Math 567 Homework 5
By Marvyn Bailly

Problem 1 *Evaluate the integrals*

$$\frac{1}{2\pi i}\oint_C f(z)dz$$

where C is the unit circle centered at the origin with $f(z)$ given below. Do these problems by both

(i) enclosing the singular points inside C

(ii) enclosing the singular points outside C (by including the point at infinity).

Show that you obtain the same result in both cases.

(a) $\frac{z^2+1}{z^2-a^2}, a^2 < 1.$

(b) $\frac{z^2+1}{z^3}.$

(c) $z^2 e^{-1/z}.$

Solution.

(a) Consider the integral

$$I = \frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C \frac{z^2+1}{z^2-a^2}dz,$$

where C is the unit circle centered at the origin and $a^2 < 1$.

(i) Observe that $f(z)$ has two simple poles at $\pm a$ and since $a^2 < 1$ they are within the C. Then by Residue theorem we have

$$I = \text{Res}(-a) + \text{Res}(a) = \frac{(-a)^2+1}{-2a} + \frac{(a)^2+1}{2a} = 0.$$

(ii) Another way to approach the problem is to consider

$$I = -\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C \frac{1}{t^2} f\left(\frac{1}{t}\right)dt,$$

using the transformation $z = \frac{1}{t}$ and $dz = \frac{1}{t^2}dt$. We know that

$$\frac{1}{t^2} f\left(\frac{1}{t}\right) = \frac{1}{t^2}\cdot\frac{1+t^2}{1-a^2t^2},$$

which has a double pole at $t = 0$. Applying the Residue theorem we have

$$\begin{aligned} I &= \lim_{t\to 0} \frac{d}{dt}\left(\frac{1+t^2}{1-a^2t^2}\right) \\ &= \lim_{t\to 0} \frac{(1-a^2t^2)2t + (1+t^2)2a^2t}{(1-a^2t^2)^2} \\ &= \lim_{t\to 0} \frac{t(2+2a^2)}{(1-a^2t^2)^2} \\ &= 0. \end{aligned}$$

Thus we have that

$$I = -\frac{1}{2\pi i}\oint_C f(z)dz = 0.$$

(b) Consider the integral

$$I = \frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C \frac{z^2+1}{z^3}dz.$$

(i) Notice that there is a triple pole at $z = 0$ which is within C. Now let's find the residue at this point,

$$\begin{aligned} \text{Res}(0) &= \frac{1}{2}\lim_{z\to 0} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{z^2+1}{z^3}\right)(z^3) \\ &= \frac{1}{2}\lim_{z\to 0} \frac{\mathrm{d}^2}{\mathrm{d}z^2}(()z^2+1) \\ &= 1. \end{aligned}$$

Thus by the residue theorem, $I = 1$.

(ii) Alternatively we can apply the transformation $z = \frac{1}{t}$ and $dz = \frac{1}{t^2}dt$,

$$I = -\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C \frac{1}{t^2} f\left(\frac{1}{t}\right).$$

Notice that

$$\frac{1}{t^2} f\left(\frac{1}{t}\right) = \frac{1}{t^2}(t+t^3) = t + \frac{1}{t},$$

which has a simple pole at $t = 0$. Now let's find the residue at this point

$$\begin{aligned} \text{Res}(0) &= \lim_{t\to 0}(t)(\frac{1}{t}+t) \\ &= \lim_{t\to 0}(1+t^2) \\ &= 1. \end{aligned}$$

Thus by the Residue Theorem we have that

$$I = 1.$$

(c) Consider the integral

$$I = \frac{1}{2\pi i}\oint_C (z^2 e^{-1/z})dz = \frac{1}{2\pi i}\oint_C z^2(1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{6z^3} + \cdots)dz,$$

and let

$$f(z) = z^2 e^{-1/z}.$$

(i) Notice that the integral has a pole at $z = 0$, we can find the residue by noting that the Taylor expansion of $f(z)$,

$$f(z) = z^2 - z + \frac{1}{2} - \frac{1}{6z} + \cdots .$$

Thus by the definition of residue, we know that $\text{Res}(0) = -\frac{1}{6}$. Therefore by the Residue Theorem we have found that

$$I = -\frac{1}{6}.$$

(ii) Alternatively consider the transformation $z = \frac{1}{t}$ and $dt = -\frac{1}{t^2}dt$,

$$I = -\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C \frac{1}{t^2} f\left(\frac{1}{t}\right).$$

