* Type of Bifu. (Transforming to the Normal Form) :- (3

a. $\dot{x} = rx - \ln(1+x)$, $r \in R$. (expand near $x=0$).

By using Maclurine Exp. about $x=0$, $f'(x) = \frac{1}{1+x}$, $f'(0) = 1$

$f(x) = \ln(1+x) = \ln 1 + \frac{1}{1!}x + \frac{1}{2!}(-1)x^2 + \frac{1}{3!}x^3 + HoT$

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + HoT$

$\dot{x} = rx - [x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + HoT]$

$= rx - x + \frac{1}{2}x^2 - \cancel{\frac{1}{6}x^3} \cancel{+HoT}$

$\dot{x} = (r-1)x + \frac{1}{2}x^2$ put $r-1 = R$

$\dot{x} = Rx + \frac{1}{2}x^2$ Put $\frac{1}{2}x = y$

$x = 2y \rightsquigarrow \dot{x} = 2\dot{y}$

$\Rightarrow 2\dot{y} = 2Ry + \frac{1}{2}(4y^2)$

$\dot{y} = Ry + y^2$ (Transcritical Bif.).

---

b. $\dot{x} = r + x - \ln(1+x)$, $r \in R$.

using Maclurine exp. for $\ln(1+x)$ about $x=0$

we get $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + HoT$

$\dot{x} = r + x - [x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + HoT]$

$= r + \cancel{x} - \cancel{x} + \frac{1}{2}x^2 - \frac{1}{6}x^3 + HoT$

$= r + \frac{1}{2}x^2 - \cancel{\frac{1}{6}x^3} \cancel{+ HoT}$

$\dot{x} = r + \frac{1}{2}x^2$ Put $\frac{1}{2}x = y$ $\rightsquigarrow$ $x = 2y$

$\dot{x} = 2\dot{y}$

$\therefore 2\dot{y} = r + \frac{1}{2}(4y^2)$

$\dot{y} = \frac{1}{2}r + y^2$ Put $\frac{1}{2}r = R$

$\therefore \dot{y} = R + y^2$ (Saddle Node Bif.).

$\dot{x} = rx - \sin x \quad , \quad r \in R.$ (4*)

By using Maclurine Exp. for $\sin x$ about $x=0$.

$f(x) = \sin x$, $\quad f'(x) = \cos x$, $\quad f'(0) = 1$

$f''(x) = -\sin x$

$\sin x = \sin(0) + \frac{1}{1!}(1)x + \frac{1}{2!}(0)x^2 + \frac{1}{3!}(-1)x^3 + HOT.$

$\sin x = x - \frac{1}{3}x^3 + HOT$

$\therefore \dot{x} = rx - [x - \frac{1}{3}x^3 + HOT]$

$= rx - x + \frac{1}{3}x^3 + HOT$

$\dot{x} = (r-1)x + \frac{1}{3}x^3$ $\qquad$ Put $r-1 = R$

$\dot{x} = Rx + \frac{1}{3}x^3$ $\qquad$ Put $\frac{1}{3}x^2 = y^2$

$\sqrt{3}\dot{y} = R(\sqrt{3}y) + \frac{1}{3}(\sqrt{3}y)^3$ $\qquad$ $x^2 = 3y^2 \rightsquigarrow x = \sqrt{3}y$

$\dot{x} = \sqrt{3}\dot{y}$

$\dot{y} = Ry + y^3$

(Normal form of Pitch fork).

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d. $\dot{x} = r - \cosh x \quad , \quad r \in R$

using Maclurin Expansion about $x=0$.

$\cosh x = F(x)$, $\quad f(0) = \cosh 0 = 1$, $\quad f'(x) = \sinh x$

$f'(0) = \sinh 0 = 0$

$\therefore \cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + HOT.$

$\therefore \dot{x} = r - [1 + \frac{x^2}{2} + \frac{x^4}{4!} + HOT]$

$= r - 1 - \frac{x^2}{2} + HOT$ $\quad \rightsquigarrow \dot{x} = (r-1) - \frac{x^2}{2}$

Put $r-1 = R$ $\quad \therefore \dot{x} = R - \frac{1}{2}x^2$ $\qquad$ Put $\frac{1}{2}x = y$

$x = 2y$,

$\dot{x} = 2\dot{y}$

$\therefore 2\dot{y} = R - \frac{1}{2}(4y^2)$

$\dot{y} = \frac{1}{2}R - y^2$ $\qquad$ Put $\frac{1}{2}R = K$

we get $\dot{y} = K - y^2$ $\quad$ (Saddle-Node Bif).

C) ~~The probl~~

e $\dot{x} = \mu x - e^{x} + 1$, $\mu \in R$.

By using Maclurin Exp. for e^{x} about $x=0$

$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\therefore \dot{x} = \mu x - \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + HoT\right] + 1$

$\dot{x} = \mu x - \cancel{1} - x - \frac{x^2}{2} - \frac{x^3}{6} + HoT + \cancel{1}$

$\dot{x} = (\mu - 1)x - \frac{x^2}{2}$ put $\mu - 1 = R$

$\dot{x} = Rx - \frac{x^2}{2}$ put $\frac{1}{2}x = y$

$2\dot{y} = R(2y) - \frac{1}{2}(4y^2)$ $x = 2y \rightsquigarrow \dot{x} = 2\dot{y}$

$\dot{y} = Ry - y^2$ (Transcritical Bif).

