"Sessional 2"

→ **Interpolation:**

Lagrange   Newton divided   Hermite

→ **Splines:**

linear   quadratic   cubic.

→ least square approximation ✓

✓ **Numerical Integration:**

Trapezoidal   Simpsons Rule   Gauss Quadrature ✓

✓ **Numerical Differentiation.**

1st Order   2nd order. ✓

✓ **Solution of Differential Equations.**

Euler's Method   Imp Euler   RK Methods. ✓

① Lagrange Interpolation.

find the value of y when $x=10$ by Lagrange Interpol —

	x	5	6	9	11
$f(x)$	y	12	13	14	16

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) \times \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13) \times$$

$$\frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) \times \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16).$$

$\boxed{\text{put } x=10}$ ✓

(2) Newton divided difference.

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) +$$
$$(x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \_\_\_\_$$

x	5	6	9	11
$f(x) = y$	12	13	14	16

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	5	12	$\frac{13-12}{6-5} = 1$	$\frac{\frac{1}{3}-1}{9-5} = -\frac{1}{6}$	$\frac{\frac{2}{15}+\frac{1}{6}}{11-5} = \frac{1}{20}$
x_1	6	13	$\frac{14-13}{9-6} = \frac{1}{3}$	$\frac{1-\frac{1}{3}}{11-6} = \frac{2}{15}$	
x_2	9	14	$\frac{16-14}{11-9} = 1$		
x_3	11	16			

$$= 12 + (x-5)(1) + (x-5)(x-6)\left(\frac{-1}{6}\right) + (x-5)(x-6)(x-9)\left(\frac{1}{20}\right)$$

(3) Hermite interpolation.

$$P(x) = \sum_{i=0}^{n} \left[1 - 2(x-x_i)L_i'(x_i)\right]\left[L_i(x)\right]^2 f(x_i) +$$
$$\sum^{n} (x-x_i)\left[L_i(x)\right]^2 f'(x_i)$$

"Hermite Interpolation"

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f'(x_1)$$

x	1	2	3
$f(x)$	3	-10	2
$f'(x)$	0	1	5

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
1	3	0		
			-13	
1	3			27
		-13		
2	-10		14	
2	-10	1		-3
			11	
3	2	+12		
3	2	5	-7	-18

CamScanner

Numerical Integration:

① **Trapezoidal Rule** → any no of intervals.

$$\int_a^b f(x)\,dx = h\left[\frac{y_0+y_n}{2} + y_1+y_2+ \text{---} + y_{n-1}\right].$$

② **Simpson 1/3 Rule** → ~~only on even intervals.~~

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(\overset{odd}{y_1+y_3+\text{---}}) + 2(y_2+y_4+\text{---})\right].$$

Example: $\int_0^1 \frac{1}{1+x^2}\,dx$. $N=6$

$h = \frac{b-a}{N} = \frac{1-0}{6} = \frac{1}{6}$.

$x_0 = 0 = a$

$x_1 = 1/6$

$x_2 = 2/6$

$x_3 = 3/6$

$x_4 = 4/6$

$x_5 = 5/6$

$x_6 = 1 = b$.

$y_0 = \frac{1}{1+x^2} = 1$

$y_1 = 0.97$

$y_2 = 0.9$

$y_3 = 0.8$

$y_4 = 0.69$

$y_5 = 0.59$

0.5

Trapezoidal.

$$= h\left[\frac{y_0+y_6}{2} + y_1+y_2+y_3+y_4+y_5\right].$$

$$= \frac{1}{6}\left[\frac{1+0.5}{2} + 0.97+0.9+0.8+0.69+0.59\right]$$

Simpson 1/3.

$$= \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right].$$

$$= \frac{1}{18}\left[(1+0.5)+4(0.97+0.8+0.59)+2(0.9+0.69)\right]$$

(3) Gauss Quadrature

→ for $n=3$

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

→ for $n=2$

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

→ for $n=1$

$$\int_{-1}^{1} f(x)\,dx = 2f(0)$$

When limits are not b/w -1 & 1

$$\boxed{\text{put } x = \frac{a+b}{2} + \left(\frac{b-a}{2}\right)t}$$

e.g. $\int_{1}^{3} f(x)\,dx \Rightarrow x = \left(\frac{1+3}{2}\right) + \left(\frac{3-1}{2}\right)t$

$$\boxed{x = 2+t} \qquad \boxed{dx = dt}$$

$$\int_{-1}^{1} f(2+t)\,dt$$

e.g. $\int_{-1}^{1} e^{-x^2}\,dx = ?$ $\quad f(x) = e^{-x^2}$

for $n=1 \rightarrow \int_{-1}^{1} f(x)\,dx = 2\cdot f(0)$ $\qquad e^{-x^2}$

$= \int_{-1}^{1} e^{-x^2}\,dx = 2\,e^{-0^2} \Rightarrow 2.$ $\qquad e^{-\left(\frac{1}{\sqrt{3}}\right)^2} + e^{-\left(\frac{-1}{\sqrt{3}}\right)^2}$

for $n=2 \rightarrow \int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$

$= e^{-\left(\frac{-1}{\sqrt{3}}\right)^2} + e^{-\left(\frac{1}{\sqrt{3}}\right)^2}$

$= e^{-1/3} + e^{-1/3} = 2e^{-1/3}$

for $n=3 \rightarrow \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{3/5}\right).$

