Math 575 Homework 1
Due 4/12/2023
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Problem 1 *Consider the system in the phase plane*

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(y) \end{aligned}$$

where $f(x)$ is a continuously differentiable real-valued function of x alone and $g(y)$ is a continuously differentiable real-valued function of y alone (and x and y are both 1-dimensional coordinates defining a plane). Define an oscillatory solution as a trajectory $(x(t), y(t))$ such that $x(t)$ and $y(t)$ are not constant it time and, for any integer N, $x(t+NT) = x(t)$ and $y(t+NT) = y(t)$. Here, T is the period of the oscillation.

(a) Answer YES or NO and give a simple proof or example: Can a system of this form produce an oscillatory solution?

(b) Then, repeat this question, but for the discrete time map

$$\begin{aligned} x_{n+1} &= f(x_n) \\ \dot{y}_{n+1} &= g(y_n) \end{aligned}$$

Solution.
Consider the system in the phase plane

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(y) \end{aligned}$$

where $f(x)$ is a continuously differentiable real-valued function of x alone and $g(y)$ is a continuously differentiable real-valued function of y alone (and x and y are both 1-dimensional coordinates defining a plane).

(a) First we wish to show that there are no oscillatory solutions to a system of this form. For the sake of contradiction, let $(x(t), y(t))$ be an oscillatory solution such that $x(t)$ and $y(t)$ are not constant in time and for any integer N, $x(t+NT) = x(t)$ and $y(t+NT) = y(t)$ where T is the period of the oscillation.Then $x(0) = x(T)$, so there exists $s \in (0,T)$ such that $x'(s) = f(x(s)) = 0$ by the Mean Value theorem. Thus $x(s)$ is an equilibrium point, and by definition $x(t) = x(s)$ for all $t > s$. Now consider when $t < s$, there exists N such that $t + NT > s$. Therefore, $x(t) = x(t+NT) = x(s)$, so x is constant which is a contradiction to the oscillatory solution assumption. We can make a similar argument for y.

(b) Now we wish to know if a discrete time map of the form

$$\begin{aligned} x_{n+1} &= f(x_n), \\ y_{n+1} &= g(y_n), \end{aligned}$$

can produce oscillatory solutions. In this case, we can have oscillatory solutions. Consider the example when $f(x_n) = -x_n$ and $g(y_n) = -y_n$. Then for any initial condition $(x_0, y_0) = (a, b)$, we have $(x_1, y_1) = (-a, -b)$ and then $(x_2, y_2) = (x_0, y_0) = (a, b)$. Thus we have an oscillatory solution with period $T = 2$.

$\square$

Problem 2 *Consider the 2-D systems below. Find all equilibria and determine where they are lyapunov stable, asymptotically stable, or neither. Here μ is an arbitrary real parameter, so make sure to give answers valid for each relevant range of μ:*

- $\dot{x} = 0,\ \dot{y} = \mu x$
- $\dot{x} = 0,\ \dot{y} = \mu y$

Solution.

(a) First let's consider the system

$$\begin{cases} x_t = 0, \\ y_t = \mu x, \end{cases}$$

which has the fixed points $(0, y_0)$ for any $y_0 \in \mathbb{R}$ and the general solution $x = x_0$ and $y = \mu x_0 t + y_0$. Now define the fixed point $\overline{v} = (0, \overline{y_0})$ which gives

$$\begin{aligned} \|v(t) - \overline{v}\|_2 &= \left(|x(t)|^2 + |y(t) - \overline{y_0}|^2\right)^{1/2} \\ &= \left(|x_0|^2 + |\mu x_0 t + y_0 - \overline{y_0}|^2\right)^{1/2} \to \infty. \end{aligned}$$

Therefore the fixed point isn't Lyapunov stable or asymptotically stable when $\mu \neq 0$. When $\mu = 0$ then the solutions are constant and thus are Lyapunov stable but not asympotically.