* Type of Bif. (without transforming)

a. $\dot{y} = ky^2 + y + 1, \quad k \in \mathbb{R}.$

fixed point $F(y) = 0 \rightarrow ky^2 + y + 1 = 0$

$$y_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightsquigarrow y_{1,2} = \frac{-1 \pm \sqrt{1-4k}}{2k}$$

$$y_1^* = \frac{-1 + \sqrt{1-4k}}{2k} \qquad y_2^* = \frac{-1 - \sqrt{1-4k}}{2k}$$

1. $1 - 4k > 0$
$\exists$ 2 fixed point y_1^*, y_2^*

2. $1 - 4k = 0$
$\exists$ one fixed pt.
$y_1^* = y_2^* = -\frac{1}{2k}, \ k \neq$

3. $1 - 4k < 0$
No fixed point

(Saddle Node).

b. $\dot{y} = y^2 + y + k, \quad k \in \mathbb{R}. \rightarrow y^2 + y + k = 0$

$$y_{1,2} = \frac{-1 \pm \sqrt{1-4k}}{2} \rightsquigarrow y_1^* = \frac{-1 + \sqrt{1-4k}}{2}, \quad y_2^* = \frac{-1 - \sqrt{1-4k}}{2}$$

1. if $1 - 4k > 0$
$\exists$ 2 fixed pt y_1^*, y_2^*

2. $1 - 4k = 0$
$\exists$ one fixed pt
$y_1^* = y_2^* = -\frac{1}{2}$
(Saddle - Node Bif.).

3. $1 - 4k < 0$
No fixed pt

Scanned with CamScanner

(iii) $\dot{y} = y(y-4)(y-k)$

① Fixed Point:

$f(y) = 0 \longrightarrow y(y-4)(y-k) = 0$

$\boxed{y_1^* = 0}$ $\boxed{y_2^* = 4}$ $\boxed{y_3^* = k}$

② Stability:

$f(y) = y(y-4)(y-k) = (y^2-4y)(y-k) = y^3 - (4+k)y^2 + 4ky$

$f'(y) = 3y^2 - 2(4+k)\,y + 4k = \boxed{3y^2 - (8+2k)y + 4k}$

$f'(y_1^*) = f'(0) = 4k$	$f'(y_2^*) = f'(4) = 16 - 4k$	$f'(y_3^*) = f'(k) = k^2 - 4k$
ⓐ $f'(0) > 0 \rightarrow$ unstable	ⓐ $f'(4) > 0 \rightarrow$ unstable	ⓐ $f'(k) > 0 \rightarrow$ unstable
ⓑ $f'(0) = 0 \rightarrow$ Can't Judge	ⓑ $f'(4) = 0 \rightarrow$ Can't Judge	ⓑ $f'(k) = 0 \rightarrow$ Can't Judge
ⓒ $f'(0) < 0 \rightarrow$ stable.	ⓒ $f'(4) < 0 \rightarrow$ stable	ⓒ $f'(k) < 0 \rightarrow$ stable

$\rightarrow$ which represent the type of Transcritical bifurcation. #

Ex: what the type of bifurcation of the following:

$$\dot{x} = r\log(x) + x - 1 \quad ; \quad x \in \mathbb{R}$$

Sol: By using Taylor exp. at $x = 1$

$\because \log(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots\cdots$

$\therefore \dot{x} = r\left[(x-1) - \frac{(x-1)^2}{2} + \underbrace{\cdots}_{\text{neglected}}\right] + (x-1)$

$$\dot{x} = (r+1)(x-1) - \frac{r}{2}(x-1)^2$$

Res calling:

let $r+1 = K$

$y = \frac{r}{2}(x-1) \longrightarrow x - 1 = \frac{2}{r}y \rightarrow \dot{x} = \frac{2}{r}\dot{y}$

$\therefore \frac{2}{r}\dot{y} = K\left(\frac{2}{r}y\right) - \frac{r}{2}\left(\frac{2}{r}\right)^2 y^2$

$\frac{2}{r}\dot{y} = K\left(\frac{2y}{r}\right) - \frac{2}{r}y^2 \qquad * \frac{r}{2}$

$\therefore \dot{y} = Ky - y^2 \rightarrow$ which is the normal form of Transcritical bifurcation.

#

* Mathematical Biology *

* Transcritical Bifurcation:

→ Normal form: $\dot{x} = rx + x^2$, $r \in R$

→ Fixed points: we solve $f(x) = 0$

$rx + x^2 = 0 \rightarrow x(r + x) = 0$

$x_1^* = 0$, $x_2^* = -r$.

→ Stability of fixed points: $\{ f'(x) = r + 2x \}$

* at $x_1^* = 0 \Rightarrow f'(x_1^*) = r$

If $r > 0$
$\therefore f'(0) > 0$
$\therefore x_1^*$ is unstable.

