Math 573 Homework 3
Due November 4
By Marvyn Bailly

Problem 1 The KdV equation for ion-acoustic waves in plasmas. *Ion-acoustic waves are low-frequency electrostatic waves in a plasma consisting of electrons and ions. We consider the case with a single ion species.*

Consider the following system of one-dimensional equations

$$\begin{aligned}\frac{\partial n}{\partial t}+\frac{\partial}{\partial z}(nv)&=0\\ \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}&=-\frac{e}{m}\frac{\partial \phi}{\partial z}\\ \frac{\partial^2\phi}{\partial z^2}&=\frac{e}{\varepsilon_0}\left[N_0\exp\left(\frac{e\phi}{\kappa T_e}\right)-n\right]\end{aligned}$$

Here n denotes the ion density, v is the ion velocity, e is the electron charge, m is the mass of an ion, ϕ is the electrostatic potential, ε_0 is the vacuum permittivity, N_0 is the equilibrium density of the ions, κ is Boltzmann's constant, and T_e is the electron temperature.

a *Verify that $c_s=\sqrt{\frac{\kappa T_e}{m}}$, $\lambda_{De}=\sqrt{\frac{\varepsilon_0\kappa T_e}{N_0e^2}}$, and $\omega_{pi}=\sqrt{\frac{N_0e^2}{\varepsilon_0 m}}$ have dimensions of velocity, length and frequency, respectively. These quantities are known as the ion acoustic speed, the Debye wavelength for the electrons, and the ion plasma frequency.*

b *Nondimensionalize the above system, using*

$$n=N_0n^*,\quad v=c_sv^*,\quad z=\lambda_{De}z^*,\quad t=\frac{t^*}{\omega_{pi}},\quad \phi=\frac{\kappa T_e}{e}\phi^*.$$

c *You have obtained the system*

$$\begin{aligned}\frac{\partial n}{\partial t}+\frac{\partial}{\partial z}(nv)&=0\\ \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}&=-\frac{\partial \phi}{\partial z}\\ \frac{\partial^2\phi}{\partial z^2}&=e^{\phi}-n\end{aligned}$$

for the dimensionless variables. Note that we have dropped the $*$'s, to ease the notation. Find the linear dispersion relation for this system, linearized around the trivial solution $n=1$, $v=0$, and $\phi=0$.

d *Rewrite the system using the "stretched variables"*

$$\xi = \epsilon^{1/2}(z - t), \quad \tau = \epsilon^{3/2} t.$$

Given that we are looking for low-frequency waves, explain how these variables are inspired by the dispersion relation.

e *Expand the dependent variables as*

$$\begin{aligned} n =& 1 + \epsilon n_1 + \epsilon^2 n_2 + \ldots, \\ v =& \epsilon v_1 + \epsilon^2 v_2 + \ldots, \\ \phi =& \epsilon \phi_1 + \epsilon^2 \phi_2 + \ldots. \end{aligned}$$

Using that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$, find a governing equation which determines how ϕ_1 depends on ξ and τ.

Solution.

Consider the following system of one-dimensional equations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{e}{m}\frac{\partial \phi}{\partial z} \tag{2}$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{e}{\varepsilon_0}\left[N_0 \exp\left(\frac{e\phi}{\kappa T_e}\right) - n\right] \tag{3}$$

Here n denotes the ion density, v is the ion velocity, e is the electron charge, m is the mass of an ion, ϕ is the electrostatic potential, ε_0 is the vacuum permittivity, N_0 is the equilibrium density of the ions, κ is Boltzmann's constant, and T_e is the electron temperature.

a We begin by verifying that $c_s = \sqrt{\frac{\kappa T_e}{m}}$, $\lambda_{De} = \sqrt{\frac{\varepsilon_0 \kappa T_e}{N_0 e^2}}$, and $\omega_{pi} = \sqrt{\frac{N_0 e^2}{\varepsilon_0 m}}$ have dimensions of velocity, length and frequency, respectively. Let's begin by listing all the SI units for the values. Recall that, κ is $\frac{\mathrm{m^2 kg}}{\mathrm{s^2 K}}$, T_e is K, N_0 is $\frac{1}{\mathrm{m}^3}$, and N_0 is $\frac{\mathrm{A^2 s^4}}{\mathrm{kg m^3}}$. Now let's check the units for $c_s = \sqrt{\frac{\kappa T_e}{m}}$,