(b) Next let's consider the system

$$\begin{cases} x_t = 0, \\ y_t = \mu y, \end{cases}$$

which has the fixed points $(x_0, 0)$ for any $x_0 \in \mathbb{R}$ and the general solution $x = x_0$ and $y = y_0 e^{\mu t}$. Now define the fixed point $\overline{v} = (\overline{x_0}, 0)$ which gives

$$\begin{aligned} \|v(t) - \overline{v}\|_2 &= \left(|x(t) - \overline{x_0}|^2 + |y(t)|^2\right)^{1/2} \\ &= \left(|x_0 - \overline{x_0}|^2 + |y_0 e^{\mu t}|^2\right)^{1/2} \\ &= \begin{cases} |x_0 - \overline{x_0}| & \text{if } \mu < 0, \\ \left(|x_0 - \overline{x_0}|^2 + |y_0|^2\right)^{1/2} & \text{if } \mu = 0, \\ \left(|x_0 - \overline{x_0}|^2 + |y_0 e^{\mu t}|^2\right)^{1/2} & \text{if } \mu > 0. \end{cases} \end{aligned}$$

Clearly when $\mu > 0$, the fixed points are unstable. To determine the stability of the fixed points when $\mu \leq 0$, let $\epsilon > 0$ and let $\delta < \epsilon$. Then observe that

$$\|v(t) - \overline{v_0}\| \leq \delta < \epsilon,$$

and since $\|v(t) - \overline{v_0}\|$ is non-increasing we have that the fixed point is Lyapunov stable for $\mu \leq 0$ but is not asymptotically stable.

□

Problem 3 *Consider the 2-D system:*

$$\begin{aligned}\dot{x} &= -y + \mu(x^2+y^2)x \\ \dot{y} &= x + \mu(x^2+y^2)y\end{aligned}$$

where μ is an arbitrary real parameter. Hint: transform to polar coordinates and obtain an exact solution.

a For all possible values of μ, find all fixed points, and determine whether they are lyapunov stable, asymptotically stable, or neither.

b For all possible values of μ, and all possible initial values, determine the maximum duration in both forward and inverse time for which a solution exists.

Solution.

(a) Consider the 2-D system given by

$$\begin{cases} x_t = -y + \mu(x^2+y^2)x, \\ y_t = x + \mu(x^2+y^2)y. \end{cases}$$

First we wish to transform the system into polar coordinates. To do so, we will use $r^2 = x^2 + y^2$ and $\tan(\theta) = \frac{y}{x}$ and taking derivative with respect to time gives $rr_t = xx_t + yy_t$ and $\sec^2(\theta)\theta_t = \frac{xy'-x'y}{x^2}$. Plugging x_t and y_t into these equations yields

$$\begin{aligned} rr_t &= xx_t + yy_t \\ r_t &= \frac{x(-y+\mu(x^2+y^2)x) + y(x+\mu(x^2+y^2)y)}{r} \\ &= \frac{\mu(x^2+y^2)x^2 + \mu(x^2+y^2)y^2}{r} \\ &= \frac{\mu r^4\cos^2(\theta) + \mu r^4\sin^2(\theta)}{r} \\ &= \mu r^3, \end{aligned}$$

and

$$\begin{aligned} \sec^2(\theta)\theta_t &= \frac{xy'-x'y}{x^2} \\ \theta_t &= \cos^2(\theta)\left(\frac{x(x+\mu(x^2+y^2)y) - y(-y+\mu(x^2+y^2)x)}{x^2}\right) \\ &= \cos^2(\theta)\left(\frac{x^2+y^2}{x^2}\right) \\ &= \cos^2(\theta)\left(\frac{r^2}{r^2\cos^2(\theta)}\right) \end{aligned}$$

$$= 1.$$

Thus the 2-D system in polar coordinates is given by

$$\begin{cases} r_t = \mu r^3, \\ \theta_t = 1. \end{cases}$$

We can find the general solution using separation of variables to get $r = \pm\frac{1}{\sqrt{r_0^{-2} - 2\mu t}}$ and $\theta = \theta_0 + t$ but since we are in Polar coordinates we can only consider $r \geq 0$. The only fixed point is at $(0, 0)$ and to find the stability of this fixed point observe that

$$\begin{aligned} \|v(t) - \overline{v_0}\|_2 &= |r(t)| \\ &= \frac{1}{\sqrt{r_0^{-2} - 2\mu t}}. \end{aligned}$$

If $\mu = 0$, then we have that $\|v(t) - \overline{v_0}\|_2 = \frac{1}{\sqrt{r_0^{-2}}}$ which means that the solution trajectories are circles and thus the fixed points Lyapunov stable. If $\mu < 0$ then $r(t) \to 0$ and thus is asymptotically stable. If $\mu > 0$, then we will have an asymptote at $t = \frac{1}{2\mu r_0^2}$ which causes the solution to blow up. Thus for $\mu > 0$, the fixed point is unstable.

