Serial No:

Final Exam

Total Time: 3 Hours

Total Marks: 110

CS-2008 Numerical Computing

BS(CS)

Tuesday, May 30, 2023

Course Instructors

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Signature of Invigilator

Asadullah Nawaz — Student Name | 20I-0761 — Roll No | C — Section | Asad — Signature

DO NOT OPEN THE QUESTION BOOK OR START UNTIL INSTRUCTED.

Instructions:

1. Attempt on question paper. Attempt all of them. Read the question carefully, understand the question, and then attempt it.
2. No additional sheet will be provided for rough work. Use the back of the last page for rough work.
3. If you need more space write on the back side of the paper and clearly mark question and part number etc.
4. After asked to commence the exam, please verify that you have 13 pages different printed pages including this title page. There are a total of 5 questions.
5. Calculator sharing is strictly prohibited.
6. Use permanent ink pens only. Any part done using soft pencil will not be marked and cannot be claimed for rechecking.
7. Fit in all your answers in the provided space. You may use extra space on the last page if required. If you do so, clearly mark question/part number on that page to avoid confusion.

Good

	Q-1	Q-2	Q-3	Q-4	Q-5	Total
Marks Obtained	08	9.5	28	20	23	88.5
Total Marks	10	16	30	24	30	110

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Question # 1[10 marks]

(a) For the following data (2.5+2.5)

x	y
0.5	0.4794
0.6	0.5646
0.7	0.442

Find $y'(0.55)$ and $y''(0.55)$

Hint: Use formulas given in formula sheet

Calculate Lagrange Polynomial,

$$P_2(x) = L_{0,2}(x)y_0 + L_{1,2}(x)y_1 + L_{2,2}(x)y_2$$

$$P_2(x) = \frac{(x-0.6)(x-0.7)}{0.02}(0.4794) + \frac{(x-0.5)(x-0.7)}{-0.01}(0.5646)$$

$$+ \frac{(x-0.5)(x-0.6)}{0.02}(0.442)$$

4

$$P_2(x) = 23.97(x-0.6)(x-0.7) - 56.46(x-0.5)(x-0.7)$$

$$+ 22.1(x-0.5)(x-0.6)$$

$$P_2(x) = 23.97(x^2-1.3x+0.42) - 56.46(x^2-1.2x+0.35) + 22.1(x^2-1.1x+0.3)$$

$$f'(x) = P_2'(x) = 23.97(2x-1.3) - 56.46(2x-1.2) + 22.1(2x-1.1)$$

$$\boxed{f'(0.55) = 11.902}$$

X

$$f''(x) = P_2''(x) = 23.97(2) - 56.46(2) + 22.1(2)$$

$$\boxed{f''(0.55) = -20.78}$$

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(b) Solve the given system (upto 4 dp) (2.5+2.5)

$$\begin{bmatrix}0.0003 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1.0002\\ 1\end{bmatrix}$$

(i) without partial pivoting

(ii) with partial pivoting

(i)

$$\left[\begin{array}{cc:c}0.0003 & 3 & 1.0002\\ 1 & 1 & 1\end{array}\right]$$

$$m_{21} = \frac{1}{0.0003} = 3333.33$$

$$E_2 - 3333.33E_1 \to E_2$$

4

$$\left[\begin{array}{cc:c}0.0003 & 3 & 1.0002\\ 0 & -9998.99 & -3332.997\end{array}\right]$$

$$\cancel{-}9998.99x_2 = \cancel{-}3332.997$$

$$\boxed{x_2 = 0.3334}$$

$$0.0003x_1 = 1.0002 - 3(0.3334)$$

$$\boxed{x_1 = 0}$$

$$x = \begin{bmatrix}0\\ 0.3334\end{bmatrix}$$

ii) with Partial Pivoting

$$\left[\begin{array}{cc:c}0.0003 & 3 & 1.0002\\ 1 & 1 & 1\end{array}\right]$$

$E_1 \leftrightarrow E_2$ (since 1st pivot max in 2nd row)

$$\left[\begin{array}{cc:c}1 & 1 & 1\\ 0.0003 & 3 & 1.0002\end{array}\right]$$

$$m_{21} = \frac{0.0003}{1} = 0.0003$$

$$E_2 - 0.0003E_1$$

$$\left[\begin{array}{cc:c} 1 & 1 & 1 \\ 0 & 2.9997 & 0.9999 \end{array}\right]$$

$$x_2 = \frac{0.9999}{2.9997} = 0.3334$$

$$x_1 = 1 - 0.3334 = 0.6666$$

$$x = \begin{bmatrix} 0.6666 \\ 0.3334 \end{bmatrix}$$

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Question # 2[16 marks]

