MATH 4107
FINAL EXAMINATION

Name	

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- All answers must be justified unless otherwise noted, and all proofs must be written in clear and grammatical English.
- You may cite any theorem, lemma, proposition, etc. proved in class or in the sections we covered in the text, in addition to any assigned homework problem (unless the exam problem itself was assigned on the homework).
- Good luck, and good work all semester!

Problem 1. [10 points]

Let G be a group and let $a, b \in G$. Prove that ab and ba are conjugate.

Solution.

$ba = b(ab)b^{-1}$

Problem 2. [10 points; 5 points each]

a) Find all generators of the cyclic group $C_{12} = \{1, x, x^2, \ldots, x^{11}\}$ of order 12.

b) Find all generators of the cyclic (multiplicative) group $\mathbf{F}_{13}^{\times}$ of units modulo 13.

Solution.

a) We know that x^i generates C_{12} if and only if i is coprime to 12, so the generators are x, x^5, x^7, and x^{11}.

b) The powers of 2 modulo 13 are, in order:

$$1, 2, 4, 8, 3, 6, -1, -2, -4, -8, -3, -6.$$

Therefore 2 generates $\mathbf{F}_{13}^{\times} \cong C_{12}$. By (a), the other generators are $2^5 = 6, 2^7 = -2$, and $2^{11} = -6$.

Problem 3. [10 points]

Prove that a group G of order 15 is cyclic.

Solution.

Let H and K be 3- and 5-Sylow subgroups, respectively. The number of 3-Sylows divides 5 and is congruent to 1 modulo 3, so there is only one such, and thus H is normal. Similarly, K is normal, so H and K are both normal. We have $H \cap K = \{1\}$ since their orders are coprime, so the map $\varphi \colon H \times K \to G$ is an injective group homomorphism. Since $|H \times K| = 15 = |G|$, φ is an isomorphism, so $G \cong H \times K$. But $H \cong C_3$ and $K \cong C_5$, so since 3 and 5 are coprime, $H \times K \cong C_{15}$.

Problem 4. [10 points]

Determine the class equation of the quaternion group

$$Q = \{\pm 1, \pm i, \pm j, \pm k\}$$

defined by the relations

$$\begin{array}{lll} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j. \end{array}$$

Solution.

Let G be any non-abelian group of order 8 (of which Q is an example). Since G is not abelian, it has a conjugacy class of size at least 2. Since G has order $8 = 2^3$, its center Z has size at least 2. Suppose there exists $x \in G$ with conjugacy class of size 4. Then the centralizer $Z(x)$ has size 2. But $Z \leq Z(x)$, so $Z = Z(x)$. But $x \notin Z$ and $x \in Z(x)$, which is a contradiction. Thus G has no conjugacy class of size 4.

If $|Z| = 4$ then G/Z has two elements, hence is cyclic. This can only happen when G is abelian, which we are assuming is not the case. Thus $|Z| = 2$, so the class equation is:

$$8 = 1 + 1 + 2 + 2 + 2.$$

Problem 5. [10 points; 5 points each]

a) Use the counting formula to determine the number of rotational symmetries of the cube.

b) Use the counting formula to determine the number of symmetries of the cube, including mirror symmetries.

Solution.

a) A cube has six faces. The stabilizer of a face consists of the four rotations by multiples of 90 degrees. All faces are in the same orbit. Hence there are $6 \times 4 = 24$ total symmetries.

b) In this case the stabilizer of a face includes flips through the center and diagonal lines, hence is isomorphic to D_8. The action on faces is still transitive, so there are $6 \times 8 = 48$ total symmetries.

Problem 6. [10 points]

Let I and J be ideals of a ring R. Prove that $I \cap J$ is an ideal.

Solution.

We have $0 \in I \cap J$, so $I \cap J \neq \emptyset$. Let $x, y \in I \cap J$. Then $x, y \in I$, so $x + y \in I$. Likewise, $x, y \in J$, so $x + y \in J$. Hence $x + y \in I \cap J$. Let $r \in R$ and $x \in I \cap J$. Then $rx \in I$ and $rx \in J$ because I and J are ideals, so $rx \in I \cap J$. Thus $I \cap J$ is an ideal.

