ÉCOLE NATIONALE DES PONTS ET CHAUSSÉES

Projet informatique
Finance : aspects mathématiques et numériques

Vasicek Model

Marwan Akrouch, Yassmine Hachani

Contents

1 introduction: interest rate models

Interest rate models can be used to model the dynamics of the yield curve, which is essential in pricing and hedging of fixed-income securities and interest rate derivatives (Bond options, swaptions..). It is also of great importance from a macro economical point where we can estimate the future short-term interest rates, inflation and economic activity. Traditionally, these models specify a stochastic process for the term structure dynamics in a continuous-time setting.

A general form of interest rate models can be expressed as follows:

$$dr(t) = A_1 dt + A_2 dW_t$$

Where W is a Weiner process under probability $\mathbb{P}$.
The first term is deterministic and called the drift-term, it depends on t and r(t). The second term describes the randomness of the process.

In general, interest rate models can be separated into two categories:

- **short-rate models:** describe the evolution of the instantaneous interest rate R(t) as a stochastic process. (see examples bellow)
- **froward-rate models:** capture dynamics of the whole forward rate curve. (e.g. the one- or two- factor Heath-Jarrow-Morton model)

Another classification separates interest rate models into:

- **Arbitrage-free models:** take the current term structure -constructed from the most liquid bonds- and are arbitrage-free with respect to the current market prices of bonds. e.g:
  - Hull-White model (extension of Vasisek model)
  - Ho-Lee model

  both are sort-rate models.
- **Equilibrium models:** do not necessarily match the current term structure. The name equilibrium comes from the fact that these models aim to achieve a balance between the supply of bonds and other interest rate derivatives and the demand of investors. e.g:
  - Vasicek model
  - Cox-Ingersoll-Ross model

  , both are short-rate models.

2 Vasicek Model

2.1 Vasicek Model

In Vasicek, the coefficient of the Wiener-Process is simply the volatility σ and thus independent of the short rate r. The diffusion process proposed by Vasicek is a mean reverting process. Vasicek's goal is to choose simplified tractable version of the stochastic differential equation which fits the actual stochastic behavior of the short rate. Basically, in Vasicek model, the short-term interest rate follows a Ornstein–Uhlenbeck process with an additional drift term, which write as follows:

$$dr(t) = a(b - r(t))dt + \sigma dW_t \tag{1}$$

Where W is a Weiner process under probability $\mathbb{P}$.

- $a(b - r_t)$ is the drift factor that represents the expected change in the interest rate at time t.
- a is speed of reversion: It gives the adjustment of speed and has to be positive in order to maintain stability around for the long-term value.

- b is long term level of the mean: The long run equilibrium value which the interest rate goes back.
- σ the volatility of the interest rate, which is supposed constant.

2.2 solving the SDE

Multiplying both sides of (1) by e^{at} we get:

$$e^{at}dr(t) = e^{at}a(b - r(t))dt + e^{at}\sigma dW_t$$

using IPP we have:

$$e^{at}dr(t) = d(e^{at}r(t)) - r(t)ae^{at}dt$$

using last two equations we get:

$$d(e^{at}r(t)) = ae^{at}bdt + e^{at}\sigma dW_t$$

We integrate from time t=0 to t:

$$e^{at}r(t) = r_0 + \int_{s=0}^{t} abe^{as}ds + \int_{s=0}^{t} e^{as}\sigma dW_s$$

which leads to:

$$r(t) = r_0 e^{-at} + \int_{s=0}^{t} abe^{a(s-t)}ds + \int_{s=0}^{t} e^{a(s-t)}\sigma dW_s$$

finally:

$$r(t) = b + e^{-at}(r_0 - b) + \int_{s=0}^{t} e^{a(s-t)}\sigma dW_s$$

Similarly, if we integrate between t and t+h, s.t. $0 \leq h$, the solution of the SDE (1) between t and t+h is

$$r(t+h) = b + e^{-ah}(r(t) - b) + \sigma e^{-a(t+h)} \int_{s=t}^{t+h} e^{as}dW_s$$

We can calculate the mean and the variance of $r(t)$:

$$\mathbb{E}[r_t] = b + e^{-at}(r_0 - b) + \sigma e^{-at}\mathbb{E}[\int_{s=0}^{t} e^{as}dW_s] = b + e^{-at}(r_0 - b)$$

