Chapter 31

Normal Forms for Symplectic Maps and Their Applications

31.1 Equivalence Relations

Def. 1.1: Let X be some (possibly abstract) set, and let $\sim$ be some relation (something that can be true or false) among pairs of elements in X. The relation $\sim$ is said to be an *equivalence* relation if it satisfies three properties:

i. $x \sim x$ for all x in X (reflexive property).

ii. $x_1 \sim x_2$ implies $x_2 \sim x_1$ for all x_1, x_2 in X (symmetric property) .

iii. $x_1 \sim x_2$ and $x_2 \sim x_3$ implies $x_1 \sim x_3$ for all x_1, x_2, x_3 in X (transitive property).

Def. 1.2: The set of all elements in X that are equivalent (under some given equivalence relation $\sim$) to a given x in X is called the *equivalence class* of x, and is denoted by the symbol $\{x\}$.

Thrm. 1.1: We have the logical relation

$$x_1 \sim x_2 \Leftrightarrow \{x_1\} = \{x_2\}. \tag{31.1.1}$$

Thrm. 1.2: Given an equivalence relation $\sim$ on some set X, show that each x in X belongs to one and only one equivalence class. Thus, under an equivalence relation, a set decomposes in a natural way into disjoint subsets: the equivalence classes produced by the equivalence relation.

Def. 1.3: Let X be some set and x some element in X. Then, given an equivalence relation $\sim$, x belongs to the equivalence class $\{x\}$. A *normal form* x_n for x is an element of $\{x\}$ that has some desired attribute such as "simplicity". See Figure 1.1.

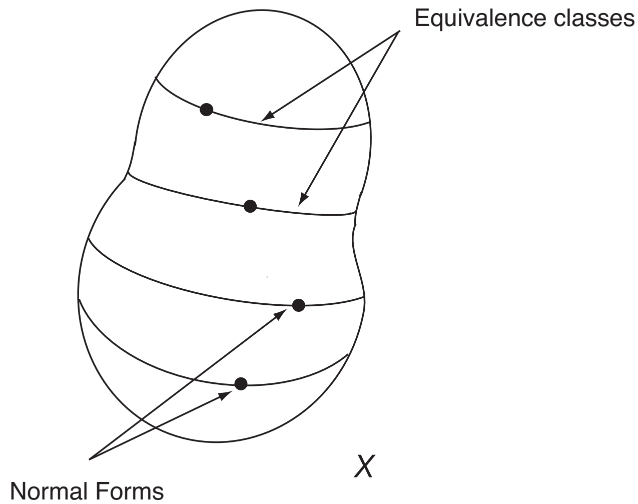

Figure 31.1.1: Decomposition of a set X into disjoint equivalence classes, with a normal form element representative for each equivalence class.

31.2 Symplectic Conjugacy of Symplectic Maps

Def. 2.1: Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two symplectic maps. These maps are said to be (symplectically) *conjugate* if there exists a third (symplectic) map $\mathcal{A}$ such that

$$\mathcal{M}_2 = \mathcal{A}\mathcal{M}_1\mathcal{A}^{-1}. \tag{31.2.1}$$

Def. 2.2: The map $\mathcal{A}$ is called the *conjugating* map.

Thrm. 2.1: Conjugacy and symplectic conjugacy are equivalence relations and therefore determine equivalence classes called *conjugacy classes.* Two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent (belong to the same conjugacy class) if a conjugating map $\mathcal{A}$ can be found such that (2.1) holds.

31.3 Normal Forms for Maps

Def. 3.1: A map *normal form* is a representative of an equivalence class (in this case a conjugacy class) selected for its maximal simplicity: Given any map $\mathcal{M}_1$, consider maps $\mathcal{N}_1$ of the form

$$\mathcal{N}_1 = \mathcal{A}_1\mathcal{M}_1\mathcal{A}_1^{-1}. \tag{31.3.1}$$

Select the map $\mathcal{A}_1$, and thereby also the map $\mathcal{N}_1$, in such a way that $\mathcal{N}_1$ is as *simple* as possible. The map $\mathcal{N}_1$ is called the normal form of $\mathcal{M}_1$, and the conjugating map $\mathcal{A}_1$ is called the *normalizing* map. Note that, by construction, we have the relations

$$\mathcal{N}_1 \sim \mathcal{M}_1 \text{ and } \{\mathcal{N}_1\} = \{\mathcal{M}_1\} \tag{31.3.2}$$

so that $\mathcal{N}_1$ is indeed a representative of the conjugacy class of $\mathcal{M}_1$

Thrm. 3.1: Given suitable specifications concerning the set of allowed conjugating maps, there is a *unique* normal form element for each conjugacy class. In other words, the normal form is unique in the sense that if $\mathcal{M}_2$ and $\mathcal{M}_1$ belong to the same conjugacy class,

$$\mathcal{M}_2 \sim \mathcal{M}_1 \text{ and } \{\mathcal{M}_2\} = \{\mathcal{M}_1\}, \tag{31.3.3}$$

then they have the same normal form,

$$\mathcal{N}_2 = \mathcal{N}_1. \tag{31.3.4}$$

Conversely, if two maps $\mathcal{M}_2$ and $\mathcal{M}_1$ have the same normal form, i.e. (3.4) holds, then they are conjugate and (3.4) holds. Thus, we have the logical relation

$$\mathcal{N}_2 = \mathcal{N}_1 \Leftrightarrow \mathcal{M}_2 \sim \mathcal{M}_1 \text{ and } \{\mathcal{M}_2\} = \{\mathcal{M}_1\}. \tag{31.3.5}$$

Proof: Stating that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the normal forms $\mathcal{N}_1$ and $\mathcal{N}_2$ means that there exist normalizing maps $\mathcal{A}_1$ and $\mathcal{A}_2$ such that

$$\mathcal{N}_1 = \mathcal{A}_1\mathcal{M}_1\mathcal{A}_1^{-1}, \tag{31.3.6}$$

$$\mathcal{N}_2 = \mathcal{A}_2\mathcal{M}_2\mathcal{A}_2^{-1}, \tag{31.3.7}$$

and both $\mathcal{N}_1$ and $\mathcal{N}_2$ have maximal simplicity. Now suppose that the left equality in (3.5) holds. This supposition, when combined with (3.6) and (3.7), gives the relation

$$\mathcal{A}_1\mathcal{M}_1\mathcal{A}_1^{-1} = \mathcal{A}_2\mathcal{M}_2\mathcal{A}_2^{-1}, \tag{31.3.8}$$

which can be rewritten in the form

$$\mathcal{M}_1 = (\mathcal{A}_1^{-1}\mathcal{A}_2)\mathcal{M}_2(\mathcal{A}_1^{-1}\mathcal{A}_2)^{-1}. \tag{31.3.9}$$

We conclude that $\mathcal{M}_2$ and $\mathcal{M}_1$ are conjugate,

$$\mathcal{M}_2 \sim \mathcal{M}_1 \text{ and } \{\mathcal{M}_2\} = \{\mathcal{M}_1\}. \tag{31.3.10}$$

Conversely, suppose that $\mathcal{M}_2$ and $\mathcal{M}_1$ are in the same conjugacy class. Then there exists a conjugating map $\mathcal{A}$ such that (2.1) holds. From the relations (2.1), (3.6), and (3.7) we deduce the results

$$\mathcal{N}_1 = \mathcal{A}_1\mathcal{M}_1\mathcal{A}_1^{-1} = \mathcal{A}_1\mathcal{A}^{-1}\mathcal{A}\mathcal{M}_1\mathcal{A}^{-1}\mathcal{A}\mathcal{A}_1^{-1} = (\mathcal{A}_1\mathcal{A}^{-1})\mathcal{M}_2(\mathcal{A}_1\mathcal{A}^{-1})^{-1}, \tag{31.3.11}$$

$$\mathcal{N}_2 = \mathcal{A}_2\mathcal{M}_2\mathcal{A}_2^{-1} = \mathcal{A}_2\mathcal{A}\mathcal{A}^{-1}\mathcal{M}_2\mathcal{A}\mathcal{A}^{-1}\mathcal{A}_2^{-1} = (\mathcal{A}_2\mathcal{A})\mathcal{M}_1(\mathcal{A}_2\mathcal{A})^{-1}. \tag{31.3.12}$$

We see that $\mathcal{M}_2$ has the normal form $\mathcal{N}_1$ when $(\mathcal{A}_1\mathcal{A}^{-1})$ is used as a normalizing map, and $\mathcal{M}_1$ has the normal form $\mathcal{N}_2$ when $(\mathcal{A}_2\mathcal{A})$ is used as a normalizing map. Next, conjugate both sides of (3.6) with the map $(\mathcal{A}_2\mathcal{A}\mathcal{A}_1^{-1})$. Doing so gives the result

$$\begin{aligned}(\mathcal{A}_2\mathcal{A}\mathcal{A}_1^{-1})\mathcal{N}_1(\mathcal{A}_2\mathcal{A}\mathcal{A}_1^{-1})^{-1} &= (\mathcal{A}_2\mathcal{A}\mathcal{A}_1^{-1})\mathcal{A}_1\mathcal{M}_1\mathcal{A}_1^{-1}(\mathcal{A}_2\mathcal{A}\mathcal{A}_1^{-1})^{-1} \\ &= \mathcal{A}_2\mathcal{A}\mathcal{M}_1\mathcal{A}^{-1}\mathcal{A}_2^{-1} \\ &= \mathcal{A}_2\mathcal{M}_2\mathcal{A}_2^{-1} = \mathcal{N}_2. \end{aligned} \tag{31.3.13}$$

Similarly, conjugating both sides of (3.7) with the map $(\mathcal{A}_1\mathcal{A}^{-1}\mathcal{A}_2^{-1})$ gives the result

$$\begin{aligned}(\mathcal{A}_1\mathcal{A}^{-1}\mathcal{A}_2^{-1})\mathcal{N}_2(\mathcal{A}_1\mathcal{A}^{-1}\mathcal{A}_2^{-1})^{-1} &= (\mathcal{A}_1\mathcal{A}^{-1}\mathcal{A}_2^{-1})\mathcal{A}_2\mathcal{M}_2\mathcal{A}_2^{-1}(\mathcal{A}_1\mathcal{A}^{-1}\mathcal{A}_2^{-1})^{-1} \\ &= \mathcal{A}_1\mathcal{A}^{-1}\mathcal{M}_2\mathcal{A}\mathcal{A}_1^{-1} \\ &= \mathcal{A}_1\mathcal{M}_1\mathcal{A}_1^{-1} = \mathcal{N}_1. \end{aligned} \tag{31.3.14}$$

We have learned that $\mathcal{M}_2$, under the assumption that it is conjugate to $\mathcal{M}_1$, can be normalized to *both* the normal forms $\mathcal{N}_1$ and $\mathcal{N}_2$. Similarly, under the same assumption, $\mathcal{M}_1$ can be normalized to both the normal forms $\mathcal{N}_1$ and $\mathcal{N}_2$. By assumption, the $\mathcal{A}_1$ used in (3.6) and the $\mathcal{A}_2$ used in (3.7) are supposed to make $\mathcal{N}_1$ and $\mathcal{N}_2$ as simple as possible. Moreover, $\mathcal{N}_1$ and $\mathcal{N}_2$ are equally simple. For if $\mathcal{N}_2$ were simpler than $\mathcal{N}_1$, then (3.12) and (3.13) show that the map $(\mathcal{A}_2\mathcal{A})$ normalizes $\mathcal{M}_1$ to the simpler form $\mathcal{N}_2$, which is contrary to the assumption that $\mathcal{A}_1$ has been properly choosen in (3.6) to make $\mathcal{N}_1$ as simple as possible. Similarly, (3.11) and (3.14) show that $\mathcal{N}_1$ cannot be simpler than $\mathcal{N}_2$. We conclude that they must be the same,

$$\mathcal{N}_1 = \mathcal{N}_2. \tag{31.3.15}$$

Rmk. 3.1: We have seen that the normal form $\mathcal{N}_1$ of a map $\mathcal{M}_1$ is unique. However we remark that, without further requirements, the normalizing map $\mathcal{A}_1$ is not unique. Suppose $\mathcal{B}_1$ is any invertible map that commutes with $\mathcal{N}_1$,

$$\mathcal{N}_1\mathcal{B}_1 = \mathcal{B}_1\mathcal{N}_1. \tag{31.3.16}$$

Use $\mathcal{B}_1$ to conjugate both sides of (3.6). Doing so and making use of (3.16) gives the result

$$\mathcal{N}_1 = \mathcal{B}_1\mathcal{N}_1\mathcal{B}_1^{-1} = \mathcal{B}_1\mathcal{A}_1\mathcal{M}_1(\mathcal{B}_1\mathcal{A}_1)^{-1}. \tag{31.3.17}$$

We see that the map $(\mathcal{B}_1\mathcal{A}_1)$ also is a normalizing map that normalizes $\mathcal{M}_1$ to $\mathcal{N}_1$.

Thrm. 3.2: Conversely, suppose $\mathcal{A}_1$ and $\tilde{\mathcal{A}}_1$ are both normalizing maps for $\mathcal{M}_1$,

$$\mathcal{N}_1 = \mathcal{A}_1\mathcal{M}_1\mathcal{A}_1^{-1} = \tilde{\mathcal{A}}_1\mathcal{M}_1\tilde{\mathcal{A}}_1^{-1}. \tag{31.3.18}$$

Then $\tilde{\mathcal{A}}_1$ and $\mathcal{A}_1$ are related by the equation

$$\tilde{\mathcal{A}}_1 = \mathcal{B}_1\mathcal{A}_1 \tag{31.3.19}$$

with

$$\mathcal{B}_1 = \tilde{\mathcal{A}}_1\mathcal{A}_1^{-1}, \tag{31.3.20}$$

where $\mathcal{B}_1$ commutes with $\mathcal{N}_1$.

31.4 Sample Normal Forms

We now describe what normal forms can be achieved in various cases when we are working in the setting of a 6-dimensional phase space with the variables x, y, τ, p_x, p_y, p_τ. As illustrated in Figure 4.1, there are four broad possibilities for general maps $\mathcal{M}$: dynamic (τ-dependent) maps with or without translation factors (characterized by the presence or absence of f_1 terms); static (τ-independent) maps with or without translation factors. These cases are listed below, and will be discussed in subsequent sections.

i. Dynamic map with an f_1 translation factor

ii. Dynamic map without an f_1 translation factor

iii. Static map with an f_1 translation factor

iv. Static map without an f_1 translation factor

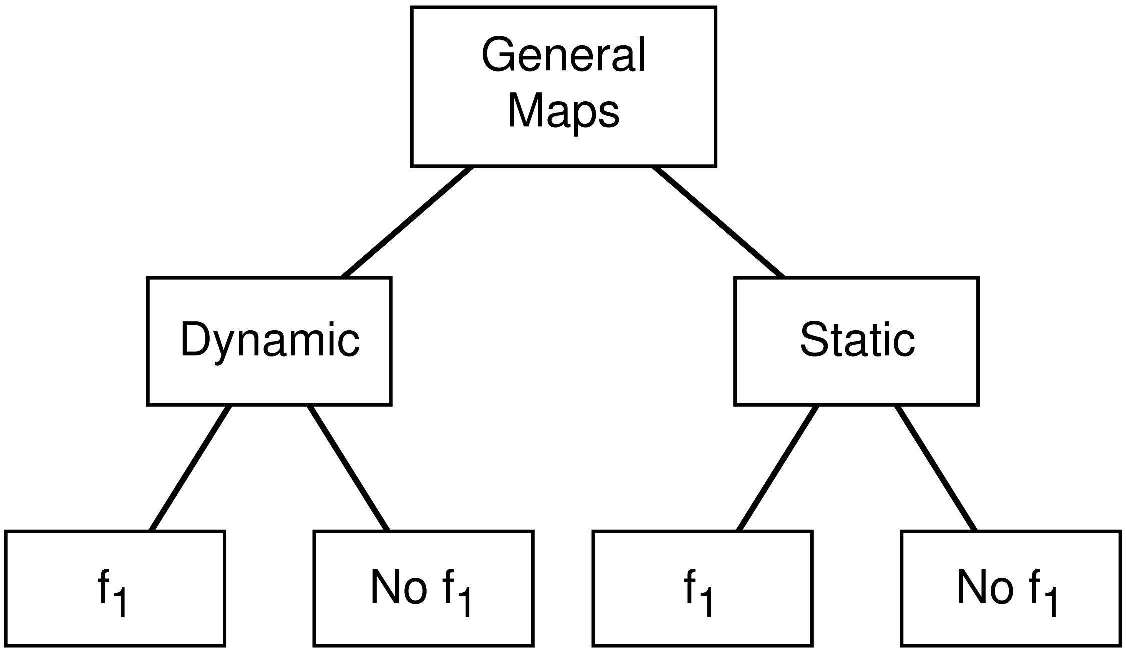

Figure 31.4.1: Four broad possibilities for general maps.

The easiest case to discuss is static maps without f_1 translation factors. We will therefore treat it first.

31.5 Static Maps Without Translation Factor

31.5.1 Properties of Linear Part

For the purposes of this section it is convenient to order the phase-space variables as

$$z = (x, p_x; y, p_y; \tau, p_\tau). \tag{31.5.1}$$

According to Section 3.2, J then takes the form (3.2.10) which, in the 6×6 case, is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{31.5.2}$$

Suppose $\mathcal{M}$ is an origin preserving symplectic map so that it has the Lie factorization

$$\mathcal{M} = \mathcal{R}\exp(:f_3:)\exp(:f_4:)\cdots . \tag{31.5.3}$$

Next assume the $\mathcal{M}$ and also has the property

$$\mathcal{M}p_\tau = p_\tau. \tag{31.5.4}$$

For reasons to become evident shortly, we will call such an $\mathcal{M}$ a *static* map. Upon using the representation (5.2) in (5.3) and equating terms of like degree, it follows that there are the relations

$$\mathcal{R}p_\tau = p_\tau. \tag{31.5.5}$$

$$0 =: f_m : p_\tau = [f_m, p_\tau] = \partial f_m/\partial\tau \text{ for } m \geq 3, \tag{31.5.6}$$

The relation (5.6) says that the generators f_m are τ independent (hence static) for $m \geq 3$, and we will see that (5.5) and the symplectic condition imply that the matrix R associated with $\mathcal{R}$ has a very special form.

To explore the properties of R, let us begin by writing it out in full using standard matrix notation,

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{pmatrix}. \tag{31.5.7}$$

Now require that $\mathcal{R}$ be a symplectic map that satisfies (5.5). From (3.1.10) we know that the symplectic condition requires the relation

$$RJR^T = J. \tag{31.5.8}$$

When written in terms of components, this relation takes the form

$$\sum_{bc} R_{ab}J_{bc}R_{dc} = J_{ad}. \tag{31.5.9}$$

As a result of these two requirements (5.5) and (5.8) we will see that many matrix elements of R are 0 or 1, and others are related.

Define the quantities Δ_1 through Δ_4 by the rules

$$\Delta_1 = R_{52}, \tag{31.5.10}$$

$$\Delta_2 = -R_{51}, \tag{31.5.11}$$

$$\Delta_3 = R_{54}, \tag{31.5.12}$$

$$\Delta_4 = -R_{53}, \tag{31.5.13}$$

and view them as the components of a vector Δ. Also define a matrix $\hat{R}$ by the rule

$$\hat{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & 0 & 0 \\ R_{21} & R_{22} & R_{23} & R_{24} & 0 & 0 \\ R_{31} & R_{32} & R_{33} & R_{34} & 0 & 0 \\ R_{41} & R_{42} & R_{43} & R_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{31.5.14}$$

and write (5.14) in the more compact form

$$\hat{R} = \begin{pmatrix} \check{R} & 0 \\ 0 & I \end{pmatrix} \tag{31.5.15}$$

where $\check{R}$ is the 4×4 matrix

$$\check{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{pmatrix}. \tag{31.5.16}$$

Finally, let $\check{J}$ be the the 4×4 version of J,

$$\check{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{31.5.17}$$

Then, it is the case that R must be of the more specific form

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & 0 & (\check{R}\Delta)_1 \\ R_{21} & R_{22} & R_{23} & R_{24} & 0 & (\check{R}\Delta)_2 \\ R_{31} & R_{32} & R_{33} & R_{34} & 0 & (\check{R}\Delta)_3 \\ R_{41} & R_{42} & R_{43} & R_{44} & 0 & (\check{R}\Delta)_4 \\ -\Delta_2 & \Delta_1 & -\Delta_4 & \Delta_3 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{31.5.18}$$

Also, the entries in the upper left 4×4 block of R, the entries in $\check{R}$, must obey the "reduced" symplectic relations

$$\check{R}\check{J}\check{R}^T = \check{J}. \tag{31.5.19}$$

Observe that the entries R_{16}, R_{26}, R_{36}, and R_{46} in R describe dispersive effects. That is, they describe how the transverse coordinates and momenta depend on p_τ. By contrast, the entries R_{51}, R_{52}, R_{53}, and R_{54} in R describe how the time of flight depends on the transverse initial conditions. From (5.18) we see that dispersive effects and time of flight effects are related by the symplectic condition! They are opposite sides of the same coin. This is an example of what we call *symplectic reciprocity*: seemingly unrelated quantities are in fact related by the symplectic condition.