(a) Find $\|\mathbf{A}\|_2$ and $\rho(\mathbf{A})$, if (3)

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

(b) Find the number of arithmatic operations required to solve the system $\mathbf{U}_{4\times4}\mathbf{x}_{4\times1} = \mathbf{b}_{4\times1}$, where $\mathbf{U}$ is upper triangular matrix with all diagonal entries equal 1. (5)

(a) For $\|A\|_2 = \sqrt{(|4|+|0|+|0|)^2 + (|0|+|-2|+|0|)^2 + (|0|+|0|+|-6|)^2}$

$= 7.4833$

$\rho(A)$ can be found by

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & -6-\lambda \end{vmatrix} = 0$$

$(4-\lambda)((-2-\lambda)(-6-\lambda)) = 0$

$(4-\lambda)(12+2\lambda+6\lambda+\lambda^2) = 0$

$(4-\lambda)(12+8\lambda+\lambda^2) = 0$

$(4-\lambda)(\lambda^2+8\lambda+12) = 0$

$(4-\lambda)(\lambda^2+8\lambda+12) = 0$

$(4-\lambda)(\lambda^2+6\lambda+2\lambda-12) = 0$

$(\lambda-4)(\lambda+6)(\lambda+2) = 0$

$\lambda = 4, \lambda = -6, \lambda = -2$

$\rho(A) = \max\{4, -6, -2\} = 4$

2.5

$(-2-\lambda)(-6-\lambda)$

$12+2\lambda+6\lambda$

$\lambda(\lambda-6$

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(b) $U_{4\times4} X_{4\times1} = b_{4\times1}$

if U is upper triangular

$$\begin{bmatrix} 1 & & & 0 \\ a_{21} & 1 & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

~~we will~~ For last ~~Row~~ Row

$x_4 = b_4$ $\qquad (\times) = 0$, $(+) = 0$

For 2nd last Row

$x_3 = b_3 - x_4 a_{34}$ $\qquad (\times) = 1$, $(+) = 1$

For 3rd last Row

$x_2 = b_2 - x_4 a_{24} - x_3 a_{23}$ $\qquad (\times) = 2$, $(+) = 2$

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For 4th last Row

$x_1 = b_1 - x_4 a_{14} - x_3 a_{13} - x_2 a_{12}$ $\qquad (\times) = 3$, $(+) = 3$

Total operation $= 0+0+1+1+2+2+3+3$
$= 12$

can also be found by generic formula

$\text{op count} = n^2 - n = (4)^2 - 4 = 16 - 4 = 12$

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(c) The table shows the pressure of the wind (Pa) measured at various heights (m) on a vertical wall. (8)

h	0	15	30	45	60
p(h)	310	425	530	575	612

Find the height of the pressure center $\bar{h}$, (by using any quadrature formula) which is defined as

$$\bar{h} = \frac{\int_0^{60} h p(h) dh}{\int_0^{60} p(h) dh}$$

~~Using Simpson~~

$\cancel{h \to}$

$$t = \frac{2h - b - a}{b - a} = \frac{2h - 60}{60}$$

$$t = \frac{2h-60}{60}$$

$$60t = 2h - 60$$

$$h = 30t + 30$$

$$dh = 30\, dt$$

$\cancel{\bar{h}} = \int_0^{60}$

Using Simpson's 1/3 Formula.

$h = \frac{60-0}{2} = 30$

2

$$\int_0^{60} p(h) = \frac{h}{3}\left[f(0) + 4f(30) + f(60)\right]$$

$$= \frac{30}{3}\left[310 + 4(530) + 612\right]$$

$$\int_0^{60} p(h) = 30420$$

$$\bar{h} = \frac{\int_0^{60} h p(h) dh}{30420}$$

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Question # 3[30 marks]

(a) What is the sufficient condition for the convergence of Jacobi and Gauss Siedel iterative method. (2)