In fact, since $\mathbb{E}(\int_{s=0}^{t}(e^{as})^2 ds) < +\infty$, we have the following result: $\int_{s=0}^{t} e^{as}dW_s$ is a martingale equal to 0 at t=0 so its expectation is 0.

$$\begin{aligned}
Var(r_t) &= \mathbb{E}((r_t - \mathbb{E}(r_t))^2) \\
&= \sigma^2 \mathbb{E}(e^{-2at}(\int_{s=0}^{t} e^{as}dW_s)^2) \\
&= \sigma^2 e^{-2at}\mathbb{E}(\int_{s=0}^{t} e^{2as}ds) \\
&= \sigma^2 \frac{1 - e^{-2at}}{2a}
\end{aligned}$$

The random variable r(t) writes: $r(t) = \mu(t) + \int_{s=0}^{t} f(s)dW_s$ where f(.) is a deterministic function s.t. $\int_{s=0}^{t} f(s)^2 ds$. so r(t) is a normal random variable for which we have calculated the mean and the variance.

Similarly, we have $r(t+h) = \mu(h, r(t)) + \int_{s=t}^{t+h} f_{t,h}(s)dW_s$, with $f_{t,h}$ deterministic.

So, the distribution of r(t+h) conditionally on r(t)=r is a normal distribution with mean:

$$\mathbb{E}_{r_t=r}(r(t+h)) = b + e^{-ah}(r-b)$$

and variance:

$$Var_{r_t=r}(r(t+h)) = \sigma^2 \frac{1-e^{-2ah}}{2a}$$

2.3 Simulation

2.3.1 algorithm

Using the previous results, we can deduce an algorithm that simulate the short-term interest rate through the vector $(r(kh), 0 \leq l \leq N)$:

Algorithm 1: Vasicek Model Simulation

Data: r_0, a, b, σ, h, N
Result: the vector $(r(kh), 0 \leq l \leq N)$
Initialization: $r = [r_0]$, $k \leftarrow 1$
while $k \leq N$ **do**
  $mean = b + e^{-ah}(r[-1] - b)$
  $variance = \sigma^2 \frac{1-e^{-2ah}}{2a}$
  r.append(Normal(mean, variance))

2.3.2 results

For numerical applications, we used $a = 10/yr$, $r_0 = b = 0.05/yr$, $\sigma = 0.1/\sqrt{yr}$

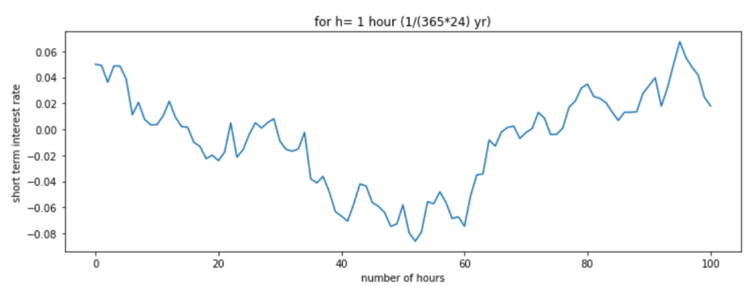

Figure 1: vasicek Model for h = 1 hour

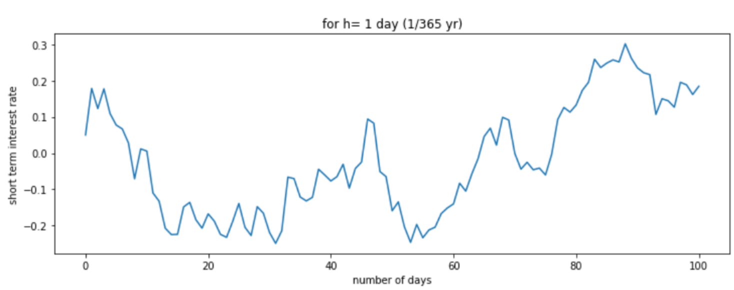

Figure 2: vasicek Model for h = 1 day

We notice that the larger h is, more r takes values further away from $r_0 = b = 0.05/an$.

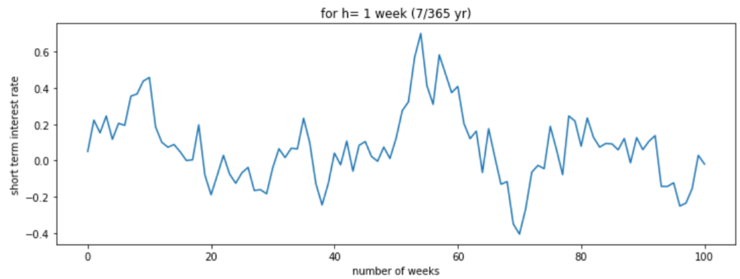

Figure 3: vasicek Model for h = 1 week

Note

The interest rate can get to negative values with a strictly positive probability, which can be seen in the simulation.