$\boxed{\text{limit not in range}}$

$\int_{0}^{2} e^{-t^2}\,dt = ?$ $\quad (b = 2,\ a = 0)$

$t = \frac{(a+b)}{2} + \left(\frac{b-a}{2}\right)t$

$t = \frac{2}{2} + \left(\frac{2-0}{2}\right)t$

$\boxed{t = 1+t}$ $\qquad dt = dx.$

$\int_{-1}^{1} e^{-(1+x)^2}\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$

$= e^{-\left(1+\left(\frac{-1}{\sqrt{3}}\right)\right)^2} + e^{-\left(1+\frac{1}{\sqrt{3}}\right)^2}$

$= e^{-\left(1-\frac{1}{\sqrt{3}}\right)^2} + e^{-\left(1+\frac{1}{\sqrt{3}}\right)^2}$ ✓

$0.836 + 0.083 \Rightarrow 0.919$

$=$ ~~[illegible]~~

"~~NUMERICAL DIFFERENTIATION~~"

" SOLUTION OF DIFFERENTIAL EQUATIONS " Stepsize = 5

① Euler's method:

step size

$$y_{n+1} = y_n + hf(x_n + y_n) \quad \text{①}$$

I.C $x_0 = 0$ & $y_0 = 1$

Solve $\frac{dy}{dx} = x + y$, Boundary condition $x = 0$ & $y = 1$

find y for $x = 0.1$

$$h = \frac{b-a}{5} = \frac{0.1}{5} = 0.02.$$

I.C

$x_0 = 0$	$y_0 = 1$
$x_1 = 0.02$	$y_1 = 1.02$
$x_2 = 0.04$	$y_2 = 1.0408$
$x_3 = 0.06$	$y_3 = 1.0624$
$x_4 = 0.08$	$y_4 = 1.0848$
$x_5 = 0.1$	$y_5 = 1.1081$

[for y_1 put $n = 0$ in ①]

$y_1 = y_0 + hf(x_0 + y_0)$.

$y_1 = 1 + 0.02(0+1)$

$\boxed{y_1 = 1.02}$

[for y_2 put $n = 1$ in ①.]

$y_2 = y_1 + hf(x_1 + y_1)$.

$y_2 = 1.02 + 0.02(0.02 + 1.02)$

$\boxed{y_2 = 1.0408}$

(Runge Kutta order 1)

② Improved Euler's method: (Runge Kutta order 2)

$$y^*_{n+1} = y_n + hf(x_n, y_n) \longrightarrow \text{Euler.}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$$

1.C

Example: $\frac{dy}{dx} = x^2+y \qquad y(0)=1$

find $y(0.02)$ & $y(0.04)$. $\boxed{h = 0.02}$.

$x_0 = 0 \qquad x_1 = 0.02 \qquad x_2 = 0.04$

$y_0 = 1$

$y^*_{n+1} = y_n + hf(x_n, y_n)$

$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right].$

(1) $y^*_1 = y_0 + hf(x^2_0, y_0)$

$= (1) + 0.02[(0)^2+1] \Rightarrow 1.02.$

1.E → $y_1 = y_0 + \frac{h}{2}\left[f(x^2_0, y_0) + f(x^2_1, y^*_1)\right].$

$y_1 = 1 + \frac{0.02}{2}\left[(0^2+1) + (0.02^2 + 1.02)\right].$

$\boxed{y_1 = 1.0202}.$

put n = 1

(2) $y^*_2 = y_1 + hf(x_1, y_1) = 1.0202 + 0.02(0.02^2+1.02$

$\boxed{y^*_2 = 1.0406.}$

put n=1

$y_2 = y_1 + \frac{h}{2}\left[f(x^2_1, y_1) + f(x^2_2, y^*_2)\right].$

$= 1.0202 + \frac{0.02}{2}\left[[(0.02)^2 + 1.0202] + [(0.04)^2 + 1.0406]\right]$

$\boxed{y_2 = 1.0408.}$

(3) ~~Runge-Kutta Method~~.

"Least Square Approximation"

$$y = a + bx.$$

$$\boxed{\begin{aligned} \Sigma y &= na + b\Sigma x. \\ \Sigma xy &= a\Sigma x + b\Sigma x^2 \end{aligned}}$$

$\boxed{n = 6}$

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6.

x	y	xy	x^2
1	2.4	2.4	1
2	3	6	4
3	3.6	10.8	9
4	4	16	16
6	5	30	36
8	6.	48	64

$\Sigma x = 24$ $\Sigma y = 24$ ~~$\Sigma 130$~~ ~~$\Sigma 113.2$~~

$\Sigma 113.2$ $\Sigma 130$

$6a + 24b = 24.$ — (1)

~~$24a + 113.2b = 130$~~

$24a + 130b = 113.2.$ — (2).

Solve Both.

$\boxed{a = 1.9765 \quad b = 0.5059}$

put the values in

$\boxed{y = a + bx}$

"NUMERICAL DIFFERENTIATION"

① **first order.** Suppose the value of a function at x_i & x_{i+1} are $f(x_i)$ & $f(x_{i+1})$.

→ **forward difference.**

x_{i-1} x_i x_{i+1}

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

→ **Backward difference**

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

→ **Central difference.**

$$f(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}$$

Taylor Polynomial with remainder.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \text{——}$$

→ $f(x+h) = f(x) + hf'(x) + O(h^2)$ → Truncation error

$$\boxed{f'(x) = \frac{f(x+h) - f(x)}{h}} \to \text{FD}$$

→ $f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \text{——}$

$f(x-h) = f(x) - hf'(x) + O(h^2)$.

$$\boxed{f'(x) = \frac{f(x) - f(x-h)}{h}} \to \text{BD}$$

Subtracting Both.

$$\boxed{f'(x) = \frac{f(x+h) - f(x-h)}{2h}} \to \text{CD}$$

Example: $f(x) = 2e^{1.5x}$ at $x = 3$

take $h = 0.1$.

