ALGEBRA QUALIFYING EXAM PROBLEMS
GROUP THEORY

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CONTENTS

GROUP THEORY

General Group Theory

1. Prove or give a counter-example:
   (a) If H_1 and H_2 are groups and $G = H_1 \times H_2$, then any subgroup of G is of the form $K_1 \times K_2$, where K_i is a subgroup of H_i for $i = 1, 2$.
   (b) If $H \trianglelefteq N$ and $N \trianglelefteq G$ then $H \trianglelefteq G$.
   (c) If $G_1 \cong H_1$ and $G_2 \cong H_2$, then $G_1 \times G_2 \cong H_1 \times H_2$.
   (d) If $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ with $N_1 \cong N_2$ and $G_1/N_1 \cong G_2/N_2$, then $G_1 \cong G_2$.
   (e) If $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ with $G_1 \cong G_2$ and $N_1 \cong N_2$, then $G_1/N_1 \cong G_2/N_2$.
   (f) If $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ with $G_1 \cong G_2$ and $G_1/N_1 \cong G_2/N_2$, then $N_1 \cong N_2$.

2. Let G be a group and let N be a normal subgroup of index n. Show that $g^n \in N$ for all $g \in G$.

3. Let G be a finite group of odd order. Show that every element of G has a unique square root; that is, for every $g \in G$, there exists a unique $a \in G$ such that $a^2 = g$.

4. Let G be a group. A subgroup H of G is called a *characteristic* subgroup of G if $\varphi(H) = H$ for every automorphism φ of G. Show that if H is a characteristic subgroup of N and N is a normal subgroup of G, then H is a normal subgroup of G.

5. Show that if H is a characteristic subgroup of N and N is a characteristic subgroup of G, then H is a characteristic subgroup of G.

6. Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Show that if the order of H is relatively prime to the index of N in G, then $H \subseteq N$.

7. Let G be a group and let $Z(G)$ be its center. Show that if $G/Z(G)$ is cyclic, then G is abelian.

8. Let G be a group and let $Z(G)$ be the center of G. Prove or disprove the following.
   (a) If $G/Z(G)$ is cyclic, then G is abelian.
   (b) If $G/Z(G)$ is abelian, then G is abelian.
   (c) If G is of order p^2, where p is a prime, then G is abelian.

9. Show that if G is a nonabelian finite group, then $|Z(G)| \leqslant \frac{1}{4}|G|$.

10. Let G be a finite group and let M be a maximal subgroup of G. Show that if M is a normal subgroup of G, then $|G : M|$ is prime.

11. Let G be a group and let A be a maximal abelian subgroup of G; i.e., A is maximal among abelian subgroups. Prove that $C_A(g) < A$ for every element $g \in G - A$.

12. Show that if $\mathcal{K}$ and $\mathcal{L}$ are conjugacy classes of groups G and H, respectively, then $\mathcal{K} \times \mathcal{L}$ is a conjugacy class of $G \times H$.

13. (a) State a formula relating orders of centralizers and cardinalities of conjugacy classes in a finite group G.
    (b) Let G be a finite group with a proper normal subgroup N that is not contained in the center of G. Prove that G has a proper subgroup H with $|H| > |G|^{1/2}$.
    [Hint: (a) applied to a noncentral element of G inside N is useful.]

14. Let H be a subgroup of G of index 2 and let g be an element of H. Show that if $C_G(g) \subseteq H$ then the conjugacy class of g in G splits into 2 conjugacy classes in H, and if $C_G(g) \nsubseteq H$, then the class of g in G remains the class of g in H.

15. Let G be a finite group, H a subgroup of G of index 2, and $x \in H$. Denote by $c\ell_G(x)$ the conjugacy class of x in G and by $c\ell_H(x)$ the conjugacy class of x in H.
    (a) Show that if $C_G(x) \leqslant H$, then $|c\ell_H(x)| = \frac{1}{2}|c\ell_G(x)|$.
    (b) Show that if $C_G(x)$ is not contained in H, then $|c\ell_H(x)| = |c\ell_G(x)|$.

    [Hint: Consider centralizer orders.]

16. Let x be in the conjugacy class k of a finite group G and let H be a subgroup of G. Show that
$$\frac{|C_G(x)| \cdot |k \cap H|}{|H|}$$
is an integer. [Hint: Show that the numerator is the cardinality of $\{g \mid gxg^{-1} \in H\}$, which is a union of cosets of H.]

17. Let H be a proper subgroup of the finite group G. Prove that the union of all the conjugates of H is a proper subset of G.

18. Let N be a normal subgroup of G and let $\mathcal{C}$ be a conjugacy class of G that is contained in N. Prove that if $|G : N| = p$ is prime, then either $\mathcal{C}$ is a conjugacy class of N or $\mathcal{C}$ is a union of p distinct conjugacy classes of N.

19. Let G be a group, $g \in G$ an element of order greater than 2 (possibly infinite) such that the conjugacy class of g has an odd number of elements. Prove that g is not conjugate to g^{-1}.

20. Let H be a subgroup of the group G. Show that the following are equivalent:
    (i) $x^{-1}y^{-1}xy \in H$ for all $x, y \in G$
    (ii) $H \trianglelefteq G$ and G/H is abelian.

21. Let H and K be subgroups of a group G, with $K \trianglelefteq G$ and $K \leqslant H$. Show that H/K is contained in the center of G/K if and only if $[H, G] \leqslant K$ (where $[H, G] = \langle h^{-1}g^{-1}hg \mid h \in H,\ g \in G \rangle$).

22. Let G be any group for which G'/G'' and G''/G''' are cyclic. Prove that $G'' = G'''$.

23. Let $\mathrm{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ matrices with complex entries. Give a complete list of conjugacy class representatives for $\mathrm{GL}_2(\mathbb{C})$ and for $\mathrm{GL}_3(\mathbb{C})$.

24. Let H be a subgroup of the group G and let T be a set of representatives for the distinct right cosets of H in G. In particular, if $t \in T$ and $g \in G$ then tg belongs to a unique coset of the form Ht' for some $t' \in T$. Write $t' = t \cdot g$. Prove that if $S \subseteq G$ generates G, then the set $\{ts(t \cdot s)^{-1} \mid t \in T, s \in S\}$ generates H.
    Suggestion: If K denotes the subgroup generated by this set, prove the stronger assertion that $KT = G$. Start by showing that KT is stable under right multiplication by elements of G.

