Maths for Cryptology

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Numbers : Some *sets* of *numbers*

1. The set of *natural* numbers : $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$
2. The set of *integers* numbers : $\mathbb{Z} = \{\cdots, -1098, -2, -1, 0, 1, 2, 3, \cdots\cdots\} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$
3. The set of *rational* numbers : $\mathbb{Q} = \{\frac{a}{b} : a, b(\neq 0) \in \mathbb{Z}\}$
4. The set of *real* numbers : $\mathbb{R}$
5. The set of *complex* numbers : $\mathbb{C}$
6. And the most important cryptographic set of numbers : the set of all *residue classes* modulo a positive integer n : $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n = \{0, 1, 2, 3, \cdots n-1\}$.

Binary operations on a set

Binary Operations

A binary operation $\circ$ on a set S is a rule that assigns to each ordered pair (a, b) of elements of S, *i.e.* : $(a, b) \in S \times S$, a unique element of S.
Exemples :

- Ordinary addition (symbol : +) : on the set $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{Z}_n$ or on an elliptic curve
- Ordinary multiplication (symbol : $*$ or $\times$ or .) : on the set $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{Z}_n$.

Problems for cryptography

Two widely used problems [Sti94, MvOV97, Vau06]

1. Let $n \in \mathbb{N}^*$ a large number : find a "non trivial' factor of n : this is the *factorization problem*. Example : try to factorize 1000000100000000000000180000009000000000000000081
2. Let $g \in G$, $(G, \times)$ is a large group generated by g, with $y \in G$, find the least integer x such that $y = g^x$; this is the *discrete logarithm problem* (DLP). Example : $G = \mathbb{Z}_p^*$, p is a large prime.

For these problems, it seems computationally difficult to solve them if parameters are well chosen. [[WIK] : *. . . there are no proofs that integer factorization is computationally difficult . . .*].

Cryptography for security : remainder

Widely used tools [Vau06, MvOV97, Sti94]

1. Find a large prime p (1024 bits, 2048 etc.)
2. Find a large group $G = \mathbb{Z}_p$ and g a generator of G (or sometimes of a large subgroup H of G).
3. Find a a square root modulo a prime p, *i.e.* solve $y^2 = a \bmod p$.
4. How to generate pseudo-random numbers with a good *quality of randomness* ?

For these problems, we can solve them for any reasonable parameters.

Find a a square root modulo a prime p, *i.e.* solve $y^2 = a \bmod p$: Easy

For example :

- If $p \equiv 3 \bmod 4$
- Compute $Y = \pm a^{(p+1)/4}$
- Y is a solution of $y^2 = a \bmod p$
- A very Quick Proof :

$$Y^2 = (\pm a^{(p+1)/4})^2 = a^{(p+1)/2} = (a^{p-1}a^2)^{1/2} = (a^2)^{1/2} = a$$

Fact :

Very useful for Cryptographic Elliptic Curves (ECCs).

Some cryptographic important problems

- Very Easy : How to compute $\gcd(a, b)$? With the *Euclidean Algorithm*. Fast algorithms are fortunaltely known.
- Quite Easy How to prove that a number is a prime ? [Primality Tests, which are probabilistic algorithms, like Miller-Rabin are very efficient]
- Quite Easy : How to generate quickly a prime of a given length (example : a 512 bits prime number) ?

Fermat's little theorem [Vau06, Sti94, MvOV97]

It's states that if p is a prime number, then for any integer a, $a^p - a$ will be evenly divisible by p : $a^p \equiv a \bmod p$
A variant : if p is a prime and a is an integer coprime to p, then $a^{p-1} - 1$ will be evenly divisible by p :

$$a^{p-1} \equiv 1 \bmod p.$$

We can generalize with the Euler totient function φ :

$$a^{\varphi(n)} \equiv 1 \bmod n.$$

Fermat's little theorem and Euler generalization is the basis for the Fermat primality test (not good) and efficient variants (Miller-Rabin) and explains why RSA works ! !

