```
mutate(block = paste0("../", block)) %>%
filter(status=="passed")
```

```
Warning: One or more parsing issues, see `problems()` for details
```

```
Rows: 157 Columns: 8
-- Column specification --------------------------------------------------------
Delimiter: ","
chr (6): block, chapter, hash, fname, status, skill
dbl (1): number
lgl (1): comment

i Use `spec()` to retrieve the full column specification for this data.
i Specify the column types or set `show_col_types = FALSE` to quiet this message.
```

# 1 Exercises

```
format_exercises(Exer_roster %>% head(100))
```

## Exer. 2.2

**Exercise 2.2** ../Preliminaries/Exercises/function-notation.Rmd

According to the notation style we use in CalcZ, which of these things is a function? Which a number?

**Question A** What kind of a thing is $\sqrt{z}$?

i. A functionNice!

ii. A numberx$z$ is one of the standard names we will use for an input to a function. If we had given the function a name, for instance $g(z) \equiv \sqrt{z}$, we would be able to tell from the $(z)$ that $g()$ is a function. But whenever we use a standard input name, the expression is a function.

**Question B** What kind of a thing is $\sqrt{y^\star}$?

i. A numberNice!

ii. A functionxAccording to our convention, the name $y^\star$ refers to a particular value as opposed to the name of a function input.

**Question C** What kind of a thing is $e^{kt}$?

i. A functionRight!

ii. A numberxThe letter $t$ is one of our standard letters for input names. $k$ is a letter, but since it is not from the end of the alphabet, you have a hint that it is intended to be a parameter, that is a symbol representing a specific numerical quantity.

**Question D** What kind of a thing is $k$ in the definition $g(k) \equiv e^k$?

i. A numberx

ii. The name of an input to a function.Correct.

iii. A functionxNotwithstanding our convention that $k$ is not on its own a name used for inputs, when we use the full function-definition style, anything in the parentheses following the function name is explicitly stated as an input name.

## Exer. 2.3

**Exercise 2.3** ../Preliminaries/Exercises/puppy-build-radio.Rmd

The following traditional-style notation is intended to define a function that is 2 times the pattern-book sinusoid. But something is wrong.

$$g(t) \equiv 2\sin(x)$$

**Question A** What's wrong with the definition?

i. $g()$ isn't an appropriate namexYou can name functions what you like.

ii. The formula should be written $2 \times \sin(x)$xYou are welcome to write things that way if you want to emphasize the multiplication, but you don't have to. On the other hand, in R it would be absolutely required to write the multiplication operator: `2 * sin(x)`.

iii. $t$ is not a good choice for the input name.xIt's a fine choice, especially when the input is time.

iv. The input name in the formula doesn't match the input name on the left side of $\equiv$.Good. That's it. Once you choose an input name (here, $t$), make sure to write the formula in terms of that name.

# Exer. 2.4

**Exercise 2.4** ../Preliminaries/Exercises/horse-sing-drawer.Rmd

Consider this expression in math notation:

$$\frac{e^{kt}}{k}$$

**Question A** Which of the following R expressions implements the math-notation expression?

i. `k exp(kt)`xYou must always express multiplication explicitly, using `*` between the quantities to be multiplied.

ii. `e^k*t / k`xUse `exp(x)` for $e^x$.

iii. `exp(k t) / k`x`k t` should be written `k*t`

iv. `exp(k*t) / k`Excellent!

v. `1/k e^kt`x1) Use `*` for multiplication. 2) Use `exp(x)` for $e^x$.

# Exer. 2.5

**Exercise 2.5** ../Preliminaries/Exercises/pine-light-mug.Rmd

Suppose you want to define a straight-line function named $f()$ such that $f(x) \equiv mx + b$. Each of the following R statements is **incorrect** for this purpose. Say why.

**Question A** `f <- m*x + b`

i. Need to use `makeFun()` to define a function.Correct.

ii. `m` is not defined.x

iii. `b` is not defined.x

iv. Should be `y <- m*x + b`.x

**Question B** `f <- makeFun(m*x + b)`

i. The first argument to `makeFun()` should be a *tilde expression*.Right!

ii. `m` is not defined.x

iii. `b` is not defined.x

iv. `makeFun()` requires **two** inputs.xNo, but it does require a tilde expression input.

**Question C** `f <- makeFun(x ~ m*x + b)`

i. The tilde expression should have the input name on the right-hand side of the `~`.Right!

ii. `m` is not defined.x

iii. `b` is not defined.x

iv. The first argument is not a tilde expression.xActually, it is a tilde expression, just not the right sort of tilde expression.

**Question D** `f <- makeFun(mx + b ~ x)`

i. The tilde expression is missing the multiplication operator `*` between `m` and `x`Correct.

ii. `m` is not defined.x

iii. `b` is not defined.x

iv. The name `f` is mis-spelled.x

**Question E** `f <- makeFun(b*x + m ~ x)`

i. The roles of `m` and `b` have been reversed.Right!

ii. `m` is not defined.x

iii. `b` is not defined.x

iv. `x` is not defined.x

# Exer. 2.6

**Exercise 2.6** ../Preliminaries/Exercises/fish-sees-tree.Rmd

Open a SANDBOX. (Just click on that link, although you may eventually be given other ways to open a sandbox.)

### Open an R console and ….

When you see a breakout box like this, it means that we're providing some computer code that you can paste into a sandbox and run. For this exercise, that code is

```
x <- 2
sin(x)*sqrt(x)
```

Paste those two lines into the sandbox and press "Run code." Verify that you get this as a result:

```
[1] 1.285941
```

Each line that you pasted in the sandbox is a ***command***. The first command gives a value to $x$. The second command uses that value for $x$ to calculate a function output. The function is $g(x) \equiv \sin(x) \times \sqrt{x}$.

### Why did you?

Why not simplify the above code to the single line `sin(2)*sqrt(2)`? This would produce the same output but would introduce an ambiguity to the human reader. We want to make it clear to the reader (and the computer) that whatever $x$ might be, it should be used as the input to **both** the sin() and the $\sqrt{\ }$ functions.

In the following questions, numbers have been rounded to two or three significant digits. Select the answer *closest* to the computer output.

**Question A** Change $x$ to 1. What's the output of $\sin(x)\sqrt{x}$

-1.51x 0.244X 0.84♡ 0.99x 2.14X `NaN`x

**Question B** Change $x$ to 3. What's the output of $\sin(x)\sqrt{x}$

-1.51x 0.244♡ 0.84X 0.99x 2.14x `NaN`x

**Question C** Change $x$ to $-5$. What's the output of $\sin(x)\sqrt{x}$

i. -1.51x
ii. 0.244x
iii. 0.84x
iv. 0.99x
v. 2.14x
vi. `NaN`Nice! This stands for Not-a-Number, which is what you get when you calculate the square root of a negative input.

In the sandbox, change the **function** to be $\sqrt{\text{pnorm}(x)}$.

**Question D** For $x = 2$, what's the output of $\sqrt{\text{pnorm}(x)}$?

-1.51x 0.244x 0.84X 0.99♡ 2.14X `NaN`x

# Exer. 2.7

**Exercise 2.7** ../Preliminaries/Exercises/goat-look-boat.Rmd

Using the R console, translate each of the following mathematical expressions into R in order to calculate the **numerical value** of the expression.

i. $(16-3)/2$
ii. $\sqrt{\frac{19}{3}}$
iii. $\cos(\frac{2\pi}{3})$
iv. $\pi^3 + 2$
v. $\pi^{3+2}$

# Exer. 2.8

**Exercise 2.8** ../Preliminaries/Exercises/pollen-fly-lamp.Rmd

Each of these attempts to define a mathematical function using R leads to an error message. Modify the statement so that it works properly.

1. `f(x) <- makeFun(2*x + 3 ~ x)`
2. `h <- makeFun(x ~ 2*x + 3)`
3. `f <- makeFun(2x + 3)`
4. `g(x) <- makeFun(4 sin(x))`
5. `h2 <- 2*x + 3 ~ x`
6. `g2 <- makeFun(2*x + 3 ~ y)`
7. `p(x,y) <- makeFun(2 x + 3 y~ x & y)`

# Exer. 2.9

**Exercise 2.9** ../Preliminaries/Exercises/seahorse-take-pen.Rmd

Make this an exercise on "when things go wrong."

When your R command is not a complete sentence, the SANDBOX will display an error like this:

Error in parse(text = x, keep.source = TRUE) : :5:0: unexpected end of input

The "unexpected end of input" is the computer's way of saying, "You haven't finished your sentence so I don't know what to do."

Each of these R expressions is incomplete. Your job, which you should do in a sandbox, is to turn each into a complete expression. Sometimes you'll have to be creative, since when a sentence is incomplete you, like the computer, don't really know what it means to say! But each of these erroneous expressions can be fixed by **adding** or **changing** text.

Open a sandbox and copy each of the items below, one at a time, into a sandbox. Press "Run code" for that item and verify that you get an error message.

For the first item, the sandbox will look like this:

NEED TO SORT OUT IMAGE INCLUSION IN PDF ::: {.cell}

:::

Then, fix the command so you get a numerical result rather than the error message.

Working through all of these will help you develop an eye and finger-memory for R commands.

i. `sin 3`
ii. `((16 - 4) + (14 + 2) / sqrt(7)`
iii. `pnorm(3; mean=2, sd=4)`
iv. `log[7]`
v. `14(3 + 7)`
vi. `e^2`
vii. `3 + 4 x + 2 x^2`

# Exer. 4.2

**Exercise 4.2** ../Preliminaries/Exercises/squirrel-hold-bulb.Rmd

The interactive figure displays a function, but we haven't shown you any formula for the function, just the graph.

No view available in PDF format.

As you place the cursor on a point on the surface, a box will display the $(x, y, z)$ coordinates are displayed.

1. Find three points on the surface where $f(x, y) = 15$. (It doesn't have to be exactly 15, just close.)
2. Find a point where $f(x = 2, y) = 12$.
3. Explain why you can find multiple input points that generate an output of 15, but only one point where $f(x = 2, y) = 12$).

# Exer. 4.3

**Exercise 4.3** ../Preliminaries/Exercises/seahorse-hear-magnet.Rmd

TURN THIS INTO AN EXERCISE that requires drawing contour graphs and interpreting them.

A triangle consists of three connected line segments: the sides. It has other properties that are related to the sides and each other, for example, the angles between sides, the perimeter, or the area enclosed by the triangle.

Here's a description of the relationship between the perimeter $p$ and the lengths of the sides, $a$, $b$, $c$, written in the form of a function:

$$p(a, b, c) \equiv a + b + c\ .$$

The mathematical expression of the relationship between area $A$ enclosed by the triangle and the side lengths goes back at least 2000 years to Heron of Alexandria (circa 10–70). As a function, it can be written

$$A(a, b, c) \equiv \frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}\ .$$

We can't readily plot out this function because there are three inputs and one output, and it's unnatural to draw a 4-dimensional space. But we can draw a **slice** through the 4-dimensional space.

Let's do that by setting, say, $a = 1$ and looking at the area as a function of the lengths of sides $b$ and $c$.

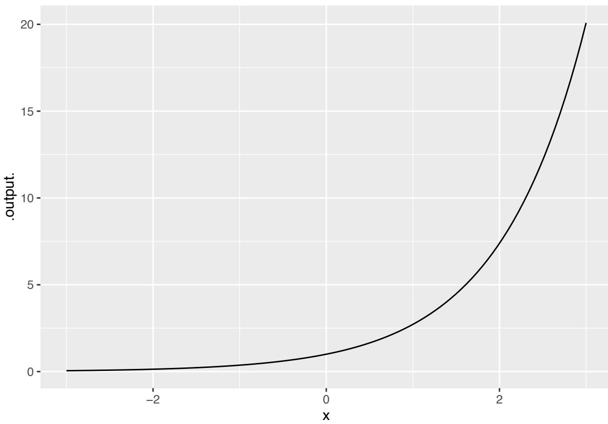
Another slice might set $a = b$, in which case we would be showing the areas of isosceles triangles.

# Exer. 5.1

**Exercise 5.1** ../Preliminaries/Exercises/pattern-book-zeros.Rmd

Copy and paste the R/mosaic command below into your R session. The command shows a simple way to graph a function, in this case `exp()`. Press "Run code" to execute the command and draw the graph.

```
slice_plot(exp(x) ~ x, domain(x=c(-3, 3)))
```




In this exercise, you'll be modifying the sandbox code to draw different functions, so you can examine their shapes.

Your task is to read and interpret the graphs of the basic modeling functions. Here, you will be looking for ***zero-crossings***: the neighborhood of a point in the function's domain where the **output** of the function is negative for inputs on one side and positive for inputs on the other side. If zero is touched but not crossed, we'll call that "touched zero."

Make a list of the pattern-book functions. For each function in the list, say whether the function crosses zero, touches zero but doesn't cross, or doesn't touch at all in the part of the domain shown in the graphic: $-3 \leq x \leq 3$. Also note if the value of the function appears to be reaching a horizontal asymptote at zero for very negative $x$, for very positive $x$, for both, or neither.

We'll show you the answers for the exponential function. You'll have to modify the computer command to graph the other pattern-book functions.

# Exer. 5.2

**Exercise 5.2** ../Preliminaries/Exercises/lobster-ride-glasses.Rmd

On a piece of paper, sketch from memory a graph of each of the nine pattern-book functions.

# Exer. 5.3

**Exercise 5.3** ../Preliminaries/Exercises/pattern-book-axis-crossing.Rmd

For each of the pattern-book functions except the reciprocal, the graph crosses either the vertical axis (that is $x = 0$) or the horizontal axis (that is, $f(x) = 0$), or both. It's helpful to know the exact quantitative value for the output where the function graph crosses the vertical axis.

To answer these questions, you will want to open a SANDBOX to try the various possibilities.

**Question A** What is the exact output of the pattern-book **exponential** function when the input is $x = 0$?

0x 0.3989423x 1/2X 1♡

**Question B** What is the exact output of the pattern-book **sine** function when the input is $x = 0$?

0♡ 0.3989423x 1/2X 1x

**Question C** What is the exact output of the pattern-book **sigmoid** function when the input is $x = 0$?

0x 0.3989423X 1/2♡ 1x

**Question D** What is the output (to several digits) of the pattern-book **hump** function when the input is $x = 0$?

0x 0.3989423♡ Right. But 0.4 will do when you're sketching a graph. 1/2x 1x

**Question E** What is the exact output of the pattern-book **constant** function?

0x 0.3989423x 1/2X 1♡

# Exer. 5.4

**Exercise 5.4** ../Preliminaries/Exercises/cat-lend-futon.Rmd

**Question A** True or false: $2^x$ is a power-law function.

i. TRUExIn a power-law function, the input is the base. In $2^x$, the input is the exponent. So it's an exponential function.

ii. FALSECorrect.

**Question B** True or false: $3/x^2$ is a power-law function.

i. TRUERight!

ii. FALSExThis is the same as $3x^{-2}$. You can see that $x$ is the base, so this is a power-law function.

**Question C** True or false: $5\sqrt{x}$ is a power-law function.

i. TRUECorrect.

ii. FALSExThis is the same as $5x^{1/2}$. The input $x$ is the base, so this is a power-law function.

**Question D** The gravitational force, F, between two bodies is inversely proportional to the square of the distance $d$ between them. Then ...

i. $F = kd^2$xInversely proportional to the square would be $d^{-2}$

ii. $F = kd^{-2}$Correct.

iii. $F = kd^{1/2}$xThis is a square-root relationship.

iv. $F = kd^{-1/2}$xThis is inversely proportional to the square root.

# Exer. 5.5

**Exercise 5.5** ../Preliminaries/Exercises/bear-lay-plant.Rmd

Some of our pattern-book functions have a distinctive property called **scale invariance**. This means the graph of the function looks the same even when plotted on very different horizontal and vertical axes. The function $\ln(x)$ plotted on two different scales in Figure @ref(fig:log-scale-invariance) shows that the graph of the function has practically the same shape.

Figure @ref(fig:square-invariance) shows a power-law function, $g(x) \equiv x^2$, which is also scale invariant.

Other pattern-book functions are not scale invariant, for example $\sin(x)$.

In contrast to scale-invariant functions, some of our pattern-book functions have a ***characteristic scale***. This is a domain length over which the whole of a characteristic feature of the function is evident. Graphing on larger domains simply squashes down the characteristic feature to a small part of the graphic domain. For instance, in the sin() function the cycle is a characteristic feature. The cycle in the pattern-book sinusoid has a characteristic length of $2\pi$, the length of the cycle. Consequently, the graph looks different depending on the length of the graphics domain in multiples of the characteristic length. You can see from Figure @ref(fig:sin-invariance) that the graph on the domain $-10 < x < 10$, that is, about 3 times the characteristic scale, looks different from the graph on the larger domain that has a length 30 times the characteristic scale.

The output of the sigmoid function runs from 0 to 1 but reaches these values only asymptotically, as $x \to \pm\infty$. In defining a characteristic scale, it would be reasonable to look at the length of the domain that takes the output from, say, 0.01 to 0.99. In other words, we want the characteristic scale to be defined in a way that captures **almost** all the action in the output of the function. For a gaussian, a reasonable definition of a characteristic scale would be the length of domain where the output falls to about, say, 1% of it's peak output.

**Question A** The gaussian (hump) function `dnorm()` has a characteristic scale. Which of these is a domain length that can encompass the characteristic shape of the gaussian?

0.1x 1x 6♡ The domain $-3 < x < 3$ supports practically everything. 16X 256x

**Question B** The sigmoid function `pnorm()` also has a characteristic scale. Which of these is a domain length that can encompass the characteristic shape of the sigmoid?

0.1x 1x 6♡ 16X 256x

Throughout science, it's common to set a standard approach to defining a characteristic scale. For instance, the characteristic scale of an aircraft could be taken as the length of body. Gaussian and sigmoids are so common throughout science that there is a convention for defining the characteristic scale called the ***standard deviation***. For the pattern book gaussian and sigmoid, the standard deviation is 1. That's much shorter than the domain that captures the bulk of action of the gaussian or sigmoid. For this reason, statisticians in practice use a characteristic scale of $\pm 2$ or $\pm 3$ standard deviations.

# Exer. 5.6

**Exercise 5.6** ../Preliminaries/Exercises/pattern-book-domain.Rmd

All but two of the pattern-book functions have a domain that runs over the whole number line: $-\infty <$

$x < \infty$.

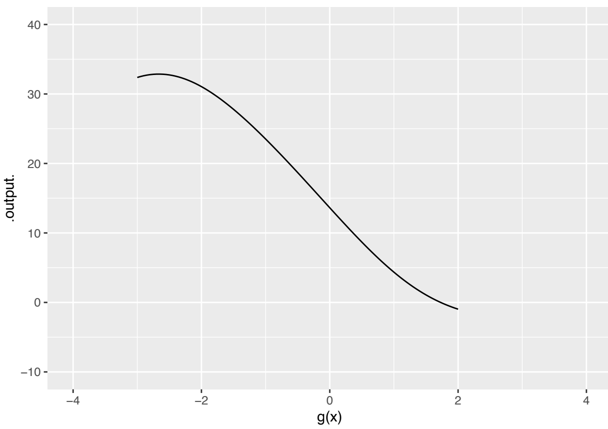
Which pattern-book function has a domain that excludes zero and negative numbers as inputs?

Which pattern-book function has just a single value missing from its domain?

# Exer. 5.7

**Exercise 5.7** ../Preliminaries/Exercises/range-domain.Rmd

Consider this graph of a function $g(x)$:




**Question A** What is the **domain** of $g(x)$?

i. $-\infty < x < \infty$x

ii. $-3 \leq x \leq 2$Correct.

iii. $-4 \leq x \leq 4$xThis might be called the "graphics" domain, yet the function graph doesn't extend over that whole interval.

iv. $-10 \leq g(x) \leq 40$xThis is the vertical extent of the **graphics frame**.

v. $-1 \leq g(x) \leq 33$xThe **domain** refers to the horizontal axis.

**Question B** What is the **range** of $g(x)$?

i. $-\infty < x < \infty$xThe **range** refers output of the function. $x$ is the input.

ii. $-3 \leq x \leq 2$xThe **range** refers output of the function. $x$ is the input.

iii. $-4 \leq y \leq 4$xYou're used to calling the function output $y$, but that's a bad habit. Break it!

iv. $-10 \leq g(x) \leq 40$xThis is the vertical extent of the **graphics frame**.

v. $-1 \leq g(x) \leq 33$Good.

# Exer. 5.8

**Exercise 5.8** ../Preliminaries/Exercises/pattern-book-range.Rmd

A function's ***domain*** is the set of possible inputs to the function. A function's ***range*** is the set of possible outputs. For each of the pattern-book functions, specify what is the range.

**Question A** What is the range of the pattern-book **exponential** function?

i. All positive outputsNice!

ii. All negative outputsx

iii. The whole number linex

iv. A closed, finite interval of possibilitiesx

**Question B** What is the range of the pattern-book **sine** function?

i. All positive outputsx

ii. All negative outputsx

iii. The whole number linex

iv. A closed, finite interval of possibilitiesExcellent! Yes. The output of pattern-book sinusoid functions is always in the interval from -1 to 1, inclusive

**Question C** What is the range of the pattern-book **logarithm** function?

i. All positive outputsx

ii. All non-negative outputsx

iii. All negative outputsx

iv. The whole number lineCorrect.

v. A closed, finite interval of possibilitiesx

**Question D** What is the range of the pattern-book **square** function?

i. All positive outputsxClose. Zero is one of the possible outputs. We can say, equivalently, that the range is all the **positive outputs plus 0** or all the **non-negative** outputs.

ii. All non-negative outputsNice!

iii. All negative outputsx

iv. The whole number linex

v. A closed, finite interval of possibilitiesx

**Question E** What is the range of the pattern-book **proportional** function?

i. All positive outputsx

ii. All negative outputsx

iii. The whole number lineCorrect.

iv. A closed, finite interval of possibilitiesxThe range extends from $-\infty$ to $\infty$.

**Question F** What is the range of the pattern-book **sigmoid** function?

i. All positive outputsx

ii. All negative outputsx

iii. The whole number linex

iv. A closed, finite interval of possibilitiesCorrect. Right. The pattern-book sigmoid function has an output that is always in the interval $0 \leq \text{pnorm}(x) \leq 1$.

# Exer. 6.1

**Exercise 6.1** ../Preliminaries/Exercises/pattern-book-descriptions.Rmd

Answer these questions about the pattern-book functions. You can refer to the graphs in Figures @ref(fig:monomial-graphs) through @ref(fig:non-integer-graphs).

**Question A** Which of these best describes the concavity of the gaussian function?

i. It's not concave.xIf it curves, it's either concave up or down.

ii. It's concave down.xIn some places, but not in others.

iii. It's concave down in the center and concave up on both flanks.Good.

iv. It's concave down on the left and concave up on the rightxLook again

**Question B** Which of these best describes the concavity of the sigmoid function?

i. It's not concave.xIf it curves, it's either concave up or down.

ii. It's concave down.xIn some places, but not in others.

iii. It's concave down on the left and concave up on the right.xLook again

iv. It's concave up on the left and concave down on the right.Nice!

**Question C** Which of these best describes the concavity of the second-order monomial $m_2(x) \equiv x^2$?

i. It's not concave.xIf it curves, it's either concave up or down.

ii. It's concave down.xIs it a smile or a frown?

iii. It's concave down on the left and concave up on the right.xLook again

iv. It's concave up everywhere.Excellent!