25. Let G be a group, H a subgroup of finite index n, G/H the set of left cosets of H in G, and $S(G/H)$ the group of permutations of G/H (with composition from right to left). Define $f : G \to S(G/H)$ by $f(g)(xH) = (gx)H$ for $g, x \in G$.
    (a) Show that f is a group homomorphism.
    (b) Show that if H is a normal subgroup of G, then H is the kernel of f.
26. Let G be an abelian group. Let $K = \{a \in G : a^2 = 1\}$ and let $H = \{x^2 : x \in G\}$. Show that $G/K \cong H$.
27. Let $N \trianglelefteq G$ such that every subgroup of N is normal in G and $C_G(N) \subseteq N$. Prove that G/N is abelian.
28. Let H be a subgroup of G having a normal complement (i.e., a normal subgroup N of G satisfying $HN = G$ and $H \cap N = \langle 1 \rangle$). Prove that if two elements of H are conjugate in G, then they are conjugate in H.
29. Let H be a subgroup of the group G with the property that whenever two elements of G are conjugate, then the conjugating element can be chosen within H. Prove that the commutator subgroup G' of G is contained in H.
30. Let $a \in G$ be fixed, where G is a group. Prove that a commutes with each of its conjugates in G if and only if a belongs to an abelian normal subgroup of G.
31. Let G be a group with subgroups H and K, both of finite index. Prove that $|H : H \cap K| \leqslant |G : K|$, with equality if and only if $G = HK$.
    (One variant of this is to prove that if $(|G : H|, |G : K|) = 1$ then $G = HK$.)
32. Show that if H and K are subgroups of a finite group G satisfying $(|G : H|, |G : K|) = 1$, then $G = HK$.
33. Let $G = A \times B$ be a direct product of the subgroups A and B. Suppose H is a subgroup of G that satisfies $HA = G = HB$ and $H \cap A = \langle 1 \rangle = H \cap B$. Prove that A is isomorphic to B.
34. Let N_1, N_2, and N_3 be normal subgroups of a group G and assume that for $i \neq j$, $N_i \cap N_j = \langle 1 \rangle$ and $N_i N_j = G$. Show that G is isomorphic to $N_1 \times N_1$ and G is abelian.
35. Show that if the size of each conjugacy class of a group G is at most 2, then $G' \leqslant Z(G)$.
36. Let N be a normal subgroup of G. Show that if $N \cap G' = \langle 1 \rangle$, then N is contained in the center of G.
37. Let G be a finite group.
    (a) Show that every proper subgroup of G is contained in a maximal subgroup.
    (b) Show that the intersection of all maximal subgroups of G is a normal subgroup.
38. Let G be a finite group that has a maximal, simple subgroup H. Prove that either G is simple or there exists a minimal normal subgroup N of G such that G/N is simple.

39. Let G be a group. Show that G has a composition series if and only if G satisfies the following two conditions:
   (i) If $G = H_0 \trianglerighteq H_1 \trianglerighteq H_2 \trianglerighteq \cdots$ is any subnormal series, then there is an n such that $H_n = H_{n+1} = \cdots$.
   (ii) If H is any subgroup of G in a subnormal series and $K_1 \leqslant K_2 \leqslant K_3 \leqslant \cdots$ is an ascending chain of normal subgroups of H, then there is an m such that $K_m = K_{m+1} = \cdots$.

40. Let G_1 and G_2 be groups, let H be a subgroup of $G_1 \times G_2$, and let $\pi_i : H \to G_i$ be the restriction to H of the natural projection map onto the ith factor. Assume π_i is surjective for $i = 1, 2$, let $N_i = \ker \pi_i$, and let e_i denote the identity element of G_i. Show that $N_1 = \{e_1\} \times K$ and $N_2 = M \times \{e_2\}$ for normal subgroups $M \triangleleft G_1$ and $K \triangleleft G_2$, and that $G_1/M \cong G_2/K$.

Cyclic Groups

41. Let φ be the Euler φ-function — that is, $\varphi(n)$ is the number of positive integers less than the integer n and relatively prime to n. Let G be a finite group of order n with at most d elements x satisfying $x^d = 1$ for each divisor d of n.
   (a) Show that in a *cyclic* group of order n, the number of elements of order d is $\varphi(d)$ for each divisor d of n. Deduce that $\displaystyle\sum_{d|n} \varphi(d) = n$.
   (b) Let $\psi(d)$ be the number of elements of G of *order* d. Show that for any d, either $\psi(d) = 0$ or $\psi(d) = \varphi(d)$.
   (c) Show that G is cyclic.
   (d) Show that any finite subgroup of the multiplicative group of a field must be cyclic.

42. Show that if G is a cyclic group then every subgroup of G is cyclic.

43. Show that if G is a finite cyclic group, then G has exactly one subgroup of order m for each positive integer m dividing $|G|$.

44. Show that if H is a cyclic normal subgroup of a finite group G, then every subgroup of H is a normal subgroup of G.

45. Let G be a cyclic group of order 12 with generator a. Find b in G such that $G/\langle b \rangle$ is isomorphic to $\langle a^{10} \rangle$. (Here $\langle x \rangle$ denotes the subgroup of G generated by $\{x\}$, for $x \in G$.)

Homomorphisms

46. State and prove the three "isomorphism theorems" (for groups).

47. Let G be a group and let K be a subgroup of G. Give necessary and sufficient conditions for K to be the kernel of a homomorphism from G to G. Prove your answer. (*N.B.:* The homomorphism must be from G to G.)

48. Let G be a group with a normal subgroup N of order 5, such that $G/N \cong S_3$. Show that $|G| = 30$, G has a normal subgroup of order 15, and G has 3 subgroups of order 10 that are not normal.

49. Let G be a group with a normal subgroup N of order 7, such that $G/N \cong D_{10}$, the dihedral group of order 10. Show that $|G| = 70$, G has a normal subgroup of order 35, and G has 5 subgroups of order 14 that are not normal.

50. Let $f : G \to H$ be a homomorphism of groups with kernel K and image I.
    (a) Show that if N is a subgroup of G then $f^{-1}(f(N)) = KN$.
    (b) Show that if L is a subgroup of H then $f(f^{-1}(L)) = I \cap L$.

51. Let G and H be finite groups with $(|G|, |H|) = 1$. Show that if $\varphi : G \to H$ is a homomorphism, then $\varphi(g) = 1_H$ for all g in G (where 1_H is the identity element of H).

52. Let $G = \mathrm{GL}_n(\mathbb{R})$ be the (multiplicative) group of nonsingular $n \times n$ matrices with real entries and let $S = \mathrm{SL}_n(\mathbb{R})$ be the subgroup of G consisting of matrices of determinant 1. Show that $S \trianglelefteq G$ and $G/S \cong \mathbb{R}^*$, the multiplicative group of real numbers.