The SOCAT *false prime*

SOCAT has used/published the following "prime" number in a new release :

```
Q=12118534197456004471544151826551243891225082214
2516953909460325096122256592599421232622191858222
6211548415987042561010916950016066101367391095659
3393441895812421291104695011867338507355055756162
6119524755353858317781412465115960056220715417743
81205194074173
```

https ://www.cryptologie.net/article/329/socat-new-dh-modulus/ : *Socat did not work in FIPS mode because 1024 instead of 512 bit DH prime is required. Thanks to Zhigang Wang for reporting and sending a patch.*

The SOCAT *false prime* : $2^{Q-1} \neq 1 \bmod Q$

Unfortunately, this number is **composed** : $Q = p\,q\,r$ with :

```
p=326657
q=2158244515383l
r=17189271752340904799250384767844462921986726O
49168207005756615043486534118170672181791990033
22774621226619762852159042769371465283010758316
56797277016743795924032046457796672351600215270
49089568545537015868322900133381117124364957545
03419
```

You can verify that $2^{Q-1} \bmod Q \neq 1$.

Three important algebraic structures

1. Groups
2. Rings
3. Fields

All (almots all) cryptographic algorithms are defined on a group, a ring or a field of numbers but always **finite**.

Groups

A group, denoted $< G, \circ >$, $(G, \circ)$ or simply G, is a nonempty set G of elements together with a binary operation $\circ$, such that the following axioms are satisfied :

1. *Closure* : $\forall a \in G, \forall b \in G, a \circ b \in G$.
2. *Existence of Identity* : there is a unique element $e \in G$ called the identity, such that $\forall a \in G, a \circ e = e \circ a = a$.
3. *Associativity* : $\forall a \in G, \forall b \in G, \forall c \in G, a \circ (b \circ c) = (a \circ b) \circ c$
4. *Existence of Inverse* : $\forall a \in G, \exists$ a unique $b \in G$ such $a \circ b = b \circ a = e$. This b is called the inverse of a nad we will write : $b = a^{-1}$ or $b = -a$.
5. *Commutativity* : A group $(G, \circ)$ is called commutative if it satisfies the following axiom : $\forall a \in G, \forall b \in G, a \circ b = b \circ a$.

Commutative Groups

Some example and counter-examples :

- $(\mathbb{Z}, +)$ is a commutative group
- $(\mathbb{Z}, *)$ is **not** a group (example : 2 has no inverse)
- $(\mathbb{Q}, +)$ is a commutative group
- $(\mathbb{Q}^*, *)$ is a commutative group
- $(\mathbb{Z}_p, *)$ is a **finite** group if and only if p is prime.
- $(\mathbb{Z}_n, +)$ is always a **finite** commutative group.

[WIK]

A ring is a set R equipped with two binary operations + and $\cdot$ satisfying the following three sets of axioms, called the ring axioms :

1. R is an abelian group under addition, meaning that
   - $(a + b) + c = a + (b + c)$ ∀ a, b, c in R (+ is associative).
   - $a + b = b + a$ ∀ a, b in R (+ is commutative).
   - There is an element 0 in R such that $a + 0 = a$ ∀ a in R (0 is the additive identity).
   - For each a in R there exists $?a$ in R such that $a + (?a) = 0$ ($-a$ is the additive inverse of a).
2. R is a monoid under multiplication, meaning that :
   - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ ∀ a, b, c in R ($\cdot$ is associative).
   - There is an element 1 in R such that $a \cdot 1 = a$ and $1 \cdot a = a$ ∀ a in R (1 is the multiplicative identity).

Ring

And the third ring axiom :

3. Multiplication is distributive with respect to addition :
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ ∀ a, b, c in R (left distributivity).
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ ∀ a, b, c in R (right distributivity).

