Math 568 Homework 2
Due January 18, 2023
By Marvyn Bailly

Problem 1 *Consider the nonhomogeneous problems of Problem 1 and 2: $\vec{x}' = A\vec{x} + \vec{g}(t)$.*

(a) Let $\vec{x} = M\vec{y}$ where the columns of M are the eigenvectors of the above problems.

(b) Write the equations in terms of $\vec{y}$ and multiply through by M^{-1}.

(c) Show the resulting equation is $\vec{y}' = D\vec{y} + \vec{h}(t)$ where $D = M^{-1}AM$ is a diagonal matrix whose diagonal elements are the eigenvalues of the problem considered and $\vec{h}(t) = M^{-1}\vec{g}(t)$.

(d) Show that this system is now decoupled so that each component of $\vec{y}$ can be solved independently of the other components.

Solution.

Consider the nonhomogeneous problem

$$\vec{x}' = A\vec{x} + \vec{g}(t). \tag{1}$$

(a) Let

$$\vec{x} = M\vec{y} = [\vec{v}_1|\cdots|\vec{v}_n]\vec{y},$$

where the columns of M are the n eigenvectors of 1 with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$.

(b) Substituting $\vec{y}$ into 1 gives

$$M\vec{y}' = AM\vec{y} + \vec{g}(t).$$

Multiplying both sides by M^{-1} yields

$$\vec{y}' = M^{-1}AM\vec{y} + M^{-1}\vec{g}(t) = D\vec{y} + \vec{h}(t).$$

(c) We wish to show that $D = M^{-1}AM$ is a diagonal matrix with $\lambda_1, \cdots \lambda_n$ along the main diagonal. Observe that since $A\vec{v}_1 = \lambda_1\vec{v}_1$, we have

$$AM = [\lambda_1\vec{v}_1|\cdots|\lambda_n\vec{v}_n] = [\vec{v}_1|\cdots|\vec{v}_n]\cdot(\lambda I) = M\lambda I,$$

where $\lambda I = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. Thus

$$D = M^{-1}AM = M^{-1}M\lambda I = \lambda I,$$

which is a diagonal matrix with the eigenvalues of 1 along the main diagonal. Therefore we have

$$\vec{y}' = D\vec{y} + \vec{h}(t),$$

as desired.

(d) Note that

$$\vec{y}' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = D\vec{y} + \vec{h}(t) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}.$$

Thus the j^{th} entry of $\vec{y}'$ is given by

$$y_j' = \lambda_j y_j + h_j,$$

which can be solved independently of the other components of the matrix problem.

□

Problem 2 *Given* $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ *find the eigenfunction expansion solution of*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2y = -10e^x, \qquad y(0) = 0, y'(1) = 0.$$

Solution.
Given $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, we wish to find the eigenfunction expansion solution of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2y = -10e^x, \qquad y(0) = 0, y'(1) = 0. \tag{2}$$

Recall that Sturm-Liouville problems take the form of the following second-order boundary value problem

$$Ly = \mu r(x) y + f(x),$$

on the domain $x \in [a, b]$ with boundary conditions

$$\alpha_1 u(a) + \beta_1 \frac{\partial u(a)}{\partial x} = 0,$$
$$\alpha_2 u(b) + \beta_2 \frac{\partial u(b)}{\partial x} = 0,$$

and the operator L taking the form

$$Ly = -\frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] + q(x)u.$$

Notice that rewriting 2 as

$$Ly = -\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y + 10e^x, \qquad y(0) = 0, y'(1) = 0.$$

gives a Sturm-Liouville problem defined on the domain $x \in [0, 1]$ with $f(x) = 10e^x, p(x) = 1, q(x) = 0, \alpha_1 = 1, \beta_1 = 0, \alpha_2 = 0, \beta_2 = 1$ and let's pick $\mu = 2$ and $r(x) = 1$. The eigenvalue problem associated with this Sturm-Liouville problem is

$$Ly_n = \lambda_n r(x) = \lambda_n y_n \implies -y_{n_{xx}} = \lambda_n y_n, \quad n = 1, 2, \ldots$$

which has the general solution

$$y_n(x) = c_1 \sin\left(\sqrt{\lambda_n} x\right) + c_2 \cos\left(\sqrt{\lambda_n} x\right).$$