We will prove this result in stages: We recall that the matrix R associated with $\mathcal{R}$ is given by the relation

$$\mathcal{R}z_a = \sum_b R_{ab} z_b. \tag{31.5.20}$$

As a result of (5.20), the condition (5.5) requires that R have the more specific form

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{31.5.21}$$

Next, impose the symplectic condition (5.8) for R of the form (5.21). Set $a = 6$ in the relation (5.9) to get the result

$$\sum_{bc} R_{6b} J_{bc} R_{dc} = J_{6d}. \tag{31.5.22}$$

But, from (5.21), we know that

$$R_{6b} = \delta_{6b}. \tag{31.5.23}$$

Therefore the sum (5.22) becomes

$$\sum_c J_{6c} R_{dc} = J_{6d}. \tag{31.5.24}$$

Also, we see from (5.2) that

$$J_{6c} = -\delta_{5c}. \tag{31.5.25}$$

Therefore the sum (5.24) reduces to the result

$$-R_{d5} = J_{6d} \tag{31.5.26}$$

from which we conclude that

$$R_{d5} = 0 \text{ for } d = 1 \text{ to } 4, \tag{31.5.27}$$

$$R_{55} = 1. \tag{31.5.28}$$

Consequently, R must have the yet more specific form

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & 0 & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & 0 & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & 0 & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & 0 & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{31.5.29}$$

Define the quantity ξ by the rule

$$\xi = R_{56}, \tag{31.5.30}$$

and associate with ξ and Δ the matrices $C(\xi)$ and $D(\Delta)$ by the rules

$$C(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \xi \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{31.5.31}$$

$$D(\Delta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \Delta_1 \\ 0 & 1 & 0 & 0 & 0 & \Delta_2 \\ 0 & 0 & 1 & 0 & 0 & \Delta_3 \\ 0 & 0 & 0 & 1 & 0 & \Delta_4 \\ -\Delta_2 & \Delta_1 & -\Delta_4 & \Delta_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{31.5.32}$$

It is easily verified that the matrices $C(\xi)$ and $D(\Delta)$ are symplectic and have the inverses

$$C^{-1}(\xi) = C(-\xi), \tag{31.5.33}$$

$$D^{-1}(\Delta) = D(-\Delta). \tag{31.5.34}$$

Indeed, $C(\xi)$ and $D(\Delta)$ are the matrices associated with the linear symplectic maps $\mathcal{C}(\xi)$ and $\mathcal{D}(\Delta)$ given by the relations

$$\mathcal{C} = \exp(: -\xi p_\tau^2/2 :), \tag{31.5.35}$$

$$\mathcal{D} = \exp(: p_\tau g_1 :), \tag{31.5.36}$$

where

$$g_1(\Delta) = \Delta_2 x - \Delta_1 p_x + \Delta_4 y - \Delta_3 p_y. \tag{31.5.37}$$

Note, for future use, that the matrix C commutes with both the matrices D and $\hat{R}$.

We now assert that R has the factorization

$$R = \hat{R} C D \tag{31.5.38}$$

or, equivalently,

$$\hat{R} = R D^{-1} C^{-1}. \tag{31.5.39}$$

The proof of this assertion involves matrix multiplication and invoking the symplectic condition for R. Define a matrix $\hat{R}'$ by the rule

$$\hat{R}' = R D^{-1} C^{-1}. \tag{31.5.40}$$

Carrying out the indicated multiplications gives the result

$$\hat{R}' = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & 0 & \epsilon_1 \\ R_{21} & R_{22} & R_{23} & R_{24} & 0 & \epsilon_2 \\ R_{31} & R_{32} & R_{33} & R_{34} & 0 & \epsilon_3 \\ R_{41} & R_{42} & R_{43} & R_{44} & 0 & \epsilon_4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{31.5.41}$$

where

$$\begin{aligned}\epsilon_1 &= R_{16} - [R_{11}R_{52} + R_{12}(-R_{51}) + R_{13}R_{54} + R_{14}(-R_{53})] \\ &= R_{16} - (\check{R}\Delta)_1,\end{aligned} \tag{31.5.42}$$

$$\begin{aligned}\epsilon_2 &= R_{26} - [R_{21}R_{32} + R_{22}(-R_{52}) + R_{23}R_{54} + R_{24}(-R_{53})] \\ &= R_{26} - (\check{R}\Delta)_2,\end{aligned} \tag{31.5.43}$$

$$\begin{aligned}\epsilon_3 &= R_{36} - [R_{31}R_{52} + R_{32}(-R_{51}) + R_{33}R_{54} + R_{34}(-R_{53})] \\ &= R_{36} - (\check{R}\Delta)_3,\end{aligned} \tag{31.5.44}$$

$$\begin{aligned}\epsilon_4 &= R_{46} - [R_{41}R_{52} + R_{42}(-R_{51}) + R_{43}R_{54} + R_{44}(-R_{53})] \\ &= R_{46} - (\check{R}\Delta)_4.\end{aligned} \tag{31.5.45}$$

Next, because R, C^{-1}, and D^{-1} are symplectic matrices, $\hat{R}'$ must a symplectic matrix. In analogy with (5.9), the symplectic condition for $\hat{R}'$ can be written in the form

$$\sum_{bc} \hat{R}'_{ab} J_{bc} \hat{R}'_{dc} = J_{ad}. \tag{31.5.46}$$

Now put $a = 5$ in (5.46) and make use of the special forms of J and $\hat{R}'$ as given by (5.2) and (5.41). Doing so gives the result

$$\sum_{bc} \hat{R}'_{5b} J_{bc} \hat{R}'_{dc} = J_{5d}, \tag{31.5.47}$$

which yields the relations

$$\hat{R}'_{d6} = J_{5d}. \tag{31.5.48}$$

From (5.2), (5.41), and (5.48), we conclude that

$$\epsilon_d = 0 \text{ for } d = 1 \text{ to } 4. \tag{31.5.49}$$

Therefore there is the relation

$$\hat{R}' = \hat{R}, \tag{31.5.50}$$

and (3.39) is correct.

Moreover, in view of (5.42) through (5.45) and (5.49), we have found the relations

$$R_{a6} = (\check{R}\Delta)_a \text{ for } a = 1 \text{ to } 4 \tag{31.5.51}$$

with Δ given by (5.10) through (5.13). Thus, (5.18) is correct. To reiterate, what we have learned from the symplectic condition is that in (5.29) the matrix elements R_{a6} for $a = 1$ to 4 are not independent of the matrix elements R_{5a} for $a = 1$ to 4, but instead are related by the conditions (5.10) through (5.13) and (5.51). Finally, because of (5.15), the remaining relations demanded by (5.46) yield the matrix relation (5.19).

We close this section by noting that the matrix relation (5.38) implies a related factorization for the map $\mathcal{R}$. Let $\hat{\mathcal{R}}$ be the symplectic map associated with $\hat{R}$. Then (5.38) is equivalent to the map factorization relation

$$\mathcal{R} = \mathcal{D}\mathcal{C}\hat{\mathcal{R}} = \mathcal{C}\mathcal{D}\hat{\mathcal{R}}. \tag{31.5.52}$$

Here we have used the fact that C and D commute.

Exercises

31.5.1. Verify that the maps $\mathcal{C}$ and $\mathcal{D}$ given by (5.35) and (5.36) are equivalent to the matrices C and D given by (5.31) and (5.32).

31.5.2. Starting with (5.40), verify that carrying out the indicated multiplications yields the results (5.41) through (5.45).

31.6 Static Maps With Translation Factor

31.7 Tunes, Phase Advances and Slips, Momentum Compaction, Chromaticities, and Anharmonicities

- Ring Analysis and Phase Advances
- Equivalent Locations
- Matched Insertions
- Floquet Theory
- Tune Footprints

31.8 Courant-Snyder Invariants and Lattice Functions

31.9 Dynamic Maps Without Translation Factor

31.10 Dynamic Maps With Translation Factor

31.11 Analysis of Tracking Data

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Chapter 32

Lattice Functions

Chapter 33

Solved and Unsolved Polynomial Orbit Problems: Invariant Theory

33.1 Introduction

As in (8.5.2), let $\mathcal{R}$ be any linear symplectic map [a map corresponding to an $Sp(2n)$ transformation] written in the general form

$$\mathcal{R} = \exp(:f_2^c:)\exp(:f_2^a:). \tag{33.1.1}$$

Suppose $\mathcal{R}$ acts on any homogeneous polynomial g_m in $\mathcal{P}_m$. Then, in view of (21.5.6), the result is a *transformed* polynomial $g_m^{\rm tr}$ that is also in $\mathcal{P}_m$,

$$g_m^{\rm tr}(z) = \mathcal{R}g_m(z) = g_m(\mathcal{R}z) = g_m(Rz). \tag{33.1.2}$$

Here we have also used (8.4.15). Indeed, we know from the work of Chapter 21 that in the two-variable case the $\mathcal{P}_m$ carry the irreducible representation $\Gamma(m)$ of $Sp(2)$; in the four-variable case the $\mathcal{P}_m$ carry the irreducible representation $\Gamma(m,0)$ of $Sp(4)$; in the six-variable case the $\mathcal{P}_m$ carry the irreducible representation $\Gamma(m,0,0)$ of $Sp(6)$; etc.

The set of polynomials $g_m^{\rm tr}$ that can be obtained from any given g_m and arbitrary $\mathcal{R}$ of the form (1.1) is called the *orbit* of g_m under the action of $Sp(2n)$. Now suppose that h_m is any other polynomial in $\mathcal{P}_m$. We will say that h_m is *equivalent* to g_m if there is some $\mathcal{R}$ of the form (1.1) that sends g_m to h_m,

$$h_m \sim g_m \Leftrightarrow h_m = \mathcal{R}g_m \text{ for some } \mathcal{R}. \tag{33.1.3}$$

It is easy to check that (1.3) is indeed an equivalence relation, and we may say that two polynomials in $\mathcal{P}_m$ are equivalent if they lie on the same orbit.

Finally, suppose we are given some polynomial g_m. Then the equivalence class of g_m, which we will denote by $\{g_m\}$, consists of all the $g_m^{\rm tr}$ given by (1.2) for all choices of $\mathcal{R}$. (Thus, the equivalence class $\{g_m\}$ is the orbit of g_m.) Among the $g_m^{\rm tr}$ produced in this fashion there will be one that has some particularly desirable form or property. Various possibilities come to mind: For example, we may attempt to drive to zero as many coefficients in $g_m^{\rm tr}$ as possible by a particular choice of $\mathcal{R}$. Or, if g_m happens to be on the orbit of some monomial, we might like to discover which monomial and determine its coefficient. Or, we might like to find a $g_m^{\rm tr}$ on the oribit of g_m that has the smallest length in the sense of minimizing the scalar product $\langle g_m^{\rm tr}, g_m^{\rm tr}\rangle$ as defined in Section 7.3. A or the $g_m^{\rm tr}$ that has some such desirable form or property, or perhaps some other property yet to be discovered, will be called the *normal form* of g_m and will be denoted by the symbols g_m^N. (We remark that in some literature a normal form is called a *canonical* form, and homogeneous polynomials or ratios of homogeneous polynomials are called *quantics*.) Put another way, a normal form of g_m is a particularly simple or pleasing point on the orbit of g_m. Exactly what a normal form for g_m should be is partly a matter of investigation, and partly a matter of choice. Given a g_m, one must first examine all the members of the equivalence class $\{g_m\}$. Then, with their properties clearly in mind, one selects a particularly pleasing $g_m^{\rm tr}$ and calls it g_m^N. Ideally one would like to have an algorithm that takes g_m as an input and provides as outputs g_m^N and the *normalizing* $\mathcal{R}$ that transforms g_m into g_m^N.

Three facts are now obvious practically as a matter of definition. First, $Sp(2n)$ acts transitively on each equivalence class. Second, we may label the equivalence class of g_m by specifying g_m^N. That is. we have the relation

$$\{g_m\} = \{g_m^N\}. \tag{33.1.4}$$

Third, suppose two polynomials g_m and h_m are known or can be shown to have the same normal form,

$$g_m^N = h_m^N. \tag{33.1.5}$$

Then, they are in the same equivalence class and there is an $\mathcal{R}$ that sends one into the other as in (1.3).

There is another terminology that is sometimes used for the situation we have been describing. In this terminology each equivalence class (orbit) is called a *leaf*, and the decomposition of $\mathcal{P}_m$ into equivalence classes is called a *foliation*.

Evidently a general homogeneous polynomial g_m is specified by giving its coefficients. It can be shown (and we will see examples) that there exist polynomial functions of these coefficients that remain unchanged under the transformation (1.2). These functions are called *invariants*. Thus if h_m and g_m are equivalent as in (1.3), each invariant function must have the same value for the coefficients of h_m and the coefficients of g_m.

Why, apart from curiosity, should one care about orbits of g_m in $\mathcal{P}_m$, normal forms g_m^N, and invariants? We will see in Chapter 33 that a knowledge of normal forms for g_2 and invariants for g_m is useful for characterising beams. In Chapter 34 we will see that normal forms for g_3, g_4, $\cdots$ might, if we knew them, be useful in the approximate but exactly symplectic numerical evaluation of the effect of a general map $\mathcal{M}$ on a general phase-space point z as in (7.6.2).

Exercises

33.1.1. Show that (1.3) defines an equivalence relation. See Exercise (5.12.7).

33.2 Solved Polynomial Orbit Problems

In this section we will describe briefly some of what is known about the orbits of g_m under the action of $Sp(2n)$ or, to be more precise, $Sp(2n, \mathbb{R})$. Our results will be fairly complete for the cases $m = 1$ and $m = 2$, and therefore these cases can be characterized as being *solved*. The cases $m > 2$ are much more difficult, and will be characterized largely in terms of what is not known. They are treated in the next section.

First, to dispell a possible false expectation, recall the action of the rotation group on ordinary 3-dimensional Euclidean space. If x, y, z are the usual Cartesian coordinates in Euclidean 3-space, we know that the group $SO(3)$ of rotations about the origin *preserves* the polynomial $(x^2 + y^2 + z^2)$ and any function of this polynomial. Does something analogous happen for the action of $Sp(2n)$ on phase space? The answer is *no*. Suppose that some g_m is preserved,

$$g_m^{\text{tr}} = g_m. \tag{33.2.1}$$

Then, from (1.2), we find the result

$$g_m(Rz) = g_m(z) \tag{33.2.2}$$

for all R in $Sp(2n)$. But, from Sections 3.6.5 and 7.2, we know that $Sp(2n)$ acts transitively on phase space. Therefore, any g_m that satisfies (2.2) for all R must have the same value everywhere in phase space, and the only such polynomial is g_0.

There is another instructive way to reach the same conclusion. From (1.1), (1.2), and (2.2) one sees that to be preserved g_m must satisfy the relation

$$\exp(:\epsilon f_2:)g_m = g_m \text{ for all } f_2. \tag{33.2.3}$$

The infinitesimal version of (2.3) is the relation

$$:f_2:g_m = 0 \text{ for all } f_2. \tag{33.2.4}$$

But, say for $sp(6)$, we know that any g_m belongs to the *irreducible* representation $\Gamma(m, 0, 0)$. See Section 1.8. Therefore, the only way that (2.4) can be satisfied is to have $m = 0$.

33.2.1 First-Order Polynomials

We have seen that there is no nontrivial preserved g_m. Thus, $Sp(2n)$ must have some genuine action on each $\mathcal{P}_m$. Let us begin with the case of $\mathcal{P}_1$. Any g_1 in $\mathcal{P}_1$ can be written in the form

$$g_1(a; z) = \sum_j a_j z_j = (a, z). \tag{33.2.5}$$

In this case use of (1.2) gives the result

$$g_1^{\text{tr}}(a; z) = (a, Rz) = (R^T a, z) = g_1(R^T a; z) = g_1(a^{\text{tr}}; z). \tag{33.2.6}$$

Here we have introduced the notation

$$a^{\text{tr}} = R^T a. \tag{33.2.7}$$

We know that R^T is symplectic if R is symplectic; and we again recall that $Sp(2n)$ acts transitively. It follows that if a is any nonzero $2n$-vector, there is a symplectic R such that $a^{\rm tr}$ is any desired vector. Therefore, $\mathcal{P}_1$ decomposes into two equivalence classes: the identically zero polynomial and all the rest. If we ignore the trivial case of the identically zero polynomial, we may say that $\mathcal{P}_1$ consists of only one equivalence class and correspondingly, a single orbit. A convenient normal form is the monomial q_1 with unit coefficient,

$$g_1^N = q_1. \tag{33.2.8}$$

33.2.2 Second-Order Polynomials

Next consider $\mathcal{P}_2$. Here the situation is more complicated. Any g_2 in $\mathcal{P}_2$ can be written in the form

$$g_2(S;z) = \sum_{jk} S_{jk} z_j z_k = (z, Sz) \tag{33.2.9}$$

where S is any symmetric matrix. In this case use of (1.2) gives the result

$$g_2^{\rm tr}(S;z) = (Rz, SRz) = (z, R^T SRz) = g_2(S^{\rm tr}; z) \tag{33.2.10}$$

where

$$S^{\rm tr} = R^T SR. \tag{33.2.11}$$

It is easily checked that $S^{\rm tr}$ is symmetric if S is.

The analysis of the relation (2.11) is facilitated by a trick. Let B denote the *Hamiltonian* matrix gotten from S by the rule

$$B = JS. \tag{33.2.12}$$

Since J is invertible, one can always find S given B, and vice versa. See Section 3.7. Next we define $B^{\rm tr}$ by the rule

$$B^{\rm tr} = JS^{\rm tr}. \tag{33.2.13}$$

With the aid of these definitions the relation (2.11) takes the form

$$B^{\rm tr} = JS^{\rm tr} = JR^T SR = JR^T J^{-1} JSR = R^{-1} BR. \tag{33.2.14}$$

Here we have used (3.1.9). With the aid of J we have turned a symplectic congruency relation (2.11) into a symplectic conjugacy (similarity) relation (2.14). What we learn from (2.14) is that the problem of finding orbits in $\mathcal{P}_2$ is equivalent to finding orbits in the space of $2n \times 2n$ real Hamiltonian matrices under the action of real symplectic similarity transformations. We know that eigenvalues are unchanged by similarity transformations, and therefore expect that eigenvalues and functions constructed from eigenvalues will play an important role.

Suppose we were allowed to make arbitrary (including complex and nonsymplectic) similarity transformations. Then we know that B, if it has distinct eigenvalues, can be diagonalized. And if the eigenvalues are not distinct, B might still be diagonalizable or, in the worst case, it could still be brought to Jordan normal form. We might define the diagonal or Jordan form for B to be $B^{\rm tr}$, and then try to form $S^{\rm tr}$ and $g_2^N = g_2^{\rm tr}$ accordingly. However, we are only allowed to use real symplectic similarity transformations, and we must see to what extent something analogous can be done using only such transformations.

32.2.2.1 Two-Dimensional Phase-Space Case

We will come to the general $2n \times 2n$ case eventually. As a warm-up exercise, consider first the 2×2 case for 2-dimensional phase space.. Then g_2 has the general form

$$g_2 = \beta p^2 + 2\alpha pq + \gamma q^2 \tag{33.2.15}$$

where α, β, γ are arbitrary constants. Correspondingly, the matrices S and B take the forms

$$S = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}, \tag{33.2.16}$$

$$B = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}. \tag{33.2.17}$$

Evidently the transformation (2.14) cannot change the determinant of B which we will call δ,

$$\delta = \det B = \beta\gamma - \alpha^2. \tag{33.2.18}$$

That is, δ is an *invariant* constructed from the coefficients of g_2. [Note that δ is just the negative of the discriminant. See (8.7.30).] However, as will be seen, we can change α, β, γ while maintaining the condition (2.18). Note that the matrix B has the characteristic polynomial

$$P(\lambda) = \det(B - \lambda I) = \lambda^2 + \beta\gamma - \alpha^2 = \lambda^2 + \delta, \tag{33.2.19}$$

and therefore has the eigenvalues

$$\lambda_\pm = \pm(\alpha^2 - \beta\gamma)^{1/2} = \pm(-\delta)^{1/2}. \tag{33.2.20}$$

It is convenient to consider separately the six cases listed below:

i. $\beta > 0$

ii. $\beta < 0$

iii. $\gamma > 0$

iv. $\gamma < 0$

v. $\beta = \gamma = 0$

vi. $\alpha = \beta = \gamma = 0$

In the next few paragraphs we will treat them one by one.

Case *i*, $\beta > 0$

Let us begin with case i by supposing $\beta > 0$. Let M be the symplectic matrix defined by the equation

$$M = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix}. \tag{33.2.21}$$

Then use of (2.11) with $R = M$ gives a transformed S that we will call S',

$$S' = M^T S M = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}. \tag{33.2.22}$$

Next suppose that $\delta > 0$. In this case we conclude from (2.18) that $\gamma > 0$ and from (2.20) that the eigenvalues $\lambda_\pm$ are pure imaginary. Let N be the symplectic matrix

$$N = \begin{pmatrix} \delta^{-1/4} & 0 \\ 0 & \delta^{1/4} \end{pmatrix}. \tag{33.2.23}$$

Use of N to transform S' to $S^{\rm tr}$ gives the result

$$S^{\rm tr} = N^T S' N = \begin{pmatrix} \delta^{-1/4} & 0 \\ 0 & \delta^{1/4} \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta^{-1/4} & 0 \\ 0 & \delta^{1/4} \end{pmatrix} = \begin{pmatrix} \delta^{1/2} & 0 \\ 0 & \delta^{1/2} \end{pmatrix}. \tag{33.2.24}$$

Correspondingly $g_2^{\rm tr}$ is given by the relation

$$g_2^{\rm tr} = (z, S^{\rm tr} z) = \delta^{1/2}(p^2 + q^2). \tag{33.2.25}$$

Suppose instead that $\delta = 0$. Now we conclude from (2.18) that $\gamma \geq 0$ and from (2.20) that the eigenvalues $\lambda_\pm$ both vanish. In this case S' as given by (2.22) can be used directly to give the result

$$g_2^{\rm tr} = (z, S' z) = p^2. \tag{33.2.26}$$

It is easily verified that p^2 and q^2 are equivalent,

$$p^2 \sim q^2. \tag{33.2.27}$$

See Exercise 2.1. Therefore, if desired, we can find and employ an R such that

$$g_2^{\rm tr} = q^2. \tag{33.2.28}$$

Finally suppose that $\delta < 0$ (in which case $\beta\gamma < \alpha^2$ and the eigenvalues $\lambda_\pm$ are real). Let N be the symplectic matrix

$$N = \begin{pmatrix} (-\delta)^{-1/4} & 0 \\ 0 & (-\delta)^{1/4} \end{pmatrix}. \tag{33.2.29}$$

Use of N to transform S' to $S^{\rm tr}$ gives the result

$$\begin{aligned} S^{\rm tr} = N^T S' N &= \begin{pmatrix} (-\delta)^{-1/4} & 0 \\ 0 & (-\delta)^{1/4} \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (-\delta)^{-1/4} & 0 \\ 0 & (-\delta)^{1/4} \end{pmatrix} \\ &= (-\delta)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{33.2.30}$$

Correspondingly $g_2^{\rm tr}$ is given by the relation

$$g_2^{\rm tr} = (z, S^{\rm tr} z) = (-\delta)^{1/2}(q^2 - p^2). \tag{33.2.31}$$

Case *ii*, $\beta < 0$

To consider case ii, suppose $\beta < 0$. Let M be the symplectic matrix defined by the equation

$$M = \begin{pmatrix} \sqrt{-\beta} & 0 \\ \alpha/\sqrt{-\beta} & 1/\sqrt{-\beta} \end{pmatrix}. \tag{33.2.32}$$

Then use of (2.11) with $R = M$ gives a transformed S which we again call S',

$$S' = M^T S M = \begin{pmatrix} -\delta & 0 \\ 0 & -1 \end{pmatrix}. \tag{33.2.33}$$

Next suppose $\delta > 0$. In this case we conclude from (2.18) that $\gamma < 0$ and from (2.20) that the eigenvalues $\lambda_\pm$ are pure imaginary. Again let N be the symplectic matrix (2.23). Then we find for $S^{\rm tr}$ the result

$$S^{\rm tr} = N^T S' N = -\delta^{1/2} I. \tag{33.2.34}$$

Correspondingly $g_2^{\rm tr}$ is given by the relation

$$g_2^{\rm tr} = -\delta^{1/2}(p^2 + q^2). \tag{33.2.35}$$

Suppose instead that $\delta = 0$. Now we conclude from (2.18) that $\gamma \leq 0$ and from (2.20) that the eigenvalues $\lambda_\pm$ both vanish. In this case use of S' directly gives the result

$$g_2^{\rm tr} = -p^2. \tag{33.2.36}$$

Alternatively, in view of (2.27), we can find an R such that

$$g_2^{\rm tr} = -q^2. \tag{33.2.37}$$

Finally suppose $\delta < 0$. Then use of (2.33) and N given by (2.29), and calling the result S'', give the relation

$$S'' = -\begin{pmatrix} (-\delta)^{1/2} & 0 \\ 0 & -(-\delta)^{1/2} \end{pmatrix} = -(-\delta)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33.2.38}$$

Also, it is easily verified that in this case use of the symplectic matrix J gives the relation

$$S^{\rm tr} = J^T S'' J = -S'' = (-\delta)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33.2.39}$$

It follows that we may take for $g_2^{\rm tr}$ the polynomial

$$g_2^{\rm tr} = (-\delta)^{1/2}(q^2 - p^2). \tag{33.2.40}$$

Cases *iii* and *iv*, $\gamma > 0$ or $\gamma < 0$

We have covered cases i and ii. Exercise 2.2 shows that cases iii and iv give results identical to those for cases i and ii, respectively. In particular, we still find the results (2.25), (2.26), (2.28), (2.31), (2.35), (2.36), (2.37), and (2.40) [which is identical to (2.31)].