Ring in a few words

Let $(R, +)$ be a commutative (abelian) group with a second law $\times$, called multiplication, that has the following properties :

1. Closure : $\forall a, b \in R$ we have $a \times b \in R$
2. Associativity : well, $\times$ is associative
3. Neutral element : there exists an element e , it is unique, so we can call it 1.
4. Distributivity : $\forall a, b, c \in R$ we have
   - $a \times (b + c) = a \times b + a \times c$
   - $(a + b) \times c = a \times c + b \times c$

Fundamentals : *Fields*

Field [WIK]

Intuitively, a field is a set F

1. that is a commutative group with respect to two compatible operations, addition and multiplication (the latter excluding zero), with "compatible" being formalized by distributivity,
2. and the caveat that the additive and the multiplicative identities are distinct ($0 \neq 1$).

Field [WIK]

The most common way to formalize this is by defining a field F as a set together with two operations, usually called addition and multiplication, and denoted by $+$ and $\cdot$, respectively, such that the following axioms hold :

1. Closure of F under addition and multiplication
2. Associativity of addition and multiplication
3. Commutativity of addition and multiplication
4. Existence of additive and multiplicative identity elements
5. Existence of additive inverses and multiplicative inverses
6. Distributivity of multiplication over addition

Groups, Rings and Fields

Some example and counter-examples :

- $(\mathbb{Z}, +, \cdot)$ is a ring (and not a Field) because $(\mathbb{Z}, \cdot)$ is **not** a group (example : 2 has no inverse)
- $(\mathbb{Q}, +)$ and $(\mathbb{Q}^*, \cdot)$ are commutative groups
- so $(\mathbb{Q}, +, \cdot)$ is a Field
- $(\mathbb{Z}_n^*, \cdot)$ is a **finite** group if and only if n is prime.
- so $(\mathbb{Z}_n, +, \cdot)$ is a **finite** Field if and only if n is prime.
- $(\mathbb{R}, +, \cdot)$ is a Field
- $(\mathbb{C}, +, \cdot)$ is a Field.

A cryptographic finite Ring

- Let p be a prime number (or a power of a prime)
- Let $\mathbb{Z}_p[x]$ be the set of polynomial (indeterminate x)
- + : classical polynomial addition (mod p for the coefficients)
- * : classical polynomial multiplication (mod p for the coefficients)
- $(\mathbb{Z}_p[x], +, \cdot)$ is a **ring**

A cryptographic finite Field [Sti94]

- Let p be a prime number (or a power of a prime)
- Let $f(x)$ a polynomial of degree n
- Let $\mathbb{Z}_p[x]/f(x)$ be the set of p^n polynomials with degree at most $n-1$
- $+$ and $\cdot$: classical polynomial addition and multiplication (but $\mod p$ and $\mod f(x)$)
- $(\mathbb{Z}_p[x], +, \cdot)$ is a **field** if and only if $f(x)$ is an irreducible polynomial.

Example : in AES we use :

- $\mathbb{F}_{2^8}$ is a field
- This field is isomorphic to $\mathbb{Z}_2[x]/(x^8+x^4+x^3+x+1)$
- Since $f(x) = x^8+x^4+x^3+x+1$ is irreducible
- Then $(\mathbb{F}_{2^8}, +, \cdot) \approx (\mathbb{Z}_2[x]/f(x), +, \cdot)$ is **a field**.

An example of a ring : ring of matrices

Let R be a ring, finite or infinite. Let $\mathcal{M}_n(R)$ be the set of all matrices of size $n \times n$ with elements in R. Then let us consider $(\mathcal{M}_n(R), +, \times)$ then :

- If the ring R is finite then $(\mathcal{M}_n(R), +, \times)$ is also finite and it is a ring
- If R is infinite then $(\mathcal{M}_n(R), +, \times)$ is also infinite and it is a ring

A matrix ring is any collection of matrices over a ring R that form a ring under matrix addition (+) and matrix multiplication (×) [WIK].