(b) For the given matrices

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}, \quad \mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(i) Find $\mathbf{T}_j$ and $\mathbf{C}_j$ such that Jacobi iterative scheme can be written as (4)

$$\mathbf{x}^{(k+1)} = \mathbf{T}_j\mathbf{x}^{(k)} + \mathbf{C}_j.$$

(ii) Approximate the solution($TOL = 10^{-3}$) of the system $\mathbf{Ax} = \mathbf{b}$, by using

- Gauss Siedel iterative method (7)
- SoR method with $\omega = 1.25$ (7)
- Conjugate gradient method, show that exact solution is obtained in three or less iterations(10)

(a) The sufficient condition for convergence of Jacobi and Gauss Siedel method is that for system $Ax=b$, matrix A should be diagonally dominant, i.e

$$|A_{ii}| \geq \sum_{j=1, j\neq i}^{n} |A_{ij}| \quad \text{where } i=1,2,\dots n$$

2/2

(b)

(i)

~~Final~~ Deriv. After Derivation 2/2

$$\boxed{\begin{aligned} T_j &= -D^{-1}(L+U) \\ C_j &= D^{-1} b \end{aligned}}$$

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ii)

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix},\ b = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix},\ x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

↳ diagonally dominant ✓

Gauss Seidel Iterative Method:-

$$x_1^{(k+1)} = \frac{1}{4}\left(2 + x_2^{(k)}\right)$$
$$x_2^{(k+1)} = \frac{1}{4}\left(4 + x_1^{(k+1)} + x_3^{(k)}\right)$$
$$x_3^{(k+1)} = \frac{1}{4}\left(10 + x_2^{(k+1)}\right)$$

k=0

$$x_1^{(1)} = \frac{1}{4}\left(2 + x_2^{(0)}\right) = 0.75$$
$$x_2^{(1)} = \frac{1}{4}\left(4 + x_1^{(1)} + x_3^{(0)}\right) = 1.4375$$
$$x_3^{(1)} = \frac{1}{4}\left(10 + x_2^{(1)}\right) = 2.859375$$

k=1

$$x_1^{(2)} = \frac{1}{4}\left(2 + x_2^{(1)}\right) = 0.8594$$
$$x_2^{(1)} = \frac{1}{4}\left(4 + x_1^{(1)} + x_3^{(1)}\right) = 1.9297$$
$$x_3^{(1)} = \frac{1}{4}\left(10 + x_2^{(1)}\right) = 2.9824$$

k=2

$$x_1^{(3)} = 0.9824$$
$$x_2^{(3)} = 1.9912$$
$$x_3^{(3)} = 2.9978$$

k=3

$$x_1^{(4)} = 0.9978$$
$$x_2^{(4)} = 1.9989$$
$$x_3^{(4)} = 2.9997$$

k=4

$$x_1^{(5)} = 0.9997$$
$$x_2^{(5)} = 1.9999$$
$$x_3^{(5)} = 3.0000$$

k=5

$$x_1^{(6)} = \cancel{0.999}\ 1.0000$$
$$x_2^{(6)} = 2.0000$$
$$x_3^{(6)} = 3.0000$$

$$\|x^{(k+1)} - x^{(k)}\|_2 = 0.0003 < tol$$

Hence,

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

SOR method with $\omega = 1.25$

$$x_1^{(k+1)} = (1-\omega)x_1^{(k)} + \omega\left(\frac{1}{4}(2+x_2^{(k)})\right)$$
$$x_2^{(k+1)} = (1-\omega)x_2^{(k)} + \omega\left(\frac{1}{4}(4+x_1^{(k+1)}+x_3^{(k)})\right)$$
$$x_3^{(k+1)} = (1-\omega)x_3^{(k)} + \omega\left(\frac{1}{4}(10+x_2^{(k+1)})\right)$$

k=0

$$x_1^{(1)} = (-0.25)x_1^{(0)} + (1.25)\left(\frac{1}{4}(2+x_2^{(0)})\right) = 0.6875$$
$$x_2^{(1)} = (-0.25)x_2^{(0)} + (1.25)\left(\frac{1}{4}(4+x_1^{(1)}+x_3^{(0)})\right) = 1.5273$$
$$x_3^{(1)} = (-0.25)x_3^{(0)} + (1.25)\left(\frac{1}{4}(10+x_2^{(1)})\right) = 3.3523$$