FD $f'(x) = \dfrac{f(x+h) - f(x)}{h}$

$f'(x) = \dfrac{f(3+0.1) - f(3)}{0.1} \Rightarrow \dfrac{2e^{1.5(3.1)} - 2e^{1.5(3)}}{0.1}$

BD $f'(x) = \dfrac{f(x) - f(x-h)}{h} = \dfrac{2e^{1.5(3)} - 2e^{1.5(2.9)}}{0.1}$

CD $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} = \dfrac{2e^{1.5(3.1)} - 2e^{1.5(2.9)}}{2(0.1)}$

② 2nd order

$f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2} f''(x) + \dfrac{h^3}{6} f'''(x) + \dfrac{h^4}{24} f^{iv}(x) + \text{---}$

$f(x-h) = f(x) - hf'(x) + \dfrac{h^2}{2} f''(x) - \dfrac{h^3}{6} f'''(x) + \dfrac{h^4}{24} f^{iv}(x) + \text{---}$

Subtracting both.

$f(x+h) - f(x-h) = 2f(x) + 2\dfrac{h^2}{2} f''(x) + O(h^4)$

$$\boxed{f''(x) = \dfrac{f(x+h) + f(x-h) - 2f(x)}{h^2}}$$

Example: $f(x) = x \ln x \rightarrow f''(x)$ at $x = 1$, $h = 0.1$

$f''(x) = \dfrac{f(x+h) - f(x-h) - 2f(x)}{h^2} = \dfrac{f(1+0.1) - f(1-0.1) - 2f(1)}{(0.1)^2}$

$= \dfrac{f(1.1) - f(0.9) - 2f(1)}{0.01} = \dfrac{1.1 \ln(1.1) - 0.9 \ln 0.9 - 2\ln 1}{0.01}$

"SPLINE INTERPOLATION"

① Linear Spline:

Simply the equation of a line.

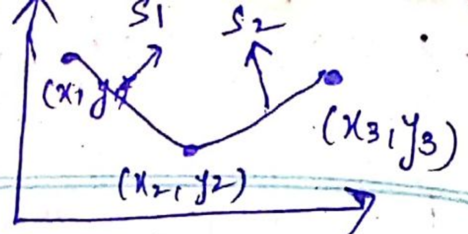
$$y = mx + c.$$

$$\boxed{P_1(x) = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)} \quad x_1 \leq x \leq x_2$$

$$\boxed{P_2(x) = y_2 + \left(\frac{y_3 - y_2}{x_3 - x_2}\right)(x - x_2).} \quad x_2 \leq x \leq x_3$$

Example : ~~x = 2, 5, 8 ; y =~~

② Quadratic Spline



→ 3 unknowns.

3 data pts → 2 splines

$$\boxed{P_i(x) = a_i x^2 + b_i x + c_i} \longleftrightarrow \text{general}$$

$$\left.\begin{aligned} P_1(x) &= a_1x^2 + b_1x + c_1 \\ \& \quad P_2(x) &= a_2x^2 + b_2x + c_2 \end{aligned}\right] \to 6 \text{ unknowns.}$$

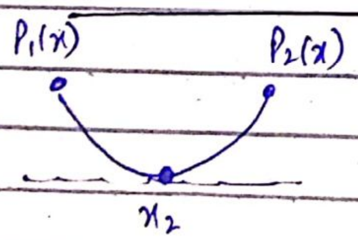
(x_1, y_1) (x_2, y_2) (x_3, y_3).

→ Process of finding the unknowns:

① Evaluation at known pts: → gives 2 eq per splines

$$\left.\begin{aligned} P_1(x_1) &= y_1 & \quad P_1(x_2) &= y_2 \\ P_2(x_2) &= y_2 & \quad P_2(x_3) &= y_3 \end{aligned}\right] \text{4 unknowns aa jayengy.}$$

② Smoothness at interior pts:



$$\left.\frac{d P_1(x_2)}{dx} = \frac{d P_2(x_2)}{dx_2}\right] \text{5th equation}$$

③ Assumption:

we assume 2nd derivative of our first spline to be zero.

$$\left.f''(P_1(x_1)) = 0.\right] \text{6th equation}$$

Example:

$\boxed{\text{Step \#01} \quad \text{Writing polynomial equations}}$

$$P_1(x) = a_1x^2 + b_1x + c_1$$
$$P_2(x) = a_2x^2 + b_2x + c_2$$
$$P_3(x) = a_3x^2 + b_3x + c_3.$$

	x	y	
x_1	1	2	y_1
x_2	3	3	y_2
x_3	5	9	y_3
x_4	8	10.	y_4

$\boxed{\text{Step \#02} \quad \text{Evaluation @ known pts}}$

P_1

$$P_1(x_1) = y_1 = a_1(1)^2 + b_1(1) + c_1 = 2.$$
$$\boxed{a_1 + b_1 + c_1 = 2.} \longrightarrow ①$$

$$P_1(x_2) = y_2 = 3 = a_1(3)^2 + b_1(3) + c_1$$
$$\boxed{9a_1^2 + 3b_1 + c_1 = 3.} \longrightarrow ②$$