Divisibility

Let a, b be integers, we say that a *divides* b and we write $a|b$ if there exists an integer c such that $b = a\,c$

Some properties of divisibility :

- $a|a$ (a divides a) (reflexivity)
- If $a|b$ and $b|c$ then $a|c$ (transitivity)
- If $a|b$ and $b|c$ then $a|(b\,x + c\,y)\quad \forall x, y \in \mathbb{Z}$
- If $a|b$ and $b|a$ then $a = \pm b$.

Definition (important)

Let $n > 0$ and $a, b \in \mathbb{Z}$:

- a is said to be congruent to b if $n|(a-b)$
- ... and we write $a \equiv b \pmod{n}$
- ... or $a - b \equiv 0 \pmod{n}$
- n is the modulus

Theorem

Let $n > 0$ be an integer. For any integer a, there exists a unique integer b such that

$$a \equiv b \pmod{n}$$

with $0 \leq b < n$, we will write $b = a \pmod{n}$.

Examples

1. $123 \equiv 4 \pmod{17}$ because $123 = 17 * 7 + 4$
2. $123 \equiv 18 \pmod{35}$ because $123 = 3 * 35 + 18$
3. $123 \equiv -17 \pmod{35}$ because $123 = 4 * 35 - 17$

Some properties

- $a \equiv a \pmod{n}$
- $a \equiv b \pmod{n}$ is equivalent to : $\exists k \in \mathbb{Z}, a = b + k.n$
- $a \equiv b \pmod{n}$ implies $b \equiv a \pmod{n}$
- $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ implies $a \equiv c \pmod{n}$

Arithmetic of congruences

If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$ then

- $a + b \equiv a' + b' \pmod{n}$
- $a \cdot b \equiv a' \cdot b' \pmod{n}$

A theorem about multiplicative inverse :

Theorem

Let n, a with $a \in \mathbb{Z}_n$, then a has a multiplicative inverse modulo n iff $GCD(a, n) = 1$.

Proof (part $\Rightarrow$) :

- if A is a multiplicative inverse of a modulo n then $a \cdot A = 1 \bmod n$
- let k such that $a \cdot A = 1 + k \cdot n$
- if $d|a$ and $d|n$ then $d|1$
- therefore $GCD(a, n) = 1$.

A theorem about multiplicative inverse :

Proof (part $\Leftarrow$) :

- if $GCD(a, n) = 1$
- then it exists k such that $a|1 + k \cdot n$ (we admit this for the moment)
- then it exists A such that $a \cdot A = 1 + k \cdot n$
- therefore A is a multiplicative inverse of a modulo n

The classical Euclidean algorithm :

Algorithm 1 : **First version of the Euclidean algorithm :**

Input : two integers a and b with $a \geq b$;
Output : $GCD(a, b)$;
Begin :
  While $b \neq 0$ **Do**
    Set $r = a \bmod b$;
    Set $a = b$;
    Set $b = r$;
  End Of While
  Return a ;
End.