3 Bond Pricing

3.1 hypotheses

In the original paper of Oldrich VASICEK: "AN EQUILIBRIUM CHARACTERIZATION OF THE TERM STRUCTURE", the mathematician made the following assumptions in order to price bonds:

- The instantaneous (spot) interest rate follows a diffusion process (continuous Markov process). which is given by the formula 1.
- The price of a discount bond depends only on the spot rate over its term.
- The market is efficient (no transaction costs, information is available to all investors simultaneously, and every investor acts rationally).

3.2 Price of a Zero Coupon Bond: formula

let denote $P(t,T)$ the price at t of a zero coupon bond with maturity T.

let $\mathbb{P}$ be the probability under which the discounted value of the bond $\widetilde{P} = e^{-\int_0^t r_s ds} P(t,T)$ is a martingale.

From the results of section 6.2.1 in the book, the fair price of the bond is:

$$P(t,T) = exp[-(T-t)R(T-t, r(t))] \tag{2}$$

where $R(T-t, r(t))$ can be seen as the mean interest rate over the perio $[t,T]$ and given by:

$$R(\theta, r) = R_\infty - \frac{1}{a\theta}((R_\infty - r)(1-e^{-a\theta}) - \frac{\sigma^2}{4a^2}(1-e^{-a\theta})^2)$$

and $R_\infty = lim_{\theta\to\infty} R(\theta, r) = b - \frac{\sigma^2}{2a^2}$ which can be interpreted as a long-term rate.

3.3 Simulation

Using the simulation of r under Vasicek Model and the formula (2) we can simulate the discretized trajectory of the bond price $(P(kh,T), 0 \le k \le N)$. We used h = 1 day, and N such that Nh = 1 year (N=365)

3.3.1 results

we used $a = 10/yr, \quad r_0 = b = 0.05/yr, \quad \sigma = 0.1/\sqrt{yr}$.

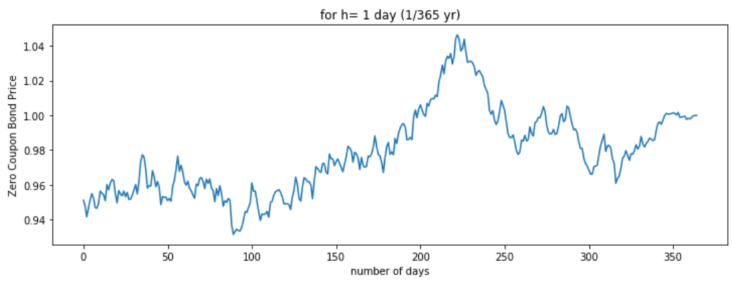

Figure 4: Zero Coupon Bond price trajectory simulation using Vasicek Model, for h = 1 day, N =365

We plotted different simulation results on the same graph as we see in Figure 5. The bond price always starts at 0.95 at $t = 0$ since $r_0 = 0.05/yr$ is fixed, and is always equal to 1 at time $t = T$ since $P(T,T) = 1$. for $t \in]0,1[$, $P(t,T)$ depends on $r(t)$ which is stochastic.

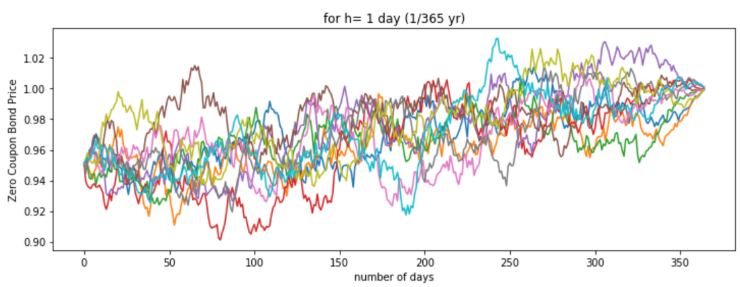

Figure 5: Many trajectories: Zero Coupon Bond price trajectory simulation using Vasicek Model, for h = 1 day, N =365