$$c_s = \sqrt{\frac{\mathrm{m^2 kg} K}{\mathrm{s^3 kg\ K}}} = \sqrt{\frac{\mathrm{m}^2}{\mathrm{s}^2}} = \frac{\mathrm{m}}{\mathrm{s}},$$

which are the units for velocity. Next let's look at $\lambda_{De} = \sqrt{\frac{\varepsilon_0 \kappa T_e}{N_0 e^2}}$,

$$\lambda_{De} = \sqrt{\frac{\mathrm{A^2 s^4}}{\mathrm{kg m^3}}\frac{\mathrm{m^2 kg}}{\mathrm{s^2 K}}\frac{\mathrm{K}}{1}\frac{\mathrm{m}^3}{1}\frac{1}{\mathrm{A^2 s^2}}} = \sqrt{\frac{\mathrm{m}^2}{1}} = m,$$

which are the units for length. Next let's look at $\omega_{pi} = \sqrt{\frac{N_0 e^2}{\varepsilon_0 m}}$,

$$w_{pi} = \sqrt{\frac{1}{\mathrm{m}^3}\frac{\mathrm{A}^2\mathrm{s}^2}{1}\frac{\mathrm{kgm}^3}{\mathrm{A}^2\mathrm{s}^4}\frac{1}{\mathrm{kg}}} = \sqrt{\frac{1}{\mathrm{s}^2}} = \frac{1}{\mathrm{s}},$$

which are the units for frequency.

b Next let's Nondimensionalize the system using

$$n = N_0 n^*, \quad v = c_s v^*, \quad z = \lambda_{De} z^*, \quad t = \frac{t^*}{\omega_{pi}}, \quad \phi = \frac{\kappa T_e}{e}\phi^*.$$

Observe plugging the nondimensionalized units into (1),

$$\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) &= 0 \\
(N_0)(\omega_{pi})\frac{\partial n^*}{\partial t^*} + (N_0)(c_s)\left(\frac{1}{\lambda_{De}}\right)\frac{\partial n^* v^*}{\partial z^*} &= 0 \\
(\omega_{pi})\frac{\partial n^*}{\partial t^*} + (c_s)\left(\frac{1}{\lambda_{De}}\right)\frac{\partial n^* v^*}{\partial z^*} &= 0 \\
\left(\sqrt{\frac{N_0 e^2}{\varepsilon_0 m}}\right)\frac{\partial n^*}{\partial t^*} + \left(\sqrt{\frac{\kappa T_e}{m}}\right)\left(\sqrt{\frac{N_0 e^2}{\epsilon_0 \kappa T_e}}\right)\frac{\partial n^* v^*}{\partial z^*} &= 0 \\
\frac{\partial n^*}{\partial t^*} + \frac{\partial n^* v^*}{\partial z^*} &= 0.
\end{aligned}$$

Next let's plug the nondimensionalized units into (2),

$$\begin{aligned}
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} &= -\frac{e}{m}\frac{\partial \phi}{\partial z} \\
(c_s)(\omega_{pi})\frac{\partial v^*}{\partial t^*} + (c_s v^*)(c_s)\left(\frac{1}{\lambda_{De}}\right)\frac{\partial v^*}{\partial z^*} &= \left(-\frac{e}{m}\right)\left(\frac{\kappa T_e}{e}\right)\left(\frac{1}{\lambda_{De}}\right)\frac{\partial \phi^*}{\partial z^*} \\
\left(\sqrt{\frac{\kappa T_e}{m}}\right)\left(\sqrt{\frac{N_0 e^2}{\varepsilon_0 m}}\right)\frac{\partial v^*}{\partial t^*} + v^*\left(\frac{\kappa T_e}{m}\right)\left(\sqrt{\frac{N_0 e^2}{\epsilon_0 \kappa T_e}}\right)\frac{\partial v^*}{\partial z^*} &= \left(-\frac{1}{m}\right)\left(\frac{\kappa T_e}{1}\right)\left(\sqrt{\frac{N_0 e^2}{\epsilon_0 \kappa T_e}}\right)\frac{\partial \phi^*}{\partial z^*} \\
(\sqrt{\kappa T_e})\left(\sqrt{\frac{N_0 e^2}{\varepsilon_0}}\right)\frac{\partial v^*}{\partial t^*} + v^*(\kappa T_e)\left(\sqrt{\frac{N_0 e^2}{\epsilon_0 \kappa T_e}}\right)\frac{\partial v^*}{\partial z^*} &= -(\kappa T_e)\left(\sqrt{\frac{N_0 e^2}{\epsilon_0 \kappa T_e}}\right)\frac{\partial \phi^*}{\partial z^*} \\
(\sqrt{\kappa T_e})\frac{\partial v^*}{\partial t^*} + v^*(\kappa T_e)\left(\sqrt{\frac{1}{\kappa T_e}}\right)\frac{\partial v^*}{\partial z^*} &= -(\kappa T_e)\left(\sqrt{\frac{1}{\kappa T_e}}\right)\frac{\partial \phi^*}{\partial z^*} \\
\frac{\partial v^*}{\partial t^*} + v^*\frac{\partial v^*}{\partial z^*} &= -\frac{\partial \phi^*}{\partial z^*}.
\end{aligned}$$