P_2

$$P_2(x_2) = y_2 = 3 = a_2(3)^2 + b_2(3) + c_2$$
$$\boxed{9a_2 + 3b_2 + c_2 = 3.} \longrightarrow ③$$

$$P_2(x_3) = y_3 = 9 = a_2(5)^2 + b_2(5) + c_2$$
$$\boxed{25a_2 + 5b_2 + c_2 = 9} \longrightarrow ④$$

P_3

$$P_3(x_3) = y_3 = \boxed{9 = 25a_3 + 5b_3 + c_3} \longrightarrow ⑤$$

$$P_3(x_4) = y_4 = \boxed{10 = 64a_3 + 8b_3 + c_3.} \longrightarrow ⑥.$$

$\boxed{\text{Step \#03} \quad \text{Smoothness at interior data pts.}}$ $(x_2 \,\&\, x_3)$

$$\left.\frac{dP_1(x)}{dx}\right|_{x=x_2} = \left.\frac{dP_2(x)}{dx}\right|_{x=x_2} \longrightarrow \text{for } x_2$$

$$\left.\frac{d}{dx}(a_1x^2 + b_1x + c_1)\right|_{x=x_2} = \left.\frac{d}{dx}(a_2x^2 + b_2x + c_2)\right|_{x=x_2}.$$

$$(2a_1x_1 + 2b_1)\Big|_{x=x_2} = (2a_2x_2 + 2b_2)\Big|_{x=x_2}.$$

$$2a_1(3) + b_1 = 2a_2(3) + b_2$$

$$6a_1 + b_1 = 6a_2 + b_2.$$

$$\boxed{6a_1 + b_1 - 6a_2 - b_2 = 0} \longrightarrow (7).$$

→ for x_3

$$\frac{d}{dx} P_2(x)\Big|_{x=x_3} = \frac{d}{dx} P_3(x)\Big|_{x=x_3}.$$

$$\frac{d}{dx}(a_2x^2 + b_2x + c_2)\Big|_{x=x_3} = \frac{d}{dx}(a_3x_3^2 + b_3x + c_3)\Big|_{x=x_3}.$$

$$(2a_2x + 2b_2)\Big|_{x=x_3} = (2a_3x + 2b_3)\Big|_{x=x_3}.$$

$$2a_2x_3 + 2b_2 = 2a_3x_3 + 2b_3.$$

$$2a_2(5) + 2b_2 = 2a_3(5) + 2b_3$$

$$10a_2 + 2b_2 = 10a_3 + 2b_3$$

$$\boxed{10a_2 + 2b_2 - 10a_3 - 2b_3 = 0.} \longrightarrow (8)$$

Step #04 | Assumption that 2nd derivative @ $x_1 = 0$ |

$$\frac{d^2}{dx^2}(P_1(x))\Big|_{x=x_1} = 0.$$

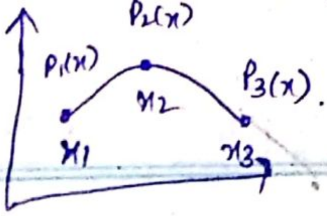
$$\frac{d}{dx}(2a_1x + 2b_1)\Big|_{x=x_1} = 0.$$

$$2a_1x\Big|_{x=x_1} = 0 \longrightarrow 2a_1x_1 = 0.$$

$$2a_1(1) = 0.$$

$$\boxed{a_1 = 0.} \longrightarrow (9)$$

(3) Cubic Spline.



3rd order polynomials.

$P_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1.$

8 unknowns.

$P_2(x_2) = a_2x^3 + b_2x^2 + c_2x + d_2.$

Step#01 Evaluation at known data pts.

$P_1(x_1) = y_1 \quad P_2(x_2) = y_2 \quad P_3(x_2) = y_3 \quad P_3(x_3) = y_3.$

Step#02. Smoothness at interior pts → excluding extreme pts.

$P_1'(x_2) = P_2'(x_2)$

Step#03 Put the 2nd derivatives at interior pts equal.

$P_1''(x_2) = P_2''(x_2)$

Step#04 Assumptions about the end pts.

Assume $P_1''(x_1) = 0$ and $P_2''(x_3) = 0$.

Example:

Step#01

$P_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1$
$P_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2$
$P_3(x) = a_3x^3 + b_3x^2 + c_3x + d_3$

x	y.
1	2
3	3
5	9
8.	10

P_1 ① ② ③

Step#02 Solving at known pts.

$P_1(x_1) = y_1 \rightarrow \boxed{a_1 + b_1 + c_1 + d_1 = 2} \rightarrow (1).$
$P_1(x_2) = y_2 \rightarrow \boxed{27a_1 + 9b_1 + 3c_1 + d_1 = 3} \rightarrow (2).$
$P_2(x_2) = y_2 \rightarrow \boxed{27a_2 + 9b_2 + 3c_2 + d_2 = 3} \rightarrow (3).$
$P_2(x_3) = y_3 \rightarrow \boxed{125a_2 + 25b_2 + 5c_2 + d_1 = 9} \rightarrow (4)$
$P_3(x_3) = y_3 \rightarrow \boxed{125a_3 + 25b_3 + 5c_3 + d_3 = 9} \rightarrow (5).$
$P_3(x_4) = y_4 \rightarrow \boxed{512a_3 + 64b_3 + 8c_3 + d_3 = 10} \rightarrow (6)$

Step#03 Interior Pts. first order. Der

$\frac{d}{dx} P_1(x) = 3a_1x^2 + 2b_1x + c_1$

$\frac{d}{dx} P_2(x) = 3a_2x^2 + 2b_2x + c_2$

$\frac{d}{dx} P_3(x) = 3a_3x^2 + 2b_3x + c_2$.