4 Bond option

4.1 preleminary

Using the fact that $\mathbb{P}$ is itself the probability under which $\tilde{P}(t,T)$ is a martingale which mean that the risk premium $-q(t)$ is null, we use the results of section 6.1.3 of the book to claim that:

$$dP(t,T) = P(t,T)(r(t)dt + \sigma_t^T dW_t) \tag{3}$$

with $\sigma_t^T = -\sigma \frac{1-e^{-a(T-t)}}{a}$, for which the equation (6.8) of the book is satisfied using formula (2).

note that equation (3) can also be written as $\frac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} = \sigma_s^T dW_t$

we introduce $P^\theta(t,T) = \frac{P(t,T)}{P(t,\theta)}$
By Integration by parts applied to the fraction above and using (3), we prove that:

$$dP^\theta(t,T) = P^\theta(t,T)(\sigma_t^T - \sigma_t^\theta)(dW_t - \sigma_t^\theta dt)$$

let $\mathbb{P}^\theta$ the probability defined by $\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \frac{e^{-\int_0^\theta r(s)ds}}{P(0,\theta)}$, under which $W_t^\theta = W_t - \int_0^t \sigma_s^\theta ds$ is a brownian motion. so:

$$\frac{dP^\theta(t,T)}{P^\theta(t,T)} = (\sigma_t^T - \sigma_t^\theta)dW_t^\theta \tag{4}$$

We use itô's Lemma and (4) to prove:

$$d\ ln(dP^\theta(t,T)) = (\sigma_t^T - \sigma_t^\theta)dW_t^\theta - (\sigma_t^T - \sigma_t^\theta)^2 dt$$

By integrating on the interval $[t,\theta]$:

$$P^\theta(\theta,T) = P^\theta(t,T)e^{\int_t^\theta (\sigma_s^T - \sigma_s^\theta)dW_s^\theta - (\sigma_s^T - \sigma_s^\theta)^2 ds}$$

Let $\Sigma^2(t,\theta) = \int_t^\theta (\sigma_s^T - \sigma_s^\theta)^2 ds$

Since $(\sigma_s^T - \sigma_s^\theta)$ is deterministic, we deduce that $\int_t^\theta (\sigma_s^T - \sigma_s^\theta)dW_s^\theta$ had normal distribution with mean 0 and variance $\Sigma^2(t,\theta)$

So, for $G \sim \mathcal{N}(0,1)$ we write:

$$P^\theta(\theta,T) = P^\theta(t,T)e^{\Sigma(t,\theta)G - \frac{1}{2}\Sigma^2(t,\theta)} \tag{5}$$

4.2 Call on a ZC bond

4.2.1 Call price formula

We consider a call of maturity θ on a zero coupon bond of maturity T, such that $T > \theta$.

the price C_t of the call at t is given by:

$$\begin{aligned} C_t &= \mathbb{E}[e^{-\int_t^\theta r(s)ds}(P(\theta,T) - K)_+ | \mathcal{F}_t] \\ &= P(t,\theta)\mathbb{E}^\theta[(P(\theta,T) - K)_+ | \mathcal{F}_t] \end{aligned} \tag{6}$$

the last equality comes from prop 6.1.7 in the book.

Using the fact that $P(\theta,T) = P^\theta(\theta,T)$ since $P(\theta,\theta) = 1$, we have:

$$C_t = P(t,\theta)\mathbb{E}^\theta[(P^\theta(\theta,T) - K)_+ | \mathcal{F}_t]$$

Using (5) we deduce that

$$C_t = P(t,\theta)B(t, P^\theta(t,T)) \tag{7}$$

Where:

$$\begin{aligned} B(t,x) &= \mathbb{E}[xe^{\Sigma(t,\theta)G - \frac{1}{2}\Sigma^2(t,\theta)}] \\ &= xN(d_1(t,x) - KN(d_2(t,x)) \end{aligned}$$

with N is the cumulative distribution function of $G \sim \mathcal{N}(0,1)$,

$$d_1(t,x) = \frac{ln(x/K) + (\Sigma^2(t,\theta)/2)}{\Sigma(t,\theta)} \quad \text{and} \quad d_2(t,x) = d_1(t,x) - \Sigma(t,\theta)$$