k=1

$$x_1^{(2)} = (-0.25)x_1^{(1)} + 1.25\left(\frac{1}{4}(2+x_2^{(1)})\right) = 0.9304$$
$$x_2^{(2)} = (-0.25)x_2^{(1)} + 1.25\left(\frac{1}{4}(4+x_1^{(2)}+x_3^{(1)})\right) = 2.2065$$
$$x_3^{(2)} = (-0.25)x_3^{(1)} + 1.25\left(\frac{1}{4}(10+x_2^{(2)})\right) = 2.9765$$

k=2

$x_1^{(3)} = 1.0819$
$x_2^{(3)} = 1.9666$
$x_3^{(3)} = 2.9954$

k=3

$x_1^{(4)} = 0.9691$
$x_2^{(4)} = 1.9973$
$x_3^{(4)} = 3.0003$

k=4

$x_1^{(5)} = 1.0069$
$x_2^{(5)} = 2.0029$
$x_3^{(5)} = 3.0008$

k=5

$x_1^{(6)} = 0.9992$
$x_2^{(6)} = 1.9993$
$x_3^{(6)} = 2.9996$

k=6

$x_1^{(7)} = 1.0000$
$x_2^{(7)} = 2.0000$
$x_3^{(7)} = 3.0001$

$\|x^{(k+1)} - x^{(k)}\|_2 = 0.00175 < tol$

Hence,

$$x = \begin{bmatrix} 1.0000 \\ 2.0000 \\ 3.0001 \end{bmatrix}$$

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Conjugate Gradient method

$$A=\begin{bmatrix}4&-1&0\\-1&4&-1\\0&-1&4\end{bmatrix},\ b=\begin{bmatrix}2\\4\\10\end{bmatrix},\ x^{(0)}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$r^{(0)}=b-Ax^{(0)}=\begin{bmatrix}2\\4\\10\end{bmatrix}-\begin{bmatrix}3\\2\\3\end{bmatrix}=\begin{bmatrix}-1\\2\\7\end{bmatrix}$$

$$d^{(0)}=r^{(0)}=\begin{bmatrix}-1\\2\\7\end{bmatrix}$$

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Iteration $k=1$

$$\lambda_0=\frac{\langle r^{(0)},r^{(0)}\rangle}{\langle d^{(0)},Ad^{(0)}\rangle}=\frac{[-1\ 2\ 7]\begin{bmatrix}-1\\2\\7\end{bmatrix}}{[-1\ 2\ 7]\begin{bmatrix}-6\\2\\26\end{bmatrix}}=0.28125$$

$$x^{(1)}=x^{(0)}+\lambda_0 d^{(0)}=\begin{bmatrix}1\\1\\1\end{bmatrix}+0.28125\begin{bmatrix}-1\\2\\7\end{bmatrix}=\begin{bmatrix}0.71875\\1.5625\\2.96875\end{bmatrix}$$

$$r^{(1)}=r^{(0)}-\lambda_0 Ad^{(0)}=\begin{bmatrix}-1\\2\\7\end{bmatrix}-0.28125\begin{bmatrix}-6\\2\\26\end{bmatrix}=\begin{bmatrix}0.6875\\1.4375\\-0.3125\end{bmatrix}$$

$$\alpha_0=\frac{\langle r^{(1)},r^{(1)}\rangle}{\langle r^{(0)},r^{(0)}\rangle}=\frac{[0.6875\ 1.4375\ -0.3125]\begin{bmatrix}0.6875\\1.4375\\-0.3125\end{bmatrix}}{[-1\ 2\ 7]\begin{bmatrix}-1\\2\\7\end{bmatrix}}=0.04883$$

$$d^{(1)}=r^{(1)}+\alpha_0 d^{(0)}=\begin{bmatrix}0.6875\\1.4375\\-0.3125\end{bmatrix}+0.04883\begin{bmatrix}-1\\2\\7\end{bmatrix}=\begin{bmatrix}0.63867\\1.53516\\0.02931\end{bmatrix}$$