Satisfying the first boundary condition $y_n(0) = 0$ gives,

$$y_n(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 = 0.$$

Then the solution is given by

$$y_n(x) = c_1 \sin\left(\sqrt{\lambda_n} x\right).$$

Applying the second boundary condition $y_n'(1) = 0$ gives

$$y_n'(1) = c_1\sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}\right) = 0 \implies \sqrt{\lambda_n} = \frac{(2n-1)\pi}{2}.$$

Thus we have found the unnormalized eigenfunctions to be

$$y_n = c_n \sin\left(\frac{2n-1}{2}\pi x\right), \quad n = 1, 2, \ldots.$$

The normalized eigenfunctions can be determined by enforcing $\langle y_n, y_n\rangle = 1$ which gives

$$\begin{aligned}
\int_0^1 \sin^2\left(\frac{2n-1}{2}\pi x\right) &= \frac{1}{2}\int_0^1 (1 - \cos((2n-1)\pi x))dx \\
&= \frac{1}{2}\int_0^1 1dx - \frac{1}{2}\int_0^1 \cos((2n-1)\pi x)dx \\
&= \frac{1}{2} - \frac{1}{2}\left.\frac{\sin(\pi(2n-1)x)}{\pi(2n-1)}\right|_0^1 \\
&= \frac{1}{2}.
\end{aligned}$$

Thus the normalized eigenfunctions are given by

$$y_n(x) = \sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right).$$

Since the eigenfunctions form a complete set, solutions can be represent by

$$y = \sum_{n=1}^{\infty} c_n y_n,$$

where c_n are determined by $f(x)$ and orthogonality. Thus we can expand $f(x)$ as

$$\frac{f(x)}{r(x)} = f(x) = \sum_{n=1}^{\infty} b_n y_n.$$

To solve for b_n, let $a_n = \sqrt{\lambda_n} = \frac{(2n-1)\pi}{2}$ and observe that

$$\begin{aligned}
b_n &= \langle f, y_n\rangle \\
&= 10\sqrt{2}\int_0^1 e^x \sin(a_n x)dx,
\end{aligned}$$

using integration by parts with $u = e^x, du = e^x dx, dv = \sin(a_n x)dx$ and $v = -\frac{1}{a_n}\cos(a_n x)$ so if

$$J = \int_0^1 e^x \sin(a_n x)dx$$

$$= \frac{-e^x \cos(a_n x)}{a_n}\bigg|_0^1 + \frac{1}{a_n}\int_0^1 e^x \cos(a_n x)dx,$$

using integration by parts with $u = e^x, du = e^x dx, dv = \cos(a_n x)dx$ and $v = \frac{1}{a_n}\sin(a_n x)$ we have

$$\begin{aligned}
J &= \frac{-e^x \cos(a_n x)}{a_n}\bigg|_0^1 + \frac{1}{a_n}\left(\frac{e^x \sin(a_n x)}{a_n}\bigg|_0^1 - \frac{1}{a_n}\int_0^1 e^x \sin(a_n x)dx\right) \\
&= \left(-\frac{e\cos(a_n)}{a_n} + \frac{1}{a_n}\right) + \frac{1}{a_n}\left(\frac{e\sin(a_n x)}{a_n} - \frac{1}{a_n}J\right) \\
&= A + \frac{1}{a_n}B - \frac{1}{a_n^2}J.
\end{aligned}$$

Then we can solve for the integral

$$J = \frac{A + \frac{1}{a_n}B}{1 + \frac{1}{a_n}},$$

and substituting our values back into the integral we get that

$$b_n = \frac{10\sqrt{2}(\sqrt{\lambda_n} + e\sin(\sqrt{\lambda_n}x))}{\lambda_n + 1}.$$