(b) Recall that the general solutions of the 2-D system in polar coordinates are $\theta = \theta_0 + t$ and $r = \frac{1}{\sqrt{r_0^{-2} - 2\mu t}}$. First observe that regardless of the choice of μ, the solution θ will always exist so lets focus on the solution r. If $\mu = 0$, the solution $r = \frac{1}{\sqrt{r_0^{-2}}} = r_0$ will always exist for all t. If $\mu < 0$ and $r_0 \neq 0$, then the solution will exist if

$$\sqrt{r_0^{-2} - 2\mu t} \neq 0 \text{and} \sqrt{r_0^{-2} - 2\mu t} \in |R \iff r_0^{-2} - 2\mu t > 0.$$

Thus for $t > 0$, the solution will always exist but for $t < 0$, the solution will exist only for $t > \frac{1}{2r_0^2\mu}$. But if $r_0 = 0$, then we have $r = \sqrt{-2\mu t}$ so the solution will always exist for $t > 0$ but will never exist for $t < 0$. Similarly for $\mu > 0$ and $r_0 \neq 0$, then the solution exits for all $t < 0$ but for $t > 0$ the solution only exists for $t < \frac{1}{2r_0^2\mu}$. But $r_0 = 0$, then we have $r = \sqrt{-2\mu t}$ so the solution will always exist for $t < 0$ but will never exist for $t > 0$.

□

Problem 4 *The van der Pol oscillator. Consider:*

$$\frac{d^2x}{dt^2} + (x^2 - v)\frac{dx}{dt} + x = 0$$

where v is a parameter that can take any real value.

(a) Find all fixed points, and the Jacobian evaluated as these fixed point(s).

(b) State whether the fixed point(s) are lyapunov stable, asymptotically stable, or neither for all possible values of v.

Solution.

Consider the Van der Pol oscillator

$$\frac{d^2x}{dt^2} + (x^2 - v)\frac{dx}{dt} + x = 0,$$

where $v \in \mathbb{R}$. If we let $y = x_t$ then we can transform the equation into the following system of equations

$$\begin{cases} x_t = y, \\ y_t = -(x^2 - v)y - x. \end{cases}$$

(a) To find the fixed points, we require that $y = x_t = 0$ which implies that

$$\begin{cases} x_t = 0, \\ y_t = -x, \end{cases}$$

and thus we can see that the only fixed point is at $(0, 0)$. Next we can find the Jacobian of the system to be

$$J(x, y) = \begin{pmatrix} \frac{\partial x_t}{\partial x} & \frac{\partial x_t}{\partial y} \\ \frac{\partial y_t}{\partial x} & \frac{\partial y_t}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2xy - 1 & -(x^2 - v) \end{pmatrix}.$$

Evaluating the Jacobian at $(0, 0)$ which gives

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}.$$

(b) To find the stability of the fixed point $(0, 0)$, let's study the eigenvalues of $J(0, 0)$ which are

$$\lambda_1 = \frac{1}{2}\left(v - \sqrt{-4 + v^2}\right) \text{ and } \lambda_2 = \frac{1}{2}\left(v + \sqrt{-4 + v^2}\right).$$

Let's first consider when the $v \in (-\infty, -2)$ and $v \in (2, \infty)$. In this case, the eigenvalue are real valued. If $v \in (-\infty, -2)$, then $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) < 0$ and thus the fixed point is asymptotically stable. On the other hand, if $v \in (2, \infty)$, then $\text{Re}(\lambda_1), \text{Re}(\lambda_2) > 0$ and thus the fixed point is unstable.

Next let's consider when $v = \pm 2$ which mean that $\lambda_1 = \lambda_2 = \pm 1$. Thus when $v = -2$, $\text{Re}(\lambda_1), \text{Re}(\lambda_2) < 0$ and thus the fixed point is asymptotically stable. And similarly when $v = 2$, $\text{Re}(\lambda_1), \text{Re}(\lambda_2) > 0$ and thus the fixed point is unstable.

Net consider when $v \in (-2, 0)$ and $v \in (0, 2)$. In these regions v is complex. If $v \in (-2, 0)$, then $\text{Re}(\lambda_1), \text{Re}(\lambda_2) < 0$ and thus the fixed point is asymptotically stable. On the other hand, if $v \in (0, 2)$, then $\text{Re}(\lambda_1), \text{Re}(\lambda_2) > 0$ and thus the fixed point is unstable.