The classical Euclidean algorithm : python2

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```
>>> from fractions import gcd
>>> gcd(20,8)
  4
>>> print inspect.getsource(gcd)
  def gcd(a, b):
  """Calculate the Greatest Common Divisor of a
and b. Unless b==0, the result will have the
same sign as b (so that when b is divided by
it, the result comes out positive).
    """
    while b:
        a, b = b, a%b
    return a
```

Bezout's theorem

Theorem

Let a and b be two integers such that $GCD(a,b) = d$, then there exist two integers u and v such that $a \times u + b \times v = d$.

A *partial reciprocal* of this theorem exist : let a and b be two integers, if there exist two integers u and v such that $a \times u + b \times v = d$, then $GCD(a,b)$ is a divisor of d.

An important case of the Bezout theorem

The particular case $GCD(a, b) = 1$ gives the following theorem :

Theorem

Let a et b be two integers
$\exists u \in \mathbb{Z}, \exists v \in \mathbb{Z}$ *tels que* $a \times u + b \times v = 1$
$\iff GCD(a, b) = 1.$

Properties

So, if $x \in \mathbb{Z}_n$, we will have the following *equivalent* properties :

- $\exists y \in \mathbb{Z}_n \; x\,y \equiv 1 \bmod n$
- $\exists y, z \in \mathbb{Z} \; x \times y + n \times z = 1$
- $\mathrm{GCD}(x, n) = 1.$

When $b = n$ and $a \in \mathbb{Z}_n$,

then u is the multiplicative inverse of a modulo n :

$$\begin{aligned} 1 &= u \times a + v \times n \equiv u \times a \bmod n \\ u &\equiv a^{-1} \bmod n \end{aligned}$$

So, the Bezout theorem proves that the multiplicative inverse of a modulo n can be computed (efficiently as we will see).
The Extended Euclidean algorithm computes the integers u, v and d such that :

$$d = \mathrm{GCD}(a, b) = u \times a + v \times b.$$

The set of invertible elements of $\mathbb{Z}_n$:

$$\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n \,|\, (\exists\, y \in \mathbb{Z}_n)\; x \times y = 1\}.$$

An element of $\mathbb{Z}_n^*$ is called an invertible element or a unit of $\mathbb{Z}_n$.
This set $\mathbb{Z}_n^*$ with the multiplication in $\mathbb{Z}_n$, i.e. $(\mathbb{Z}_n^*, \times)$ is a commutative group.

The Extended Euclidean algorithm

Let $a > 0$ and $b \geq 0$ be two integers with $GCD(a, b) = d$, how can we compute u and v such that

$$d = a \times u + b \times v$$

This is done with the Extended Euclidean algorithm

The Extended Euclidean algorithm

- we will define three sequences r_i, u_i and v_i
- we begin with $r_0 = a$ and $r_1 = b$
- we define for $i \geq 2$ $r_i = q_i \cdot r_{i+1} + r_{i+2}$ (q_0 is well defined)
- we define $u_0 = 1$ and $v_0 = 0$ and
- we define $u_1 = 0$ and $v_1 = 1$
- for $i \geq 2$ we define $u_i = u_{i-2} - q_{i-2} \cdot u_{i-1}$ and
- we define $v_i = v_{i-2} - q_{i-2} \cdot v_{i-1}$ and

Then, there exists $k > 0$ such that $r_k = 0$, this gives

$$GCD(a,b) = r_{k-1} = a \cdot u_{k-1} + b \cdot v_{k-1}.$$

The Extended Euclidean algorithm (continue)

Short Proof We always have $r_i = a \times u_i + b \times v_i$

- $r_0 = a = 1 \times a + 0 \times b$
- $r_1 = b = 0 \times a + 1 \times b$
- if $r_{i-2} = a \cdot u_{i-2} + b \cdot v_{i-2}$ and
- $r_{i-1} = a \cdot u_{i-1} + b \cdot v_{i-1}$

then :

$$\begin{cases} a \cdot u_i + b \cdot v_i & = a \cdot (u_{i-2} - q_{i-2} \cdot u_{i-1}) + \\ & \quad b \cdot (v_{i-2} - q_{i-2} \cdot v_{i-1}) \\ & = r_{i-2} - q_{i-2} \cdot r_{i-1} \\ & = r_i \end{cases}$$