Case *v*, $\beta = \gamma = 0$

For case v we immediately have the result

$$S = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} = (-\delta)^{1/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{33.2.41}$$

and g_2 takes the form

$$g_2 = 2(-\delta)^{1/2} qp. \tag{33.2.42}$$

Let O be the symplectic matrix

$$O = (1/\sqrt{2}) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{33.2.43}$$

Use of O to transform S gives the result

$$S^{\rm tr} = O^T S O = (-\delta)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33.2.44}$$

Correspondingly, we find for $g_2^{\rm tr}$ the identical result (2.40). Note that comparison of (2.41) and (2.44) reveals that there are the equivalence relations

$$+(-\delta)^{1/2}(q^2 - p^2) \sim -(-\delta)^{1/2}(q^2 - p^2) \sim -2(-\delta)^{-1/2} qp \sim +2(-\delta)^{1/2} qp. \tag{33.2.45}$$

Case *vi*, $\alpha = \beta = \gamma = 0$

Finally, case vi gives the zero polynomial.

Normal Forms

Evidently, we may take the various $g_2^{\rm tr}$ discovered for cases i through vi to be normal forms. We see that, for the most part, the normal form is labeled by the value of the invariant δ with additional qualifications for the sign of β or γ in the cases $\delta \geq 0$. Therefore, as mentioned before, a necessary condition for $h_2 \sim g_2$ is that they have the *same* invariant. All these results are summarized in Figure 2.1 below.

$\delta < 0$	$\delta = 0$	$\delta > 0$
$(-\delta)^{1/2}(q^2 - p^2)$	q^2 or p^2 if β or $\gamma > 0$	$(\delta)^{1/2}(p^2 + q^2)$ if β or $\gamma > 0$
or	$-\,q^2$ or $-\,p^2$ if β or $\gamma < 0$	$-\,(\delta)^{1/2}(p^2 + q^2)$ if β or $\gamma < 0$
$2(-\delta)^{1/2}pq$		

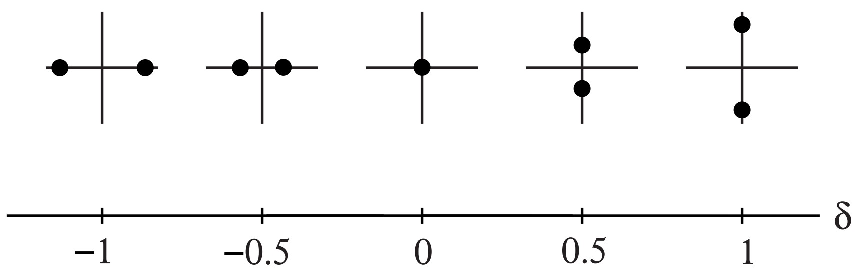

Figure 33.2.1: Normal forms g_2^N and eigenvalue spectrum of associated Hamiltonian matrices in the case of 2-dimensional phase space. The normal forms given in the three columns above are for the cases $\delta < 0$, $\delta = 0$, and $\delta > 0$, respectively.

Geometrical Description

It is also instructive to examine the surfaces $\beta\gamma - \alpha^2 = \delta$ for various values of δ. We will see that each such surface is an orbit. To do so, it is convenient to perform a 45° rotation in the β, γ plane, and to scale α, by introducing new variables ξ, η, ζ by the definitions

$$\beta = (1/\sqrt{2})(\xi - \eta), \tag{33.2.46}$$

$$\gamma = (1/\sqrt{2})(\xi + \eta), \tag{33.2.47}$$

$$\alpha = (1/\sqrt{2})\zeta. \tag{33.2.48}$$

In terms of these variables the relation (2.18) becomes

$$\xi^2 - \eta^2 - \zeta^2 = 2\delta. \tag{33.2.49}$$

Also, use of the scalar product of Section 7.3 and (2.15) gives the result

$$\langle g_2, g_2 \rangle = 2\beta^2 + 4\alpha^2 + 2\gamma^2 = 2(\xi^2 + \eta^2 + \zeta^2). \tag{33.2.50}$$

Thus, polynomials of any given norm are *spheres* in ξ, η, ζ space.

In the case that $\delta \le 0$, the relation (2.49) can be rewritten in the form

$$\eta^2 + \zeta^2 - \xi^2 = -2\delta. \tag{33.2.51}$$

For $\delta < 0$ this equation yields hyperboloids of one sheet with symmetry axis ξ. See Figure 2.2a below. Evidently for $\delta < 0$ the points on a given hyperboloid that are closest to the origin lie on the plane $\xi = 0$, in which case (2.51) becomes the circle

$$\eta^2 + \zeta^2 = -2\delta. \tag{33.2.52}$$

According to (2.50) all polynomials on this circle have the same squared norm, namely -4δ. The normal form (2.31) lies on this circle and has the coordinates

$$\xi = 0,\ \eta = (-2\delta)^{1/2},\ \zeta = 0, \tag{33.2.53}$$

or, equivalently,

$$-\beta = \gamma = (-\delta)^{1/2},\ \alpha = 0. \tag{33.2.54}$$

Here use has been made of (2.46) through (2.48). Evidently all points on the hyperboloid corresponding to a given value of $\delta < 0$ lie on the same orbit. That is, there is one equivalence class for each value of $\delta < 0$.

The case $\delta = 0$ produces two cones with a common vertex at the origin. Again see Figure 2.2a. Shortly we will discuss it more.

For $\delta > 0$ the equation (2.49) yields a hyperboloid of two sheets with symmetry axis ξ. See Figure 2.2b. On the upper sheet $\xi > 0$, and on the lower sheet $\xi < 0$. Also from (2.49) we conclude that

$$\xi^2 - \eta^2 = \zeta^2 + 2\delta > 0 \text{ when } \delta > 0. \tag{33.2.55}$$

It follows from (2.46) and (2.47) that $\beta, \gamma > 0$ on the upper sheet and $\beta, \gamma < 0$ on the lower sheet. Also, on the upper sheet and for a given value of $\delta > 0$, there is a single point closest to the origin; from (2.49) it has the coordinates

$$\eta = \zeta = 0,\ \xi = (2\delta)^{1/2}, \tag{33.2.56}$$

or, equivalently,

$$\beta = \gamma = \delta^{1/2},\ \alpha = 0. \tag{33.2.57}$$

This point corresponds to the normal form given by (2.25). Similarly, on the corresponding lower sheet, there is also a single point closest to the origin with the coordinates

$$\eta = \zeta = 0,\ \xi = -(2\delta)^{1/2}, \tag{33.2.58}$$

or, equivalently,

$$\beta = \gamma = -\delta^{1/2},\ \alpha = 0. \tag{33.2.59}$$

This point corresponds to the normal form given by (2.35). Evidently, for a fixed value of $\delta > 0$, all points on the upper sheet lie on the same orbit, and those on the lower sheet lie on a second distinct orbit. Consequently, there are two equivalence classes for each positive value of δ. The upper sheet gives the case $\beta, \gamma > 0$, and the lower sheet gives the case $\beta, \gamma < 0$.

There remains the case $\delta = 0$ for which, as already mentioned, the relation (2.49) yields two cones. These cones have no point in common save the origin which corresponds to the single-element equivalence class $g_2 = 0$. Moreover, points on the upper and lower cones belong to separate equivalence classes. It is easy to check that β or $\gamma > 0$ on the upper cone and β or $\gamma < 0$ on the lower cone. (Here the origin is to be excluded.) The monomial $+q^2$ given by (2.28) provides a normal form for polynomials corresponding to points on the upper cone. Its coordinates are given by the relations

$$\xi = \eta = 1/\sqrt{2},\ \zeta = 0, \tag{33.2.60}$$

or, equivalently,

$$\alpha = \beta = 0,\ \gamma = 1. \tag{33.2.61}$$

The monomial $-q^2$ given by (2.37) provides a normal form for polynomials corresponding to points on the lower cone. Its coordinates are given by the relations

$$\xi = \eta = -1/\sqrt{2},\ \zeta = 0, \tag{33.2.62}$$

or, equivalently,

$$\alpha = \beta = 0,\ \gamma = -1. \tag{33.2.63}$$

It follows that each cone is a separate orbit.

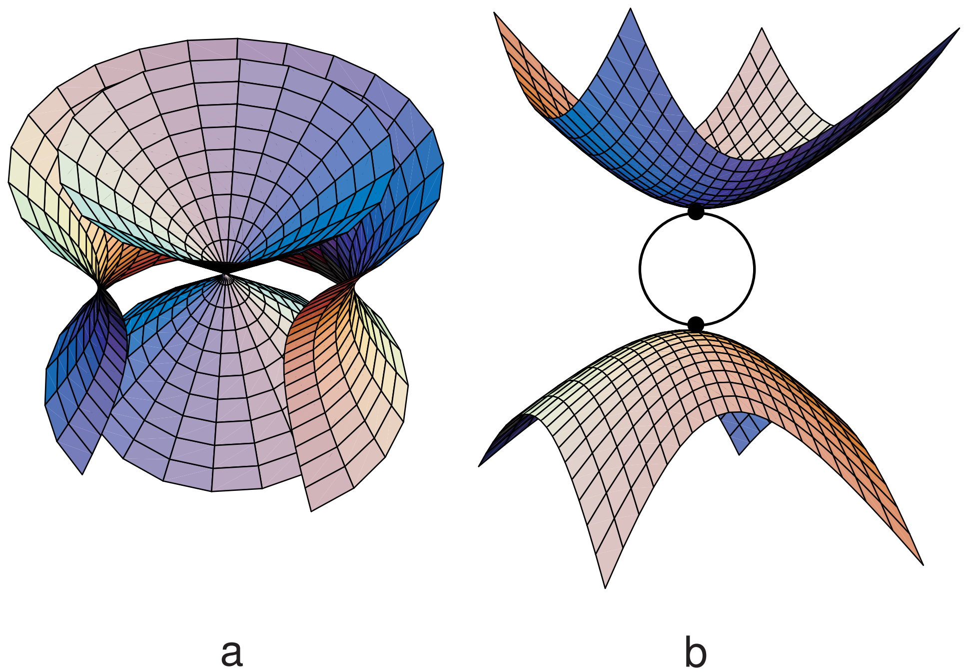

Figure 33.2.2: Equivalence classes (orbits/leaves) for the space $\mathcal{P}_2$ of second-order polynomials in two variables. They are displayed in terms of the variables ξ, η, ζ. In this figure the ξ axis points upward, the η axis points out of the page, and the ζ axis points to the right. Case a, for which $\delta < 0$, shows a typical one-sheeted hyperboloid. The point on the equator given by (2.53) and (2.54) corresponds to the normal form $(-\delta)^{1/2}(q^2 - p^2)$ for that value of δ. Also shown on this diagram are the two cones for $\delta = 0$. The point on the front of the top cone given by ((2.60) and (2.61) corresponds to the normal form $+q^2$ and the point on the rear of the bottom cone given (2.62) and (2.63) by corresponds to the normal form $-q^2$. The origin where the cones meet is the single-element equivalence class $g_2 = 0$. Case b, for which $\delta > 0$, shows a typical two-sheeted hyperboloid. Also shown is the sphere (2.50) that just kisses the hyperboloid. The two kissing points (the points on the upper and lower sheets that are closest to the origin) correspond to the normal forms $\pm\delta^{1/2}(p^2 + q^2)$. For simplicity, cases a and b are shown separately. They should actually be superimposed along with many other such hyperboloids to show all the one-sheeted and two-sheeted hyperboloids for all values of δ.

Observations

Note that generically each equivalence class is two dimensional. However, the origin is zero dimensional. There are four other observations to be drawn from the simple 2-dimensional phase-space example we have been studying.

First, we know that exponentiating Hamiltonian matrices produces symplectic matrices. Therefore the normal form problem for quadratic polynommials is related to the normal form problem for symplectic matrices. For example, exponentiating Hamiltonian matrices corresponding to the quadratic polynomials associated with points on a one-sheeted hyperboloid such as that shown in Figure 2.2a for $\delta < 0$ produces symplectic matrices whose spectrum corresponds to that shown in case 1 of Figure 3.4.1. Also, exponentiating Hamiltonian matrices corresponding to the quadratic polynomials associated with points on either sheet of the two-sheeted hyperboloid such as that shown in Figure 2.2b for $\delta > 0$ produces symplectic matrices whose spectrum corresponds to that shown in case 3 of Figure 3.4.1. In particular, exponentiating the Hamiltonian matrix corresponding to a normal form g_2^N for $\delta > 0$ produces symplectic matrices of the form (3.5.58). Finally, exponentiating Hamiltonian matrices corresponding to the quadratic polynomials associated with points on either cone as shown in Figure 2.2a for $\delta = 0$ produces symplectic matrices whose spectrum corresponds to that shown in case 4 of Figure 3.4.1. Because of the relation between quadratic polynomial and symplectic matrix normal forms, the normal form references cited at the end of Chapter 3 are also relevant to the polynomial case. However, as learned in Section 8.7.2, not every symplectic matrix can be written in single exponential form. Therefore, the classification of symplectic matrices is more complicated than the classification of quadratic polynomials (Hamiltonian matrices).

The second observation is that all quadratic polynomials associated with points on the upper-sheet of the two-sheeted hyperboloid for $\delta > 0$, see Figure 2.2b, are *positive definite*. That is, such polynomials obey $g_2(z) > 0$ for any nonzero z. From this perspective, (2.25) is the normal form for positive-definite quadratic polynomials. Correspondingly, all quadratic polynomials associated with points on the lower sheet of the two-sheeted hyperboloid are *negative definite*. They obey $g_2(z) < 0$ for any nonzero z. Their normal form is given by (2.35).

The third observation is that the normal form problem for quadratic polynomials is identical to that of classifying quadratic Hamiltonians: Given two quadratic Hamiltonians, is there a linear canonical transformation that will send one into the other? Given a quadratic Hamiltonian, how "simple" can it be made by applying a suitable linear canonical transformation? Given a quadratic Hamiltonian, what will be the nature of the motion it generates? For the case of a 2-dimensional phase space we have learned that all quadratic Hamiltonians with $\delta < 0$ can be brought to the form (2.31), and they generate *exponentially unbounded* motion. All nonzero Hamiltonians with $\delta = 0$ can be brought to one of the forms (2.26), (2.36), and they generate *linearly unbounded* motion. All Hamiltonians with $\delta > 0$ can be brought to one of the forms (2.25), (2.35), and they generate *bounded* motion. See Exercise 2.4.

The fourth observation is that we may view the g_2 as elements of the Lie algebra $sp(2)$. From this perspective, we have been studying what elements in the Lie algebra $sp(2)$ can be transformed into each other under the action of the group $Sp(2)$. In the case of a $2n$-dimensional phase space we may view the g_2 as elements of the Lie algebra $sp(2n)$, and studying the action of $Sp(2n)$ on the g_2 is equivalent to studying the action of $Sp(2n)$ on $sp(2n)$. See Exercise 2.6 for a preliminary effort in this direction, mostly devoted to $sp(4)$.

32.2.2.2 Case of Four-Dimensional Phase Space

Having mastered the case of 2-dimensional phase space, we consider the next more complicated case, namely 4-dimensional phase space. Now S takes the general form

$$S = \begin{pmatrix} a & b & c & d \\ b & e & r & s \\ c & r & t & u \\ d & s & u & v \end{pmatrix}. \tag{33.2.64}$$

Invariants

The Hamiltonian matrix $B = JS$ has the characteristic polynomial

$$P(\lambda) = \det(B - \lambda I) = \lambda^4 + C\lambda^2 + D \tag{33.2.65}$$

where

$$C = -b^2 + ae - 2dr + 2cs - u^2 + tv, \tag{33.2.66}$$

$$\begin{aligned} D &= d^2r^2 - 2cdrs + c^2s^2 - d^2et + 2bdst - as^2t \\ &+ 2cdeu - 2bdru - 2bcsu + 2arsu + b^2u^2 \\ &- aeu^2 - c^2ev + 2bcrv - ar^2v - b^2tv + aetv. \end{aligned} \tag{33.2.67}$$

Here we have used the form (3.2.10) for J. Note that $P(\lambda)$ has only even powers of λ as expected for the characteristic polynomial of a Hamiltonian matrix. See Exercise 3.7.14.

It is well known that the coefficients of the characteristic polynomial are invariant under similarity transformations; and, according to (2.14), it is similarity transformations that are being made. Therefore C and D are invariants.

Do they have any interpretation? From (2.65) we have the relation

$$D = \det(B) = \det(JS) = [\det(J)][\det(S)] = \det(S). \tag{33.2.68}$$

Here we have used (3.1.4). It follows that D (modulo sign conventions) is the *discriminant* of the quadratic form g_2. From this perspective the invariance of D follows directly from taking the determinant of both sides of (2.11) and using (3.1.8). Also see Exercise 2.5.

The interpretation of C is a bit more complicated. Since B is traceless (see Exercise 3.7.10), use of (3.7.136), and (3.7.137), and (3.7.143) gives the result

$$C = -(1/2)\ \mathrm{tr}(B^2) = -(1/2)(B, B)_F. \tag{33.2.69}$$

Here we have also used (21.11.15) and employed the subscript F to denote the *fundamental* representation. Since C is constructed from the invariant metric, and the quadratic Casimir operator is also constructed from this metric, by mental association C (as a function of the coefficients in S) is sometimes called the *Casimir* polynomial. From this perspective, the invariance of C is a special case of (21.11.25).

We have seen that there are two invariants for the case of 4-dimensional phase space, namely C and D. Since the set of all symmetric 4×4 matrices S is 10 dimensional, it follows that the orbit space (each equivalence class) is generically 8 dimensional. However, at certain points it has smaller dimension. See Exercise 2.6. In various regions it may also be expected to be multi-sheeted.

Eigenvalues

The eigenvalues of B [the roots of $P(\lambda) = 0$] are of the form

$$\lambda = \pm\sqrt{w} \tag{33.2.70}$$

where w is a root of the equadratic equation

$$w^2 + Cw + D = 0. \tag{33.2.71}$$

That is, w is given by the relation

$$w = [-C \pm (C^2 - 4D)^{1/2}]/2. \tag{33.2.72}$$

Figure 2.3 below shows possible eigenvalue configurations for 4×4 real Hamiltonian matrices depending on the values of C and D. Evidently, there are nine cases to be considered:

i. Complex quartet of eigenvalues of the form $\pm\alpha \pm i\beta$.

ii. Two pairs of pure imaginary eigenvalues $\pm i\alpha$ and $\pm i\beta$.

iii. Two pairs of real eigenvalues $\pm\alpha$ and $\pm\beta$.

iv. One pair of real eigenvalues $\pm\alpha$ and one pair of pure imaginary eigenvalues $\pm i\beta$.

v. A pair of repeated real eigenvalues $\pm\alpha$.

vi. A pair of repeated pure imaginary eigenvalues $\pm i\beta$.

vii. A pair of real eigenvalues $\pm\alpha$ and repeated zero eigenvalues $0, 0$.

viii. A pair of imaginary eigenvalues $\pm i\alpha$ and repeated zero eigenvalues $0, 0$.

ix. A quartet of zero eigenvalues $0, 0, 0, 0$.

Cases i through iv are generic, and cases v through ix are degenerate. As usual, transitions between generic configurations can only occur by passage through a degenerate configuration. Note that the results we have found are in accord with those of Exercise 3.7.14. Finally, It is instructive to compare the pair of figures 27.2.1, 3.4.3, the pair of figures 27.2.3, 3.4.4, and the pair of figures 27.2.4, 3.5.1. The first figure in each pair displays the spectrum (eigenvalues) of elements in the Lie algebra $sp(2n, \mathbb{R})$, and the second displays the spectrum of elements in the associated Lie group $Sp(2n, \mathbb{R})$. See also Figures 3.4.1 and 3.4.2.