If $r = 0$
$\therefore f'(0) = 0$
we can't judge stability

If $r < 0$
$\therefore f'(0) < 0$
$\therefore x_1^*$ is stable.

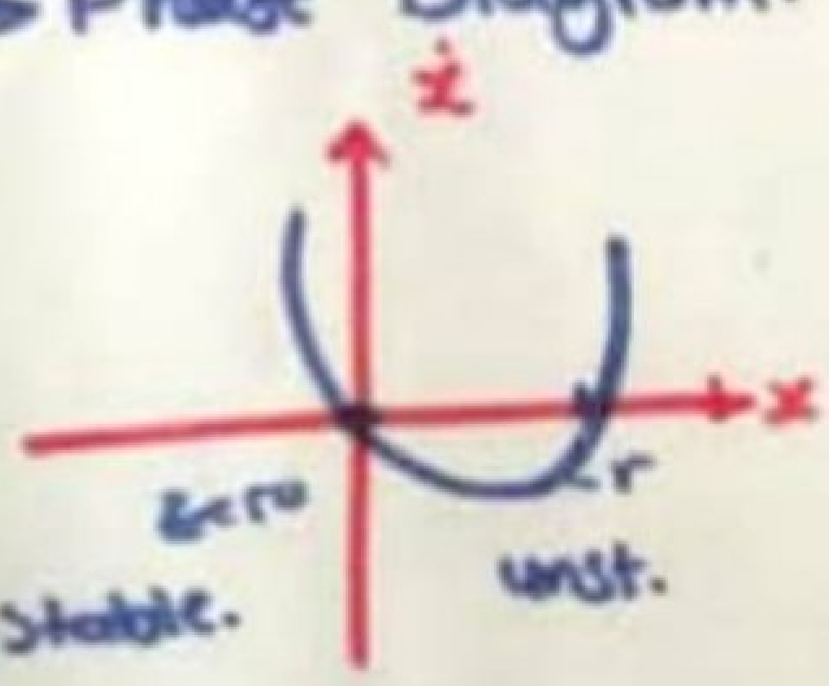
* at $x_2^* = -r \Rightarrow f'(x_2^*) = r + 2(-r) = -r$

If $r > 0$
$\therefore f'(-r) < 0$
$\therefore x_2^*$ is stable.

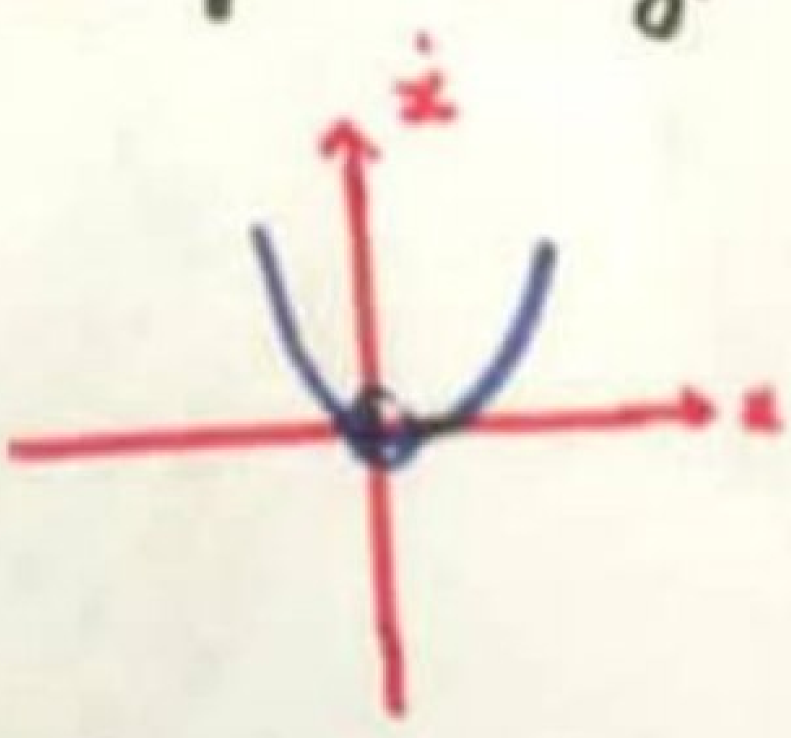
If $r = 0$
$\therefore f'(r) = 0$
$\therefore$ we can't judge stability.

