Math 567 Homework 1
Due October 12 2022
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**Problem 1** *Express each of the following in polar exponential form:*

*b.* $-i$

*c.* $1+i$

*d.* $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

*Solution.*

b. Let $r=1$ and $\theta=\frac{3\pi}{2}$. Then $z=-i=e^{\frac{3\pi i}{2}+2\pi ik}$ for $k=0,\pm1,\pm2,\dots$.

c. Let $r=\sqrt{2}$ and $\theta=\frac{\pi}{4}$. Then $z=\sqrt{2}e^{\frac{\pi}{4}i+2k\pi i}$ for $k=0,\pm1,\pm2,\dots$.

d. Let $r=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$ and $\theta=\frac{\pi}{3}$. Then $z=e^{\frac{\pi}{3}i+2k\pi i}$ for $k=0,\pm1,\pm2,\dots$.

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**Problem 2** *Express each of the following in the form of $a+bi$, where $a$ and $b$ are real.*

*a.* $e^{2+i\pi/2}$

*b.* $\frac{1}{1+i}$

*c.* $(1+i)^3$

*d.* $|3+4i|$

*e.* $\cos(i\pi/4+c)$ *for some real* $c$

*Solution.*

a. $e^{2+i\pi/2}=e^2e^{i\pi/2}=e^2\left(\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})\right)=ie^2$.

b. $\frac{1}{1+i}=\frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{1-i}{2}=\frac{1}{2}-i\frac{1}{2}$.

c. $(1+i)^3=(1+i)(1+i)(1+i)=(1+2i-1)(1+i)=2i(1+i)=2(i-1)=-2+i2$.

d. $|3+4i|=\sqrt{\Re(3+4i)^2+\Im(3+4i)}=\sqrt{3^2+4^2}=\sqrt{25}=5$

e. Let c be real, then we can rewrite as following,

$$\begin{aligned}
\cos(i\frac{\pi}{4}+c) &= \frac{e^{i(i\frac{\pi}{4}+c)}+e^{-i(i\frac{\pi}{4}+c)}}{2} \\
&= \frac{1}{2}\left(e^{(-\frac{\pi}{4}+ic)}+e^{(\frac{\pi}{4}-ic)}\right) \\
&= \frac{1}{2}\left(e^{(-\frac{\pi}{4})}e^{(ic)}+e^{(\frac{\pi}{4})}e^{(-ic)}\right) \\
&= \frac{1}{2}\left(e^{(-\frac{\pi}{4})}(\cos(c)+i\sin(c))+e^{(\frac{\pi}{4})}(\cos(c)-i\sin(c))\right) \\
&= \frac{1}{2}\left(e^{(-\frac{\pi}{4})}\cos(c)+i\sin(c)e^{(-\frac{\pi}{4})}+e^{(\frac{\pi}{4})}\cos(c)-ie^{(\frac{\pi}{4})}\sin(c)\right) \\
&= \frac{1}{2}\cos(c)\left(e^{(-\frac{\pi}{4})}+e^{(\frac{\pi}{4})}\right)+i\frac{1}{2}\sin(c)\left(e^{(-\frac{\pi}{4})}-e^{(\frac{\pi}{4})}\right) \\
&= \frac{1}{2}\left(\cos(c)\left(e^{(-\frac{\pi}{4})}+e^{(\frac{\pi}{4})}\right)+i\sin(c)\left(e^{(-\frac{\pi}{4})}-e^{(\frac{\pi}{4})}\right)\right) \\
&= \cos(c)\cosh\left(\frac{\pi}{4}\right)-i\sin(c)\sinh\left(\frac{\pi}{4}\right)
\end{aligned}$$