Normal Forms

For each of the cases i through ix there is a corresponding normal form. They are listed below. Note that cases i, v, vi, and ix are special in that the two degrees of freedom cannot be uncoupled by a suitable choice of coordinates. In all other cases the normal form (when viewed as a Hamiltonian) is a sum of two terms involving different degrees of freedom, and therefore the two terms are in involution. But here is an amazing thing: In each of the

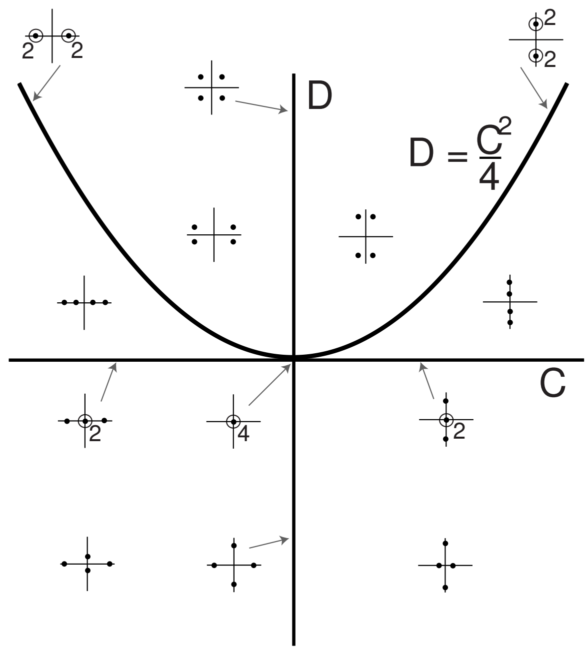

Figure 33.2.3: Eigenvalues of a 4×4 real Hamiltonian matrix as a function of the coefficients C and D in its characteristic polynomial.

coupled cases i, v, vi, and ix the normal form g_2^N is also the sum of two terms, and it can be verified in each case that the two terms are also in involution.

Case i:

$$\begin{aligned}
&\lambda = \pm\alpha \pm i\beta,\ \text{all signs taken independently},\ \text{with } \alpha, \beta > 0,\\
&C = -2\alpha^2 + 2\beta^2,\\
&D = (\alpha^2 + \beta^2)^2,\\
&g_2^N = 2\alpha(q_1 p_1 + q_2 p_2) + 2\beta(q_1 p_2 - q_2 p_1).
\end{aligned} \tag{33.2.73}$$

Case ii:

$$\begin{aligned}
&\lambda = \pm i\alpha \text{ and } \pm i\beta,\ \text{all signs taken independently},\ \text{with } \alpha, \beta > 0,\\
&C = \alpha^2 + \beta^2,\\
&D = \alpha^2 \beta^2,\\
&g_2^N = \pm\alpha(p_1^2 + q_1^2) \pm \beta(p_2^2 + q_2^2),\ \text{all signs taken independently}.
\end{aligned} \tag{33.2.74}$$

Case iii:

$$\begin{aligned}
&\lambda = \pm\alpha \text{ and } \pm\beta,\ \text{all signs taken independently},\ \text{with } \alpha, \beta > 0,\\
&C = -\alpha^2 - \beta^2,\\
&D = \alpha^2 \beta^2,\\
&g_2^N = 2\alpha q_1 p_1 + 2\beta q_2 p_2.
\end{aligned} \tag{33.2.75}$$

Case *iv*:

$$\begin{aligned}
&\lambda = \pm\alpha \text{ and } \pm i\beta, \text{ all signs taken independently, with } \alpha, \beta > 0,\\
&C = -\alpha^2 + \beta^2,\\
&D = -\alpha^2\beta^2,\\
&g_2^N = 2\alpha p_1 q_1 \pm \beta(p_2^2 + q_2^2). \qquad (33.2.76)
\end{aligned}$$

Case *v*:

$$\begin{aligned}
&\lambda = \text{ repeated pair } \pm\alpha \text{ with } \alpha > 0,\\
&C = -2\alpha^2,\\
&D = \alpha^4,\\
&g_2^N = 2\alpha(q_1p_1 + q_2p_2) + 2q_1p_2 \text{ or case } iii \text{ with } \alpha = \beta. \qquad (33.2.77)
\end{aligned}$$

Case *vi*:

$$\begin{aligned}
&\lambda = \text{ repeated pair } \pm i\beta \text{ with } \beta > 0,\\
&C = 2\beta^2,\\
&D = \beta^4,\\
&g_2^N = 2\beta(q_1p_2 - q_2p_1) \pm (q_1^2 + q_2^2) \text{ or case } ii \text{ with } \alpha = \beta. \qquad (33.2.78)
\end{aligned}$$

Case *vii*:

$$\begin{aligned}
&\lambda = \pm\alpha \text{ and } 0, 0 \text{ with } \alpha > 0,\\
&C = -\alpha^2,\\
&D = 0,\\
&g_2^N = 2\alpha q_1p_1 \pm q_2^2 \text{ or case } iii \text{ with } \beta = 0. \qquad (33.2.79)
\end{aligned}$$

Case *viii*:

$$\begin{aligned}
&\lambda = \pm i\alpha \text{ and } 0, 0 \text{ with } \alpha > 0,\\
&C = \alpha^2,\\
&D = 0,\\
&g_2^N = \pm\alpha(q_1^2 + p_1^2) \pm q_2^2 \text{ or case } ii \text{ with } \beta = 0. \qquad (33.2.80)
\end{aligned}$$

Case *ix*:

$$\begin{aligned}
&\lambda = 0, 0, 0, 0,\\
&C = 0,\\
&D = 0,\\
&g_2^N = 2q_1p_2 \pm q_2^2, \text{ or} \qquad (33.2.81)\\
&g_2^N = 2q_1p_2, \text{ or} \qquad (33.2.82)\\
&g_2^N = \pm q_1^2 \pm q_2^2, \text{ all signs taken independently, or} \qquad (33.2.83)\\
&g_2^N = \pm q_1^2. \qquad (33.2.84)
\end{aligned}$$

Observations

We are ready for four more observations. The first is that, up to $\pm$ signs and degeneracy complications that depend on the Jordan block structure of B, the normal form depends primarily on the eigenvalue spectrum. And, since the spectrum depends only on the invariants C and D, it follows that the normal form depends primarily on the values of C and D. Thus, as before, a necessary condition for $h_2 \sim g_2$ is that they have the same invariants.

The second observation is that, unlike the 2-dimensional case, we have not exhibited the transformation $\mathcal{R}$ that brings a g_2 to its normal form g_2^N. In general this must be done numerically, and enough is known about the problem to write computer programs for this purpose. Exercise 2.7 treats an interesting example where some of the required mathematical concepts are illustrated.

The third observation is that we have not exhibited the geometric nature of the various equivalence classes and their representative normal forms in analogy to Figure 2.2. To do so is difficult because, as already described earlier, we need to study 8-dimensional surfaces embedded in a 10-dimensional Euclidean space. See Exercise 2.6. Also, as indicted by the $\pm$ signs in cases *ii*, *iv*, and *vi* through *ix*, there is a multi-sheeted structure in parts of the space. However, in the generic situation when the eigenvalues are distinct (no degeneracy), there is one attribute of the normal forms that we can verify without too much effort. As Figure 2.2 illustrates, in the 2-dimensional generic case there is no point on a given orbit that is closer to the origin then the normal form. We may guess that the same is true in the $2n$-dimensional case. What we can easily check is that a g_2^N is at least a local minimum of $\langle g_2^{\rm tr}, g_2^{\rm tr}\rangle$.

Consider elements $g_2^{\rm tr}$ of the form

$$g_2^{\rm tr} = \mathcal{R} g_2^N \tag{33.2.85}$$

with $\mathcal{R}$ given by (1.1). These are just the elements of the equivalence class $\{g_2^N\}$. From (7.3.29) we know that transformations of the form $\exp(: f_2^c :)$ do not change the distance of an element from the origin. Therefore, we may restrict our attention to transformations of the form

$$\mathcal{R}_\epsilon = \exp(\epsilon : f_2^a :) \tag{33.2.86}$$

where, for convenience, we have explicitly included a scaling parameter ϵ. Let us define elements g_2^ϵ by the relation

$$g_2^\epsilon = \mathcal{R}_\epsilon g_2^N. \tag{33.2.87}$$

Then we find the results

$$\begin{aligned}\langle g_2^\epsilon, g_2^\epsilon\rangle &= \langle \mathcal{R}_\epsilon g_2^N, \mathcal{R}_\epsilon g_2^N\rangle = \langle g_2^N, \mathcal{R}_\epsilon^\dagger \mathcal{R}_\epsilon g_2^N\rangle = \langle g_2^N, \mathcal{R}_\epsilon^2 g_2^N\rangle = \langle g_2^N, \exp(2\epsilon : f_2^a :) g_2^N\rangle \\ &= \langle g_2^N, g_2^N\rangle + 2\epsilon \langle g_2^N, : f_2^a : g_2^N\rangle + (4\epsilon^2/2!)\langle g_2^N, : f_2^a :^2 g_2^N\rangle + O(\epsilon^3).\end{aligned} \tag{33.2.88}$$

Here we have used (7.3.30), which states that $: f_2^a :$ is Hermitian. Because $: f_2^a :$ is Hermitian, we may also write

$$\langle g_2^N, : f_2^a :^2 g_2^N\rangle = \langle : f_2^a : g_2^N, : f_2^a : g_2^N\rangle \geq 0. \tag{33.2.89}$$

It follows from (2.88) that g_2^N is at least a local minimum if we have the relation

$$\langle g_2^N, : f_2^a : g_2^N\rangle = 0. \tag{33.2.90}$$

It can be verified by explicit calculation that (2.90) holds for all the normal forms in the generic cases i through iv. Consider, for example, case i. Then, using (2.73) and the notation of Section 5.5.7, we may write

$$g_2^N = -2\alpha f^2 - 2\beta b^2, \tag{33.2.91}$$

$$f_2^a = \phi_1 f^1 + \phi_2 f^2 + \phi_3 f^3 + \gamma_1 g^1 + \gamma_2 g^2 + \gamma_3 g^3. \tag{33.2.92}$$

Here the ϕ_j and γ_j are arbitrary parameters. See (5.7.4) and (5.7.31). From the Poisson bracket rules (5.7.32) through (5.7.37) we find the result

$$\begin{aligned} :f_2^a: g_2^N &= [f_2^a, g_2^N] = -[g_2^N, f_2^a] \\ &= 4\beta\phi_1 f^3 - 4\beta\phi_3 f^1 + 4\beta\gamma_1 g^3 - 4\beta\gamma_3 g^1 \\ &- 4\alpha\phi_1 b^3 + 4\alpha\phi_3 b^1 + 4\alpha\gamma_2 b^0. \end{aligned} \tag{33.2.93}$$

Finally, all the terms on the right of (2.91) are orthogonal to all the terms on the right of (2.93). See Exercise 7.3.8. It follows that (2.90) is true, and therefore g_2^N is indeed at least a local minimum.

32.2.2.3 Case of General $2n$-Dimensional Phase Space

For the general case of $2n$ dimensional phase space normal-form results are also fully known, but considerably more complicated. However, based on our experience with the two and four dimensional cases, we are prepared for some statements about the general $2n$-dimensional case. The first is that nothing new, beyond what has already been seen for the 4-dimensional case, happens in the general case providing all the eigenvalues are distinct (as is generically true). When the eigenvalues are distinct (no degeneracy), the normal form g_2^N can always be taken to be a sum of terms of the form (2.73) through (2.76). Also, if zero occurs only as a doubly repeated eigenvalue, then the normal form g_2^N can be taken to be a sum of terms of the form (2.73) through (2.76) possibly augmented by a term of the form $\pm q_j^2$, as occurs for example in (2.79) and (2.80).

The second observation is that, as is already clear in the 4-dimensional case, repeated (degenerate) eigenvalues can cause complications. When the eigenvalues are degenerate there is the possibility of having the Hamiltonian analog of Jordan blocks. Fortunately, these complications are completely understood in the general case, and detailed results can be found in the literature. Moreover, it is important to note that there are two common cases where degeneracy causes no problems. In the first case, suppose we are working with a Hamiltonian that can be written in the form $h_2 = T_2(p) + V_2(q)$ where the kinetic energy term T_2 is known to be positive definite. Then standard normal mode theory shows that the Hamiltonian can always be diagonalized by a linear canonical transformation [$Sp(2n)$ element] even if some or all eigenvalue pairs are degenerate. Second, suppose that g_2 is known to be positive definite. Then g_2^N can always be taken to be a sum of harmonic oscillators, for example as in (2.74), with all signs positive even if some or all eigenvalue pairs are degenerate. This result will be proved and used in Chapter 33.

Exercises

33.2.1. Verify the equivalence relation (2.27). Find an $\mathcal{R}$ such that $\mathcal{R}q = p$ and $\mathcal{R}p = -q$.

33.2.2. Study cases *iii* and *iv* for 2-dimensional phase space and show that your results are identical to those obtained for cases *i* and *ii*. In particular, exhibit transforming matrices M analogous to (2.21) and (2.32). For example, show that for $\gamma > 0$ one may use the matrix

$$M = \begin{pmatrix} 1/\sqrt{\gamma} & -\alpha/\sqrt{\gamma} \\ 0 & \sqrt{\gamma} \end{pmatrix}. \tag{33.2.94}$$

33.2.3. Verify (2.50).

33.2.4. Solve Hamilton's equations of motion using the normal forms (2.25), (2.35), (2.26), (2.36), and (2.31) as Hamiltonians.

33.2.5. Equation (2.69) shows that the invariant C can be expressed in terms of the Lie element B and various manifestly invariant trace operations. What can be said about the discriminant invariants δ and D?

Use (3.7.115) to show that

$$\delta = \det(B) = -(1/2)\ \mathrm{tr}(B^2) = -(1/2)(B, B)_F \tag{33.2.95}$$

in the 2×2 case. (Recall that B is traceless.) Observe that *stability* is determined by the value of $(B, B)_F$. Show, by examining Figure 2.1, that stability occurs when $(B, B)_F < 0$.

Use (3.7.117) to show that

$$D = \det(B) = (1/8)[\mathrm{tr}(B^2)]^2 - (1/4)\ \mathrm{tr}(B^4) = (1/2)C^2 - (1/4)\ \mathrm{tr}(B^4) \tag{33.2.96}$$

in the 4×4 case. Show, by employing (2.69) and examining Figure 2.3, that, unlike the 2×2 case, knowledge of more that $(B, B)_F$ is required to determine stability.

In view of the ingredients of (2.95) and (2.96), δ and D could also be called Casimir polynomials.

33.2.6. This Exercise studies the dimensionality of the orbits $\{g_2^N\}$ for various normal forms g_2^N. Suppose g_2^N is some normal form. Then the orbit of g_2^N consists of all elements of the form

$$g_2^{\mathrm{tr}} = \mathcal{R} g_2^N. \tag{33.2.97}$$

Since $\mathcal{R}$ is a continuous and invertible mapping (a homeomorphism) of $\mathcal{P}_2$ into itself, the dimensionality of $\{g_2^N\}$ will be the same at every point on the orbit, and it suffices to determine the dimensionality in the vicinity of g_2^N. In this case we may take $\mathcal{R}$ to be near the identity, which means that it can be written in the form

$$\mathcal{R} = \exp(\epsilon : f_2 :) \tag{33.2.98}$$

for some small, but finite, ϵ. Correspondingly we may rewrite (2.94) in the form

$$g_2^{\mathrm{tr}} = \exp(\epsilon : f_2 :) g_2^N = g_2^N + \epsilon : f_2 : g_2^N + O(\epsilon^2). \tag{33.2.99}$$

Thus, infinitesimally, the dimensionality of $\{g_2^N\}$ is given by the number of linearly independent elements produced by terms of the form $: f_2 : g_2^N$ for arbitrary choices of f_2. Note that there is the relation

$$: f_2 : g_2^N = [f_2, g_2^N] = -[g_2^N, f_2] = - : g_2^N : f_2. \tag{33.2.100}$$

All terms of the form $(: g_2^N : f_2)$ for arbitrary f_2 comprise what is called the *range* (in $\mathcal{P}_2$) of the operator $: g_2^N :$. It is easy to check that the range of a linear operator is a linear vector space. (Check it!) Consequently, we may also say that the dimensionality of $\{g_2^N\}$ as a manifold is equal to the linear vector space dimensionality of the range of $: g_2^N :$.

Now carry out the calculations described below:

a) Suppose g_2^N is given by (2.73) as in case i. We are going to study the range of this $: g_2^N :$ in $\mathcal{P}_2$. Let λ^j be the eigenvalues for this case labeled by the scheme

$$\lambda^1 = \alpha + i\beta, \tag{33.2.101}$$

$$\lambda^2 = -\alpha + i\beta, \tag{33.2.102}$$

$$\lambda^3 = -\alpha - i\beta, \tag{33.2.103}$$

$$\lambda^4 = \alpha - i\beta. \tag{33.2.104}$$

Let g_1^j be the first-order polynomials defined by the relations

$$g_1^1 = p_1 - ip_2, \tag{33.2.105}$$

$$g_1^2 = q_1 - iq_2, \tag{33.2.106}$$

$$g_1^3 = q_1 + iq_2, \tag{33.2.107}$$

$$g_1^4 = p_1 + ip_2. \tag{33.2.108}$$

Evidently they are linearly independent. Verify the eigen relations

$$: g_2^N : g_1^j = 2\lambda^j g_1^j. \tag{33.2.109}$$

Next let g_2^{jk} be the second-order polynomials defined by the relations

$$g_2^{jk} = g_1^j g_1^k. \tag{33.2.110}$$

Show that there are 10 such polynomials (since there is symmetry in the j, k indices), and that they are all linearly independent. They consequently form a basis for $\mathcal{P}_2$ [and, therefore, also for $sp(4)$] in the case of a four-dimensional phase space. Show that these polynomials satisfy the eigen relations

$$: g_2^N : g_2^{jk} = 2(\lambda^j + \lambda^k) g_2^{jk}. \tag{33.2.111}$$

Thus, $: g_2^N :$ is diagonal in this basis. Verify that g_2^{13} and g_2^{24} are eigenvectors with eigenvalue zero, and verify that all the other eight eigenvectors have nonzero eigenvalues.

The g_2^{jk} form a complex basis for $\mathcal{P}_2$. Verify that a real basis is provided by the elements

$$g_2^1 = \ \mathrm{Re}(g_2^{11}) = \ \mathrm{Re}(g_2^{44}) = p_1^2 - p_2^2, \tag{33.2.112}$$

$$g_2^2 = \ \mathrm{Im}(g_2^{11}) = \ -\mathrm{Im}(g_2^{44}) = -2p_1p_2, \tag{33.2.113}$$

$$g_2^3 = \ \mathrm{Re}(g_2^{12}) = \ \mathrm{Re}(g_2^{34}) = q_1p_1 - q_2p_2, \tag{33.2.114}$$

$$g_2^4 = \ \mathrm{Im}(g_2^{12}) = \ -\mathrm{Im}(g_2^{34}) = -q_1p_2 - q_2p_1, \tag{33.2.115}$$

$$g_2^5 = \ \mathrm{Re}(g_2^{13}) = \ \mathrm{Re}(g_2^{24}) = q_1p_1 + q_2p_2, \tag{33.2.116}$$

$$g_2^6 = \ \mathrm{Im}(g_2^{13}) = \ -\mathrm{Im}(g_2^{24}) = q_2p_1 - q_1p_2, \tag{33.2.117}$$

$$g_2^7 = g_2^{14} = p_1^2 + p_2^2, \tag{33.2.118}$$

$$g_2^8 = \ \mathrm{Re}(g_2^{22}) = \ \mathrm{Re}(g_2^{33}) = q_1^2 - q_2^2, \tag{33.2.119}$$

$$g_2^9 = \ \mathrm{Im}(g_2^{22}) = \ -\mathrm{Im}(g_2^{33}) = -2q_1q_2, \tag{33.2.120}$$

$$g_2^{10} = g_2^{23} = q_1^2 + q_2^2. \tag{33.2.121}$$

Indeed, verify that these elements are mutually orthogonal for the inner product of Section 7.3.

Using this basis, write an arbitrary f_2 in the form

$$f_2 = \sum_{j=1}^{10} a_j g_2^j \tag{33.2.122}$$

where the a_j are arbitrary coefficients. Verify the relation

$$\begin{aligned} :g_2^N: f_2 \ &= \ 4(\alpha a_1 + \beta a_2)g_2^1 + 4(-\beta a_1 + \alpha a_2)g_2^2 + 4\beta a_4 g_2^3 - 4\beta a_3 g_2^4 \\ &+ \ 4\alpha a_7 g_2^7 + 4(-\alpha a_8 + \beta a_9)g_2^8 + 4(-\beta a_8 - \alpha a_9)g_2^9 \\ &- \ 4\alpha a_{10} g_2^{10}. \end{aligned} \tag{33.2.123}$$

Observe that all the g_2^j except g_2^5 and g_2^6 appear on the right side of (2.123). Therefore, the range of $:g_2^N:$ is potentially 8 dimensional. To be sure we must show that any linear combination of g_2^1, g_2^2, g_2^3, g_2^4, and g_2^7, g_2^8, g_2^9, g_2^{10} can be obtained on the right side of (2.123) for a suitable choice of the a_j in (2.122). Evidently there are no problems with g_2^3, g_2^4, g_2^7, and g_2^{10} since their coefficients are simply $4\beta a_4$, $-4\beta a_3$, $4\alpha a_7$, and $-4\alpha a_{10}$, respectively, and a_3, a_4, a_7, and a_{10} appear nowhere else on the right side of (2.123). For the coefficients of g_2^1 and g_2^2 we may write the matrix relation

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \alpha a_1 + \beta a_2 \\ -\beta a_1 + \alpha a_2 \end{pmatrix}. \tag{33.2.124}$$

The determinant of the matrix appearing on the left side of (2.124) has the value $(\alpha^2 + \beta^2)$. Therefore the matrix is always invertible, and we can achieve any desired combination of g_2^1 and g_2^2 on the right side of (2.123). Show that a similar argument holds for the coefficients of g_2^8 and g_2^9. We conclude that the range of $:g_2^N:$ is indeed 8 dimensional. Correspondingly, $\{g_2^N\}$ is 8 dimensional.