53. Let H and K be normal subgroups of a finite group G.
    (a) Show that there exists a one-to-one homomorphism
    $$\varphi : G/H \cap K \to G/H \times G/K.$$
    (b) Show that φ is an isomorphism if and only if $G = HK$.

54. (a) Suppose H and K are normal subgroups of a group G.
    Show that there exists a one-to-one homomorphism
    $$\varphi : G/H \cap K \to G/H \times G/K.$$
    (b) Use part (a) to show that if $(m, n) = 1$ then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

55. Prove that the commutator subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is *proper* in $\mathrm{SL}_2(\mathbb{Z})$. (Hint: Any homomorphism of rings $R \to S$ induces a homomorphism of groups $\mathrm{SL}_2(R) \to \mathrm{SL}_2(S)$.)

56. Let H and K be subgroups of a finite group G and assume H is isomorphic to K. Prove that there exists a group $\tilde{G}$ containing G as a subgroup, such that H and K are conjugate in $\tilde{G}$.

Automorphism Groups

57. Let $\mathrm{Inn}(G)$ be the group of inner automorphisms of the group G and let $\mathrm{Aut}(G)$ be the full automorphism group.
    (a) Show that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.
    (b) Show that if $Z(G)$ is the center of G, then $\mathrm{Inn}(G) \cong G/Z(G)$.

58. Show that if H is a subgroup of G, then $C_G(H) \trianglelefteq N_G(H)$ and $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

59. Let G be a simple group of order greater than 2 and let $\mathrm{Aut}(G)$ be its automorphism group. Show that the center of $\mathrm{Aut}(G)$ is trivial if and only if G is non-abelian.

60. Let G be a finite group with a normal subgroup $N \cong S_3$. Show that there is a subgroup H of G such that $G = N \times H$.

61. A group N is said to be *complete* if the center of N is trivial and every automorphism of N is inner. Show that if G is a group, $N \trianglelefteq G$, and N is complete, then $G = N \times C_G(N)$.

62. Let H be a normal subgroup of G, $K \leqslant H$, and assume every automorphism of H is inner. Prove that $G = HN_G(K)$, where $N_G(K)$ is the normalizer of K in G.

63. Let $K \leqslant H \lhd G$ and assume every automorphism of H is inner. Prove that $G = H\, N_G(K)$, where $N_G(K)$ is the normalizer of K in G.

Abelian Groups

64. Let A be an abelian group with the following property:
(*) If $B \leqslant A$ then there is a $C \leqslant A$ with $A = B \oplus C$.
Show the following.
   (a) Each subgroup of A satisfies (*).
   (b) Each element of A has finite order.
   (c) If p is a prime, then A has no element of order p^2.

65. Let A be an abelian p-group of exponent p^m. Show that if B is a subgroup of A of order p^m and both B and A/B are cyclic, then there is a subgroup C of A such that $A = B + C$ and $B \cap C = \{0\}$.

66. (a) List all abelian groups of order 360 (up to isomorphism).
   (b) Find the invariant factors and elementary divisors of the group

$$G = \mathbb{Z}_{25} \oplus \mathbb{Z}_{45} \oplus \mathbb{Z}_{48} \oplus \mathbb{Z}_{300}.$$

67. Consider the property (*) of abelian groups G:

   (*) If H is any subgroup of G then there exists a subgroup F of G such that $G/H \cong F$.

   Show that if G is a finitely generated abelian group then G has property (*) if and only if G is finite.

68. Let n be a positive integer and let $A = \mathbb{Z}^n$. Prove that if B is any subgroup of A that is generated by fewer than n elements, then the index $[A : B]$ is infinite.

69. Show that if A, B, and C are abelian groups, then

$$\mathrm{Hom}(A, B \oplus C) \cong \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(A, C).$$

70. Show that if A, B, and C are abelian groups, then

$$\mathrm{Hom}(A \oplus B, C) \cong \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(B, C).$$

71. Let A, B, A_α ($\alpha \in I$) and B_β ($\beta \in J$) be abelian groups. Prove the following:

$$\mathrm{Hom}\Big(\bigoplus_{\alpha \in I} A_\alpha, B\Big) \cong \prod_{\alpha \in I} \mathrm{Hom}(A_\alpha, B)$$

$$\mathrm{Hom}\Big(A, \prod_{\beta \in J} B_\beta\Big) \cong \prod_{\beta \in J} \mathrm{Hom}(A, B_\beta).$$

72. Let:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of Abelian groups and homomorphisms in which both rows are exact. If α, β, δ, and ϵ are isomorphisms, prove that γ is an isomorphism also.

73. Let A, U, V, W, X, and Y be abelian groups.
If $\alpha \in \operatorname{Hom}(X, Y)$ define $\alpha_* : \operatorname{Hom}(A, X) \to \operatorname{Hom}(A, Y)$ by $\alpha_*(f) = \alpha \circ f$. If

$$0 \to U \overset{\alpha}{\to} V \overset{\beta}{\to} W \to 0$$

is exact, to what extent is

$$0 \to \operatorname{Hom}(A, U) \xrightarrow{\alpha_*} \operatorname{Hom}(A, V) \xrightarrow{\beta_*} \operatorname{Hom}(A, W) \to 0$$

exact? Prove your assertions.

74. Same as the previous problem, except use $\operatorname{Hom}(-, A)$ instead, making the obvious modifications.

Symmetric Groups

75. (a) Find the centralizer in S_7 of $(\ 1\ \ 2\ \ 3\ )(\ 4\ \ 5\ \ 6\ \ 7\ )$.
    (b) How many elements of order 12 are there in S_7?

76. (a) Give an example of two nonconjugate elements of S_7 that have the same order.
    (b) If $g \in S_7$ has maximal order, what is $o(g)$?
    (c) Does the element g that you found in part (b) lie in A_7?
    (d) Is the set $\{h \in S_7 \mid o(h) = o(g)\}$ a single conjugacy class in S_7, where g is the element found in part (b)?

77. (a) Give a representative for each conjugacy class of elements of order 6 in S_6.
    (b) Find the order of the centralizer in S_6 of each element from part (a).

78. How many elements of order 6 are there in S_6? How many in A_6?

79. (a) Write $\sigma = (\ 4\ \ 5\ \ 6\ )(\ 2\ \ 3\ )(\ 1\ \ 2\ )(\ 6\ \ 7\ \ 8\ )$ as a product of disjoint cycles and find the order of σ.
    (b) Let $n > 1$ be an odd integer. Show that S_n has an element of order $2(n - 2)$.