Finally we consider when $v = 0$, which gives that $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = 0$ and thus the fixed point is undetermined. To find the stability, consider the function

$$V(x, y) = x^2 + y^2,$$

where $x, y \in \mathbb{R}$. To show that the $V(x, y)$ is a Lyapunov function for the fixed point $(0, 0)$ observe that $V(x, y) > 0, \forall (x, y) \in \mathbb{R} \setminus (0, 0)$, $V(0, 0) = 0$ and

$$\begin{aligned} V'(x, y) &= \begin{pmatrix} x' \\ y' \end{pmatrix}^T \nabla V \\ &= \begin{pmatrix} y \\ -x^2 y - x \end{pmatrix} \begin{pmatrix} 2x \\ 2y \end{pmatrix} \\ &= -2x^2 y^2, \end{aligned}$$

and thus $V'(x, y) \leq 0, \forall (x, y) \in \mathbb{R} \setminus (0, 0)$. Thus the fixed point is Lyapunov stable.

$\square$

Problem 5 *A general description of a network of N nonlinearly coupled units is given by*

$$\frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} w_{ij} g(u_j) \tag{1}$$

Here, u_i is the activity of the i^{th} unit, and the matrix w gives the connection weights among these units; in particular, w_{ij} is the connection weight between unit j and unit i. Finally, $g(\cdot)$ is a monotonically increasing function that describes how the strength of interaction between units depends on their activities.

- *For LINEAR interactions: $g(y) = y$, write down a simple bound on the entries w_{ij}, based on the Gershgorin circle theorem, that guarantees that the origin will be a stable equilibrium.*

- *For NONLINEAR interactions $g(\cdot)$, consider the "energy function"*

$$H = -1/2 \sum_{ij} w_{ij} V_i V_j + \sum_i \int_0^{V_i} g^{-1}(V) dV \tag{2}$$

where $V_i = g(u_i)$. [a] If the matrix w is symmetric, show the following bound on the time evolution of the energy:

$$\frac{dH}{dt} \leq 0 \ \ .$$

Your answer should be valid for a smooth monotonically increasing function $g(\cdot)$. [b] Then, let $\bar{u}$ be an equilibrium for the system. What additional requirements on H would imply that $\bar{u}$ is asympotically stable? [c] Take $g(x) = \tanh(x)$. Construct a simple example of w for which you can find an equilibrium $\bar{u}$ and demonstrate that your function H implies that it is asymptotically stable. A numerical approach is suggested, and rigorous arguments are not needed, though of course if you wish to use analysis instead that is just fine. [d] Finally, does the bound

$$\frac{dH}{dt} \leq 0$$

also hold in general for anti-symmetric w?

Solution.

(a) **Linear:**

We consider the network of N nonlinearly coupled units given by

$$\frac{du_i}{dt} = -u_i + \sum_{j=1}^{N} w_{ij} g(u_j).$$

In the linear interaction case we have that $g(y) = y$. We wish to find a bound on the weights w_{ij} such that the origin is guaranteed to be a stable equilibrium. Let the origin be $\overline{u}$. First we will compute the Jacobian of the system and evaluate it at $\overline{u}$ which yields

$$J(\overline{u}) = \begin{bmatrix} -1+w_{11} & w_{12} & \cdots & w_{1N} \\ w_{21} & -1+w_{22} & & w_{2N} \\ \vdots & & \ddots & \vdots \\ w_{N1} & \cdots & & -1+w_{NN} \end{bmatrix}.$$

Now we can apply the Gershgorin Circle Theorem which states that every eigenvalue of $J(\overline{u})$ lie within at least one of the N closed discs $D(-1+w_{ii}, R_i) \in \mathbb{C}, i \leq N$ where the origin is at $-1+w_{ii}$ and the radius is given by

$$R_i = \sum_{j \neq i} |w_{ji}|.$$

Note that the right most point of the disk will lay at $-1 + w_{ii} + R_i$ on the real axis. Thus if we are to require the origin to be stable, the real part of the eigenvalues of $J(\overline{u})$ must be less than zero. This is achieved when $-1 + w_{ii} + R_i < 0$ by Gershgorin Circle Theorem. The restriction is achieved by bounding the weights by $|w_{ij}| < \frac{1}{N}$. Therefore, the origin will be a stable equilibrium point if $|w_{ij}| < \frac{1}{N}$.