Then we have that,

$$\frac{1}{t^2} f\left(\frac{1}{t}\right) = \frac{1}{t^4}(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots) = \frac{1}{t^4} - \frac{1}{t^3} + \frac{1}{2t^2} - \frac{1}{6t} + \cdots .$$

Once again we have a simple pole at $t = 0$ which we know has residue $-\frac{1}{6}$. Thus by the residue theorem we know that

$$I = -\frac{1}{6}.$$

□

Problem 2 *Find the Fourier transformation of*

$$f(t) = \begin{cases} 1 & \textit{for } -a < t < a \\ 0 & \textit{otherwise} \end{cases}.$$

Then, do the inverse transform using techniques of contour integration.

Solution. Consider the function

$$f(t) = \begin{cases} 1 & \text{for } -a < t < a \\ 0 & \text{otherwise} \end{cases}.$$

We wish to take the Fourier transformation of $f(t)$ and then apply the inverse. First let's take the Fourier transformation of $f(t)$

$$\begin{aligned}
\mathcal{F}[f(t)] &= F(\lambda) \\
&= \int_{-\infty}^{\infty} f(t) e^{i\lambda t} dt \\
&= \int_{-a}^{a} e^{i\lambda t} dt \\
&= \frac{1}{\lambda i} \left[e^{i\lambda t} \right]_{-a}^{a} \\
&= \frac{1}{\lambda i} \left(e^{i\lambda a} - e^{-i\lambda a} \right) \\
&= \frac{2}{\lambda} \sin(\lambda a).
\end{aligned}$$

Thus we have that

$$\mathcal{F}[f(t)] = F(\lambda) = \frac{2}{\lambda} \sin(\lambda a).$$

Next we wish to take the inverse Fourier transformation of $F(\lambda)$ which is given by

$$\mathcal{F}^{-1}[F(\lambda)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{2}{\lambda} \sin(\lambda a) d\lambda.$$

We note that this integral is proper and thus we have that

$$\begin{aligned}
f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{2}{\lambda} \sin(\lambda a) d\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{2}{\lambda} \sin(\lambda a) d\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{e^{i\lambda a} - e^{-i\lambda a}}{\lambda} d\lambda
\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\lambda(a-t)} - e^{-i\lambda(a+t)}}{\lambda} d\lambda$$

.

Now if we define

$$I(y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\lambda y}}{\lambda} d\lambda$$

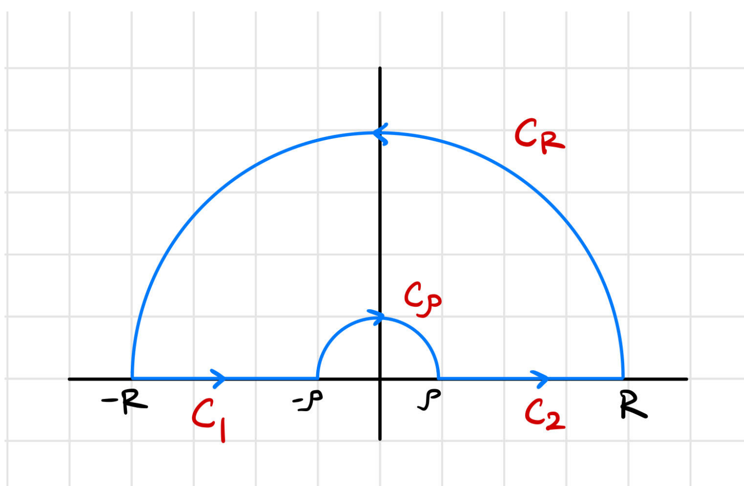
we can rewrite $\mathcal{F}^{-1}$ as

$$\mathcal{F}^{-1}[F(\lambda)] = I(a-t) - I(-(a+t)).$$

This gives us three cases to consider, when $y = 0, y < 0$, and $y > 0$.
First let's consider when $y = 0$. This gives that

$$\begin{aligned} I(y) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\lambda 0}}{\lambda} d\lambda \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\lambda} d\lambda \\ &= \frac{1}{2\pi i}\left[\frac{-1}{\lambda^2}\right]_{-\infty}^{\infty} \\ &= 0. \end{aligned}$$