Then we have that

$$f(x) = \sum_{n=1}^{\infty} \frac{10\sqrt{2}(\sqrt{\lambda_n} + e\sin(\sqrt{\lambda_n}x))}{\lambda_n + 1}\sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right),$$

and since $\mu \neq \lambda_n$ for all n, we have the solution $y(x) = \sum_{n=1}^{\infty} \frac{\langle f, u_n\rangle}{\lambda_n - \mu} u_n(x)$ which gives

$$\boxed{y(x) = \sum_{n=1}^{\infty} 20\frac{\left(\frac{(2n-1)\pi}{2}\right) + e\sin\left(\frac{(2n-1)\pi}{2}x\right)}{\left(\left(\frac{(2n-1)\pi}{2}\right)^2 + 1\right)\left(\left(\frac{(2n-1)\pi}{2}\right)^2 - 2\right)} \cdot \sin\left(\frac{(2n-1)\pi}{2}x\right)}$$

$\square$

Problem 3 *Given* $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ *find the eigenfunction expansion solution of*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2y = -x, \qquad y(0) = 0, y(1) + y'(1) = 0.$$

Solution.
Given $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, we wish to find the eigenfunction expansion solution of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2y = -x, \qquad y(0) = 0, y(1) + y'(1) = 0. \tag{3}$$

Recall that Sturm-Liouville problems take the form of the following second-order boundary value problem

$$Ly = \mu r(x) y + f(x),$$

on the domain $x \in [a, b]$ with boundary conditions

$$\alpha_1 u(a) + \beta_1 \frac{\partial u(a)}{\partial x} = 0,$$
$$\alpha_2 u(b) + \beta_2 \frac{\partial u(b)}{\partial x} = 0,$$

and the operator L taking the form

$$Ly = -\frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] + q(x)u.$$

Notice that rewriting 3 as

$$Ly = -\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y + x, \qquad y(0) = 0, y(1) + y'(1) = 0.$$

gives a Sturm-Liouville problem defined on the domain $x \in [0, 1]$ with $f(x) = x, p(x) = 1, q(x) = 0, \alpha_1 = 1, \beta_1 = 0, \alpha_2 = 1, \beta_2 = 1$ and let's pick $\mu = 2$ and $r(x) = 1$. The eigenvalue problem associated with this Sturm-Liouville problem is

$$Ly_n = \lambda_n r(x) = \lambda_n y_n \implies -y_{n_{xx}} = \lambda_n y_n, \quad n = 1, 2, \ldots$$

which has the general solution

$$y_n(x) = c_1 \sin\left(\sqrt{\lambda_n} x\right) + c_2 \cos\left(\sqrt{\lambda_n} x\right).$$

Satisfying the first boundary condition $y_n(0) = 0$ gives,

$$y_n(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 = 0.$$

Then the solution is given by

$$y_n(x) = c_n \sin\left(\sqrt{\lambda_n} x\right).$$

Applying the second boundary condition $y(1) + y'(1) = 0$ gives the transcendental equation for the eigenvalues,

$$\begin{aligned} c_1 \sin\left(\sqrt{\lambda_n}\right) + c_1\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}) &= 0 \\ \tan\left(\sqrt{\lambda_n}\right) + \sqrt{\lambda_n} &= 0. \end{aligned}$$

Eigenvalues can be determined by root finding algorithms but in this case let's keep it general. Now let's normalize the eigenfunction by enforcing $\langle y_n, y_n \rangle$ which gives

$$\begin{aligned} \int_0^1 \sin^2\left(2\sqrt{\lambda}\right) dx &= \frac{1}{2}\int_0^1 \left(1 - \cos\left(2\sqrt{\lambda}x\right)\right) dx \\ &= \frac{1}{2} - \frac{1}{2}\int_0^1 \cos\left(2\sqrt{\lambda}x\right) dx \\ &= \frac{1}{2} - \frac{1}{4\sqrt{\lambda}} \sin\left(2\sqrt{\lambda}\right). \end{aligned}$$