$\rightarrow P_1'(x_2) = P_2'(x_2)$.

$3a_1x^2 + 2b_1x + c_1 \Big|_{x=x_2} = 3a_2x^2 + 2b_2x + c_2 \Big|_{x=x_2}$

$\boxed{-27a_1 - 6b_1 - c_1 + 27a_2 + 6b_2 + c_2 = 0} \rightarrow (7)$.

$\rightarrow P_2'(x_3) = P_2'(x_3)$

$3a_2x^2 + 2b_2x + c_2 \Big|_{x=x_3} = 3a_3x^2 + 2b_3x + c_2 \Big|_{x=x_2}$

$\boxed{\quad} \rightarrow (8)$.

Now, for 2nd order.

$P_1''(x) = 6a_1x + 2b_1$

$P_2''(x) = 6a_2x + 2b_2$

$P_3''(x) = 6a_3x + 2b_2$.

$\rightarrow P_1''(x_2) = P_2''(x_2)$.

$\boxed{\quad} \rightarrow (9)$.

$\rightarrow P_2''(x_3) = P_3''(x_3)$

$\boxed{\quad} \rightarrow (10)$.

Step #04 End pts assumptions.

Put double derivative of the extreme pts = 0.

$$P_1''(x_1) = 0 \quad (11)$$

$$\boxed{P_3''(x_4) = 0} \quad (12)$$

$P_1(x_1) = y_1$ $P_1(x_2) = y_2$

"CHAPTER #09"

Gauss Elimination.
LU factorization.
Cholesky factorization.
Jacobi Method.
Gauss-Siedel method.

① Gauss Elimination (Direct Method).

$x - y + 2z = 3$
$x + 2y + 3z = 5$
$3x - 4y - 5z = -13$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}$$

Augmented Matrix → Row Reduction.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 3 & -4 & -5 & -13 \end{array}\right] \xleftrightarrow{R} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -32 & -64 \end{array}\right]$$

$x - y + 2z = 3$
$3y + z = 2$
$-32z = -64$

$\boxed{x = 1}$ $\boxed{y = 0}$ $\boxed{z = 2}$

② LU factorization.

$$x + 5y + z = 14$$
$$2x + y + 3z = 13$$
$$3x + y + 4z = 17$$

$$\underset{A}{\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}}.$$

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = LU$$

L → lower triangular.
U → upper triangular.

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12}+U_{22} & l_{21}U_{13}+U_{23} \\ l_{31}U_{11} & l_{31}U_{12}+l_{32}U_{22} & l_{31}U_{13}+l_{32}U_{23}+U_{33} \end{bmatrix}.$$

$U_{11} = 1$ $\quad U_{12} = 5$ $\quad U_{13} = 1.$

$l_{21} = 2$

$(2)(5) + U_{22} = 1$

$\boxed{U_{22} = -9}$

$2 + U_{23} = 3$

$\boxed{U_{23} = 1}$

$\boxed{l_{31} = 3.}$

$(3)(5) + l_{32}(-9) = 1$

$15 - 9l_{32} = 1$

$-9l_{32} = -14$

$\boxed{l_{32} = 14/9.}$

$(3)(1) + \left(\frac{14}{9}\right)(1) + U_{33} = 4$

$3 + \frac{14}{9} + U_{33} = 4$

$U_{33} = 4 - 3 - \frac{14}{9}$

$U_{33} = 1 - \frac{14}{9}$

$\boxed{U_{33} = -5/9}$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 14/9 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix}.$$

$$A = LU.$$

We know that $AX=B$. & $A = LU$.

So, $LUX = B$

put $UX = Y$.

<u>finding Y.</u>

AX = B

LUX = B

$$LY = B$$

LY = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 14/9 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

UX = Y

A = LU

AX = B

LUX = B

LY = B

$$y_1 = 14$$
$$2y_1 + y_2 = 13.$$
$$3y_1 + \frac{14}{9}y_2 + y_3 = 17$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix}$$

<u>Now</u>, <u>finding X</u>

UX = Y

using $UX = Y$.

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix}.$$

$$x + 5y + z = 14$$
$$-9y + z = -15$$
$$-5/9\, z = \frac{-5}{3}.$$

$$\boxed{x = 1 \quad y = 2 \quad z = 3.}$$

③ "Cholesky factorization"

Matrix A should be:

① symmetric
② positive definite.

$AX = B$.

Example:

$$4x + 2y + 14z = 14$$
$$2x + 17y - 5z = -101$$
$$14x - 5y + 83z = 155$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}.$$

we know that $A = LU$.

when A is symmetric & +ve definite.

put $U = L^T$.

then $= A = LL^T$

put $A = LL^T$ in $AX = B$.

$LL^T X = B$.

put $L^T X = Y$.

then $\boxed{LY = B}$ ⟶ solve it for Y.