(a) **Nonlinear:**

Next we will consider nonlinear interactions where $g(\cdot)$ is subject to

$$H = -1/2 \sum_{ij} w_{ij} V_i V_j + \sum_i \int_0^{V_i} g^{-1}(V) dV$$

where $V_i = g(u_i)$. We wish to show that if the matrix W is symmetric, then $\frac{dH}{dt} < 0$. First let's compute

$$H' = -\frac{1}{2} \sum_{i,j} w_{ij}(g(u_i)g'(u_j)u_j' + g(u_j)g'(u_i)u_i') + \sum_i u_i g'(u_i)u_i'.$$

Then since W is symmetric, $w_{ij} = w_{ji}$ and thus

$$-\frac{1}{2} \sum_{i,j} w_{ij} g(u_i) g'(u_j) u_j' - \frac{1}{2} \sum_{i,j} w_{ji} g(u_j) g'(u_i) u_i' = -\sum_{i,j} w_{ij} g(u_i) g'(u_j) u_j'.$$

Using the above equality and the fact that $u' = -u_i + \sum_{j=1}^N w_{ij} g(u_j)$ in H' yields

$$H' = -\sum_i \sum_j w_{ij} g'(u_i) u_i' g(u_j) + \sum_i u_i g'(u_i) u_i'$$

$$
\begin{aligned}
&= -\sum_i \left(g'(u_i)u_i' \left(\sum_j w_{ij} g(u_j) \right) - u_i g'(u_i) u_i' \right) \\
&= -\sum_i \left(g'(u_i)u_i' \left(\sum_j w_{ij} g(u_j) - u_i \right) \right) \\
&= -\sum_i g'(u_i)(u_i')^2 \\
&\leq 0,
\end{aligned}
$$

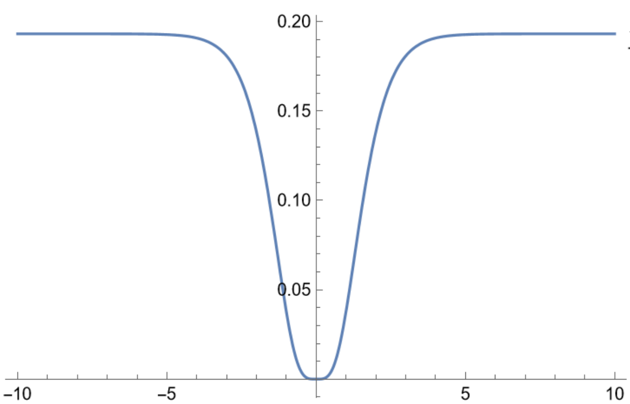
since $g(\cdot)$ is monotonically increasing.

(b) If we let $\overline{u}$ be an equilibrium for the system, then it will be asymptotically stable if there exists a Lyapunov function that meets the requirements for asymptotic stability. If we require H to satisfy the following conditions: $H(\overline{u}) = 0$, $H(u) > 0$ for all $u \neq \overline{u}$, and $\frac{dH}{dt} < 0$, then H will be a Lyapunov function that shows that $\overline{u}$ is asymptotically stable.

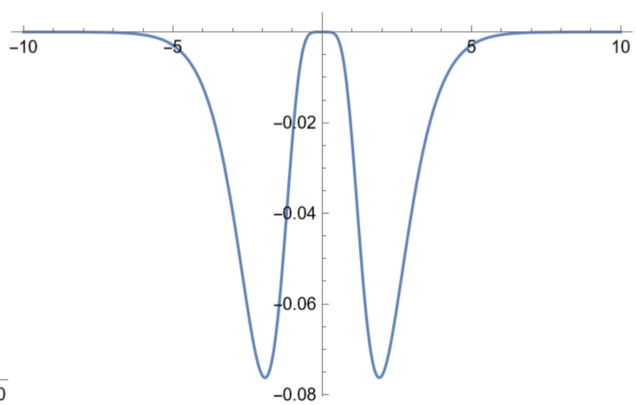
(c) If we take $g(x) = \tanh(x), w = 1,$ and $N = 1$, then we have

$$
H(u) = -\frac{1}{2}\tanh^2(u) + \int_0^{\tanh(u)} \tanh^{-1}(v)\mathrm{d}v.
$$

Graphing $H(u)$ in Figure 1 (a) we see that $H(u) > 0, \ \forall u \neq \vec{0}$. Graphing $H'(u)$ in Figure 1 (b) we see that $H'(u) > 0, \ \forall u \neq \vec{0}$. We clearly have that $H(\vec{0}) = 0$. Thus $H(u)$ is a Lyapunov function that shows that the equilibrium $\overline{u} = \vec{0}$ of the system is asymptotically stable.



(a) Graph of $H(u)$.



(b) Graph of $H'(u)$.