# Exer. 6.2

**Exercise 6.2** ../Preliminaries/Exercises/pattern-book-concave.Rmd

In this activity, you will be examining the various pattern-book functions to look for two different features:

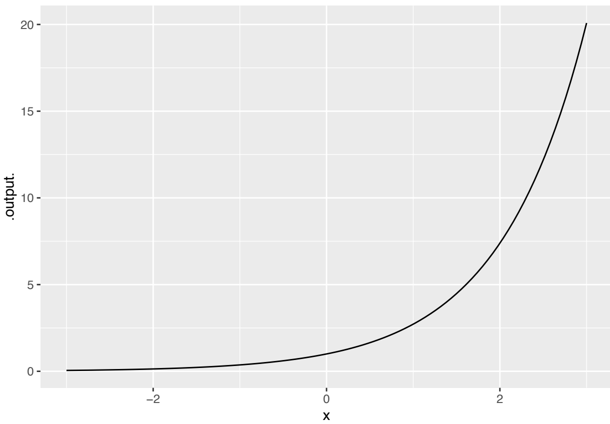
1. ***Slope***: whether the graph has a slope that is consistently positive, negative, both, or neither, and
2. ***Concavity***: whether the function being graphed is concave up, concave down, neither, or both (i.e., concave up in some regions of the domain and down for others).

---

Open an R console and ....

Copy and paste the R/mosaic command below in a SANDBOX to draw a function graph. Remember to press “Run code.” ::: {.cell}

```
slice_plot(exp(x) ~ x, domain(x=c(-3, 3)))
```




To modify the command to draw another function, replace the `exp(x)` with another formula, for instance `1/x`.

:::

Make a list of the pattern-book functions. For each function in the list, write down the R expression for the function, say whether the function has a consistently positive or negative slope, whether it is consistently concave up or down, and if the value of the function appears to be reaching a horizontal asymptote at zero for very negative $x$, for very positive $x$, for both, or neither.

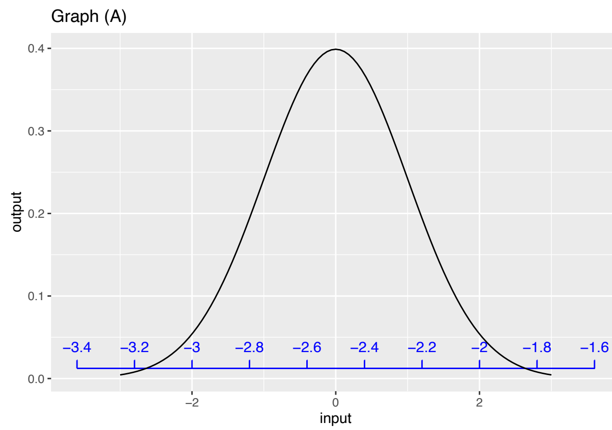
We’ll show you the answers for the exponential and sinusoid functions. You’ll have to modify the computer command to graph the other pattern-book functions.

| function name | R formula | slope | concavity | horiz. asymptote |
|---|---|---|---|---|
| exponential | `exp(x)` | positive | concave up | $x \to -\infty$ |
| logarithm | | | | |
| sinusoid | `sin(x)` | both | both | neither |
| square | | | | |
| proportional | | | | |
| constant | | | | |
| reciprocal | | | | |
| gaussian | | | | |
| sigmoid | | | | |

# Exer. 8.2

**Exercise 8.2** ../Modeling/Exercises/scale-input-1.Rmd

Each of the graphs shows two horizontal scales and one of the basic modeling functions. Which horizontal scale (black or blue) corresponds to the pattern-book function?



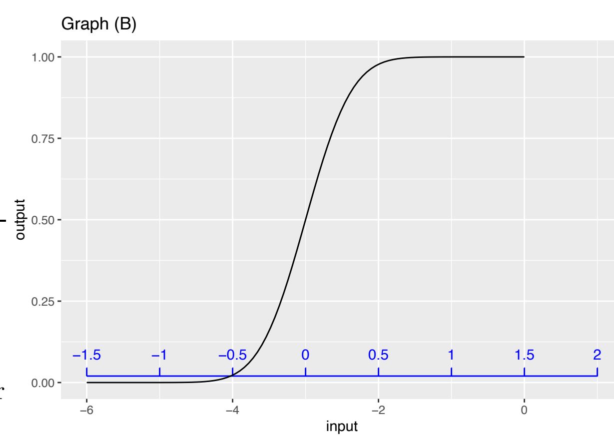

**Question A** For graph (A), which scale corresponds to the pattern-book function?

i. blackCorrect.

ii. bluex

iii. neitherx

iv. bothxIt can’t be both. There’s only one pattern-book function. When you scale the input, it becomes a “basic modeling function”.



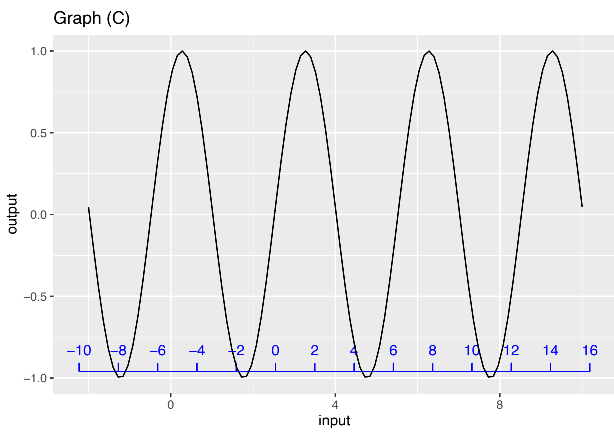

**Question B** For graph (B), which scale corresponds to the pattern-book function?

i. blackx

ii. blueRight! Right. The pattern-book function has an output of 1/2 when the output is zero. That's what the blue scale shows.

iii. neitherx

iv. bothxIt can't be both. There's only one pattern-book function. When you scale the input, it becomes a "basic modeling function".




**Question C** For graph (C), which scale corresponds to the pattern-book function?

i. blackx

ii. blueCorrect. The pattern-book sinusoid has a positive-going zero crossing at $x = 0$. That's the blue scale.

iii. neitherx

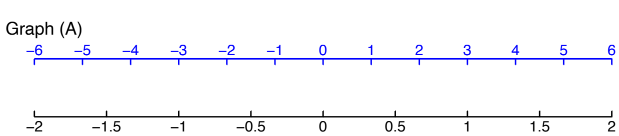
iv. bothxIt can't be both. There's only one pattern-book function. When you scale the input, it becomes a "basic modeling function".

# Exer. 8.3

**Exercise 8.3** ../Modeling/Exercises/scale-input-2.Rmd

Find the straight-line function that will give the value on the black scale for each point $x$ on the blue scale. The function will take the blue-scale reading as input and produce the black-scale reading as output, that is:

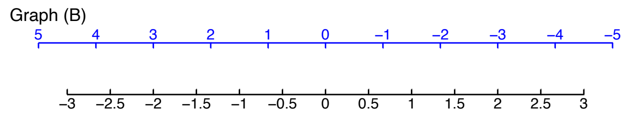
$$\text{black}(x) \equiv a(x - x_0)$$




**Question A** For Graph A, which function maps blue $x$ to the value on the black scale?

i. ${black}({blue}) $\equiv \frac{1}{3}x$$*Nice*!
${black}({blue}) $\equiv 3, x$$*xYou'regoingthewrongway, fromblack*

ii. ${black}({blue}) $\equiv x{+}3$$*xIsthereahoriztontalshift?*
${black}({blue}) $\equiv x - 3$$*xIsthereahorizontalshift?*

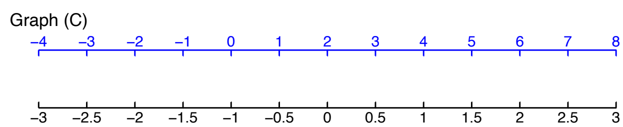



**Question B** For Graph B, which function maps blue $x$ to the value on the black scale?

iii. ${black}({blue}) $\equiv -\frac{2}{3}, x$$*Nice*!
${black}({blue}) $\equiv \frac{3}{2}x$$*xLookcarefullyatthe*$\pm$ signs on the scales.

iii. ${black}({blue}) $\equiv \frac{2}{3}x$$*xLookcarefullyatthe*$\pm$ signs on the scales.

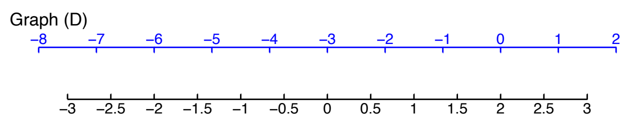
iv. ${black}({blue}) $\equiv -\frac{3}{2}x$$*xYou'regoingthewrongway, fromblacktoblu*




**Question C** For Graph C, which function maps blue $x$ to the value on the black scale?

i. ${black}({blue}) $\equiv \frac{1}{2}(x{-}2)$$*Excellent*! *Good.Anintervalofleng*
${black}({blue}) $\equiv 3, x$$*xIsthereashift*

ii. ${black}({blue}) $\equiv 2, x$$*xIsthereashift?*
${black}({blue}) $\equiv 2, (x + 2)$$*xYou'regoingthewrongway, fromblack*




**Question D** For Graph D, which function maps blue $x$ to the value on the black scale?

iii. ${black}({blue}) $\equiv \frac{2}{3}(x + 3)$$*Excellent*!
${black}({blue}) $\equiv \frac{3}{2}(x - 3)$$*x*

ii. ${black}({blue}) $\equiv \frac{3}{2}(x + 1)$$*x*
${black}({blue}) $\equiv$

iii. rac{3}{2}(x - 2)$xYou're going the wrong way, from black to blue.

# Exer. 8.1

**Exercise 8.1** ../Modeling/Exercises/fiducial-point.Rmd

Recall that each ***basic modeling function*** is constructed from the corresponding pattern-book function by scaling the input.

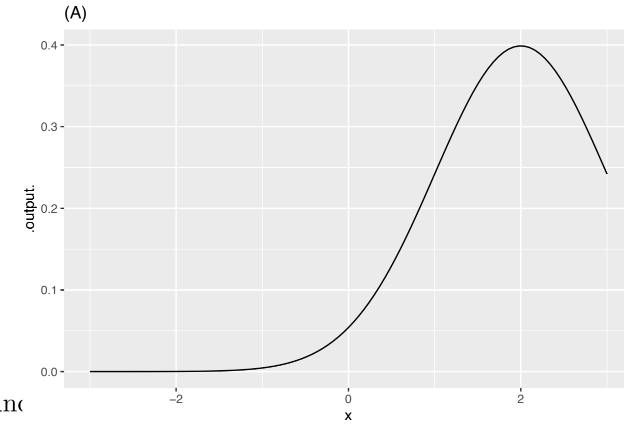
$$\text{pattern-book function} \underset{x \to a(x - x_0)}{\overset{\text{input scaling}}{\Longrightarrow}} \text{basic modeling func}$$

Figure @ref(fig:fid-points) shows the pattern-book functions with some added annotations. When the function has horizontal or vertical asymptotes, the location is shown by orange arrows. There is also a blue dot placed on the graph of functions with asymptotes. For functions without asymptotes, there are two blue dots. The location of the asymptotes and blue dots mark characteristic features of each function. The positions of the blue dot and asymptotes, or the positions of the two blue dots, are useful for figuring out the values of parameters in basic modeling functions.

For example, the basic modeling reciprocal function is $g(x) \equiv \frac{1}{m(x-x_0)} + C$. The parameter $C$ will be the value where the horizontal asymptote crosses the vertical axis. The parameter $x_0$ will be the value where the vertical asymptote crosses the horizontal axis. As for the parameter $m$: find the input where the function value is $C+1$. Let's call that $x^\star$. The $m = 1/(x^\star - x_0)$.

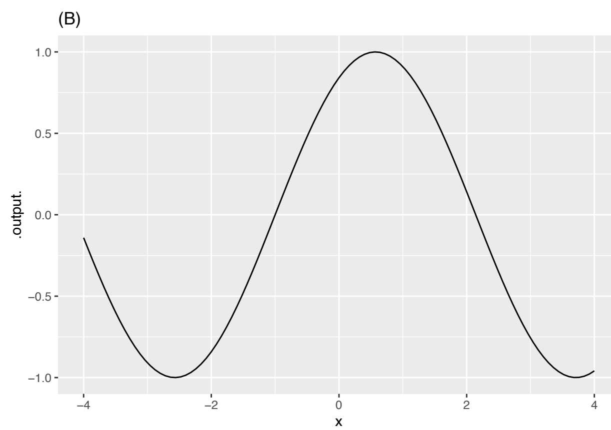
For the sinusoid, the blue dots mark the positive-going zero crossings of the baseline. The horizontal distance between the blue dots is the period parameter, $P$. The horizontal position of either of the two dots tells the phase offset $x_0$.

Each of the following plots shows a basic modeling function whose input scaling has the form $x - x_0$. Your job is to figure out from the graph what is the numerical value of $x_0$. (Hint: For simplicity, $x_0$ in the questions will always be an integer.)



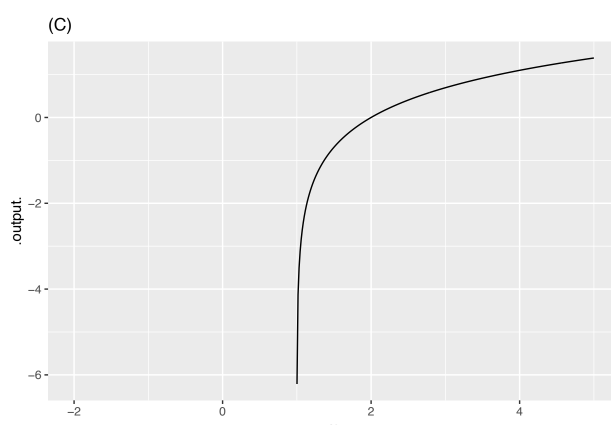

**Question A** In plot (A), what is $x_0$?

-2x  -1x  0x  1X  2♡ Right. Look for the input that generates the peak output value.



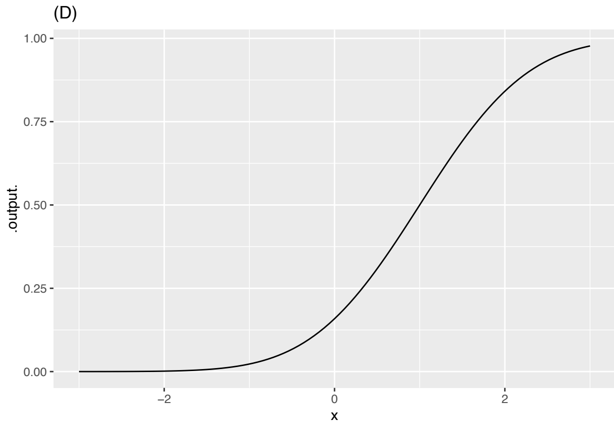

**Question B** In plot (B), what is $x_0$?

-2x  -1♡ The fiducial point is a positive-going zero crossing.  0x  1X  2x




**Question C** In plot (C), what is $x_0$?

-2x  -1x  0X  1♡ The vertical asymptote is the clue.  2x




**Question D** In plot (D), what is $x_0$?

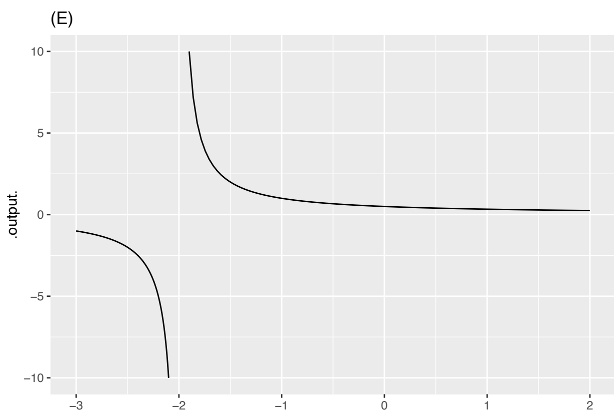
i. -2x

ii. -1x

iii. 0x

iv. 1Excellent! The input where the output is half way between the two horizontal asymptotes

v. 2x




**Question E** In plot (E), what is $x_0$?

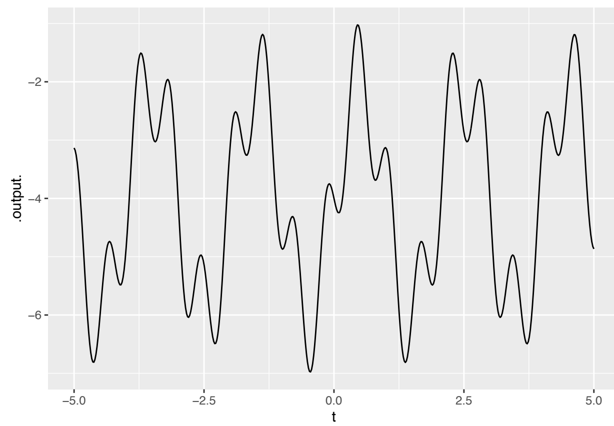
-2♡ Right. The location of the vertical asymtote is the clue. -1x 0x 1x 2x

# Exer. 8.5

**Exercise 8.5** ../Modeling/Exercises/two-sines.Rmd

The graph shows a linear combination of two sinusoids, one of period 0.6 and the other of period 2. There is also a baseline shift. That is, the graph shows the function:

$$A_1 \sin\left(\frac{2\pi}{2}t\right) + A_2 \sin\left(\frac{2\pi}{0.6}(t - .3)\right) + A_3$$




**Question A** What is $A_3$?

-4♡ -2x 0x 2X 4x

**Question B** What is $A_1$?

0x 1x 2♡ 3.5x

**Question C** What is $A_2$?

0x 1♡ 2x 3.5x

# Exer. 8.7

**Exercise 8.7** ../Modeling/Exercises/walnut-run-dish.Rmd

Turn this into a guided problem to find the input and output scaling that turns sin() into the day-length versus day-of-year.

To illustrate, suppose that $f(x)$ is one of the pattern-book function, say, sin(). The input to pattern-book sin() must always be a pure number and the output will always be a pure number. Consider a phenomenon that shows oscillatory behavior, for instance the length of daylight (in hours) as a function of the day-of-the-year (1 to 365, in days). The output of the modeling function is a quantity in hours, the input is a quantity in days. Neither this input nor the output are pure numbers.

To use the pattern sin() as a basis for modeling, we replace the input name $x$ with a straight-line function: $x(t) = at + b$. This gives us the function

$$\sin(at + b)$$

where $a$ and $b$ are parameters. If $t$ is to be the day-of-year in units of days, then the parameter $a$ will have units "per day," so that $at$ will be a pure number.

The output of the function $\sin(at + b)$ will be a pure number. In order to translate this into a quantity such as length of daylight, we apply another straight-line function, for instance

$$\text{daylight}(y) \equiv Ay + B$$

where $y$ stands for the output from $\sin(at + b)$ for any input $t$. Putting this all together, we have the function

$$\text{daylight}(t) = A\sin(at + b) + B\ ,$$

a function with four parameters: $a$, $b$, $A$, and $B$.

For example, for a location at latitude 40°N, the length of daylight is approximately

$$\text{daylight}_{40^\circ}(t) = 2.75\sin(0.0173t - 0.155) + 12\ ,$$

where $t$ is in days (January 1 is $t = 1$ and December 31 is $t = 365$), 2.75 and 12 are in hours, 0.0173 is "per day" and 0.155 is a pure number.

Keep in mind that the straight-line function is often written $\text{line}(t) = a\,(t - t_0)$. In this form, the daylight() function would be written

$$\text{daylight}_{40^\circ}(t) = 2.75\sin(0.0173\,(t - 9) + 12\ ,$$

where the 9 is in days.

# Exer. 8.9

**Exercise 8.9** ../Modeling/Exercises/boy-put-sofa.Rmd

Convert this to a guided exercise on finding the appropriate scale conversion parameters.

Figure @ref(fig:covid-scale) shows the model we fit to the COVID-19 data for the cumulative number of confirmed cases for each day in March: $\text{cases}(\text{day}) = e^{0.19(\text{day}--32)}$

The function being drawn is simply $e^x$: a function from the pattern book. The black horizontal scale shows $x$, the input to the pattern-book function. Where does that value of $x$ come from? It's $0.19(\text{day} - -32)$, where day is the number of the day in March. For instance, on March 20, day$=10$ and $0.19 * (\text{day} - -32) = 9.88$. You can see that 20 on the blue scale matches 10 on the black scale. The model says that on day 20 (blue scale) the input to the pattern-book function will be 9.88 (black scale). Plugging the input 9.88 into the pattern-book exponential gives $e^{9.88} = 19536 \approx 20,000$ cases.

The pattern-book function does not give a good model of the COVID case numbers. But if we scale the input before applying the pattern-book function, we are effectively laying a new axis, the blue one in Figure @ref(fig:covid-scale), that is stretched and shifted from the pattern-book input (blackscale). Using the blue axis lets us read off the number of cases as a function of the day in March.

Input scaling empowers the pattern-book functions to model a huge variety of phenomena. There's just one exponential function and it always looks the same. But there is a huge variety of ways to draw a blue axis, that is, to scale the input. With input scaling, the pattern-book function is tailored to become one of our basic modeling functions.

$$\underbrace{e^x}_{\text{pattern-book function}} \quad \text{versus} \quad \underbrace{e^{k(x-x_0)}}_{\text{basic modeling function}}$$

# Exer. 9.1

**Exercise 9.1** ../Modeling/Exercises/seahorse-build-pot.Rmd

Turn this into a programming activity, where they need to introduce the function `deg2rad()`.

---

R/MOSAIC COMMANDS

Composing functions is very common in computer programming. Consider these two functions

---

A computer implementation must look different, since $L$ and $\delta$ are typically provided in degrees while the `tan()` and other trigonometric functions in most computer languages expect units of radians. The conversion is easy: $\text{deg2rad}(d) \equiv \frac{\pi}{180} d$. The conversion the other way is $\text{rad2deg}(r) \equiv \frac{180}{\pi} r$.

In order to get the day-length formula to work in a computer, we can compose the tan() function with `deg2rad()`. The output of `acos()` is in radians, so we have to convert it back to degrees. Like this:

```
day_length <- makeFun(
  (2/15)*rad2deg(
    acos(
      -tan(deg2rad(L))*tan(deg2rad(d))
    )

  ) ~ L & d)
```

Now to make a plot of day length as a function of day of the year. Of course, `day_length(L, d)` does not take day of the year into account. What's missing is to know the declination of the sun as a function of calendar day.

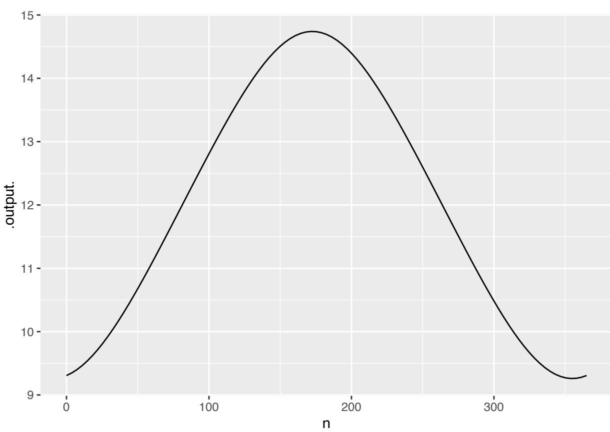
The input is a number $n$ that runs from 0 at the start of January 1st to 365 at the end of December 31. In terms of this input, the declination of the sun is known to be approximately ::: {.cell}

```
delta_sun <- makeFun(-23.44*cos((2*pi/365)*(n+10)) ~ n)
```

:::

Composing `day_length()` with `delta_sun()` (on the `d` argument only), and setting the latitude to be, say, 39°N, we get a function of day of year `n`: ::: {.cell}

```
slice_plot(
  day_length(39, delta_sun(n)) ~ n,
  domain(n=c(0,365))
  )
```




:::

## Exer. 9.2

**Exercise 9.2** ../Modeling/Exercises/dog-dive-kitchen.Rmd

Make this about identifying the functions being composed, multiplied, or linearly combined.

