac{b - a}{ab(b - a)^2} \right] \\
&= \frac{\pi}{ab(b + a)}.
\end{aligned}$$

Now let's consider the case when $a = b$. In this case, we have a double pole at $z_1 = ai$. then we can compute the residue to be,

$$\begin{aligned}
\lim_{z \to ai} \frac{d}{dz} \left(\frac{(z - ai)^2}{(z - ai)^2 (a + ai)^2} \right) &= \lim_{z \to ai} \left(\frac{1}{(z + ai)^2} \right) \\
&= \lim_{z \to ai} \frac{-2}{(z + ai)^3} \\
&= \frac{-2}{(2ai)^3}
\end{aligned}$$

$$= \frac{1}{4a^3 i}.$$

Thus by the residue theorem,

$$\begin{aligned}
I_R &= 2\pi i \frac{1}{4a^3 i} \\
&= \frac{\pi}{2a^3} \\
&= \frac{\pi}{a^3 + a^3} \\
&= \frac{\pi}{a^2 b + ab^2} \\
&= \frac{\pi}{ab(b+a)}
\end{aligned}$$

and thus we have the same result in both cases. Next let's show that $I_{C_2} = 0$. Observe that

$$\begin{aligned}
\lim_{|Z|\to\infty} \left| \frac{z}{(z^2+a^2)(z^2+b^2)} \right| &= \lim_{R\to\infty} \frac{R}{|(R^2 e^{i2\theta} + a^2)(R^2 e^{2i\theta} + b^2)|} \\
&= \lim_{R\to\infty} \frac{R}{|a^2 b^2 + a^2 R^2 e^{2it} + b^2 R^2 e^{2it} + R^4 e^{4it}|} \\
&= \lim_{R\to\infty} \frac{1}{\left|\frac{a^2 b^2}{R} + (a^2+b^2) R e^{2it} + R^3 e^{4it}\right|} \\
&\leq \lim_{R\to\infty} \frac{1}{\left||R^3 e^{4it}| - \left|\frac{a^2b^2}{R} + (a^2+b^2)Re^{2it}\right|\right|} \\
&\leq \lim_{R\to\infty} \frac{1}{\left||R^3 e^{4it}| - \left|\frac{a^2b^2}{R}\right| - |(a^2+b^2)Re^{2it}|\right|} \\
&= \lim_{R\to\infty} \frac{1}{\left|R^3 - R(a^2+b^2) - \frac{a^2b^2}{R}\right|} \\
&= 0
\end{aligned}$$

Thus $\lim_{R\to\infty} |I_{C_2}| = 0$ which means that $I_{C_2} = 0$. Therefore

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \int_\infty^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} I_R = \frac{1}{2} I_{C_1} = \frac{\pi}{2ab(b+a)}$$

d Consider the integral

$$\int_0^\infty \frac{dx}{x^6+1}$$

where the integrand is an even function and thus

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{x^6+1}.$$

Next lets look at the integral $\int_{-\infty}^{\infty} \frac{dx}{x^6+1}$ in the complex plane to get,

$$I_R = I_{C_1} + I_{C_2} = \oint_{C_R} \frac{dz}{z^6+1},$$

where C_R is the contour made from C_1 which runs along the real line from $-R$ to R and C_2 which is the upper semi circle from R to $-R$. Next let's find the singularities of $\frac{1}{z^6+1}$ using the sixth root of unity which is of the form $e^{\frac{1}{6}i(\pi+2\pi k)}$ which,

$$\{e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{2}}, e^{i\frac{5\pi}{6}}, e^{i\frac{7\pi}{6}}, e^{i\frac{3\pi}{2}}, e^{i\frac{11\pi}{6}}\}.$$

Of these roots, only $e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{2}}, e^{i\frac{5\pi}{6}}$ are within C_R and they are all singular poles. By Residue Theorem,

$$I_c = 2\pi i \sum \text{of the residues.}$$

Now let's compute the residues using the fact that,

$$\text{Res}(z_0) = \frac{P(z_0)}{Q'(z_0)}$$

Thus we can compute the residues to be,

$$\begin{aligned}
\text{Res}(e^{i\pi/6}) &= \frac{1}{6(e^{i\pi/6})^5} \\
&= \frac{1}{6e^{i5\pi/6}} \\
\text{Res}(e^{i\pi/2}) &= \frac{1}{6(e^{i\pi/2})^5} \\
&= \frac{1}{6e^{i5\pi/2}} \\
&= \frac{1}{6e^{i\pi/2}} \\
\text{Res}(e^{i5\pi/6}) &= \frac{1}{6(e^{i5\pi/6})^5} \\
&= \frac{1}{6e^{i25\pi/6}} \\
&= \frac{1}{6e^{i\pi/6}}
\end{aligned}$$

thus we have that,

$$I_c = 2\pi i \left(\frac{1}{6} \left(e^{-i5\pi/6} + e^{-i\pi/2} + e^{-i\pi/6} \right) \right)$$