Figure 1: Graphs of $H(u)$ and $H'(u)$

(d) If W is antisymmetric, then the bound $\frac{\mathrm{d}H}{\mathrm{d}t} \leq 0$ does not hold in general. For example consider when

$$W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -W^T,$$

and $g(u_i) = \tanh(u_i)$. Then H is given by

$$\begin{aligned} H = &-\frac{1}{2}(0 + \tanh u_1 \tanh u_2 - \tanh u_2 \tanh u_1 + 0) \\ &+ \int_0^{\tanh(u_1)} \tanh^{-1}(V)\mathrm{d}V + \int_0^{\tanh(u_2)} \tanh^{-1}(V)\mathrm{d}V, \end{aligned}$$

and thus H' is

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= \tanh^{-1}(\tanh(u_1))(\tanh(u_1))'u_1' + \tanh^{-1}(\tanh(u_2))(\tanh(u_2))'u_2' \\ &= u_1 u_1' \mathrm{sech}^2(u_1) + u_2 u_2' \mathrm{sech}^2(u_2) \\ &= u_1(-u_1 + \tanh(u_2))\mathrm{sech}^2(u_1) + u_2(-u_2 + \tanh(u_1))\mathrm{sech}^2(u_2) \end{aligned}$$

Thus we can see that $H'(u_1, u_2)$ is not less than or equal to zero for all (u_1, u_2). For example, consider $H(0.25, 5) = 0.31985 \not\leq 0$. Thus we no longer have the bound $\frac{\mathrm{d}H}{\mathrm{d}t} \leq 0$ when W is antisymmetric.

□

Problem 6 *A specific case of the Lorenz equations is given by*

$$\begin{cases} x' = 10(-x+y) \\ y' = rx - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases} \tag{3}$$

a For varying r, find all equilibrium points and discuss their stability.

b Calculate up to second order terms the local invariant manifolds W^u, W^s and W^c for the fixed point at the origin of the Lorenz equations when $r = 1$.

Solution.

(a) We consider the system

$$\begin{cases} x' = 10(-x+y) \\ y' = rx - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases}$$

We can find the equilibrium points when $x' = y' = z' = 0$. Using Mathematica we can compute the fixed points to be:

$$\overline{(x,y,z)} \in \left\{ (0,0,0), \left(-2\sqrt{\frac{2}{3}}\sqrt{r-1}, -2\sqrt{\frac{2}{3}}\sqrt{r-1}, r-1 \right), \right.$$
$$\left. \left(2\sqrt{\frac{2}{3}}\sqrt{r-1}, 2\sqrt{\frac{2}{3}}\sqrt{r-1}, r-1 \right) \right\}.$$

Next we can compute the Jacobian of the system to be

$$J\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ r-z & -1 & -x \\ y & x & -\frac{8}{3} \end{pmatrix}.$$

Evaluating the Jacobian at the fixed point $(0,0,0)$ gives

$$J\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix},$$

which has eigenvalues

$$\lambda \in \left\{ -\frac{8}{3}, \frac{1}{2}\left(-\sqrt{40r+81} - 11\right), \frac{1}{2}\left(\sqrt{40r+81} - 11\right) \right\}.$$

Using Mathematica we can determine that $\mathrm{Re}(\lambda_i) < 0$ when $r < 1$, which shows that the fixed point $(0,0,0)$ is asymptotically stable when $r < 1$. Furthermore, when $r > 1$,

$\mathrm{Re}(\lambda_i) > 0$ which means that the fixed point is unstable. When $r = 1$, $\mathrm{Re}(\lambda_i) = 0$ and thus the fixed point is undetermined in this case.

Evaluating the Jacobian at the fixed point $\left(-2\sqrt{\frac{2}{3}}\sqrt{r-1}, -2\sqrt{\frac{2}{3}}\sqrt{r-1}, r-1\right)$ gives

$$J\begin{pmatrix} -2\sqrt{\frac{2}{3}}\sqrt{r-1} \\ -2\sqrt{\frac{2}{3}}\sqrt{r-1} \\ r-1 \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & 2\sqrt{\frac{2}{3}}\sqrt{r-1} \\ -2\sqrt{\frac{2}{3}}\sqrt{r-1} & -2\sqrt{\frac{2}{3}}\sqrt{r-1} & -\frac{8}{3} \end{pmatrix},$$

which has eigenvalues

$$\lambda \in \Bigg\{ -\frac{1}{9}\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201} + \frac{8r-\frac{961}{9}}{\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201}} - \frac{41}{9},$$
$$\frac{1}{18}(1-i\sqrt{3})\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201} - \frac{(1+i\sqrt{3})\left(8r-\frac{961}{9}\right)}{2\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201}} - \frac{41}{9},$$
$$\frac{1}{18}(1+i\sqrt{3})\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201} - \frac{(1-i\sqrt{3})\left(8r-\frac{961}{9}\right)}{2\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201}} - \frac{41}{9}\Bigg\}.$$