Next let's consider when $y > 0$. Let's form the contour shown in the following image (kindly drawn by Rohin Gilman)



where $C = C_1 + C_\rho + C_2 + C_R$. Note that $\frac{e^{i\lambda y}}{\lambda}$ is analytic inside and on C and thus by Cauchy's Theorem we have that

$$\frac{1}{2\pi i}\oint_C \frac{e^{i\lambda y}}{\lambda} d\lambda = 0.$$

For C_R we have that

$$\left|\frac{1}{\lambda}\right| \to 0 \quad \text{as} \quad |\lambda| \to \infty,$$

and thus by Jordan's Lemma

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{C_R} \frac{e^{i\lambda y}}{\lambda} d\lambda = 0.$$

Next let's consider C_ρ which has a simple pole at $\lambda = 0$ which is within C_ρ. We can compute the residue to be

$$\begin{aligned}
\text{Res}(0) &= \lim_{\lambda\to 0} \frac{\lambda e^{i\lambda y}}{\lambda} \\
&= \lim_{\lambda\to 0} e^{i\lambda y} \\
&= e^0 \\
&= 1.
\end{aligned}$$

Thus by Residue theorem we have that

$$\frac{1}{2\pi i}\int_{C_R} \frac{e^{i\lambda y}}{\lambda} d\lambda = \frac{1}{2\pi i}(-i\pi)(1) = -\frac{1}{2}.$$

Next let's look at C_1 and C_2, observe that

$$\frac{1}{2\pi i}\left(\int_{C_1} + \int_{C_2}\right)\frac{e^{i\lambda y}}{\lambda} d\lambda = \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{e^{i\lambda y}}{\lambda} d\lambda = I(y).$$

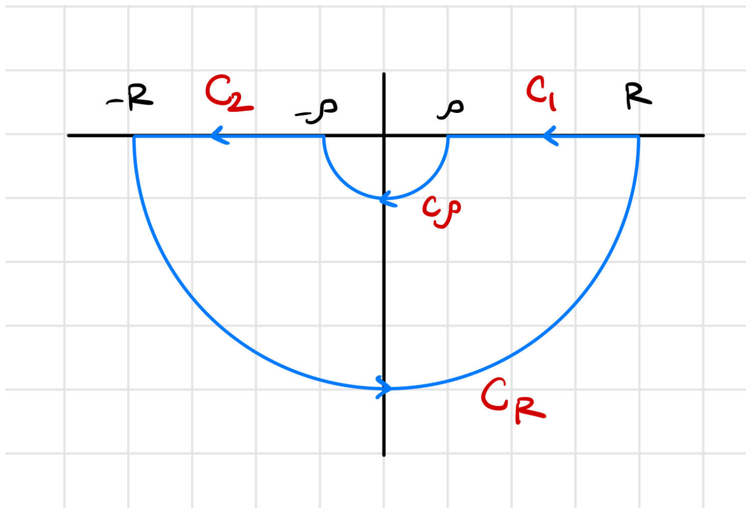
Now if we collect all our terms we see that

$$\frac{1}{2\pi i}\oint_C \frac{e^{i\lambda y}}{\lambda} d\lambda = I(y) - \frac{1}{2} = 0,$$

and thus

$$I(y) = \frac{1}{2}.$$

Next let's consider when $y < 0$. Let's form the contour shown in the following image (kindly drawn by Rohin Gilman)



where $C = C_1 + C_\rho + C_2 + C_R$. Note that $\frac{e^{i\lambda y}}{\lambda}$ is analytic inside and on C and thus by Cauchy's Theorem we have that

$$\frac{1}{2\pi i}\oint_C \frac{e^{i\lambda y}}{\lambda} d\lambda = 0.$$

For C_R we have that

$$\left|\frac{1}{\lambda}\right| \to 0 \quad \text{as} \quad |\lambda| \to \infty,$$

and thus by Jordan's Lemma

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{C_R} \frac{e^{i\lambda y}}{\lambda} d\lambda = 0.$$