$L^T X = Y$. ⟶ solve it for X.

$$A = L L^T$$

$$\begin{array}{ccc} & L^T & L \\ \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = & \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{21} & l_{22} & l_{32} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} & \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{21} & l_{22} & l_{32} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

$$l_{11}^2 = 4 \longrightarrow l_{11} = 2$$

$$l_{11}l_{21} = 2 \longrightarrow l_{21} \; 1$$

$$l_{11}l_{31} = 14 \longrightarrow l_{31} = 7$$

$$l_{21}^2 + l_{22}^2 = 17 \longrightarrow l_{22} = \sqrt{17 - l_{21}^2}$$

$$= \sqrt{17 - 1} = 4$$

$$l_{21}l_{31} + l_{22}l_{32} = -5 \longrightarrow (1)(7) + (4)\,l_{32} = -5$$

$$7 + 4l_{32} = -5$$

$A = LU \quad U = L^T$

$A = LL^T$

$$-12 = 4\,l_{32}$$

$$\boxed{l_{32} = -3}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 83$$

$$l_{33} = \sqrt{83 - l_{31}^2 - l_{32}^2}$$

$$l_{33} = \sqrt{83 - 7 - (-3)^2}$$

$$= \sqrt{83 - 49 + 9} = \sqrt{25}$$

$$\boxed{l_{33} = 5}$$

$AX = B$

$LL^T X = B$

$LY = B \quad L^T X = Y$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \overset{L}{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}} \overset{L^T}{\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}}$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}.$$

$A = LL^T$

$AX = B$ in ~~AB = X~~ $AX = B$.

put $A = LL^T$

$A = LL^T$

$ALL^TX = B$ $\quad LL^TX = B$.

put $L^TX = Y$.

$LY = B$

$L^TX = Y$ $\quad LY = B$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}.$$

$2y_1 = 14$

$y_1 + 4y_2 = -101$

$7y_1 - 3y_2 + 5y_3 = 155$ $\quad$ find $y_1\ y_2\ y_3$.

Then Use $L^TX = Y$ To find X.

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow \text{value}$$

find $x\ y\ z$.

④ Jacobi Method (iterative Eqn Solving)

$27x + 6y - z = 85$
$6x + 15y + 2z = 72$
$x + y + 54z = 110.$

$|27| > |6| + |-1|$
$|15| > |6| + |2|.$
$|54| > |1| + |1|.$

* kisi ek ka coefficient baki 2 ke coefficient sy hamesha bara hoga.

Jiska coefficient bara hy usko left side py rakh lein gy.

$x = \frac{1}{27}(85 - 6y + z)$

$y = \frac{1}{15}(72 - 6x - 2z).$

$z = \frac{1}{54}(110 - x - y).$

$x_0 = 0 \quad y_0 = 0 \quad z_0 = 0.$

$\boxed{\text{first iteration.}}$

$x_1 = \frac{1}{27}(85) = 3.148 \longrightarrow x_1$

$y_1 = \frac{72}{15} = 4.8 \longrightarrow y_1$

$z_1 = \frac{110}{54} = 2.037. \longrightarrow z_1$

$\boxed{\text{2nd iteration.}}$

$x_2 = \frac{1}{27}(85 - 6(4.8) + 2.037) = 2.157$

$y_2 = \frac{1}{15}(72 - 6(3.148) - 2(2.037)) = 3.269$

$z_2 = \frac{1}{54}(110 - 3.148 - 4.8) = 1.89$

$\boxed{\text{Third iteration.}}$

$x_3 = \frac{1}{27}(85 - 6(3.269) + 1.89)$

$y_3 = \frac{1}{15}(72 - 6(2.157) - 2(1.89)).$

$z_4 = \frac{1}{54}(110 - 2.157 - 3.269)$

repeat until you get repetitive values

(5) Gauss Siedel. Method.

$2x_1 - x_2 + 0x_3 = 7$
$-x_1 + 2x_2 - x_3 = 1$
$0x_1 - x_2 + 2x_3 = 1$

(*) Wohi pichli baar ki tarah coefficients wala kaam.

$x_1 = \frac{1}{2}[7 + x_2]$.

$x_2 = \frac{1}{2}[1 + x_1 + x_3]$.

(*) Last iteration me latest calculated value use hoti hai.

$x_3 = \frac{1}{2}[1 + x_2]$.

$\boxed{\text{First iteration.}}$

$x_1 = 0 \quad x_2 = 0 \quad x_3 = 0$

~~$x_0 = 0 \quad y_0 = 0 \quad z_0 = 0$.~~

$x_1 = 7/2 = 3.5$

$x_2 = \frac{1}{2}[1 + 3.5 + 0] = 2.25$

$x_3 = \frac{1}{2}[1 + 2.25] = 1.625$

$\boxed{\text{2nd iteration.}}$

$(x_1)_2 = \frac{1}{2}[7 + 2.25] = 4.625$

$(x_2)_2 = \frac{1}{2}[1 + 4.625 + 1.625] = 3.625$

$(x_3)_2 = \frac{1}{2}[1 + 3.625] = 2.3125$

repeat until you get repetitive values.

"CHAPTER #10"

linear least squares → hw 20
Gram-Schmidt → hw 17 & 18, 19
QR factorization → hw 21
Singular Value Decomposition → hw 27 & 28

① Linear Least Squares

$$(A^T A)\hat{x} = A^T b$$

we are given matrix A & b.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\to A^T A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\to A^T b = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$

Now, As $(A^T A)\hat{x} = A^T b$.