For any g_2^N, consider the set of all polynomials h_2^0 such that

$$: g_2^N : h_2^0 = 0. \tag{33.2.125}$$

They comprise what is called the *null space* or *kernel* (in $\mathcal{P}_2$) of $: g_2^N :$. Show that these polynomials form a Lie subalgebra with the Poisson bracket as a Lie product. (Hint: use the Jacobi identity.) Show, for the g_2^N given by (2.73), that this Lie algebra is two dimensional and is spanned by the elements $(q_1p_1 + q_2p_2)$ and $(q_1p_2 - q_2p_1)$, which are the real and imaginary parts of g_2^{13} and g_2^{24}. [Hint: Write h_2^0 in the form (2.122) and use (2.123).] Verify that g_2^N is constructed from them and that they are in involution.

Let H be the corresponding subgroup of $Sp(4)$ generated by the $: h_2^0 :$ and consider the coset space $Sp(4)/H$. Show that in some finite neighborhood of the identity any element in $\mathcal{R}$ can be written in the factored form

$$\mathcal{R} = \exp(: h_2^R :) \exp(: h_2^0 :) \tag{33.2.126}$$

where h_2^R is in the range of $: g_2^N :$ and h_2^0 is in the null space of $: g_2^N :$. The factor $\exp(: h_2^0 :)$ corresponds to an element in H, and the factor $\exp(: h_2^R :)$ corresponds to some element in the coset $Sp(4)/H$. From the relation

$$: h_2^0 : g_2^N = [h_2^0, g_2^N] = -[g_2^N, h_2^0] = - : g_2^N : h_2^0 = 0 \tag{33.2.127}$$

show that

$$\exp(: h_2^0 :) g_2^N = g_2^N \tag{33.2.128}$$

and

$$g_2^{\text{tr}} = \mathcal{R} g_2^N = \exp(: h_2^R :) g_2^N. \tag{33.2.129}$$

This construction goes beyond the infinitesimal, and shows that the orbit $\{g_2^N\}$ may be identified with the coset space $Sp(4)/H$. Evidently, an analogous result holds for any phase space dimension. In the terminology of Section 5.12, $\{g_2^N\}$ is a homogeneous space and H is the stability group for g_2^N.

b) Carry out similar calculations for the remaining generic cases *ii* through *iv*. You should find that $\{g_2^N\}$ is 8 dimensional in each case.

c) As an example of a degenerate case, consider the specific subcase of case *ix* for which g_2^N is given by the relation

$$g_2^N = q_1^2. \tag{33.2.130}$$

Following the notation of (2.64), let us write an arbitrary f_2 in the form

$$\begin{aligned} f_2 &= aq_1^2 + 2bq_1p_1 + 2cq_1q_2 + 2dq_1p_2 + ep_1^2 \\ &+ 2rp_1q_2 + 2sp_1p_2 + tq_2^2 + 2uq_2p_2 + vp_2^2. \end{aligned} \tag{33.2.131}$$

Show that

$$: g_2^N : f_2 = 4bq_1^2 + 4eq_1p_1 + 4rq_1q_2 + 4sq_1p_2. \tag{33.2.132}$$

Since all the monomials on the right of (2.132) are linearly independent, inspection indicates that the range of $: g_2^N :$ is 4 dimensional. Corresponding, $\{g_2^N\}$ is 4 dimensional. Since $\mathcal{P}_2$ is 10 dimensional, in this case there must be 4 more invariants in addition to C and D.

Show that the condition $: g_2^N : h_2 = 0$, with h_2 written in the form (2.131), requires that $b = e = r = s = 0$. Therefore, the null space of $: g_2^N :$ is 6 dimensional. Verify, as expected, that the null space is a Lie subalgebra.

Show that

$$: g_2^N :^2 z_a = 0, \tag{33.2.133}$$

which implies that

$$B^2 = 0. \tag{33.2.134}$$

We say that B is *nilpotent*. Correspondingly, $\{g_2^N\}$ in this case is called a nilpotent orbit. Show that if $g_2^{\rm tr}$ is any element lying on a nilpotent orbit, it must satisfy the relation

$$: g_2^{\rm tr} :^2 z_a = 0, \tag{33.2.135}$$

and conversely.

Show that the other normal forms of case *ix* also produce nilpotent orbits, and find their dimensions.

Show that each of the ladder elements $\tilde{r}(\boldsymbol{\mu})$ given by (21.5.11) through (21.5.18) lies on a nilpotent orbit. Show that any linear combination of ladder elements of the form $[a_\alpha \tilde{r}(\boldsymbol{\alpha}) + a_\beta \tilde{r}(\boldsymbol{\beta}) + a_\gamma \tilde{r}(\boldsymbol{\gamma})]$ lies on a nilpotent orbit. Are there other linear combinations of ladder operators that lie on nilpotent orbits? Represent a general element g_2 in $sp(4)$ as a linear combination of the basis elements given by (21.5.9) through (21.5.18). What are the necessary and sufficient conditions on the expansion coefficients for g_2 to lie on a nilpotent orbit?

d) Determine the dimension of $\{g_2^N\}$ for some of the other normal forms in cases v through $viii$. For example, show for

$$g_2^N = 2\alpha q_1 p_1 \tag{33.2.136}$$

that $\{g_2^N\}$ is 6 dimensional, and that the null space of $: g_2^N :$ has dimension 4. Is this $\{g_2^N\}$ nilpotent?

33.2.7. Consider the motion of a nonrelativistic particle of rest mass m and charge q in a uniform magnetic field $\boldsymbol{B}$ with

$$\boldsymbol{B} = \tilde{B}\boldsymbol{e}_z. \tag{33.2.137}$$

Show that this field can be generated by the vector potential

$$\boldsymbol{A} = -(1/2)(\boldsymbol{r} \times \boldsymbol{B}) = (\tilde{B}/2)(x\boldsymbol{e}_y - y\boldsymbol{e}_x). \tag{33.2.138}$$

[This choice of vector potential for $\boldsymbol{B}$ is sometimes called the *symmetric* gauge. In view of the facts that $\nabla \cdot \boldsymbol{A} = 0$ and $\boldsymbol{r} \cdot \boldsymbol{A} = 0$, this choice can also be called the Poincaré-Coulomb gauge. See Section 15.2.4 and (16.1.14).] Show that motion in this field is governed by the Hamiltonian

$$\begin{aligned} H &= (p_x + q\tilde{B}y/2)^2/(2m) + (p_y - q\tilde{B}x/2)^2/(2m) + p_z^2/(2m) \\ &= (p_x^2 + p_y^2)/(2m) + [q^2\tilde{B}^2/(8m)](x^2+y^2) - [q\tilde{B}/(2m)](xp_y - yp_x) + p_z^2/(2m) \\ &= (p_x^2 + p_y^2)/(2m) + [q^2\tilde{B}^2/(8m)](x^2+y^2) - [q\tilde{B}/(2m)]L_z + p_z^2/(2m). \end{aligned} \tag{33.2.139}$$

Here $L_z = xp_y - yp_x$ is the z component of the canonical angular momentum. Verify that L_z is an integral of motion.

Evidently the motion in the z direction is uncoupled from the motion in the x, y plane, and we can devote our attention to the latter. Show that this motion is governed by the Hamiltonian

$$H_{xy} = (z, Sz) \tag{33.2.140}$$

where the symbol z now stands for the phase-space variables $z = (x, p_x, y, p_y)$ and S is the symmetric matrix

$$S = \begin{pmatrix} q^2\tilde{B}^2/(8m) & 0 & 0 & -q\tilde{B}/(4m) \\ 0 & 1/(2m) & q\tilde{B}/(4m) & 0 \\ 0 & q\tilde{B}/(4m) & q^2\tilde{B}^2/(8m) & 0 \\ -q\tilde{B}/(4m) & 0 & 0 & 1/(2m) \end{pmatrix}. \tag{33.2.141}$$

Show that the corresponding Hamiltonian matrix $B = JS$ is given by the relation

$$B = \begin{pmatrix} 0 & 1/(2m) & q\tilde{B}/(4m) & 0 \\ -q^2\tilde{B}^2/(8m) & 0 & 0 & q\tilde{B}/(4m) \\ -q\tilde{B}/(4m) & 0 & 0 & 1/(2m) \\ 0 & -q\tilde{B}/(4m) & -q^2\tilde{B}^2/(8m) & 0 \end{pmatrix}. \tag{33.2.142}$$

Show that for this B the invariants C and D have the values

$$C = [q\tilde{B}/(2m)]^2\ ,\ D = 0. \tag{33.2.143}$$

Show that these invariant values correspond to case *viii* or case *ii* with $\beta = 0$, and that the eigenvalues of B are

$$\lambda = \pm iq\tilde{B}/(2m)\ ,\ 0\ ,\ 0. \tag{33.2.144}$$

We now want to find the transformation $\mathcal{R}$ that brings H_{xy} to its normal form.

Introduce the notation

$$\lambda_\pm = \pm iq\tilde{B}/(2m). \tag{33.2.145}$$

Show that the vectors $w_\pm$ given by

$$w_\pm = \begin{pmatrix} 1 \\ \pm iq\tilde{B}/2 \\ \pm i \\ -q\tilde{B}/2 \end{pmatrix} \tag{33.2.146}$$

satisfy the relations

$$w_{\mp} = \overline{w}_{\pm}, \tag{33.2.147}$$

and are eigenvectors of B with eigenvalues $\lambda_{\pm}$,

$$Bw_{\pm} = \lambda_{\pm} w_{\pm}. \tag{33.2.148}$$

Show that the vectors r and s given by

$$r = \begin{pmatrix} 0 \\ -q\tilde{B}/2 \\ 1 \\ 0 \end{pmatrix}, \tag{33.2.149}$$

$$s = \begin{pmatrix} 1 \\ 0 \\ 0 \\ q\tilde{B}/2 \end{pmatrix}, \tag{33.2.150}$$

are eigenvectors of B with eigenvalue 0,

$$Br = 0\ ,\ \ Bs = 0. \tag{33.2.151}$$

Define scaled vectors $\hat{u}$, $\hat{v}$, $\hat{r}$, $\hat{s}$ by the relation

$$\hat{u} = (q\tilde{B})^{-1/2}\ \mathrm{Re}(w_+) = \begin{pmatrix} (q\tilde{B})^{-1/2} \\ 0 \\ 0 \\ -(q\tilde{B})^{1/2}/2 \end{pmatrix}, \tag{33.2.152}$$

$$\hat{v} = (q\tilde{B})^{-1/2}\ \mathrm{Im}(w_+) = \begin{pmatrix} 0 \\ (q\tilde{B})^{1/2}/2 \\ (q\tilde{B})^{-1/2} \\ 0 \end{pmatrix}, \tag{33.2.153}$$

$$\hat{r} = (q\tilde{B})^{-1/2} r = \begin{pmatrix} 0 \\ -(q\tilde{B})^{1/2}/2 \\ (q\tilde{B})^{-1/2} \\ 0 \end{pmatrix}, \tag{33.2.154}$$

$$\hat{s} = (q\tilde{B})^{-1/2} s = \begin{pmatrix} (q\tilde{B})^{-1/2} \\ 0 \\ 0 \\ (q\tilde{B})^{1/2}/2 \end{pmatrix}. \tag{33.2.155}$$

Show that these vectors obey the "symplectic" orthonormality conditions

$$(\hat{u}, J\hat{v}) = 1, \tag{33.2.156}$$

$$(\hat{r}, J\hat{s}) = 1, \tag{33.2.157}$$

$$(\hat{u}, J\hat{r}) = (\hat{v}, J\hat{r}) = (\hat{u}, J\hat{s}) = (\hat{v}, J\hat{s}) = 0. \tag{33.2.158}$$

See Sections 3.5 and 4.6 for similar constructions.

Let R be the matrix defined by the relation

$$R = (\hat{u}, \hat{v}, \hat{r}, \hat{s}) = \begin{pmatrix} (q\tilde{B})^{-1/2} & 0 & 0 & (q\tilde{B})^{-1/2} \\ 0 & (q\tilde{B})^{1/2}/2 & -(q\tilde{B})^{1/2}/2 & 0 \\ 0 & (q\tilde{B})^{-1/2} & (q\tilde{B})^{-1/2} & 0 \\ -(q\tilde{B})^{1/2}/2 & 0 & 0 & (q\tilde{B})^{1/2}/2 \end{pmatrix}. \tag{33.2.159}$$

Here each vector $\hat{u}$, $\hat{v}$, $\hat{r}$, $\hat{s}$ is to be viewed as a column vector so that the collection (2.159) forms a real 4×4 matrix. Show that R, as a consequence of (2.156) through (2.158), is a symplectic matrix. Verify the relations

$$B(\hat{u} \pm i\hat{v}) = \pm i[q\tilde{B}/(2m)](\hat{u} \pm i\hat{v}), \tag{33.2.160}$$

$$B\hat{u} = -[q\tilde{B}/(2m)]\hat{v}, \tag{33.2.161}$$

$$B\hat{v} = [q\tilde{B}/(2m)]\hat{u}. \tag{33.2.162}$$

Show that

$$\begin{aligned} BR &= (B\hat{u}, B\hat{v}, B\hat{r}, B\hat{s}) = (-[q\tilde{B}/(2m)]\hat{v}, [q\tilde{B}/(2m)]\hat{u}, 0, 0) \\ &= \begin{pmatrix} 0 & (q\tilde{B})^{1/2}/(2m) & 0 & 0 \\ -(q\tilde{B})^{3/2}/(4m) & 0 & 0 & 0 \\ -(q\tilde{B})^{1/2}/(2m) & 0 & 0 & 0 \\ 0 & -(q\tilde{B})^{3/2}/(4m) & 0 & 0 \end{pmatrix} \\ &= R \begin{pmatrix} 0 & q\tilde{B}/(2m) & 0 & 0 \\ -q\tilde{B}/(2m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= RJ \begin{pmatrix} q\tilde{B}/(2m) & 0 & 0 & 0 \\ 0 & q\tilde{B}/(2m) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{33.2.163}$$

Show that

$$B^{\mathrm{tr}} = R^{-1}BR = \begin{pmatrix} 0 & q\tilde{B}/(2m) & 0 & 0 \\ -q\tilde{B}/(2m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{33.2.164}$$

and

$$S^{\mathrm{tr}} = J^{-1}B^{\mathrm{tr}} = R^T S R = \begin{pmatrix} q\tilde{B}/(2m) & 0 & 0 & 0 \\ 0 & q\tilde{B}/(2m) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33.2.165}$$

It follows that H_{xy} belongs to case *ii* with $\beta = 0$, and has the normal form

$$H_{xy}^N = (\omega/2)(p_1^2 + q_1^2) \tag{33.2.166}$$

where ω is the *cyclotron* frequency,

$$\omega = (q\tilde{B}/m). \tag{33.2.167}$$

Note that H_{xy}^N does not depend on q_2, p_2 at all!

Let q_1^i, p_1^i, q_2^i, p_2^i be *initial* conditions at $t = 0$. For these initial conditions show that the Hamiltonian H_{yt}^N generates the trajectory

$$q_1(t) = q_1^i \cos(\omega t) + p_1^i \sin(\omega t), \tag{33.2.168}$$

$$p_1(t) = -q_1^i \sin(\omega t) + p_1^i \cos(\omega t), \tag{33.2.169}$$

$$q_2(t) = q_2^i, \tag{33.2.170}$$

$$p_2(t) = p_2^i. \tag{33.2.171}$$

Show that the old and new variables are related by the equation

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = R \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}, \tag{33.2.172}$$

and therefore the trajectory in the original phase-space variables is given by the equations

$$\begin{aligned} x(t) &= (q\tilde{B})^{-1/2}[q_1(t) + p_2(t)] \\ &= (q\tilde{B})^{-1/2}[q_1^i \cos(\omega t) + p_1^i \sin(\omega t) + p_2^i], \end{aligned} \tag{33.2.173}$$

$$\begin{aligned} p_x(t) &= [(q\tilde{B})^{1/2}/2][p_1(t) - q_2(t)] \\ &= [(q\tilde{B})^{1/2}/2][-q_1^i \sin(\omega t) + p_1^i \cos(\omega t) - q_2^i], \end{aligned} \tag{33.2.174}$$

$$\begin{aligned} y(t) &= (q\tilde{B})^{-1/2}[p_1(t) + q_2(t)] \\ &= (q\tilde{B})^{-1/2}[-q_1^i \sin(\omega t) + p_1^i \cos(\omega t) + q_2^i], \end{aligned} \tag{33.2.175}$$

$$\begin{aligned} p_y(t) &= [(q\tilde{B})^{1/2}/2][-q_1(t) + p_2(t)] \\ &= [(q\tilde{B})^{1/2}/2][-q_1^i \cos(\omega t) - p_1^i \sin(\omega t) + p_2^i]. \end{aligned} \tag{33.2.176}$$

Show that the orbit in the x, y plane is a circle with a radius ρ given by the relation

$$\rho^2 = [(q_1^i)^2 + (p_1^i)^2]/(q\tilde{B}), \tag{33.2.177}$$

and that the *center* of the circle has coordinates x_c, y_c given by the relations

$$x_c = (q\tilde{B})^{-1/2} p_2^i, \tag{33.2.178}$$

$$y_c = (q\tilde{B})^{-1/2} q_2^i. \tag{33.2.179}$$

Note the curious fact that x_c and y_c do *not* "commute". Evaluate the Poisson bracket $[x_c, y_c]$.

What modifications are required if the particle is relativistic?

33.2.8. Verify that (2.90) holds for all the generic normal forms.

33.2.9. This is an exercise on Krein collisions. Our aim is to study a simple example for which Krein collisions are avoided (as expected) when phase advances have the same sign, and can be seen to occur when phase advances have opposite signs. Consider the Hamiltonian

$$H = (\omega_1/2)(p_1^2 + q_1^2) + (\omega_2/2)(p_2^2 + q_2^2) + \epsilon q_1 q_2 \tag{33.2.180}$$

and the associated map $\mathcal{R}$ given by

$$\mathcal{R} = \exp(- : H :). \tag{33.2.181}$$

Since $\mathcal{R}$ is a linear map, its action on phase space is described by a matrix R. Following the discussion surrounding (10.4.24), show that this Hamiltonian can be written in the form

$$H = (1/2)(z, Sz) \tag{33.2.182}$$

where the symbol z now stands for the phase-space variables $z = (q_1, p_1, q_2, p_2)$ and S is the symmetric matrix

$$S = \begin{pmatrix} \omega_1 & 0 & \epsilon & 0 \\ 0 & \omega_1 & 0 & 0 \\ \epsilon & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix}. \tag{33.2.183}$$

Show that the corresponding Hamiltonian matrix $B = JS$ is given by the relation

$$B = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & -\epsilon & 0 \\ 0 & 0 & 0 & \omega_2 \\ -\epsilon & 0 & -\omega_2 & 0 \end{pmatrix}. \tag{33.2.184}$$

According to (10.4.8) there is the relation

$$R = \exp(B). \tag{33.2.185}$$

Show that, when $\epsilon = 0$, R is the matrix

$$R = \begin{pmatrix} \cos(\omega_1) & \sin(\omega_1) & 0 & 0 \\ -\sin(\omega_1) & \cos(\omega_1) & 0 & 0 \\ 0 & 0 & \cos(\omega_2) & \sin(\omega_2) \\ 0 & 0 & -\sin(\omega_2) & \cos(\omega_2) \end{pmatrix}. \tag{33.2.186}$$

Therefore, when there is no perturbation, the eigenvalues of R are $\exp(\pm i\omega_1)$ and $\exp(\pm i\omega_2)$, and the phase advances of R are ω_1 and ω_2. See Example 5.1 in Section 3.5.

What concerns us are the eigenvalues of B. Once we know them, we will also know the eigenvalues of R. In particular, if the eigenvalues of B are pure imaginary, then the eigenvalues of R will lie on the unit circle. Moreover if, under perturbation, the eigenvalues of B leave the imaginary axis to become a complex quartet, then the eigenvalues of R will leave the unit circle to form a Krein quartet as in Figure 3.5.1.

Show that for this B the invariants C and D have the values

$$C = (\omega_1^2 + \omega_2^2), \tag{33.2.187}$$

$$D = \omega_1\omega_2(\omega_1\omega_2 - \epsilon^2). \tag{33.2.188}$$

Evidently, for finite values of ω_1, ω_2 and ϵ sufficiently small, both C and D are positive. Thus, our attention should be turned to the upper right quadrant of Figure 2.3; and we are concerned with cases i and ii and the transition between them.

With (2.72) and Figure 2.3 in mind, show that

$$C^2 - 4D = (\omega_1^2 - \omega_2^2)^2 + 4\epsilon^2\omega_1\omega_2. \tag{33.2.189}$$

Show that, when $\epsilon = 0$, the eigenvalues of B are given by

$$\lambda = \pm i\omega_1,\ \pm i\omega_2. \tag{33.2.190}$$

Evidently they are pure imaginary, and they are distinct unless $\omega_1 = \omega_2$ or $\omega_1 = -\omega_2$. Suppose that ω_1 and ω_2 have the same sign. Show that in this case

$$C^2 - 4D \geq 0 \tag{33.2.191}$$

no matter what the value of ϵ. Prove, consequently, that if ω_1, ω_2 are finite and of the same sign, then the eigenvalues remain pure imaginary for sufficiently small ϵ.[1] Indeed, suppose that

$$\omega_1 = \omega_2 = \Omega. \tag{33.2.192}$$

Show that in this case that

$$\lambda = \pm i\Omega[1 \pm \epsilon/\Omega]^{1/2} \tag{33.2.193}$$

where all $\pm$ signs are to be taken independently. Thus, in this case, the eigenvalues remain pure imaginary under perturbation ($\epsilon \neq 0$ but sufficiently small) if ω_1, ω_2 are finite and of the same sign. Correspondingly, there is no Krein collision of the eigenvalues of R. Suppose, instead, that

$$\omega_1 = -\omega_2 = \Omega. \tag{33.2.194}$$

Show that in this case that

$$\lambda = \pm i\Omega[1 \pm i\epsilon/\Omega]^{1/2} \tag{33.2.195}$$

where all $\pm$ signs are to be taken independently. Now, under perturbation, the eigenvalues leave the imaginary axis to become a complex quartet. Correspondingly, the eigenvalues of R leave the unit circle to become a Krein quartet.

