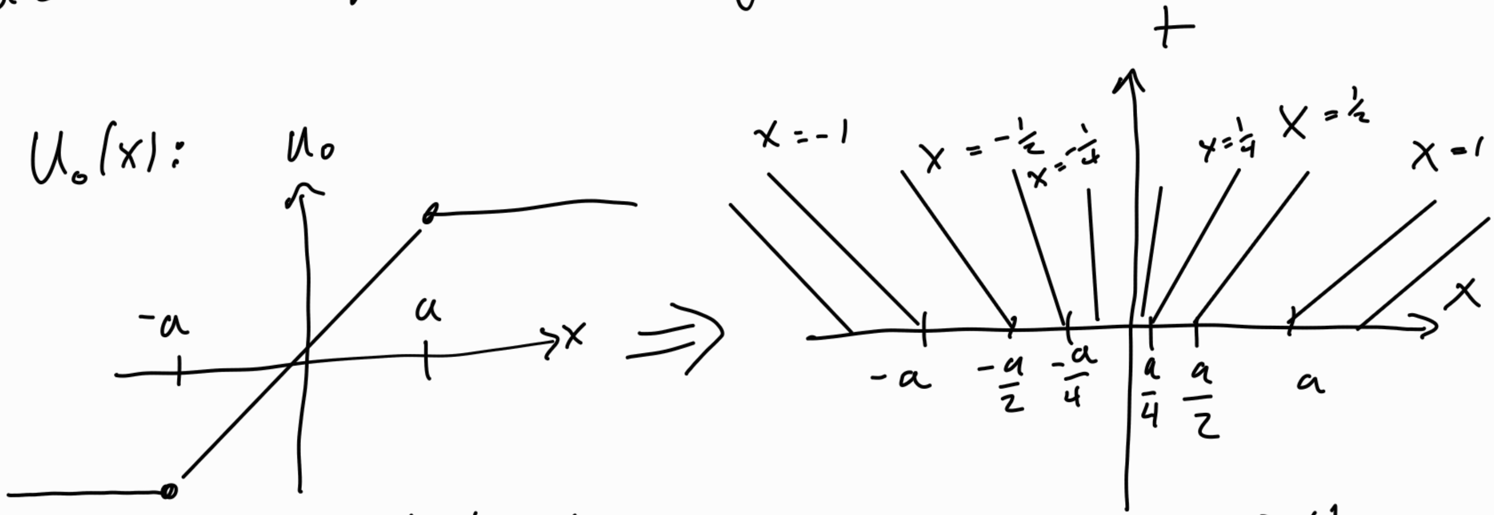
1) $u_t + u u_x = 0, \quad -\infty < x < \infty, \quad t > 0,$

with I.C.

$$u(x,0) = u_0(x) = \begin{cases} -1, & -\infty < x \le -a, \quad a > 0, \\ \frac{x}{a}, & -a < x < a, \\ 1, & a \le x < \infty \end{cases}$$

Let's use the method of Characteristics:



We can see that there are no shocks $\forall t$

$$X_t = U_0(\xi) \Rightarrow x(t) = U_0(\xi)t + x(0)$$
$$= U_0(\xi)t + \xi$$
$$\Rightarrow \xi = x - U_0(\xi)t$$

where ξ is a constant of integration. Plug in this into $u(x,t)$ yields:

$$u(x,t) = U_0(\xi)$$
$$= U_0(x - U_0(\xi)\,t) = U_0(x - ct)$$

We know that $u_0(\xi) = \begin{cases} -1, & -\infty < \xi < -a \\ \xi/a, & -a \le \xi \le a \\ 1, & a < \xi < \infty \end{cases}$

when $\xi < -a$, then $\xi = x + t \Rightarrow x + t < -a \Rightarrow x < -a - t$