Now we have to show that I_{C_2},

$$\begin{aligned}
\lim_{|z|\to\infty} \frac{1}{z^6+1} &= \lim_{R\to\infty} \frac{R}{|R^6 e^{i6\pi}+1|} \\
&= \lim_{R\to\infty} \frac{1}{|R^5 e^{i6\pi}+1|} \\
&\leq \lim_{R\to\infty} \frac{1}{||R^5 e^{i6\pi}| - |1||} \\
&= 0
\end{aligned}$$

thus $\lim_{R\to\infty} |I_{C_2}| = 0$ which means that $I_{C_2} = 0$. Thus we have,

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^6+1} = \frac{1}{2}\int_\infty^\infty \frac{dx}{x^6+1} = \frac{1}{2} I_R = \frac{1}{2} I_{C_1} &= \pi i \left(\frac{1}{6} \left(e^{-i5\pi/6} + e^{-i\pi/2} + e^{-i\pi/6} \right) \right) \\
&= \frac{\pi i}{6} \left(\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i \right) - i + \left(-\frac{-\sqrt{3}}{2} - \frac{1}{2} i \right) \right) \\
&= \frac{\pi i}{6} (-2i) \\
&= \frac{\pi}{3}
\end{aligned}$$

$\square$

Problem 2 *AF: 4.2.2: a,b, and h*

Solution.

a Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2}$$

for $a^2 > 0$. For the sake of applying Jordan's Lemma, consider

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2} = \text{Im}\frac{ze^{iz}dz}{a^2+z^2},$$

and let

$$f(z) = \frac{z}{a^2+z^2}.$$

Now let's show that,

$$\begin{aligned}
\lim_{|z|\to\infty} \frac{z}{z^2+a^2} &= \lim_{R\to\infty} \frac{R}{|R^2e^{i2\theta}+a^2|} \\
&= \lim_{R\to\infty} \frac{1}{|Re^{i2\theta}+\frac{a^2}{R}|} \\
&\leq \lim_{R\to\infty} \frac{1}{||Re^{i2\theta}|-|\frac{a^2}{R}||} \\
&= \lim_{R\to\infty} \frac{1}{R-\frac{a^2}{R}} \\
&= 0
\end{aligned}$$

and thus $|f(z)| \to 0$ as $R \to \infty$. Without loss of generality assume that $a > 0$ and by Jordan's Lemma ,

$$\begin{aligned}
I &= \text{Im}\oint_{C_R} \frac{z}{a^2+z^2}e^{iz}dz \\
&= \text{Im}(2\pi i \;\text{ Res of}\frac{ze^{iz}}{a^2+z^2} \;\text{ at } z = ia) \\
&= \text{Im}(2\pi\frac{aie^{-a}}{2ai}) \\
&= \pi e^{-a}
\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2} = \pi e^{-a}$$

b Consider the integral,

$$I = \int_{-\infty}^{\infty} \frac{\cos(kx)dx}{(x^2+a^2)(x^2+b^2)},$$

and $a^2, b^2, k > 0$. Without loss of generality, assume that $a, b > 0$. Notice that,

$$I = \text{Re} \int_{-\infty}^{\infty} \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)} dz,$$

and let

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}.$$

Recall from problem 1 part a, that $\lim_{R\to\infty} \frac{z}{(z^2+a^2)(z^2+b^2)} = 0$ and since $f(z) \leq \frac{z}{(z^2+a^2)(z^2+b^2)}$, we can apply Jordan's Lemma to get,

$$\begin{aligned}
I &= \text{Im} \oint \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)} dz \\
&= \text{Im} \left\{ 2\pi i \sum \text{ res of } \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)} \right\}
\end{aligned}$$

Similarly to problem 1 part a, we know that there are two single poles at ai and bi when $a \neq b$ and a double pole at ai when $a = b$. First let's consider when $a \neq b$,

$$\begin{aligned}
\lim z \to ai \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)} &= \frac{e^{ikia}}{(zia)(b^2-a^2)} \\
&= \frac{e^- ka}{(2ia)(b^2-a^2)},
\end{aligned}$$

and

$$\lim z \to bi \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)} = \frac{e^- kb}{(2ib)(b^2-a^2)}.$$