Finally lets nondimensionalize (3),

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial z^2} &= \frac{e}{\varepsilon_0}\left[N_0 \exp\left(\frac{e\phi}{\kappa T_e}\right) - n\right] \\
\frac{\kappa T_e}{e}\left(\frac{N_0 e^2}{\epsilon_0 \kappa T_e}\right)\frac{\partial^2 \phi^*}{\partial z^{*2}} &= \frac{eN_0}{\epsilon_0}\left[e^{\phi^*} - n^*\right] \\
\frac{\partial^2 \phi^*}{\partial z^{*2}} &= e^{\phi^*} - n^*
\end{aligned}$$

Thus we have the nondimensionalized system,

$$\begin{aligned}
\frac{\partial n^*}{\partial t^*} + \frac{\partial}{\partial z^*}(n^* v^*) &= 0 \\
\frac{\partial v^*}{\partial t^*} + v^* \frac{\partial v^*}{\partial z^*} &= -\frac{\partial \phi^*}{\partial z^*} \\
\frac{\partial^2 \phi^*}{\partial z^{*2}} &= e^{\phi^*} - n^*
\end{aligned}$$

c Now let's drop the $*$'s in the nondimensionalized system to get,

$$\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) &= 0 \\
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} &= -\frac{\partial \phi}{\partial z} \\
\frac{\partial^2 \phi}{\partial z^2} &= e^{\phi} - n.
\end{aligned}$$

Next we will find the linear dispersion relation. Let's begin by linearizing around the trivial solution $n = 1$, $v = 0$, and $\phi = 0$. If we let,

$$\begin{aligned}
n &= 1 + \epsilon a + \mathcal{O}(\epsilon^2) \\
v &= \epsilon b + \mathcal{O}(\epsilon^2) \\
\phi &= 1\epsilon c + \mathcal{O}(\epsilon^2)
\end{aligned}$$

where ϵ is some small parameter. Plugging this into our system of equation we get,

$$\begin{aligned}
\epsilon a_t + \frac{\mathrm{d}}{\mathrm{d}z}\left[(1+\epsilon a)(\epsilon b)\right] + \mathcal{O}(\epsilon^2) &= 0 \\
\epsilon b_t + (\epsilon b)(\epsilon b_z) &= -\epsilon c_z + \mathcal{O}(\epsilon^2) \\
\epsilon c_{zz} &= \left(1 + \epsilon c + \frac{\epsilon^2 c^2}{2} + \cdots\right) - 1 - \epsilon a + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Taking the first order terms, we have our linearized system,

$$a_t + b_z = 0 \tag{4}$$

$$b_t = -c_z \tag{5}$$
$$c_{zz} = c - a. \tag{6}$$

Next let's find the dispersion relation by taking the partial of (4) with respect to t to get,

$$b_{zt} = -a_{tt}.$$

And by taking the partial of (5) with respect to z we have,

$$b_{tz} = -c_{zz},$$

which allows us to combine these equation to get,

$$-a_{tt} = -c_{zz} \implies a_{tt} = c_{tt}. \tag{7}$$

Next let's taking the second partial of (6) with respect to t to get,

$$c_{zztt} = c_{tt} - a_{tt},$$

and plugging (7) into that we have,

$$c_{zztt} = c_{tt} - c_{zz}.$$

If we let $c = e^{ikz - i\omega t}$ be a solution we can find the dispersion by observing,