Iteration $k=2$

$$\lambda_1=\frac{\langle r^{(1)},r^{(1)}\rangle}{\langle d^{(1)},Ad^{(1)}\rangle}=\frac{[0.6875\ 1.4375\ -0.3125]\begin{bmatrix}0.6875\\1.4375\\-0.3125\end{bmatrix}}{[0.63867\ 1.53516\ 0.02931]\begin{bmatrix}1.01952\\5.47266\\-1.41792\end{bmatrix}}=0.28994$$

$$x^{(2)}=x^{(1)}+\lambda_1 d^{(1)}=\begin{bmatrix}0.71875\\1.5625\\2.96875\end{bmatrix}+0.28994\begin{bmatrix}0.63867\\1.53516\\0.02931\end{bmatrix}=\begin{bmatrix}0.9039\\2.0076\\2.9772\end{bmatrix}$$

$$r^{(2)} = r^{(1)} - \lambda_1 d^{(1)} = \begin{bmatrix} 0.6875 \\ 1.4375 \\ -0.3125 \end{bmatrix} - 0.28994 \begin{bmatrix} 1.01952 \\ 5.47266 \\ -1.41792 \end{bmatrix} = \begin{bmatrix} 0.3919 \\ -0.1492 \\ 0.0986 \end{bmatrix}$$

$$\alpha_1 = \frac{\langle r^{(2)}, r^{(2)} \rangle}{\langle r^{(1)}, r^{(1)} \rangle} = 0.07038$$

$$d^{(2)} = r^{(2)} + \alpha_1 d^{(1)} = \begin{bmatrix} 0.3919 \\ -0.1492 \\ 0.0986 \end{bmatrix} + 0.07036 \begin{bmatrix} 0.63867 \\ 1.53516 \\ 0.02931 \end{bmatrix} = \begin{bmatrix} 0.4366 \\ -0.041 \\ 0.100 \end{bmatrix}$$

<u>Iteration k=3</u>

$$\lambda_2 = \frac{\langle r^{(2)}, r^{(2)} \rangle}{\langle d^{(2)}, Ad^{(2)} \rangle} = \frac{\begin{bmatrix} 0.3919 & -0.1492 & 0.0986 \end{bmatrix} \begin{bmatrix} 0.3919 \\ -0.1492 \\ 0.0986 \end{bmatrix}}{\begin{bmatrix} 0.43665 & -0.04115 & 0.10066 \end{bmatrix} \begin{bmatrix} 1.78855 \\ -0.37291 \\ 0.44379 \end{bmatrix}} = 0.22056$$

$$x^{(3)} = x^{(2)} + \lambda_2 d^{(2)} = \begin{bmatrix} 0.9039 \\ 2.0076 \\ 2.9772 \end{bmatrix} + 0.22056 \begin{bmatrix} 0.43685 \\ -0.04115 \\ 0.10066 \end{bmatrix} = \begin{bmatrix} 1.0003 \\ 1.9985 \\ 2.9994 \end{bmatrix}$$

$$r^{(3)} = r^{(2)} - \lambda_2 d^{(2)} = \begin{bmatrix} 0.3919 \\ -0.1492 \\ 0.0986 \end{bmatrix} - 0.22056 \begin{bmatrix} 1.78855 \\ -0.37291 \\ 0.44379 \end{bmatrix} = \begin{bmatrix} -0.0026 \\ -0.067 \\ 0.0007 \end{bmatrix}$$

Since $\|r\| < tol$, we stop

Exact Solution is obtained in 3 iterations

$$x = \begin{bmatrix} 1.0003 \\ 1.9985 \\ 2.9994 \end{bmatrix}$$

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Question # 4[24 marks]

(a) Discuss the limitations of power method (3)

(b) For the matrix

$$\mathbf{A} = \begin{pmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(i) Use the power method to approximate the dominant eigenvalue with $TOL = 10^{-2}$ of the matrix (7)

(ii) Apply Aitken's Δ^2 method to the approximations to the eigenvalue of the matrix to accelerate the convergence. (7)

(iii) Find the eigenvalue of A nearest to 4 with $TOL = 10^{-2}$. (7)

(a) Limitations of Power Method :-

* Convergence is guaranteed but convergence rate is unknown, sometimes it converges fast and sometimes too slow.

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* It can only give dominant eigenvalue if max eigenvalue is non-repeating for matrix.