Thus we have that,

$$\begin{aligned}
I &= \text{Im} \left\{ 2\pi i \left(\frac{e^{-ka}}{(2ia)(b^2-a^2)} + \frac{e^{-kb}}{(2ib)(b^2-a^2)} \right) \right\} \\
&= \pi \left(\frac{e^{-ka} + e^{-kb}}{ab(b^2-a^2)} \right)
\end{aligned}$$

when $a \neq b$. Next let's consider when $a = b$. First let's find the residue at $z = ai$,

$$\begin{aligned}
\lim_{z\to ai} \frac{d}{dz} \left(\frac{e^{ikz}}{(z+ai)^2} \right) &= \lim_{z\to ai} \frac{(z+ai)^2 ike^{ikz} - 2(z+ai)e^{ikz}}{(z+ai)^4} \\
&= \lim_{z\to ai} \frac{(z+ai)ike^{ikz} - 2e^{ikz}}{(z+ai)^3}
\end{aligned}$$

$$= \frac{-ake^{-ka} - 2e^{-ka}}{-8a^3 i}$$
$$= \frac{k^{-ka}(ak+1)}{4a^3}.$$

Thus we have that,

$$I = \mathrm{Im}\left\{2\pi i \frac{k^{-ka}(ak+1)}{4a^3}\right\}$$
$$= \pi k^{-ka}\left(\frac{ak+1}{2a^3}\right)$$

when $a = b$.

h Consider the integral

$$I = \int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2},$$

and under the transformation $d\theta = \frac{dz}{iz}$ and $\sin(\theta) = \frac{1}{2i}(z - \frac{1}{z})$ we get,

$$I = \int_0^{2\pi} \frac{dz}{(5 - 3(\frac{1}{2i})(z - \frac{1}{z}))^2 iz}$$
$$= \int_0^{2\pi} \frac{4i^2 z^2 dz}{(10iz - 3z^2 + 3)^2 iz}$$
$$= \int_0^{2\pi} \frac{4izdz}{(10iz - 3z^2 + 3)^2}$$
$$= \int_0^{2\pi} \frac{4izdz}{((z - 3i)(3z - i))^2}$$

which shows that there are singularities at $z = 3i$ and $z = \frac{i}{3}$ of which only the latter is within our contour. Now let's find the residue at this point,

$$\lim_{z\to i/3} \frac{d}{dz}\left[f(z)(z - \frac{i}{3})^2\right] = \lim_{z\to i/3} \frac{d}{dz}\left[\frac{4iz(z - \frac{i}{3})^2}{((z - 3i)(3z - i))^2}\right]$$
$$= \lim_{z\to i/3} \frac{d}{dz}\left[\frac{4iz}{(3(z - 3i))^2}\right]$$
$$= \frac{1}{9} \lim_{z\to i/3} \frac{d}{dz}\left[\frac{4iz}{(z - 3i)^2}\right]$$
$$= \frac{1}{9} \lim_{z\to i/3} \frac{(z - 3i)^2(4i) - (4iz)2(z - 3i)}{(z - 3i)^4}$$
$$= \frac{1}{9} \lim_{z\to i/3} \frac{(z^2 - 6iz - 9)(4i) - (8iz)(z - 3i)}{(z - 3i)^4}$$

$$
\begin{aligned}
&= \frac{1}{9}\lim_{z\to i/3}\frac{i4z^2+24z-36i-24z-8iz^2}{(z-3i)^4}\\
&= \frac{1}{9}\lim_{z\to i/3}\frac{-4iz^2-36i}{(z-3i)^4}\\
&= \frac{1}{9}\left(\frac{-4i(\frac{i}{3})^2-36i}{(\frac{i}{3}-3i)^4}\right)\\
&= \frac{1}{9}\left(\frac{\frac{4i}{9}-36i}{(\frac{i}{3}-3i)^4}\right)\\
&= \frac{1}{9}\left(\frac{\frac{-(320i)}{9}}{(\frac{-8i}{3})^4}\right)\\
&= \frac{1}{9}\left(\frac{\frac{-(320i)}{9}}{\frac{4096}{81}}\right)\\
&= -\frac{1}{9}\frac{45i}{64}\\
&= -\frac{5i}{64}
\end{aligned}
$$