Using Mathematica we can determine that $\mathrm{Re}(\lambda_i) < 0$ when $1 < r < \frac{470}{19}$, which shows that the fixed point $\left(-2\sqrt{\frac{2}{3}}\sqrt{r-1}, -2\sqrt{\frac{2}{3}}\sqrt{r-1}, r-1\right)$ is asymptotically stable when $1 < r < \frac{470}{19}$. Furthermore, when $r < 1$ or $r > \frac{470}{19}$, $\mathrm{Re}(\lambda_i) > 0$ which means that the fixed point is unstable. When $r = 1$ or $r = \frac{470}{19}$, $\mathrm{Re}(\lambda_i) = 0$ and thus the fixed point is undetermined in this case.

Evaluating the Jacobian at the fixed point $\left(2\sqrt{\frac{2}{3}}\sqrt{r-1}, 2\sqrt{\frac{2}{3}}\sqrt{r-1}, r-1\right)$ gives

$$J\begin{pmatrix} 2\sqrt{\frac{2}{3}}\sqrt{r-1} \\ 2\sqrt{\frac{2}{3}}\sqrt{r-1} \\ r-1 \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -2\sqrt{\frac{2}{3}}\sqrt{r-1} \\ 2\sqrt{\frac{2}{3}}\sqrt{r-1} & 2\sqrt{\frac{2}{3}}\sqrt{r-1} & -\frac{8}{3} \end{pmatrix},$$

which has eigenvalues

$$\lambda \in \Bigg\{ -\frac{1}{9}\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201} + \frac{8r-\frac{961}{9}}{\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201}} - \frac{41}{9},$$
$$\frac{1}{18}(1-i\sqrt{3})\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201} - \frac{(1+i\sqrt{3})\left(8r-\frac{961}{9}\right)}{2\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201}} - \frac{41}{9},$$
$$\frac{1}{18}(1+i\sqrt{3})\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201} - \frac{(1-i\sqrt{3})\left(8r-\frac{961}{9}\right)}{2\sqrt[3]{36\sqrt{3}\sqrt{96r^3+54119r^2+91470r-221310}+15012r+5201}} - \frac{41}{9}\Bigg\}.$$

Using Mathematica we can determine that $\mathrm{Re}(\lambda_i) < 0$ when $1 < r < \frac{470}{19}$, which shows that the fixed point $\left(-2\sqrt{\frac{2}{3}}\sqrt{r-1}, -2\sqrt{\frac{2}{3}}\sqrt{r-1}, r-1\right)$ is asymptotically stable when $1 < r < \frac{470}{19}$. Furthermore, when $r < 1$ or $r > \frac{470}{19}$, $\mathrm{Re}(\lambda_i) > 0$ which means that the fixed point is unstable. When $r = 1$ or $r = \frac{470}{19}$, $\mathrm{Re}(\lambda_i) = 0$ and thus the fixed point is undetermined in this case.

(b) Next we wish to calculate up to second order terms the local invariant manifolds for the fixed point at the origin with $r = 1$. First notice that the Jacobian evaluated at the origin is

$$J\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix},$$

which has the eigenpairs:

$$\lambda_1 = -11, \qquad v_1 = \begin{bmatrix}-10\\1\\0\end{bmatrix},$$

$$\lambda_2 = -\frac{8}{3}, \qquad v_2 = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$\lambda_3 = 0, \qquad v_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

Since $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) < 0$ they correspond to stable parts of the solution while $\mathrm{Re}(\lambda_3) = 0$ corresponds to center part of the solution. Thus we are expecting

$$\dim(W^u) = 0, \;\; \dim(W^s) = 2, \;\; \text{and} \;\; \dim(W^c) = 1.$$

Now we introduce a coordinate transform

$$\begin{pmatrix}u'\\v'\\w'\end{pmatrix} = T^{-1}AT\begin{pmatrix}u\\v\\w\end{pmatrix} + T^{-1}R\left(T\begin{pmatrix}u\\v\\w\end{pmatrix}\right) =,$$

where $A = J((0,0,0)^T)$, $T = \begin{pmatrix} -10 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and $R = (0, -xz, xy)^T$ which correspond to the nonlinear terms in the system of equations, thus we get

$$\begin{pmatrix}u'\\v'\\w'\end{pmatrix} = \begin{pmatrix} -\frac{1}{11}v(w-10u) - 11u \\ (w-10u)(u+w) - \frac{8v}{3} \\ \frac{-10}{11}v(w-10u) \end{pmatrix}.$$