when $\xi > a$, then $\xi = x - t \Rightarrow x - t < a \Rightarrow x < a + t$

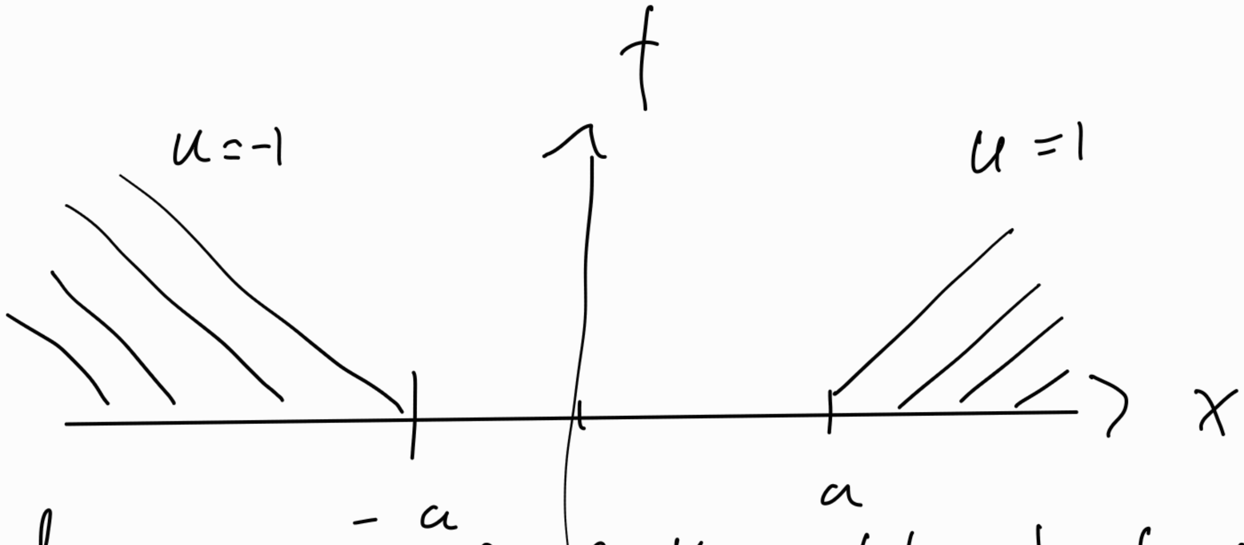
When $-a \le \xi \le a$, we have $\xi = x - \xi/a\, t$

$$\Rightarrow \xi + \xi/a\, t = x$$

$$\Rightarrow \xi\left(1 + \frac{t}{a}\right) = x$$

$$\Rightarrow \xi = x\left(1 + \frac{a}{t}\right)$$

$$\Rightarrow \xi = \frac{ax}{a+t}$$



Therefore we have found the solution to be:

$$u(x,t) = \begin{cases} -1 & x < -a - t \\ \dfrac{x}{a+t} & -a \le x \le a \\ 1 & x > a + t \end{cases} \qquad a, t > 0$$

2) Consider $u_t + uu_x = 0$ with the initial condition

$$u(x,0) = u_0(x) = \begin{cases} 1, & x \le 0 \\ 1-x, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$$

a) We wish to find where and when a shock first forms

We know a shock occurs when there is a discontinuity in u. From class we know that

$$x_t = u = u_0(\xi) \Rightarrow x = \xi + tu_0(\xi)$$
$$\Rightarrow \xi = x - tu_0(\xi)$$
$$\therefore u(x,t) = u_0(\xi) = u_0(x - tu_0(\xi))$$

Consider the chain rules:

$u_x = u_0'(\xi)\xi_x$ and $u_t = u_0'(\xi)\xi_t$.

As $\xi = x - tu_0(\xi)$,

$$\xi_x = 1 - tu_{0\xi}\xi_x \Rightarrow \xi_x = \frac{1}{1 + tu_{0\xi}}$$

showed in class that ξ_t has same denominator as ξ_t

Thus there will be a discontinuity when

$$1 + tu_{0\xi} = 0 \Rightarrow t^* = \frac{-1}{u_{0\xi}(\xi^*)} \quad (*)$$

We defined the shock to occur at minimum of $(*)$ s.t. t^* is positive. Thus we pick ξ^*

to correspond to the characteristic for which $u_{0\xi} < 0$ and $|u_{0\xi}|$ is maxed. In our case

$$u_0(\xi) = \begin{cases} 1, & \xi \leq 0 \\ 1-\xi, & 0 < \xi < 1 \\ 0, & \xi \geq 1 \end{cases}$$

$$\Rightarrow u_{0\xi}(\xi) = \begin{cases} 0, & \xi \leq 0 \\ -1, & 0 < \xi < 1 \\ 0, & \xi \geq 1 \end{cases}$$

