MATH 4107
MIDTERM EXAMINATION 1

Name	

1	2	3	4	5	6	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 60 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- All answers must be justified unless otherwise noted, and all proofs must be written in clear and grammatical English.
- You may cite any theorem, lemma, proposition, etc. proved in class or in the sections we covered in the text, in addition to any assigned homework problem (unless the exam problem itself was assigned on the homework).
- Good luck!

Problem 1. [10 points]

Let G be a group such that $|G| = p^n$ for p prime and $n \geq 1$. Prove that G contains an element of order p.

Solution.

Let $x \in G$, $x \neq 1$. Then the order of x is a power of p, say p^r. Let $y = x^{p^{r-1}}$. Then y has order p.

Problem 2. [10 points; 2 points each]

Use the first isomorphism theorem to identify the following quotient groups. In other words, what common group is the quotient isomorphic to?

a) $\mathbf{R}^\times/\{\pm 1\}$

b) $\mathbf{C}^\times/S^1$ where $S^1 = \{e^{2\pi i t} \mid t \in \mathbf{R}\}$

c) $\mathbf{R}^+/2\pi\mathbf{Z}$

d) $\mathbf{C}^\times/\langle e^{2\pi i/n}\rangle$

e) G/H where

$$G = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathrm{GL}_2(\mathbf{R})\right\} \qquad H = \left\{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbf{R})\right\}$$

Solution.

a) The homomorphism $|\cdot|\colon \mathbf{R}^\times \twoheadrightarrow \mathbf{R}_{>0}$ has kernel $\{\pm 1\}$, so $\mathbf{R}^\times/\{\pm 1\} \xrightarrow{\sim} \mathbf{R}_{>0}$.

b) The homomorphism $|\cdot|\colon \mathbf{C}^\times \twoheadrightarrow \mathbf{R}_{>0}$ has kernel S^1, so $\mathbf{C}^\times/S^1 \xrightarrow{\sim} \mathbf{R}_{>0}$.

c) The homomorphism $t \mapsto e^{it}\colon \mathbf{R}^+ \to \mathbf{C}^\times$ has image S^1 and kernel $2\pi\mathbf{Z}$, so

$$\mathbf{R}^+/2\pi\mathbf{Z} \xrightarrow{\sim} S^1.$$

d) The homomorphism $z \mapsto z^n\colon \mathbf{C}^\times \twoheadrightarrow \mathbf{C}^\times$ has kernel $\langle e^{2\pi i/n}\rangle$, so $\mathbf{C}^\times/\langle e^{2\pi i/n}\rangle \xrightarrow{\sim} \mathbf{C}^\times$.

e) The map $\varphi\colon G \to \mathbf{R}^\times \times \mathbf{R}^\times$ defined by

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = (a, c)$$

is a surjective homomorphism with kernel H. Therefore $G/H \xrightarrow{\sim} \mathbf{R}^\times \times \mathbf{R}^\times$.

Problem 3. [10 points]

Let $G = \langle x \rangle$ be a cyclic group and let $H \leq G$ be a subgroup. Prove that G/H is cyclic.

Solution.

Let $\pi\colon G \to G/H$ be the quotient homomorphism. We claim that $G/H = \langle \pi(x) \rangle$. Indeed, let $\overline{y} \in G/H$, and choose $y \in G$ with $\pi(y) = \overline{y}$. Then $y = x^n$ for some n because G is cyclic, so $\overline{y} = \pi(x^n) = \pi(x)^n$.

Problem 4. [10 points]

Determine all homomorphisms from S_3 to $\mathbf{Z}^+$. Justify your answer.

Solution.

Let $\varphi : S_3 \to \mathbf{Z}^+$ be a homomorphism. Let $x \in S_3$. Since $|S_3| = 6$, we have $x^6 = 1$, so $\varphi(x^6) = 6\varphi(x) = 0$, and hence $\varphi(x) = 0$. It follows that φ is the trivial homomorphism.

Problem 5. [10 points]

Let G be a group and let $H, H' \leq G$ be subgroups of order 5. Prove that either $H \cap H' = \{1\}$ or $H = H'$.

Solution.

First note that since H and H' have prime order, they are cyclic and are generated by any non-identity element. Suppose that $H \cap H' \neq \{1\}$. Let $x \in H \cap H'$, $x \neq 1$. Then $H = \langle x \rangle = H'$.

Problem 6.

[10 points]

Let G be a group that contains normal subgroups H and K of orders 3 and 5, respectively. Prove that G contains an element of order 15.

Solution.

Let $f : H \times K \to HK$ be the multiplication map, and let $G' = HK$ be its image. Note that G' is a subgroup of G because H and K are normal. The non-identity elements of H and K have orders 3 and 5, respectively, so $H \cap K = \{1\}$. Clearly H and K are normal in G', so by the product group criterion, $G' \cong H \times K$. Since H and K have prime orders, they are cyclic, and since their orders are coprime, $H \times K$ is cyclic of order 15.

[Scratch work]