(b) (i) 7

$$A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \quad x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y^{(1)} = Ax^{(0)} = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -1 \end{bmatrix} = 5\begin{bmatrix} -0.8 \\ -1 \\ -0.2 \end{bmatrix} \Rightarrow x^{(1)} = \begin{bmatrix} 0.8 \\ -1 \\ -0.2 \end{bmatrix}, \lambda_0 = 5$$

$$y^{(2)} = Ax^{(1)} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix} -0.8 \\ -1 \\ -0.2 \end{bmatrix} = \begin{bmatrix} -10.8 \\ -9 \\ 0.4 \end{bmatrix} = 10.8\begin{bmatrix} -1 \\ -0.8334 \\ 0.03704 \end{bmatrix} \Rightarrow x^{(2)} = \begin{bmatrix} -1 \\ -0.8334 \\ 0.03704 \end{bmatrix}, \lambda_1 = 10.8$$

$$y^{(3)} = Ax^{(2)} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix} -1 \\ -0.8334 \\ 0.3704 \end{bmatrix} = \begin{bmatrix} -7.6676 \\ -5.8342 \\ 1.7408 \end{bmatrix} = 7.6676\begin{bmatrix} -1 \\ -0.7609 \\ 0.2270 \end{bmatrix} \Rightarrow x^{(3)} = \begin{bmatrix} -1 \\ -0.7609 \\ 0.2270 \end{bmatrix}, \lambda_2 = 7.6676$$

$$y^{(4)} = Ax^{(3)} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix} -1 \\ 0.7609 \\ 0.2270 \end{bmatrix} = \begin{bmatrix} -6.6526 \\ -4.847 \\ 1.454 \end{bmatrix} = 6.6526\begin{bmatrix} -1 \\ -0.7353 \\ 0.2186 \end{bmatrix} \Rightarrow x^{(4)} = \begin{bmatrix} -1 \\ 0.7353 \\ 0.2186 \end{bmatrix}, \lambda_3 = 6.6526$$

$$y^{(5)} = Ax^{(4)} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix}-1\\-0.7353\\0.2136\end{bmatrix} = \begin{bmatrix}-6.2942\\-4.5584\\1.4372\end{bmatrix} = 6.2942\begin{bmatrix}-1\\-0.7243\\0.2283\end{bmatrix} \Rightarrow x^{(5)} = \begin{bmatrix}-1\\-0.724\\0.2283\end{bmatrix}$$

$\lambda_4 = 6.2942$

$$y^{(6)} = Ax^{(5)} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix}-1\\-0.7243\\0.2283\end{bmatrix} = \begin{bmatrix}-6.1402\\-4.4159\\1.4566\end{bmatrix} = 6.1402\begin{bmatrix}-1\\-0.7192\\0.2372\end{bmatrix} \Rightarrow x^{(6)} = \begin{bmatrix}-1\\-0.7192\\0.2372\end{bmatrix}$$

$\lambda_5 = 6.1402$

$$y^{(7)} = Ax^{(6)} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix}-1\\-0.7192\\0.2372\end{bmatrix} = \begin{bmatrix}-6.0688\\-4.3446\\1.4744\end{bmatrix} = 6.0688\begin{bmatrix}-1\\0.7167\\0.2429\end{bmatrix} \Rightarrow x^{(7)} = \begin{bmatrix}-1\\-0.7167\\0.2429\end{bmatrix}$$

$\lambda_6 = 6.0688$

$$y^{(8)} = Ax^{(7)} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix}-1\\-0.7167\\0.2429\end{bmatrix} = \begin{bmatrix}-6.0338\\-4.3171\\1.4858\end{bmatrix} = 6.0338\begin{bmatrix}-1\\0.7155\\0.2462\end{bmatrix} \Rightarrow x^{(8)} = \begin{bmatrix}-1\\-0.7155\\0.2462\end{bmatrix}$$

$\lambda_7 = 6.0338$

$$y^{(9)} = Ax^{(8)} = \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix}-1\\-0.7155\\0.2462\end{bmatrix} = \begin{bmatrix}-6.017\\-4.3015\\1.4934\end{bmatrix} = 6.017\begin{bmatrix}-1\\-0.7149\\0.2480\end{bmatrix} \Rightarrow x^{(9)} = \begin{bmatrix}-1\\-0.7149\\0.2480\end{bmatrix}$$

$\lambda_8 = 6.017$

Since, $|\lambda_8 - \lambda_7| = 0.017 < tol$, we stop

and dominant eigen value i.e $\boxed{\lambda_{max} = 6.017}$

ii) Use formula $\hat{\lambda}_n = \lambda_n - \dfrac{(\lambda_{n+1} - \lambda_n)^2}{\lambda_{n+2} - 2\lambda_{n+1} + \lambda_n}$

n	Power (λ_n)	Aitken's Method (use formula) $\hat{\lambda}_n$
0	5	8.7661
1	10.8	6.16605
2	7.6676	6.0986
3	6.6526	6.0242
4	6.2942	6.0071
5	6.1402	6.0001
6	6.0688	6.0012
7	6.0338	
8	6.017	

