Separation of Variables and Exact Solutions to Nonlinear PDEs

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1 Intorduction

Nonlinear partial differential equations (PDEs) of the second and higher orders (nonlinear equations of mathematical physics) often arise in various fields of mathematics, physics, mechanics, chemistry, biology, and in numerous applications.
Thegeneral solution of nonlinear equations of mathematical physics can only be obtained in exceptional cases.
Therefore, one usually has to confine themselves to the search and analysis of particular solutions, which are usually called exact solutions.

Methods of Generalized Separation of Variables

2 Simple Separable Solutions

2.1 Multiplicative and Additive Separable Solutions

Linear equations of mathematical physics.
The method of generalized separation of variables is the most common analytical method for solving linear equations of mathematical physics [1, 2].
For equations with two independent variables, x and t, and one unknown function, $u = u(x, t)$, this method suggests searching for exact solutions as the product of functions with different arguments:

$$u = \varphi(x)\psi(t). \tag{2.1.1}$$

The functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are described by linear ordinary differential equations (ODEs) and determined in the course of a subsequent analysis.

■ **Example 2.1.1** Let us look at the linear heat equation

$$u_t = u_{xx}. \tag{2.1.2}$$

Its exact solutions are sought in the product form (2.1.1). Substituting (2.1.1) into (2.1.2) gives

$$\varphi\psi'_t = \psi\varphi''_{xx}. \tag{2.1.3}$$

Separating the variables by dividing both sides of this equations by $\varphi\psi$, we get

$$\frac{\psi'_t}{\psi} = \frac{\varphi''_{xx}}{\varphi}. \tag{2.1.4}$$

The left-hand side of this equation only depends on the variable t, while the right-hand side depends on x alone.
This is possible only if both sides of equation (2.1.4) are individually equal to the same constant quantity, so that

$$\frac{\psi'_t}{\psi} = C, \frac{\varphi''_{xx}}{\varphi} = C. \tag{2.1.5}$$

where C is the so-called constant of separation, which is a free parameter. For $C = -\lambda^2 < 0$, the general solutions to ODEs (2.1.5) are given by

$$\varphi = A_1 cos(\lambda x) + A_2 sin(\lambda x), \quad \psi = A_3 exp(-\lambda^2 t), \tag{2.1.6}$$

where A_1, A_2, and A_3 are arbitrary constants.
Since φ and ψ appear in solution (2.1.1) as a product, the constant A_3 can be set, without loss of generality, equal to unity.
To the different values of λ in (2.1.6), e.g., $\lambda = \lambda_1, \ . \ . \ . \ , \lambda = \lambda_n$, there correspond different solutions.
Since equation (2.1.2) is linear, these solutions can be added together by virtue of the linear superposition principle. As a result, an exact solution to equation (2.1.2) can be written as the sum

$$u = \sum_{k=1}^{n} \varphi_k(x)\psi_k(t), \tag{2.1.7}$$

where

$$\varphi_k(x) = A_{k1} cos(\lambda_k x) + A_{k2} sin(\lambda_k x), \quad \psi_k(t) = A_{k3} exp(-\lambda_k^2 t), \tag{2.1.8}$$

with A_{k1}, A_{k2}, and A_{k3} being arbitrary constants.
The solution of initial-boundary value problems for equation (2.1.2) on a finite closed interval $x_1 \leq x \leq x_2$ is sought in the form of the infinite series (2.1.7) with $n = \infty$.
The constants λ_k, A_{k1}, and A_{k2} are determined from the boundary conditions (as well as a normalization condition), while the constants A_{k3} are determined from the initial condition [1, 2]. ∎

Nonlinear first-order partial differential equations.
The integration of isolated classes of nonlinear first-order partial differential equations (PDEs) is based on seeking exact solutions in the form of the sum of functions with different arguments [1]:

$$u = \varphi(x) + \psi(t). \tag{2.1.9}$$

Many nonlinear first-, second-, and higher-order PDEs also admit solutions of the from (2.1.9).

2.2 Simple Cases of Separation of Variables in Nonlinear Partial Differential Equations

Nonlinear equations in two independent variables.
In simple cases, separation of variables in nonlinear partial differential equations in two independent variables is carried out following the same scheme as in linear equations.
Exact solutions are sought as the product or sum of functions with different arguments.
Substituting (2.1.1) or (2.1.9) into the equation concerned and performing a simple rearrangement, one arrives at a single equation (for equations in two independent variables) with either side dependent on a single variable.
The equality is possible only if both sides equal the same constant quantity.
As a result, one obtains two ordinary differential equations for the unknown functions, one for $\varphi = \varphi(x)$ and the other one for $\psi = \psi(t)$.
Exact solutions obtained using this kind of separation of variables will be referred to as simple separable solutions.
Let us illustrate the above with specific examples.