$\therefore$ $0 < \xi^* < 1$, $\xi^* = -1$. Thus the shock forms at

$$t^* = -\frac{1}{u_{0\xi}(\xi^*)} = 1$$

and to find the location recall that

$$x = t u_0(\xi) + \xi$$

and if $0 < \xi^* < 1$, then $u_0(\xi^*) = 1 - \xi$. Thus

$$x(\xi^*; t) = u_0(\xi^*) + \xi^* = (1 - \xi^*) + \xi^* = 1$$

Thus the shock happens at $(x,t) = (1,1)$.

b) Now we wish to solve the problem before the shock forms, i.e. $t < t^*$ where we have

$$\xi = x - t u_0(\xi)$$

If $\xi \leq 0$, then $u_0(\xi) = 1$ and $\xi = x - t$

If $0 < \xi < 1$, then $u_0(\xi) = 1 - \xi$ and

$$\xi = x - (1-\xi)t$$

$$\Rightarrow \xi = \frac{x-t}{1-t}$$

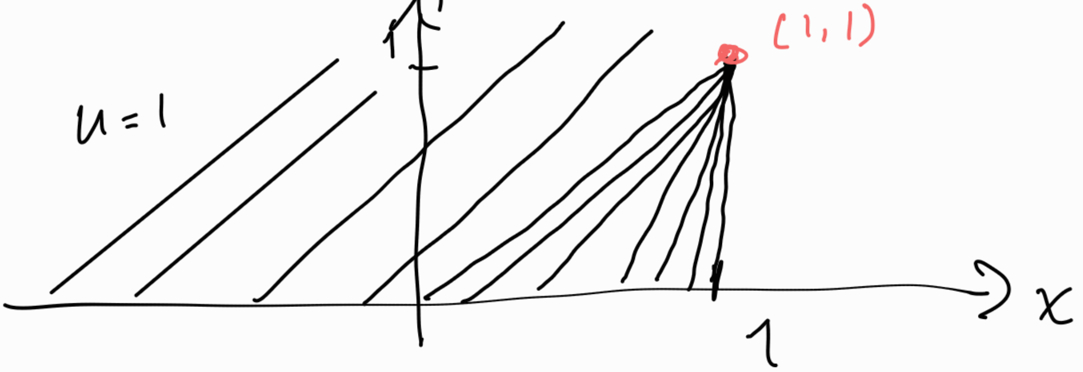
and if $\xi \geq 1$, then $u_0(\xi) = 0$ and $\xi = x$

Thus

$$u(x,t) = \begin{cases} 1, & x - t \leq 0 \\ 1 - \frac{x-t}{1-t}, & 0 < \frac{x-t}{1-t} < 1 \\ 0, & x \geq 1 \end{cases}$$