4.2.2 Simulation: call price

We implemented the formulas proved in the previous section and plotted the price of the call at time 0 as a function of the strike K, we obtained the results in figure 6. we used $a = 10/yr$, $\quad r_0 = b = 0.05/yr$, $\sigma = 0.1/\sqrt{yr}$, $T = 1$, $\theta = .75$

note that, in the cas of Vasicek Model, we have: $\Sigma^2(t,\theta) = \frac{\sigma^2}{2a^3}(e^{-aT} - e^{-a\theta})^2(e^{2a\theta} - e^{2at})$

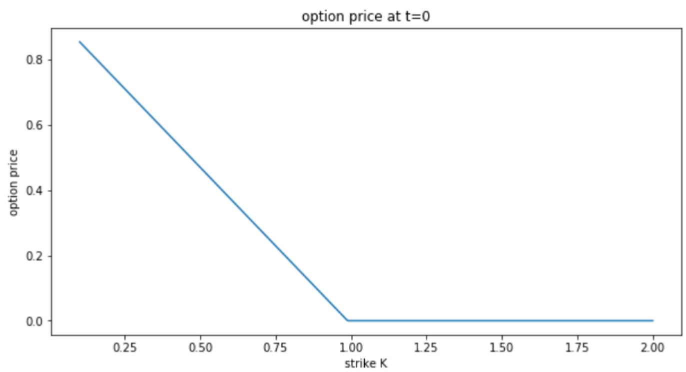

Figure 6: price of the call at time 0 as a function of the strike K, we took $a = 10/yr$, $\quad r_0 = b = 0.05/yr$, $\quad \sigma = 0.1/\sqrt{yr}$, $T = 1$, $\theta = .75$

We get a call price that is convex decreasing function of K, which was expected if we use the formula (6)

As we raise σ, the plot becomes smooth, for exemple the Figure 7 is obtained for $\sigma = 2$.

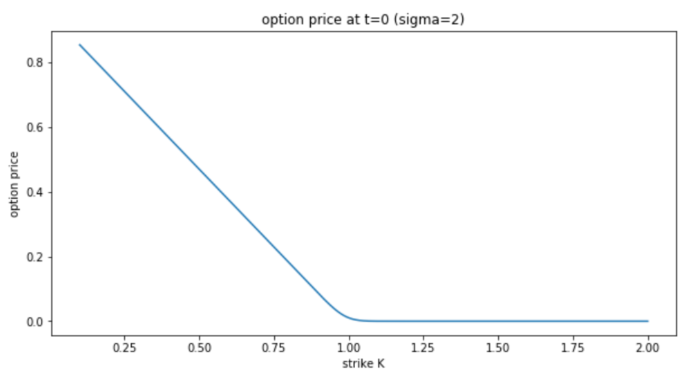

Figure 7: Case of a more volatile bond: price of the call at time 0 as a function of the strike K, we took $a = 10/yr$, $\quad r_0 = b = 0.05/yr$, $\quad \sigma = 2/\sqrt{yr}$, $T = 1$, $\theta = .75$

5 Hedging Portfolio:

5.1 Self-financed hedging portfolio

Using (7), we have:

$$\begin{aligned} C_t &= P(t,\theta)B(t, P^\theta(t,T)) \\ &= P(t,T)H_t^T + P(t,\theta)H_t^\theta \end{aligned} \tag{8}$$

Where

$$H_t^T = N(d_1(t, \frac{P(t,T)}{P(t,\theta)})) \quad \text{and} \quad H_t^\theta = -KN(d_2(t, \frac{P(t,T)}{P(t,\theta)})) \tag{9}$$

We define $C_t^\theta := \frac{C_t}{P(t,\theta)} = B(t, P^\theta(t,T))$, the last equality comes from (7). We use Itô's Lemma and the fact that C_t^θ is a martingale under $\mathbb{P}^\theta$ (comes from eq (6)) to prove that: $dC_t^\theta = \frac{\partial B}{\partial x}(t, P^\theta(t,T))P^\theta(t,T)$

We also have $\frac{\partial B}{\partial x}(t,x) = N(d_1(t,x))$. It is a result of the dominated convergence theorem to differentiate $B(t,x) = \mathbb{E}^*[xe^{\Sigma(t,\theta)G - \frac{1}{2}\Sigma^2(t,\theta)}]$ w.r.t x.

So $dC_t^\theta = H_t^T dP^\theta(t,T)$. On the other hand, we have $C_t^\theta = H_t^T P^\theta(t,T) + H_t^\theta$.