80. Let $\sigma = (\ 1\ \ 2\ \ 3\ )(\ 4\ \ 5\ \ 6\ ) \in S_6$.
    (a) Determine the size of the conjugacy class of σ and the order of the centralizer of σ in S_6.
    (b) Determine if $C_{S_6}(\sigma)$ is abelian or non-abelian. Prove your answer.

81. Let G be a subgroup of the symmetric group S_n. Show that if G contains an odd permutation, then $G \cap A_n$ is of index 2 in G.

82. Show that if G is a non-abelian simple subgroup of S_n, then G is contained in A_n.

83. Show that if G is a subgroup of S_n of index 2, then $G = A_n$.

84. Let $n \geqslant 3$ be an integer and let k be n or $n-1$, whichever is odd. Prove that the set of k-cycles in A_n is not a conjugacy class of A_n.

85. For $i = 1, \ldots, n-1$, let x_i be the transposition $(i \quad i+1)$ in the symmetric group S_n. Show that $S_n = \langle x_1, \ldots, x_{n-1} \rangle$.

86. Let H be a subgroup of S_n. Show that if H is a transitive subgroup of S_n and H is generated by some set of transpositions, then $H = S_n$.

87. Prove that the symmetric group S_n is a maximal subgroup of S_{n+1}.
[Hint: Show that if $g \in S_{n+1} - S_n$, then $S_{n+1} = S_n \cup S_n g S_n$.]

88. (a) If $n = k + \ell$ with $k \neq \ell$, then $S_k \times S_\ell$ is a maximal subgroup of S_n in the natural embedding.
    (b) If $n = 2k$, then $S_k \times S_k$ is not a maximal subgroup of S_n in the natural embedding.

89. (a) Prove that if A is a transitive abelian subgroup of the symmetric group S_n, then $|A| = n$.
    (b) Give an example of n, A_1, A_2, where A_1 and A_2 are transitive abelian subgroups of S_n, but A_1 is not isomorphic to A_2.

90. Let $g \in S_n$ (the symmetric group on n letters) be a product of two disjoint cycles, one a k-cycle and the other an ℓ-cycle where $k < \ell$ and $k + \ell = n$.
Prove that if $H = C_{S_n}(g) = \{h \in S_n \mid hg = gh\}$, then H is not a transitive subgroup of S_n.

91. Let A be an abelian, transitive subgroup of S_n. Show that for all $\alpha \in \{1, \ldots, n\}$, the stabilizer A_α of α in A is trivial.

92. Let H be a subgroup of index n in a group G. Let S_n be the symmetric group on n letters and let $S_{n-1} \subseteq S_n$ be the usual embedding. Show that $H = f^{-1}(S_{n-1})$ for some homomorphism $f : G \to S_n$. (Hint: Let G act on the cosets of H.)

93. Show that if $\sigma = \rho\lambda \in S_{m+n}$ is the product of an m-cycle ρ and an n-cycle λ, with ρ and λ disjoint and $m \neq n$, then the centralizer in S_{m+n} of σ is $\langle \rho, \lambda \rangle$.

94. Let τ be an element of the symmetric group S_n and let $\sigma \in S_n$ be a transposition. Show that the number of cycles in the cycle decomposition of $\sigma\tau$ is either one more or one less than the number of cycles in the cycle decomposition of τ.

95. Show that if $\sigma \in S_n$ is an $(n-1)$-cycle, where $n \geqslant 3$, then $C(\sigma) = \langle \sigma \rangle$.

96. Let g and h be elements of the alternating group A_n that have the same cycle structure. Assume that in a cycle decomposition of g (and hence also of h), two cycles have the same length. Prove that g and h are conjugate in A_n.

Infinite Groups

97. Let A and B be subgroups of the additive group of rationals $\mathbb{Q}$. Show that if A is isomorphic to B and $f : A \to B$ is an isomorphism, then there exists $q \in \mathbb{Q}$ such that $f(x) = qx$ for all $x \in A$.

98. (a) Prove that the additive group of the rational numbers is not cyclic.
    (b) Prove that a finitely generated subgroup of the additive group of the rational numbers must be cyclic.

99. If G is a finitely generated group and n is a positive integer, prove that there are at most finitely many subgroups of index n in G. (HINT: Consider maps into the symmetric group S_n.)

100. Let G be a group with a proper subgroup of finite index. Show that G has a proper normal subgroup of finite index.

101. Let $\mathbb{Q}$ be the additive group of rationals and $\mathbb{Z}$ its subgroup of integers. Prove the following.
    (a) If n is a positive integer, then $\mathbb{Q}/\mathbb{Z}$ has an element of order n.
    (b) If n is a positive integer, then $\mathbb{Q}/\mathbb{Z}$ has a unique subgroup of order n.
    (c) Every finite subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic.

102. Let G have the presentation $G = \langle a, b \mid a^2 = 1,\, a^{-1}bab = 1 \rangle$. Prove that G is infinite but the commutator subgroup of G is of finite index in G.

103. Let N be a normal subgroup of G with the order of N finite. Prove there is a normal subgroup M of G such that $[G : M]$ is finite and $nm = mn$ for all $n \in N$ and $m \in M$.

104. Let G be a finitely presented group in which there are fewer relations than generators. Prove that G is necessarily infinite.

p-Groups

105. Show that the center of a finite p-group is non-trivial.

106. Show that if P is a finite p-group and $\langle 1 \rangle \neq N \trianglelefteq P$, then $N \cap Z(P) \neq \langle 1 \rangle$.

107. Let P be a finite p-group and let H be a proper subgroup of P. Prove that H is a proper subgroup of its normalizer $N_P(H)$.

108. Show that a group of order p^2, where p is a prime, must be abelian.

109. Let p be a prime and let G be a non-abelian group of order p^3.
    (a) Show that the center $Z(G)$ of G and the commutator subgroup of G are equal and of order p.
    (b) Show that $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

110. Let p be a prime and let G be a group of order p^n satisfying the following property:
    (*) If A and B are subgroups of G then $A \leqslant B$ or $B \leqslant A$.
    Prove that G is a cyclic group.
    [Note: This statement is also true without the assumption that G is a p-group.]

111. Let G be a finite group. Prove that G is a cyclic p-group, for some prime p, if and only if G has exactly one conjugacy class of maximal subgroups.

112. Let G be a finite p-group for some prime p. Show that if G is not cyclic, then G has at least $p + 1$ maximal subgroups.

113. Let P be a finite p-group in which all the non-identity elements of the center $Z(P)$ have order p. If $\{Z_i(P)\}$ is the upper central series of P, prove that for every i, every non-identity element of $Z_{i+1}(P)/Z_i(P)$ has order p.