**All together now!**

Two or all three of the techniques for combining functions—linear combinations, function composition, and function multiplication—can be used in the same function.

Consider the function for the length of the day

$$\text{daylight}(L, \delta) \equiv \frac{2}{15} \arccos\left(-\tan(L) * \tan(\delta)\right)$$

The 2/15 is scaling the output of arccos(). The arccos() is being composed with an interior function that is itself a scaled product of two functions.

## Exer. 9.3

**Exercise 9.3** ../Modeling/Exercises/pine-bring-dish.Rmd

Make this a computer exercise about multiplying functions. Try to play the sounds they create.

This example concerns a bit of familiar technology: music synthesis. Generating a pure tone electronically is easily done using a sinusoid. Generating a note with rich instrumental timbre can be accomplished by a linear combination of sinusoids. Of course, the note will be localized in time. This could be accomplished by multiplying the sinusoids by a gaussian function envelope.

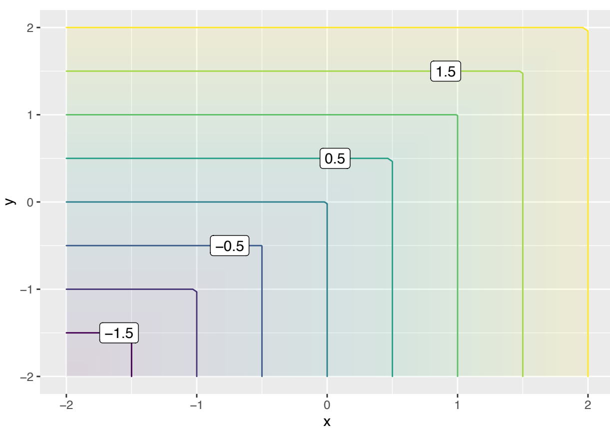
It turns out that the gaussian function, `dnorm()`, does not generate a realistic sound. Instead, a more complicated envelope is used, such as the ADSR function shown in Figure @ref(fig:ADSR). The function has six (!) parameters: the time the key is pressed, the duration A of the "attack" phase when the sound amplitude is increasing in response to the impulse imposed on the key, a decay of duration D to an output level S that lasts until the key is released, then a decay to zero over duration R. It's reasonable to think of the D and S phases as a piecewise linear approximation to exponential decay.

## Exer. 9.4

**Exercise 9.4** ../Modeling/Exercises/bigger-two.Rmd

The function `bigger()` is defined piecewise in terms of two extremely simple functions. Each of the two simple functions has a contour plot with contours that are parallel. The piecewise combination of the simple functions has a more complicated contour plot, with each simple function's parallel contours showing up in half of the domain. We'll call these "pieces" of the domain.

```
bigger <- makeFun(ifelse(y > x, y, x) ~ x + y)
contour_plot(bigger(x,y) ~ x+y, domain(x=c(-2,2), y=c(-2,2)))
```




**Question A** Which of the following best describes the two pieces of the domain?

i. One is above and to the left of the line of identity (that is, $y = x$) and the other is below and to the right of that line.Correct.

ii. One is $x > 0$ and the other $x \leq 0$x

iii. One is $x > 0$ and the other $y \leq 0$x

## Exer. 9.5

**Exercise 9.5** ../Modeling/Exercises/beech-ride-table.Rmd

The Heaviside function has two asymptotes.

1. Are they horizontal or vertical asymptotes?

2. What are their values?

## Exer. 9.6

**Exercise 9.6** ../Modeling/Exercises/shark-rise-kitchen.Rmd

TURN THIS INTO AN EXERCISE

Open an R console and see what happens when you make the plot.

```
slice_plot(log(x) ~ x, domain(x=c(-5,5)))
```

## Exer. 10.1

**Exercise 10.1** ../Modeling/Exercises/NOAA.Rmd

Many printed tables are meant to be used as functions; you plug in the input values and read off the output. Here's a table published by the National Oceanic and Atmospheric Administration for the heat index, a way of summarizing the perceived comfort (or discomfort) of summer-like weather conditions.

**Question A** What are the inputs to the heat-index function

i. temperature and relative humidityNice!

ii. temperature and wind speedxThose are the inputs to the wind-chill function, not the heat index.

iii. temperature, latitude, and longitudexThe heat index doesn't depend on location.

The table shows three different functions:

1. The heat index in ° F.
2. The heat index in ° C.
3. A caution warning level.

**Question B** For inputs of 70% relative humidity and 88° F, what are the outputs of the three functions?

i. 100° F, 38° C, and "extreme caution".Good.

ii. 100° F, 38° C, and "danger".xCheck again!

iii. 100° F, 33° C, and "extreme caution".x33C does is not the same temperature as 100F.

**Question C** Holding the relative humidity at 70%, how much would the ambient temperature have to increase (from 88° F) to change the caution-level output to "dangerous"?

i. Increase by 2° FGood.

ii. Increase by 6° FxIt looks like you're increasing the humidity to the point where the heat index is $106^circ$ F. But we asked you how much the temperature *input* has to change, not the heat-index output.

iii. Increase relative humidity to 80%.xIt's true that at 100° F and 80% humidity, the caution-index is "dangerous". But the problem specified holding humidity constant.

**Question D** From a starting point of 88° F and 70%

humidity, what is the slope of the increase in heat index when moving to 80% humidity.

i. 6° F per 10 percentage points humidityNice!

ii. 6° FxA slope is always "rise over run". You've got the rise right, but what about the run?

iii. 6° F per 80% humidity.xThe slope is the change in output divided by the change in input, i.e. "rise over run". 80% is the humidity at the endpoint, but the run is the change in humidity from the starting point to the endpoint.

**Question E** What is the heat-index output when the inputs are 52% relative humidity and 91° F? Choose the best answer.

i. 98.4° FGood. Of course, the 4 in the last digit is sketchy, but it's reasonable to calculate the interpolated output by averaging over neighboring outputs.

ii. 101° FxThat's the output at 55% humidity and 92° F.

iii. The table doesn't say.xWhile it's true that there is no table entry specifically for 52% and 91° F, you can make a very reasonable guess by *interpolation*, that is, reading between the rows and columns.

**Question F** True or false: The caution-level output could have been presented as a function of just one input, rather than needing both temperature and humidity.

i. TRUEGood. The caution-level output is not a function of ambient temperature alone or of humidity alone. But if you know the heat-index, you know that caution level exactly.

ii. FALSExNotice that the caution-level output is the same for any given level of the heat index, regardless of the ambient temperature or humidity separately.

The US National Weather Service also publishes a heat index graphic, the one below.

```
Warning in normalizePath("www/weather-service-heatindex.png"): path[1]="www/
weather-service-heatindex.png": No such file or directory
```

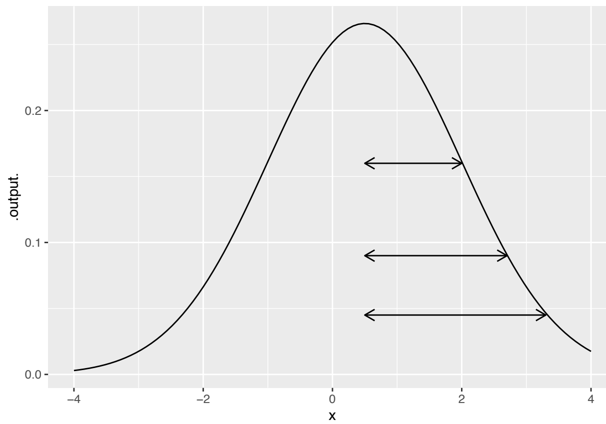
Source link

# Exer. 11.1

**Exercise 11.1** ../Modeling/Exercises/lion-chew-bowl.Rmd

The table shows eight of the pattern-book function shapes.

In the plot below, one of the double-headed arrows represents the `width` parameter. The others are misleading.




**Question A** Which arrow shows correctly the `width` parameter of the gaussian function in the graph with arrows?

top♡ middlex bottomX none of themx

**Question B** What is the value of `center` in the graph with arrows?

i. -2xThe `center` parameter is the argmax of the function.

ii. -1xThe `center` parameter is the argmax of the function.

iii. -0.5xThe `center` parameter is the argmax of the function.

iv. 0xThe `center` parameter is the argmax of the function.

v. 0.5Right!

vi. 1xThe `center` parameter is the argmax of the function.

vii. 2xThe `center` parameter is the argmax of the function.

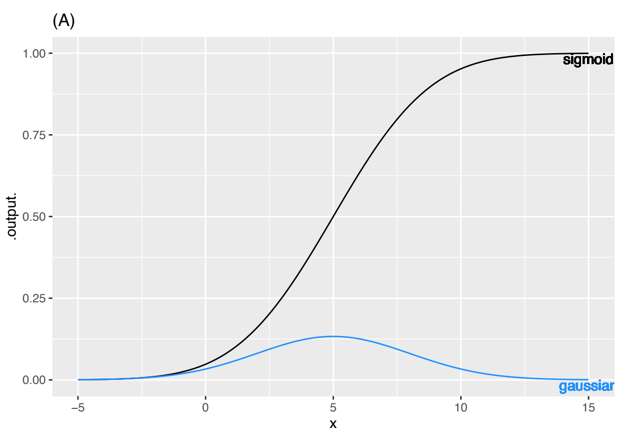
## Exer. 11.15

**Exercise 11.15** ../Modeling/Exercises/sigmoid-intro.Rmd

Gaussian functions and sigmoidal functions come in pairs. For every possible sigmoid, there is a corresponding gaussian that gives, for each value of the input, the slope of the sigmoid.

Each of the following graphs shows a sigmoid and a gaussian function. The two might or might not correspond to one another. That is, the output of the gaussian might be the slope of the sigmoid, or the gaussian might correspond to some other sigmoid. Remember, you're comparing the *output* of the gaussian to the *slope* of the sigmoid.

For each graph, say whether the gaussian and the sigmoid correspond to one another. If not, choose one of the reasons why not.

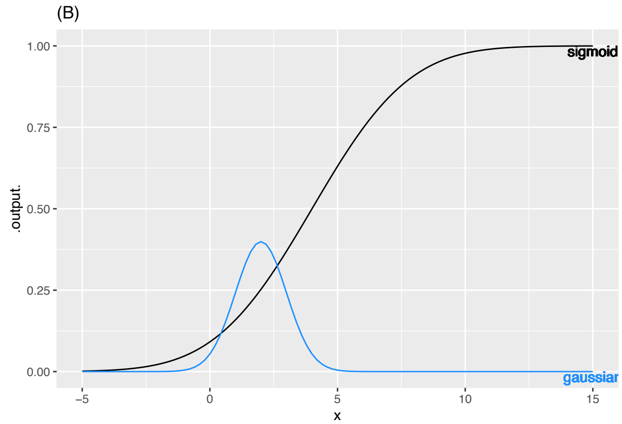



**Question A** Graph (A)

i. The gaussian and sigmoid correspond.Good.

ii. The peak of the gaussian does not occur at the same value of $x$ at which the sigmoid is steepest.xFor what $x$ is the sigmoid the steepest? For what $x$ is the gaussian the highest?

iii. The numerical value of the output of the gaussian function is, for all $x$, much larger than the numerical value of the *slope* of the sigmoid.xDid you calculate the numerical value of the slope of the sigmoid?

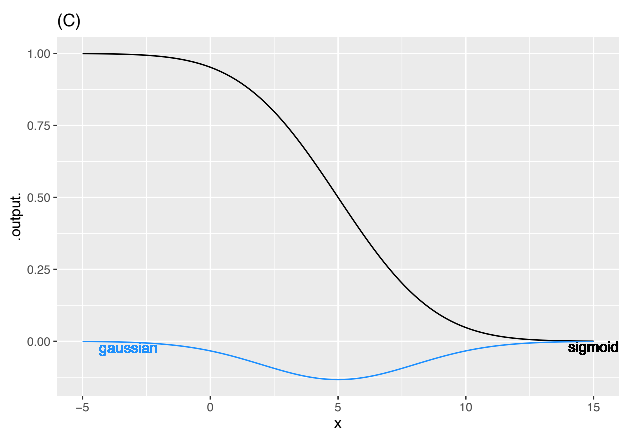



**Question B** Graph (B)

i. The gaussian and sigmoid correspond.x

ii. The peak of the gaussian does not occur at the same value of $x$ at which the sigmoid is steepest.Excellent! The gaussian peaks at about $x = 2$ while the steepest part of the sigmoid is at about $x = 4$

iii. The numerical value of the output of the gaussian function is much larger than the numerical value of the *slope* of the sigmoid.X Did you calculate the numerical value of the slope of the sigmoid?



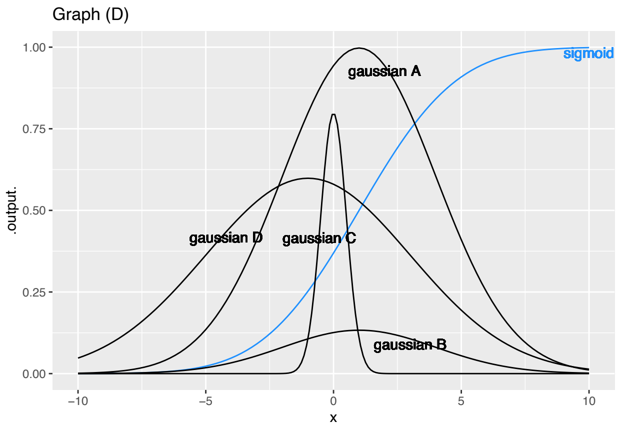

**Question C** Graph (C)

i. The gaussian and sigmoid correspond.Nice!

ii. The peak of the gaussian does not occur at the same value of $x$ at which the sigmoid is steepest.xFor what $x$ is the sigmoid the steepest? For what $x$ is the gaussian the highest?

iii. The numerical value of the output of the gaussian function is, for all $x$, much larger than the numerical value of the *slope* of the sigmoid.xDid you calculate the numerical value of the slope of the sigmoid?

In the graph $D$, there are several gaussian functions shown, only one of which corresponds to the sigmoid.




**Question D** Which gaussian corresponds to the sigmoid?

i. AxThe value of the gaussian output is much larger than the slope of the sigmoid.

ii. BRight! Right! The gaussian is centered on the steepest part of the sigmoid and falls to zero where the sigmoid levels out.

iii. CxThe gaussian is too narrow.

iv. DxThe gaussian is too broad and shifted to the left.

# Exer. 11.17

**Exercise 11.17** ../Modeling/Exercises/sigmoid-bath.Rmd

Have in mind a gaussian function and a sigmoid function that form a corresponding pair.

**Question A** Which of these stories is consistent with the relationship between a gaussian and its corresponding sigmoid?

i. The gaussian is the amount of water in a bathtub while the sigmoid is the time you spend in the bath.x

ii. The gaussian is the amount of water in the bathtub while the sigmoid is the rate at which water flows from the tap.xYou turn the tap on and off after a while. That's not what the sigmoid looks like.

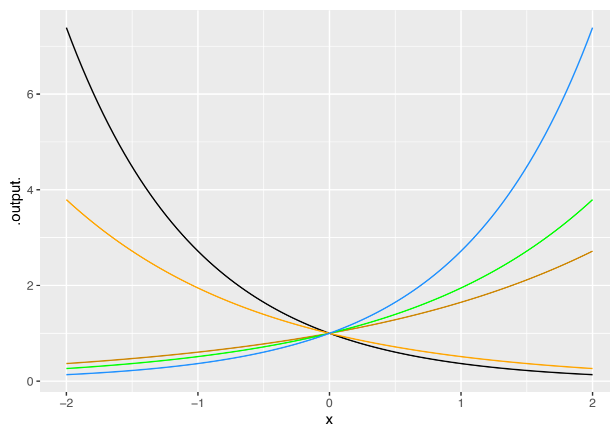
iii. The gaussian is the rate at which water flows from the tap and the sigmoid is the amount of water in the bathtub.Good.

iv. The gaussian indicates the amount the drain is open and the sigmoid is the amount of water in the bathtub.xShouldn't the amount of water go down when the drain is open?

# Exer. 11.19

**Exercise 11.19** ../Modeling/Exercises/flipping-2.Rmd

Each of the curves in the graph is an exponential function $e^{kt}$, for various values of $k$.




**Question A** What is the order from $k$ smallest (most negative) to k largest?

i. black, orange, red, green, blueCorrect. Exponential functions that grow slowly have $k$ with a small absolute value

ii. black, orange, blue, green, redxSorry. Notice the red curve grows the most slowly. This means it has the smallest $|k|$.

iii. red, green, blue, orange, blackxThe orange and black curves have negative $k$, so they will be smaller than the other curves with positive sign.

# Exer. 11.21

**Exercise 11.21** ../Modeling/Exercises/ebola-sigmoid.Rmd

*Put aside for the moment that the Ebola data plotted in Figure @ref(fig:ebola-data) doesn't look exactly like the standard sigmoid function. Follow the fitting procedure as best you can.*

**Question A** Where is the top plateau?

i. About Day 600.xMeasure the height of the plateau, not where it starts horizontally.

ii. About 14,000 casesGood.

iii. About 20,000 casesxRead the vertical axis markings more carefully.

iv. None of the abovexOne of the above answers is pretty good.

**Question B** Where is the centerline?

i. Near Day 200Right!

ii. Near Day 300xThat's the center of the vertical scale, not the day at which the curve reaches half-way to the eventual plateau.

iii. At about 7000 casesxThat's half-way up to the plateau, but the answer you want is the day at which the curve reaches that point.

**Question C** Now to find the `width` parameter. The curve looks more classically sigmoidal to the left of the centerline than to the right, so follow the curve *downward* as in Step 4 of the algorithm to find the parameters. What's a good estimate for `width`?

i. About 50 daysCorrect.

ii. About 100 daysxToo wide!

iii. About 10 daysxToo small

iv. About 2500 casesxThe width is measured along the horizontal axis, not the vertical

## Exer. 14.2

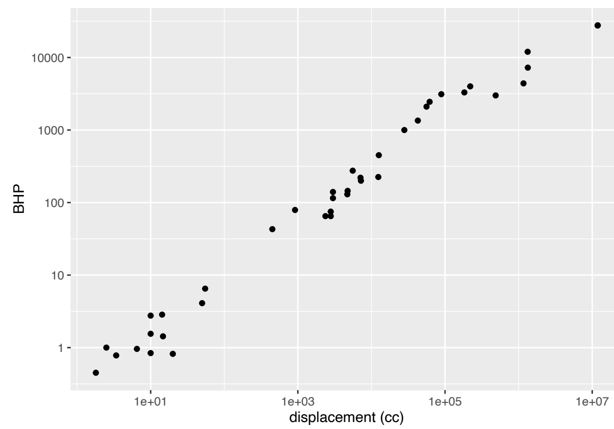
**Exercise 14.2** ../Modeling/Exercises/mag-blue.Rmd

Open a SANDBOX and make the following log-log plot of horsepower (`BHP`) versus displacement (in cc, cubic-centimeters) of the internal combustion engines listed in the `Engines` data frame.

---

Open an R console and ....

```
gf_point(BHP ~ displacement, data = Engines) %>%
  gf_refine(scale_x_log10(), scale_y_log10()) %>%
  gf_labs(x = "displacement (cc)")
```




---

In the plot, you'll see that the vertical axis has labels at 1, 10, 100, 1000, 10000. These numbers are hardly spaced evenly when plotted on a linear scale, but on the log scale they are evenly spaced. Since there is a factor of ten between consecutive labels, the interval between the labels is called a ***decade***. On the horizontal axis, the labels are at 10, 1000, 100,000, and 10,000,000. Each of those intervals spans a factor of one hundred. For instance, from 1000 is one-hundred times 10, 100,000 is one-hundred times 1000, and so on. An interval of size 100 is said to span ***two decades***, not 20 years but a factor of 100.

Based on the log-log plot, answer these questions.

**Question A** How many engines have a displacement of 1 liter or less?

i. nonexPerhaps you recognized that the left-most tick mark corresponds to a value of 1, and that no data points are 1 or smaller. But one liter corresponds to 1000 cc.

ii. 7xThis is the number of engines with displacement of 10 cc or smaller. But one liter corresponds to 1000 cc.

iii. 14Correct. Right. It's the $10^3$ tick that marks 1 liter, since 1 liter is 1000 cc.

iv. 25xThat would be true if the cut-off were 10 liters. But it's not.

**Question B** Using the log-log plot, how many decades of BHP are spanned by the data?

i. 4xNot a bad answer, but not the best one either. Notice that the smallest engine is about half a decade below 1 BPM, and the largest is about half a decade above 10,000 BPH

ii. 5Good.

iii. 100xThe number $10^{100}$ is called a *googol* and is roughly how many particles (including photons, neutrinos, etc.) are in the universe. Imagine, quite contrary to fact, that 1 BHP could be generated by burning one molecule of fuel per second. Then as many fuel molecules as there are particles in the universe would have to be burned each second to power an engine at the high end of a span of 100 decades.

# Exer. 14.4

**Exercise 14.4** ../Modeling/Exercises/maple-hit-saucer.Rmd

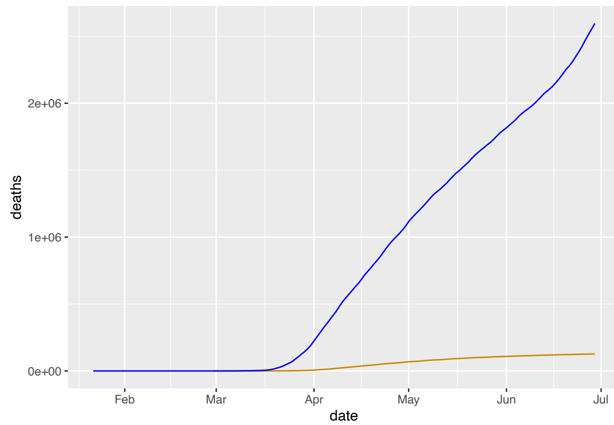
You have likely heard the phrase "exponential growth" used to describe the COVID-19 pandemic. Let's explore this idea using actual data.

The COVID-19 Data Hub is a collaborative effort of universities, government agencies, and non-governmental organizaions (NGOs) to provide up-to-date information about the pandemic. We're going to use the data about the US at the whole-country level. (There's also data at state and county levels. Documentation is available via the link above.)

Perhaps the simplest display is to show the number of cumulative cases (the `confirmed` variable) and deaths as a function of time. We'll focus on the data up to June 30, 2020.

The plot shows confirmed cases in blue and deaths in tan. ::: {.cell}

```
gf_line(deaths ~ date,
        data = Covid_US %>% filter(date < as.Date("2020-06-30")),
        color = "orange3") %>%
  gf_line(confirmed ~ date, color = "blue")
```




::: :::

**Question A** As of mid June, 2020 about how many confirmed cases were there? (Note that the labeled tick marks refer to the beginning of the month, so the point labeled `Feb` is February 1.)

i. about 50,000xThe number 1e6 means 1,000,000, that is, six zeros following the 1.

ii. about 200,000xThe number 1e6 means 1,000,000, that is, six zeros following the 1.

iii. about 500,000xThe number 1e6 means 1,000,000, that is, six zeros following the 1.