Thus the normalized eigenfunctions are given by

$$y_n(x) = c_n \sin\sqrt{\lambda_n}x = \left(\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} - \sin\left(2\sqrt{\lambda_n}\right)}\right)^{\frac{1}{2}} \sin\sqrt{\lambda_n}x.$$

Computing b_n by solving $b_n = \langle f, y_n \rangle$ yields

$$\begin{aligned} \langle f, y_n \rangle &= c_n \int_0^1 x \sin\sqrt{\lambda_n}x dx \\ &= c_n\left(-\frac{x}{\sqrt{\lambda_n}}\cos\left(\sqrt{\lambda_n}x\right)\Big|_0^1 + \int_0^1 \frac{1}{\sqrt{\lambda_n}}\cos\left(\sqrt{\lambda_n}x\right) dx\right) \\ &= c_n\left(-\frac{1}{\sqrt{\lambda_n}}\cos\left(\sqrt{\lambda_n}\right) + \frac{1}{\lambda_n}\sin\left(\sqrt{\lambda_n}\right)\right) \\ &= \left(\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} - \sin\left(2\sqrt{\lambda_n}\right)}\right)^{\frac{1}{2}} \left(\frac{\sin\left(\sqrt{\lambda_n}\right) - \sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}\right)}{\lambda_n}\right). \end{aligned}$$

Since $\mu \neq \lambda_n$ for all n, we have the solution $y(x) = \sum_{n=1}^{\infty} \frac{\langle f, u_n \rangle}{\lambda_n - \mu} u_n(x)$ which gives

$$\boxed{y(x) = \sum_{n=1}^{\infty} \left(\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} - \sin\left(2\sqrt{\lambda_n}\right)}\right) \frac{\sin\left(\sqrt{\lambda_n}\right) - \sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}\right)}{\lambda(\lambda_n - 2)} \cdot \sin\left(\sqrt{\lambda_n}x\right)}$$

□

Problem 4 *Consider the Sturm-Liouville eigenvalues problem:*

$$Lu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q(x)u = \lambda\rho(x)u \quad 0 < x < l,$$

with the boundary conditions

$$\alpha_1 u(0) + \beta_1 u'(0) = 0,$$
$$\alpha_2 u(l) + \beta_2 u'(l) = 0,$$

and with $p(x) > 0, \rho(x) > 0$, *and* $q(x) \geq 0$ *and with* $p(x), \rho(x), q(x)$ *and* $p'(x)$ *continuous over* $0 < x < l$. *With the inner product* $(\phi, \psi) = \int_0^l \rho(x)\psi(x)\psi(x)^* dx$, *show the following:*

(a) L is a self-adjoint operator.

(b) Eigenfunctions corresponding to different eigenvalues are orthogonal.

(c) Eigenvalues are real, non-negative, and eigenfunctions may be chosen to be real valued.

(d) Each eigenvalue is simple.

Solution.

Consider the Sturm-Liouville eigenvalues problem:

$$Lu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q(x)u = \lambda\rho(x)u \quad 0 < x < l,$$

with the boundary conditions

$$\alpha_1 u(0) + \beta_1 u'(0) = 0,$$
$$\alpha_2 u(l) + \beta_2 u'(l) = 0,$$

and with $p(x) > 0, \rho(x) > 0$, and $q(x) \geq 0$ and with $p(x), \rho(x), q(x)$ and $p'(x)$ continuous over $0 < x < l$. With the inner product $(\phi, \psi) = \int_0^l \rho(x)\psi(x)\psi(x)^* dx$.