114. Let P be a p-group satisfying $[P : Z(P)] = p^n$. Show that $|P'| \leqslant p^{\frac{n(n-1)}{2}}$.
(Hint: Use induction on n. Apply the inductive hypothesis to a maximal subgroup of P.)

115. Let G be a group of order 16 with an element g of order 4. Prove that the subgroup of G generated by g^2 is normal in G.

Group Actions

116. Show that if the center of a group G is of index n in G, then every conjugacy class of G has at most n elements.

117. Let $G_n = \mathrm{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ matrices with complex entries and let $M_n = \mathrm{M}_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices.
   (a) Show that for $g \in G_n$ and $m \in M_n$, $g \cdot m = gmg^{-1}$ defines a (left) action of G_n on M_n.
   (b) For $n = 2$ and $n = 3$, find a complete set of orbit representatives.

118. Let G be a finite group acting on a set A and suppose that for any two ordered pairs (a_1, a_2) and (b_1, b_2) of elements of A, there is an element $g \in G$ such that $g \cdot a_i = b_i$ for $i = 1, 2$. Show that if $|A| = n$, then $|G|$ is divisible by $n(n-1)$. [Hint: Show that if $a \in A$ then G_a acts transitively on $A - \{a\}$.]

119. Let G be a group acting *transitively* on a set Ω. Show that the following are equivalent.
   (i) The action is doubly transitive (i.e., for any two ordered pairs (α_1, β_1), (α_2, β_2) of elements of Ω with $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, there is an element g in G such that $g \cdot \alpha_1 = \alpha_2$ and $g \cdot \beta_1 = \beta_2$).
   (ii) For all $\alpha \in \Omega$, the stabilizer G_α acts transitively on $\Omega - \{\alpha\}$.

120. Let G be a group acting transitively on the set Ω. Show that if $\alpha \neq \beta$ are elements of Ω, then $G_\alpha G_\beta$ is a proper subset of G.

121. Let G be a group acting transitively on a set A. Show that if there is an element $a \in A$ such that $G_a = \{1\}$, then $G_b = \{1\}$ for all $b \in A$.

122. Let the group G act transitively on the set Ω, and let N be a normal subgroup of G. Prove that G permutes the N-orbits of Ω and that these orbits all have the same size.

123. Let G act on a set A and let B be a subset of A. For $g \in G$, let $g \cdot B = \{g \cdot b \,:\, b \in B\}$. Show that $H = \{g \in G \,:\, g \cdot B = B\}$ is a subgroup of G.

124. Let G be a group acting on the set S and let H be a subgroup of G acting transitively on S. Show that if $t \in S$ then $G = G_t H$, where G_t is the stabilizer of t in G.

125. Let G be a finite group. Show that if G has a normal subgroup N of order 3 that is not contained in the center of G, then G has a subgroup of index 2. [Hint: The group G acts on N by conjugation.]

126. (a) Let G be a finite group acting on the finite set S. For $g \in G$, let

$$F(g) = |\{x \in S \,:\, g \cdot x = x\}|.$$

Show that the number of orbits is $\dfrac{1}{|G|} \displaystyle\sum_{g \in G} F(g)$.

(b) Show that the number of conjugacy classes of a finite group G is $\dfrac{1}{|G|} \displaystyle\sum_{g \in G} |C_G(g)|$.

127. Let G be a subgroup of S_n that acts transitively on $\{1, 2, \ldots, n\}$.
(a) Show that if $G_1 = \{g \in G \mid g \cdot 1 = 1\}$ then $[G : G_1] = n$.
(b) Show that if G is abelian then G is of order n.

128. Let G be a finite group acting transitively on a set Ω. Fix $\alpha \in \Omega$ and let G_α be the stabilizer of α in G. Let Δ be the set of points fixed by G_α, i.e., $\Delta = \{\beta \in \Omega \mid \beta \cdot x = \beta \,\forall x \in G_\alpha\}$. Show that Δ is stabilized by $N_G(G_\alpha)$ and that $N_G(G_\alpha)$ acts transitively on Δ.

129. Let G act transitively on a set Ω, fix $\alpha \in \Omega$, and let $H = G_\alpha$. Show that the orbits of H on Ω are in one-to-one correspondence with the $H - H$ double cosets in G.

130. Let G act on a set Ω and assume N is a normal subgroup of G that is contained in the kernel of the action. Show that there is a natural action of G/N on Ω which satisfies the property that G is transitive if and only if G/N is transitive.

131. Let G be a group with a subgroup H of finite index n. Show that there is a homomorphism $\varphi : G \to S_n$ with $\ker\varphi \subseteq H$.

132. Suppose a group G has a subgroup H with $|G : H| = n < \infty$. Prove that G has a normal subgroup N with $N \subseteq H$ and $|G : N| \leqslant n!$.

133. **[NEW]**
Let G be a finite group of order mn where m and n are reltively prime. Assume that there exist subgroups M and N of orders m and n, respectively. Prove that G is isomorphic to a subgroup of the symmetric group S_{m+n}.

134. Let $n > 1$ be a fixed integer. Prove that there are only finitely many simple groups (up to isomorphism) containing a proper subgroup of index less than or equal to n.

135. Let $n = p^m r$ where p is prime and r is an integer greater than 1 such that p does not divide r. Show that if there is a simple group of order n, then p^m divides $(r-1)!$.

136. Show that if G is a simple group of order greater than 60, then G has no proper subgroup of index less than or equal to 5.

137. Let G be a group of order $2016 = 2^5 \cdot 3^2 \cdot 7$ in which all elements of order 7 are conjugate. Prove that G has a normal subgroup of index 2.

138. Prove that if G is a simple group containing an element of order 45, then every proper subgroup of G has index at least 14.

139. Let G be a finite simple group containing an element of order 21. Show that every proper subgroup of G has index at least 10.

140. Let G be a finite group and let K be a subgroup of index p, where p is the smallest prime dividing the order of G. Show that K is a normal subgroup of G.

141. Let G be a nonabelian finite simple group and let H be a subgroup of index p, where p is a prime. Prove that the number of distinct conjugates of H in G is p.

142. Let G be a finite simple group with a subgroup H of prime index p. Show that p must be the largest prime dividing the order of G.

143. Let G be a finite simple group and p a prime such that p^2 divides the order of G. Show that G has no subgroup of index p.

144. Let G be a finite group in which a Sylow 2-subgroup is cyclic. Prove that there exists a normal subgroup N of odd order such that the index $[G : N]$ is a power of 2. [Hint: Generalize the previous problem.]