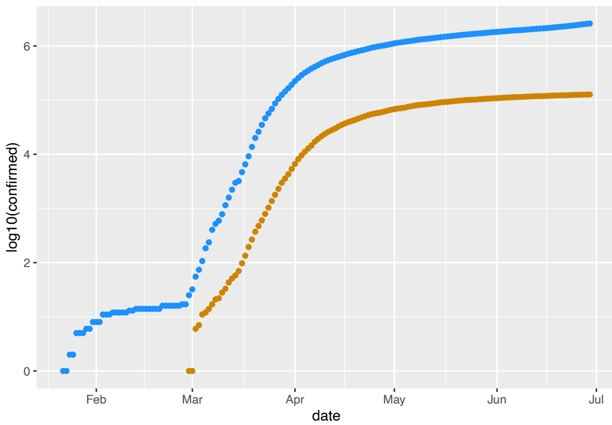
iv. about 1,000,000xMid June is the tick mark *after* the mark labelled `Jun`.

v. about 2,000,000Nice!

vi. about 5,000,000xMid June is the tick mark *after* the mark labelled `Jun`.

Here's the same graphic as above, but taking the *logarithm* (base 10) of the number of cases (that is, `confirmed`) and of the number of deaths. Since we're taking the logarithm of only the y-variable, this is called a "semi-log" plot.

```
gf_point(log10(confirmed) ~ date,
         data = Covid_US %>% filter(date < as.Date("202
         color = "dodgerblue") %>%
  gf_point(log10(deaths) ~ date, color = "orange3")
```




Up through the beginning of March in the US, it is thought that most US cases were in people travelling into the US from hot spots such as China and Italy and the UK, as opposed to contagion between people within the US. (Such contagion is called "community spread.") So let's look at the data representing community spread, from the start of March onward.

Exponential growth appears as a straight-line on a semi-log plot. Obviously, the overall pattern of the curves is not a straight line. The explanation for this is that the exponential growth rate changes over time, perhaps due to public health measures (like business closures, mask mandates, etc.)

The first (official) US death from Covid-19 was recorded was recorded on Feb. 29, 2020. Five more deaths occurred two days later, bringing the cumulative number to 6.

**Question B** The tan data points for Feb 29/March 1 show up at zero on the vertical scale for the semi-log plot. The tan data point for March 2 is at around 2 on the vertical scale. Is this consistent with the facts stated above?

i. No. The data contradict the facts.xThink about what it means to be 0 on the vertical scale.

ii. Yes. The vertical scale is in log (base 10) units, so 0 corresponds to 1 death, since $\log_{10} 1 = 0$.Right!

iii. No. The vertical scale doesn't mean anything.xYou can see from the plotting command what the quantity on the vertical axis is: `log10(confirmed)` for the blue dots and `log10(deaths)` for the tan.

One of the purposes of making a semi-log plot is to enable you to compare very large numbers with very small numbers on the same graph. For instance, in the semi-log plot, you can easily see when the first death occurred, a fact that is invisible in the plot of the raw counts (the first plot in this exercise).

Another feature of semi-log plots is that they preserve proportionality. Look at the linear plot of raw counts and note that the curve for the number of deaths is much shallower than the curve for the number of (confirmed) cases. Yet on the semi-log plot, the two curves are practically parallel.

On a semi-log plot, the arithmetic difference between the two curves tells you what the proportion is between those curves. The parallel curves mean that the proportion is practically constant. Calculate what the proportion between deaths and cases was in the month of May. Here's a mathematical hint: $\log_{10} \frac{a}{b} == \log_{10} a - \log_{10} b$. We are interested in $\frac{a}{b}$.

**Question C** What is the proportion of deaths to cases during the month of May?

i. about 1%xThis would correspond to a (vertical) difference between the curves of about 2 log10 units. Is it really that big?

ii. about 2%xThis would correspond to a (vertical) difference between the curves of about 1.7 log units. Is it really that big?

iii. about 5%Right! On the semi-log plot, the deaths curve is about 1.2 log10 units lower than the cases curve. $10^{-1.2} = 0.063 = 6.3\%$) separates the two curves.

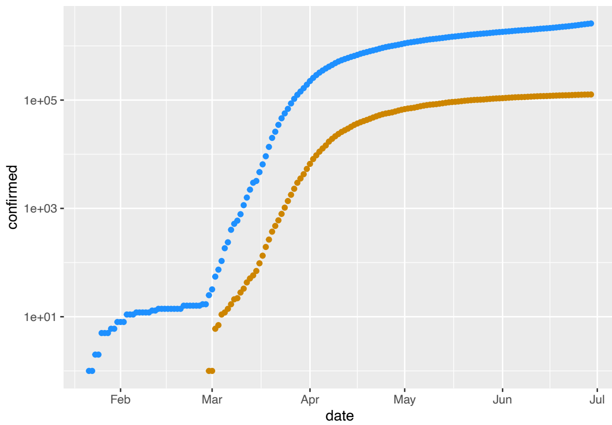
iv. about 25%xI'm not really sure what could lead you to this answer. You're making a mistake that I didn't anticipate.

v. about 75%xIt's true that in May log10(deaths) is about 5, and log10(cases) is about 6, and 5/6 is indeed roughly 75%. But, on a log scale, the proportion relates to the difference between logs, not the ratio of logs.

In many applications, people use semi-log plots to see whether a pattern is exponential or to compare very small and very large numbers. Often, people find it easier if the vertical scale is written in the original units rather than the log units. To accomplish both, the vertical scale can be ruled with raw units spaced logarithmically, like this:

```
gf_point(confirmed ~ date,
         data = Covid_US %>% filter(date < as.Date("202
```

```
         color = "dodgerblue") %>%
  gf_point(deaths ~ date, color = "orange3") %>%
  gf_refine(scale_y_log10())
```




The **labels** on the vertical axis show the raw numbers, while the **position** shows the logarithm of those numbers.

The next question has to do with the meaning of the interval between grid lines on the vertical axis. Note that on the *horizontal* axis, the spacing between adjacent grid lines is half a month.

**Question D** What is the numerical spacing (in terms of raw counts) between adjacent grid lines on the vertical axis? (Note: Two numbers are different by a "factor of 10" when one number is 10 times the other." Similarly, "a factor of 100" means that one number is 100 times the other.

i. 10 casesxIf this were true, moving up from the lowest label (`1e+01`, that is, 10) the next grid line would be at 20, then 30, then 40.

ii. 100 casesxIf this were true, moving up from the lowest label (`1e+01`) the next grid line would be at 110, then 210, then 310.

iii. A factor of 10.Excellent! Right. Every time you move up by one grid line, the raw number increases ten-fold, so 10, 100, 1000, 10,000, and so on. The phrase `a factor of 10` means to multiply by 10, not to add 10.

iv. A factor of 100.xYou're thinking along the right lines, but this is the difference between every second grid line, not adjacent grid lines.

# Exer. 14.5

**Exercise 14.5** ../Modeling/Exercises/Boyles-data.Rmd

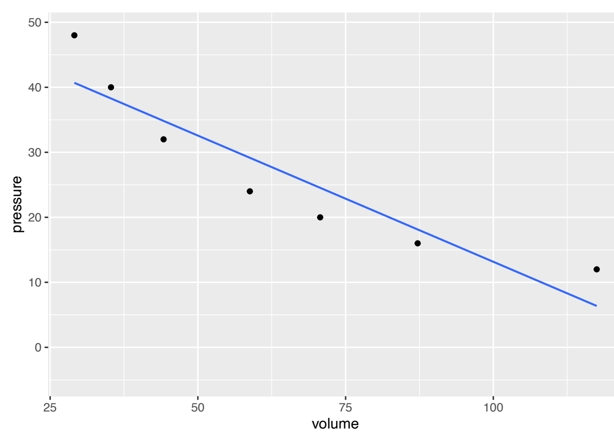
Open a sandbox to carry out some calculations with Boyle's data. To see how the data frame is organized, use the **`head(Boyle)`** and **`names(Boyle)`** commands.

---

## Open an R console and ....

The scaffolding here contains a command for plotting out Boyle's data. It also includes a command, **`gf_lm()`** that will add a graph of the best straight-line model to the plotted points. Recall that the **`#`** symbol turns what follows on the line into a *comment*, which is ignored by R. By removing the **`#`** selectively you can turn on the display of log axes.

```
gf_point(pressure ~ volume, data = Boyle) %>%
  gf_refine(
     # scale_x_log10(),
     # scale_y_log10()
  ) %>%
  gf_lm()
```




---

**Question A** In a sandbox, plot pressure versus volume using linear, semi-log, and log-log axes. Based on the plot, and the straight-line function drawn, which of these is a good model of the relationship between pressure and volume?

i. linearxThis would look like a straight line on linear axes.

ii. exponentialxThis would look like a straight line on semi-log axes.

iii. power-lawGood.

## Exer. 14.6

**Exercise 14.6** ../Modeling/Exercises/lion-jump-kitchen.Rmd

Recall Robert Boyle's data on pressure and volume of a fixed mass of gas held at constant pressure. In **?@sec-magnitude-graphics** of the text you saw a graphical analysis that enabled you to identify Boyle's Law with a power-law relationship between pressure and volume:

$$P(V) = aV^n$$

On log-log axes, a power-law relationship shows up as a straight-line graphically.

Taking logarithms translates the relationship to a straight-line function:

$$\text{lnP(lnV)} = \ln(a) + n\ \ln(V)$$

To find the parameter $n$, you can fit the model to the data. This R command will do the job:

```
fitModel(log(pressure) ~ log(a) + n*log(volume), data = Boyle) %>%
  coefficients()
```

Open a SANDBOX and run the model-fitting command. Then, interpret the parameters.

**Question A** What is the slope produced by `fitModel()` when fitting a power law model?

i. Roughly -1xYou must be a very precise person!

ii. Almost exactly -1Right!

iii. About -1.5xI'm not sure how you arrived at this answer.

iv. Slope > 0xYou should be able to see from the graph you made in part (1) that the slope is negative.

According to the appropriate model that you found in (A) and (B), interpret the function you found relating pressure and volume.

**Question B** As the volume becomes very large, what happens to the pressure?

i. Pressure becomes very small.Good.

ii. Pressure stays constantxYou can see from the graph in part (A) that pressure does change with volume.

iii. Pressure also becomes large.xYou can see from the graph in part (A) that pressure goes down as volume goes up.

iv. None of the abovex

Return to your use of `fitModel()` to find the slope of the straight-line fit to the appropriately log-transformed model. When you carried out the log transformation, you used the so-called "natural logarithm" with expressions like `log(pressure)`. Alternatively, you could have used the log base-10 or the log base-2, with expressions like `log(pressure, base = 10)` or `log(volume, base = 2)`. Whichever you use, you should use the same base for all the logarithmic transformations when finding the straight-line parameters.

**Question C (D)** Does the **slope** of the straight line found by `fitModel()` depend on which base is used?

i. NoCorrect.

ii. YesxDid you use the same base for both logarithms in your `fitModel()` expression?

iii. There's no way to tell.xYes, there is. Try using `fitModel()` with the different bases of log.

**Question D (E)** Does the **intercept** of the straight line found by `fitModel()` depend on which base is used?

i. YesCorrect. Good. But this will come out in the wash when you calculate the parameter $C$ in $Cx^b$, since $C$ will be either $2^{\text{intercept}}$ or $10^{\text{intercept}}$ or $e^{\text{intercept}}$ depending on the base log you use.

ii. NoxAre you sure you tried different bases?

## Exer. 14.7

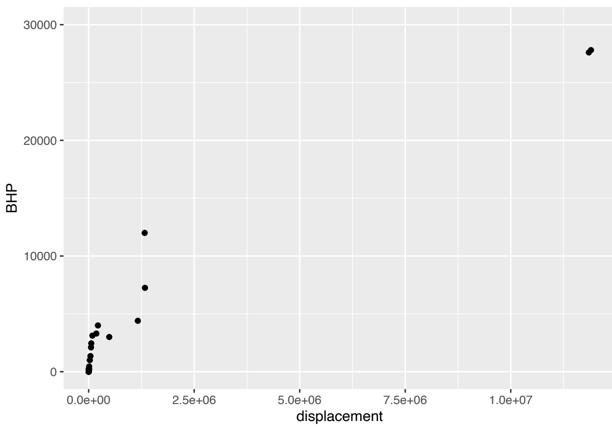
**Exercise 14.7** ../Modeling/Exercises/engine-magnitude-new.Rmd

Here is a plot of the power output (BHP) versus displacement (in cc) of 39 internal combustion engines.

---

Open an R console and ....

```
gf_point(BHP ~ displacement, data = Engines
  gf_lims(y  = c(0, 30000))
```




---

**Question A** Your study partner claims that the smallest engine in the data has a displacement of 2000 cc (that is, 2.0 liters) and 100 horsepower. Based only on the graph, is this claim plausible?

i. Yes, because 2000 cc and 100 hp would look like (0, 0) on the scale of this graph.Good.

ii. Yes, because that size engine is typical for a small car.xThat may be, but certainly you've encountered lawn mower engines that are much smaller.

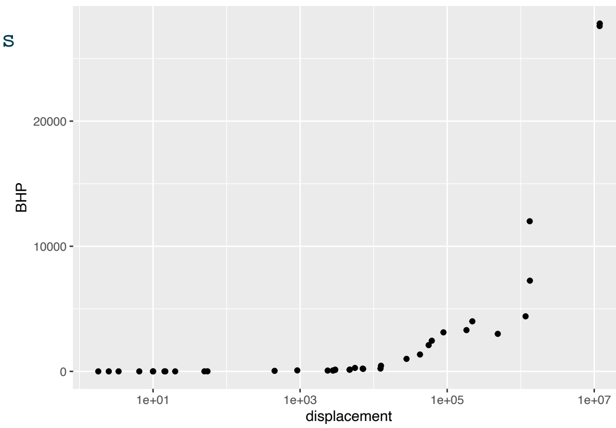
iii. No, the smallest engine is close to 0 cc.xWould you be able to distinguish visually an engine of 1 cc from an engine of 1000cc on this graph? Both these values would lie on the same horizontal pixel in the graph.

iv. No, my study partner is always wrong.xBe that as it may, we're looking for a principled answer, not an *ad hominem* one.

**Semi-log scales**

The next command will make a graph of the same engine data as before, but with a log scale on the horizontal axis. The vertical axis is still linear.

```
gf_point(BHP ~ displacement, data = Engines) %>%
  gf_refine(scale_x_log10())
```




**Question B** Using just the graph, answer this question: The engines range over how many decades of displacement? (Remember, a decade is a factor of 10.)

i. 7 decadesCorrect.

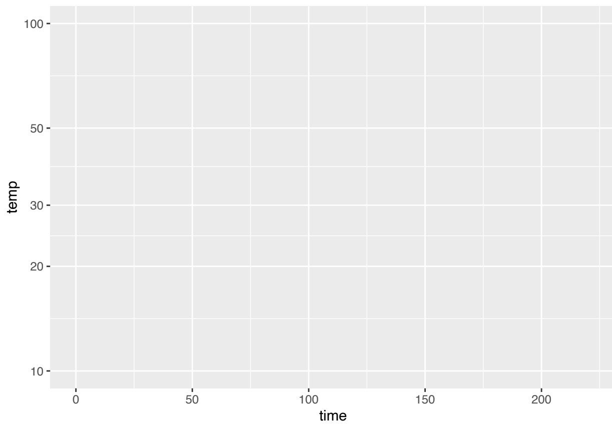
ii. Can't tellxYes, you can. Figure out what one decade corresponds to in terms of distance on the log axes.

iii. $10^7$ decadesxThe estimated volume of the entire universe is about $4 \times 10^{86}$ cc. The volume of a neutron is about $6 \times 10^{-81}$ cc. The range between a neutron and the universe is therefore about $86 - -81 = 167$ decades. Do you think it likely that there is an internal combustion engine smaller than a neutron or larger than the universe?

iv. About 3.5 decadesxPerhaps you're treating the distance between axis labels as one decade. Look carefully and you see that it's a factor of 100, that is, two decades.

# Exer. 14.8

**Exercise 14.8** ../Modeling/Exercises/fish-walk-green.Rmd




**Question A** Consider the axis scales shown above. Which kind of scale is the horizontal axis?

i. linearExcellent! You can see this because a given length along the axis corresponds to the same arithmetic difference regardless of where you are on the axis. the distance between 0 and 50 is the same as the difference between 50 and 100, or the distance between 150 and 200.

ii. logarithmicxA clue that an axis is **not** logarithmic is that there is a zero marked. The log of zero is $-\infty$, which can't appear on any actual graph. Another key is whether the scale shows doubling behavior. The distance between 50 and 100 represents one doubling: 100 is twice 50. If the scale were logarithmic, moving forward that same distance from 100 would bring you to 200. But that's not what happens here.

iii. semi-logarithmicx"Semi-logarithmic" is not about a single axis but about two axes: horizontal and vertical. It means that one axis is linear while the other is logarithmic.

iv. log-logx"Log-log" is not about a single axis but about two axes. It means that both the horizontal and vertical axes are logarithmic.

**Question B** Which kind of scale is the vertical axis?

i. linearxMeasure the distance from 30 to 50. If the scale were linear, then moving that same distance from 50 would bring you to 70, and moving that distance again would bring you to 90. But you can see that instead of reaching 90, you'd reach something greater than 100 on the scale. So the scale is not linear.

ii. logarithmicExcellent!

iii. semi-logarithmicx"Semi-logarithmic" is not about a single axis but about two axes: horizontal and vertical. It means that one axis is linear while the other is logarithmic.

iv. log-logx"Log-log" is not about a single axis but about two axes. It means that both the horizontal and vertical axes are logarithmic.

**Question C** Given your answers to the previous two questions, what kind of plot would be made in the frame being displayed at the top of this question?

i. semi-logNice!

ii. log-logxA log-log plot has log scales for both axes. The horizontal axis here is linear.

iii. linear-linearxNo, the vertical axis is logarithmic.

# Exer. 14.9

**Exercise 14.9** ../Modeling/Exercises/tiger-have-fork.Rmd

The data frame `SSA_2007` comes from the US Social Security Administration and contains mortality in the US as a function of age and sex. ("Mortality" refers to the probability of dying in the next year.)

---

Open an R console and ....

Open a sandbox and copy in the following scaffolding to see the organization of the data. ::: {.cell}

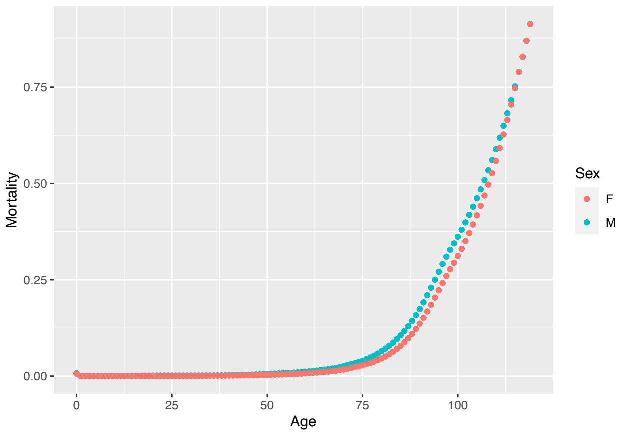
```
data(SSA_2007)
SSA_2007
```

---

:::

Once you understand the data organization, delete the old scaffolding and insert this:

```
data(SSA_2007)
gf_point(Mortality ~ Age, color = ~ Sex, data = SSA_200
```




There is a *slight* mistake in the way the command is written, so an error message will be generated. To figure out what's wrong, read the error message, check the variable names, and so on until you successfully make the plot.

**Question A** What was the mistake in the plotting command in the above code box?

i. Variable names didn't match the ones in the data.Nice!

ii. The *tilde* in the argument `color = ~ sex`xThe `color =` argument is right. The value being used, `~ sex`, is a one-sided formula and is used for things like color, shape, transparency, . . . .

iii. The data frame name is spelled wrong.xNo.

iv. There is no function `gf_point()`.X No. `gf_point()` is one of the more commonly used plotting functions

**Essay**: What's the obvious (simple) message of the above plot?

Now you are going to use semi-log and log-log scales to look at the mortality data again. To do this, you will use the `gf_refine()` function.

```
gf_point( __and_so_on__) %>%
  gf_refine(
    scale_y_log10(),
    scale_x_log10()
  )
```

Fill in the `__and_so_on__` details correctly and run the command in your sandbox.

As written, both vertical and horizontal axes will be on log scales. This may not be what you want in the end.

Arrange the plotting command to make a semi-log plot of mortality versus age. Interpret the plot to answer the following questions. Note that labels such as those along the vertical axis are often called "decade labels."

**Question B** The level of mortality in year 0 of life is how much greater than in year 1 and after?

i. About twice as large.xHint: How much is the change between successive labels on the y axis?

ii. About five times as largexHint: How much is the change between successive labels on the y axis?

iii. About 10 times as largeGood.

iv. About 100 times as largexHint: How much is the change between successive labels on the y axis?

**Question C** Near age 20, the mortality of males is how much compared to females?

i. Less than twice as large.xHint: Due to the nature of logs, a difference of half a decade corresponds to a change of $\sqrt{10}$.

ii. A bit more than three times as largeExcellent!

iii. About 8 times as largexHint: Due to the nature of logs, a difference of half a decade corresponds to a change of $\sqrt{10}$.

iv. About 12 times as largexHint: Due to the nature of logs, a difference of half a decade corresponds to a change of $\sqrt{10}$.

**Question D** Between the ages of about 40 and 80, how does mortality change with age?

i. It stays about the same.xBut the curve is sloping up, isn't it?

ii. It increases as a straight line.xIt would be fair to say this about the logarithm of mortality. But a straight line in log mortality means that mortality itself is increasing exponentially.

iii. It increases exponentially.Correct.

iv. It increases, then decreases, then increases again.xInteresting that you would say this when the function in clearly monotonically increasing above age 30.

Remake the plot of mortality vs age once again, but this time put it on log-log axes. The sign of a power-law relationship is that it shows up as a straight line on log-log axes.

**Question E** Between the ages of about 40 and 80 is the increase in mortality better modeled by an exponential or a power-law process?

i. Power-lawxBut it's hard to find any straight line on the log-log plot.

ii. ExponentialGood. Right. The graph is much closer to a straight line on semi-log scales than on log-log scales.

iii. No reason to prefer one or the other.xOne is much closer to a straight line than the other.

## Exer. 15.1

**Exercise 15.1** ../Modeling/Exercises/frog-win-fridge.Rmd

You are designing a pendulum for a planned joint NASA/ESA mission to Mars. From the orbital period and radius of Mars, its mass is known. From the mass and the observed diameter of the planet, gravitational acceleration at the surface is calculated as 3.721 m/s$^2$. According to **?@sec-pendulum-dimensions**, the period is $\text{Period} = 2\pi\sqrt{\frac{\text{Length}}{\text{Gravity}}}$.

The length of your pendulum is 3 feet.

**Question A** What will be the period of your pendulum when it eventually gets to Mars? (Hint: Don't make the mistake of the engineers working on the Mars Polar Lander and forget to resolve the different units of length presented in the problem.)

1.3 secondsx 1.9 secondsx 3.1 seconds♡ 9.1 secondsx

**Question B** What is the period of your pendulum on Earth?

1.3 secondsx 1.9 seconds♡ 3.1 secondsx 9.1 secondsx

## Exer. 15.2

**Exercise 15.2** ../Modeling/Exercises/dim-formulas.Rmd

For each mathematical operation, identify the operation as valid or invalid according to the rules of dimensional arithmetic.

**Question A** In this formula

$$\frac{8\text{m} - 2.5\text{km}}{2\text{min} - 32\text{s}}$$

choose which rule (if any) is violated.

i. Addition or Subtraction rulexBoth the numerator and denominator are valid subtractions, with dimension L and T respectively.

ii. Multiplication or Division rulexThere are no restrictions for multiplication and division, so a formula can hardly violate them!

iii. ExponentialxThere's no exponent here.

iv. It's valid. No rules are violated.Nice!