(a) First we wish to show that L is a self-adjoint operator. In other words, we must show that $\langle Lu, v\rangle = \langle u, Lv\rangle$ $(L = L^\dagger)$ for all u, v. Observe that

$$\begin{aligned}
\langle Lu, v\rangle &= \int_0^l (-(pu')' + qu)v^* dx \\
&= \int_0^l -p'u'v^* - pu''v^* + quv^* dx \\
&= \int_0^l -pv^*u'' dx + \int_0^l p'v^*u' dx + \int_0^l quv^* dx.
\end{aligned}$$

Applying integration by parts twice to the first integral and once to the second gives

$$\begin{aligned}
\langle Lu, v\rangle &= -pv^*u'\big|_0^l + (pv^*)'u\big|_0^l - p'v^*u\big|_0^l + \int_0^l u(-(pv^*)'' + (p'v^*)' + qv^*)dx \\
&= J(u,v) + \int_0^l u(-pv''^* - p'v'^* + qv^*)dx \\
&= J(u,v) + \int_0^l u(-(pv''^* + p'v'^*) + qv^*)dx \\
&= J(u,v) + \int_0^l u(-(pv'^*)' + qv^*)dx \\
&= J(u,v) + \int_0^l u(Lv)^* dx \\
&= J(u,v) + \langle u, Lv\rangle.
\end{aligned}$$

Notice that

$$\begin{aligned}
J(u,v) &= -pv^*u'\big|_0^l + (pv^*)'u\big|_0^l - p'v^*u\big|_0^l \\
&= v'^*(l)p(l)u(l) - p(l)v^*(l)u'(l) - v'^*(0)p(0)u(0) + p(0)v^*(0)u'(0),
\end{aligned}$$

and applying the given boundary condition on u gives that

$$J(u,v) = (v'^*(l)u(l) - p(l)v^*(l)u'(l)) + (-v'^*(0)p(0)u(0) + p(0)v^*(0)u'(0)) = 0.$$

Thus we have that v^* has the same boundary conditions as u, and that $\langle Lu, v\rangle = \langle u, Lv\rangle$. Therefore L is self-adjoint.

(b) Next we wish to show that the eigenfunctions corresponding to different eigenvalues are orthogonal. Let u_i and u_j be eigenfunction corresponding to different eigenvalues λ_i and λ_j. Observe that

$$\begin{aligned}
&\langle u_n, Lu_m\rangle = \langle Lu_n, u_m\rangle \\
\iff\ &\langle u_n, \lambda_m \rho u_m\rangle = \langle \lambda_n \rho u_n, u_m\rangle \\
\iff\ &\lambda_m^* \langle u_n, \rho u_m\rangle = \lambda_n \langle \rho u_n, u_m\rangle \\
\iff\ &\lambda_m^* \langle u_n, u_m\rangle_\rho = \lambda_n \langle u_n, u_m\rangle_\rho.
\end{aligned}$$

Thus we have that

$$0 = (\lambda_m^* - \lambda_n)\langle u_n, u_m\rangle_\rho.$$

Note that by part (c), we have that eigenvalues are real so $\lambda_m = \lambda_m^*$ and since $\lambda_m \neq \lambda_n \implies \langle u_n, u_m\rangle_\rho = 0$. Thus u_n and u_m are orthogonal. Therefore eigenfunctions corresponding to different eigenvalues are orthogonal.

(c) Next we wish to show that eigenvalues are real ($\lambda = \lambda^*$), non-negative and eigenfunctions may be picked to be real valued. First let's show that the eigenvalues are real. Consider the eigenvalue λ and its corresponding eigenfunction u. Observe that

$$\begin{aligned}
\lambda \langle u, u \rangle_\rho &= \langle \lambda \rho u, u \rangle \\
&= \langle Lu, u \rangle \\
&= \langle u, Lu \rangle \\
&= \int_0^l u (Lu)^* dx \\
&= \int_0^l u (\lambda \rho u)^* dx \\
&= \lambda^* \int_0^l u \rho u^* dx \\
&= \lambda^* \langle u, u \rangle_\rho .
\end{aligned}$$

Thus we have that $\lambda = \lambda^*$ since $\langle u, u \rangle_\rho \neq 0$ which means that λ is real. Next let's show that the eigenvalues are non-negative. Observe that

$$\langle -(pu')', u \rangle + \langle qu, u \rangle = \lambda \langle \lambda \rho u, u \rangle .$$