$$\begin{aligned}
(ik)^2(-i\omega)^2 &= (-iw)^2 - (ik)^2 \\
(-k^2)(-\omega^2) &= -w^2 + k^2 \\
k^2\omega^2 &= -w^2 + k^2 \\
k^2\omega^2 + w^2 &= k^2 \\
\omega^2(k^2+1) &= k^2 \\
\omega^2 &= \frac{k^2}{k^2+1} \\
\omega &= \pm\sqrt{\frac{k^2}{k^2+1}},
\end{aligned}$$

which gives us our dispersion relation.

d To study longs waves, let's rewrite our system in "stretched variables." Let's begin by Taylor Expanding our dispersion relation around $k = 0$. Using Mathematica I found that,

$$\sqrt{2}k + \frac{k^3}{2\sqrt{2}} - \frac{k^5}{16\sqrt{(2)}} + \cdots .$$

Plugging this into $e^{ikz - iwt}$,

$$e^{ikz - it\left(\sqrt{2}k + \frac{k^3}{2\sqrt{2}} - \frac{k^5}{16\sqrt{(2)}} + \cdots\right)} = e^{ikz - i\sqrt{2}tk + it\frac{k^3}{2\sqrt{2}} - it\frac{k^5}{16\sqrt{2}} + \cdots}$$

$$= e^{ik\left(\sqrt{2}(z-t)+t\frac{k^2}{2\sqrt{2}}-t\frac{k^4}{16\sqrt{2}}+\cdots\right)}$$

and by letting $e = k^2$

$$e^{i\epsilon^{1/2}\left(\sqrt{2}(z-t)+\frac{1}{2\sqrt{2}}\epsilon t-\frac{1}{16\sqrt{2}}t\epsilon^2+\cdots\right)}.$$

Thus for long waves, we can look at the first few terms of the expansion to find that,

$$\xi = \epsilon^{1/2}(z-t), \quad \tau = \epsilon^{3/2}t.$$

to be our stretched variables. Next let's rewrite the system stretched variables, First let's compute the partials

$$\begin{aligned}
\frac{\partial \xi}{\partial z} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial}{\partial \xi}(\epsilon^{\frac{1}{2}}) \\
\frac{\partial^2 \xi}{\partial z^2} &= \frac{\partial}{\partial z}\left(\epsilon^{\frac{1}{2}}\frac{\partial}{\partial \xi}\right) = \epsilon\frac{\partial^2}{\partial \xi^2} \\
\frac{\partial \tau}{\partial t} &= \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} = \epsilon^{\frac{3}{2}}\frac{\partial}{\partial \tau} - \epsilon^{\frac{1}{2}}\frac{\partial}{\partial \xi}.
\end{aligned}$$

Plugging these into the nondimensionalized system we get

$$\begin{aligned}
\epsilon^{\frac{3}{2}}\frac{\partial n}{\partial \tau} - \epsilon^{\frac{1}{2}}\frac{\partial n}{\partial \xi} + \epsilon^{\frac{1}{2}}\frac{\partial nv}{\partial \xi} &= 0 \\
\epsilon^{\frac{3}{2}}\frac{\partial v}{\partial \tau} - \epsilon^{\frac{1}{2}}\frac{\partial v}{\partial \xi} + v\epsilon^{\frac{1}{2}}\frac{\partial v}{\partial \tau} &= -\epsilon^{\frac{1}{2}}\frac{\partial \phi}{\partial \xi} \\
\epsilon\frac{\partial^2 \phi}{\partial \xi^2} &= e^{\phi} - n,
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\epsilon\frac{\partial n}{\partial \tau} - \frac{\partial n}{\partial \xi} + \frac{\partial nv}{\partial \xi} &= 0 \\
\epsilon\frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial \xi} + v\frac{\partial v}{\partial \xi} &= 0 \\
\epsilon\frac{\partial^2 \phi}{\partial \xi^2} &= e^{\phi} - n.
\end{aligned}$$

e Now let's expand this system using the dependent variables as

$$\begin{aligned}
n &= 1 + \epsilon n_1 + \epsilon^2 n_2 + \ldots, \\
v &= \epsilon v_1 + \epsilon^2 v_2 + \ldots, \\
\phi &= \epsilon \phi_1 + \epsilon^2 \phi_2 + \ldots.
\end{aligned}$$

We also note that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$.