If $r < 0$
$\therefore f'(r) > 0$
$\therefore x_2^*$ is unstable

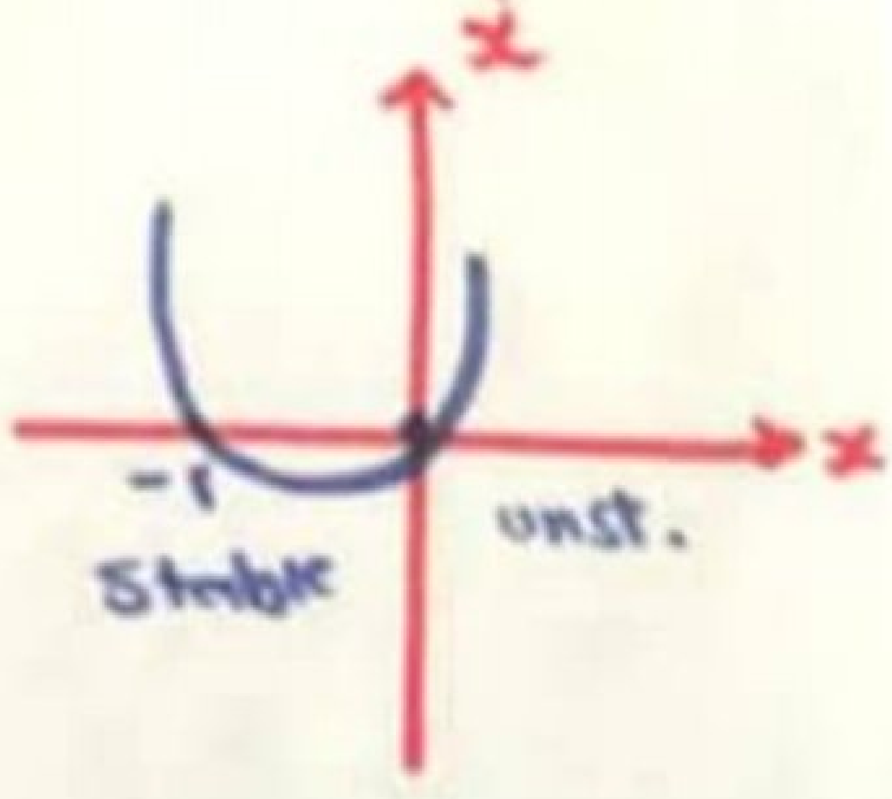
→ Phase Diagram:



$\boxed{r<0}$ Before



$\boxed{r=0}$ during Bif.



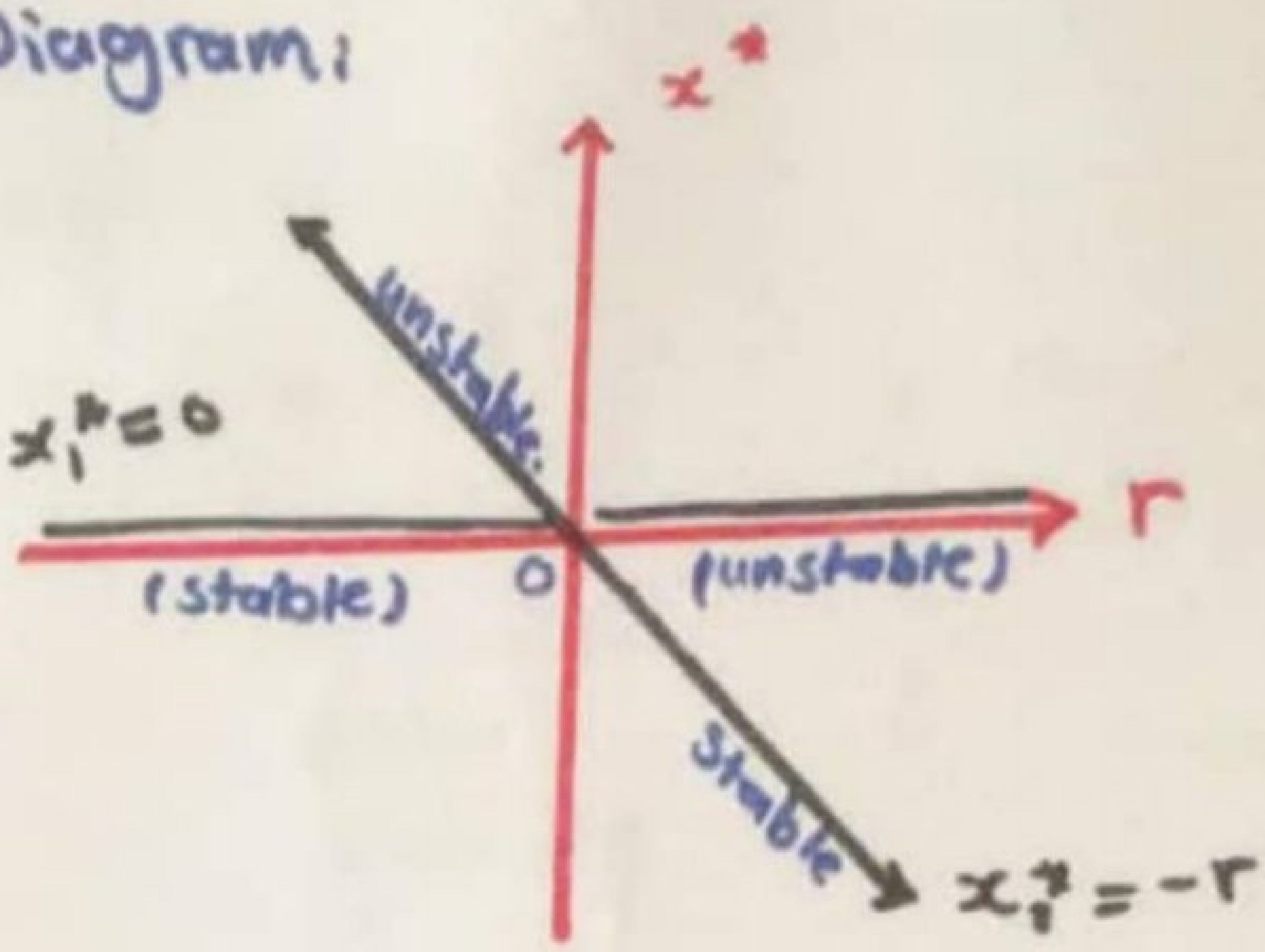
$\boxed{r>0}$ After Bif.