145. (a) Let G be a subgroup of the symmetric group S_n. Show that if G contains an odd permutation then $G \cap A_n$ is of index 2 in G.
(b) Let G be a group of order $2r$, where $r > 1$ is an odd integer. Show that in the regular permutation representation of G, an element t of G of order 2 corresponds to an odd permutation.
(c) Show that a group of order $2r$, with $r > 1$ an odd integer, cannot be simple.

146. Let G be a finite cyclic group and H a subgroup of index p, p a prime. Suppose G acts on a set S and the restriction of the action to H is transitive. Let G_x, H_x be the stabilizer of $x \in S$ in G, H, respectively. Show the following.
(a) $H_x = G_x \cap H$
(b) $[H : H_x] = [G : G_x] = |S|$
(c) $|S|$ is not divisible by p.

147. Let G be a finite group and p a prime. Then G acts on $\mathrm{Syl}_p(G)$ by conjugation; let $\rho : G \longrightarrow \mathrm{Sym}(\mathrm{Syl}_p(G))$ be the homomorphism corresponding to this action.
(a) $\rho(P)$ fixes exactly one point (element of $\mathrm{Syl}_p(G)$).
(b) If $P \in \mathrm{Syl}_p(G)$ has order p, then $\rho(x)$ is a product of one 1-cycle and a certain number of p-cycles, for $x \in P - \{1\}$.
(c) If $P \in \mathrm{Syl}_p(G)$ has order p and $y \in N_G(P) - C_G(P)$ then $\rho(y)$ fixes at most r points, where r is the number of orbits under the action of $\rho(P)$ (including the fixed point of part (a)).

148. Let G be a finite group acting faithfully and transitively on a set Ω. Assume that there exists a normal subgroup N such that N acts regularly on Ω (i.e., $G = G_\alpha N$ and $G_\alpha \cap N = 1$ for all $\alpha \in \Omega$). Prove that G_α embeds as a subgroup of $\mathrm{Aut}(N)$.

Sylow Theorems

149. (a) Let G be a finite p-group acting on the finite set S. Let S_0 be the set of all elements of S fixed by G. Show that $|S| \equiv |S_0| \pmod{p}$.
(b) Show that if H is a p-subgroup of a finite group G, then $[N_G(H) : H] \equiv [G : H] \pmod{p}$.
(c) State and prove Sylow's theorems.

150. Let G be a finite group and let P be a Sylow p-subgroup of G. Prove the following.
   (a) If M is any normal p-subgroup of G then $M \leqslant P$.
   (b) There is a normal p-subgroup N of G that contains all normal p-subgroups of G.

151. Let n be an integer and p a prime dividing n. Assume that there exists exactly one divisor d of n satisfying both $d > 1$ and $d \equiv 1 \pmod{p}$. Prove that if G is any finite group of order n and P is a Sylow p-subgroup of G, then either $P \trianglelefteq G$ or else $N_G(P)$ is a maximal subgroup of G.

152. Let P be a Sylow p-subgroup of the finite group G and let H be a subgroup of G containing the normalizer $N_G(P)$ of P. Prove that $N_G(H) = H$.

153. Let G be a group of order 168 and let P be a Sylow 7-subgroup of G. Show that either P is a normal subgroup of G or else the normalizer of P is a maximal subgroup of G.

154. Show that if G is a simple group of order 60 then $G \cong A_5$.

155. Show that a group of order $2001 = 3 \cdot 23 \cdot 29$ contains a normal cyclic subgroup of index 3.

156. Show that if G is a group of order $2002 = 2 \cdot 7 \cdot 11 \cdot 13$, then G has an abelian subgroup of index 2.

157. Show that a group of order $2004 = 2^2 \cdot 3 \cdot 167$ must be solvable. Give an example of a group of order 2004 in which a Sylow 3-subgroup is not a normal subgroup.

158. Determine all groups of order $2009 = 7^2 \cdot 41$, up to isomorphism.

159. Show that if G is a group of order $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, then G has a normal subgroup of order 5.

160. Show that if G is a group of order $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, then G is solvable.

161. Prove or disprove: Every group of order $14077 = 7 \cdot 2011$ is cyclic. Use Sylow's Theorems.

162. Determine, up to isomorphism, all groups of order 2012. (Note that $2012 = 2^2 \cdot 503$ and 503 is a prime.)

163. Prove that a group G of order 36 must have a normal subgroup of order 3 or 9.

164. Show that a group of order 96 must have a normal subgroup of order 16 or 32.

165. Show that a group of order $160 = 2^5 \cdot 5$ must contain a nontrivial normal 2-subgroup.

166. Show that if G is a group of order $392 = 2^3 \cdot 7^2$, then G has a normal subgroup of order 7 or a normal subgroup of order 49.

167. Let G be a finite simple group containing an element of order 9. Show that every proper subgroup of G has index at least 9.

168. Show that there is no simple group of order 120.

169. (a) Show that S_6 has no simple subgroup of index 4 (i.e. order 180).
   (b) Show that a group of order $180 = 2^2 \cdot 3^2 \cdot 5$ cannot be simple.

170. (a) Show that $|\mathrm{Aut}(\mathbb{Z}_7)| = 6$.
    (b) Show that a group of order 63 must contain an element of order 21.

171. Show that a simple group of order 168 must be isomorphic to a subgroup of the alternating group A_8.

172. Let G be a simple group of order 168. Determine the number of elements of G of order 7. Explain your answer.

173. Let $p > q$ be primes. Show that if $p - 1$ is not divisible by q, then there is exactly one group of order pq.

174. Let G be a group of order pqr, where $p > q > r$ are primes. Prove that a Sylow subgroup for one of these primes is normal.

175. Let G be a group of order pqr, where $p > q > r$ are primes. Let P be a Sylow p-subgroup of G and assume P is not normal in G. Show that a Sylow q-subgroup of G must be normal.

176. Let G be a group of order pqr, where $p > q > r$ are primes. Show that if $p - 1$ is not divisible by q, then a Sylow p-subgroup of G must be normal.

177. Let G be a group of order pqr, where $p > q > r$ are primes. Show that if $p - 1$ is not divisible by q or r and $q - 1$ is not divisible by r, then G must be abelian (hence cyclic).
[Hint: Show that G' must be contained in a Sylow subgroup for two different primes.]

178. Let G be a group of order $105 = 3 \cdot 5 \cdot 7$. Prove that a Sylow 7-subgroup of G is normal.

179. Show that a group of order $3 \cdot 5 \cdot 7$ must be solvable.

180. Prove that a group of order $29 \cdot 30$ has a normal Sylow 29-subgroup.

181. Show that a group G of order $255 = 3 \cdot 5 \cdot 17$ must be abelian.

182. Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Prove that a Sylow 11-subgroup is contained in the center of G.