□

Problem 2 Obtaining the KdV equation from the NLS equation. *We have shown that the NLS equation may be used to describe the slow modulation of periodic wave trains of the KdV equation. In this problem we show that the KdV equation describes the dynamics of long-wave solutions of the NLS equation.*

Consider the defocusing NLS equation

$$ia_t = -a_{xx} + |a|^2 a.$$

a. *Let*

$$a(x,t) = e^{i \int V \, dx} \rho^{1/2}.$$

Derive a system of equations for the phase function $V(x,t)$ and for the amplitude function $\rho(x,t)$, by substituting this form of $a(x,t)$ in the NLS equation, dividing out the exponential, and separating real and imaginary parts. Write your equations in the form $\rho_t = \ldots$, and $V_t = \ldots$. Due to their similarity with the equations of hydrodynamics, this new form of the NLS equation is referred to as its hydrodynamic form.

b. *Find the linear dispersion relation for the hydrodynamic form of the defocusing NLS equation, linearized around the trivial solution $V = 0$, $\rho = 1$. In other words, we are examining perturbations of the so-called Stokes wave solution of the NLS equation, which is given by a signal of constant amplitude.*

c. *Rewrite the system using the "stretched variables"*

$$\xi = \epsilon(x - \beta t), \quad \tau = \epsilon^3 t.$$

Given that we are looking for long waves, explain how these variables are inspired by the dispersion relation. What should the value of β be?

d. *Expand the dependent variables as*

$$\begin{aligned} V =& \epsilon^2 V_1 + \epsilon^4 V_2 + \ldots, \\ \rho =& 1 + \epsilon^2 \rho_1 + \epsilon^4 \rho_2 + \ldots. \end{aligned}$$

Using that all disturbances return to their equilibrium values as $\xi \to \pm\infty$, $\tau \to \infty$, find a governing equation which determines how V_1 depends on ξ and τ. This equation should be equivalent to the KdV equation.

Solution.

Consider the defocusing NLS equation

$$ia_t = -a_{xx} + |a|a^2.$$

a. Let

$$a(x,t) = e^{i\int V\,dx}\rho^{1/2}.$$

Now let's find the partial,

$$\begin{aligned}
a_t &= i\frac{\partial}{\partial t}\left(\int vdx\right)e^{i\int vdx}\rho^{\frac{1}{2}} + e^{i\int vdx}\frac{1}{2\sqrt{\rho}}\rho_t \\
a_x &= ive^{i\int vdx}\rho^{\frac{1}{2}} + e^{i\int vdx}\rho^{-\frac{1}{2}}\left(\frac{1}{2}\rho_x\right) \\
a_{xx} &= \rho^{\frac{1}{2}}\left(ivde^{i\int vdx}(iv) + iv_x e^{i\int vdx}\right) + \frac{1}{2}\rho^{-\frac{1}{2}}(\rho_x)e^{i\int vdx} \\
&\quad + \frac{1}{2}\left(\rho^{-\frac{1}{2}}\left(e^{i\int vdx}(-\rho_x)(\rho_{xx}) + \rho_x(iv)e^{i\int vdx}\right) + e^{i\int vdx}(-\frac{1}{2}\rho^{-\frac{3}{2}}\rho_x)(\frac{1}{2}\rho_x)\right).
\end{aligned}$$

Plugging these values into the defocusing NLS equation and dividing out the exponential we get

$$\begin{aligned}
-\frac{\partial}{\partial t}\left(\int vdx\right)\rho^{\frac{1}{2}} + \frac{i}{2\sqrt{\rho}}\rho_t = iv_x\rho^{\frac{1}{2}} + v^2\rho^{\frac{1}{2}} - \frac{i}{2}v\rho^{-\frac{1}{2}}\rho_x - \frac{i}{2}vp^{-\frac{1}{2}}\rho_x \\
+ \frac{1}{4}\rho^{-\frac{3}{2}}\rho_x^2 - \frac{1}{2}\rho^{-\frac{1}{2}}\rho_{xx} + \left|e^{i\int vdx}\rho^{\frac{1}{2}}\right|^2\rho^{\frac{1}{2}}.
\end{aligned}$$