▪ **Example 2.2.1** The heat equation with a power-law nonlinearity

$$u_t = a(u^k u_x)_x \tag{2.2.1}$$

admits an exact solution as the product of functions with different arguments. Indeed, substituting (2.1.1) into (2.2.1), we get

$$\varphi\psi'_t = a\psi^{k+1}(\varphi^k\varphi'_x)'_x.$$

Dividing both sides by $\varphi\psi^{k+1}$ gives

$$\frac{\psi'_t}{\psi^{k+1}} = \frac{a(\varphi^k\varphi'_x)'_x}{\varphi}$$

The left-hand side of this equation only depends on t, while the right-hand side depends on x alone. The equality is possible only if

$$\frac{\psi_t'}{\psi^{k+1}} = C, \quad \frac{a(\varphi^k \varphi_x')_x'}{\varphi} = C, \tag{2.2.2}$$

where C is an arbitrary constant.
The solution of the first ODE in (2.2.2) is expressed in terms of elementary functions.
The other ODE can be solved in implicit form.
The procedure for finding separable solutions of the form (2.1.1) to the non-linear equation (2.2.1) is very much the same as that for solving the linear heat equation (2.1.2) and other linear partial differential equations.
The fundamental difference between linear and nonlinear differential equations is that the superposition principle does not apply to solutions of nonlinear equations.
This means that solutions (2.1.1) of equations (2.2.1), obtained by integrating ODE (2.2.2) and taken at different C, cannot be added together. ■

■ **Example 2.2.2** The heat equation in an anisotropic medium with a logarithmic source

$$[f(x)u_x]_x + [g(y)u_y]_y = auln(u) \tag{2.2.3}$$

admits a multiplicative separable solution

$$u = \varphi(x)\psi(t). \tag{2.2.4}$$

Indeed, substituting (2.2.4) into (2.2.3) followed by dividing the resulting equation by $\varphi\psi$ and moving some terms to the left- or right-hand side, we obtain

$$\frac{1}{\varphi}[f(x)\varphi_x']_x' - aln\varphi = -\frac{1}{\psi}[g(y)\psi_y']_y' + aln\psi.$$

The left-hand side of this equation only depends on x, while the right-hand side depends on y alone. Equating these with the same constant, one obtains ordinary differential equations for $\varphi(x)$and $\psi(y)$. ■

Table gives other examples of simple, additive or multiplicative, separable solutions for some nonlinear equations.

Table Some nonlinear equations of mathematical physics that admit simple separable solutions (C, C_1, and C_2 are arbitrary constants)