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$

$5x_1 + 3x_2 = 0 \quad \to x_1 = -3, \; x_2 = 5$
$3x_1 + 3x_2 = 6.$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

(b) To find the error:

$\|A\hat{x} - b\| = ?$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}.$$

$$= \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \sqrt{(-1)^2 + (2)^2 (-1)^2} = \sqrt{6}.$$

② $\boxed{\text{Gram Schmidt}}$:

~~$v_1 = u_1$~~

~~$v_2 = u_2 - \text{Proj}_{v_1} u_2$~~

~~$v_3 =$~~

$U_1 = V_1$

$U_2 = V_2 - \text{Proj}_{U_1} V_2$

$U_3 = V_3 - \text{Proj}_{U_1} V_3 - \text{Proj}_{U_2} V_3$.

$$U_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad U_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

$\rightarrow V_1 = U_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$

$\rightarrow V_2 = U_2 - \text{Proj}_{V_1} U_2 \rightarrow U_2 - \dfrac{U_2 V_1}{V_1 \cdot V_1} V_1$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}.$$

$$V_3 = U_3 - \mathrm{Proj}_{V_1} U_3 - \mathrm{Proj}_{V_2} U_3 .$$

$$\mathrm{Proj}_{V_1} U_3 = \frac{U_3 V_1}{V_1 . V_1} V_1 = \frac{\begin{pmatrix}1\\1\\2\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}} \begin{pmatrix}1\\-1\\1\end{pmatrix} = \frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

$$\mathrm{Proj}_{V_2} U_3 = \frac{U_3 V_2}{V_2 . V_2} * V_2 = \frac{\begin{pmatrix}1\\1\\2\end{pmatrix}\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix}}{\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix}\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix}} . \begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} = \frac{5}{2}\begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix}.$$

Now,

$$V_3 = \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} - \frac{5}{2}\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} . \Rightarrow \begin{pmatrix}-1/3\\0\\1/3\end{pmatrix}.$$

$$\|V_1\| = \sqrt{3}.$$
$$\|V_2\| = \sqrt{6}/3$$
$$\|V_3\| = \sqrt{2}/9$$

~~Orthogonal~~ Orthonormal Basis : $\left\{ \begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}, \begin{pmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}, \begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} \right\}$..

Projection Matrix : $\dfrac{V_1 V_1^T}{V_1^T V_1} + \dfrac{V_2 V_2^T}{V_2^T V_2} + \dfrac{V_3 V_3^T}{V_3^T V_3}.$

③ QR factorization.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad 0+2+2 = 4 \neq 0.$$

$A = QR.$

$Q \rightarrow$ Orthonormal matrix A from Gram Schmidt process.

$R = Q^T A.$

$$U_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

$$V_1 = U_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

$$V_2 = U_2 - \mathrm{Proj}_{V_1} U_2 \left(\frac{U_2 V_1}{V_1 V_1} V_1 \right) = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$V_2 = \begin{bmatrix} 1 \\ 6/5 \\ -3/5 \end{bmatrix} \qquad V_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$\|V_1\| = \sqrt{5}$$

$$\|V_2\| = \sqrt{1 + (6/5)^2 + (-3/5)^2} = \sqrt{70}/5$$

$$Q = \begin{bmatrix} 0 & 5/\sqrt{70} \\ 1/\sqrt{5} & 6/\sqrt{70} \\ 2/\sqrt{5} & -3/\sqrt{70} \end{bmatrix}.$$

$$R = Q^T A.$$

$$R = \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 5/\sqrt{70} & 6/\sqrt{70} & -3/\sqrt{70} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 + 1/\sqrt{5} + 2(2/\sqrt{5}). & 2(1/\sqrt{5}) + 2/\sqrt{5} \\ 6/\sqrt{70} - 6/\sqrt{70} & 5/\sqrt{70} + \frac{12}{\sqrt{70}} - \frac{3}{\sqrt{70}} \end{bmatrix}$$

$$R = \begin{bmatrix} 5/\sqrt{5} & 4/\sqrt{5} \\ 0 & 14/\sqrt{70} \end{bmatrix}.$$ $A = U\Sigma V^T$

(4) $\boxed{\text{Singular Value Decomposition.}}$

$$\boxed{A = U\Sigma V^T}.$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2\times3}. \qquad A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\underset{2\times3}{A} = \underset{2\times2}{U}\ \underset{2\times3}{\Sigma}\ \underset{3\times3}{V^T}.$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, $(A^TA - \lambda I) = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

Expanding R_1,

$$1-\lambda \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 0 & 1-\lambda \end{vmatrix} + 0 = 0.$$

$$= 1-\lambda(1-\lambda)^2 - (1-\lambda) = 0$$
$$(1-\lambda)^3 - 1 + \lambda = 0.$$

$$\boxed{\lambda = 1, 2, 0.}$$

When $\lambda = 0$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ By row reducing. $R_1 \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$

$x_1 + x_2 = 0.$

$x_3 = 0.$

$x_1 = -x_2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} V_1.$$

$$\boxed{\|V_1\| = \sqrt{2}}$$

when $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0.$

$x_2 = 0.$

$x_3 = x_3.$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} v_2.$$

$\boxed{\text{Norm of } v_2 = 1}$.

when $\lambda = 2$.

Runge Kutta of order 4.

$$K_1 = hf(x_n, y_n).$$

$$K_2 = hf(x_n + h/2, y_n + K_1/2)$$

$$K_3 = hf(x_n + h/2, y_n + K_2/2).$$

$$K_4 = hf(x_n + h, y_n + K_3).$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4).$$

$$y_{n+1} = y_n + K.$$

question.