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iii) For eigenvalue nearest to 4, we need to find

$$A-4I = \begin{bmatrix} 0 & 14 & 0 \\ -5 & 9 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

~~We need~~ we need λ_{min} of $A-4I$ so find λ_{max} of $(A-4I)^{-1}$

$$adj(A-4I) = \begin{bmatrix} -18 & -10 & 9 \\ 28 & 0 & -14 \\ 0 & 0 & 70 \end{bmatrix}^T = \begin{bmatrix} -18 & 28 & 0 \\ -10 & 0 & 0 \\ 9 & -14 & 70 \end{bmatrix}$$

$$\det(A-4I) = -140$$

04

$$B = (A-4I)^{-1} = \frac{adj(A-4I)}{\det(A-4I)} = \begin{bmatrix} 9/70 & -1/5 & 0 \\ 1/14 & 0 & 0 \\ -9/140 & 1/10 & -1/2 \end{bmatrix}$$

Use power method on B with initial guess $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$y^{(1)} = Bx^{(0)} = \begin{bmatrix} 9/70 & -1/5 & 0 \\ 1/14 & 0 & 0 \\ -9/140 & 1/10 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1286 \\ 0.0714 \\ -0.0643 \end{bmatrix} = 0.1286\begin{bmatrix} 1 \\ 0.5552 \\ -0.5 \end{bmatrix} \Rightarrow x^{(1)} =$$

$\lambda_0 = 0.1286$

$$= \begin{bmatrix} 0.0175 \\ 0.0714 \\ 0.2412 \end{bmatrix} = 0.2412\begin{bmatrix} 0.07255 \\ 0.2960 \\ 1 \end{bmatrix} \rightarrow$$

$\lambda_1 = 0.2412$

$$= \begin{bmatrix} -0.0499 \\ 0.0052 \\ -0.4751 \end{bmatrix} = 0.4751\begin{bmatrix} 0.1050 \\ 0.0109 \\ -1 \end{bmatrix} \Rightarrow$$

$\lambda_2 = 0.4751$

$$= \begin{bmatrix} -0.01568 \\ -0.0075 \\ 0.5078 \end{bmatrix} = 0.5078\begin{bmatrix} -0.0313 \\ -0.0148 \\ 1 \end{bmatrix} \Rightarrow$$

$\lambda_3 = 0.5078$

$$= \begin{bmatrix} -0.0011 \\ -0.0023 \\ -0.4995 \end{bmatrix} = 0.4995\begin{bmatrix} 0.0022 \\ -0.0046 \\ -1 \end{bmatrix} \Rightarrow$$

$\lambda_4 = 0.4995$

$$= \begin{bmatrix} 0.0006 \\ -0.00016 \\ 0.4997 \end{bmatrix} = 0.4997\begin{bmatrix} 0.0012 \\ 0.0032 \\ 1 \end{bmatrix} \Rightarrow$$

$\lambda_5 = 0.4997$

Since $|\lambda_5 - \lambda_4| < tol$

$$S_{max} = 0.4997$$

So $$M_{min} = \frac{1}{S_{max}} = \frac{1}{0.4997} = 2.0012$$

Here, eigen value of A closest to 4 is

$$\lambda_0 = M_{min} + 4 = 2.0012 + 4 = \cancel{6.0012}$$

$$\boxed{\lambda = 6.0012}$$

$y + h(yt^2 - 1.1y)$
$y + hyt^2 - 0.077$
$y(1 + ht^2 - (1.1)(\ldots$

Question # 5[30 marks]

(a) Can the Euler's method be considered as RK-1 method? Explain your argument. (2)

(b) Solve the following initial value problem

$$\frac{dy}{dt} = yt^2 - 1.1y, \quad 0 \le t \le 1, \quad y(0) = 1.$$

(i) Euler's method with $h = 0.25$ (7)

(ii) Heunn's method with $h = 0.25$ (7)

(iii) Fourth-order RK method with $h = 0.5$ (8)

(iv) Use any interpolation technique, for data obtained in the part (i), to get a polynomial approximation of the solution of given initial value problem (6)

02

(a) Yes, it can be considered as RK-1 method with global error, $O(n)$ and local truncation error $O(n^2)$.