Taking the real part of this equation and evaluating $-\frac{\partial}{\partial t}\left(\int vdx\right) = -\int v_t dx$ using Leibniz's integral we get

$$\begin{aligned}
-\int v_t dx\rho^{\frac{1}{2}} &= v^2\rho^{\frac{1}{2}} + \frac{1}{4}\rho^{\frac{-3}{2}}\rho_x^2 - \frac{1}{2}\rho^{-\frac{1}{2}}\rho_{xx} + |\rho|\rho^{\frac{1}{2}} \\
-\int v_t dx &= v^2 + \frac{1}{4}\rho^{-2}\rho_x^2 - \frac{1}{2}\rho^{-1}\rho_{xx} + |\rho| \\
v_t &= -2vv_x - \rho^{-2}\rho_x\rho_{xx} + \frac{1}{2}\rho^{-3}\rho_x^3 + \frac{1}{2}\rho^{-1}\rho_{xxx} - |p|p^{-1}p_x.
\end{aligned}$$

Next we can find the imaginary part to be

$$\begin{aligned}
\frac{1}{2}\rho^{-\frac{1}{2}}\rho_t &= -v_x\rho^{-\frac{1}{2}} - \frac{1}{2}v_x\rho^{-\frac{1}{2}}\rho_x - \frac{1}{2}v^{-\frac{1}{2}}\rho\rho_x \\
\frac{1}{2}\rho^{-\frac{1}{2}}\rho_t &= -v_x\rho^{-\frac{1}{2}} - v\rho^{-\frac{1}{2}}\rho_x \\
\rho_t &= -2v_x\rho - 2v\rho_x.
\end{aligned}$$

Thus we get that the hydrodynamic equations to be

$$\begin{aligned}
v_t &= -2vv_x - \rho^{-2}\rho_x\rho_{xx} + \frac{1}{2}\rho^{-3}\rho_x^3 + \frac{1}{2}\rho^{-1}\rho_{xxx} - |p|p^{-1}p_x \\
\rho_t &= -2v_x\rho - 2v\rho_x.
\end{aligned}$$

b.

c.

d.

□

Problem 3 *Consider the previous problem, but with the focusing NLS equation*

$$ia_t = -a_{xx} - |a|^2 a.$$

The method presented in the previous problem does not allow one to describe the dynamics of long-wave solutions of the focusing NLS equation using the KdV equation. How does this show up in the calculations?

Solution. This is a solution $\square$

Problem 4 *The mKdV equation considered in the text is known as the* focusing *mKdV equation, because of the behavior of its soliton solutions. This behavior is similar to that of the focusing NLS equation. In this problem, we study the* defocusing *mKdV equation*

$$4u_t = -6u^2 u_x + u_{xxx}.$$

You have already seen that you can scale the coefficients of this equation to your favorite values, except for the ratio of the signs of the two terms on the right-hand side.

1. *Examine, using the potential energy method and phase plane analysis, the traveling-wave solutions.*

2. *If you have found any homoclinic or heteroclinic connections, find the explicit form of the profiles corresponding to these connections.*

Solution. This is a solution □

Problem 5 *Consider the so-called Derivative NLS equation (DNLS)*

$$b_t + \alpha \left(b|b|^2\right)_x - ib_{xx} = 0.$$

This equation arises in the description of quasi-parallel waves in space plasmas. Here $b(x,t)$ is a complex-valued function.

1. *Using a polar decomposition*

$$b(x,t) = B(x,t)e^{i\theta(x,t)},$$

with B and θ real-valued functions, and separating real and imaginary parts (after dividing by the exponential), show that you obtain the system

$$B_t + 3\alpha B^2 B_x + \frac{1}{B}(B^2\theta_x)_x = 0,$$
$$\theta_t + \alpha B^2\theta_x + \theta_x^2 - \frac{1}{B}B_{xx} = 0.$$

2. *Assuming a traveling-wave envelope, $B(x,t) = R(z)$, with $z = x - vt$ and constant v, show that $\theta(x,t) = \Phi(z) - \Omega t$, with constant Ω, is consistent with these equations. You can show (but you don't have to) that assuming a traveling-wave amplitude results in only this possibility for $\theta(x,t)$. At this point, we have reduced the problem of finding solutions with traveling envelope to that of finding two one-variable functions $R(z)$ and $\Phi(x)$. The problem also depends on two parameters v (envelope speed) and Ω (frequency like).*

3. *Substituting these ansatz in the first equation of the above system, show that*

$$\Phi' = \frac{C + vs - 3s^2}{2s},$$

where C is a constant and $s = \alpha R^2/2$.