Suppose that ω_1 and ω_2 are opposite in sign, but not exactly equal in magnitude. What happens then under perturbation? Show that the eigenvalues of B leave the imaginary axis to become a complex quartet when

$$\epsilon \geq (1/2)|\omega_1^2 - \omega_2^2|/(|\omega_1\omega_2|)^{1/2} \tag{33.2.196}$$

[1] For sufficiently large ϵ they can be driven to the situation depicted in the lower right quadrant of Figure 2.3.

or

$$\epsilon \leq -(1/2)|\omega_1^2 - \omega_2^2|/(|\omega_1\omega_2|)^{1/2}. \tag{33.2.197}$$

We conclude from this example that, as might be expected, Krein collisions are imminent when phase advances are of opposite sign and nearly equal (even if not exactly equal) in magnitude.

Consider the case, corresponding to tunes of approximately $\pm 1/4$, for which

$$\omega_1 = 1.5,\ \omega_2 = -1.6. \tag{33.2.198}$$

Figure 2.4 displays the eigenvalues λ of B as a function of ϵ for this case. When $\epsilon = 0$, the eigenvalues have values of $\pm 1.5i$, $\pm 1.6i$ with all signs taken independently. As ϵ is increased, they merge in pairs and then leave the imaginary axis. Verify analytically that they merge and then leave the imaginary axis when $\epsilon \approx \pm .10$. See Figure 2.4 and the upper-right quadrant of Figure 2.3.

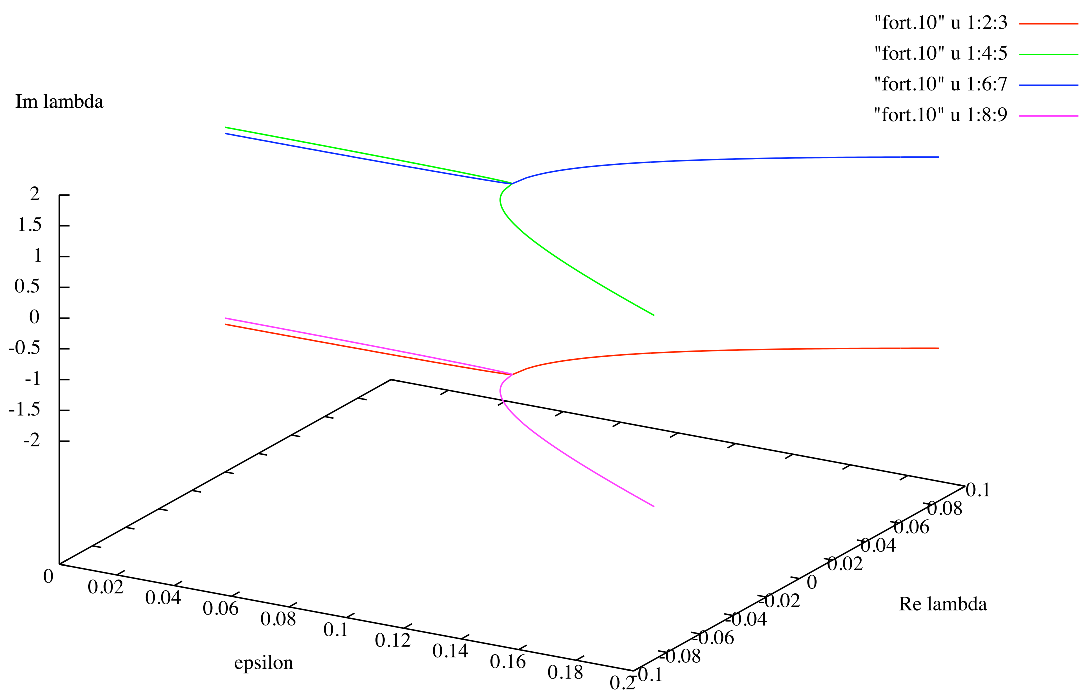

Figure 33.2.4: Eigenvalues of B as a function of ϵ when $\omega_1 = 1.5$ and $\omega_2 = -1.6$.

33.2.10. Consider quadratic polynomials in the phase-space variables for the case of a 4-dimensional phase space. For each of the normal-form cases i through ix, find the Hamiltonian matrix B associated with the specified g_2^N, verify that the invariants C and D have the indicated values, and that the eigenvalues λ have the indicated values.

33.2.11. Consider quadratic polynomials in the phase-space variables for the case of a 4-dimensional phase space. In each of the coupled cases i, v, vi, and ix the normal form g_2^N is the sum of two terms. Verify, in each case, that the two terms are in involution.

33.2.12. Consider quadratic polynomials in the phase-space variables for the case of a 4-dimensional phase space. For each of the cases i through ix, consider the motion generated by g_2^N. Do this by studying the behavior of $z'(t)$ defined by

$$z'_a(t) = \exp(-t : g_2^N :)z_a. \tag{33.2.199}$$

Show that the motion is unbounded (the origin is an unstable equilibrium point) for large $|t|$ in all cases except ii.

33.2.13. Review Exercise 2.12 above. A matrix with distinct eigenvalues can always be diagonalized by a similarity transformation. When the eigenvalues are not distinct, there are matrices for which the best that can be done by a similarity transformation is to bring them to Jordan normal form. In the 2×2 case the eigenvalues for the symplectic matrix

$$M = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \tag{33.2.200}$$

are not distinct (they are both $+1$), and moreover M cannot be brought to diagonal form when $\ell \neq 0$.

When an eigenvalue collision occurs on the unit circle in the 4×4 symplectic case as in Figure 3.5.1, the eigenvalues are not distinct. Are there real 4×4 symplectic matrices for which the eigenvalues are complex, lie on the unit circle but are not distinct, and which cannot be diagonalized? What happens at the moment of collision for the two cases of $\omega_1 \simeq \omega_2$ and $\omega_1 \simeq -\omega_2$ in Exercise 2.9 above? Is R diagonalizable?

33.3 Mostly Unsolved Polynomial Orbit Problems

In this section we will describe briefly the case of orbits in $\mathcal{P}_m$ with $m > 2$. Now the situation is far more complicated because we do not have the matrix trick simplification that led to (2.14), and only limited results are available. Some results are known for 2-dimensional phase space. Much less is known for higher-dimensional phase spaces. Even the 2-dimensional case is very difficult for large m.

For the case of 4 and higher dimensional phase space no normal forms seem to be known even for g_3. However, some few invariants are known for any $\mathcal{P}_m$ and any phase-space dimension. They are sufficiently complicated that it requires several pages to write out any one of them explicitly. It is also known that in principle there are many such invariants, and an empirical estimation of their number is available. Finally, for any $\mathcal{P}_m$ and any phase-space dimension, it is known that all invariants can be computed in terms of the monomial coefficients and (when viewed as a tensor) the entries of J.

The previous section described equivalence classes and normal forms, under the action of $Sp(2n, \mathbb{R})$, for the cases of $\mathcal{P}_1$ and $\mathcal{P}_2$. The next few paragraphs provide a sample of some known results for $\mathcal{P}_3$ and $\mathcal{P}_4$.

33.3.1 Cubic Polynomials

For the case of $\mathcal{P}_3$ and 2-dimensional phase space we may write the most general g_3 in the form

$$g_3 = a_0 q^3 + 3a_1 q^2 p + 3a_2 q p^2 + a_3 p^3 \tag{33.3.1}$$

where the a_j are arbitrary coefficients. Note, as in often convenient, we have multiplied the coefficients of q^3, q^2p, qp^2, p^3 by the factors 1, 3, 3, 1. These factors are the binomial coefficients in the expansion of $(q+p)^3$. [We remark that homogeneous polynomials in two variables are called *binary forms* in the Mathematics literature. Thus, (3.1) is called a *cubic* binary form.] It can be shown in this case that there is the invariant D, called the *discriminant* of the cubic form, given by the relation

$$D = a_0^2 a_3^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3 - 3a_1^2 a_2^2. \tag{33.3.2}$$

(The *discriminant* of a binary form g_m is an invariant with the special property that its vanishing indicates that the equation $g_m = 0$ has at least one repeated root.) Associated with g_3 is a quadratic form H, called the *Hessian* of g_3, defined by the equation

$$\begin{aligned} H &= (1/36)\det(\partial^2 g_3/\partial z_i \partial z_j) \\ &= (a_0 a_2 - a_1^2)q^2 + (a_0 a_3 - a_1 a_2)qp + (a_1 a_3 - a_2^2)p^2. \end{aligned} \tag{33.3.3}$$

Here we have used our customary notation $z = (q, p)$.

With this background in mind, it can be shown for the case $D > 0$ that g_3 has the normal form

$$g_3^N = D^{1/4}(q^3 + p^3) \text{ for } D > 0. \tag{33.3.4}$$

And, if $D < 0$, g_3 has the normal form

$$g_3^N = (-D/4)^{1/4}(q^3 - 3qp^2) \text{ for } D < 0. \tag{33.3.5}$$

If the discriminant vanishes but the coefficients of the Hessian are not all zero, g_3 has the normal form

$$g_3^N = qp^2 \text{ for } D = 0 \text{ and } H \neq 0. \tag{33.3.6}$$

Finally, if both the discriminant and all coefficients in the Hessian vanish, g_3 has the normal form

$$g_3^N = q^3 \text{ for } D = 0 \text{ and } H = 0. \tag{33.3.7}$$

Evidently cases (3.4) and (3.5) are generic while cases (3.6) and (3.7) are increasingly specific. It is interesting to note that the generic normal forms (3.4) and (3.5) minimize $\langle g_3^{\rm tr}, g_3^{\rm tr}\rangle$. See Exercise 3.1. Note also that, under the canonical transformation $q \to p$ and $p \to -q$, (3.5) takes monkey-saddle form. See Exercise 22.5.4.

33.3.2 Quartic Polynomials

For the case of $\mathcal{P}_4$ and 2-dimensional phase space we may write the most general g_4 in the form

$$g_4 = a_0 q^4 + 4a_1 q^3 p + 6a_2 q^2 p^2 + 4a_3 q p^3 + a_4 p^4. \tag{33.3.8}$$

It can be shown that in this case there are two functionally independent invariants S and T,

$$S = a_0a_4 - 4a_1a_3 + 3a_2^2, \tag{33.3.9}$$

$$T = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3. \tag{33.3.10}$$

For even degree forms there is always an invariant (with some algebraic/geometrical significance that need not concern us here) given the wonderful name *catalecticant*. In this case T is the catalecticant of g_4. The discriminant D of g_4 is functionally dependent on S and T and is given by the relation

$$D = S^3 - 27T^2. \tag{33.3.11}$$

In the case that D is positive g_4 has the normal form

$$g_4^N = \pm a(q^4 + p^4) + 6bq^2p^2 \text{ for } D > 0 \tag{33.3.12}$$

with

$$a > 0, \tag{33.3.13}$$

$$S = a^2 + 3b^2, \tag{33.3.14}$$

$$T = a^2b - b^3, \tag{33.3.15}$$

$$D = a^2(a^2 - 9b^2)^2. \tag{33.3.16}$$

In the case that D is negative g_4 has the normal form

$$g_4^N = a(q^4 - p^4) + 6bq^2p^2 \text{ for } D < 0 \tag{33.3.17}$$

with

$$a > 0, \tag{33.3.18}$$

$$S = -a^2 + 3b^2, \tag{33.3.19}$$

$$T = -a^2b - b^3, \tag{33.3.20}$$

$$D = -a^2(a^2 - 9b^2)^2. \tag{33.3.21}$$

These are the generic cases. Like g_3, there are also several specific cases for which $D = 0$, and each such case has its own normal form. We will not record them here, but results are available in the literature.

Exercises

33.3.1. Exercise on minimization.

33.4 Application to Analytic Properties

Let $\mathcal{N}$ be the two-dimensional nonlinear map defined by the relation

$$\mathcal{N} = \exp(: g_3 :) \tag{33.4.1}$$

with g_3 given by (3.1). It has the action

$$Z(z) = \exp(: g_3 :)z. \tag{33.4.2}$$

As usual, we write $z = (q; p)$ and $Z = (Q; P)$. What we wish to determine in this section are the analytic properties $\mathcal{N}$. That is, we wish to study the analytic properties of the quantities $Q(q, p)$ and $P(q, p)$ as functions of the variables q and p. We will see that the answer to this question depends on the normal form of g_3.

Let $\mathcal{R}$ be a linear symplectic map, and consider transformed maps of the form

$$\mathcal{N}^{\rm tr} = \mathcal{R}\mathcal{N}\mathcal{R}^{-1} = \mathcal{R}\exp(: g_3 :)\mathcal{R}^{-1} = \exp(: \mathcal{R}g_3 :) = \exp(: g_3^{\rm tr} :). \tag{33.4.3}$$

We see that a study of the analytic properties of $\mathcal{N}$ is equivalent to studying the analytic properties of $\exp(: g_3^{\rm tr} :)$ where $g_3^{\rm tr}$ is any of the normal form polynomials given by (3.4) through (3.7).

Consider these cases one at a time and in order of increasing complexity. The simplest case is (3.7), for which we find that

$$\bar{q} = \exp(: q^3 :)q = q, \tag{33.4.4}$$

$$\bar{p} = \exp(: q^3 :)p = p + 3q^2. \tag{33.4.5}$$

Evidently $\mathcal{N}^{\rm tr}$ in this case has no singularities save at infinity, and therefore is entire. Correspondingly, all maps in its equivalence class are also entire.

The next simplest case is (3.6), for which we find that

$$\bar{q} = \exp(: qp^2 :)q = q(1 - p)^2, \tag{33.4.6}$$

$$\bar{p} = \exp(: qp^2 :)p = p/(1 - p). \tag{33.4.7}$$

Here we have used results from Section 1.4.2. In this case $\mathcal{N}^{\rm tr}$ has a pole on the surface $p = 1$. Correspondingly, maps in its equivalence class also have pole singularities.

The case (3.4) is next in order of increasing difficulty. Now we have

$$g_3^{\rm tr} = \lambda(q^3 + p^3) \text{ with } \lambda = D^{1/4} \tag{33.4.8}$$

so that

$$\mathcal{N}^{\rm tr} = \exp(\lambda : q^3 + p^3 :). \tag{33.4.9}$$

Define a parameter dependent map $\mathcal{N}^{\rm tr}(t)$ by the relation

$$\mathcal{N}^{\rm tr}(t) = \exp(t : g_3^{\rm tr} :) = \exp(t\lambda : q^3 + p^3 :) \tag{33.4.10}$$

and write

$$\bar{q}(t) = \mathcal{N}^{\rm tr}(t)q = \exp(t\lambda : q^3 + p^3 :)q, \tag{33.4.11}$$

$$\bar{p}(t) = \mathcal{N}^{\rm tr}(t)p = \exp(t\lambda : q^3 + p^3 :)p. \tag{33.4.12}$$

We will show that there is a curve in the real q, p plane such that $\bar{q}(1) = -\infty$ and $\bar{p}(1) = +\infty$. Therefore $\mathcal{N}^{\rm tr} = \mathcal{N}^{\rm tr}(1)$ is singular on this curve.

Differentiate (4.11) and (4.12) with respect to t to obtain the equations of motion

$$\dot{\bar{q}} = \exp(t\lambda : q^3 + p^3 :)\lambda : q^3 + p^3 : q = \exp(t\lambda : q^3 + p^3 :)(-3\lambda p^2) = -3\lambda\bar{p}^2, \tag{33.4.13}$$

$$\dot{\bar{p}} = \exp(t\lambda : q^3 + p^3 :)\lambda : q^3 + p^3 : p = \exp(t\lambda : q^3 + p^3 :)(3\lambda q^2) = 3\lambda\bar{q}^2. \tag{33.4.14}$$

They have the integral

$$-\lambda(\bar{q}^3 + \bar{p}^3) = \Lambda. \tag{33.4.15}$$

Figure 4.1 shows the curves of constant Λ/λ. Also shown as arrows are the directions of the flow that follow from the equations of motion (4.13) and (4.14). Evidently $\bar{q} = \bar{p} = 0$ is the only equilibrium point, and it lies on the curve $\Lambda = 0$. Moreover, all points on the flow line $\bar{p} = -\bar{q}$ and having $\bar{q} > 0$ flow to the origin. All other points flow asymptotically to the point at infinity $\bar{q} = -\infty$, $\bar{p} = +\infty$.

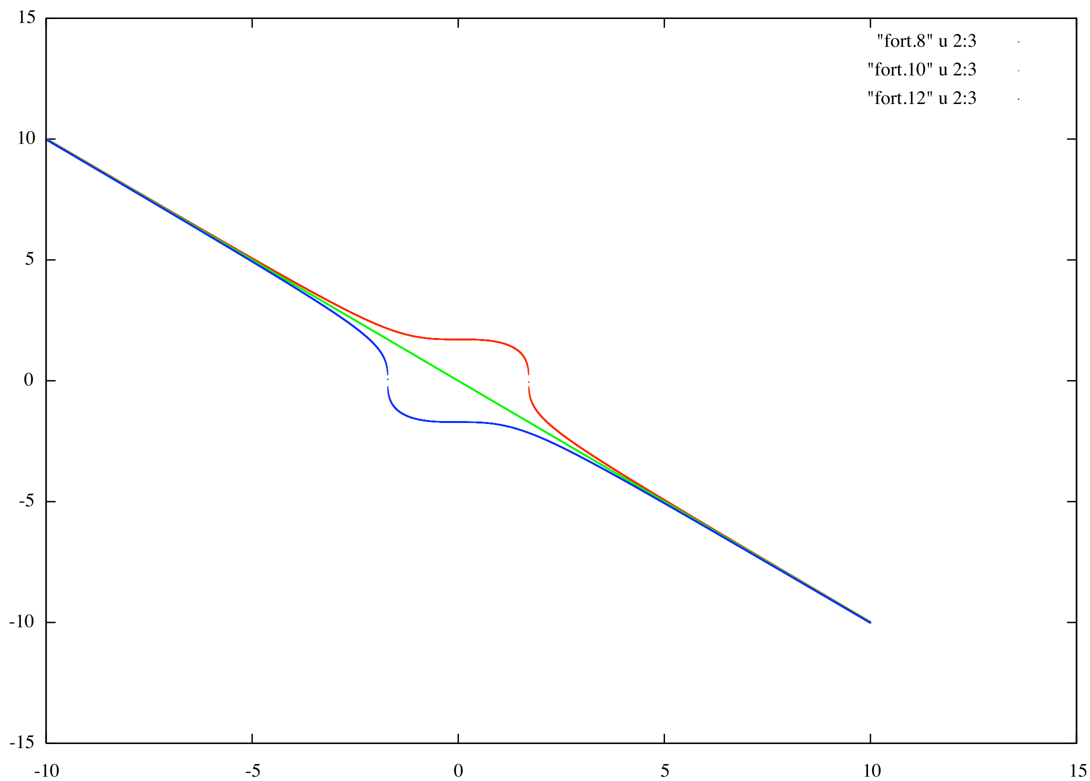

Figure 33.4.1: Curves of constant Λ/λ and flow directions for the equations of motion (4.13) and (4.14). These curves were made with $\lambda = 1$, which simply sets the scale for q and p, and $\Lambda = 0, \pm 5$.

Let us compute the *speed* at which points move along flow lines. The Euclidean distance in the $\bar{q}, \bar{p}$ plane is given by the relation

$$(ds)^2 = (d\bar{q})^2 + (d\bar{p})^2 \tag{33.4.16}$$

and therefore

$$(ds/dt)^2 = (\dot{\bar{q}})^2 + (\dot{\bar{p}})^2 = 9\lambda^2(\bar{q}^4 + \bar{p}^4). \tag{33.4.17}$$

Here we have used the equations of motion (4.13) and (4.14). We see that the speed is always positive save for the equilibrium point at the origin.

Evidently the right sides of the differential equations (4.13) and (4.14) are analytic save at infinity. Therefore, according to Section 1.3, the final conditions will be analytic functions of the initial conditions as long as the intermediate points on the flow are finite. What we wish to compute are the points for which, when regarded as initial conditions at $t = 0$, the flow reaches $\bar{q} = -\infty$, $\bar{p} = +\infty$ when $t = 1$. These points will be the frontier at which $\mathcal{N}^{\mathrm{tr}}(1)$ becomes singular.

Consider first the case $\Lambda = 0$ for which

$$\bar{p}(t) = -\bar{q}(t). \tag{33.4.18}$$

As stated earlier, points on the line $\bar{p} = -\bar{q}$ with $\bar{q} > 0$ flow along the line and into the origin; and points on the line with $\bar{q} < 0$ flow along the line to $\bar{q} = -\infty$ and $\bar{p} = +\infty$. What we will show is that on the line there is an initial condition $\bar{q}(0)$, $\bar{p}(0)$ with $\bar{q}(0) < 0$ such that

$$\bar{q}(1) = -\infty \tag{33.4.19}$$

and

$$\bar{p}(1) = +\infty \tag{33.4.20}$$

To do so, employ (4.18) in (4.13) to find the relation

$$\dot{\bar{q}} = -3\lambda\bar{q}^2 \tag{33.4.21}$$

from which it follows that

$$dt = -d\bar{q}/(3\lambda\bar{q}^2). \tag{33.4.22}$$

Integrate both sides of (4.22) to find the result

$$\int_{t^i}^{t^f} dt = -\int_{\bar{q}^i}^{\bar{q}^f} d\bar{q}/(3\lambda\bar{q}^2), \tag{33.4.23}$$

from which it follows that

$$t^f - t^i = [1/(3\lambda)](1/\bar{q})|_{\bar{q}^i}^{\bar{q}^f} = [1/(3\lambda)](1/\bar{q}^f - 1/\bar{q}^i). \tag{33.4.24}$$

Now set

$$t^i = 0,\ t^f = 1,\ \bar{q}^f = -\infty \tag{33.4.25}$$

to obtain the result

$$1 = [1/(3\lambda)](-1/\bar{q}^i) \tag{33.4.26}$$

from which it follows that

$$\bar{q}(0) = \bar{q}^i = -1/(3\lambda) \tag{33.4.27}$$

and

$$\bar{p}(0) = -\bar{q}(0) = 1/(3\lambda). \tag{33.4.28}$$

Next consider the case $\Lambda > 0$. We will again find that there is an initial condition on the curve (4.15) such that (4.19) and (4.20) hold. In this case (4.15) can be solved for $\bar{p}$ to give the result

$$\bar{p} = (-\Lambda/\lambda - \bar{q}^3)^{1/3}. \tag{33.4.29}$$

Here the negative cube root is to be extracted if the quantity $(-\Lambda/\lambda - \bar{q}^3)$ is negative, and the positive square root is to be extracted if the quantity is positive. Thus,

$$\bar{p} = -(\Lambda/\lambda + \bar{q}^3)^{1/3} < 0 \text{ when } \bar{q} > -(\Lambda/\lambda)^{1/3}, \tag{33.4.30}$$

$$\bar{p} = 0 \text{ when } \bar{q} = -(\Lambda/\lambda)^{1/3}, \tag{33.4.31}$$

$$\bar{p} = (-\Lambda/\lambda - \bar{q}^3)^{1/3} > 0 \text{ when } \bar{q} < -(\Lambda/\lambda)^{1/3}. \tag{33.4.32}$$

Observe that $\bar{p}^2$ is always ≥ 0 and is given by the relation

$$\bar{p}^2 = [(\Lambda/\lambda + \bar{q}^3)^2]^{1/3}. \tag{33.4.33}$$

Now the differential equation (4.13) takes the form

$$\dot{\bar{q}} = -3\lambda[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3}, \tag{33.4.34}$$

from which we conclude

$$dt = -[1/(3\lambda)]d\bar{q}/[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3}. \tag{33.4.35}$$

And integrating both sides of (4.35) yields the result

$$\int_{t^i}^{t^f} dt = -\int_{\bar{q}^i}^{\bar{q}^f} [1/(3\lambda)]d\bar{q}/[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3} \tag{33.4.36}$$

from which it follows that

$$1 = -\int_{\bar{q}^i}^{-\infty} [1/(3\lambda)]d\bar{q}/[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3} = \int_{-\infty}^{\bar{q}^i} [1/(3\lambda)]d\bar{q}/[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3}. \tag{33.4.37}$$

Here we have again used (4.25).