$y_{k+1} = y_k + K_1$ ← RK-1 form for Euler
where $K_1 = hf(t_k, y_k)$

(b)
i) Euler's Method

$f(t,y) = yt^2 - 1.1y$, $t_0 = 0$, $y_0 = 1$, $h = 0.25$

Using euler method

$t_{k+1} = t_k + h$
$y_{k+1} = y_k + hf(t_k, y_k)$
$= y_k + 0.25(y_k t_k^2 - 1.1y_k)$
$= y_k(1 + 0.25t_k^2 - 0.275)$

07

For $k=0$

$t_1 = t_0 + h = 0 + 0.25 = 0.25$
$y_1 = y_0(1 + 0.25t_0^2 - 0.275) = 0.725$

For $k=1$

$t_2 = t_1 + h = 0.25 + 0.25 = 0.5$
$y_2 = y_1(1 + 0.25t_1^2 - 0.275) = 0.5369$

For $k=2$

$$t_3 = t_2 + h = 0.5 + 0.25 = 0.75$$

$$y_3 = y_2(1 + 0.25t_2^2 - 0.275) = 0.4228$$

For $k=3$

$$t_4 = t_3 + h = 0.75 + h = 1$$

$$y_4 = y_3(1 + 0.25t_3^2 - 0.275) = 0.3650$$

ii) Heunn's method

$$y_{k+1} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k))\right]$$

For $k=0$

$$y_1 = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0))\right]$$

$$y_1 = \cancel{y_0 + h}\, y_0 + \frac{h}{2}\left[y_0t_0^2 - 1.1y_0 + (y_0 + h(y_0t_0^2 - 1.1y_0))t_1^2 - 1.1(y_0 + h(y_0t_0^2 - 1.1y_0))\right]$$

$y_1 = 0.7685$ $\quad t_1 = 0.25$

$k=1$ $\quad y_2 = 0.6084$ $\quad t_2 = 0.5$

$k=2$ $\quad y_3 = 0.5116$ $\quad t_3 = 0.75$

$k=3$ $\quad y_4 = 0.4717$ $\quad t_4 = 1$

iii)

$f(t_n, y_n) = yt^2 - 1.1y$, $t_0 = 0$, $y_0 = 1$, $h = 0.5$

For $k = 0$

$t_1 = t_0 + h = 0 + 0.5 = 0.5$

$k_1 = hf(t_0, y_0) = 0.5f(0,1) = -0.55$

$k_2 = hf(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.5f(0.25, 0.725) = -0.3761$

$k_3 = hf(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.5f(0.25, 0.8119) = -0.4212$

$k_4 = hf(t_0 + h, y_0 + k_3) = 0.5f(0.5, 0.5788) = -0.24599$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\boxed{y_1 = 0.6016}$

8

For $k = 1$

$t_2 = t_1 + h = 0.5 + h = 1$

$k_1 = hf(t_1, y_1) = 0.5f(0.5, 0.6016) = -0.2557$

$k_2 = hf(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.5f(0.75, 0.47375) = -0.1273$

$k_3 = hf(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.5f(0.75, 0.53795) = -0.1446$

$k_4 = hf(t_1 + h, y_1 + k_3) = 0.5f(1, 0.457) = -0.02285$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\boxed{y_2 = 0.4645}$

iv)

Data from (a)

x	y
0	1
0.25	0.725
0.5	0.5369
0.75	0.4228
1	0.3660

using lagrange Interpolation :-

10/

$$P_4(x) = \frac{(x-0.25)(x-0.5)(x-0.75)(x-1)}{(0-0.25)(0-0.5)(0-0.75)(0-1)}(1) + \frac{(x-0)(x-0.5)(x-0.75)(x-1)}{(0.25-0)(0.25-0.5)(0.25-0.75)(0.25-1)}(0.725)$$

$$+ \frac{(x-0)(x-0.25)(x-0.75)(x-1)}{(0.5-0)(0.5-0.25)(0.5-0.75)(0.5-1)}(0.5369) + \frac{(x-0)(x-0.25)(x-0.5)(x-1)}{(0.75-0)(0.75-0.25)(0.75-0.5)(0.75-1)}(0.4228)$$

$$+ \frac{(x-0)(x-0.25)(x-0.5)(x-0.75)}{(1-0)(1-0.25)(1-0.5)(1-0.75)}(0.3660)$$