183. Show that a group of order $10000 = 2^4 \cdot 5^4$ cannot be simple.

184. Show that a group of order $3^3 \cdot 5 \cdot 13$ must have a normal Sylow 13-subgroup or a normal Sylow 5-subgroup. [Hint: Show that if a Sylow 13-subgroup is not normal, then a Sylow 13-subgroup must normalize a Sylow 5-subgroup. Consider the normalizer of a Sylow 5-subgroup.]

185. Let G be a group of order $3 \cdot 5 \cdot 7 \cdot 13$. Prove that G is not a simple group. [Hint: If a Sylow 7-subgroup is not normal, then some Sylow 13-subgroup will centralize it. Now compute the number of Sylow 13-subgroups.]

186. Let G be a group of order $p^n q$, where p and q are distinct primes, and assume $q \nmid p^i - 1$ for $1 \leqslant i \leqslant n - 1$. Prove that G is solvable.

187. Let p and q be distinct primes. Show that a group of order $p^2 q$ has a normal Sylow p-subgroup or a normal Sylow q-subgroup.

188. Let G be a group of order $(p+1)p(p-1)$ where p is a prime. Prove that the number of Sylow p-subgroups is either 1 or $p + 1$.

189. **[NEW]**
Let p be a prime number and G a group of order $p(p+1)$. Prove that G has a normal subgroup of order p or $p+1$.

190. **[NEW]**
Show that if G is a finite group of even order, then G has an odd number of elements of order 2.

191. **[NEW]**
Prove that if the prime p divides the order of the finite group G, then the number of elements of order p in G is congruent to -1 modulo p.

192. Let G be a finite group with exactly $p+1$ Sylow p-subgroups. Prove that if P and Q are two distinct Sylow p-subgroups, then $P \cap Q$ is a normal subgroup of G.
[Hint: First show $|P : P \cap Q| = p$.]

193. Show that a group of order $2^3 \cdot 3 \cdot 7^2$ is not simple.

194. Show that a group of order $380 = 2^2 \cdot 5 \cdot 19$ must be solvable.

195. Show that a group of order $2 \cdot 7 \cdot 13$ must be solvable.

196. Show that a group of order $1960 = 2^3 \cdot 5 \cdot 7^2$ must be solvable.

197. Prove that a group of order $1995 = 3 \cdot 5 \cdot 7 \cdot 19$ must be solvable.

198. Show that a group of order $1998 = 2 \cdot 3^3 \cdot 37$ must be solvable.

199. Show that every group of order $2015 = 5 \cdot 13 \cdot 31$ must have a normal cyclic subgroup of index 5.

200. Show that if G is a group of order $2020 = 2^2 \cdot 5 \cdot 101$, then G is solvable.

201. Show that a group of order $2021 = 43 \cdot 47$ is solvable.

202. Determine, up to isomorphism, the groups of order $2022 = 2 \cdot 3 \cdot 337$.

203. Suppose the finite group G has exactly 61 Sylow 3-subgroups. Prove that there exist two Sylow 3-subgroups P and Q satisfying $|P : P \cap Q| = 3$.

204. Let G be a group with exactly 31 Sylow 3-subgroups. Prove that there exist Sylow 3-subgroups P and Q satisfying $[P : P \cap Q] = [Q : P \cap Q] = 3$.

205. Let G be a finite group, p a prime divisor of $|G|$ and assume there are k distinct Sylow p-subgroups of G. Let $f : G \to S_k$ be the homomorphism of G into the symmetric group induced by the natural action of G by conjugation on the set of Sylow p-subgroups of G, and let $\overline{G} = f(G)$. Prove that $\overline{G}$ has k distinct Sylow p-subgroups.

206. (a) Show that if K is a subgroup of G then the number of distinct conjugates of K in G is $[G : N_G(K)]$.
    (b) Show that if G has n_p Sylow p-subgroups, then G has a subgroup of index n_p.

207. Let G be a finite group and p a prime. Show that the intersection of all Sylow p-subgroups of G is a normal subgroup of G.

208. Let K be a normal subgroup of G and let P be a Sylow p-subgroup of K. Show that if $P \trianglelefteq K$ then $P \trianglelefteq G$.

209. Let G be a finite group and let P be a *normal* Sylow p-subgroup of G. Show that P is a characteristic subgroup of G.

210. A subgroup H of a group G is subnormal if there exists a chain $H = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_k = G$ such that H_i is a normal subgroup of H_{i+1} for every i. Prove that if P is a Sylow p-subgroup of a finite group G, then P is a subnormal in G if and only if P is normal in G.

211. Let G be a finite group and p a prime. Let N be a normal subgroup of G and H a Sylow p-subgroup of G. Show that
    (a) HN/N is a Sylow p-subgroup of G/N, and
    (b) $H \cap N$ is a Sylow p-subgroup of N.

212. Let G be a finite group with subgroups H, K such that $G = HK$. Show that if p is any prime number, then there exist $P \in \mathrm{Syl}_p(H)$ and $Q \in \mathrm{Syl}_p(K)$ such that $PQ \in \mathrm{Syl}_p(G)$.

213. Let G be a finite group, p a prime, and P a Sylow p-subgroup of G. Let H be a subgroup of G that contains the normalizer $N_G(P)$ of P in G. Show that if g is an element of G such that $g^{-1}Pg \leqslant H$, then g is an element of H.

214. Let G be a finite group, H be a subgroup of G, and P be a Sylow p-subgroup of H for some prime p. Show that if H contains the normalizer $N_G(P)$ of P, then P is a Sylow p-subgroup of G.

215. A subgroup H of a group G is called *pronormal* if, for any $g \in G$, H is conjugate to H^g in $\langle H, H^g \rangle$.
    (a) Show that if $H \leqslant N \trianglelefteq G$ with H pronormal in G, then $G = N_G(H)N$.
    (b) Show that if P is a Sylow p-subgroup of G, then P is pronormal in G.

216. Let G be a finite group and H a normal subgroup. Show that if P is a Sylow p-subgroup of H, then $G = HN_G(P)$.

217. Let P be a Sylow p-subgroup of a group G and let K be a subgroup of G containing $N_G(P)$. Show that $N_G(K) = K$.

218. Let x and y be two elements of $Z(P)$ where P is a Sylow p-subgroup of G. If x and y are conjugate in G, prove that x is conjugate to y in $N_G(P)$.

219. (a) Let p be a prime and let H be a p-subgroup of the finite group G. Show that

$$[N_G(H) : H] \equiv [G : H] \pmod{p}.$$

    (Hint: Let H act on G/H by left multiplication.)
    (b) Let P be a p-subgroup of G. Show that P is a Sylow p-subgroup of G if and only if P is a Sylow p-subgroup of $N_G(P)$.