The Extended Euclidean algorithm (recursion version) :

FONCTION 1 : **Extended-Euclidean**(a, b) **:**

Input : two integers a and b with $a \geq b$;

Output : (d, u, v) such that $d = a \times u + b \times v$;

Begin :

  If (b==0) then **Return** (a,1,0) ;

  (d', u', v') = Extended-Euclidean$(b, a \bmod b)$;

  $(d, u, v) = (d, v', u' - \lfloor a/b \rfloor v')$;

  Return (d, u, v).

End.

Divisions algorithm for integers (Euclidean division)

Let a, $b > 1$ be integers, there exists integers q and r such that

$$a = q\,b + r$$

- q is the Euclidean quotient
- r is the Euclidean remainder
- integers q and r are unique if we ask $0 \leq r < b$.

The remainder r is denoted $a \bmod b$ and the quotient $a\ div\ b$. Remark : in C we write $a\%b$ for $a \bmod b$ and a/b for $a\ div\ b$, with the declaration **int a,b ;**

Greatest common divisors

Let a, b be integers, a non-negative integer d is called the *greatest common divisor* of integers a and b, denoted $d = \mathrm{GCD}(a, b)$ if

1. d is a common divisor of a and b (i.e. $d|a$ and $d|b$) ;
2. whenever $c|a$ and $c|b$ then $c|d$

$\mathrm{GCD}(a, b)$ is the *largest* positive integer that divides both a and b, with an exception : $\mathrm{GCD}(0, 0) = 0$.

Least common multiple

Let a, b be integers, a non-negative integer m is called the *least common multiple* of integers a and b , denoted $m = \text{LCM}(a, b)$ if

1. $a|m$ and $b|m$;
2. whenever $a|c$ and $b|c$ then $m|c$

$\text{LCM}(a, b)$ is the *smallest* non-negative integer divisible by both a and b.

Some facts

Let a, b be positive integers :

$$\mathrm{LCM}(a,b) = \frac{a \cdot b}{\mathrm{GCD}(a,b)}$$

or

$$\mathrm{GCD}(a,b) \cdot \mathrm{LCM}(a,b) = a \cdot b$$

Coprimes and Primes

Coprimes

Let a, b be positive integers, a and b are said to be *coprime* or relatively prime if $\mathrm{GCD}(a, b) = 1$. a and b have no common divisor except the number 1.

Primes

Let p be an integer, p is said to be *prime* if its only positive divisors are 1 and p.
Otherwise, p is said *composite*.

Permutations [WIK, Ros99]

- A permutation of a set of distincts objects is an ordered arrangement of these objects.
- Permutations occur, in more or less prominent ways, in almost every area of cryptography.
- They often arise when different orderings on certain finite sets are considered, possibly only because one wants to ignore such orderings and needs to know how many configurations are thus identified.
- For similar reasons permutations arise in the study of sorting algorithms in computer science.
- Reminder : The number of permutations of n distinct objects is n factorial usually written as $n!$.
- An ordered arrangement of r elements of a set is called an r-permutation.

Examples

Let X be the set $\{a, b, c, d\}$. and let σ be a permutation :

- We can define σ by :
  $\sigma(a) = d, \sigma(b) = c, \sigma(c) = b, \sigma(d) = a$.
- We can define σ^k with $k \geq 0$: $\sigma^k(x) = \sigma(\sigma^{k-1}(x))$.
- For example : σ^2 is defined by :
  $\sigma^2(a) = a, \sigma^2(b) = b, \sigma^2(c) = c, \sigma^2(d) = d$.
- So $\sigma^2 = \sigma$

A permutation that exchanges only two elements is called a *transposition* . A permutation σ that verifies $\sigma^2 = \sigma$ is called an *involution*.

Why permutations in cryptography ?