Equation	Equation name	Form of solutions	Determining equations
$u_t = au_{xx} + bu\ln u$	Heat equation with source	$u = \varphi(x)\psi(t)$	$a\varphi''_{xx}/\varphi - b\ln\varphi = -\psi'_t/\psi + b\ln\psi = C$
$u_t = a(u^k u_x)_x + bu$	Heat equation with source	$u = \varphi(x)\psi(t)$	$(\psi'_t - b\psi)/\psi^{k+1} = a(\varphi^k\varphi'_x)'_x/\varphi = C$
$u_t = a(u^k u_x)_x + bu^{k+1}$	Heat equation with source	$u = \varphi(x)\psi(t)$	$\psi'_t/\psi^{k+1} = a(\varphi^k\varphi'_x)'_x/\varphi + b\varphi^k = C$
$u_t = a(e^{\lambda u}u_x)_x + b$	Heat equation with source	$u = \varphi(x) + \psi(t)$	$e^{-\lambda\psi}(\psi'_t - b) = a(e^{\lambda\varphi}\varphi'_x)'_x = C$
$u_t = a(e^u u_x)_x + be^u$	Heat equation with source	$u = \varphi(x) + \psi(t)$	$e^{-\psi}\psi'_t = a(e^{\varphi}\varphi'_x)'_x + be^{\varphi} = C$
$u_t = au_{xx} + bu_x^2$	Potential Burgers equation	$u = \varphi(x) + \psi(t)$	$\psi'_t = a\varphi''_{xx} + b(\varphi'_x)^2 = C$
$u_t = au_x^k u_{xx}$	Filtration equation	$u = \varphi(x) + \psi(t)$, $u = f(x)g(t)$	$\psi'_t = a(\varphi'_x)^k\varphi''_{xx} = C_1$, $g'_t/g^{k+1} = a(f'_x)^k f''_{xx}/f = C_2$
$u_t = f(u_x)u_{xx}$	Filtration equation	$u = \varphi(x) + \psi(t)$	$\psi'_t = f(\varphi'_x)\varphi''_{xx} = C$
$u_{tt} = a(u^k u_x)_x$	Wave equation	$u = \varphi(x)\psi(t)$	$\psi''_{tt}/\psi^{k+1} = a(\varphi^k\varphi'_x)'_x/\varphi = C$
$u_{tt} = a(e^{\lambda u}u_x)_x$	Wave equation	$u = \varphi(x) + \psi(t)$	$e^{-\lambda\psi}\psi''_{tt} = a(e^{\lambda\varphi}\varphi'_x)'_x = C$
$u_{tt} = au_{xx} + bu\ln u$	Wave equation with source	$u = \varphi(x)\psi(t)$	$\psi''_{tt}/\psi - b\ln\psi = a\varphi''_{xx}/\varphi + b\ln\varphi = C$
$u_{xx} + a(u^k u_y)_y = 0$	Anisotropic steady heat equation	$u = \varphi(x)\psi(y)$	$\varphi''_{xx}/\varphi^{k+1} = -a(\psi^k\psi'_y)'_y/\psi = C$
$u_{xx} + au_y u_{yy} = 0$	Equation of steady transonic gas flow	$u = \varphi(x) + \psi(y)$, $u = f(x)g(y)$	$\varphi''_{xx} = -a\psi'_y\psi''_{yy} = C_1$, $f''_{xx}/f = -ag'_y g''_{yy}/g = C_2$
$u_{xy}^2 = u_{xx}u_{yy}$	Monge–Ampère equation	$u = \varphi(x) + \psi(y)$, $u = f(x)g(y)$	$\varphi''_{xx} = 0$ or $\psi''_{yy} = 0$, $(f'_x)^2/(ff''_{xx}) = gg''_{yy}/(g'_y)^2 = C$
$u_t = au_{xxx} + bu_x^2$	Potential Korteweg–de Vries equation	$u = \varphi(x) + \psi(t)$	$\psi'_t = a\varphi'''_{xxx} + b(\varphi'_x)^2 = C$
$u_y u_{xy} - u_x u_{yy} = au_{yyy}$	Boundary layer equation	$u = \varphi(x) + \psi(y)$, $u = f(x)g(y)$	$-\varphi'_x = a\psi'''_{yyy}/\psi''_{yy} = C_1$, $f'_x = ag'''_{yyy}[(g'_y)^2 - gg''_{yy}]^{-1} = C_2$

Some generalizations. Below is proposition that allow one to generalize simple separation solutions of special forms.

Proposition 2.1 — **Suppose that the equation.**

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0 \tag{2.2.5}$$

admits a simple separable solution of the special form

$$u = t^\beta \varphi(x), \quad \beta \neq 0, \tag{2.2.6}$$

which is invariant (remains the same) under the dilation transformation

$$t \longmapsto \lambda t, \quad u \longmapsto \lambda^\beta u \quad (\lambda > 0 \textit{ is an arbitrary constant}). \tag{2.2.7}$$

Suppose equation (2.2.5) is also invariant under transformation (2.2.7). Then equation (2.2.5) admits a more complicated solution of the form

$$u = (t + C_1)^\beta \theta(z), \quad z = x + C_2 ln|t + C_1| + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

■ **Example 2.2.3** The nonlinear wave equation

$$u_{tt} = a(u^k u_x)_x \tag{2.2.8}$$

has a multiplicative separable solution of the form (2.2.6) with $\beta = -2/k$.
Therefore, equation (2.2.8) also has a more complicated solution

$$u = (t + C_1)^{-2/k} \theta(z), \quad z = x + C_2 ln|t + C_1| + C_3,$$

The function $\theta = \theta(z)$ is determined by the ordinary differential equation

$$\frac{2(k+2)}{k^2}\theta - \frac{(k+4)}{k} C_2 \theta'_z + C_2^2 \theta''_{zz} = a(\theta^k \theta'_z)'_z$$

■

Proposition 2.2 — Suppose that equation (2.2.5) does not change under scaling of the unknown function.

$$u \longmapsto \lambda u, \tag{2.2.9}$$

where $\lambda > 0$ is an arbitrary constant.
Then equation (2.2.5) admits an exact solution of the form

$$u = e^{kt}\theta(z), \quad z = px + qt, \tag{2.2.10}$$

where k, p, and q are arbitrary constants ($pq \neq 0$).

