Chapter-08 Numerical Differentiation

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tag	

▼ First and Second Order Difference Approximations

Forward Differences

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

Steps:

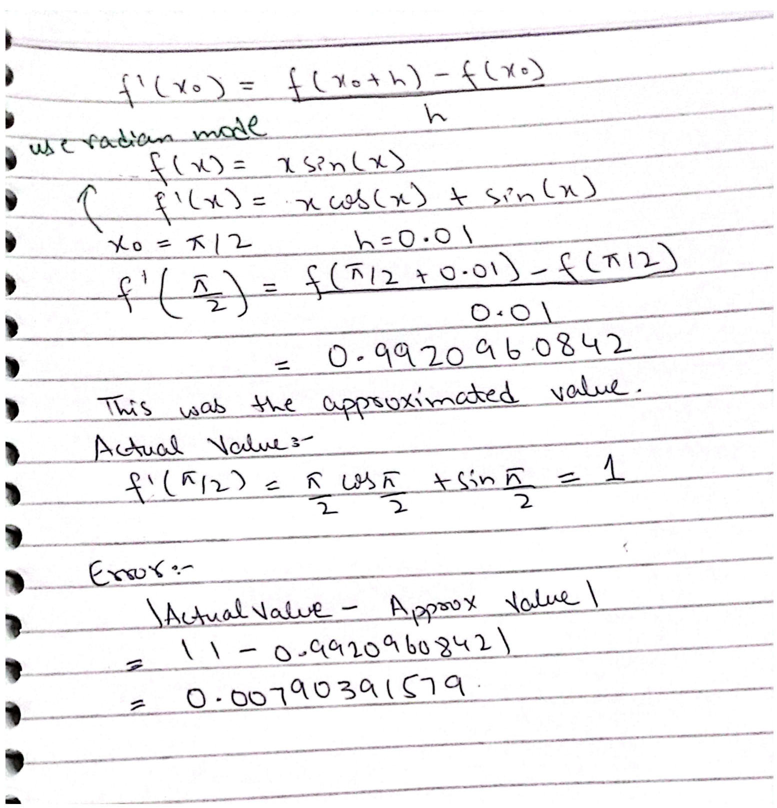
- Find Approximate Value using above formula
- Find Actual Derivative Value
- Compute the Error using |Actiual Value - Aprrox Value|

```
import numpy as np
def f(x):
    y = x*np.sin(x)
    return y

x0 = np.pi/2

h = 0.01

derivative = ( f(x0+h) - f(x0) ) / h
print('df(x) =',derivative, 'Error =',np.abs(derivative-1.0)
```

df(x) = 0.992096084232319 Error = 0.007903915767681013

> The smaller the h, the lesser the error, the closer the approximation!

if h=0.001 → Error = 0.0007858980980426367



$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

use radian mode

$f(x) = x \sin(x)$

$f'(x) = x \cos(x) + \sin(x)$

$x_0 = \pi/2 \qquad h = 0.01$

$$f'\left(\frac{\pi}{2}\right) = \frac{f(\pi/2 + 0.01) - f(\pi/2)}{0.01}$$

$= 0.9920960842$

This was the approximated value.

Actual Value:-

$$f'(\pi/2) = \frac{\pi}{2} \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$$

Error:-

$|\text{Actual Value} - \text{Approx Value}|$

$= |1 - 0.9920960842|$

$= 0.00790391579$

Backward Differences

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h}$$

```
x0 = np.pi/2
h = 0.01

derivative = ( f(x0) - f(x0-h) ) / h
print('df(x) =',derivative, 'Error =',np.abs(derivative-1.0))
```

df(x) = 1.0078039166010244 Error = 0.007803916601024419

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h}$$

$$f'(\pi/2) = \frac{f(\pi/2) - f(1.560796327)}{0.01}$$

$$= 1.007803917$$

Error:-

$$= 1 - 1.007803917$$

$$= 0.0078039 \sim$$

Central Differences/ Second Order Finite Difference

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

```
x0 = np.pi/2.0
h = 0.01
```

```
derivative = (f(x0+h) - f(x0-h))/(2.0*h)
print('df(x)=',derivative, 'Error=',np.abs(derivative-1.0))
```

df(x)= 0.9999500004166717 Error= 4.999958332829735e-05

> Observe that for h=0.01 the forward and backward differences give the derivative with 2 correct decimal digits, while the central difference approximation gives the derivative with 4 correct decimal digits. The reason for that is that the central difference approximation has convergence rate 2 while for the other approximations is 1.

$$f'(\pi/2) = \frac{f(\pi/2 + 0.01) - f(\pi/2 - 0.01)}{2(0.01)}$$

$$= 0.99995000004$$

$$\text{Error} = 4.9999 \times 10^{-5}$$

▼ Second-Order Derivatives

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + \mathcal{O}(h^2)$$

$f''(x) = 2\cos(x) - x\sin(x)$

$f''(\pi/2) = -\pi/2$

```
x0 = np.pi/2
h = 0.01

derivative2 = (f(x0+h) - 2.0*f(x0) + f(x0-h))/(h**2)
print('d2f(x) =',derivative2, 'Error =',np.abs(derivative2-(-
```

d2f(x) = -1.5707832368705432 Error = 1.3089924353337778e-05

▼ Richardson's Extrapolation

Forward Difference

```
#forward
x0 = np.pi/2
h = 0.01

d1 = (f(x0+h) - f(x0))/h
d2 = (f(x0+h/2.0) - f(x0))/(h/2.0)

derivative = 2.0*d2-d1
print('df(x) =',derivative, 'Error =',np.abs(derivative-1.0))
```

$$f'(x_0, h_1) = \frac{f(x_0 + h_1) - f(x_0)}{h_1}$$

$$f'(x_0, h_2) = \frac{f(x_0 + h_2) - f(x_0)}{h_2}$$

$h_1 = 0.01 \qquad h_2 = 0.005$

$$f'(\pi/2, 0.01) = \frac{f(\pi/2 + 0.01) - f(\pi/2)}{0.01}$$

$$= 0.9920960842$$

$$f'(\pi/2, 0.005) = \frac{f(\pi/2 + 0.005) - f(\pi/2)}{0.005}$$

$$= 0.9960605174$$

$$f'(\pi/2) = 2f'(\pi/2, 0.005) - f'(\pi/2, 0.01)$$

$$= 1.000024951$$

Backward Difference

$$f'(\pi/2, 0.01) = \frac{f(\pi/2) - f(\pi/2 - 0.01)}{0.01}$$

$$= 1.007803917$$

$$f'(\pi/2, 0.005) = \frac{f(\pi/2) - f(\pi/2 - 0.005)}{0.005}$$

$$= 1.003914483$$

$$f'(\pi/2) = 2f'(\pi/2, 0.01) - f'(\pi/2, 0.005)$$

$$= 1.000025048.$$

Central Difference

$$f'(\pi/2, 0.01) = 0.9999500004$$

$$f'(\pi/2, 0.005) = \frac{f(\pi/2 + 0.005) - f(\pi/2 - 0.005)}{2(0.005)}$$

$$= 0.9999875$$

$$f'(\pi/2) = 2f'(\pi/2, 0.005) - f'(\pi/2, 0.01)$$

$$= 1.000025$$

FORMULA

$$f'(x_0) \approx \frac{f(x_0 + h) + f(x_0 + h/2) - 2f(x_0)}{h}$$

O(h^2)

💡 above formula not working?????

$$D = \frac{2^p d(h_1/2) - d(h_1)}{2^p - 1}$$

Taking p=1

$$D = 2d(h_1/2) - d(h_1)$$