**Question B** In this formula

$$\frac{3\text{g} \times 2\text{m}}{3\text{km}^2}$$

choose which rule (if any) is violated.

i. Addition or Subtraction rulexNo addition or subtraction here.

ii. Multiplication or Division rulexThere are no restrictions for multiplication and division, so a formula can hardly violate them!

iii. ExponentialxThere's no exponent here.

iv. It's valid. No rules are violated.Excellent!

**Question C** For this formula

$$10^{\frac{4\text{hr}}{3\text{g}}}$$

choose which rule (if any) is violated.

i. Addition or Subtraction rulexNo addition in this formula.

ii. Multiplication or Division rulexThere are no restrictions for multiplication and division, so a formula can hardly violate them!

iii. ExponentialRight! The exponent is 4 ft / 3 g, which has dimension L / M. Exponents must *always* have dimension [1].

iv. It's valid. No rules are violated.x

**Question D** In this formula

$$6^{\frac{2\text{hr}}{3\text{min}}}$$

choose which rule (if any) is violated.

i. Addition or Subtraction rulexNo addition or subtraction in this formula.

ii. Multiplication or Division rulexThere are no restrictions for multiplication and division, so a formula can hardly violate them!

iii. ExponentialxThe exponent is 4 hr/3 min, which has dimension T/T = [1]. So the rule is satisfied.

iv. It's valid. No rules are violated.Correct.

**Question E** In this formula

$$5\text{g} \times 3\text{kg} - 7\text{lbs}$$

choose which rule (if any) is violated.

i. Addition or Subtraction ruleGood. You can't subtract M from $\text{M}^2$. (Strictly speaking, lbs has dimension of force, $ML^2/T^2$, but you can't subtract force from $\text{M}^2$ either.

ii. Multiplication or Division rulexThere are no restrictions for multiplication and division, so a formula can hardly violate them!

iii. ExponentialxThere's no exponent here.

iv. It's valid. No rules are violated.x

**Question F** In this formula

$$\sqrt[3]{8m^3 + 27\text{ft}^2}$$

choose which rule (if any) is violated.

i. Addition or Subtraction ruleGood. You can't add $\text{L}^3$ to $\text{L}^2$.

ii. Multiplication or Division rulexThere are no restrictions for multiplication and division, so a formula can hardly violate them!

iii. ExponentialxMaybe you're thinking that the cube-root rule is violated, but since the quantity in the cube root is invalid, the root doesn't do anything additionally wrong.

iv. It's valid. No rules are violated.x

# Exer. 15.4

**Exercise 15.4** ../Modeling/Exercises/crow-mean-dress.Rmd

The surface area $S$ of a mammal is reasonably well approximated by the function

$$S(M) \equiv kM^{2/3}$$

where $M$ is the body mass (in kg) and the constant $k$ depends on the particular species under consideration.

Note that $M^{2/3}$ is **not an allowed arithmetic operation**. $[M]$ = mass, and mass, like any other dimension, cannot be raised to a non-integer power. More properly, the expression should be written

$$\left(\frac{M}{1\ kg}\right)^{2/3}$$

The division by "1 kg" renders dimensionless the quantity in the parentheses:

$$\left[\frac{M}{1\ kg}\right] = 1$$

In order to render the quantity both dimensionless and *unitless*, $M$ should be specified in kg. The usual practice is to skip the "1 kg" business and simply say, "Where $M$ is in kg." You will see such notation frequently in your career and should take care to use the indicated units.

You'll need to open a computing sandbox to do the calculations.

**Question A** Consider a baby and an adult. The adult's mass is 8 times greater than the baby's. Then the adult's surface area is ...?

i. The same as the baby'sx

ii. 1.5 times of the baby'sx

iii. 4 times the baby'sCorrect.

iv. 8 times the baby'sx

**Question B** Consider a human of body mass 70 kg with a skin surface area of 18,600 $\text{cm}^2$. Which of the following units for the constant of proportionality $k$ is correct?

i. $\text{cm}^2\ \text{kg}^{-2/3}$xKilograms to a fractional power is not a sensible unit.

ii. $\text{cm}^2$Nice!

iii. cm$^2$ kg$^{2/3}$xWhen you multiply $(70kg/kg)^{2/3}$ by $k$, you need to get a result in $cm^2$.

iv. kg$^{-1}$xWould this produce $cm^2$ for the result?

**Question C** In the units of part (B), which value is $k$ closest to?

1x      10x      100X      1000♡

The numerical value of the constant $k$ changes depending on what units you want to express it in. The value you found in part (C) works for masses stated in kg and skin areas in cm$^2$.

Suppose you want to figure out a value of $k'$ that you can use in the formula for people who are used to talking about skin area in square inches and mass in pounds. The units of $k$ are cm$^2$, and we want the units of $k'$ to be in$^2$. That part is easy: just multiply $k$ by two flavors of one to change the units from cm to inches, like this:

$$k' = k \underbrace{\frac{\text{in}}{2.6\text{cm}}}_{\text{flavor of 1}} \underbrace{\frac{\text{in}}{2.6\text{cm}}}_{\text{flavor of 1}} = \frac{k}{2.6^2}$$

where the flavor of 1 reflects that 1 inch is 2.6 cm.

**But this is not the whole story**. We have to be very careful in dealing with the $\left(\frac{M}{1kg}\right)^{2/3}$. Translated to pounds, $M$ = 70 kg = 154 lbs, since, in the rough-and-ready way everyday people express themselves, 1 kg ≈ 2.2 lbs.[2]

Plugging in $M$ = 154 lbs makes the power-law part of the formula for skin area

$$\left(\frac{154 \text{ lbs}}{1 \text{ kg}}\right)^{2/3}$$

You can't take (pounds)$^{2/3}$ or (kg)$^{2/3}$; you won't get a sensible unit in either case. But [pounds/kg] = [1], so taking the two-thirds power of the ratio is perfectly legitimate.

Still, there's a problem. Multiplying $k$ $154^{2/3}$(lbs/kg)$^{2/3}$ has the right dimension, but strange-looking units that have nothing to do with skin area.

The resolution to this paradox is to multiply $\frac{154 \text{ lbs}}{1 \text{ kg}}$ by an appropriate flavor of 1 to render the dimensionless quantity unitless as well as dimensionless. This flavor will be $\frac{1\text{kg}}{2.2lbs}$, giving a formula for skin area in square inches:

```
Warning in normalizePath("www/EQ-cancel.png"): path[1]="www/EQ-cancel.png": No
such file or directory
```

Preliminaries/www/EQ-cancel.png

**Question D** Optional challenge) Assuming that $k$ = 1000cm$^2$ when specifying mass in kilograms, what should be the numerical value of $k'$ in square-inches that should be used when body mass is given in pounds?

8.7x      87♡      870X      8700x      87000x

# Exer. 15.5

**Exercise 15.5** ../Modeling/Exercises/Boyd-1.Rmd

The "Energy-maneuverability Theory" (E-M) of aircraft performance was developed by renowned fighter pilot Col John Boyd and mathematician Thomas Christie in the 1960s. The theory posits that the available maneuverability of an aircraft is closely related to its *specific energy* $E_s$, that is, the kinetic plus potential energy of the aircraft divided by aircraft weight. To be highly maneuverable, an aircraft must be able to change it's specific energy rapidly in time. Let's call this ability the *specific power* (that is, power divided by mass), $P_s$. An aircraft with large $P_s$ is more maneuverable than one with small $P_s$.

An important formula in E-M Theory is

$$P_s = \frac{T - D}{W} V$$

where $T$ is aircraft thrust, $D$ is drag, $W$ is weight, and $V$ is velocity. $(T - D)$, thrust minus drag, is the net forward force on the aircraft.

Recall these facts about the dimension of physical quantities:

- Velocity has dimension $L^1T^{-1}$ (e.g. meters per second)
- Acceleration has dimension $L^1T^{-2}$ = [Velocity] × $T^{-1}$ (e.g. meters per second-squared)
- Force has dimension $M\times$ [Acceleration]
- Energy has dimension [Force] × $L$
- Power has dimension [Energy] × $T^{-1}$.

**Question A** Which of the following is a correct dimensional formulation of power?

[Force] [Velocity]♡      [Energy] [Velocity]X
[Force] / [Velocity]x      [Energy] / [Velocity]x

**Question B** What is the dimension of $P_s$ in E-M Theory?

i. [Power] $\times$ $M^{-1}$Nice! In other words, specific power, that is, power per mass.

ii. [Force] $\times$ [Acceleration]xJust so you know, such a dimension is rarely, if ever, encountered in practice.

iii. [Force] $\times$ [Velocity]xYou're leaving out the division by $W$ in the E-M Theory formula.

iv. [Power]x

# Exer. 15.6

**Exercise 15.6** ../Modeling/Exercises/snake-leave-candy.Rmd

Newton's law of universal gravitation—also known as the inverse square law—is generally written

$$F = G\frac{m_1\ m_2}{r^2}.$$

$m_1$ and $m_2$ are the masses of the two objects (say, Earth and Sun). $r$ is the distance between the two objects (about 150,000,000 km). $F$ is the gravitational force and $G$ is a fixed quantity called the "gravitational constant."

Of course, you already know the dimension of force, mass, and distance.

**Question A** What is the dimension of the gravitational constant, $G$?

i. $L^3\ M^{-1}\ T^{-2}$Right!

ii. $L^2\ M^{-2}$xRemember, $Gm_1 m_2/r^2$ has to have the same dimensions as force.

iii. $L^2\ M^2\ T^{-2}$xCheck your signs!

iv. $L^2\ M^{-2}\ T^{-2}$x

**Question B** The quantity $G$ is $6.674 \times 10^{11}$ when $L$ is in meters, $M$ is in kilograms, and $T$ in seconds. What is the gravitational force between Earth (mass $6 \times 10^{24}$ kg) and Sun (mass $2 \times 10^{30}$ kg) ?

i. $3.6 \times 10^{28}$ NewtonsCorrect.

ii. $3.6 \times 10^{31}$ metersx

iii. $3.6 \times 10^{28}$ meters per second-squaredx

iv. $3.6 \times 10^{31}$ meter seconds per kgx

# Exer. 15.7

**Exercise 15.7** ../Modeling/Exercises/frog-grow-pantry.Rmd

In this book, we are parameterizing the sinusoid using the ***period*** $P$, the duration a cycle. In many settings, such as communications engineering and physics, it is preferable to parameterize in terms of the ***frequency***, often written with the Greek letter $\omega$ ("omega").

Here's the relationship:

$$\sin\left(\frac{2\pi}{P}t\right) = \sin(2\pi\omega t)$$

**Question A** When the input quantity $t$ represents time, it has dimension T. The period P has the same dimension so that the overall argument to sin() is dimensionless, as required. What is the dimension of $\omega$?

Tx    $\text{T}^2$X    $\text{T}^{-1}$♡    $\text{T}^{-2}$x

In an earlier exercise, we looked at human breathing. The period of a breathing cycle differs from hour to hour and from person to person. (It's also somewhat, but not completely, under conscious control.) A reasonable scale for the period of normal human breathing is 3 seconds.

**Question B** Given a respiratory period of 3 seconds/breath, what is the respiratory **frequency** in units of breaths/minute?

i. 20 breaths/minuteRight! Right. Each breath takes 1/20th of a minute, which is 3 seconds, the period specified in the question.

ii. 3 breaths/minutexIf this were true, each breath would take 20 seconds to complete.

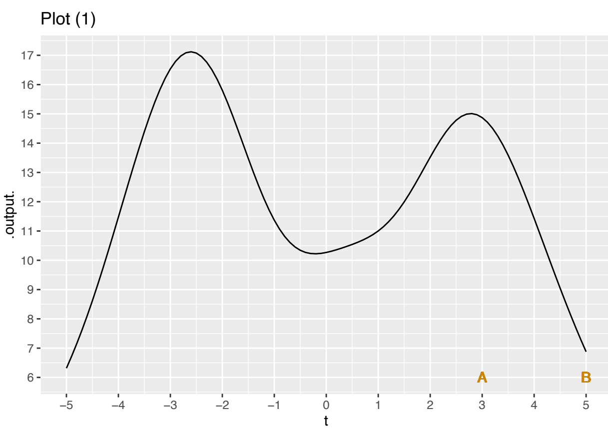
iii. 1/3 breath per minutexWith breaths completed every three seconds, 1/3 of a breath is completed each second. But the problem asked for breaths per minute.

iv. 20 seconds per breathxThe *period* is in the units of seconds per breath, but the *frequency* will have units of breaths per second. Frequency is the reciprocal of period (and vice versa).

# Exer. 17.2

**Exercise 17.2** ../Differentiation/Exercises/duck-takes-chair.Rmd

Each of the plots shows the graph of a function with two inputs, $A$ and $B$, marked.




**Question A** In plot (1), what is the **rate of change** over the interval $A$ to $B$? (Pick the closest answer.)

i. -15/2xRemember to take the *difference*: $f(B) - f(A)$.

ii. 15/2xThe order of A and B is significant!

iii. 2/15xIt's "rise" over "run", not the other way around.

iv. -2/15xIt's "rise" over "run", not the other way around.

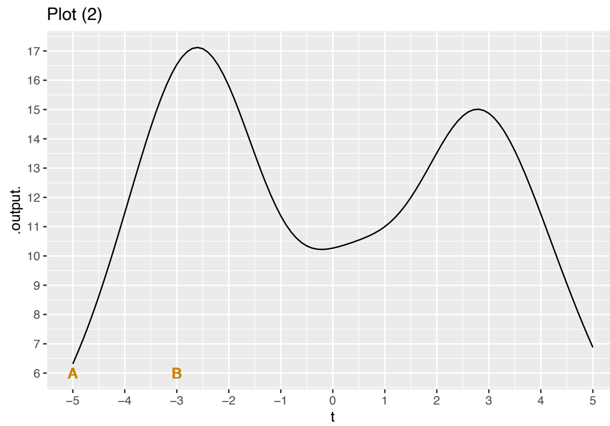
v. 8/2xThe order of A and B is significant!

vi. -8/2Nice!

vii. 2/8xThe order of A and B is significant!

viii. -2/8xIt's "rise" over "run", not the other way around.

ix. -15/5xRemember to take the *difference*: $B - A$




**Question B** In plot (2), what is the **rate of change** over the interval $A$ to $B$? (Pick the closest answer.)

i. -17/2xRemember to take the *difference*: $f(B) - f(A)$.

ii. 2/17xIt's "rise" over "run", not the other way around.

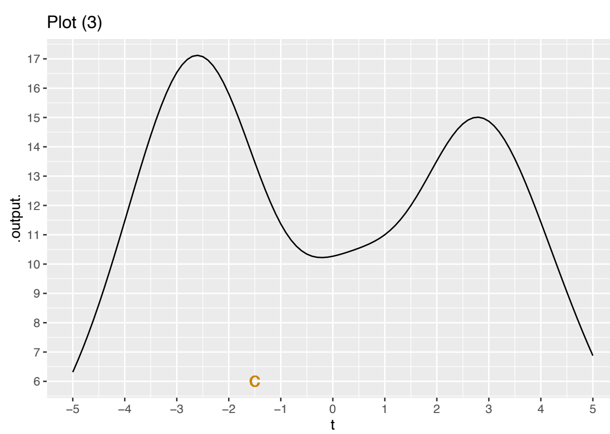
iii. -10/2xThe order of A and B is significant!

iv. 10/2Nice!

v. 2/10xThe order of A and B is significant!

vi. -2/10xIt's "rise" over "run", not the other way around.

vii. -17/-5xRemember to take the *difference*: $B - A$




**Question C** We haven't told you exactly how to do this yet, but give it a try. What is the rate of change near the point marked $C$? (Pick the closest answer.))

-1/2x    -1x    -2x    -3X    -3/2x    -4x    -5♡

# Exer. 17.4

**Exercise 17.4** ../Differentiation/Exercises/goat-dive-saucer.Rmd

From the graph in Figure @ref(fig:water-average), compute the average rate of change over the interval $10 \leq t \leq 200$. How does it compare to the average rate of change over the interval $10 \leq t \leq 125$?

# Exer. 17.5

**Exercise 17.5** ../Differentiation/Exercises/snail-tear-blanket.Rmd

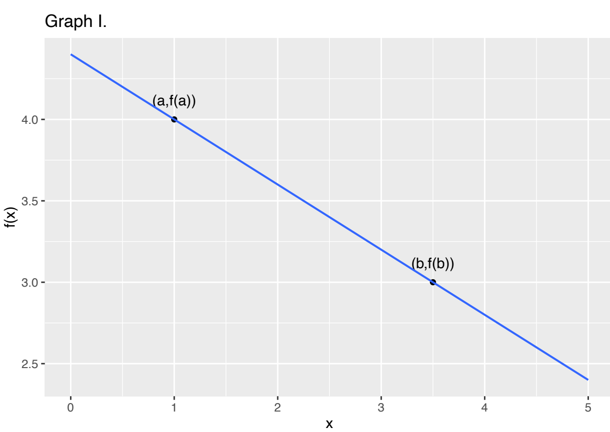
We will be working extensively with the *change* in output value of a function when the *input value changes*.

- The change in the output value of a function $f()$ when the input changes from $x = a$ to $x = b$ is

$$f(x = b) - f(x = a)$$

  Notice that when we talk about the change from $x = a$ to $x = b$ we subtract $f(a)$ from $f(b)$. That change is sometimes called the **rise** in the value of the function. Rise always compares (by subtraction) the two **output values** corresponding to two specific input values. Remember that $a$ and $b$ stand for specific numbers.

- Corresponding with the idea of the change in output being $f(b) - f(a)$ the change in the input value to a function is $b - a$. This is often called the *run* in the value of the input.



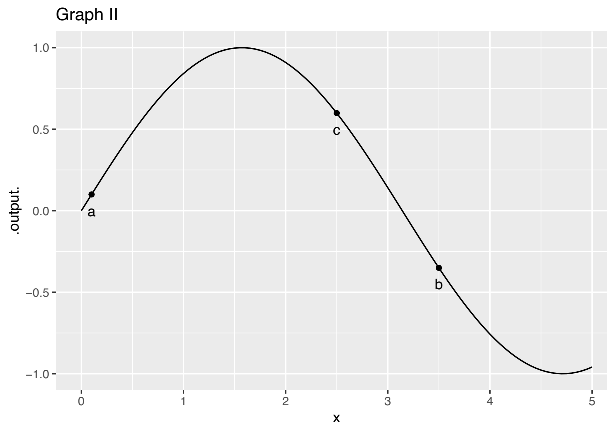

**Question A** True or false: In Graph I, the *rise* from a to b is positive.

TRUEx$f(x = a) > f(x = b)$, so the rise $f(x = b) - f(x = a)$ is *negative.* FALSE♡

**Question B** True or false: In Graph I, the *run* from $x = a$ to $x = b$ is positive.

i. TRUENice!

ii. FALSExThe *run* is about the relative positions of $x = a$ and $x = b$ on the x-axis. Since $a < b$, the run from $x = a$ to $x = b$ is positive.




**Question C** True or false: In Graph II, the *run* from a to b is positive.

i. TRUERight!

ii. FALSExThe *run* is about the relative positions of a and b on the x-axis. Since a is to the left of b, the run from a to b is positive.

**Question D** True or false: In Graph II, the *rise* from a to b is positive.

i. TRUExRemember, the *rise* from $x = a$ to $x = b$ is $f(x = b) - f(x = a)$

ii. FALSEGood.

**Question E** True or false: In Graph II, the *run* from b to c is positive.

i. TRUExThe *run* from $x = b$ to $x = c$ is $c - b$. Since $b > c$ b to c is negative.

ii. FALSECorrect.

**Question F** True or false: In Graph II, the *rise* from b to c is positive.

i. TRUEExcellent!

ii. FALSExThe *rise* from $x = b$ to $x = c$ is $f(x = c) - f(x = b)$. Since $f(x = c) > f(x = b)$, the rise is positive.

**Question G** For an interval [2, 6] what is the value of the run? (The answer is independent of any particular graph/function.)

i. 4Correct. The run is always the second number in the interval minus the first number. That's $6 - 2$ here.

ii. -4xYou got it backwards! The second number in the interval, 6, is numerically to the right of 2, so the run is positive.

**Question H** Which is the run of the interval [6, 2]? (Again, the answer is independent of any particular graph/function.)

i. 4xSorry. The run from $x = 6$ to $x = 2$ is $2 - 6$ which is $-4$.

ii. -4Good. The run is $2 - 6$, the second number in the interval minus the first number.

# Exer. 17.6

**Exercise 17.6** ../Differentiation/Exercises/chicken-show-map.Rmd

---

Open an R console and ….

Open an R sandbox. You can use these function definitions to help you in your calculations. ::: {.cell}

```
f <- makeFun(2*exp(x+1) ~ x)
g <- makeFun(3*exp(-x) ~ x)
h <- makeFun(x*exp(x) ~ x)
```

---

:::

Using R, compute the *average* rate of change of the function over the given interval. Choose the closest answer for each problem.

**Question A** $f(x) \equiv 2e^{x+1}$ over [-2, 2]

-2.99x 1.54x 2.72x 4.68X 9.85♡ 11.32x

**Question B** $g(x) \equiv 3e^{-x}$ over [-1, 1.5]

-2.99♡ 1.54x 2.72X 4.68x 9.85x 11.32x

**Question C** $h(x) \equiv xe^{x}$ over [0, 1]

-3x 1.54X 2.72♡ 4.68x 9.85X 11.32x

It's much less work if we use the R function `c()` to define the interval, and the R function `diff()` to calculate differences. The next sandbox contains an example asking you to compute the average rate of change of $f(x) \equiv e^x$ over the interval [0, 4]. You only need lines 1, 3, and 5. The other lines show intermediate results to help you understand what `diff()` is doing.

---

Open an R console and ….

```
interval <- c(0, 4)  # creates the interval
diff(interval)       # calculate the run
f <- makeFun(exp(x) ~ x)   #  create the function
f(interval)      # evaluate  function at the endpoints o
diff(f(interval)) / diff(interval)  # complete answer
```

---

**Question D** True or false: The average rate of change of $f(x) \equiv e^{x^2}$ over [0.0, 0.1] is 0.1005017

TRUE♡ FALSEx

**Question E** True or false: The average rate of change of $f(x) \equiv \log(x)$ over [2, 3] is 0.5062353.
(Hint: Change the code above so the interval goes from 2 to 3 and f becomes the function $f(x) \equiv \log(x)$)

TRUEx FALSE♡

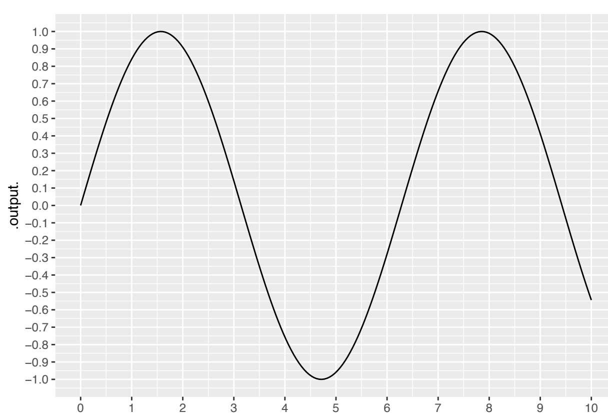
**Question F** True or false: The average rate of change of $f(x) \equiv \sin(x)$ over [0.0, 0.5] is 0.9588511

TRUE♡ FALSEx

# Exer. 17.7

**Exercise 17.7** ../Differentiation/Exercises/eagle-bring-car.Rmd

Consider the sinusoid function, graphed below.