What about the case $\Lambda < 0$? Here (4.29) and (4.33) continue to hold. Now we must use

$$\bar{p} = -(\Lambda/\lambda + \bar{q}^3)^{1/3} < 0 \text{ when } \bar{q} > (-\Lambda/\lambda)^{1/3}, \tag{33.4.38}$$

$$\bar{p} = 0 \text{ when } \bar{q} = (-\Lambda/\lambda)^{1/3}, \tag{33.4.39}$$

$$\bar{p} = (-\Lambda/\lambda - \bar{q}^3)^{1/3} > 0 \text{ when } \bar{q} < (-\Lambda/\lambda)^{1/3}. \tag{33.4.40}$$

With this understanding, (4.37) also continues to hold.

Taken together, (4.27) and (4.37) yield $\bar{q}^i$ as a function of Λ. And, when $\bar{q}^i$ is known and Λ is specified, (4.30) through (4.32) and (4.38) through (4.41) give $\bar{p}^i$. This curve, shown in Figure 4.2 superimposed on an enlarged portion of Figure 4.1, provides (in the real $\bar{q}, \bar{p}$ plane) the set of points at which $\mathcal{N}^{\rm tr}(1)$ becomes singular. The map is well defined and analytic in the initial conditions for initial conditions to the right of this curve. Its status for points to left of this curve is not yet established. To do so would require, among other things some compactification procedure analogous to the Riemann sphere but in four dimensions. At this stage we do not know whether the map has only a pole or poles as was the case for (4.7), or has more complicated singularities, such as branch points that preclude single-valued analytic continuation, or singularities that preclude any analytic continuation at all. Finally we note that this one-dimensional curve is the real intersection of a two-dimensional manifold in the full four-dimensional domain of two complex variables. This manifold is specified by letting Λ be complex in (4.37) and in the relations for $\bar{p}^i$ in terms of $\bar{q}^i$ and Λ.

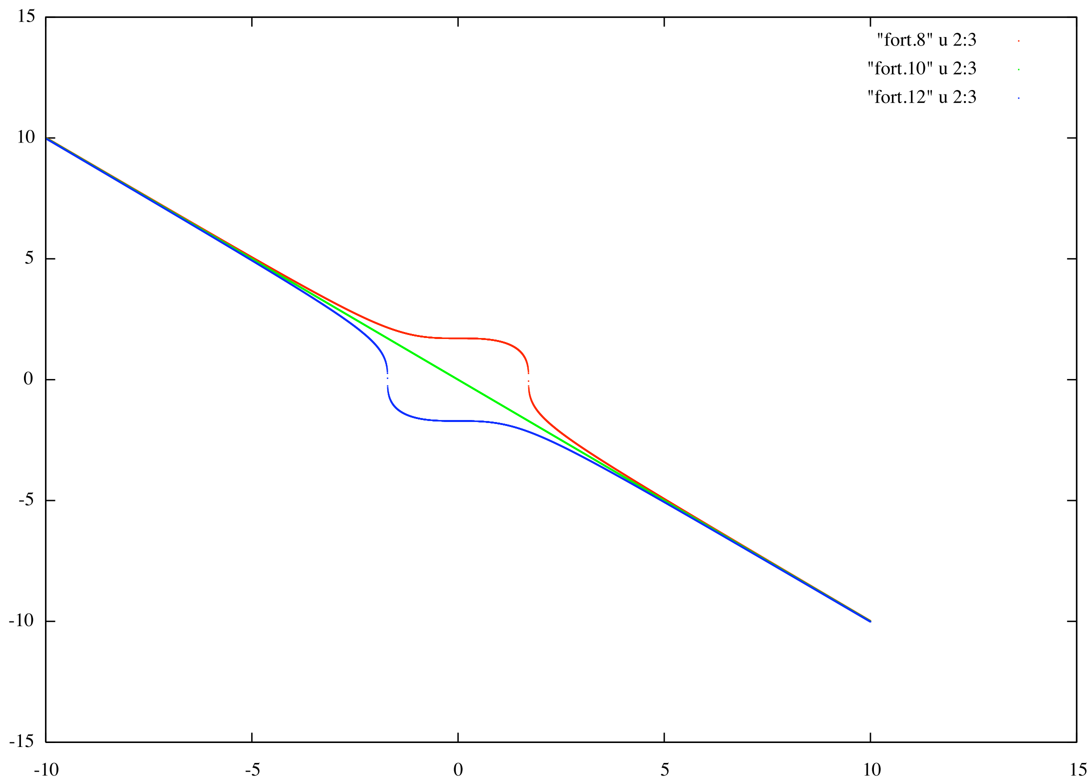

Figure 33.4.2: (Place Holder) Curve on which the map $\mathcal{N}^{\rm tr}(1)$ becomes singular. The map is well defined and analytic for phase-space points to the right of this curve. Points on the curve are sent to infinity. The possible action of the map on points to the left of the curve is unknown.

The last and most difficult case is (3.5). Now we have

$$g_3^{\rm tr} = \lambda(q^3 - 3qp^2) \text{ with } \lambda = (-D)^{1/4} \tag{33.4.41}$$

so that

$$\mathcal{N}^{\rm tr} = \exp(\lambda : q^3 - 3qp^2 :). \tag{33.4.42}$$

Again define a parameter dependent map $\mathcal{N}^{\rm tr}(t)$, now by the relation

$$\mathcal{N}^{\rm tr}(t) = \exp(t : g_3^{\rm tr} :) = \exp(t\lambda : q^3 - 3qp^2 :) \tag{33.4.43}$$

and write

$$\bar{q}(t) = \mathcal{N}^{\rm tr}(t)q = \exp(t\lambda : q^3 - 3qp^2 :)q, \tag{33.4.44}$$

$$\bar{p}(t) = \mathcal{N}^{\rm tr}(t)p = \exp(t\lambda : q^3 - 3qp^2 :)p. \tag{33.4.45}$$

We will again show that there is a curve in the real q, p plane on which $\mathcal{N}^{\rm tr} = \mathcal{N}^{\rm tr}(1)$ is singular.

Differentiate (4.44) and (4.45) with respect to t to obtain the equations of motion

$$\dot{\bar{q}} = \exp(t\lambda : q^3 - 3qp^2 :)\lambda : q^3 - 3qp^2 : q = \exp(t\lambda : q^3 - 3p^2 :)(3\lambda qp) = 3\lambda\bar{q}\bar{p}, \tag{33.4.46}$$

$$\dot{\bar{p}} = \exp(t\lambda : q^3 - 3qp^2 :)\lambda : q^3 - 3qp^2 : p = \exp(t\lambda : q^3 - 3qp^2 :)3\lambda(q^2 - p^2) = 3\lambda(\bar{q}^2 - \bar{p}^2). \tag{33.4.47}$$

They have the integral

$$-\lambda(\bar{q}^3 - 3\bar{q}\bar{p}^2) = \Lambda. \tag{33.4.48}$$

Observe that if we set $\Lambda = 0$ we find the three lines

$$\bar{q} = 0, \tag{33.4.49}$$

$$\bar{q} = (\sqrt{3})\bar{p}, \tag{33.4.50}$$

$$\bar{q} = -(\sqrt{3})\bar{p}. \tag{33.4.51}$$

Figure 4.3 shows these lines and the remaining curves of constant Λ/λ. Also shown as arrows are the directions of the flow that follow from the equations of motion (4.46) and (4.47). For the speed along these flow lines we find the result

$$(ds/dt)^2 = (\dot{\bar{q}})^2 + (\dot{\bar{p}})^2 = 9\lambda^2[(\bar{q}^2 - \bar{p}^2)^2 + (\bar{q}\bar{p})^2]. \tag{33.4.52}$$

Evidently $\bar{q} = \bar{p} = 0$ is the only equilibrium point, and it lies on the intersection of the lines (4.49) through (4.51). Moreover, we can conclude the following:

- All points on the flow line (4.49)with $p > 0$ flow into the origin. All points on that line with $p < 0$ flow to the point at infinity $(\bar{q}, \bar{p}) = (0, -\infty)$.
- All points on the flow line (4.50)with $p < 0$ flow into the origin. All points on that line with $p > 0$ flow to the point at infinity $(\bar{q}, \bar{p}) = (\infty\sqrt{3}, \infty)$.
- All points on the flow line (4.51)with $p < 0$ flow into the origin. All points on that line with $p > 0$ flow to the point at infinity $(\bar{q}, \bar{p}) = (-\infty\sqrt{3}, \infty)$.
- All other points on all other flow lines eventually flow into one of the three points at infinity listed above.

The text below requires further work

Evidently the right sides of the differential equations (3.13) and (3.14) are analytic save at infinity. Therefore, according to Section 1.3, the final conditions will be analytic functions of the initial conditions as long as the intermediate points on the flow are finite. What we wish to compute are the points for which, when regarded as initial conditions at $t = 0$, the

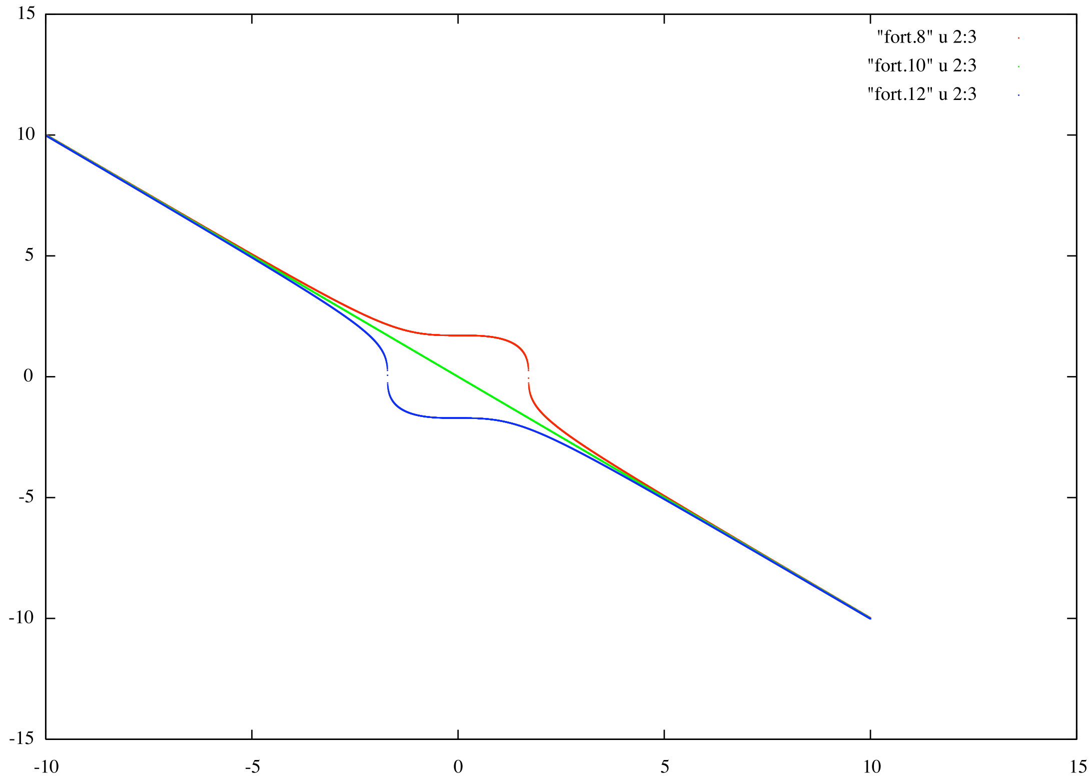

Figure 33.4.3: (Place Holder) Curves of constant Λ/λ and flow directions for the equations of motion (4.44) and (4.45). These curves were made with $\lambda = 1$, which simply sets the scale for q and p, and $\Lambda = 0, \pm 5$.

flow reaches $\bar{q} = -\infty$, $\bar{p} = +\infty$ when $t = 1$. These points will be the frontier at which $\mathcal{N}^{\rm tr}(1)$ becomes singular.

Consider first the case $\Lambda = 0$ for which

$$\bar{p}(t) = -\bar{q}(t). \tag{33.4.53}$$

As stated earlier, points on the line $\bar{p} = -\bar{q}$ with $\bar{q} > 0$ flow along the line and into the origin; and points on the line with $\bar{q} < 0$ flow along the line to $\bar{q} = -\infty$ and $\bar{p} = +\infty$. What we will show is that on the line there is an initial condition $\bar{q}(0)$, $\bar{p}(0)$ with $\bar{q}(0) < 0$ such that

$$\bar{q}(1) = -\infty \tag{33.4.54}$$

and

$$\bar{p}(1) = +\infty \tag{33.4.55}$$

To do so, employ (3.18) in (3.13) to find the relation

$$\dot{\bar{q}} = -3\lambda\bar{q}^2 \tag{33.4.56}$$

from which it follows that

$$dt = -d\bar{q}/(3\lambda\bar{q}^2). \tag{33.4.57}$$

Integrate both sides of (3.22) to find the result

$$\int_{t^i}^{t^f} dt = -\int_{\bar{q}^i}^{\bar{q}^f} d\bar{q}/(3\lambda\bar{q}^2). \tag{33.4.58}$$

from which it follows that

$$t^f - t^i = [1/(3\lambda)](1/\bar{q})|_{\bar{q}^i}^{\bar{q}^f} = [1/(3\lambda)](1/\bar{q}^f - 1/\bar{q}^i). \tag{33.4.59}$$

Now set

$$t^i = 0,\ t^f = 1,\ \bar{q}^f = -\infty \tag{33.4.60}$$

to obtain the result

$$1 = [1/(3\lambda)](-1/\bar{q}^i) \tag{33.4.61}$$

from which it follows that

$$\bar{q}(0) = \bar{q}^i = -1/(3\lambda) \tag{33.4.62}$$

and

$$\bar{p}(0) = -\bar{q}(0) = 1/(3\lambda). \tag{33.4.63}$$

Next consider the case $\Lambda > 0$. We will again find that there is an initial condition on the curve (3.15) such that (3.19) and (3.20) hold. In this case (3.15) can be solved for $\bar{p}$ to give the result

$$\bar{p} = (-\Lambda/\lambda - \bar{q}^3)^{1/3}. \tag{33.4.64}$$

Here the negative cube root is to be extracted if the quantity $(-\Lambda/\lambda - \bar{q}^3)$ is negative, and the positive square root is to be extracted if the quantity is positive. Thus,

$$\bar{p} = -(\Lambda/\lambda + \bar{q}^3)^{1/3} < 0 \text{ when } \bar{q} > -(\Lambda/\lambda)^{1/3}, \tag{33.4.65}$$

$$\bar{p} = 0 \text{ when } \bar{q} = -(\Lambda/\lambda)^{1/3}, \tag{33.4.66}$$

$$\bar{p} = (-\Lambda/\lambda - \bar{q}^3)^{1/3} > 0 \text{ when } \bar{q} < -(\Lambda/\lambda)^{1/3}. \tag{33.4.67}$$

Observe that $\bar{p}^2$ is always ≥ 0 and is given by the relation

$$\bar{p}^2 = [(\Lambda/\lambda + \bar{q}^3)^2]^{1/3}. \tag{33.4.68}$$

Now the differential equation (3.13) takes the form

$$\dot{\bar{q}} = -3\lambda[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3}, \tag{33.4.69}$$

from which we conclude

$$dt = -[1/(3\lambda)]d\bar{q}/[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3}. \tag{33.4.70}$$

And integrating both sides of (3.35) yields the result

$$\int_{t^i}^{t^f} dt = -\int_{\bar{q}^i}^{\bar{q}^f} [1/(3\lambda)]d\bar{q}/[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3} \tag{33.4.71}$$

from which it follows that

$$1 = -\int_{\bar{q}^i}^{-\infty} [1/(3\lambda)]d\bar{q}/[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3} = \int_{-\infty}^{\bar{q}^i} [1/(3\lambda)]d\bar{q}/[(\Lambda/\lambda + \bar{q}^3)^2]^{1/3} \tag{33.4.72}$$

Here we have again used (3.25).

What about the case $\Lambda < 0$? Here (3.29) and (3.33) continue to hold. Now we must use

$$\bar{p} = -(\Lambda/\lambda + \bar{q}^3)^{1/3} < 0 \text{ when } \bar{q} > (-\Lambda/\lambda)^{1/3}, \tag{33.4.73}$$

$$\bar{p} = 0 \text{ when } \bar{q} = (-\Lambda/\lambda)^{1/3}, \tag{33.4.74}$$

$$\bar{p} = (-\Lambda/\lambda - \bar{q}^3)^{1/3} > 0 \text{ when } \bar{q} < (-\Lambda/\lambda)^{1/3}. \tag{33.4.75}$$

With this understanding, (3.37) also continues to hold.

Taken together, (3.27) and (3.37) yield $\bar{q}^i$ as a function of Λ. And, when $\bar{q}^i$ is known and Λ is specified, (3.30) through (3.32) and (3.38) through (3.41) give $\bar{p}^i$. This curve, shown in Figure 3.2 superimposed on an enlarged portion of Figure 3.1, provides (in the real $\bar{q}, \bar{q}$ plane) the set of points at which $\mathcal{N}^{\rm tr}(1)$ becomes singular. The map is well defined and analytic in the initial conditions for initial conditions to the right of this curve. Its status for points to left of this curve is not yet established. To do so would require, among other things some compactification procedure analogous to the Riemann sphere but in four dimensions. At this stage we do not know whether the map has only a pole or poles as was the case for (3.7), or has more complicated singularities, such as branch points that preclude single-valued analytic continuation, or singularities that preclude any analytic continuation at all. Finally we note that this one-dimensional curve is the real intersection of a two-dimensional manifold in the full four-dimensional domain of two complex variables. This manifold is specified by letting Λ be complex in (3.37) and in the relations for $\bar{p}^i$ in terms of $\bar{q}^i$ and Λ.

Exercises

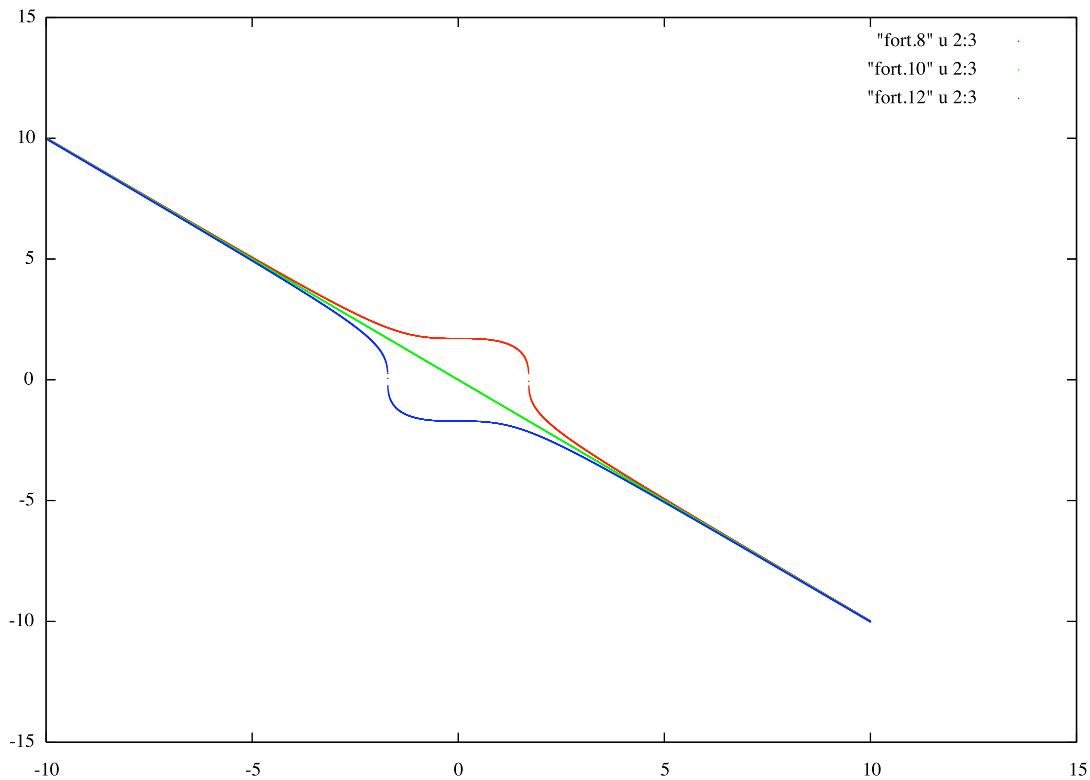

Figure 33.4.4: (Place Holder) Curve on which the map $\mathcal{N}^{\rm tr}(1)$ becomes singular. The map is well defined and analytic for phase-space points to the right of this curve. Points on the curve are sent to infinity. The possible action of the map on points to the left of the curve is unknown.