■ **Example 2.2.4** The nonlinear heat-type equation

$$u_t = au_{xx} + uf(u_x/u) \tag{2.2.11}$$

is invariant under the scaling transformation (2.2.9).
Therefore, by virtue of Proposition 2, this equation has a solution of the form (2.2.10), where the function $\theta = \theta(z)$ satisfies the nonlinear ordinary differential equation

$$k\theta + q\theta'_z = ap^2\theta''_{zz} + \theta f(p\theta'_z/\theta).$$

■

3 Structure of Generalized Separable Solutions

3.1 General Form of Solutions. The Classes of Nonlinear Differential Equations of Interest

For clarity and simplicity, we will restrict ourselves to the consideration of nonlinear equations of mathematical physics in two independent variables, x and y, and one dependent variable, u; one of the independent variables may be treated as time.

Linear partial differential equations. It is well known that linear equations of mathematical physics with constant coefficients and many linear equations with variable coefficients can have solutions as the sum of pairwise products of functions with different arguments as in Example 1.1(see also[1, 2]):

$$u(x,y) = \varphi_1(x)\psi_1(y) + \varphi_2(x)\psi_2(y) + ... + \varphi_n(x)\psi_n(y), \tag{3.1.1}$$

where $u_i = \varphi_i(x)\psi_i(y)$ are particular solutions.

The functions $\varphi_i(x)$, as well as thefunctions $\psi_i(y)$, with the different subscripts i are unrelated to one another.

It is noteworthy that linear partial differential equations often also admit exact solutions of the form (3.1.1) in which the pairwise products of functions with different arguments,$\varphi_i(x)\psi_i(y)$, are not particular solutions of these equations.

■ **Example 3.1.1** The linear heat equation

$$u_t = au_{xx},$$

with $y = t$, admits the following solutions [2]:

$$\begin{aligned}
&u = x^2 + 2at,\\
&u = x^3 + 6atx,\\
&u = x^4 + 12atx^2 + 12a^2t^2,\\
&u = x^{2n} + \sum_{k=1}^{n} \frac{(2n)(2n-1)...(2n-2k+1)}{k!}(at)^k x^{2n-2k},\\
&u = x^{2n+1} + \sum_{k=1}^{n} \frac{(2n+1)(2n)...(2n-2k+2)}{k!}(at)^k x^{2n-2k+1},\\
&u = e^{-\mu x}cos(\mu x)cos(2a\mu^2 t) + e^{-\mu x}sin(\mu x)sin(2a\mu^2 t),
\end{aligned}$$

where n is a positive integer and μ is an arbitrary constant.

Individual terms in these solutions do not solve the heat equations. ■

Nonlinear partial differential equations. Many nonlinear partial differential equations of mathematical physics with quadratic or power-law nonlinearities can be written as

$$f_1(x)g_1(y)\Pi_1[u] + f_2(x)g_2(y)\Pi_2[u] + ... + f_m(x)g_m(y)\Pi_m[u] = 0, \quad (3.1.2)$$

where $\Pi_i[u]$ are differential forms that represent products of nonnegative integer powers of u and its partial derivatives $u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}$,etc. Such equations also admit exact solutions of the form (3.1.1) (e.g., see[3,4,5]).

These solutions will be referred to as generalized separable solutions.

Unlike linear equations, the functions $\varphi_i(x)$ with different subscripts i in nonlinear equations are often related to one another (and possibly to $\psi_j(y)$).

In general, the functions $\varphi_i(x)$ and $\psi_j(y)$ are not known in advance and are subject to determination in a subsequent analysis.

3.2 Functional Differential Equations Arising in Generalized Separation of Variables

In general, after substituting solution (3.1.1) into the differential equation (3.1.2), one arrives at the following functional differential equation for determining $\varphi_i(x)$ and $\psi_j(y)$:

$$\Phi_1[X]\Psi_1[Y] + \Phi_2[X]\Psi_2[Y] + ... + \Phi_k[X]\Psi_k[Y] = 0, \quad (3.2.1)$$

where $\Phi_j[X]$ and $\Psi_j[Y]$ are functionals that depend on x and y, respectively:

$$\begin{aligned} \Phi_j[X] &\equiv \Phi_j(x, \varphi_1, \varphi_1', \varphi_1'', ..., \varphi_n, \varphi_n', \varphi_n''), \\ \Psi_j[Y] &\equiv \Psi_j(y, \psi_1, \psi_1', \psi_1'', ..., \psi_n, \psi_n', \psi_n'') \end{aligned} \quad (3.2.2)$$

For clarity, the formulas are written out for a second-order equation (3.1.2).