Applying integration by parts to the first inner product yields

$$-puu_x|_0^l + \langle pu_x, u_x \rangle + \langle qu, u \rangle = \lambda \langle \rho u, u \rangle .$$

Now $\langle pu_x, u_x \rangle \geq 0$ as $p > 0$, $\langle qu, u \rangle \geq 0$ as $q \geq 0$, and $\langle \rho u, u \rangle$ as $\rho > 0$. All that is left to be shown for $\lambda \geq 0$, is that $\left| -puu_x|_0^l \right| < \langle pu_x, u_x \rangle + \langle qu, u \rangle$. Observe that

$$-puu_x|_0^l = -\left(p(l)(u'(l)u(l)) - p(0)(u'(0)u(0))\right) = -\left(p(l)\frac{\alpha_2}{\beta_2}u^2(l) - p(0)\frac{\alpha_1}{\beta_1}u^2(0)\right).$$

So if $-p(l)\frac{\alpha_2}{\beta_2}u^2(l) + p(0)\frac{\alpha_1}{\beta_1}u^2(0) \geq -\langle pu_x, u_x \rangle - \langle qu, u \rangle$ the eigenvalues are non-negative (eigenvalue sign is dependent on the boundary conditions). Finally we wish to show that eigenfunctions can be chosen to be real valued. If $\frac{1}{2}Lu = \frac{1}{2}\lambda \rho u$ then $\frac{1}{2}Lu^* = \frac{1}{2}\lambda \rho u^*$. Now we have that

$$L(\mathrm{Re}(u)) = L\left(\frac{u + u^*}{2}\right) = \frac{1}{2}\lambda \rho (u + u^*) = \lambda \rho \mathrm{Re}(u).$$

From part (d) we know that the eigenvalues are simple, so each eigenvalue has only one eigenfunction and thus we can choose the eigenfunction to be real valued.

(d) Next we wish to show that the eigenvalues are simple. We know that if two functions are linearly dependent on an interval, then the Wronskian will be zero for all values within the interval. Consider the two eigenfunctions u_1 and u_2 corresponding to the same eigenvalue λ. Observe that

$$\begin{aligned} u_2 L(u_1) - u_1 L(u_2) &= u_2(-(pu_1')' + qu_1) - u_1(-(pu_2') + qu_2) \\ &= -u_2 p u_1'' - p' u_1' u_2 + u_1 p u_2'' + u_1 p' u_2' \\ &= p(-u_2 u_1'' + u_1 u_2'') + p'(-u_1' u_2 + u_1 u_2') \\ &= (p(-u_2 u_1' + u_1 u_2'))' \\ &= (p(W(u_1, u_2)))', \end{aligned}$$

where $W(u_1, u_2)$ is the Wronskian of u_1 and u_2. Note that we also have

$$u_2 L(u_1) - u_1 L(u_2) = u_2(\lambda \rho u_1) - u_1(\lambda \rho u_2) = 0.$$

Thus we get that

$$u_2 L(u_1) - u_1 L(u_2) = (pW(u_1, u_2))' = 0,$$

which implies that $W(u_1, u_2)$ is a constant. If we enforce the given boundary conditions then

$$\begin{aligned} W(u_1(0), u_2(0)) &= u_1(0) u_2(0)' - u_2(0) u_1(0)' = 0, \\ W(u_1(l), u_2(l)) &= u_1(l) u_2(l)' - u_2(l) u_1(l)' = 0. \end{aligned}$$

Thus $pW(u_1, u_2) = 0$ and since $p > 0$ by construction, then $W(u_1, u_2) = 0$ for all x within the interval $[0, l]$. Since the Wronskian for u_1 and u_2 is zero for all values within the interval, the vectors are linearly dependent. Thus we see that eigenvectors with the same eigenvectors are linearly dependent and thus eigenvalues are simple.

□