220. Let G be a finite group with $|G| = p^a m$, where p is a prime and $p \nmid m$. Assume that whenever P and Q are Sylow p-subgroups of G, either $P = Q$ or $P \cap Q = 1$. Show that the number of Sylow p-subgroups of G is congruent to 1 modulo p^a.

221. Let P be a Sylow p-subgroup of the finite group G, and assume $|P| = p^a$. Suppose that $P \cap P^g = \{1\}$ whenever $g \in G$ does not normalize P. Prove that the number of Sylow p-subgroups of G is congruent to 1 mod p^a.

222. Let p be a prime and let P be a p-subgroup of the finite group G. Show that P is a Sylow p-subgroup of G if and only if P is a Sylow p-subgroup of $PC_G(P)$ and $[N_G(P) : PC_G(P)]$ is not divisible by p.

223. Let G be a finite group, and p a prime. Let n_p be the number of Sylow p-subgroups of G and suppose that p^e does *not* divide $n_p - 1$. Prove that there exist two distinct Sylow p-subgroups of G, say P and Q, satisfying $[P : P \cap Q] \leqslant p^e$.

224. Let P and Q be distinct Sylow p-subgroups of a finite group G. Prove that the number of Sylow p-subgroups of G is strictly greater than $[P : P \cap Q]$.

225. Let X and G be finite groups. We say that X is *involved* in G if there exist subgroups K and H of G, with K normal in H, such that X is isomorphic to H/K. Suppose X is a p-group, P is a Sylow p-subgroup of G, and X is involved in G. Prove that X is involved in P.

Solvable and Nilpotent Groups, Commutator and Frattini Subgroups

226. Show that the following statements are equivalent.
    (i) Every finite group of odd order is solvable.
    (ii) Every non-abelian finite simple group is of even order.

227. Let H and K be subgroups of a group G with $K \trianglelefteq G$. Show that if H and K are solvable, then HK is solvable.

228. Let G be a solvable group and N a nontrivial normal subgroup of G. Show that there is a nontrivial abelian subgroup A of N with A normal in G.

229. Prove that a minimal normal subgroup of a finite solvable group is abelian.

230. Let G be a finite non-solvable group, each of whose proper subgroups is solvable. Show that $G/\Phi(G)$ is a non-abelian simple group, where $\Phi(G)$ denotes the Frattini subgroup of G.

231. We say that a group X is *involved* in a group G if X is isomorphic to H/K for some subgroups K, H of G with $K \trianglelefteq H$. Prove that if X is solvable and X is involved in the finite group G, then X is involved in a solvable subgroup of G.

232. Let G be a finite group satisfying the following property:
    (*) If A, B are subgroups of G then AB is a subgroup of G.
    Prove that G is a solvable group.

233. Let X be a set of operators for the group G and assume that G is a finite solvable group. Prove that every X-composition factor in any X-composition series for G is an elementary abelian p-group for some prime p.

234. Show that if G is a nilpotent group and $\langle 1 \rangle \neq N \trianglelefteq G$, then $N \cap Z(G) \neq \langle 1 \rangle$.

235. Show that if G is a nilpotent finite group, then every subgroup of prime index is a normal subgroup.

236. Let G be a group and let $Z \leqslant Z(G)$ be a central subgroup. Prove that if G/Z is nilpotent, then G is nilpotent.

237. (a) Show that if G is a group and H, K are subgroups of G such that $HK \subseteq KH$, then HK is a subgroup of G.

  (b) Suppose G is finite and $HK \subseteq KH$ for all subgroups H and K of G. Show that if p is a prime divisor of $|G|$, then there is a subgroup N of G such that $|G : N|$ is a power of p and $p \nmid |N|$.

238. Let G be a finite group and let $\Phi(G)$ be its Frattini subgroup. Show that $\Phi(G)$ is precisely the set of non-generators of G. (An element g of G is called a non-generator if for any subset S of G containing g and generating G, the set $S - \{g\}$ also generates G.)

239. Let $\langle 1 \rangle = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ be a central series for the nilpotent group G. Prove that $G_i \leqslant Z_i(G)$ for all i, where $\{Z_i(G)\}$ is the upper central series of G. Thus, among all central series for a nilpotent group, the upper central series ascends the fastest.

240. Let G be a finite group, let $\Phi(G)$ be the Frattini subgroup of G (that is, the intersection of all maximal subgroups of G), and let G' be the commutator subgroup of G.
Show that the following are equivalent.

  (i) The group G is nilpotent.

  (ii) If H is a proper subgroup of G, then H is a proper subgroup of its normalizer in G.

  (iii) Every maximal subgroup of G is a normal subgroup of G.

  (iv) $G' \leqslant \Phi(G)$.

  (v) Every Sylow subgroup of G is a normal subgroup of G.

  (vi) The group G is a direct product of its Sylow subgroups.

241. Let G be a finite group. Show that each of the following conditions is equivalent to the nilpotence of G.

  (a) Whenever $x, y \in G$ satisfy $(|x|, |y|) = 1$, then $xy = yx$.

  (b) Whenever p and q are distinct primes and $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, then P centralizes Q.

242. Show that if G is a finite nilpotent group and m is a positive integer such that m divides the order of G, then G has a subgroup of order m.

243. Let G be a finite nilpotent group and G' its commutator subgroup.
Show that if G/G' is cyclic then G is cyclic.

244. A finite group G is called an *N-group* if the normalizer $N_G(P)$ of every non-identity p-subgroup P of G is solvable. Prove that if G is an N-group, then either (i) G is solvable, or (ii) G has a unique minimal normal subgroup K, the factor group G/K is solvable, and K is simple.

245. Let G be a finite group and let N be a normal subgroup of G with the property that G/N is nilpotent. Prove that there exists a nilpotent subgroup H of G satisfying $G = HN$.

246. Let G be a finite solvable group. Prove that the index of every maximal subgroup is a prime power.

247. Let G be a group. Show that if $g \in G$, then the conjugacy class of g is contained in gG'.

248. Let G be a group of odd order. Let $g_1, \ldots, g_n$ be the elements of G, listed in any order. Show that $\prod_{i=1}^{n} g_i$ is an element of the commutator subgroup G' of G.

249. Let G be a finite group and let M be a maximal subgroup of G.
   (a) Show that if $Z(G)$ is not contained in M, then $M \trianglelefteq G$.
   (b) Show that either $Z(G) \leqslant M$ or $G' \leqslant M$.
   (c) Show that $Z(G) \cap G' \leqslant \Phi(G)$.