↑ ① **Bisection Method**

$f(x) = 0$ $\quad f(a) \cdot f(b) < 0.$

$$f(x) = x^3 - x - 1 = 0.$$

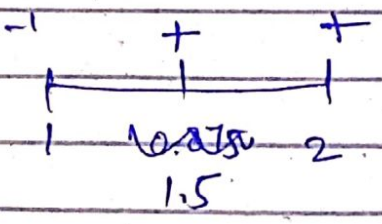
Take the initial intervals $[a, b]$ such that

$$f(a) \cdot f(b) < 0.$$

Therefore, $[1, 2]$.

$f(1) = -1 \qquad f(2) = 4$

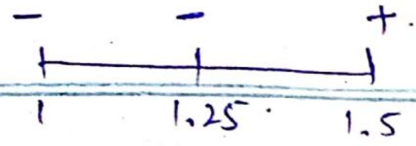
→ first iteration. $c_1 = \frac{1+2}{2} \to 1.5$



$$f(1.5) = (1.5)^3 - (1.5) - 1$$

$$f(1.5) = 0.875 > 0$$

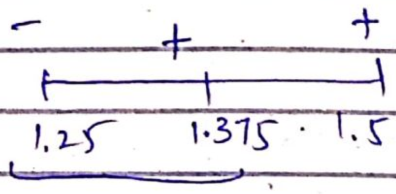
2nd iteration. $[1, 1.5]$.



$$C_2 = \frac{1+1.5}{2} = 1.25$$

$$f(1.25) = (1.25)^3 - (1.25) - 1 = -0.2968 < 0$$

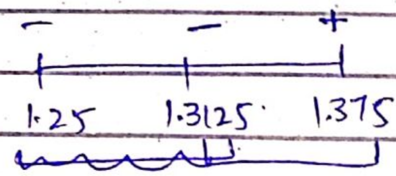
3rd iteration. $[1.25, 1.5]$.



$$C_3 = \frac{1.25+1.5}{2} = 1.375$$

$$f(1.375) = 0.2246 > 0$$

4th iteration. $[1.25, 1.375]$.



$$C_4 = \frac{1.25 + 1.375}{2} = 1.3125$$

$$f(1.3125) = -0.0515$$

How to find the iterations.

Approximation value $= 10^{-3}$ → given logi.

$$\frac{1}{2^n} < 10^{-3}$$

$$2^{-n} < 10^{-3}$$

$$\log 2^{-n} < \log 10^{-3}$$

$$-n \log 2 < -3 \log 10$$

$$-n < \frac{-3}{\log 2}$$

$$n > 9.96 \simeq 10$$

② Fixed point iteration Method.

$f(x) = x^2 + 4x - 5 = 0 \longrightarrow$ converges when $|g'(x)| < 1$

→ we have to write this in terms of x.

$4x = 5 - x^2$

$x = \frac{5 - x^2}{4} \longrightarrow x_{n+1} = g(x_n)$.

$x_{n+1} = \frac{5 - x_n^2}{4}$

$\boxed{\text{put } x_0 = 0.}$ put $n=0$

$x_1 = \frac{5}{4} = 1.25$

put $n=1$.

$x_2 = \frac{5 - x_1^2}{4} = \frac{5 - (1.25)^2}{4} = 0.8593$.

put $n=2$

$x_3 = \frac{5 - x_2^2}{4} = \frac{5 - (0.8593)^2}{4} = 1.0654$.

put $n=3$

$x_4 = \frac{5 - x_3^2}{4} = \frac{5 - (1.0654)^2}{4}$.

(3) Newton Raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

let $f(x) = x^3 + x - 1.$

$f'(x) = 3x^2 + 1.$

(1) put $x_0 = 0.$

$$\rightarrow \boxed{x_1 = 0 - \frac{(-1)}{1} = 1.}$$

(2) put $x_1 = 1.$

$$\rightarrow x_2 = x_1 - \frac{(x^3 + x - 1)}{(3x^2 + 1)}.$$

→ stops whenever two iterations are same

$$x_2 = 1 - \frac{(1^3 + 1 - 1)}{3(1)^2 + 1}$$

$$\boxed{x_2 = 1 - \frac{1}{4} \Rightarrow 3/4.}$$

(3) put $x_2 = 3/4 = 0.75$

$$x_3 = x_2 - \frac{(x^3 + x - 1)}{3x^2 + 1}$$

$$= 0.75 - \frac{(0.75^3 + 0.75 - 1)}{3(0.75)^2 + 1}$$

$$\boxed{x_3 = 0.6861}$$

(4) Secant method $\qquad$ $f(x_1) \cdot f(x_2)$ Not necessarily should < 1

$$x_{n+1} = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}$$

→ Convergence is not guaranteed

→ We need two initial guesses.

e.g. $\quad x^3 - x - 1 = 0$

$x_0 = 1 \qquad x_1 = 2$

$$x_{n+1} = \frac{x_n f x_{n-1} - x_{n-1} f x_n}{f x_{n-1} - f x_n}$$

$f(x_0) = f(1) = -1 \qquad$ put $n=0$

$f(x_1) = f(2) = 5 \qquad$ put

$$x_2 = \frac{x_1 f x_0 - x_0 f x_1}{f x_0 - f x_1} \qquad \boxed{f(x_2) = -0.5786}$$

$$= \frac{2(-1) - (1)(5)}{-1-5} = \frac{-2-5}{-6} = 7/6$$

$\boxed{= 1.1667}$

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{(1.1667)(5) - (2)(-0.5786)}{5 - (-0.5786)}$$