→ Bifurcation Diagram:

Fixed pts:

$x_1^* = 0$

$x_2^* = -r$



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* Bitch fork Bifurcation:

→ Normal form: $\dot{x} = rx + x^3$, $r \in R$.

→ Fixed Points: $f(x)=0 \rightarrow rx - x^3 = 0$

$x(r+x^2)=0 \rightarrow x_1^* = 0$, $x_2^* = \sqrt{r}$, $x_3^* = -\sqrt{r}$

exist only if $r<0$.

→ Stability of fixed points:

$f'(x) = r + 3x^2$.

* $x_1^* = 0 \rightarrow f'(0) = r$.

if $r>0$
$\therefore f'(0)>0$
$\therefore x_1^*$ unstable.

if $r=0$
$\therefore f'(0)=0$
$\therefore$ we can't judge stability

if $r<0$
$\therefore f'(0)<0$
$\therefore x_1^*$ is stable.

* at $x_2^* = \sqrt{-r}$

$f'(\sqrt{-r}) = r + 3(-r) = -2r$

* If $r>0$

$\nexists$ Fixed point

* If $r=0$

$f'(\sqrt{-r})=0$

we can't judge stability

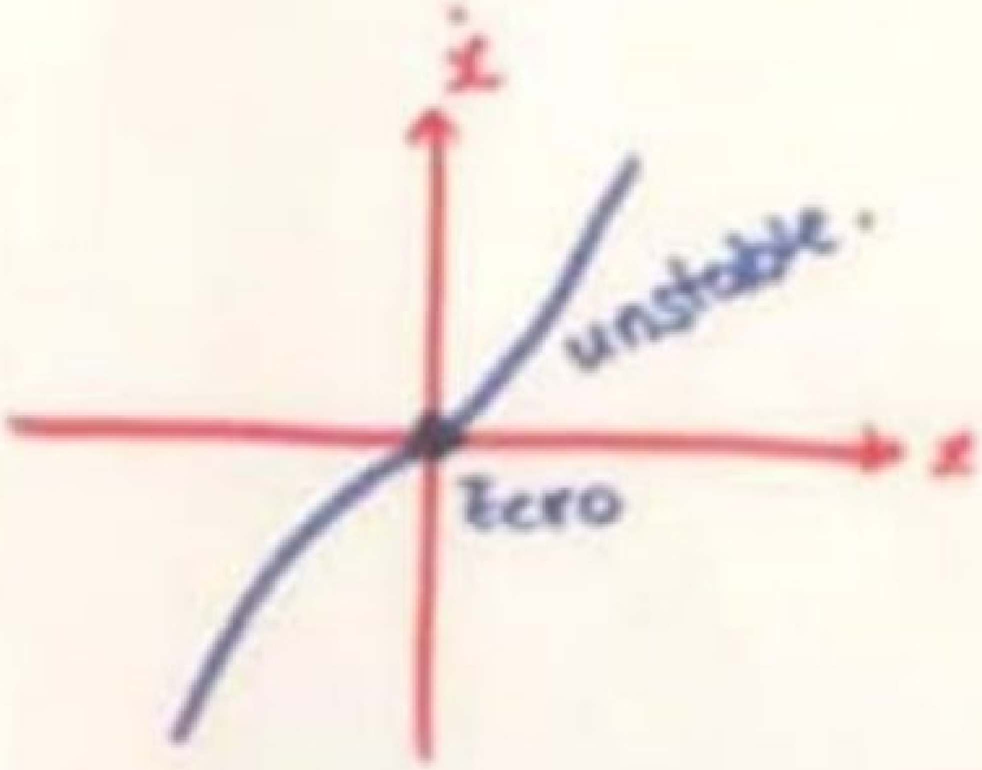
* If $r<0$

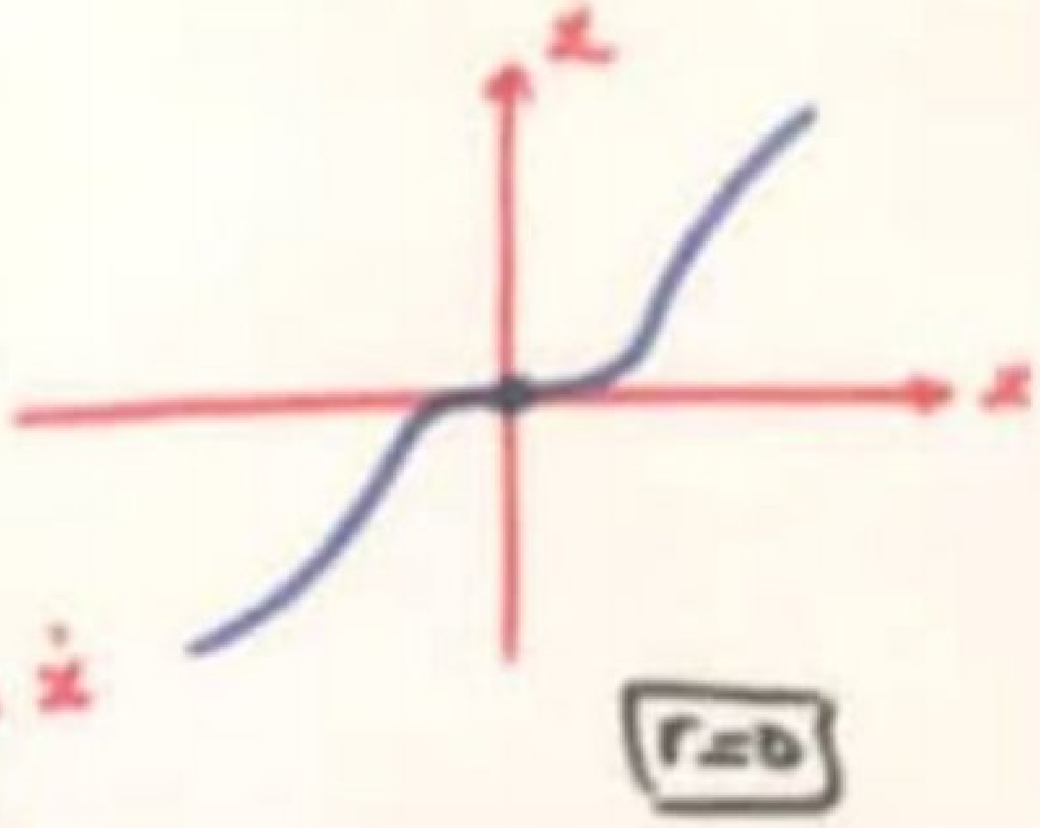
$\therefore f'(\sqrt{-r})>0$

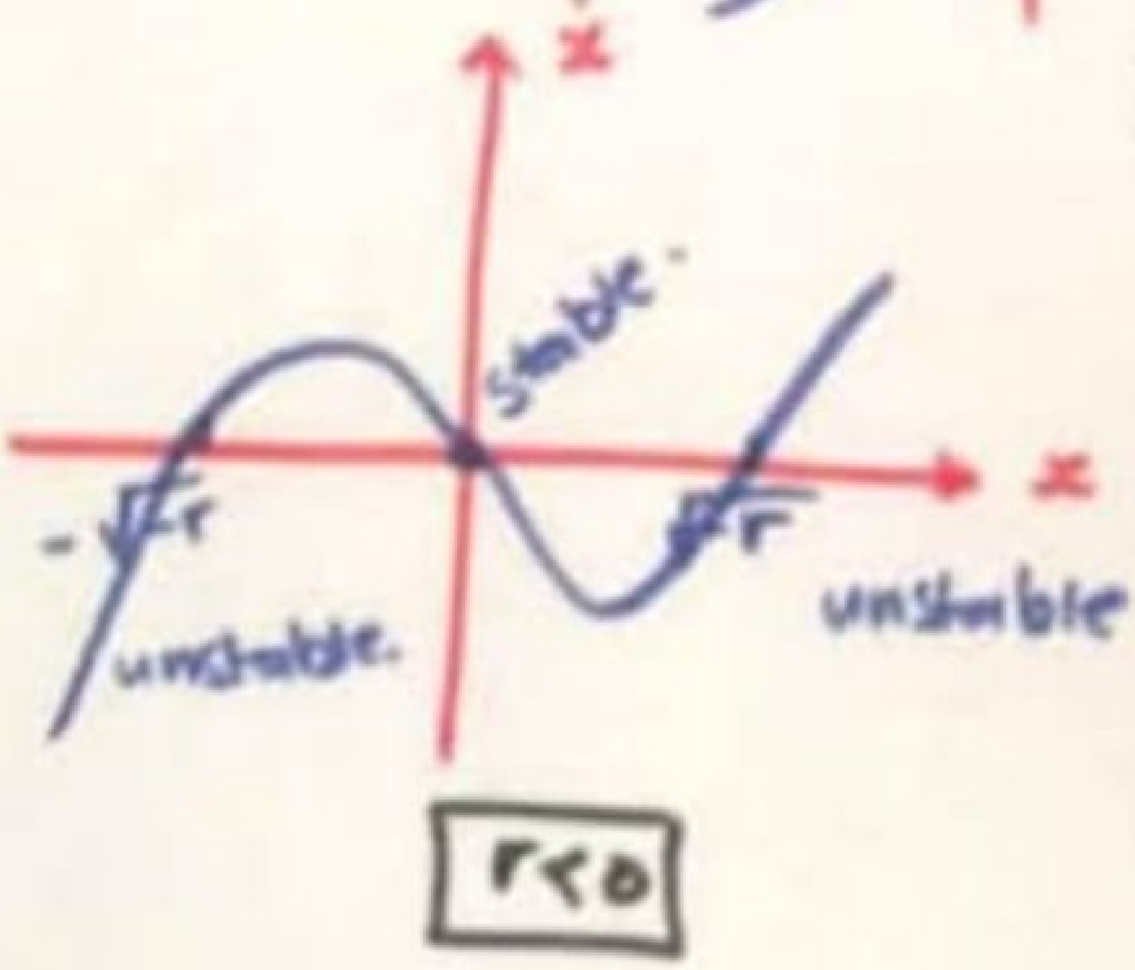
$\therefore x_2^*$ is unstable

* x_3^* By similar.