Problem 7.

[10 points]

Which of the following rings are isomorphic?

(1) $R_1 = \mathbf{Q}[x]/(x)$
(2) $R_2 = \mathbf{Q}[x]/(x-1)$.
(3) $R_3 = \mathbf{Q}[x]/(x^2)$.
(4) $R_4 = \mathbf{Q}[x]/(x^2-1)$.
(5) $R_5 = \mathbf{Q}[x]/(x^2+1)$.

Solution.

The first and second rings are both isomorphic to $\mathbf{Q}$. None of the other rings are isomorphic to each other. The third has three ideals, which correspond (via the correspondence theorem) to the ideals $(x^2), (x), (1) \subset \mathbf{Q}[x]$. The fourth has four ideals, corresponding to the ideals $(x^2-1), (x-1), (x+1), (1) \subset \mathbf{Q}[x]$. The fifth is also a field, but unlike $\mathbf{Q}$ it contains a square root of -1, hence cannot be isomorphic to $\mathbf{Q}$.

Problem 8.

[10 points]

Let $f = x^3 - x^2 + 1 \in \mathbf{F}_3[x]$, let $K = \mathbf{F}_3[x]/(f)$, and let α be the residue of x.

a) [3 points] Prove that f is irreducible over $\mathbf{F}_3$. Hence $K \cong \mathbf{F}_{27}$.

b) [7 points] Find the inverse of $1+\alpha$ in terms of the basis $1, \alpha, \alpha^2$ of K over $\mathbf{F}_3$.

Solution.

a) We evaluate

$$f(0) = 1 \quad f(1) = 1 \quad f(-1) = -1,$$

so f has no roots. Since $\deg(f) = 3$, we conclude that f is irreducible.

b) Let $\beta = a + b\alpha + c\alpha^2$ be the inverse of $1+\alpha$. We have $\alpha^3 = \alpha^2 - 1$ by construction, so

$$\begin{aligned} 1 = (1+\alpha)\beta &= a + (a+b)\alpha + (b+c)\alpha^2 + c\alpha^3 \\ &= (a-c) + (a+b)\alpha + (b+2c)\alpha^2. \end{aligned}$$

This yields the system of equations

$$1 = a - c \qquad 0 = a + b \qquad 0 = b + 2c = b - c,$$

from which it follows that $b = -a = c$ and $1 = a - c = 2a = -a$. Hence $a = -1$ and $b = c = 1$, so

$$(1+\alpha)^{-1} = -1 + \alpha + \alpha^2.$$

Problem 9. [10 points]

Let R be a ring and let $a \in R$. Find an isomorphism $R[x]/(x-a) \xrightarrow{\sim} R$.

Solution.

Define $\varphi\colon R[x] \to R$ by $\varphi(f) = f(a)$. Then φ sends constants to themselves, so it is a surjective homomorphism. On the other hand $\varphi(f) = f(a) = 0$ if and only if $x - a \mid f$, so the kernel of φ is $(x-a)$. Hence φ determines an isomorphism $R[x]/(x-a) \xrightarrow{\sim} R$.

Problem 10.

[10 points; 5 points each]

Let $\alpha \in \mathbf{R}$ be the real cube root of 2 and let $\zeta = e^{2\pi i/3}$.

a) What is the irreducible polynomial for α over $\mathbf{Q}$? (You need not justify your answer.)

b) Are the subfields $\mathbf{Q}(\alpha) \subset \mathbf{C}$ and $\mathbf{Q}(\zeta\alpha) \subset \mathbf{C}$ isomorphic?

Solution.

a) $x^3 - 2$.

b) Yes, α and $\zeta\alpha$ have the same minimal polynomial $x^3 - 2$.

[Scratch work]

[Scratch work]