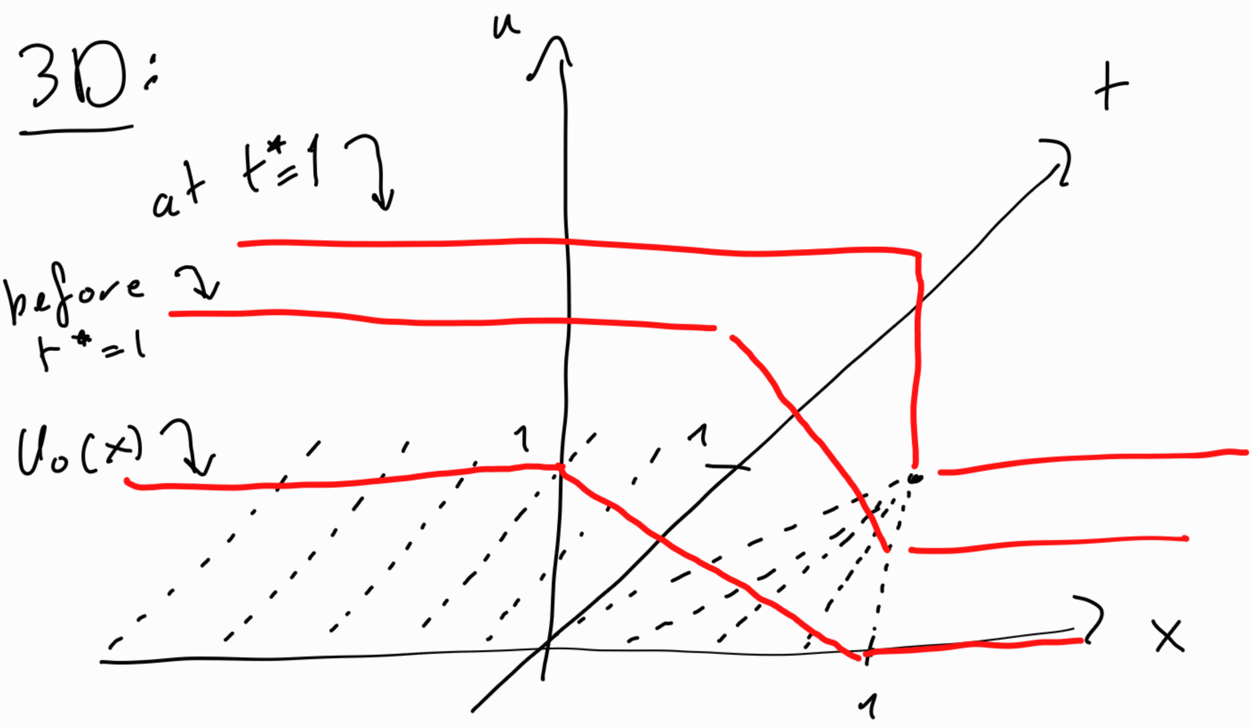
$$\Leftrightarrow u(x,t) = \begin{cases} 1, & x \leq t, \\ \frac{1-x}{1-t}, & t < x < 1, \quad 0 < t < 1 \\ 0, & x \geq 1 \end{cases}$$

We can sketch the solution as:

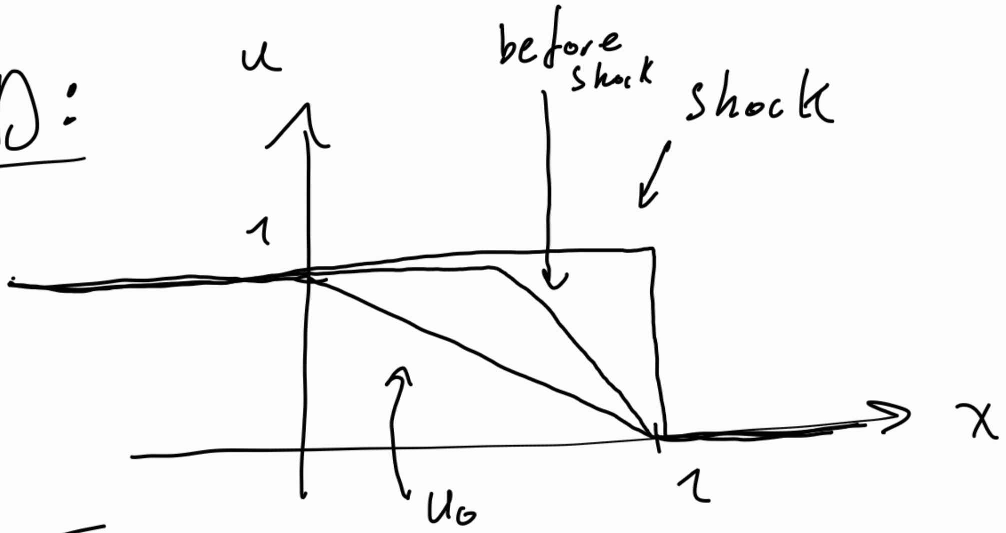


note that $x(0) \in (0,1)$ have to pass through $(1,1)$.