Next let's consider C_ρ which has a simple pole at $\lambda = 0$ which is within C_ρ. We can compute the residue to be

$$\begin{aligned} \text{Res}(0) &= \lim_{\lambda\to 0} \frac{\lambda e^{i\lambda y}}{\lambda} \\ &= \lim_{\lambda\to 0} e^{i\lambda y} \\ &= e^0 \\ &= 1. \end{aligned}$$

Thus by Residue theorem we have that

$$\frac{1}{2\pi i}\int_{C_R} \frac{e^{i\lambda y}}{\lambda} d\lambda = \frac{1}{2\pi i}(-i\pi)(1) = -\frac{1}{2}.$$

Next let's look at C_1 and C_2, observe that

$$\frac{1}{2\pi i}\left(\int_{C_1} + \int_{C_2}\right)\frac{e^{i\lambda y}}{\lambda} d\lambda = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\lambda y}}{\lambda} d\lambda = -I(y).$$

Now if we collect all our terms we see that

$$\frac{1}{2\pi i}\oint_C \frac{e^{i\lambda y}}{\lambda} d\lambda = -I(y) - \frac{1}{2} = 0,$$

and thus

$$I(y) = -\frac{1}{2}.$$

Considering all three cases of y we get that $I(y) = \frac{1}{2}\text{sgn}(y)$. This means that

$$\mathcal{F}^{-1}[F(\lambda)] = \frac{1}{2}(\text{sgn}(a-t) + \text{sgn}(a+t)).$$

This means that within the interval $-a < t < a$ we have that $\text{sgn}(a-t) =$ a+t $= 1$ which implies that $\mathcal{F}^{-1} = f(t) = 1$ within the interval. Considering the end points $\pm a$ we get that,

$$\begin{aligned} f(\pm a) &= \frac{1}{2}(\text{sgn}(a \mp a) + \text{sgn}(a \pm a)) \\ &= \frac{1}{2}(1) \\ &= \frac{1}{2}. \end{aligned}$$

Thus we have that

$$\mathcal{F}^{-1}[F(\lambda)] = \begin{cases} 1 & \text{for } -a < t < a \\ \frac{1}{2} & \text{for } -t = \pm a \\ 0 & \text{otherwise} \end{cases},$$

which is the expected solution. $\square$

Problem 3 *Consider the function*

$$f(z) = \ln(z^2 - 1),$$

made single-valued by restricting the angles in the following ways, with $z_1 = z - 1 = r_1 e^{i\theta_1}$ *and* $z_2 = z + 1 = r_2 e^{i\theta_2}$ *of*

(a) $-3\pi/2 < \theta_1 \leq \pi/2, -3\pi/2 < \theta_2 \leq \pi/2$

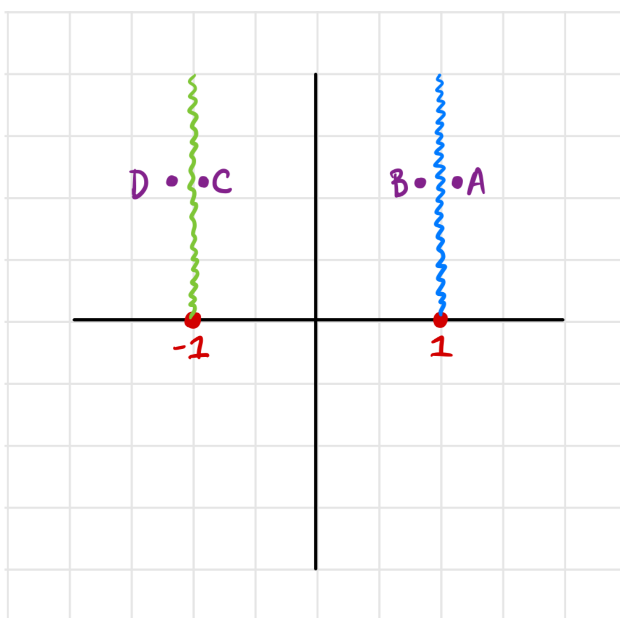
(b) $0 < \theta_1 \leq 2\pi, 0 < \theta_2 \leq 2\pi$

(c) $-\pi < \theta_1 \leq \pi, 0 < \theta_2 \leq 2\pi.$

Find where the branch cuts are for each case by locating where the function is discontinuous. Use the AB tests ad show your results.