□

**Problem 3** *Solve for the roots of the following equation:*

*a.* $z^3 = 4$

*b.* $z^4 = -1$

*Solution.*

a. Consider $z^3 = 4$. To find the roots we can use the roots of unity method which gives that the roots will be of form,

$$z = \sqrt[3]{4}e^{2k\pi i/3}$$

for $n = 1, 2, 3$. Thus the roots are $\sqrt[3]{4}e^{2\pi i/3}$, $\sqrt[3]{4}e^{4\pi i/3}$, $\sqrt[3]{4}$

b. Consider $z^4 = -1 \implies z^4 + 1 = 0$. To find the roots, we can rewrite the complex number in the form $z^4 = -1 = |-1|e^{i\pi}e^{i2\pi n}$ where $n \in \mathbb{Z}$. Then we get,

$$z = e^{i\pi\frac{(2n+1)}{4}}$$

which gives the roots to be,

$$e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$$

□

**Problem 4** *Establish the following result:*

*a.* $(z+w)^* = z^* + w^*$

*d.* $Re(z) \leq |z|$

*e.* $|wz^* + w^*z| \leq 2|wz|$

*f.* $|z_1 z_2| = |z_1||z_2|$

*Solution.*

a. Let $z, w \in \mathbb{C}$ such that $z = a + ib$ and $w = c + id$. Then,

$$\begin{aligned}(z+w)^* &= ((a+ib)+(c+id))^* \\ &= ((a+c)+i(b+d))^* \\ &= (a+c) - i(b+d) \\ &= (a+c) + (-ib - id) \\ &= (a-ib) + (c-id) \\ &= z^* + w^*\end{aligned}$$

Thus we have shown that $(z+w)^* = z^* + w^*$.

d. Let $z \in \mathbb{C}$ such that $z = a + ib$. Then we have that,

$$\Re(z) = a \leq |a| = \sqrt{a^2} \leq \sqrt{a^2+b^2} = |z|.$$

e. Let $w, z \in \mathbb{C}$ such that $z = a + ib$ and $w = c + id$. Then,

$$\begin{aligned}|wz^* + w^*z| &= |(c+id)(a+ib)^* + (c+id)^*(a+ib)| \\ &= |(c+id)(a-ib) + (c-id)(a+ib)| \\ &= |ca - icb + iad + db + ca + icb - iad + db| \\ &= |2ca + 2db| \\ &= 2\sqrt{(ca+db)^2} \\ &= 2\sqrt{(ca)^2 + 2cadb + (db)^2}\end{aligned}$$

$$\begin{aligned}2|wz| &= 2|(c+id)(a+ib)| \\ &= 2|ac - bd + i(ad+bc)| \\ &= 2\sqrt{(ac-bd)^2 + (ad+bc)^2} \\ &= 2\sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2} \\ &= 2\sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}\end{aligned}$$

Thus it is left to show that $2cabd \leq (bd)^2 + (ad)^2$. If we let $A = ad$ and $B = bd$ then we want to show that $2AB \leq A^2 + B^2$. But we know that $(A-B)^2 = A^2 - 2AB + B^2 \geq 0$ and thus $2AB \leq A^2 + B^2$ and we have shown that $(z+w)^* = z^* + w^*$.

f. $|z_1z_2| = |z_1||z_2|$.

Let $z_1, z_2 \in \mathbb{C}$ such that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$$\begin{aligned}
|z_1z_2| &= |(x_1 + iy_1)(x_2 + iy_2)| \\
&= |ix_2y_1 + ix_1y_2 + x_1x_2 - y_1y_2| \\
&= |(x_1x_2 - y_1y_2) + i(x_2y_1 + x_1y_2)| \\
&= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_2y_1 + x_1y_2)^2} \\
&= \sqrt{-2x_1x_2y_1y_2 + x_1^2x_2^2 + y_1^2y_2^2 + x_2^2y_1^2 + 2x_1x_2y_2y_1 + x_1^2y_2^2} \\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_2^2y_1^2 + x_1^2y_2^2} \\
&= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\
&= \sqrt{(x_1^2 + y_1^2)}\sqrt{(x_2^2 + y_2^2)} \\
&= |z_1||z_2|
\end{aligned}$$

$\square$