- Because a lot of encryption algorithms (almost all) are permutations of the (finite) set of clear texts.
- Which means that for such an algorithm $\mathcal{A}$, there exists an integer m such that $\mathcal{A}^m = \mathcal{A}$.
- A fixed point of a permutation P on a finite set $\mathcal{A}$ is an element $x \in \mathcal{A}$ that verifies $P(x) = x$.
- For a finite set $\mathcal{A}$ and a permutation P, for any $a \in \mathcal{A}$ there exists an integer k such that a is a fixed point of $P^k : P^k(x) = x$.

[WIK, Sti94, Vau06]

- Elliptic curve cryptography (ECC) is an approach to public-key cryptography based on the algebraic structure of elliptic curves over finite fields. One of the main benefits in comparison with non-ECC cryptography (with plain Galois fields as a basis) is the same level of security provided by keys of smaller size.
- Elliptic curves are applicable for : encryption, digital signatures, pseudo-random generators and other tasks.
- They are also used in several integer factorization algorithms, such as Lenstra elliptic curve factorization.

[WIK, Sti94, Vau06]

- For current cryptographic purposes, an elliptic curve is a plane curve over a finite field (rather than the real numbers)
- It consists of the points satisfying the equation $y^2 = x^3 + ax + b$
- Along with a distinguished point at infinity, denoted O.
- The coordinates here are to be chosen from a fixed finite field of characteristic not equal to 2 or 3, or the curve equation will be somewhat more complicated.

Cryptographic Elliptic modulo p

- Let p be a prime
- We will call $\mathcal{E}_p(a,b)$ (elliptic curve modulo p) or $E(\mathbb{Z}_p)(a,b)$ the (finite) set of points $(x,y) \in (\mathbb{Z}_p, \mathbb{Z}_p)$ $(\bigcup \mathcal{O})$ satisfying the modular equation :
- $y^2 = x^3 + ax + b \bmod p$
- This last equation is called the *Weirstrass* form
- With $a, b \in \mathbb{F}_p, 4a^3 + 27b^2 \neq 0$ (the Discriminant) (to ensure a solution to the modular equation).
- If we take a finite Field $\mathbb{F}_q$ with $q = p^n$ we will write $E(\mathbb{F}_q)(a,b)$ or $E(\mathbb{F}_{p^n})(a,b)$.

Does a modular Elliptic Curve generate a Group ?

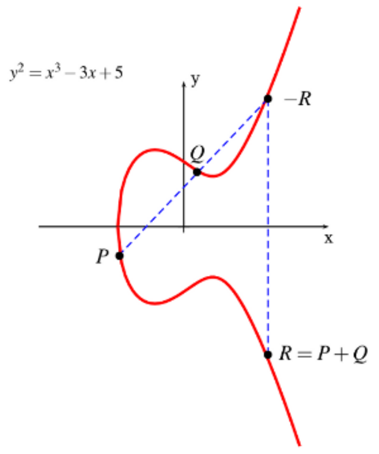
- Point addition : $(X_R, Y_R) = (X_P, Y_P) + (X_Q, Y_Q)$
- Let s the slope of the line (P,Q))
  1. $s = (Y_P - Y_Q)(X_P - X_Q)^{-1}$ mod p if $X_P \neq X_Q$
  2. $s = (3X_P^2 + a)(2Y_P)^{-1}$ mod p in the case $X_P = X_Q$
- We now define :
  1. $X_R = s^2 - X_P - X_Q$
  2. $Y_R = s(X_P - X_R) - Y_P$
- If $Q = -P$ then $P + Q = \mathcal{O}$ (the point at infinity).
- "Exponentiation" : $[k]P = P + P \cdots P$ (k times).
- $\forall P$: we have $P + \mathcal{O} = \mathcal{O} + P = P$

Have we a Group ?

Yes : $(\mathcal{E}_p(a, b), +)$ is **a commutative group**.

Point Addition : graphical illustation [a]

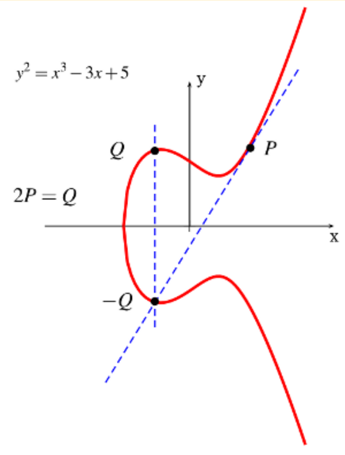
a. http ://www.purplealienplanet.com/node/27



Point Doubling : Q=2P

Point Doubling : graphical illustation [a]

a. http ://www.purplealienplanet.com/node/27



Some algorithms using EC

How to use EC in cryptography ?

- ECDSA : Elliptic Curve Digital Signature Algorithm : the EC variant of DSA
- Elliptic Curve RSA : the EC variant of RSA
- ECDH : Elliptic Curve DH the EC variant of DH.
- *In fact quite all classical PKC algorithms have an EC variant* : signature, encryption, key exchange