→ Phase Diagram:







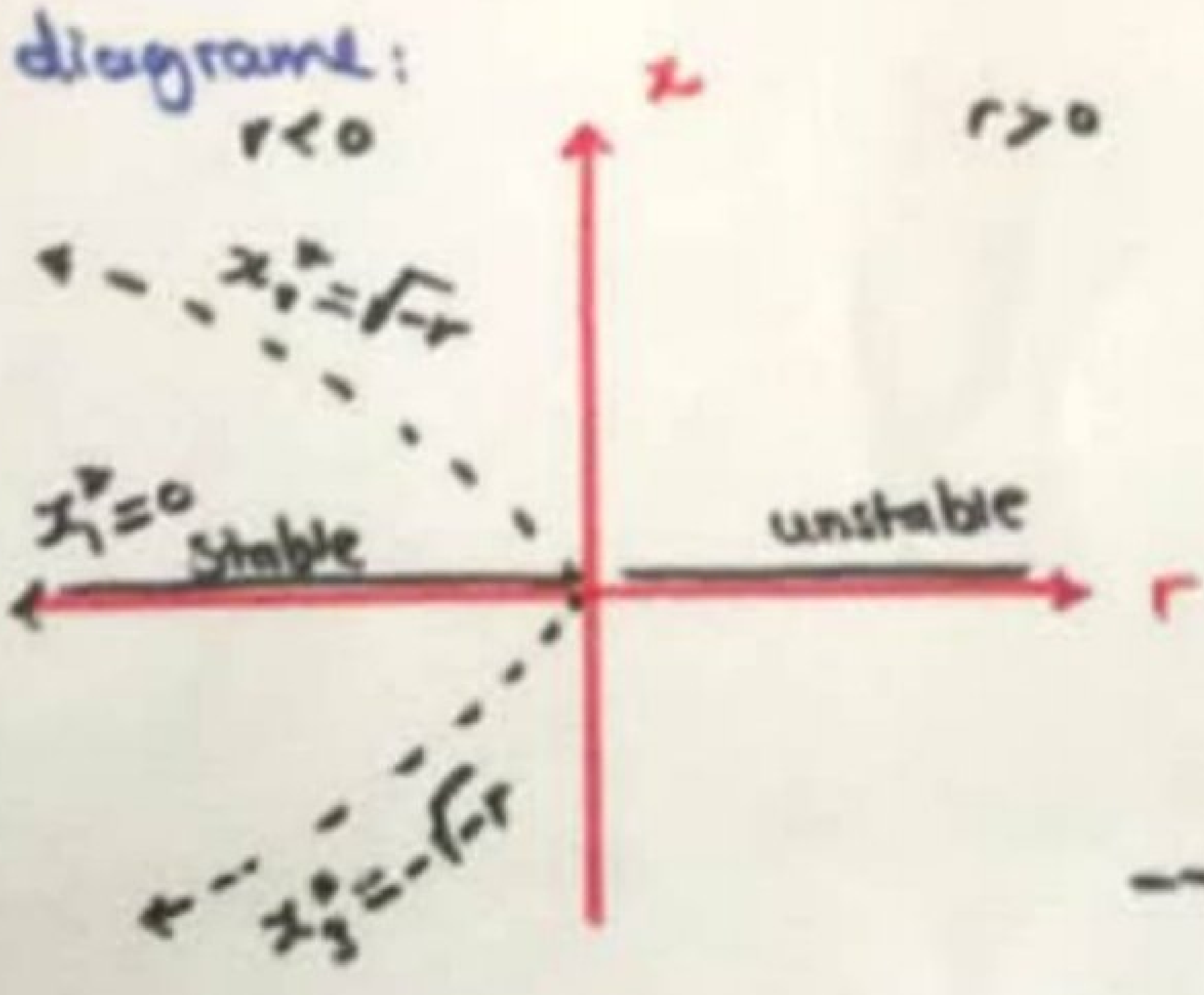
→ Bifurcation diagrame:

Fixed points:

$x_1^*=0$

$x_2^*=\sqrt{-r}$

$x_3^*=-\sqrt{-r}$



---- (unstable

* Mathematical Biology *

* Consider the following predator-prey system

$$\dot{u} = u\left(1 - \frac{u}{k}\right) - \frac{muv}{1+u}.$$

$$\dot{v} = -dv + \frac{muv}{1+u}.$$

where; u: prey pop. , v: predator pop.

k: Carrying Capacity of prey , d: death rate of preda

m: interacting parameter.

a There are 3 fixed points, find them all:

b When will the Coexisting fixed point exists:

c Discuss the stability of all fixed points:

d Give the biological meaning of the stability.