For higher-order equations, the right-hand sides of formulas (3.2.2) will contain higher-order derivatives of φ_i and ψ_i.

Sections 4 and 5 below will describe two fairly simple general methods for solving functional differential equations of the form (3.2.1)-(3.2.2).

Section 4 will also present a simple but less general method that reduces the amount of computation.

4 Simplified Method for Constructing Generalized Separable Solutions

4.1 Examples of Constructing Exact Solutions to Nonlinear Equations in Two Independent Variables

we will give several specific examples demonstrating the application of the simplified method to constructing generalized separable solutions of nonlinear second- and third-order partial differential equations.

▪ **Example 4.1.1** Consider the Guderley equation

$$u_{xx} = au_y u_{yy}, \tag{4.1.1}$$

which is employed to describe transonic gas flows, where
$\gamma = a - 1$ is the adiabatic index.
$1\circ$. First group of solutions.
We note right away that equation (4.1.1) has the obvious degenerate generalized separable solution

$$u = (C_1 x + C_2)y + C_3 x + C_4, \tag{4.1.2}$$

where $C_1, \ldots, C_4$ are arbitrary constants, which follows from the condition that $u_{xx} = u_{yy} = 0$.
We will look for generalized separable solutions

$$u(x, y) = \varphi(x)y^k + \psi(x) \tag{4.1.3}$$

other than the degenerate solution (4.1.2). The functions $\varphi(x)$ and $\psi(x)$ and constant $k \neq 0$ are to be determined.
It is noteworthy that similar two-term solutions to PDEs appear quite frequently in practice and are the simplest generalized separable solutions, along with solutions that have $e^{\lambda y}$ instead of y^k.
On substituting (4.1.3) into (4.1.1) and on rearranging, we arrive at the equation

$$\varphi''_{xx} y^k - ak^2(k-1)\varphi^2 y^{2k-3} + \psi''_{xx} = 0, \tag{4.1.4}$$

which must hold identically for any y.
Let us look at the cases $\psi''_{xx} = 0$ and $\psi''_{xx} \neq 0$.
(i) First case. For $\psi''_{xx} = 0$, we get a two-term separable equation, which is satisfied if we set

$$k = 3, \quad \varphi''_{xx} - 18a\varphi^2 = 0. \tag{4.1.5}$$

The general solution to the autonomous equation for $\varphi(x)$ can be written in the implicit form $x = \pm \int (12a\varphi^3 + C_1)^{-1} d\varphi + C_2$.
Furthermore, the equation admits a power-law particular solution, $\varphi = \frac{1}{3a}(x + C_1)^{-2}$, which generates a three-parameter exact solution to equation (4.1.1):

$$u = \frac{1}{3a}(x + C_1)^{-2} y^3 + C_2 x + C_3. \tag{4.1.6}$$

(ii) Second case. The function $\psi''_{xx} \neq 0$ can be balanced with the second term in (4.1.4) by setting $k = 3/2$.
This results in a two-term equation, which can be satisfied with

$$\varphi''_{xx} = 0, \quad \psi''_{xx} = \frac{9}{8} a\varphi^2. \tag{4.1.7}$$

These equations are easy to integrate resulting in a four-parameter exact solution:

$$u = (C_1 x + C_2) y^{\frac{3}{2}} + \frac{3a}{32C_1^2}(C_1 x + C_2)^4 + C_3 x + C4. \tag{4.1.8}$$

$2\circ$.Titov's solution (composite solution). It follows from (4.1.6) and (4.1.8) that equation ($u_t = a(uu_x)_x - bu^2$) has two similar solutions, $u_1 = \varphi_1 y^{3/2} + \psi_1$ and $u_2 = \varphi_2 y^3 + \psi_2$, whose structures differ only in the exponent of y.
This fact suggests that a more general solution to equation ($u_t = a(uu_x)_x - bu^2$) may exist that would involve both terms with the different exponents. To check out this hypothesis, we substitute the proposed combined solution

$$u(x, y) = \varphi_1(x) y^3 + \varphi_2(x) y^{3/2} + \psi(x) \tag{4.1.9}$$

into Guderley's equation (4.1.1). By collecting the coefficients of the different $y^{3n/2} (n = 0, 1, 2)$, we get

$$(\varphi_1'' - 18a\varphi_1^2) y^3 + (\varphi_2'' - \frac{45}{4} a\varphi_1\varphi_2) y^{3/2} + \psi'' - \frac{9}{8} a\varphi_2^2 = 0. \tag{4.1.10}$$