4. *Lastly, by substituting your results in the second equation of the system, show that $s(z)$ satisfies*

$$\frac{1}{2}s'^2 + V(s) = E,$$

the equation for the motion of a particle with potential $V(s)$. Find the expression for $V(s)$ and for E.

You can (don't do this; it would take a lot of work; there's a lot of cases) use this observation to classify the traveling-envelope solutions of the DNLS, in the same vein that we did at the beginning of this chapter for KdV.

Solution. This is a solution □

Problem 6 *Consider example 5.2 in the notes. Check that* $y = x^2/t$ *and* $t^{1/2}q$ *are both scaling invariant. Find the ordinary differential equation satisfied by* $G(y)$*, for similarity solutions of the form* $q(x,t) = t^{-1/2}G(y)$*. Show that this results in the same similarity solutions as in the example.*

Solution. This is a solution $\square$

Problem 7 *One way to write the* **Toda Lattice** *is*

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = a_n(b_{n+1} - b_n),$$
$$\frac{\mathrm{d}b_n}{\mathrm{d}t} = 2(a_n^2 - a_{n-1}^2),$$

where a_n*,* b_n*,* $n \in \mathbb{Z}$*, are functions of* t*.*

1. *Find a scaling symmetry of this form of the Toda lattice,* i.e., *let* $a_n = \alpha A_n$*,* $b_n = \beta B_n$*,* $t = \gamma\tau$*, and determine relations between* α*,* β *and* γ *so that the equations for the Toda lattice in the* (A_n, B_n, t) *variables are identical to those using the* (a_n, b_n, τ) *variables.*
2. *Using this scaling symmetry, find a two-parameter family of similarity solutions of the Toda lattice. If necessary, find relations among the parameters that guarantee the solutions you found are real for all* n *and for* $t > 0$*.*

The Toda Lattice was introduced originally by Toda in 1967 in the form

$$\frac{\mathrm{d}q_n}{\mathrm{d}t} = p_n,$$
$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)},$$

where q_n*,* p_n*,* $n \in \mathbb{Z}$*, are functions of* t*. It is clear that this form does not lend itself to a scaling symmetry: the quantities* q_n *show up as arguments of the exponential function, and they cannot be scaled. This can be remedied by returning to the physical setting of the derivation, where a constant would multiply these exponents. This constant, being a dimensional quantity, scales in its own way under a scaling transformation.*

Solution. This is a solution $\square$

Problem 8 *Consider the equation*

$$u_t = 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x},$$

which we will encounter more in later chapters, due to its relation to the KdV equation. Show that it has a scaling symmetry.

When we look for the scaling symmetry of the KdV equation, we have two equations for three unknowns: we have three quantities (x, t, u) to scale, and after normalizing one coefficient to 1, two remaining terms that need to remain invariant. Thus it is no surprise that we find a one-parameter family of scaling symmetries. The above equation has two more terms, and it should be clear that some "luck" is needed in order for there to be a scaling symmetry.

Solution. This is a solution □

Problem 9 *Consider a Modified KdV equation*

$$u_t - 6u^2 u_x + u_{xxx} = 0.$$

1. *Find its scaling symmetry.*
2. *Using the scaling symmetry, write down an ansatz for any similarity solutions of the equation.*
3. *Show that your ansatz is compatible with $u = (3t)^{-1/3} w(z)$, with $z = x/(3t)^{1/3}$.*
4. *Use the above form of u to find an ordinary differential equation for $w(z)$. This equation will be of third order. It can be integrated once (do this) to obtain a second-order equation. The second-order equation you obtain this way is known as the second of the Painlevé equations. We will see more about these later.*

Solution. This is a solution □