Soln. To find the fixed points; we put the R.H.S of the two eqs $=0$.

$$\therefore \quad u\left(1 - \frac{u}{k}\right) - \frac{muv}{1+u} = 0 \qquad (1)$$

$$-dv + \frac{muv}{1+u} = 0 \qquad (2)$$

$$v\left(-d + \frac{mu}{1+u}\right) = 0 \rightarrow \boxed{v=0}, \quad d = \frac{mu}{1+u} \quad \boxed{(\text{into } (1))}$$

$$\text{so } mu = d + ud \rightarrow mu - du = d \rightarrow \boxed{u = \frac{d}{m-d}}$$

So; when we use $v=0$ into eqn ① we get;

$u=0$, $u=k$ $\rightarrow$ $P_1(0,0)$, $P_2(k,0)$

Now using $u=\frac{d}{m-d}$ in ① we get,

$$u\left(1-\frac{u}{k}\right)(1+u)-muv=0$$

$$u\left[\left(1-\frac{u}{k}\right)(1+u)-mv\right]=0$$

$$u=0 \quad , \quad v=\frac{\left(1-\frac{u}{k}\right)(1+u)}{m}=\frac{\left(1-\frac{d}{k(m-d)}\right)\left(1+\frac{d}{m-d}\right)}{m}.$$

$P_1(0,0)$ extinct fixed point (No prey- No predator)

$P_2(k,0)$ existing fixed point (prey exist).

$$P_3(u^*,v^*)=\left(\frac{d}{m-d}, \frac{\left(1-\frac{d}{k(m-d)}\right)\left(1+\frac{d}{m-d}\right)}{m}\right)$$

Co-existing fixed point (Both prey and predator exist).

Now we are going to study Stability,

$$J=\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}=\begin{pmatrix} 1-\frac{2u}{k}-\frac{m[v(1+u)-uv]}{(1+u)^2} & \frac{-mu}{1+u} \\ \frac{m[v(1+u)-uv]}{(1+u)^2} & -d+\frac{mu}{1+u} \end{pmatrix}$$

* At $(0,0)$:

$$J_{(0,0)}=\begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix} \rightarrow \text{eigen value}: \lambda_1=1 , \lambda_2=-d$$

$Re(\lambda_1)>0$, $Re(\lambda_2)<0$ So we can say that

$(0,0)$ is unstable Fixed point.

$\therefore$ extinction will not happen.

* At $(K,0)$ $\longrightarrow$ $J_{(K,0)} = \begin{pmatrix} -1 & \frac{-mK}{1+K} \\ 0 & -d+\frac{mK}{1+K} \end{pmatrix}$

$\lambda_1 = -1$, $\lambda_2 = -d + \frac{mK}{1+K}$

So, we have 3 cases:

1. if $-d + \frac{mK}{1+K} > 0$ $\therefore$ $Re(\lambda_1) < 0$, $Re(\lambda_1) > 0$

   $\therefore$ unstable fixed point.

   $(u,v) \not\to (K,0)$ as $t \to \infty$

   prey will not extinct.

2. if $-d + \frac{mK}{1+K} < 0$ $\to$ $Re(\lambda_1)$ and $Re(\lambda_2) < 0$

   Stable fixed point.

   $(u,v) \longrightarrow (K,0)$ as $t \to \infty$ prey will extinct alone without predator.

3. if $-d + \frac{mK}{1+K} = 0$ $\to$ $Re(\lambda_1) = 0$

   So, we can't judge stability.

* At $P_3(u^*,v^*)$

By using trace and determinate:

$$J_{(u^*,v^*)} = \begin{pmatrix} 1-\frac{2u^*}{k}-\frac{m[v^*(1+u^*)-u^*v^*]}{(1+u^*)^2} & \frac{-mu^*}{1+u^*} \\ \frac{m[v^*(1+u^*)-u^*v^*]}{(1+u^*)^2} & \frac{-d+mu^*}{1+u^*} \end{pmatrix}$$

$\mathrm{Det}(J(u^*,v^*)) =$

$$\left[1-\frac{2u^*}{k}-\frac{m[v^*(1+u^*)-u^*v^*]}{(1+u^*)^2}\right]\left[-d+\frac{mu^*}{1+u^*}\right]+$$

$$\left[\frac{m(v^*(1+u^*)-u^*v^*)}{(1+u^*)^2}\right]\left(\frac{mu^*}{1+u^*}\right) \rightarrow ①$$

$$Tr(J(u^*,v^*)) = 1-\frac{2u^*}{k}-m\frac{[v^*(1+u^*)-u^*v^*]}{(1+u^*)^2}-d+\frac{mu^*}{1+u^*} \quad ②$$

If ② <0 and ① >0

then (u^*,v^*) will be stable fixed point

$(u,v)\rightarrow(u^*,v^*)$ as $t\rightarrow\infty$.