- It is supposed that a EC key of 160-256 bits has the same level of security than a RSA/DH key of 1024-3072 bits.
- ECC is slowly killing classical PKC algorithms (RSA, DH, etc.) : well RSA & DH are still living but . . . ...
- ECM : Elliptic Curve Method of *integer factorisation* (Lenstra) : efficient to find small or small enough factors

Curve25519 was first released by Daniel J. Bernstein in 2005 [WIK]

1. Curve25519 is an elliptic curve offering 128 bits of security
2. Designed for use with the elliptic curve Diffie-Hellman (ECDH) key agreement scheme
3. One of the fastest ECC curves
4. Not covered by any known patents, and it avoids problems with poor quality pseudo random number generators.

The curve used is defined by $y^2 = x^3 + 486662x^2 + x$, a Montgomery curve, over the prime field defined by the prime number $p = 2^{255} - 19$, and it uses the base point x = 9. [To compute y : solve a Square Root problem]

[WIK]

Curve25519 was first released by Daniel J. Bernstein in 2005, but interest increased considerably after 2013 when it was discovered that the NSA had backdoored Dual_EC_DRBG. While not directly related, suspicious aspects of the NIST's Pxxx [a] curve constants led to concerns that the NSA had chosen values that gave them an advantage in factoring public keys,
I no longer trust the constants. I believe the NSA has manipulated them through their relationships with industry.
Bruce Schneier, prominent security researcher.
Since then, Curve25519 has become the de-facto alternative to P-256. In 2014, both OpenSSH and GPG default to Curve25519-based ECDH.

a. A set of curves proposed by NIST and calculated by NSA : see [Som]

Some "not safe" ECCs [Som]

The NIST P-224 ECC : with $p = 2^{224} - 2^{96} + 1$ the curve is

$$y^2 = x^3 - 3x + a \bmod p$$

Where a is :

18958286285566608000408668544493926415504680968679321075787234672564

ECOH : Elliptic Curve Only Hash, ; not so secure

1. The ECOH-n algorithms family was a candidate for SHA-3.
2. n can be : 224, 256, 384, 512 (the length of the hash result)
3. Example : ECHO-224 uses the ECC B-233 (with a point G given) which is a NIST/NSA curve !
4. However, it was rejected in the beginning of the competition since a second pre-image attack was found.
5. Given n, ECOH divides the message M into n blocks $M_0, \ldots, M_{n-1}$. If the last block is incomplete, it is padded with single 1 and then appropriate number of 0.

ECOH : Elliptic Curve Only Hash : not so secure [Dan08]

1. Let furthermore $\mathcal{P}$ be a function that maps a message block and an integer to an elliptic curve point.
2. Then using the mapping $\mathcal{P}$, each block is transformed to an elliptic curve point P_i, and these points are added together with two more points : X_1 and X_2.
3. The first additional point X_1 contains the padding and depends only on the message length.
4. The second additional point X_2 depends on the message length and the XOR of all message blocks.
5. The result is truncated to get the hash H.

ECOH : Elliptic Curve Only Hash : not so secure [WIK, Dan08]

1. $P_i = \mathcal{P}(M_i, i)$
2. $X_1 = \mathcal{P}'(n)$
3. $X_2 = \mathcal{P}^*(M_i, n)$
4. $Q = X_1 + X_2 + \sum_{i=0}^{n-1} P_i$
5. $H = f(Q)$.
6. $f(Q) := \lfloor x(Q + \lfloor x(Q)/2 \rfloor G)/2 \rfloor \bmod 2^n$ outputs the n-bit result.

Daniel R. L. Brown.
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