To ensure that this relation holds for any y, we have to set the coefficients of $y^{3n/2}$ to zero.
This results in the system of ODEs

$$\begin{aligned} \varphi_1'' - 18a\varphi_1^2 &= 0, \\ \varphi_2'' - \frac{45}{4}a\varphi_1\varphi_2 &= 0, \\ \psi'' - \frac{9}{8}a\varphi_2^2 &= 0, \end{aligned} \tag{4.1.11}$$

This proves that equation (4.1.1) admits solution (4.1.9), which was obtained in [5].
It can be shown that system (4.1.11) admits the exact solution

$\varphi_1 = \frac{1}{3a}(x + C_1)^{-2}$,
$\varphi_2 = C_2(x + C_1)^{5/2} + C_3(x + C_1)^{-3/2}$,
$\psi = \frac{3a}{112}C_2^2(x + C_1)^7 + \frac{3}{8}aC_2C_3(x + C_1)^3 + \frac{9}{16}aC_3^2(x + C_1)^{-1} + C_4x + C_5$.

$3\circ$. Second group of solutions. Guderley's equation also has polynomial solutions in y, which can be obtained from the following considerations. We will seek solutions as a polynomial of degree n in y with coefficients dependent on x:

$$u = P_n, \quad P_n = \sum_{i=0}^{n} \psi_i(x)y^i. \tag{4.1.12}$$

Differentiating (4.1.12) with respect to both variables and assuming that $(\psi_n)''_{xx} \neq 0$, we find that

$$u_{xx} = Q_n, \quad u_y = P'_{n-1}, \quad u_{yy} = P''_{n-2}, \tag{4.1.13}$$

where Q_n, P'_{n-1}, and P''_{n-2} are polynomials in y of degree $n, n-1$, and $n-2$, respectively.
Substituting (4.1.13) into (4.1.1), we see that the left-hand side of the resulting equation is a polynomial of degree n, while the right-hand side is a polynomial of $2n-3$ (when polynomials are multiplied, their degrees are added up).
For the polynomial solution to exist, both left- and right-hand sides must be polynomials of the same degree; it follows that $n = 3$.
Consequently, solution (4.1.12) to Guderley's equation can only be a cubic polynomial in y:

$$u = \psi_1(x) + \psi_2(x)y + \psi_3(x)y^2 + \psi_4(x)y^3. \tag{4.1.14}$$

By direct verification, it is easy to see that expression (4.1.14) is indeed a solution to equation (4.1.1).
The determining functions $\psi_i = \psi_i(x)$ $(i = 1, ..., 4)$ are described by the following system of ordinary differential equations [3]:

$$\begin{aligned}
\psi_1'' &= 2a\psi_2\psi_3,\\
\psi_2'' &= 2a(3\psi_2\psi_4 + 2a\psi_3^2),\\
\psi_3'' &= 18a\psi_3\psi_4,\\
\psi_4'' &= 18a\psi_4^2.
\end{aligned}$$

This system is easy to integrate for $\psi_4 = 0$ to obtain two simple solutions quadratic in y,

$$\begin{aligned}
u &= C_1y^2 + 2aC_1^2x^2y + \frac{1}{3}a^2C_1^3x^4, \qquad (4.1.15)\\
u &= C_1xy^2 + (\frac{1}{3}aC_1^2x^4 + C_2x + C_3)y\\
&\quad + \frac{1}{63}a^2C_1^3x^7 + \frac{1}{6}aC_1C_2x^4 + \frac{1}{3}aC_1C_3X^3 + C_4x + C_5,
\end{aligned}$$

which are special cases of solution (4.1.14); $C_1, ..., C_5$ are arbitrary constants.
A more complicated solution to (4.1.14) with $\psi_4 \neq 0$ can be found in [4].
In conclusion, we give two additive separable solutions to equation (4.1.1):

$$u = \frac{1}{2}aC_1x^2 + C_2x + C_3 \pm \frac{1}{3C_1}(2C_1y + C_4)^{3/2}$$

■

5 Solution of Functional Differential Equations by the Splitting Method

5.1 Preliminary Remarks. Description of the Method.The Splitting Principle

Although the number of terms in the functional differential equation (3.2.1) decreases when one employs the differentiation method, some redundant constants of integrations arise.
These constants must be removed in the final step.
Moreover, the order of the resulting equation can be higher than that of the original one.
To avoid these difficulties, instead of solving the functional differential equation, it is often easier to solve a standard bilinear functional equation followed by solving a system of ordinary differential equations.
This way, the original problem splits into two simpler problems.
Below we outline the main steps of this method.
$1\circ$. First, we treat equation (3.2.1) as a bilinear functional equation:

$$\sum_{n=1}^{k} \Phi_n \Psi_n = 0, \tag{5.1.1}$$

where $\Phi_n = \Phi_n[X]$ and $\Psi_n = \Psi_n[Y]$ are the unknown variables ($n = 1, ..., k$), while X and Y are independent variables.
The splitting principle. All solutions to the bilinear functional equation (5.1.1) can be represented as a number of linear combinations of $\Phi_1, ..., \Phi_k$ together with linear combinations of $\Psi_1, ..., \Psi_k$:

$$\begin{aligned} &\sum_{n=1}^{k} \alpha_{ni}\Phi_n = 0, \quad i = 1, \dots, l; \\ &\sum_{n=1}^{k} \beta_{nj}\Psi_n = 0, \quad j = 1, \dots, m; \end{aligned} \tag{5.1.2}$$

where $1 \le l \le k-1$ and $1 \le m \le k-1$.
The constants α_{ni} and β_{nj} in (5.1.2) are chosen so that the bilinear relation (5.1.1) holds identically (this can always be done as shown later in Subsection 5.2).
Importantly, relations (5.1.2) are purely algebraic in nature and are independent of any particular expressions of the differential forms (3.2.2).

2 ∘ . In the second stage, we successively replace the variables Φ_i and Ψ_j in solutions (5.1.2) with the differential forms $\Phi_i[X]$ and $\Psi_j[Y]$ from (3.2.2).
As a result, we obtain systems of ordinary differential equations, which are often overdetermined, to find the functions $\varphi_p(x)$ and $\psi_q(y)$.
By solving these systems, one obtains generalized separable solutions of the form (3.1.1).
3 ∘ . Finally, in addition to the linear relations (5.1.2), one must treat separately the degenerate cases where one or more of the differential forms Φ_n and/or Ψ_n vanish.
For clarity, **Fig(*)** displays the main steps of finding generalized separable solutions by the splitting method. Find it below:
Figure (*) The schematic of constructing generalized separable solutions by the splitting method.

$\boxed{\textbf{Original equation: } F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, ...) = 0}$

$\Downarrow$ Look for generalized separable solutions

$\boxed{\textbf{Set the form of solution: } u(x, y) = \varphi_1(x)\psi_1(y) + ... + \varphi_n(x)\psi_n(y)}$

$\Downarrow$ Substitute into the original equation

$\boxed{\textbf{Arrive at functional differential equation (3.2.1)}}$

$\Downarrow$ Treat it as the bilinear functional equation (5.1.1)

$\boxed{\textbf{Use the splitting principle}}$

$\Downarrow$ Seek solutions in the form (5.1.2)

$\boxed{\textbf{Find solutions } \Phi_i \textbf{ and } \Psi_j}$

$\Downarrow$ Replace Φ_i and Ψ_j in these solutions with $\Phi_i[X]$ and $\Psi_j[Y]$ from (3.2.2)

$\boxed{\textbf{Obtain determining systems of ordinary differential equations}}$

$\Downarrow$ Solve these systems to find $\varphi_i = \varphi_i(x)$ and $\psi_i = \psi_i(y)$

$\boxed{\textbf{Obtain exact solutions to the original equation}}$

5.2 Solutions of Bilinear Functional Equations

$1\circ$. In practice, to obtain solutions to the bilinear functional equation (5.1.1), one should proceed as follows.

First, one chooses a few first elements $\Phi_1, ..., \Phi_p$ $(p < k)$ out of the entire set $\Phi_1, ..., \Phi_k$ and represents them as linear combinations of the remaining elements $\Phi_{p+1}, ..., \Phi_k$.

This defines the first set of relations (5.1.2).

On replacing $\Phi_1, ..., \Phi_p$ in (5.1.1) with their linear combinations in terms of $\Phi_{p+1}, ..., \Phi_k$, one arrives at a relation of the form

$$\sum_{q=p+1}^{k} \Omega_q \Phi_q = 0, \quad \Omega_q = \sum_{s=1}^{k} a_{qs} \Psi_s,$$

where a_{qs} are some constants.

Setting the functional coefficients Ω_q $(q = p+1, ..., k)$ to zero gives the second set of relations (5.1.2).

Due to the symmetry of equations (5.1.1) with respect to the Φ's and Ψ's, one can start with choosing elements from the set $\Psi_1, ..., \Psi_k$ rather than $\Phi_1, ..., \Phi_k$.