Then by residue theorem, we get

$$I = 2\pi i(-\frac{5i}{64}) = \frac{5\pi}{32}$$

□

Problem 3 *AF: 4.2.7:*

Solution.
Consider the integral,

$$\int_0^\infty \frac{dx}{x^5+a^5}.$$

To find the solution, let's consider it in the complex plane as,

$$I_R = \oint_{C_R} \frac{dz}{z^5+a^5},$$

where C_R is the sector contour with radius R centered at the origin with angle $0 \le \theta \le \frac{2\pi}{5}$. Let C_1 be the line segment from the origin to R, C_2 be the radial curve to θ and C_3 be the line segment returning to the origin. So we have that,

$$I_R = I_1 + I_2 +_3 = \oint_{C_1} \frac{dz}{z^5+a^5} + \oint_{C_2} \frac{dz}{z^5+a^5} + \oint_{C_3} \frac{dz}{z^5+a^5}.$$

We can find the singularities by looking at the fifth root of unity which is of the form

$$ae^{1/5i(\pi+2\pi k)} \qquad k = 1, \cdots, 5.$$

Thus the roots are,

$$\{ae^{i\pi}, ae^{i\pi/5}, ae^{3i\pi/5}, ae^{7i\pi/5}, ae^{9i\pi/5}\},$$

of which only $ae^{i\pi/5}$ is within C_R. First let's find the residue at this point,

$$\begin{aligned}
\mathrm{Re}(ae^{i\pi/5}) &= \frac{1}{5(ae^{i\pi/5})^4} \\
&= \frac{1}{5a^4e^{i4\pi/5}}.
\end{aligned}$$

By the Residue theorem we now know that,

$$I_R = 2\pi i\left(\frac{1}{5a^4e^{i4\pi/5}}\right)$$

Now let's consider I_2 We can show that,

$$\begin{aligned}
\lim_{|z|\to\infty}\left|\frac{z}{z^5+a^5}\right| &= \lim_{R\to\infty}\frac{R}{|R^5e^{i5\theta}+a^5|} \\
&= \lim_{R\to\infty}\frac{1}{|R^4e^{i5\theta}+\frac{a^5}{R}|} \\
&\leq \lim_{R\to\infty}\frac{1}{||R^4e^{i5\theta}|-|\frac{a^5}{R}||} \\
&= \lim_{R\to\infty}\frac{1}{R^4-\frac{a^5}{R}} \\
&= 0.
\end{aligned}$$

Thus $\lim_{R\to\infty}|I_2| \leq \lim_{R\to\infty}\frac{1}{R^4-\frac{a^5}{R}} = 0$ which implies that $I_2 = 0$. Therefore we have,

$$I_R = I_{C_1} + I_{C_3} = 2\pi i\left(\frac{1}{5a^4e^{i4\pi/5}}\right).$$

Next let's consider I_3,

$$\begin{aligned}
I_3 &= \int_R^0 \frac{dz}{z^5+a^5} \\
&= \int_R^0 \frac{1}{R^5e^2\pi+a^5}e^{2\pi/5}dR \qquad \text{where } z = Re^{i2\pi/5} \implies dz = e^{2i\pi/5}dR \\
&= e^{2\pi i/5}\int_R^0 \frac{1}{R^5+a^5}dR
\end{aligned}$$

$$= -e^{2\pi i/5}\int_0^R \frac{1}{R^5+a^5}dR$$
$$= -e^{2\pi i/5}\int_{C_{R_1}} \frac{1}{R^5+a^5}dR$$

Which finally gives us that,

$$\lim_{R\to\infty}\oint_{C_R}\frac{dz}{z^5+a^5} = \lim_{R\to\infty} I_1 + I_2 = \lim_{R\to\infty}(1-e^{2\pi i/5})\int_{C_{R_1}}\frac{dz}{z^5+a^5}$$

which implies that,

$$\begin{aligned}
2\pi i(\frac{1}{5a^4e^{i4\pi/5}}) &= (1-e^{2\pi i/5})\int_0^\infty \frac{dx}{x^5+a^5}\\
\implies \int_0^\infty \frac{dx}{x^5+a^5} &= 2\pi i(\frac{1}{5a^4e^{i4\pi/5}})(1-e^{2\pi i/5})^{-1}\\
&= \frac{2\pi i}{5a^4}\left(\frac{1}{e^{i4\pi/5}-e^{i6\pi/5}}\right)\\
&= \frac{2\pi i}{5a^4}\left(\frac{2i}{-e^{i\pi 5}-e^{i2\pi/5}}\right)\\
&= \frac{2\pi i}{5a^4}\left(\frac{2i}{-e^{i\pi 5}-e^{i\pi/5}}\right)\\
&= \frac{\pi}{5a^4\sin(\pi/5)}
\end{aligned}$$

Therefore

$$\int_0^\infty \frac{dx}{x^5+a^5} = \frac{\pi}{5a^4\sin(\pi/5)}.$$

□