We use the last two results and (8) to deduce that:

$$dC_t = H_t^T dP(t,T) + H_t^\theta dP(t,\theta)$$

We've just proved that the portfolio that consists in detaining at time t:

- H_t^T ZC bonds with maturity T.
- H_t^θ ZC bonds with maturity θ (since $H_t^\theta \leq 0$ we re shorting the bond)

is self-financing, that replicates the call (by (8)). At time $t = \theta$, where the option is either exercised or dies worthless, we have $C_\theta = P(\theta,\theta)\mathbb{E}^\theta[(P(\theta,T) - K)_+ | \mathcal{F}_\theta] = (P(\theta,T) - K)_+$ since $P(\theta,\theta) = 1$ and $(P(\theta,T) - K)_+$ is $\mathcal{F}_\theta$-mes.
The portfolio we constructed is a perfect hedging strategy of the call.

5.2 Simulation: hedging portfolio at $t = 0$

we implement H_0^T and H_0^θ using formula (9) using the same numerical values for the parameters as before: $a = 10/yr$, $r_0 = b = 0.05/yr$, $\sigma = 0.1/\sqrt{yr}$, $T = 1$, $\theta = .75$. We get the result if Figure

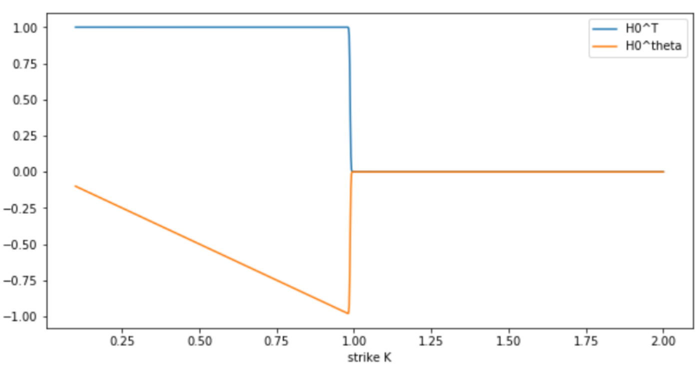

Figure 8: Quantities H_0^T and H_0^θ of how many zc bonds to hold (resp. with maturity T and θ) at time 0 as a function of the strike K, we took $a = 10/yr$, $r_0 = b = 0.05/yr$, $\sigma = .1/\sqrt{yr}$, $T = 1$, $\theta = .75$

We notice that H_0^θ is negative (we short sell the bond with maturity θ) and is a decreasing function of K for $K < .95 = P(0,T)$. and $H_0^T = 1 \quad \forall K < .95 = P(0,T)$. which means that if we want to hedge a call that we sold with a strike $K < .95$, we hold one zc bond with maturity T and we short sell $|H_0^\theta|$ bond with maturity θ

As we raise σ, the plots become smooth, for exemple the Figure 9 is obtained for $\sigma = 2$.

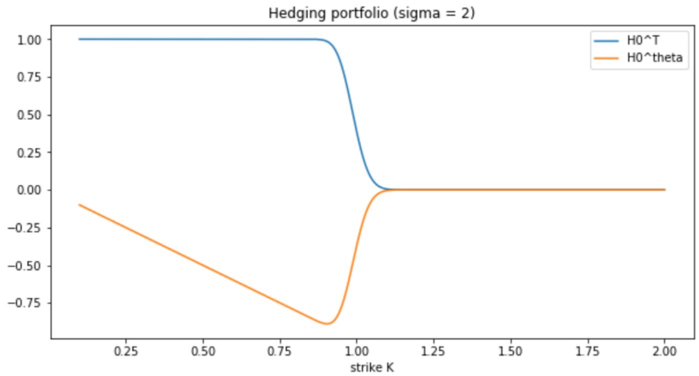

Figure 9: Quantities H_0^T and H_0^θ of how many zc bonds to hold (resp. with maturity T and θ) at time 0 as a function of the strike K, we took $a = 10/yr$, $r_0 = b = 0.05/yr$, $\sigma = 2/\sqrt{yr}$, $T = 1$, $\theta = .75$

5.3 Simulation: discrete approximation

We consider a hedging portfolio composed of $\bar{H}_s^T$ zc bonds of maturity T maintained constant on the interval $[kh, (k+1)h]$ and equal to H_{kh}^T. and $\bar{H}_s^\theta$ the quantity of zc bonds with maturity θ obtained from the discrete case self-financed condition at time kh:

$$H_{(k+1)h}^\theta = H_{kh}^\theta - \frac{P_{kh}^T}{P_{kh}^\theta}(H_{(k+1)}^T - H_{kh}^T)$$

For different values of h we calculated the residual risque of this portfolio, that is the difference between the value of out portfolio at time t and the value of the call at that same time. we get the following results.