**Question A** What is the average rate of change over the interval $0 < x < \frac{1}{2}$? (Choose the closest value.)

i. 0xFor this to be true, the function output would need to be the same at the two endpoints of the interval.

ii. 0.5xDid you forget to divide the rise by the run?

iii. 1Correct. Right. From the interval you have that the run is 1/2. The rise over that interval is from 0 to 1/2, so a rise of 1/2.

**Question B** What is the average rate of change over the interval $0 < x < 6.25$? (Choose the closest value.)

i. -0.5x

ii. 0Right! Right. The function output is zero at both endpoints of the interval, so the rise is zero. Hence, the run is zero.

iii. 0.5x

**Question C** What is the average rate of change over the interval $0 < x < 10$? (Choose the closest value.)

i. -0.05Excellent! Right. The rise is -0.5 and the run is 10.

ii. -0.5xThis is the "rise" over the interval, but it's not the average rate of change. You'll need to divide the rise by another quantity to get the average rate of change.

iii. 0x

iv. 0.5xThe rise will be $f(10) - f(0) pprox -0.5 - 0 = -0.5$. You've got the sign wrong.

# Exer. 17.8

**Exercise 17.8** ../Differentiation/Exercises/pig-look-rug.Rmd

For each of the following, compute the *average* rate of change of the function over the given interval.

**Question A** The average rate of change of $f(x) \equiv x + 5$ over [3, 5] is

i. -2xRemember, the *difference* from $x = 3$ to 5 is $f(5) - f(3)$, not the other way around. And the average *rate* of change is the difference divided by the length of the interval.

ii. -1xRemember, the *difference* from $x = 3$ to 5 is $f(5) - f(3)$, not the other way around. Similarly, the length of the interval from $x = 3$ to 5 is $5 - 3$, not $3 - 5$.

iii. 1Correct. $f(x = 5) = 10$ and $f(x = 3) = 8$, so the *difference* in values is 2. Since this difference occurs over an interval of length 2 (that is, from #x=3$ to 5), the average *rate* of change is 2/2.

iv. 2xAre you sure you took the **rate** of change rather than simply the change?

**Question B** The average rate of change of $f(x) \equiv 3 - 2x$ over [-4, -2] is

i. -4xThis is the total difference over the interval, not the *rate* of change.

ii. -2Right! The difference from $x = -4$ to $-2$ is $f(-2) - f(-4) = (7) - (11) = -4$. The length of the interval is $(-2) - (-4) = 2$. So the rate is -4/2, just as you got.

iii. 0x

iv. 1xWhen $x$ changes by two units, $f(x)$ changes by 4. The *rate* of change is 4/2 = -2.

v. 2xCheck your +/- signs.

vi. 4xThis is the negative total difference over the interval, not the *rate* of change. Also check your +/- signs.

**Question C** The average rate of change of $f(x) \equiv -3x^2$ over [0, 4] is

i. -24xPerhaps you are looking at the derivative at $x = 4$ and not the average rate of change from $x = 0$ to 4.

ii. -12Nice! The difference in $f()$ over the interval is $f(4) - f(2) = (-48) - 0 = -48$. The length of the interval is $4 - 0$, so the average rate of change over the interval is $(-48)/4 = -12$.

iii. 0xPerhaps you are looking at the derivative at $x = 0$ and not the average rate of change from $x = 0$ to 4.

iv. 1x

v. 2x

vi. 12xCheck the signs in your arithmetic.

vii. 24x

**Question D** The average rate of change of $f(x) \equiv x^3 - 2x + 1$ over [0, 2] is

i. -2xEither check your +/- signs or perhaps you are looking at the derivative at $x = 0$ and not the average rate of change from $x = 0$ to 2.

ii. 1.5x

iii. 2Excellent! The difference in $f()$ over the interval is $f(2) - f(0) = 5 - 1 = 4$. The length of the interval is $2 - 0$ so the average *rate* of change is $4/2 = 2$.

iv. 7x

v. 10xI think you are looking at the derivative at $x = 2$ rather than the average rate of change from 0 to 2.

# Exer. 17.9

**Exercise 17.9** ../Differentiation/Exercises/exponentials.Rmd

In this question, we ask you to estimate the slope from a graph of the function. But the function is **exponential**, so not a straight line.

A fundamental idea in calculus is that even a function with a curved graph, if you zoom in closely around a given point, will look like a straight line. And you know how to calculate the slope of a straight line.

When the graph is curved, the slope will be different at different points along the graph. So there's not a single slope for the function. Still, we can talk about the "slope at a point."

One way to specify a point on a function's graph is to give the horizontal coordinate: the **input** to the function. But here we're going to give you the **output** of the function. So long as the function passes the ***horizontal-line test***, as the exponential does, specifying any particular output in the function's range *uniquely* identifies a corresponding input.

Estimate the slope of the exponential function $g(x) \equiv e^x$ at several inputs, which we'll call $x_1$, $x_2$, $x_3$ and $x_4$. We won't give you numerical values for the $x_i$ points, but we will tell you the output of the function at each of those inputs. the values of $x$ where:

a. $g(x_1) = 1$
b. $g(x_2) = 5$
c. $g(x_3) = 10$
d. $g(x_4) = 0.1$

For each of (a)-(d), use a sandbox to plot the exponential function $e^x$ on a domain zoomed in around around the appropriate value of $x_i$. Then calculate the slope of the curve at that $x_i$.

**Question A** Using your answers for the slope at the points given in (a)-(d), choose the best answer to this question: What is the pattern in the slope as $x$ varies?

i. The slope at each value $x_i$ is the same as $e^{x_i}$.Good. This property of the exponential function becomes important when describing a wide range of phenomena, from nuclear isotope decay to population growth.

ii. The slope at each value $x_i$ is the same as $x_i$.xThat would being saying the slope at $x_3$ is $\approx 2.30$. Is that what you got?

iii. The slope at each value of $x_i$ is the same as $x_i^2$.xThat would being saying the slope at the $x_3$ is $\approx 5.30$. Is that what you got?

iv. The slope at each value of $x$ is the same as $\sqrt{x}$.xThat would being saying the slope at the $x_i$ is $\approx 1.52$. Is that what you got?

# Exer. 17.1

**Exercise 17.1** ../Differentiation/Exercises/lion-find-bowl.Rmd

For each exercise, you are given a series of intervals that get smaller and smaller. Your job is to calculate the *average rate of change* of the function $f(x) \equiv x^2$ for each of the intervals. As the width of the intervals approach zero, our *average* rates of change become better approximations of the *instantaneous* rate of change. You should use the results you calculate to make an informed *estimate* of the **instantaneous rate of change**.

Open an R console and ....

```
interval <- c(__start__ , __end__ )
f <- makeFun(x^2 ~ x)
diff(f(interval)) / diff(interval)
```

A. Use these three intervals to estimate the instantaneous rate of change $\partial_x f(x = 3)$ - [3, 3.1] - [3, 3.01] - [3, 3.001]

B. Use these three intervals to estimate the instantaneous rate of change $\partial_x f(x = 5)$ - [4.9, 5] - [4.99, 5] - [4.999, 5]

C. Use these three intervals to estimate the instantaneous rate of change $\partial_x f(x = -2)$ - [-2, -1.9] - [-2, -1.99] - [-2, -1.999]

# Exer. 17.12

**Exercise 17.12** ../Differentiation/Exercises/slope-web-1.Rmd

There is a web of connections between the pattern-book functions and their slopes.

**Question A** 1. Which pattern-book function has a slope function that is simply a input-shifted version of itself? (For small enough $h$.)

exponentialX sinusoid♡ logarithmX power-law $x^{-1}$x

**Question B** 2. Which pattern-book function has a slope function that is identical to itself? (For small enough $h$.)

exponential♡ sinusoidX logarithmx power-law $x^{-1}$x

**Question C** 3. Which pattern-book function has a slope function that is another pattern-book function? (Hint: The other function is also listed among the choices.)

exponentialx sinusoidX logarithm♡ power-law $x^{-1}$x

# Exer. 17.13

**Exercise 17.13** ../Differentiation/Exercises/lamb-put-bulb.Rmd

Sometimes a bit of algebra can help us see what's going on with the instantaneous rate of change. Consider the exponential function $e^x$.

Rather than writing the slope function definition with a 0.1, let's substitute in the symbol $h$. This gives

$$\mathcal{D}\exp(x) = \frac{e^{x+h} - e^x}{h}$$

Extracting out the common term $e^x$ in the numerator, we arrive at

$$\mathcal{D}\exp(x) = e^x \left[\frac{e^h - 1}{h}\right]$$

Since $h$ is a number, $[e^h - 1]/h$ is a number, not a function of $x$. So, for any given value of $h$, the slope function of the exponential is proportional to the exponential itself.

**Question A** Using a sandbox, find the value of $[e^h - 1]/h$ when $h = 0.1$. Which of these is it?

1.271828x 1.694343X 1.718282♡ e = 2.718282x

The instantaneous rate of change involves making $h$ very small, but not quite zero. If you make $h$ exactly zero, the result will be ambiguous.

**Question B** Using a sandbox, compute $[e^0 - 1]/0$. What's the result?

i. `Inf`xThis would be the result if the numerator were positive, however small. But both the numerator and the denominator are zero.

ii. `NaN`Nice! Meaning, 'not a number'.

iii. `Bogus`xThis is **not** a value and can't be used in arithmetic. But `Inf` and `NaN` each can be legitimately used in arithmetic.

iv. An error message result.xThe point of `Inf` and `NaN` is to avoid creating error conditions while still signalling that the result is ambiguous.

**Question C** Using a sandbox, compute $[e^h - 1]/h$. Make $h$ as small as you can, for instance 0.00001 or 0.00000001. What's the result?

i. 1Excellent! Amazingly, the instantaneous rate of change of $e^x$ is exactly $e^x$.

ii. It varies with $h$, but is always around 1.5 for $h$ small enoughx

iii. There's no particular result.x

# Exer. 18.02

**Exercise 18.02** ../Differentiation/Exercises/kid-mean-table.Rmd

Each question involves a pair of quantities that are a function of time and that might or might not be a quantity/rate-of-change pair. If they are, say which quantity is which. Feel free to look up a dictionary definition of words you are uncertain about.

**Question A** Deficit and debt

i. Deficit is the rate of change of debt with respect to time.Correct.

ii. Debt is the rate of change of deficit with respect to time.x

iii. They are not a rate of change pair.x

**Question B** water contained and flow

i. Flow is the rate of change of water contained with respect to time.Good.

ii. Water contained is the rate of change of flow with respect to time.x

iii. They are not a rate of change pair.x

**Question C** Interest rate and debt owed on credit card

i. Interest rate is the rate of change of credit card debt with respect to time.Excellent!

ii. Credit card debt is the rate of change of interest rate with respect to time.x

iii. They are not a rate of change pair.x

**Question D** Rain intensity and total rainfall

i. Rain intensity is the rate of change of total rainfall with respect to time.Excellent!

ii. Total rainfall is the rate of change of rain intensity with respect to time.x

iii. They are not a rate of change pair.x

**Question E** Force and acceleration

i. Force is the rate of change of acceleration with respect to time.X

ii. Acceleration is the rate of change of force with respect to time.X

iii. They are not a rate of change pair.Good. The dimension of force is $ML/T^2$. The dimension of acceleration is $L/T^2$. A rate of change with respect to time should have an extra T in the denominator of the dimensions.

**Question F** Position and acceleration

i. Position is the rate of change of acceleration with repect to time.x

ii. Acceleration is the rate of change of position with respect to time.x

iii. They are not a rate of change pair.Nice! The dimension of position is $L$. The dimension of acceleration is $L/T^2$. The rate of change of position would have dimension $L/T$. That's called 'velocity'.

**Question G** Velocity and air resistence

i. Velocity is the rate of change of air resistence with repect to time.x

ii. Air resistence is the rate of change of velocity with respect to time.x

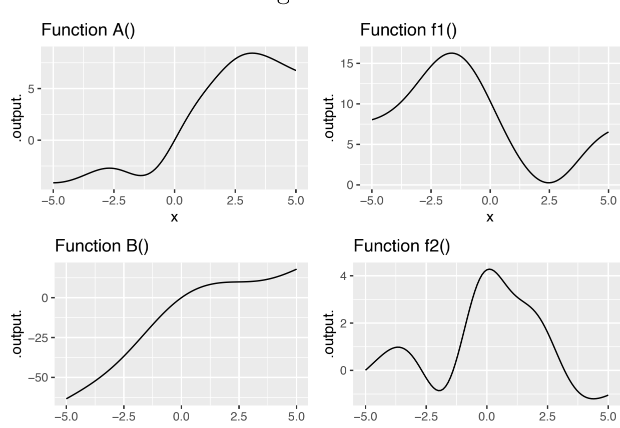
iii. They are not a rate of change pair.Right! Air resistence is a force, with dimension $ML/T^2$. Velocity has dimension $L/T$. The rate of change of velocity with respect to time is acceleration, which has dimension $L/T^2$.

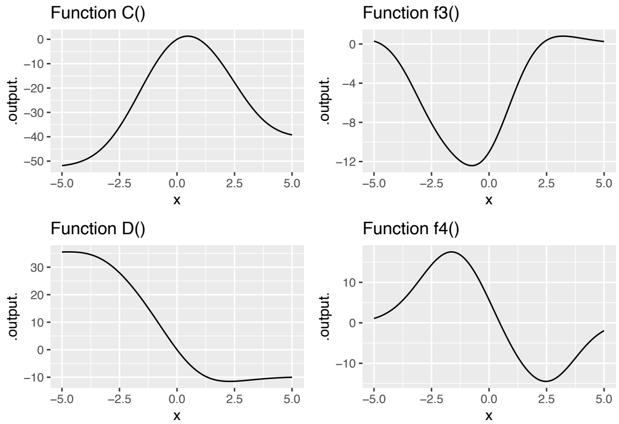
Problem with ../Differentiation Exercises/pine-lead-car.Rmd

# Exer. 18.06

**Exercise 18.06** ../Differentiation/Exercises/robin-row-boat.Rmd

Here are graphs of various functions. The right column shows functions named $f_1()$, $f_2()$, and so on. The left column shows functions $A()$, $B()$, $C()$, and so on. Most of the functions on the right are the derivative of some function on the left, and most of the functions on the left have their corresponding derivative on the right. Your task: Match the function on the left to it's derivative on the right.






**Question A** The derivative of Function A() is which of the following:

f1()x f2()♡ f3()X f4()x not shownx

**Question B** The derivative of Function B() is which of the following:

f1()♡ f2()x f3()X f4()x not shownx

**Question C** The derivative of Function C() is which of the following:

f1()x f2()x f3()X f4()♡ not shownx

**Question D** The derivative of Function D() is which of the following:

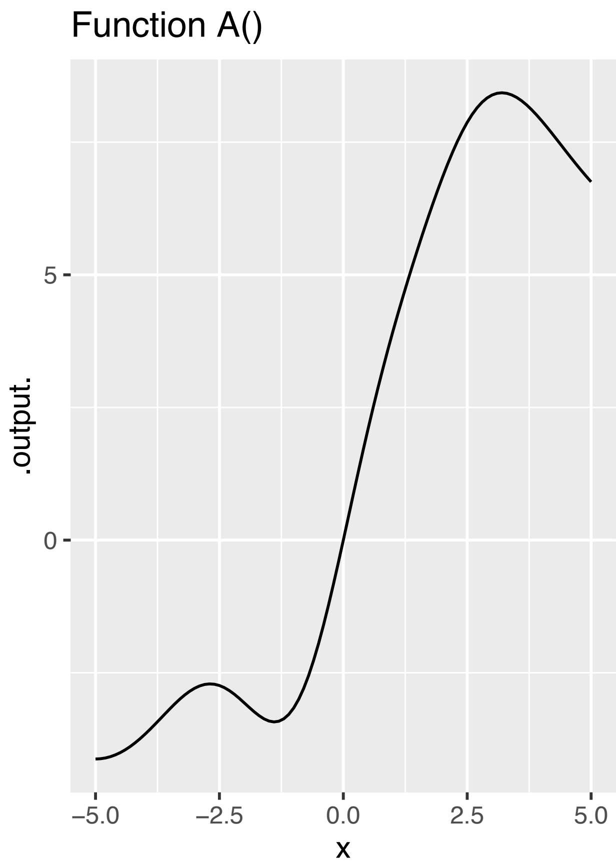
f1()x f2()x f3()X f4()x not shown♡

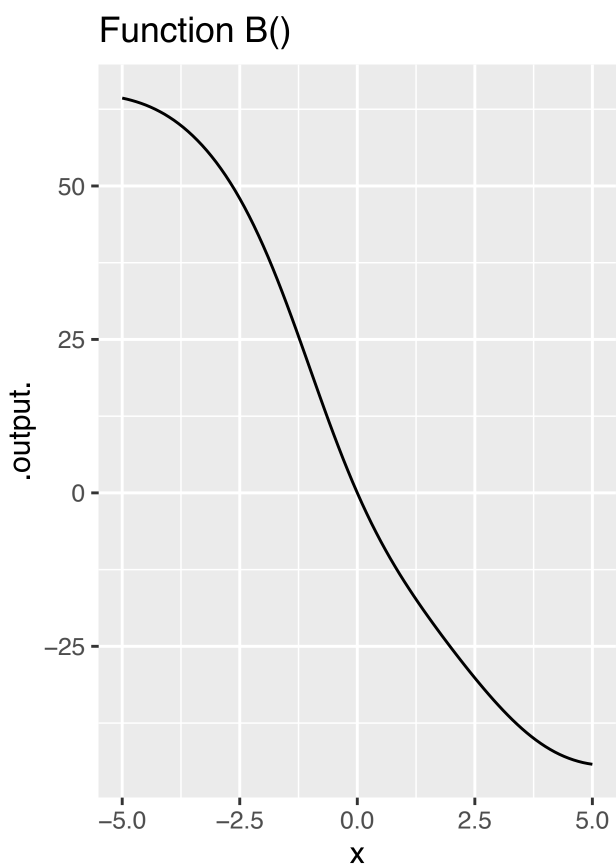
## Exer. 18.08

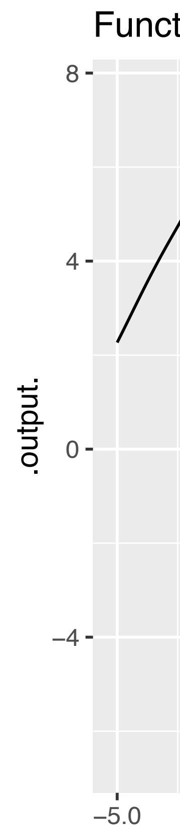
**Exercise 18.08** ../Differentiation/Exercises/deer-pitch-saw.Rmd

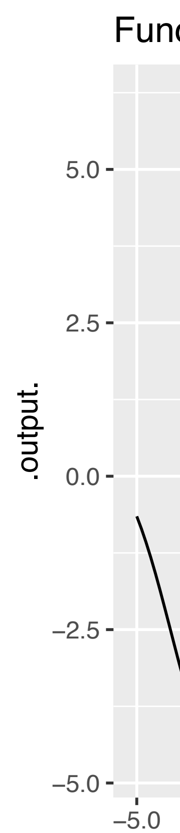
The left column of graphs shows functions A(), B(), C(), and D(). The right column shows functions dd1(), dd2(), and so on. Find which function (if any) in the right column corresponds to the 2nd derivative of a function in the left column.

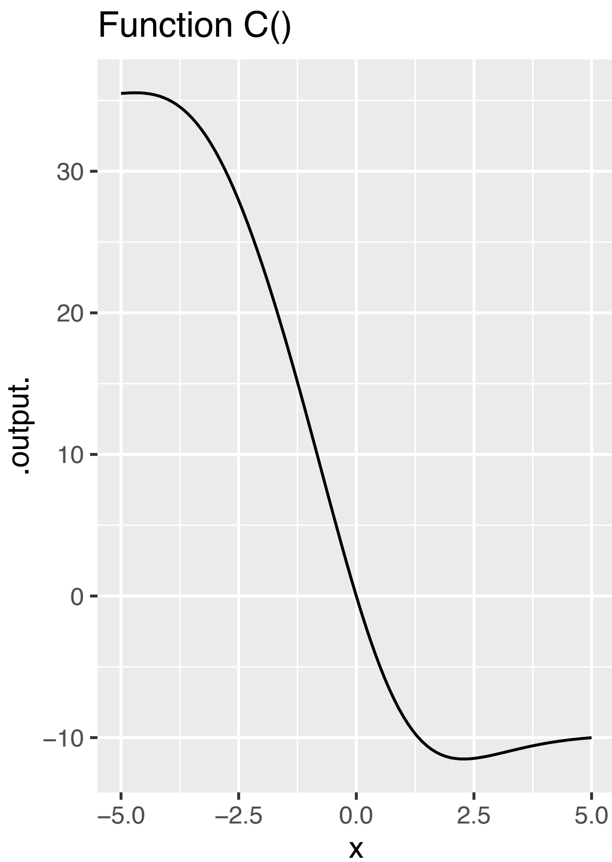
Remember the concepts of "concave up" (a smile!) and "concave down" (a frown). At those values of $x$ for which the 2nd derivative of a given function is positive, the given function will be concave up. When the 2nd derivative is negative, the given function will be concave down.