3D:



2D:



c) To find the shock speed, we will use the Rankine-Hugoniot condition for Burger's equation which states:

For a shock at $s(t)$, $s'(t) = \frac{1}{2}(u^- + u^+)$

where $u^- = \lim_{x \to s(t)^-} u(x,t)$ and $u^+ = \lim_{x \to s(t)^+} u(x,t)$.

Thus the speed is

$$s' = \frac{1}{2}(1+0) = 1/2.$$

d) Finally we wish to solve the problem after the shock has formed. The shock forms at $(1,1)$ with speed $1/2$. An equation that satisfies this is

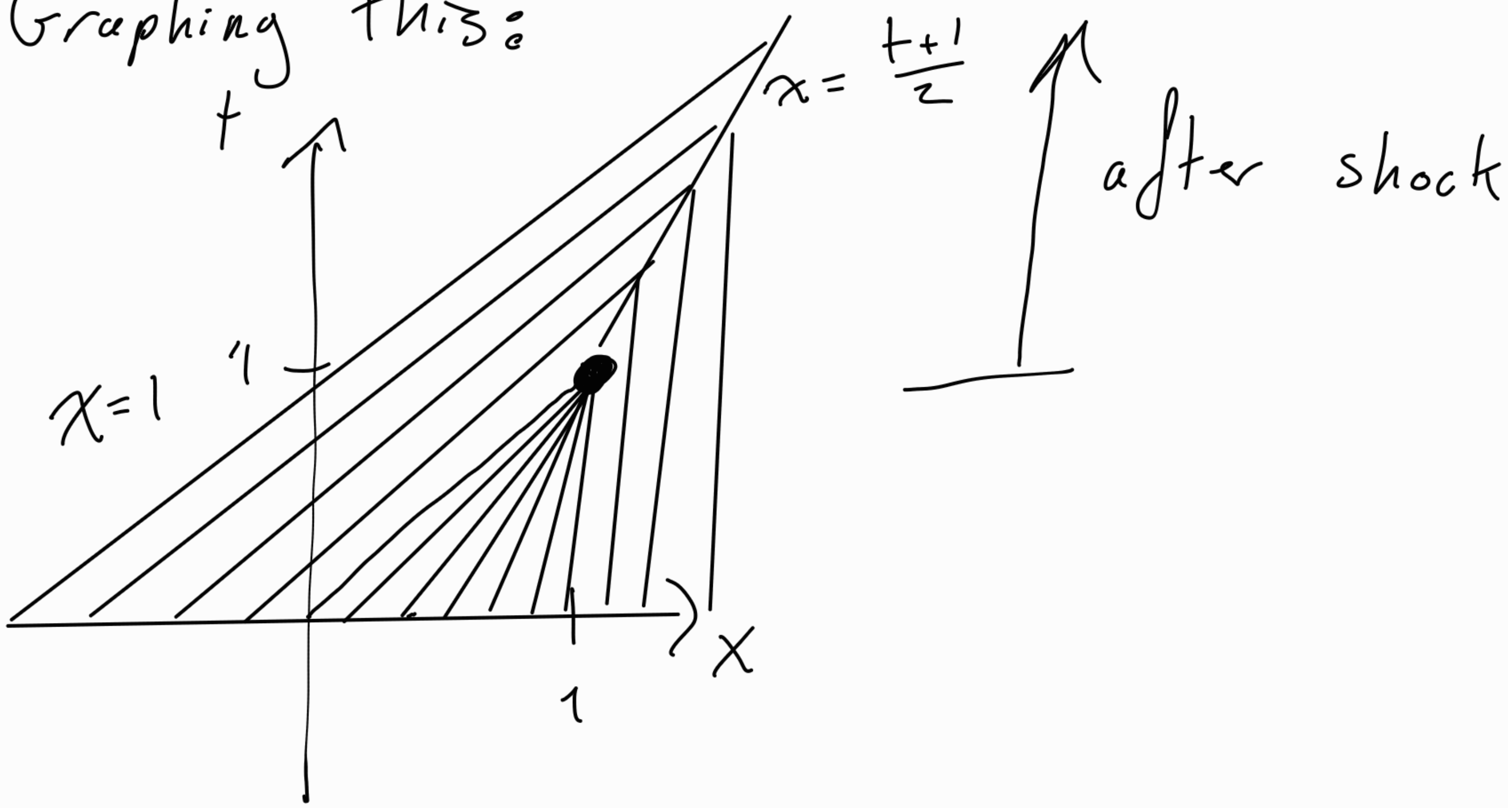
$$x - 1 = \frac{1}{2}(t-1)$$

$$\Rightarrow x = \frac{1}{2}t + \frac{1}{2}$$

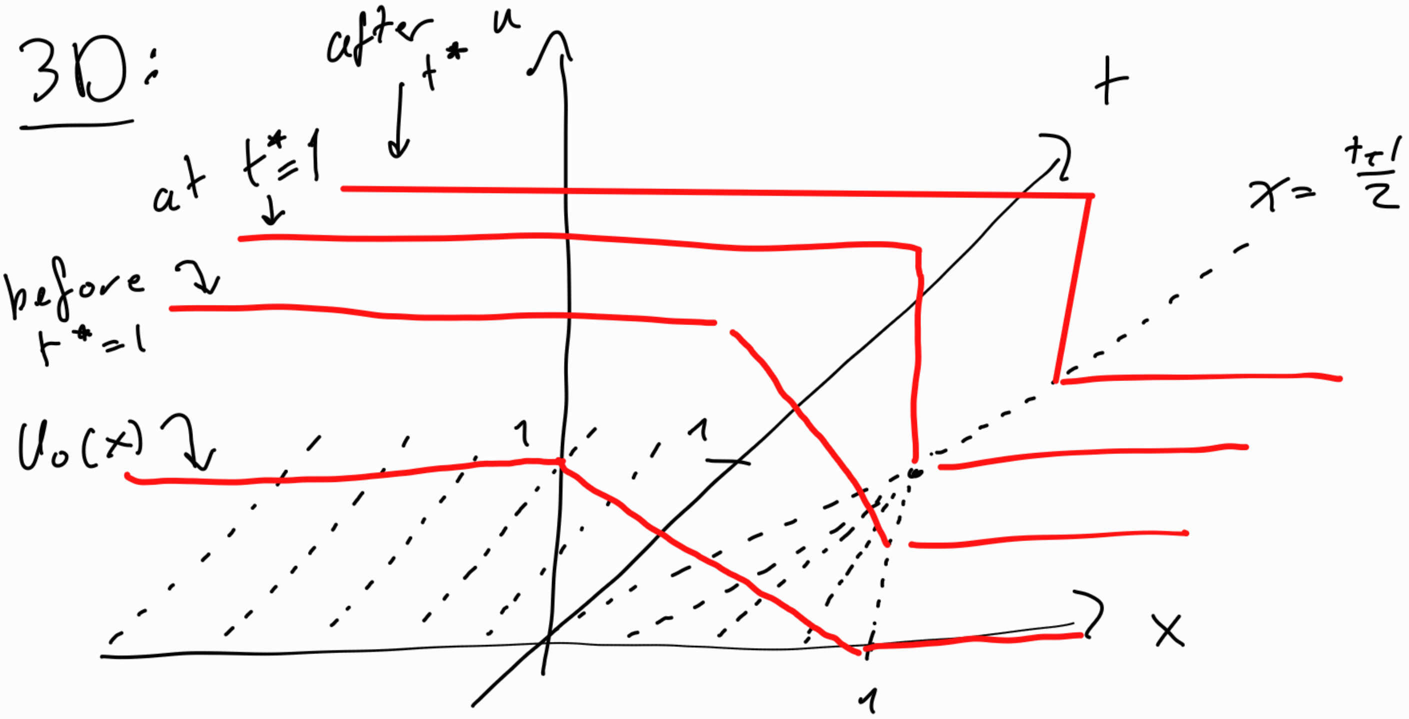
Thus the shock follows $x = \frac{1}{2}t + \frac{1}{2}$ after it's formation. Thus the solution is given by

$$u(x,t) = \begin{cases} 1, & x < \frac{t+1}{2} \\ 0, & x > \frac{t+1}{2} \end{cases}$$