5.3 Examples of Constructing Generalized Separable Solutions by the Splitting Method

Below we give a few specific examples of utilizing the splitting method to construct generalized separable solutions to nonlinear PDEs.

▪ **Example 5.3.1** Let us look at the nonlinear hyperbolic type equation

$$u_{tt} = a(uu_x)_x + f(t)u + g(t), \tag{5.3.1}$$

where $f(t)$ and $g(t)$ are arbitrary functions.
We seek generalized separable solutions of the form

$$u = \varphi(t)\theta(x) + \psi(t). \tag{5.3.2}$$

Substituting (5.3.2) into (5.3.1) and rearranging, we get

$$a\varphi^2(\theta\theta'_x)'_x + a\varphi\psi\theta''_{xx} + (f\varphi - \varphi''_{tt})\theta + f\psi + g - \psi''_{tt} = 0.$$

This equation can be represented as the four-term bilinear functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 + \Phi_4\Psi_4 = 0,$$

with

$$\begin{aligned} &\Phi_1 = (\theta\theta'_x)'_x, \quad \Phi_2 = \theta''_{xx}, \quad \Phi_3 = \theta, \quad \Phi_4 = 1, \\ &\Psi_1 = a\varphi^2, \quad \Psi_2 = a\varphi\psi, \quad \Psi_3 = f\varphi - \varphi''_{tt}, \quad \Psi_4 = f\psi + g - \psi''_{tt}. \end{aligned} \tag{5.3.3}$$

Substituting (5.3.3) into solution

$$\begin{aligned} &\Phi_1 = A_1\Phi_3 + A_2\Phi_4, \quad \Phi_2 = A_3\Phi_3 + A_4\Phi_4, \\ &\Psi_3 = -A_1\Psi_1 - A_3\Psi_2, \quad \Psi_4 = -A_2\Psi_1 - A_4\Psi_2. \end{aligned}$$

, we arrive at a system of ordinary differential equations for the unknown functions $\theta = \theta(x), \varphi = \varphi(t)$, and $\psi = \psi(t)$:

$$\begin{aligned} &(\theta\theta'_x)'_x = A_1\theta + A_2, \quad \theta''_{xx} = A_3\theta + A_4, \\ &f\varphi - \varphi''_{tt} = -A_1 a\varphi^2 - A_3 a\varphi\psi, \quad f\psi + g - \psi''_{tt} = -A_2 a\varphi^2 - A_4 a\varphi\psi. \end{aligned} \tag{5.3.4}$$

where $A_1, ..., A_4$ are arbitrary constants.
The first two equations in (5.3.4) make up an overdetermined system for the single function θ.
The second equation is linear and so is easy to integrate.
Depending on the value of A_3, its solution is expressed in terms of

trigonometric functions (for $A_3 < 0$),
hyperbolic functions (for $A_3 > 0$),
or a quadratic polynomial (for $A_3 = 0$).
Substituting one by one these solutions into the first equation, we conclude that the only compatible solution to both these equations is the quadratic polynomial

$$\theta(x) = B_2 x^2 + B_1 x + B_0, \tag{5.3.5}$$

in which the constants B_0, B_1, and B_2 are related to the constants $A_1, ..., A_4$ as follows:

$$A_1 = 6B_2, \quad A_2 = B_1^2 - 4B_0B_2, \quad A_3 = 0, \quad A_4 = 2B_2, \tag{5.3.6}$$

Inserting the expressions (5.3.6) into the last two equations in (5.3.4), we obtain the following system for $\varphi(t)$ and $\psi(t)$:

$$\begin{aligned} &\varphi''_{tt} = 6aB_2\varphi^2 + f(t)\varphi, \\ &\psi''_{tt} = [2aB_2\varphi + f(t)]\psi + a(B_1^2 - 4B_0B_2)\varphi^2 + g(t). \end{aligned} \tag{5.3.7}$$

Formulas (5.3.2) and (5.3.5) together with system (5.3.7) define a generalized separable solution to equation (5.3.1).
The first equation in (5.3.7) is solved independently; it is linear if $B_2 = 0$ and integrable by quadrature if $f(t) = const$.
The second equation in (5.3.7) is linear in ψ, once φ is known.
For $\theta \not\equiv 0$, $\varphi \not\equiv 0$, and $\psi \not\equiv 0$ as well as arbitrary f and g, equation (5.3.1) has no other solutions of the form (5.3.2).
Note that the constant a in equations (5.3.1), (5.3.4), and (5.3.7) can be replaced with $a(t)$. ■

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