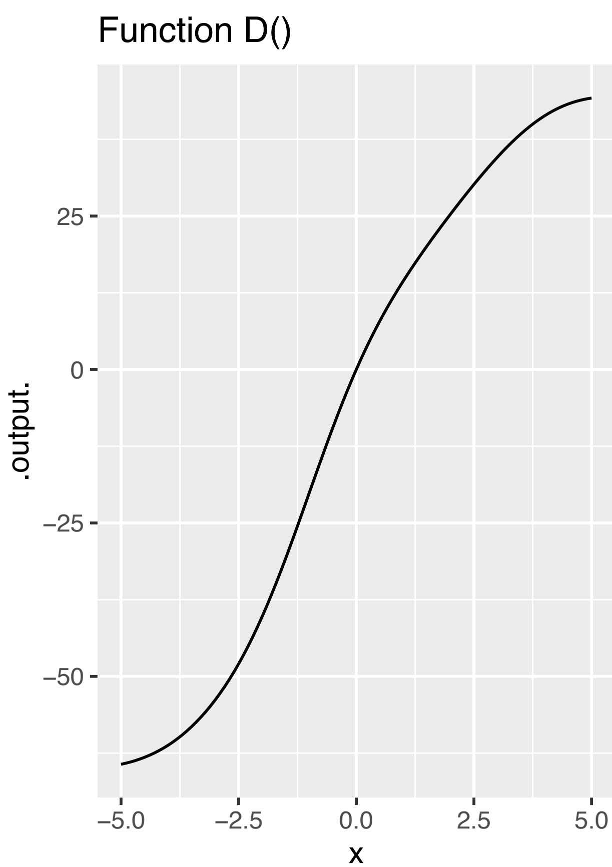


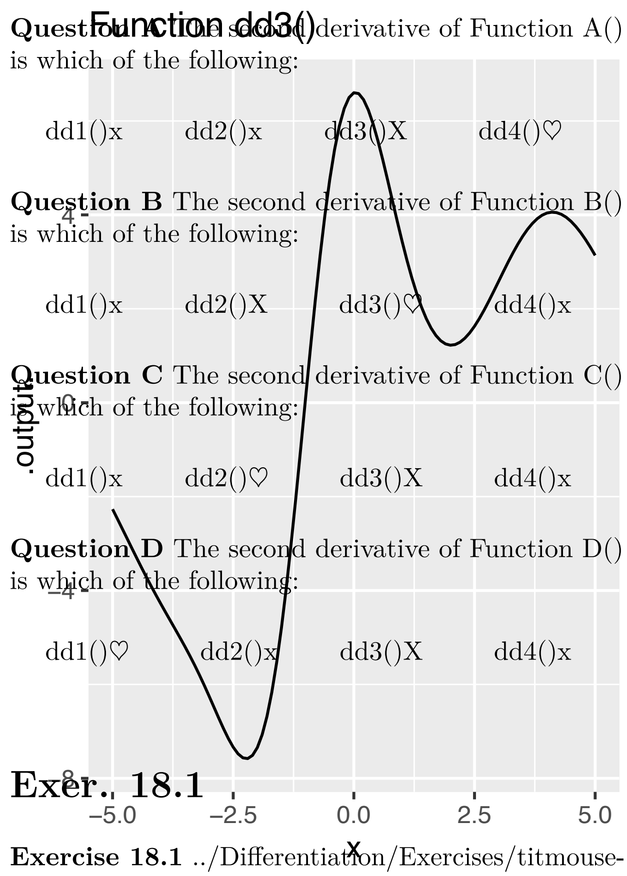








**Question A** The second derivative of Function A() is which of the following:

dd1()x    dd2()x    dd3()X    dd4()♡

**Question B** The second derivative of Function B() is which of the following:

dd1()x    dd2()X    dd3()♡    dd4()x

**Question C** The second derivative of Function C() is which of the following:

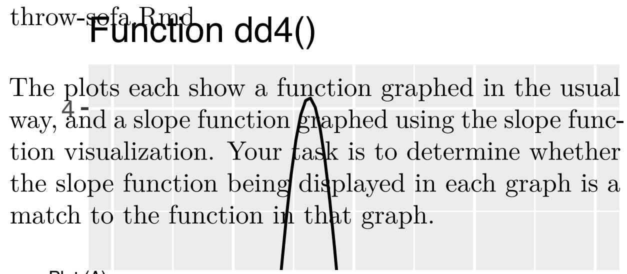
dd1()x    dd2()♡    dd3()X    dd4()x

**Question D** The second derivative of Function D() is which of the following:

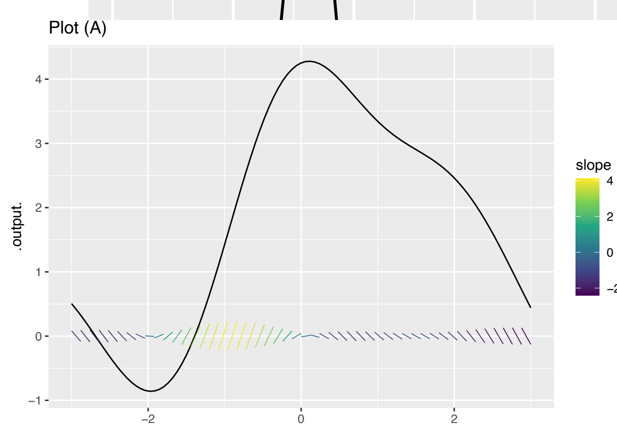
dd1()♡    dd2()x    dd3()X    dd4()x

## Exer. 18.1

**Exercise 18.1** ../Differentiation/Exercises/titmouse-throw-sofa.Rmd

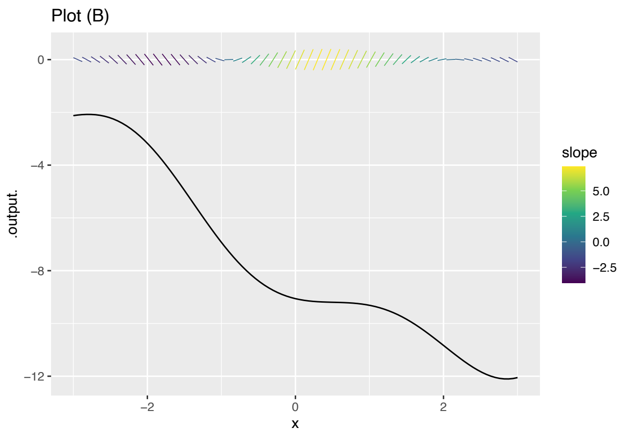



The plots each show a function graphed in the usual way, and a slope function graphed using the slope function visualization. Your task is to determine whether the slope function being displayed in each graph is a match to the function in that graph.



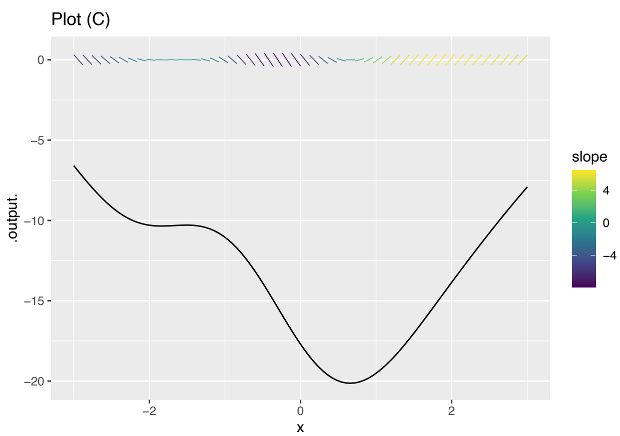

**Question A** In plot (A), does the slope function displayed correspond to the function that's graphed?

Yes♡    Nox



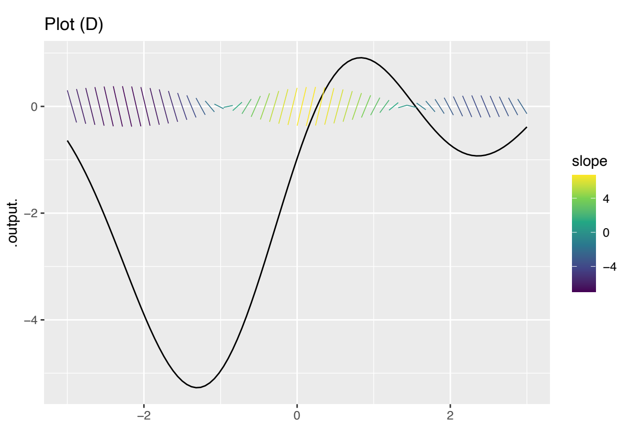

**Question B** In plot (B), does the slope function displayed correspond to the function that's graphed?

Yesx  No♡




**Question C** In plot (C), does the slope function displayed correspond to the function that's graphed?

Yes♡  Nox




**Question D** In plot (D), does the slope function displayed correspond to the function that's graphed?

Yesx  No♡

# Exer. 18.12

**Exercise 18.12** ../Differentiation/Exercises/kitten-put-kayak.Rmd

In the following, three different functions are described. Your task is to write down the **dimension** of the input and of the output. Do this both for the function itself, and for the derivative of the function. For example, the dimension of the output of $N(y)$ given below is P, for population. The input has dimension T, for time.

A. The given function is $N(y)$, the population of the Netherlands in year $y$.

i. Dimension of input to $N(y)$?
ii. Dimension of output from $N(y)$?
iii. Dimension of input to $\partial_y N(y)$?
iv. Dimension of output from $\partial_y N(y)$?

B. The given function is $p(u)$, the net profit from a manufactured good as a function of the number of units manufactured.

i. Dimension of input to $p(u)$?
ii. Dimension of output from $p(u)$?
iii. Dimension of input to $\partial_u p(u)$?
iv. Dimension of output from $\partial_u p(u)$?

C. The given function is $w(t)$, the amount of water in a leaky bucket at any time after the bucket was filled.

i. Dimension of input to $w(t)$?
ii. Dimension of output from $w(t)$?
iii. Dimension of input to $\partial_t w(t)$?
iv. Dimension of output from $\partial_t w(t)$?

# Exer. 18.14

**Exercise 18.14** ../Differentiation/Exercises/rat-take-fork.Rmd

**Question A** Tanks for bulk storage of natural gas are typically large cylinders with a cap that can move up and down. The volume of the tank is a function of the position of the cap. What is the dimension of the derivative of cylinder volume with respect to cap position?

$L^2$♡  $L$x  $L^3$X  $L^3/T$x  $T/L^3$x

# Exer. 18.16

**Exercise 18.16** ../Differentiation/Exercises/SIR-dimensions.Rmd

The standard model of epidemics used in public health planning is called the ***SIR model***. (SIR stands for

"Susceptible (S), Infective (I), Recovered (R)", the sequence that a person starts in, moves to, and ends up in (hopefully!) in an epidemic.)

One of the equations in the SIR model is

$$\frac{dS}{dt} = -aSI$$

The notation $dS/dt$ means "the rate of change of number of susceptibles, S, with respect to time." This has dimension "people/T". The dimensions $[S]$ and $[I]$ are each simply "people."

**Question A** What is $[a]$?

i. Tx

ii. $\text{T}^{-1}$x

iii. people/TxThen $[aSI]$ would be $\text{people}^3/\text{T}$, but that's not the same as $[dS/dt]$.

iv. $\text{people}^{-1}\ \text{T}^{-1}$Correct. This correctly gives $[aSI]$ as people/T, which is the same as $[dS/dt]$.

v. people × Tx

vi. None of the above.x

Another equation in the SIR model describes how the number of infective people changes over time:

$$\frac{dI}{dt} = -aSI - bI$$

where $[\frac{dI}{dt}]$ = people/T.

**Question B** What is $[b]$?

i. Tx

ii. $\text{T}^{-1}$Right!

iii. people/TxThen $[aSI]$ would be $\text{people}^3/\text{T}$, but that's not the same as $[dS/dt]$.

iv. $\text{people}^{-1}\ \text{T}^{-1}$xIf this were true, $[bI]$ would be $\text{T}^{-1}$. But $[bI]$ has to be the same as $[dI/dt]$, which is people $T^{-1}$.

v. people × Tx

vi. None of the above.x

# Exer. 20.1

**Exercise 20.1** ../Differentiation/Exercises/tilde-function.Rmd

The most common programming pattern in the R/mosaic calculus commands is:

> **Operator**(*tilde_expression*, [optional details])

Some operators: `slice_plot()`, `contour_plot`, `make_Fun()`, `D()`, `antiD()`, `Zeros()`

For each of the R/mosaic expressions, determine which kind of thing is being created. Feel free to run the expressions in a SANDBOX.

**Question A**

`makeFun(a*x - b ~ x)`

i. a function of `x`xFair enough. But the function also has arguments `a` and `b`

ii. a function of `x`, `a`, and `b`Right!

iii. a tilde expressionxThe tilde expression is the **input** to the operator. The operator translates the tilde expression into something else.

iv. a plotx

v. a data framex

vi. an errorx

**Question B**

`D(a*x - b ~ x)`

i. a function of `a`x

ii. a function of `x`, `a`, and `b`Good.

iii. a tilde expressionx

iv. a plotx

v. a data framex

vi. an errorx

**Question C**

`antiD(a*x - b ~ x)`

i. a function of `a`x

ii. a function of `x`, `a`, and `b`Right!

iii. a tilde expressionx

iv. a plotx

v. a data framex

vi. an errorx

**Question D**

`slice_plot(a*x - b ~ x, domain(x=0:5))`

i. a function of `x`x

ii. a function of `x`, `a`, and `b`x

iii. a tilde expressionx

iv. a plotxThe expression is intended to make a plot, but it doesn't work. Specific numerical values would need to be provided for `a` and `b`. You could have provided numerical values to `a` and `b` by assigning them in previous statements. If that's what you had in mind, you deserve full credit.

v. a data framex

vi. an errorGood.

**Question E**

`f <- makeFun(a*x + b ~ x, a=2, b=-4)`

`slice_plot(f(x) ~ x, domain(x=0:5))`

i. a function of `x`x

ii. a function of `x`, `a`, and `b`x

iii. a tilde expressionx

iv. a plotExcellent! This works because there are specific values provided for the `a` and `b` parameters.

v. a data framex

vi. an errorx

**Question F**

`Zeros(a*x - b ~ x, domain(x=0:5))`

i. a function of `x`x

ii. a function of `x`, `a`, and `b`x

iii. a tilde expressionx

iv. a plotxThe expression is intended to make a data frame, but it doesn't work. Specific numerical values would need to be provided for `a` and `b`.

v. a data framex

vi. an errorCorrect.

**Question G**

`a*x - b ~ x`

i. a function of `x`x

ii. a function of `x`, `a`, and `b`x

iii. a tilde expressionRight!

iv. a plotxThe expression is intended to make a plot, but it doesn't work. Specific numerical values would need to be provided for `a` and `b`.

v. a data framex

vi. an errorx

**Question H**

`f <- makeFun(a*x + b ~ x, a=2, b=-4)`

`Zeros(f(x) ~ x)`

i. a function of `x`x

ii. a function of `x`, `a`, and `b`x

iii. a tilde expressionx

iv. a plotx

v. a data frameGood.

vi. an errorx

**Question I** Suppose you create a function in the usual way, e.g. `f <- makeFun(a*x + b ~ x, a=2, b=-4)`. Which of the following will plot a straight-line function with a slope of 5.

i. `slice_plot(f(x) ~ x, domain(x=-5:5))`X The default value of `a` is 2, so the line would have a slope of 2.

ii. `slice_plot(f(x, b=2), domain(x=-5:5)`xIt's `a` that is the slope parameter.

iii. `slice_plot(f(x, a=5), domain(x=-5:5)`Right!

# Exer. 20.2

**Exercise 20.2** ../Differentiation/Exercises/fly-speak-canoe.Rmd

As you know, given a function $g(x)$ it's easy to construct a new function $\mathcal{D}_x g(x)$ that will be an approximation to the derivative $\partial_x g(x)$. The approximation function, which we call the slope function, can be

$$\mathcal{D}_x g(x) \equiv \frac{g(x+0.1) - g(x)}{0.1}$$

Open a SANDBOX and use `makeFun()` to create a function $g(x) \equiv \sin(x)$ and another that will be the slope function, called it `slope_of_g()`.

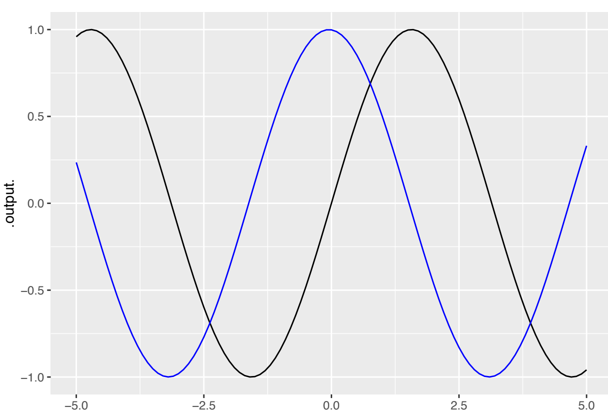
```
g <- makeFun(sin(x) ~ x)
slope_of_g <- makeFun( _your_tilde_expression_here_)
```

**Question A** What's the value of `slope_of_g(1)`?

0.3749x 0.4973♡ 1.3749X 1.4973x

Using your sandbox, plot both `g()` and `slope_of_g()` (in blue) on a domain $-5 \leq x \leq 5$. This can be done with `slicePlot()` in the following way:

```
slice_plot(g(x) ~ x, domain(x=-5:5)) %>%
  slice_plot(slope_of_g(x) ~ x, color="blue")
```




**Question B** Which of these statements best describes the graph of $g()$ compared to `slope_of_g()`?

i. `slope_of_g()` is shifted left by about $\pi/2$ compared to `g(x)`.Nice!

ii. `slope_of_g()` is shifted left by about $\pi$ compared to `g(x)`.x

iii. `slope_of_g()` has a larger amplitude than `g()`X

iv. The output of `slope_of_g()` is always positivex

v. `slope_of_g()` is practically the same function as `g()`. That is, for any input the output of the two functions is practically the same.x

# Exer. 20.5

**Exercise 20.5** ../Differentiation/Exercises/doe-pay-kitchen.Rmd

Recall the differentiation rules for three of the pattern-book functions as presented in **?@sec-symbolic-differentiation**:

| Function name | Formula | Formula for derivative | power-law exponent $p$ |
|---|---|---|---|
| Identity | $x$ | 1 | 1 |
| Square | $x^2$ | $2\,x$ | 2 |
| Reciprocal | $1/x$ | $-1/x^2$ | -1 |

All three of these pattern-book functions are members of the power-law family: $x^p$. They differ only in the value of $p$.

There is a differentiation rule for the power-law family generally. The next question offers several formulas for this rule, only one of which is correct. You can figure out which one by trying the pattern-book functions in the table above and seeing which formula gives the correct answer for the derivative.

**Question A** Which of these formulas gives the correct differentiation rule for the power-law family $x^p$?

i. $px^{p-1}$Correct.

ii. $(p-1)x^{p+1}$xIf this were true, the derivative of $x^2$ would be $x^3$.

iii. $x^{p-1}$x

iv. $(p-1)x^{p-1}$xIf this were true, the derivative of the identity function $(p=1)$ would be 0.

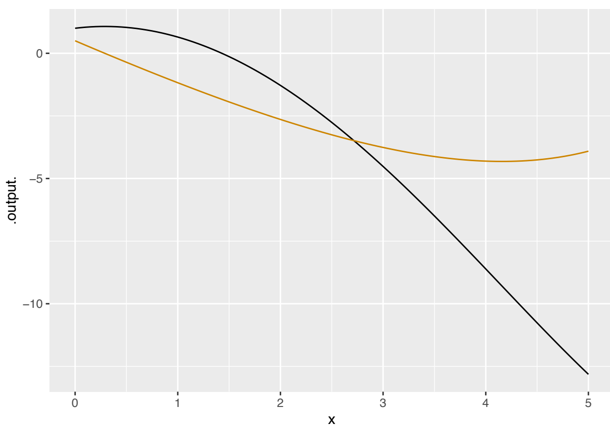
# Exer. 20.6

**Exercise 20.6** ../Differentiation/Exercises/finch-trim-kayak.Rmd

Although we created an R function named `slopeFun()` for the purposes of demonstration, it's better to use the R/mosaic operator `D()` which calculates the derivative, sometimes using symbolic methods and sometimes using a finite-difference method.

As an example of the use of `D()`, here is some more R code that defines a function `f()` and finds $\partial_x f()$, calling it `d_f()`. Then a slice plot is made of both `f()` and `d_f()`.

```
f   <- makeFun(sqrt(exp(x)) - x^2 ~ x)
d_f <- D(f(x) ~ x)
slice_plot(f(x) ~ x, domain(x=c(0, 5))) %>%
  slice_plot(d_f(x) ~ x, color = "orange3")
```




:::

For each of the following functions, write a brief comparison of the function to it's differenced version. You can combine phrases such as "same shape", "different shape. larger in amplitude", "smaller in amplitude", "same period", "shorter period", "longer period", or whatever seems appropriate. For instance, for the original example in the sandbox, a reasonable comparison might be, "f() is concave down but Diff(f) is concave up."

A. For the function $f(x) \equiv 3x$, compare $f()$ to $\partial_x f$.

B. For the function $f(x) \equiv x^2$, compare $f()$ to $\partial_x f$.

C. For the function $f(x) \equiv e^x$, compare $f()$ to $\partial_x f$.

D. For the function $f(x) \equiv e^{-0.3x}$, compare $f()$ to $\partial_x f$.

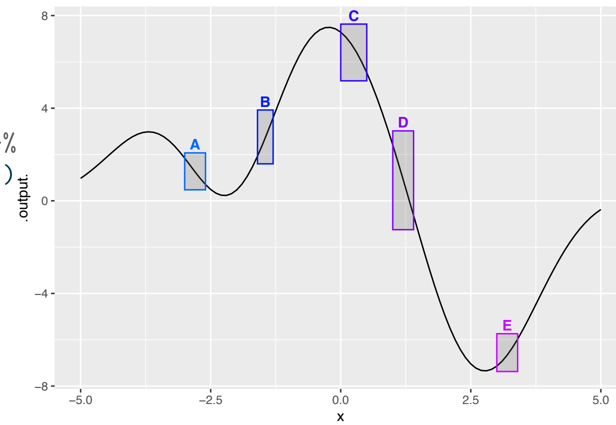
E. For the function $f(x) \equiv \sin(x)$, compare $f()$ to $\partial_x f$.

F. For the function $f(x) \equiv \sin(2\pi x)$, compare $f()$ to $\partial_x f)$.

G. For the function $f(x) \equiv \sin(\frac{2\pi}{20}x)$, compare $f()$ to $\partial_x f)$.

# Exer. 21.1

**Exercise 21.1** ../Differentiation/Exercises/frog-bid-bed.Rmd

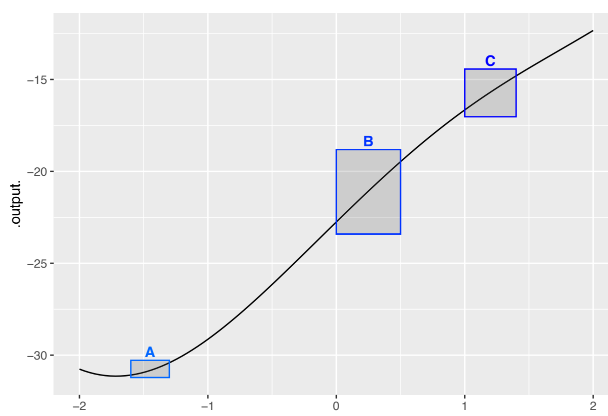



**Question A** Glance at the graph. In which boxes is the slope negative?

A, B, Cx  B, C, Dx  A, C, D♡

# Exer. 21.2

**Exercise 21.2** ../Differentiation/Exercises/turtle-send-pot.Rmd



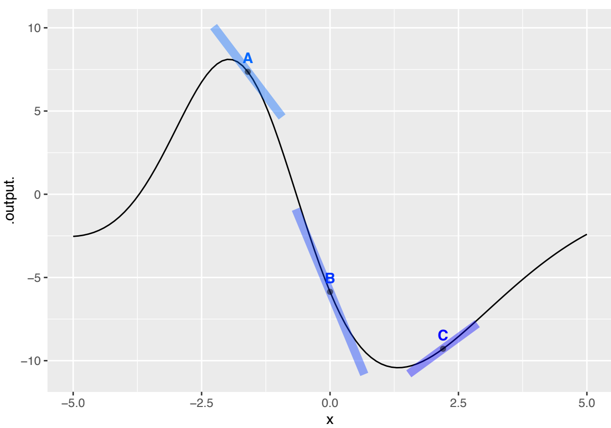

**Question A** Consider the slope of the function in the domains marked by the boxes. What is the order of boxes from least steep to steepest?