First let's find the local invariant stable manifold W^s. From class we know that we are searching for a polynomial of the form

$$w = h(u, v),$$

by the Implicit Function Theorem. Let's let w be some general polynomial of the form

$$h(u,v) = au + bv + cuv + du^2 + \rho v^2 + \dots$$

and since the manifold is invariant we may take a derivative to get

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial h}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial h}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}.$$

We get the LHS to be

$$\frac{1}{11}(-10)v\left(au + bv + cuv + du^2 - 10u + \rho v^2\right) + \dots$$

and the RHS to be

$$(a + cv + 2du)\left(-\frac{1}{11}v\left(au + bv + cuv + du^2 - 10u + \rho v^2\right) - 11u\right)$$
$$+ (b + cu + 2\rho v)\left(\left(au + bv + cuv + du^2 - 10u + \rho v^2\right)\left(au + bv + cuv + du^2 + u + \rho v^2\right) - \frac{8v}{3}\right) + \dots$$

We can drop the linear terms because there are none in w' and thus $b = 0$ and $a = 0$. We are also searching for a solution up to second order so we can toss out all the higher order terms. We can set the uv term on the LHS equal to the one on the RHS to find that

$$\frac{100}{11} = -11c - \frac{8}{3}c \implies c = -\frac{300}{451}.$$

Next we can set the u^2 term on the LHS equal to the one on the RHS to find that

$$-22u^2 d = 0 \implies d = 0.$$

Next we can set the v^2 term on the LHS equal to the one on the RHS to find that

$$-\frac{16}{3}\rho = 0 \implies \rho = 0.$$

Thus we have found that

$$h(u,v) = -\frac{300}{451}uv + \dots,$$

and transforming this back into cartesian coordinates using

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} z \\ \frac{y-x}{11} \\ \frac{x+10y}{11} \end{bmatrix}$$

we find that the local invariant stable manifold W^s is characterized by

$$\frac{x+10y}{11} = -\frac{300}{451}(z)\left(\frac{y-x}{11}\right) + \cdots \implies x + 10y + \frac{300}{451}(zy - zx) = 0 + \dots.$$

Next let's find the local invariant center manifold W^c. To do so, let's consider the system of equations

$$\begin{aligned} u &= h_1(w) = a_1 + b_1 w + c_1 w^2 + \dots, \\ v &= h_2(w) = a_2 + b_2 w + c_2 w^2 + \dots. \end{aligned}$$

Since we know that the manifold passes through the fixed point $(0,0,0)$ and must be tangent to E^c, we can set $a_1 = b_1 = a_2 = b_2 = 0$ and since the manifold is invariant we can take a derivative to get

$$\begin{cases} u' = \frac{\partial h_1}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t} + \dots \\ v' = \frac{\partial h_2}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t} + \dots. \end{cases}$$

$$\implies \begin{cases} -11c_1w^2 - \frac{1}{11}c_2w^2(-10c_1w^2 + w) & = (2c_1w)(-\frac{10}{11}c_2w^2(-10c_1w^2 + w)) + \dots \\ -\frac{8}{3}c_2w^2 + (-10c_1w^2 + w)(c_1w^2 + w) & = (2c_2w)(-\frac{10}{11}c_2w^2(-10c)1w^2 + w) + \dots \end{cases}$$

Since we are searching for solutions up to second order, let's drop all higher order terms yielding

$$\begin{cases} 11c_1w^2 = 0 \\ -\frac{8}{3}c_2w^2 + w^2 = 0 \end{cases} \implies \begin{cases} -11c_1 = 0 \\ -\frac{8}{3}c_2 + 1 = 0 \end{cases} \implies \begin{cases} c_1 = 0 \\ c_2 = \frac{3}{8}. \end{cases}$$

Thus we have found that

$$\begin{cases} v = \frac{3}{8}w^2 \\ u = 0 \end{cases}$$

and transforming back into cartesian coordinates, we find that the local invariant center manifold is characterized by

$$\begin{cases} \frac{-x+y}{11} = \frac{3}{8}\left(\frac{x+10y}{11}\right) + \dots \\ z = 0 + \dots \end{cases}$$

□