Solution.
Let $f(z) = \ln(z^2 - 1)$, $z_1 = z - 1 = r_1 e^{i\theta_1}$ and $z_2 = z + 1 = r_2 e^{i\theta_2}$.

(a) Consider the interval $-3\pi/2 < \theta_1 \leq \pi/2, -3\pi/2 < \theta_2 \leq \pi/2$ on which $f(z)$ is single-valued. The following figure, kindly provided by Rohin Gilman, shows the branch cuts at $\pm 1 + ai$ of $f(z)$.



Now let's consider the branch cut at $1 + ai$. At the point A, we can say that $\theta_1 = -\frac{3\pi}{2}$ and $\theta_2 = \arcsin\left(\frac{r_1}{r_2}\right)$. Then we have that

$$\ln((z-1)(z+1)) = \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2})$$

$$= \ln(r_1 r_2) + i\left(-\frac{3\pi}{2} + \arcsin\left(\frac{r_1}{r_2}\right)\right).$$

Next consider the point B on the other side of the branch cut. We can say that $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \arcsin\left(\frac{r_1}{r_2}\right)$. Then we have that

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i\left(\frac{\pi}{2} + \arcsin\left(\frac{r_1}{r_2}\right)\right).\end{aligned}$$

Since the values at points A and B are not equal, we know that f is discontinuous along $1 + ai$ and thus the branch cut does not cancel and remains. Next let's consider the branch cut along $-1+ai$. First let's look at the point C. If we let $\theta_1 = \arcsin\left(\frac{r_2}{r_1}\right) - \pi$ and $\theta_2 = -\frac{3\pi}{2}$. From this we get

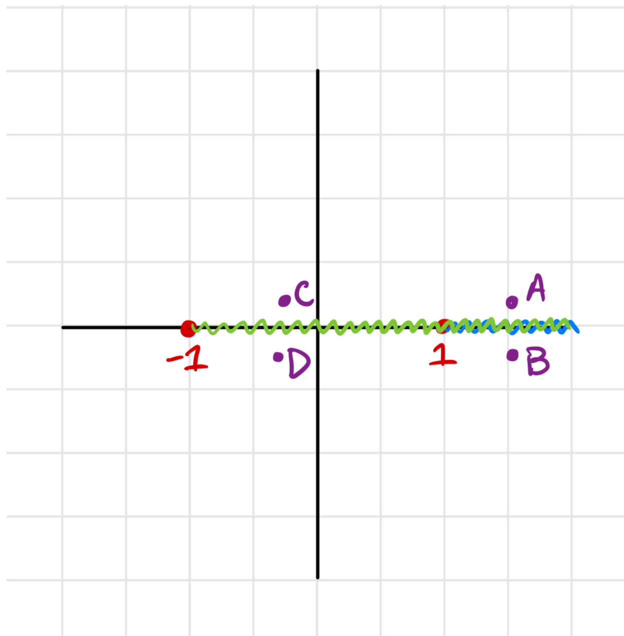
$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i\left(\arcsin\left(\frac{r_2}{r_1}\right) - \pi - \frac{3\pi}{2}\right).\end{aligned}$$

Next consider the point D, if we let $\theta_1 = \arcsin\left(\frac{r_2}{r_1}\right) - \pi$ and $\theta_2 = \frac{\pi}{2}$. From this we get

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i\left(\arcsin\left(\frac{r_2}{r_1}\right) - \pi - \frac{\pi}{2}\right).\end{aligned}$$

Since the values at C and D are different, we have that f is discontinuous along $-1+ai$ and thus the branch cut remains.

(b) Consider the interval $0 < \theta_1 \leq 2\pi, 0 < \theta_2 \leq 2\pi$ on which $f(z)$ is single-valued. The following figure, kindly provided by Rohin Gilman, shows the branch cuts at $1 < x < \infty$ and $-1 \leq x \leq 1$ of $f(z)$.