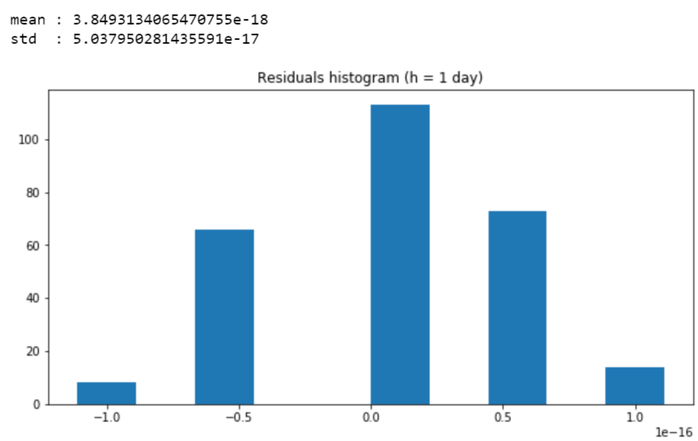

Figure 10: **Case h = 1day**: residual risque of the portfolio constructed by discretizing: (Ox) axis diffrence between C_t and the hedging portfolio, (Oy) axis is the number of occurence of this difference . we took $a = 10/yr$, $r_0 = b = 0.05/yr$, $\sigma = 0.1/\sqrt{yr}$, $T = 1$, $\theta = .75$, K=.9

Interpretation

As h tends to 0, both the mean and the standard deviation of the residual risk tends to 0. Which is expected since $h \to 0$ means we're in the continuous case, where the hedge is perfect: the portfolio replicates exactly the call.

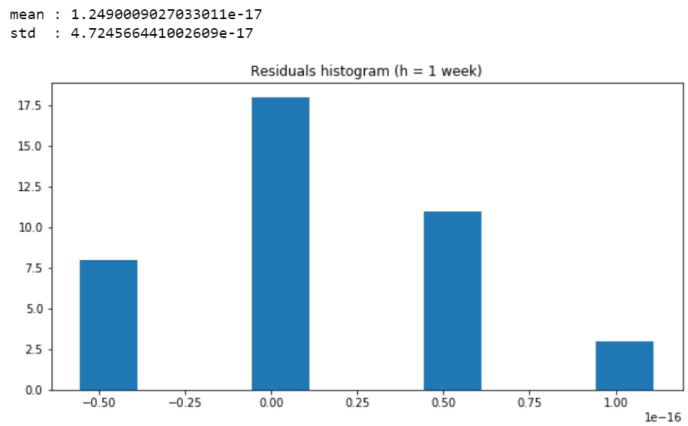

Figure 11: **Case h = 1week**: residual risque of the portfolio constructed by discretizing. we took $a = 10/yr, \quad r_0 = b = 0.05/yr, \quad \sigma = 0.1/\sqrt{yr}$, $T = 1$, $\theta = .75$

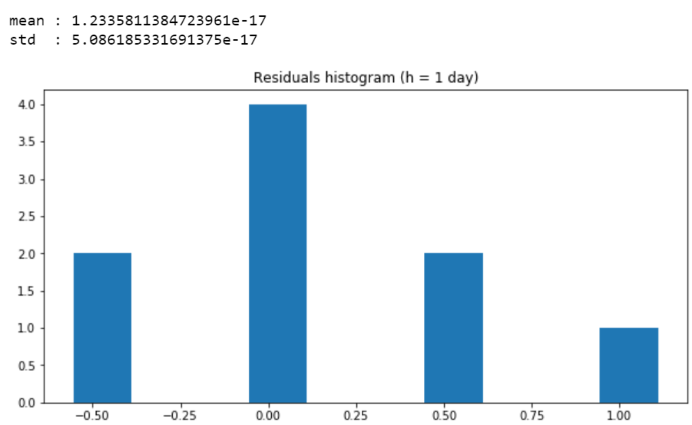

Figure 12: **Case h = 1month**: residual risque of the portfolio constructed by discretizing. we took $a = 10/yr, \quad r_0 = b = 0.05/yr, \quad \sigma = 0.1/\sqrt{yr}$, $T = 1$, $\theta = .75$