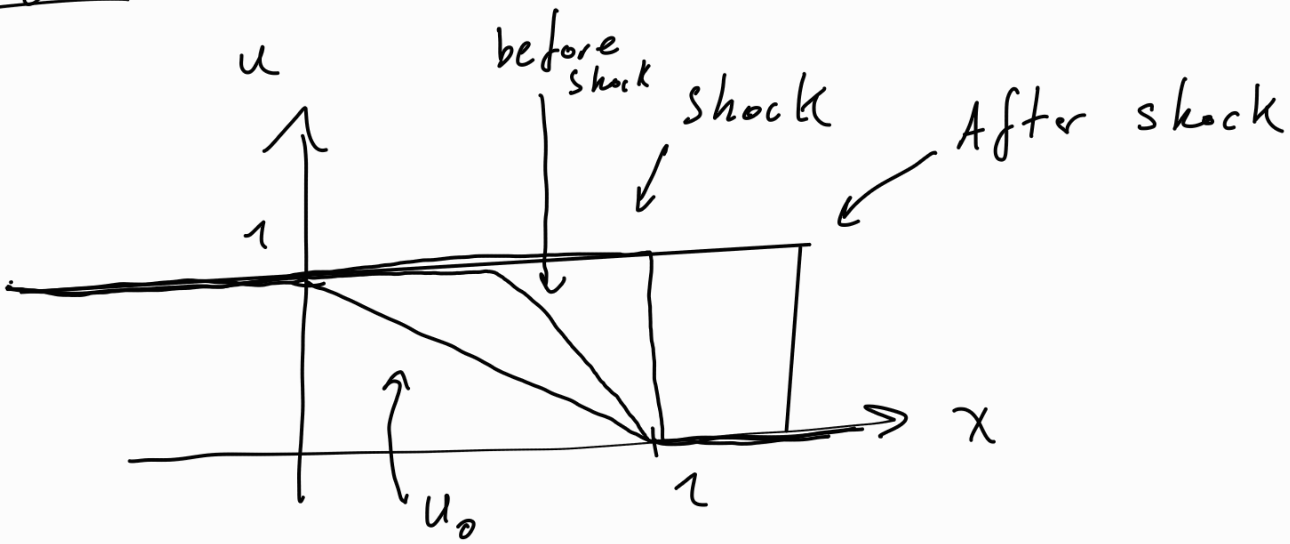
Graphing this:



3D:


after t^*
u
t
at $t^*=1$
$x = \frac{t+1}{2}$
before $t^*=1$
$u_0(x)$
1
1
1
x
1

2D:


u
before shock
shock
After shock
1
x
u_0
1

Q3:

Consider $u_t + u u_x = u \quad -\infty < x < \infty, \ t \geq 0$

with I.C. $u(x,0) = u_0(x) = 2x, \ 0 \leq x \leq 2$

We wish to know where in the x-t plane is the solution valid.

Let's use the method of characteristics:

The characteristics are given by

$$\frac{dx}{dt} = u \qquad (1)$$

and the solution along characteristic satisfies

$$\frac{du}{dt} = u \qquad (2)$$

Solving (2) yields

$$u = c e^{t}$$

and enforcing the IC

$$u(\xi, 0) = c e^{0} = 2\xi$$

$$\Rightarrow c = 2\xi$$

$$\therefore \ u = 2\xi e^{t} \quad \text{for } \xi \in [0, 2]$$

Plugging this solution into (1)

$$\frac{dx}{dt} = 2\xi e^t$$

which has the solution

$$x = 2\xi e^t + C$$

We want this to be the equation of the characteristic curve that passes through $x = \xi$ and $t = 0$, thus

$$\xi = 2\xi + C \Rightarrow C = -\xi$$

$$\therefore x = 2\xi e^t - \xi$$

$$\Rightarrow \xi = \frac{x}{2e^t - 1} \qquad \text{for } \xi \in [0, 2]$$

Thus

$$u(x,t) = \frac{2x}{2e^t - 1} e^t$$

for $x \in [0, 2(2e^t - 1)]$
$t \in \mathbb{R}$