Now let's consider the branch cut along $1 < x < \infty$. At the point A, we can have $\theta_1 = 0$ and $\theta_2 = 0$. This gives us that

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2).\end{aligned}$$

Next let's consider the point B, where $\theta_1 = 2\pi$ and $\theta_2 = 2\pi$. Thus we have that

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i(4\pi).\end{aligned}$$

We can see that the values at point A and B are different and thus f is discontinuous along the branch cut $1 < x < \infty$. Therefore the branch cut remains. Next let's consider the branch cut $1 \leq x \leq 1$. First consider the point C, then $\theta_1 = \pi$ and $\theta_2 = 0$ and we have that

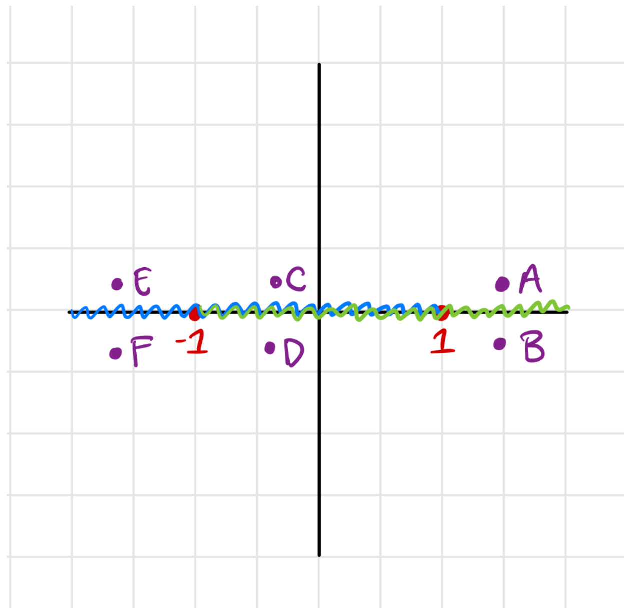
$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i(\pi).\end{aligned}$$

Finally consider the point D, where $\theta_1 = \pi$ and $\theta_2 = 2\pi$. Then we have that

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i(3\pi).\end{aligned}$$

Since the values at C and D are not equal, f is discontinuous along the branch cut $-1 \leq x \leq 1$. Therefore the branch cut remains.

(c) Finally consider the interval $-\pi < \theta_1 \leq \pi, 0 < \theta_2 \leq 2\pi$ on which f is discontinuous along the branch cuts $-1 < x < \infty$ and $-\infty < x < 1$. The following figure, kindly provided by Rohin Gilman, shows this behavior.



First let's consider the points A and B. At the point A, we can let $\theta_1 = 0$ and $\theta_2 = 0$. Then we have that

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2).\end{aligned}$$

Next consider the point B, where $\theta_1 = 0$ and $\theta_2 = 2\pi$. Then we have that

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i(2\pi).\end{aligned}$$

Since the values at are not equal, we know that the f is discontinuous along $1 < x < \infty$ and this section of the branch cut remains. Next let's consider the points C and D. At C we have that $\theta_1 = \pi$ and $\theta_2 = 0$ which gives that

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i(\pi).\end{aligned}$$

Next let's consider the point D then $\theta_1 = -\pi$ and $\theta_2 = 2\pi$ which gives

$$\begin{aligned}\ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i(\pi).\end{aligned}$$

Since these values are equal, f is continuous here. Thus the branch cuts going from $-1 \leq x \leq 1$ cancel each other. Finally let's consider the points E and F. At point E, we say that $\theta_1 = \pi$ and $\theta_2 = \pi$. Thus we have that

$$\begin{aligned} \ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2) + i(2\pi). \end{aligned}$$

Next consider F where $\theta_1 = -\pi$ and $\theta_2 = \pi$. From this we know that

$$\begin{aligned} \ln((z-1)(z+1)) &= \ln(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) \\ &= \ln(r_1 r_2). \end{aligned}$$

Since these two values are different, f is discontinuous along $-\infty < x < -1$ and thus the branch cut remains.

$\square$