Homework 3

The following problems, whenever coming from the textbook, refer to the book "Numerical Analysis" (second edition) by Walter Gautschi.

Theoretical problems:

T1. Problem 11 on page 294 of [Gautschi].

T2. Consider the function $g(x) = x^2 + \frac{3}{16}$.

(a) This function has two fixed points. What are they?

(b) Consider the fixed point iteration $x_{n+1} = g(x_n)$ for this g. For which of the points you have found in (a) can you be sure that the iterations will converge to that fixed point? Briefly justify your answer. You may assume that the initial guess is sufficiently close to the fixed point.

(c) For the point or points you found in (b), roughly how many iterations will be required to reduce the convergence error by a factor of 10?

T3. Assume $f(x)$ is sufficiently smooth. α is a root of $f(x)$ of multiplicity m, $m \geq 2$.

(a) Show that Newton's method converges locally with order one.

(b) Set $g(x) = \frac{f(x)}{f'(x)}$, show that Newton's method applied to function $g(x)$ converges locally with (at least) order two.

(c) Consider the iteration

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)},$$

show that this method converges locally with (at least) order 2.

Coding problems (attach the code you used to generate the results):

C1. For the equation

$$\frac{1}{2}x - \sin x = 0,$$

it is easy to see that the only positive root is located in the interval $[\pi/2, \pi]$.

(a) Use the method of bisection on $[\pi/2, \pi]$ to approximate the root to 3, 7 and 15 decimal places. Report the number of iterations needed in each case.

(b) Repeat the same task as in (a) but using Newton's method with $x_0 = \pi$ as the initial guess.

(c) Repeat the same task as in (b) but using the secant method with $x_0 = \pi/2$ and $x_1 = \pi$ as the initial guess.

C2. Consider the nonlinear system

$$\begin{cases} (x_1+3)(x_2^3-7)+18=0, \\ \sin(x_2 e^{x_1}-1)=0. \end{cases}$$

(a) Solve it using Newton's method. Set the initial guess as $\mathbf{x}_0 = (-0.5, 1.4)^T$. Use reasonable stopping criteria.

(b) Solve it using Broyden's method. Set the same initial guess and stopping criteria.

(c) Suppose you know the exact solution $\mathbf{x}^* = (0, 1)^T$, compute the error $\|\mathbf{x}_k - \mathbf{x}^*\|$ during the iterations of both methods and compare their rate of convergence.