A, B, Cx    C, A, Bx    A, C, B♡    none of thesex

## Exer. 21.3

**Exercise 21.3** ../Differentiation/Exercises/reptile-put-kitchen.Rmd

```
Warning in validate_domain(domain, free_args):
```



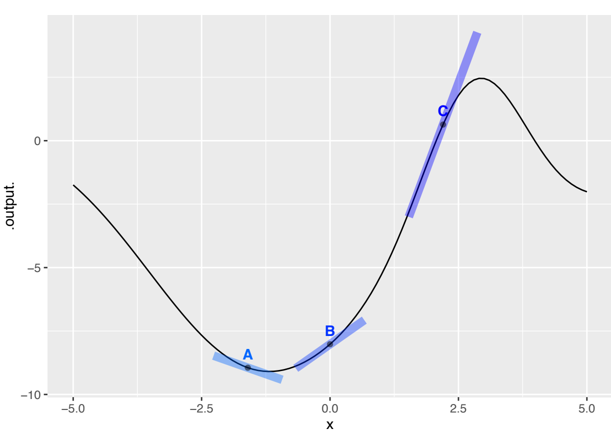

**Question A** Which of the line segments is tangent to the curve at the point marked with a dot?

Ax    Bx    Cx    all of them♡    none of themx

## Exer. 21.4

**Exercise 21.4** ../Differentiation/Exercises/goat-pay-pot.Rmd

```
Warning in validate_domain(domain, free_args): Missing domain names: x
```

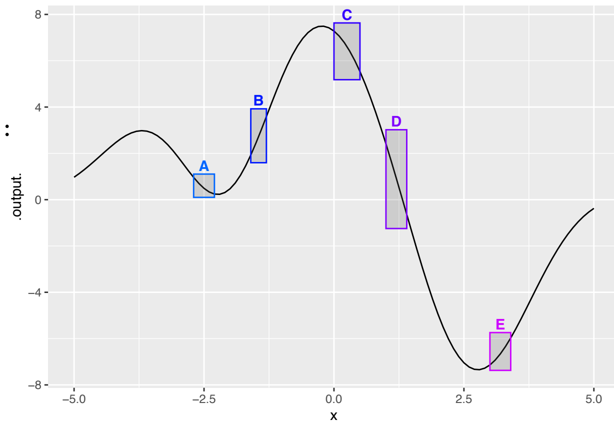



**Question A** Which of the line segments is tangent to the curve at the point marked with a dot?

A♡    Bxtoo shallow    Cxtoo steep    all of themx    none of themx

## Exer. 21.5

**Exercise 21.5** ../Differentiation/Exercises/seahorse-speak-saucer.Rmd




**Question A** In which of the boxes is the function concave up?

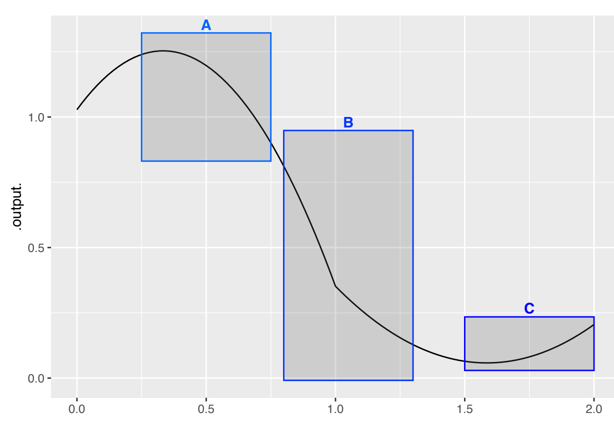
A and E♡    B and Dx    C and DX

## Exer. 21.6

**Exercise 21.6** ../Differentiation/Exercises/panda-drive-shirt.Rmd

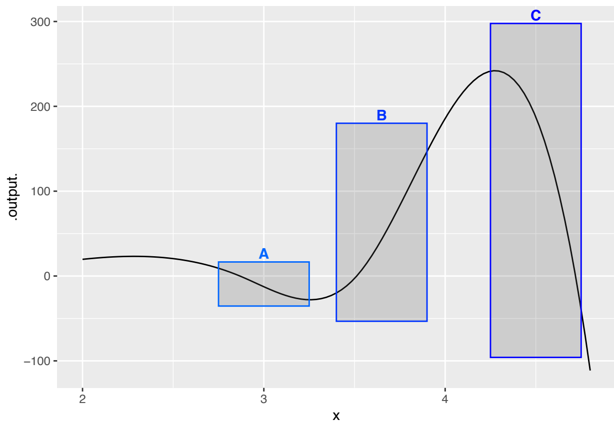
```
Warning in valVec[n] <- as.character(dots[[n]]): number of
not a multiple of replacement length

Warning in valVec[n] <- as.character(dots[[n]]): number of
not a multiple of replacement length
```




**Question A** In which boxes is the function smooth?

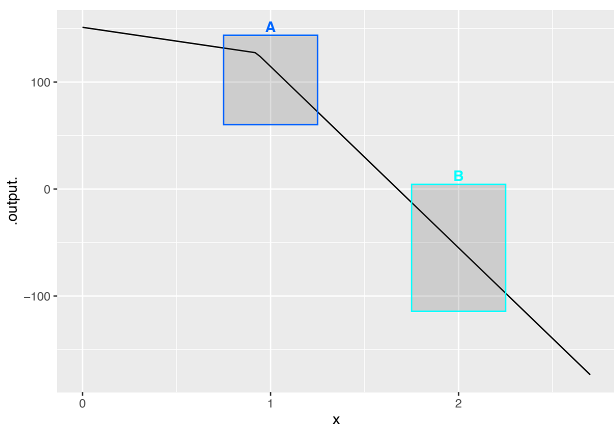
A and Bx    B and Cx    A and C♡    none of themx    all of themx




**Question B** In which boxes is the function smooth?

A and Bx B and Cx A and Cx none of themx all of them♡

---




**Question C** In which boxes is the function smooth?

Ax B♡ neither of themX both of themx

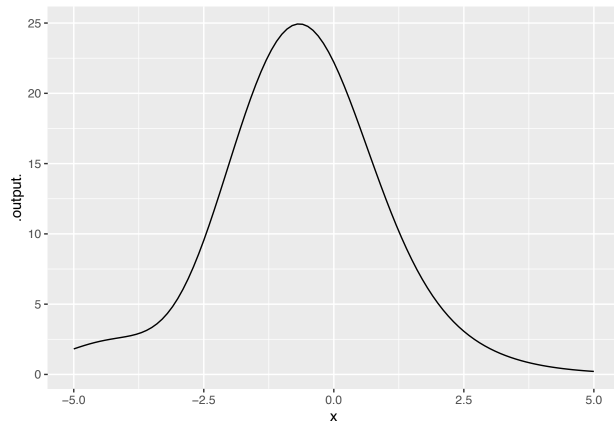
# Exer. 21.7

**Exercise 21.7** ../Differentiation/Exercises/crow-write-chair.Rmd

We introduced concavity graphically and used the terms "concave up" and "concave down." Now we can compute the concavity quantitatively using the second derivative.

In a sandbox, create this function and plot it. (Note: `rfun()` generates random functions in the same way you might by moving a pencil smoothly on a piece of paper. The `seed = 8427` effectively chooses which one of infinitely many functions is being generated. Different seeds give different functions. )

```
f <- rfun( ~ z, seed = 8427)
slice_plot(f(x) ~ x, domain(x=c(-5,5)))
```




You can see that in the region near $x = -1$ the function is concave down. While near $x = 2.5$ the function is concave up.

In your sandbox, compute the ***second derivative*** of $f(x)$ and evaluate it at $x = -1$ and $x = 2.5$.

```
dxx_f <- D(f(x) ~ x & x)
dxx_f(-1)
dxx_f(2.5)
```

Using these results, and perhaps experimenting a little with different values of $x$, you should be able to answer this question:

**Question A** Which of these is a correct statement of "concave up" in terms of the value of $\partial_{xx} f(x)$?

i. A function is concave-up at input $x_0$ when $\partial_{xx} f(x_0) > 0$Nice!

ii. A function is concave-up at input $x_0$ when $\partial_{xx} f(x_0) < 0$x

iii. A function is concave-up at input $x_0$ when $\partial_{xx} f(x_0) < 0$ **and** $\partial_x f(x_0) < 0$X The first derivative has nothing to do with it.

iv. A function is concave-up at input $x_0$ when $\partial_{xx} f(x_0) > 0$ **and** $\partial_x f(x_0) > 0$X The first derivative has nothing to do with it.

Recall that an ***inflection point*** is a value for the input $x$ at which $f(x)$ changes from concave up to concave down, or *vice versa*. Add a statement to your sandbox to graph $\partial_{xx} f(x)$.

**Question B** From reading the graph of $\partial_{xx} f(x)$, say which of these is nearest to an inflection point for $f(x)$.

i. $x = 0.0$x

ii. $x = -4.0$Nice! The inflection point nearest $x = -4$ occurs at $x = -4.156$.

iii. $x = 2.5$x

iv. $x = -3$x

**Question C** How many inflection points are there for $f(x)$ in the domain $-5 \leq x \leq 5$?

i. 1x

ii. 2x

iii. 3Excellent! You can see this by graphing $\partial_{xx} f(x)$ and counting the zero crossings.

iv. 4x

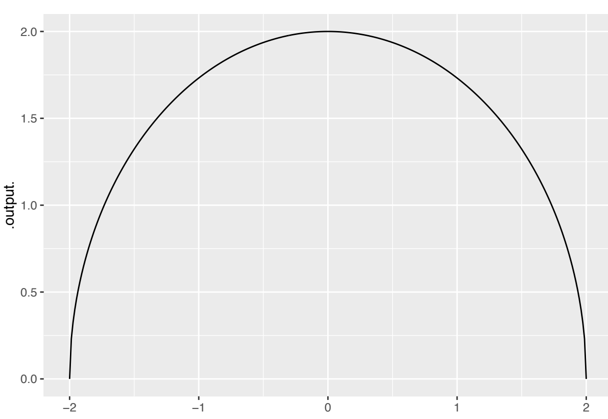
v. 5x

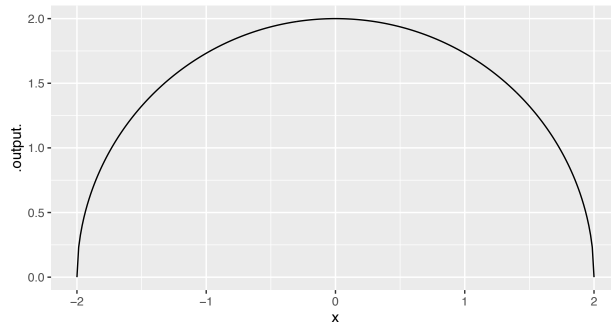
# Exer. 21.8

**Exercise 21.8** ../Differentiation/Exercises/goat-come-bed.Rmd

The graph of the function $g(x) \equiv \sqrt{R^2 - x^2}$ has the shape of a semi-circle of radius $R$, e.g.

```
g <- makeFun(sqrt(R^2 - x^2) ~ x, R=2)
slice_plot(g(x) ~ x, domain(x=-2:2), npts=300)
```




Intuition suggests that the radius of an enscribed circle for $g()$ should match the radius of the graph of the function.

In a SANDBOX, create a function to calculate the curvature of $g()$ at any input $x$. Then plot that curvature function over the domain $-2 < x < 2$. Is the curvature of $g()$ indeed constant? To help you get started, here is a bit of scaffolding for your sandbox.

---

Open an R console and ....

```
g <- makeFun(sqrt(R^2 - x^2) ~ x, R = 2) # define g()
dg  <- D(g(x) ~ x)          # first derivative of g()
ddg <- D(g(x) ~ x & x)      # second derivative of g()
curvature <- makeFun(abs(ddg(x)) / abs(__fill_in_the_fo
slice_plot(curvature(x) ~ x, domain(x=-2:2))
```

---

We set the default value of the parameter $R$ to be 2.

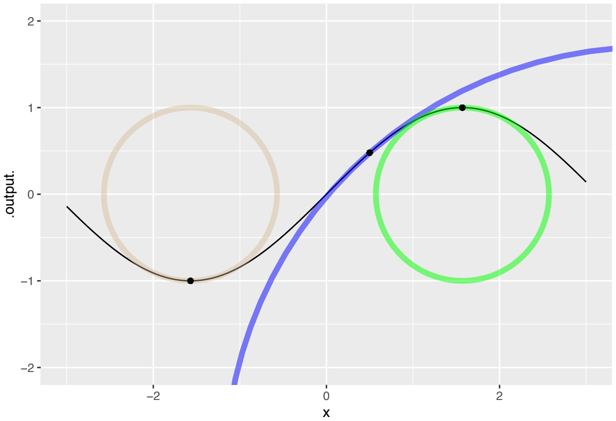
**Question A** What is the curvature of $g(x)$?

0x 0.5♡ 1X 1.5x 2x

# Exer. 21.9

**Exercise 21.9** ../Differentiation/Exercises/chicken-sleep-knife.Rmd

Here is a graph of $\sin(x)$ with points marked at $x = -\pi/2$, $x = 0.923$, and $x = \pi/2$. At each of those points, an inscribed circle has been drawn, tangent to the function at that point.




You're task is to calculate the curvature $\mathcal{K}$ at each of those three input points. This is a matter of calculating the first and second derivatives of the sine function, evaluating those derivatives at the input values, and plugging them in to the formula in **?@sec-curvature-definition**.

**Question A** What is the curvature $\mathcal{K}$ of $\sin(x = -\pi/2)$?

-1x$\mathcal{K}$ can be negative! 0X 0.5X 1♡ 2x

**Question B** What is the curvature $\mathcal{K}$ of $\sin(x = -0.923)$?

-1x$\mathcal{K}$ can be negative! 0X 0.5♡ 1X 2x

**Question C** What is the curvature $\mathcal{K}$ of $\sin(x = \pi/2)$?

-1x$\mathcal{K}$ can be negative! 0X 0.5X 1♡ 2x

**Question D** What is the curvature $\mathcal{K}$ of $\sin(x = 0)$? (Hint: You can tell straight from the graph, even though no enscribed circle has been drawn.

i. 0Excellent! The graph is straight at $x = 0$, so no curvature

ii. 0.5x

iii. 1x

iv. 2x

# Exer. 21.11

**Exercise 21.11** ../Differentiation/Exercises/boy-send-book.Rmd

DRAFT

Show that the dimension of the curvature only makes sense if the $[x]$ and $[f(x)]$ are the same.

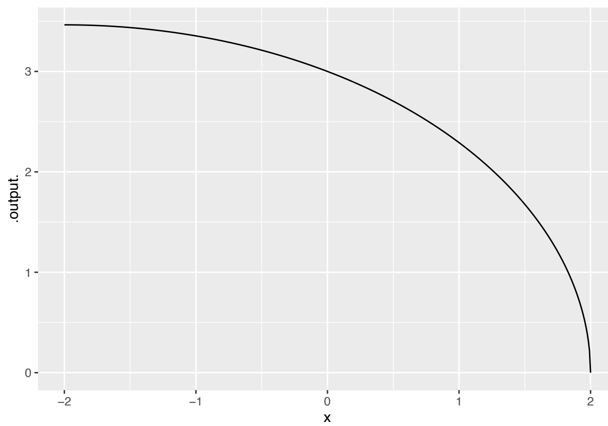
What does the 1 mean? Does it have dimension? Does it have units?

# Exer. 21.1

**Exercise 21.1** ../Differentiation/Exercises/tiger-blow-futon.Rmd

The function `road(x)` has been constructed to correspond to a curved road of gradually tighter radius from left to right

```
R <- makeFun(3 - x/2 ~ x)
road <- makeFun(sqrt(R(x)^2 - x^2) ~ x)
slice_plot(road(x) ~ x, domain(x=c(-2, 2)), npts=500)
```




Using a SANDBOX, calculate the curvature of this road for each value of $x$.

**Question A** What is the curvature of the road at $x = -1$?

0.22♡ 0.24X 0.27x 0.31X 0.35x

**Question B** What is the curvature of the road at $x = 1$?

0.22x 0.24X 0.27♡ 0.31X 0.35x

**Question C** What is the curvature of the road at $x = 0$?

0.22X 0.24♡ 0.27X 0.31x 0.35x

# Exer. 22.1

**Exercise 22.1** ../Differentiation/Exercises/spider-tug-gloves.Rmd

1. Draw the graph of a function from your imagination over the domain $-5 \leq x \leq 5$. The function should be continuous everywhere **except** at $x = -2, 1, 3$.

2. Draw the graph of a second function from your imagination over the same domain as in (1). The second function should be continuous everywhere in domain: no discontinuities. It should also have a derivative everywhere **except** at $x = -2, 1, 3$.

# Exer. 22.2

**Exercise 22.2** ../Differentiation/Exercises/moose-hears-door.Rmd

For the sketched functions below, decide what level of smoothness—$C^0, C^1, C^2, ...$— best describes the function. (We make no tricks in the drawings. Where a function looks like it's broken–that is, the function locally has a V-shape or a Λ-shape–it is meant to be broken.)

**Question A** What's the smoothness level of function A(x)? (Hint: A quadratic function has a first derivative that changes with x but a second derivative that is constant for all x.)

discontinuousx $C^0$X $C^1$♡ $C^2$ or higherx

**Question B** What's the smoothness level of function B(x)?

discontinuous♡ $C^0$X $C^1$x $C^2$ or higherx

**Question C** What's the smoothness level of function C(x)?

discontinuousX $C^0$♡ $C^1$x $C^2$ or higherx

**Question D** What's the smoothness level of function D(x)?

discontinuousx $C^0$x $C^1$X $C^2$ or higher♡

# Exer. 22.3

**Exercise 22.3** ../Differentiation/Exercises/maple-sail-boat.Rmd

The ramp function is defined algebraically as

$$\text{ramp}(x) \equiv \begin{cases} 0 & \text{for } x < 0 \\ x & \text{otherwise} \end{cases}$$

or in R as ::: {.cell}

```
ramp <- makeFun(ifelse(x<0, 0, x) ~ x)
```

:::

Evaluate these three different forms for the definition of the instantaneous rate of change at $x = 0$ using $h = 0.1$.

- version 1:

$$\mathcal{D}_x f(x) \equiv \frac{f(x+h) - f(x)}{h}$$

- version 2:

$$\mathcal{D}_x f(x) \equiv \frac{f(x) - f(x-h)}{h}$$

- version 3:

$$\mathcal{D}_x f(x) \equiv \frac{f(x+h) - f(x-h)}{2h}$$

**Question A** Do the three versions give different numerical results at $x = 0$ for $h = 0.01$?

i. They all give the same result.x

ii. Versions 1 and 3 give the same result, but 2 is different.x

iii. Versions 1 and 2 give the same result, but 3 is different.x

iv. All three are differentCorrect.

**Question B** For much smaller $h$ (say, $h = 0.0001$), do the three versions give different numerical results at $x = 0$?

Yes♡ Nox

# Exer. 22.4

**Exercise 22.4** ../Differentiation/Exercises/camel-begin-piano.Rmd

Consider this function, defined piecewise:

1. Write the R command to create this function. (Hint: Remember `ifelse` from **?@sec-fun-piecewise**.)
2. Using a SANDBOX, plot $h(x)$ over the domain $-1 \leq x \leq 1$, then sketch a copy of the graph on your paper.
3. Create the function $\partial_x h(x)$ by differentiating separately each piece of the piecewise function $h()$. Write down $\partial_x h(x)$ using mathematical notation similar to the definition of $h(x)$ given above.
4. Sketch a graph of $\partial_x h(x)$ over the domain $-1 \leq x \leq 1$. You're welcome to use a SANDBOX, but you may be able to figure out the shape of the graph yourself.
5. The shape of the function you sketched in (4) has a name, given in the text in Section @ref(continuity). What is that name?

Now you are going to do much the same as in items (3), (4), and (5), but instead of the first derivative $\partial_x h(x)$, create, sketch, and name the second derivative $\partial_{xx} h(x)$.

6. Create and write down $\partial_{xx} h(x)$ in mathematical notation.
7. Sketch $\partial_{xx} h(x)$
8. Classify the smoothness of $h(x)$ using the following table:

| Smoothness | Criterion |
|---|---|
| $C^0$ | $\partial_x h(x)$ is discontinuous |
| $C^1$ | $\partial_x h(x)$ is continuous |
| $C^2$ | $\partial_{xx} h(x)$ is continuous |
| $C^3$ | $\partial_{xxx} h(x)$ is continuous |
| $\vdots$ | and so on. |
| $C^\infty$ | All orders of derivative of $h(x)$ are continous. |

# Exer. 22.5

**Exercise 22.5** ../Differentiation/Exercises/fox-know-dress.Rmd

Consider the following functions $f_A(x)$, $f_B(x)$, ... all of which involve a domain split at $x = 0$ and the pasting together of two individually $C^\infty$ functions:

**Question A** How smooth is $f_A(x)$?

discontinuousx $C^0$X $C^1$♡ $C^2$X $C^3$x $C^\infty$x

**Question B** How smooth is $f_B(x)$?

discontinuousx $C^0$x $C^1$X $C^2$♡ $C^3$X $C^\infty$x

$$f_C(x) \equiv \begin{cases} x^3 & \text{for } 0 \leq x \\ x^3 & \text{otherwise} \end{cases}$$

**Question C** How smooth is $f_C(x)$?)

discontinuousx $C^0$x $C^1$X $C^2$x $C^3$X $C^\infty$♡

**Question D** How smooth is $f_D(x)$?

discontinuousx $C^0$X $C^1$♡ $C^2$X $C^3$x $C^\infty$x

**Question E** How smooth is $f_E(x)$?

discontinuousx $C^0$X $C^1$♡ $C^2$X $C^3$x $C^\infty$x

**Question F** How smooth is $f_F(x)$?

discontinuous♡ $C^0$X $C^1$x $C^2$x $C^3$X $C^\infty$x

**Question G** How smooth is $f_G(x)$?

discontinuousX $C^0$♡ $C^1$X $C^2$x $C^3$x $C^\infty$x

**Question H** How smooth is $f_H(x)$?

discontinuousx $C^0$x $C^1$X $C^2$♡ $C^3$X $C^\infty$x

# Exer. 23.01

**Exercise 23.01** ../Differentiation/Exercises/ant-give-room.Rmd

**?@sec-using-the-rules** explains that in differentiating a linear combination of two functions, or a product of two functions, or one function composed with another, your first task is to identify the two functions $f()$ and $g()$ involved. Second, compute the derivative of each of those functions on its own: $\partial_x f(x)$ and $\partial_x g(x)$.

Carry out these two tasks for each of the combined functions shown in the table. (The first row has been done for you as an example.)

| Combination | $f()$ | $g()$ | $\partial_x f()$ | $\partial_x g()$ |
|---|---|---|---|---|
| $e^x \ln(x)$ | $\ln(x)$ | $e^x$ | recip (that is $1/x$) | $e^x$ |
| $sin(e^x)$ | | | | |
| $x + x^2$ | | | | |
| $1/\sin(x)$ | | | | |
| $\text{pnorm}(x)^2$ | | | | |
| $\sqrt{\text{pnorm}(x)}$ | | | | |
| $\text{pnorm}(x^2)$ | | | | |
| $pnorm(sin(x))$ | | | | |

# Exer. 23.02

**Exercise 23.02** ../Differentiation/Exercises/lamb-rise-sofa.Rmd

For each of the following, say whether the function is a composition $f(g(x))$ or a product $f(x)g(x)$, or neither.

**Question A** What sort of combination is $h_1(x) \equiv \ln(x)e^x$?

product♡ compositionX neitherx

**Question B** What sort of combination is $h_2(x) \equiv \sin(x)\cos(x)$?

product♡ compositionX neitherx

**Question C** What sort of combination is $h_3(x) \equiv \sin(\ln(x))$?

productX composition♡ neitherx

**Question D** What sort of combination is $h_4(x) \equiv e^{\ln(x)}$?

productX composition♡ neitherx

**Question E** What sort of combination is $h_5(x) \equiv \sin(x) - \text{dnorm}(x)$?

productx compositionX neither♡

**Question F** What sort of combination is is $h_6(x) \equiv e^{x^2}$?

productX composition♡ neitherx

**Question G** What sort of combination is $h_7(x) \equiv \text{pnorm}(x^2)$?

productX composition♡ neitherx

**Question H** What sort of combination is $h_8(x) \equiv \text{pnorm}(x)\text{dnorm}(x)$?

product♡ compositionX neitherx

**Question I** What sort of combination is $h_9(x) \equiv 1/\sin(x)$?

i. productx

ii. compositionNice! Remember, $1/\sin(x)$ is the same as recip($\sin(x)$).

iii. neitherx