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Normal Forms

See also the Normal Form references given at the end of Chapter 3.

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Chapter 34

Beam Description and Moment Transport

34.1 Preliminaries

The previous chapters dealt with *single*-particle orbit theory. In this chapter we will treat *many*-particle distributions in the approximation that the particles are *noninteracting*. Under this noninteracting assumption all results about particle distributions are derivable from properties of the single-particle transfer map $\mathcal{M}$. Here we will find it convenient to use the phase-space variable ordering

$$z = (z_1, z_2, \ldots, z_{2n-1}, z_{2n}) = (q_1, p_1, q_2, p_2, \ldots, q_n, p_n). \tag{34.1.1}$$

That is, we will employ the ordering (3.2.20) presented in Exercise 3.2.6, but will omit the prime for notational simplicity. Also, we will use the matrix J' given by (3.2.10) and (3.2.11), but will again omit the prime. See Section 3.2.

Suppose $h(z)$ is some *density* function describing a collection of particles in phase space. That is d^6N, the number of particles in a phase-space volume d^6z, is given by the relation

$$d^6N = h(z)d^6z, \tag{34.1.2}$$

and there is the result

$$N = \int d^6z\, h(z) \tag{34.1.3}$$

where N is the number of particles under consideration.

More specifically, suppose $h^i(z)$ is a function describing some *initial* distribution of particles in phase space. Next suppose the particle distribution is transported through some system described by a map $\mathcal{M}$. Then, by Liouville's theorem, the *final* distribution $h^f(z)$ at the end of the system is given by the relation

$$h^f(z) = h^i(\mathcal{M}^{-1}z). \tag{34.1.4}$$

See Subsection 6.8.1 and Exercise 6.8.2. Also recall that, as sketched in Section 6.8, the problem of determining what distribution can be sent into what under the action of some symplectic map $\mathcal{M}$, which is what (1.4) describes, is deep and only partially understood. See also Chapter 29.

34.2 Moments and Moment Transport

Suppose, as before, that $h^i(z)$ is a function describing some *initial* distribution of particles in phase space. Since $h^i(z)$ is a function, generally an infinite number of parameters are required for its specification. One way to *characterize* $h^i(z)$ is in terms of initial moments $Z^i_{abc\cdots}$ defined by the rule

$$Z^i_{abc\cdots} = \langle z_a z_b z_c \cdots \rangle^i = (1/N) \int d^6z \; h^i(z) z_a z_b z_c \cdots . \tag{34.2.1}$$

We use the term *characterize* advisedly rather than *specify* because in general the problem of reconstructing (uniquely determining) a function given its moments is ill posed. Nevertheless, moments may provide some useful information about $h^i(z)$.

How are initial and final moments related? To answer this question it is useful to employ a different notation for moments. Let $P_\alpha(z)$, where α is some running index, denote a complete set of homogeneous polynomials in z through terms of some fixed degree. See Chapter 36. Then one can define initial moments m^i_α by the rule

$$m^i_\alpha = (1/N) \int d^6z \; h^i(z) P_\alpha(z). \tag{34.2.2}$$

Correspondingly, the final moments are given by the relation

$$\begin{aligned} m^f_\alpha &= (1/N) \int d^6z \; h^f(z) P_\alpha(z) = (1/N) \int d^6z \; h^i(\mathcal{M}^{-1}z) P_\alpha(z) \\ &= (1/N) \int d^6\bar{z} \; h^i(\bar{z}) P_\alpha(\mathcal{M}\bar{z}). \end{aligned} \tag{34.2.3}$$

Here we have used (1.4). And, to obtain the last line, we have changed variables of integration by the rule

$$\bar{z} = \mathcal{M}^{-1} z. \tag{34.2.4}$$

Doing so required calculation of the determinant of the Jacobi matrix M associated with $\mathcal{M}$. However, it is a property of symplectic matrices that they all have determinant +1. Therefore the determinant of M is +1 and need not appear explicitly in (2.3).

Since the P_α are complete, there is an expansion of the form

$$P_\alpha(\mathcal{M}\bar{z}) = \sum_\beta \mathcal{D}_{\alpha\beta}(\mathcal{M}) P_\beta(\bar{z}) \tag{34.2.5}$$

where the $\mathcal{D}_{\alpha\beta}(\mathcal{M})$ are coefficients that can be calculated for any transfer map $\mathcal{M}$. Employing (2.5) in (2.3) gives the intermediate result

$$\begin{aligned} m^f_\alpha &= (1/N) \int d^6\bar{z} \; h^i(\bar{z}) P_\alpha(\mathcal{M}\bar{z}) = (1/N) \int d^6\bar{z} \; h^i(\bar{z}) \sum_\beta \mathcal{D}_{\alpha\beta}(\mathcal{M}) P_\beta(\bar{z}) \\ &= \sum_\beta \mathcal{D}_{\alpha\beta}(\mathcal{M}) (1/N) \int d^6\bar{z} \; h^i(\bar{z}) P_\beta(\bar{z}). \end{aligned} \tag{34.2.6}$$

It follows that moments transform *linearly* according to the rule

$$m_\alpha^f = \sum_\beta D_{\alpha\beta}(\mathcal{M}) m_\beta^i. \quad (34.2.7)$$

Note that by this method one can find the evolution of moments *without* tracking (following orbits of individual particles in) particle distributions.

34.3 Various Beam Distributions and Beam Matching

34.4 Some Properties of First-Order Moments

34.4.1 Transformation Properties

For reasons that will become clear later, see Section 5, in this subsection we will examine the transformation properties of first-order moments under the action of the inhomogeneous symplectic group $ISp(2n)$. Recall Section 9.2 for a discussion of $ISp(2n)$.

Properties under Translations

Let us first find the transformation properties of first-order moments under the action of translations. Let $\mathcal{T}$ be the *translation* map given by

$$\mathcal{T} = \exp : g_1 : \quad (34.4.1)$$

with

$$g_1(z) = -(\delta, Jz). \quad (34.4.2)$$

It has the property that

$$\mathcal{T} z_a = z_a + \delta_a. \quad (34.4.3)$$

See Section 7.7. Conversely, there is the inverse relation

$$\mathcal{T}^{-1} z_a = z_a - \delta_a. \quad (34.4.4)$$

As a special case of the Liouville relation (1.4), under the action of $\mathcal{T}$ the distribution function h becomes a transformed distribution function h' with

$$h'(z) = h(\mathcal{T}^{-1} z). \quad (34.4.5)$$

From the definition (2.1) we see that the transformed moments $\langle z_a \rangle'$ are given by the relation

$$\langle z_a \rangle' = (1/N) \int d^6\, h'(z) z_a = (1/N) \int d^6 z\, h(\mathcal{T}^{-1} z) z_a. \quad (34.4.6)$$

Introduce new variables $\bar{z}$ by the rule

$$z = \mathcal{T} \bar{z} \quad (34.4.7)$$

or, equivalently,

$$\bar{z} = \mathcal{T}^{-1} z. \tag{34.4.8}$$

The relation (4.7) implies the component relations

$$z_a = \bar{z}_a + \delta_a \tag{34.4.9}$$

Also, we see that

$$d^6 z = d^6 \bar{z}. \tag{34.4.10}$$

With these facts in mind, we see that (4.6) can be rewritten in the form

$$\begin{aligned} \langle z_a \rangle' &= (1/N) \int d^6 \bar{z}\ (\bar{z}_a + \delta_a) h(\bar{z}) \\ &= (1/N) \int d^6 \bar{z}\ \bar{z}_a h(\bar{z}) + (1/N) \int d^6 \bar{z}\ \delta_a h(\bar{z}) \\ &= \langle z_a \rangle + \delta_a, \end{aligned} \tag{34.4.11}$$

which has the more compact vector form

$$\langle z \rangle' = \langle z \rangle + \delta. \tag{34.4.12}$$

We may view the first-order moments of a distribution as specifying the *centroid* of a distribution. According to (4.12), under the action of a translation, the centroid transforms like the coordinates of a particle located at the centroid. The centroid is simply translated, as expected. We observe also that the transformation rule is the same for all distributions having the same first-order moments. Finally note that the steps leading from (4.6) to (4.12) are simply (for the translation case and for first-order moments) a more detailed recapitulation of the steps (2.3) through (2.7).

Properties under Linear Symplectic Maps

Next let us find the transformation properties of first-order moments under the action of a linear symplectic map $\mathcal{R}$ described by the symplectic matrix R. Now the Liouville relation (1.4) relating the initial distribution function h and the transformed distribution function h' takes the form

$$h'(z) = h(R^{-1} z). \tag{34.4.13}$$

From the definition (2.1) we see that the transformed moments $\langle z_a \rangle'$ are given by the relation

$$\langle z_a \rangle' = (1/N) \int d^6 z\ h'(z) z_a = (1/N) \int d^6 z\ h(R^{-1} z) z_a. \tag{34.4.14}$$

Introduce new variables $\bar{z}$ by the rule

$$z = R\bar{z} \tag{34.4.15}$$

or, equivalently,

$$\bar{z} = R^{-1} z. \tag{34.4.16}$$

The relation (4.15) implies the component relations

$$z_a = \sum_c R_{ac} \bar{z}_c. \tag{34.4.17}$$

Also, because R is symplectic and therefore must have determinant one, we find that

$$d^6 z = [\det(R)] d^6 \bar{z} = d^6 \bar{z}. \tag{34.4.18}$$

With these facts in mind, we see that (4.14) can be rewritten in the form

$$\begin{aligned} \langle z_a \rangle' &= (1/N) \int d^6 \bar{z} \, \sum_c R_{ac} \bar{z}_c h(\bar{z}) \\ &= \sum_c R_{ac} (1/N) \int d^6 \bar{z} \; \bar{z}_c h(\bar{z}) \\ &= \sum_c R_{ac} \langle z_c \rangle, \end{aligned} \tag{34.4.19}$$

which has the more compact matrix form

$$\langle z \rangle' = R \langle z \rangle. \tag{34.4.20}$$

Recall that we may view the first-order moments of a distribution as specifying the centroid of a distribution. According to (4.20), under the action of a linear symplectic map, the centroid transforms like the coordinates of a particle located at the centroid. We observe, in particular, that the transformation rule is the same for all distributions having the same first-order moments. Note also that the steps leading from (4.14) to (4.20) are again (for the linear case and for first-order moments) simply a more detailed recapitulation of the steps (2.3) through (2.7).

34.4.2 Normal Form

From the work of Subsection 3.6.5 we know that $Sp(2n)$ acts transitively on phase space. See (3.6.114) through (3.6.116). Therefore, unless $\langle z \rangle = 0$, there is a sympletic matrix R such that

$$\langle z \rangle' = R \langle z \rangle = e^1. \tag{34.4.21}$$

That is, all the components of $\langle z \rangle'$ vanish save for the first, which has the value 1. Alternatively, if $\langle z \rangle$ vanishes, then $\langle z \rangle'$ also vanishes. Thus the set of first-order moments consists, under the action of $Sp(2n)$, of two *equivalence* classes: the elements that are equivalent to e^1 (which is the set all nonzero $\langle z \rangle$) and the zero element $\langle z \rangle = 0$. Correspondingly, we may view the vectors e^1 and 0 as being *normal* forms for the set of all first-order moments.

34.5 Kinematic Moment Invariants

Definition

Let m be a vector with components m_α, and let $D(\mathcal{M})$ be a matrix with entries $D_{\alpha\beta}(\mathcal{M})$. Write (2.7) in the more compact form

$$m^f = D(\mathcal{M}) m^i. \tag{34.5.1}$$

A function of moments $I[m]$ is said to be a *kinematic moment invariant* if it obeys the relations

$$I[m^f] = I[m^i], \tag{34.5.2}$$

or

$$I[D(\mathcal{M})m] = I[m], \tag{34.5.3}$$

for *all* symplectic maps $\mathcal{M}$.

Rather little is known about the existence and properties of kinematic moment invariants for the set of all symplectic maps. However, kinematic moment invariants are known to exist and all kinematic moment invariants have been found when the symplectic maps $\mathcal{M}$ are restricted to be those associated with the inhomogeneous symplectic group $ISp(2n)$. Moreover, their existence is a consequence of group theory applied to $ISp(2n)$. We note that, if deviation variables are employed, the full $\mathcal{M}$ is well approximated by translation and linear maps provided excursions about the design orbit are sufficiently small.

First-Order Moments

At this point we can observe that there are no *significant* kinematic invariants in the case of first-order moments. Evidently, by definition, a kinematic invariant has the same value for all moments that belong to the same equivalence class. If only translations $\mathcal{T}$ are considered, (4.12) shows that any set of first-order moments can be transformed to any other, and therefore there is only one equivalence class. Consequently I must have a constant value. And, if only linear symplectic transformations $\mathcal{R}$ are considered, we have seen that for the case of first-order moments there are only two equivalence classes. Consequently, in this case I can have only two possible values.

Second-Order Moments

Of particular interest are kinematic moment invariants that can be constructed from the second-order moments Z_{ab} with

$$Z_{ab} = \langle z_a z_b \rangle = (1/N) \int d^6 z \; h(z) z_a z_b. \tag{34.5.4}$$

For a given particle distribution, let Z be the matrix with entries Z_{ab}. In the case of a 1-degree of freedom system phase space is 2 dimensional, and the matrix Z in this case is 2×2. It is easily verified that in this case a kinematic moment invariant [under the action of $Sp(2)$] is given by the rule

$$I[Z] = \mathrm{tr}[(ZJ_2)^2] = 2[(Z_{12})^2 - Z_{11}Z_{22}] = -2(\langle q^2\rangle\langle p^2\rangle - \langle qp\rangle^2). \tag{34.5.5}$$

See Exercise 6.1 where this result is verified and shown to be a consequence of group theory applied to $Sp(2)$. Note that in this case I is proportional to the *mean square emittance* ϵ^2 *defined* by the rule

$$\epsilon^2 = \langle q^2\rangle\langle p^2\rangle - \langle qp\rangle^2. \tag{34.5.6}$$

In the case of a 3-degree of freedom system it can be shown that there are 3 such functionally independent invariants given by the rules

$$I^{(n)}[Z] = \mathrm{tr}[(ZJ)^n], \quad n = 2, 4, 6; \tag{34.5.7}$$

and all other invariants constructed from second-order moments are functions of these invariants. See Exercises 6.2 and 6.3.

34.6 Some Properties of Second-Order Moments

In this section we will explore various properties of Z.

34.6.1 Positive Definite Property

We begin by showing that the matrix Z, which is obviously real and symmetric, is also positive definite. Since $h(z)$ is a phase-space density, it is positive or zero for all z,

$$h(z) \geq 0 \text{ for all } z; \tag{34.6.1}$$

and it follows from (6.1) and continuity that there must be some finite phase-space volume for which $h(z) > 0$. Next let u be any real six-dimensional nonzero vector. Form the function $(u, z)^2$. It has the property

$$(u, z)^2 \geq 0 \text{ for all } z. \tag{34.6.2}$$

Moreover, in the volume where $h(z) > 0$, there must be some subvolume where $(u, z)^2 > 0$. It follows that there is the result

$$\begin{aligned} (u, Zu) &= \sum_{ab} u_a Z_{ab} u_b = (1/N) \int d^6z \; h(z) \sum_{ab} u_a z_a u_b z_b \\ &= (1/N) \int d^6z \; h(z)(u, z)^2 > 0. \end{aligned} \tag{34.6.3}$$

34.6.2 Transformation Properties

Properties under Translations

From the definition (2.1) we see that under translations the transformed moments $\langle z_a z_b \rangle'$ are given by the relation

$$\langle z_a z_b \rangle' = (1/N) \int d^6 \; h'(z) z_a z_b = (1/N) \int d^6z \; h(\mathcal{T}^{-1} z) z_a z_b. \tag{34.6.4}$$

As in Subsection 4.1, introduce new variables $\bar{z}$ by the rules (4.7) through (4.9) and employ (4.10). So doing reveals that (6.4) can be rewritten in the form

$$\begin{aligned}
\langle z_a z_b\rangle' &= (1/N)\int d^6\bar{z}\ (\bar{z}_a+\delta_a)(\bar{z}_b+\delta_b)h(\bar{z}) \\
&= (1/N)\int d^6\bar{z}\ \bar{z}_a\bar{z}_b h(\bar{z}) + (1/N)\int d^6\bar{z}\ \bar{z}_a\delta_b h(\bar{z}) \\
&\quad +(1/N)\int d^6\bar{z}\ \delta_a\bar{z}_b h(\bar{z}) + (1/N)\int d^6\bar{z}\ \delta_a\delta_b h(\bar{z}) \\
&= \langle z_a z_b\rangle + \langle z_a\rangle\delta_b + \delta_a\langle z_b\rangle + \delta_a\delta_b. \qquad (34.6.5)
\end{aligned}$$

Properties under Linear Symplectic Maps

Let us next find the transformation properties of second-order moments under the action of a linear symplectic map $\mathcal{R}$ described by the symplectic matrix R. From the definition (4.1) and (4.13) we see that the transformed moments $\langle z_a z_b\rangle'$ are given by the relation

$$\langle z_a z_b\rangle' = (1/N)\int d^6z\ h'(z)z_a z_b = (1/N)\int d^6z\ h(R^{-1}z)z_a z_b. \qquad (34.6.6)$$

Again introduce new variables $\bar{z}$ by the rules (4.15) through (4.17) and supplement (4.17) with the relation

$$z_b = \sum_d R_{bd}\bar{z}_d. \qquad (34.6.7)$$

Also employ the relation (4.18). With these tools we see that (6.6) can be rewritten in the form

$$\begin{aligned}
\langle z_a z_b\rangle' &= (1/N)\int d^6\bar{z}\ \sum_{cd} R_{ac}R_{bd}\bar{z}_c\bar{z}_d h(\bar{z}) \\
&= \sum_{cd} R_{ac}R_{bd}(1/N)\int d^6\bar{z}\ \bar{z}_c\bar{z}_d h(\bar{z}) \\
&= \sum_{cd} R_{ac}R_{bd}\langle z_c z_d\rangle. \qquad (34.6.8)
\end{aligned}$$

In terms of the notation employed in (5.4), the relation (6.8) can be rewritten in the component form

$$Z'_{ab} = \sum_{cd} R_{ac}R_{bd}Z_{cd} = \sum_{cd} R_{ac}Z_{cd}(R^T)_{db}, \qquad (34.6.9)$$

which has the more compact matrix form

$$Z' = RZR^T. \qquad (34.6.10)$$

This matrix relation specifies how second-order moments transform under a linear symplectic map. We observe, in particular, that the transformation rule is the same for all distributions having the same second-order moments.

34.6.3 Williamson Normal Form

Even more can be said. Since the matrix Z is real, symmetric, and positive definite, according to a theorem of Williamson there is a symplectic matrix A such that

$$AZA^T = D \tag{34.6.11}$$

where D is the *diagonal* matrix

$$D = \text{diag}\{\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3\} \tag{34.6.12}$$

with all $\lambda_j > 0$. The right side of (6.11) is called the *Williamson normal form* of Z.

Two things should be noted about this remarkable result. Define the matrix Z^{norm} by the rule

$$Z^{\text{norm}} = AZA^T = D. \tag{34.6.13}$$

Then, from (6.12 and (6.13), we see that there are the results

$$\langle q_j q_k \rangle^{\text{norm}} = \langle p_j p_k \rangle^{\text{norm}} = 0 \text{ if } j \neq k, \tag{34.6.14}$$

$$\langle q_j^2 \rangle^{\text{norm}} = \langle p_j^2 \rangle^{\text{norm}} = \lambda_j, \tag{34.6.15}$$

$$\langle q_j p_k \rangle^{\text{norm}} = 0. \tag{34.6.16}$$

Also, we observe that (6.11) is of the form (6.10) with $R = A$. Thus, if a beam transport system can be found whose transfer matrix is A, then this transport system will bring the second-order moments to the normal form given by (6.14) through (6.16).

34.6.4 Eigen Emittances

We will next see that two second-order moment matrices Z' and Z have the *same* Williamson normal form if they are connected by a relation of the form (6.10). Indeed, observe that we may write the relation

$$\begin{aligned} AR^{-1}Z'(AR^{-1})^T &= AR^{-1}RZR^T(R^T)^{-1}A^T \\ &= AZA^T = D. \end{aligned} \tag{34.6.17}$$

[Here we have used the result $(R^{-1})^T = (R^T)^{-1}$ which holds for any invertible matrix.] But, by the group property of symplectic matrices, the matrix AR^{-1} is symplectic if the matrices A and R are symplectic. We see from (6.17) that the symplectic matrix AR^{-1} brings Z' to Williamson normal form and, according to (6.13), this normal form is the same as that for Z. We will call the quantities λ_j^2 *eigen emittances*, and we will call the quantities λ_j *root* eigen emittances. It follows that while the entries in Z evolve as a particle distribution propagates through various elements, see (6.10), the eigen emittances remain *unchanged* (in the linear approximation). Thus, given an initial particle distribution, one can compute the initial second moments $\langle z_a z_b \rangle^i$, and from them the eigen emitances. And these eigen emittances will remain unchanged (in the linear approximation) as the particle distribution evolves.

It can be shown that the eigen emittances generalize the 1-degree of freedom mean-square emittance given by (5.6) to the fully coupled case. Indeed, it can be shown that in terms of the λ_j the kinematic invariants $I^{(n)}$ given by (5.7) have the values

$$I^{(n)} = 2(-1)^{n/2}(\lambda_1^n + \lambda_2^n + \lambda_3^n). \tag{34.6.18}$$

See Exercise 6.3.

There are symplectic matrix routines that, given Z, find A and the λ_j. If only the λ_j are required, they can be found from the eigenvalues of JZ. Note that JZ is a Hamiltonian matrix. See Exercise 3.17.14.

To see that the λ_j can be found from the eigenvalues of JZ, suppose both sides of (6.11) are multiplied by J to give the result

$$JAZA^T = JD. \tag{34.6.19}$$

From the symplectic condition for A it follows that there is the relation

$$JA = (A^T)^{-1}J. \tag{34.6.20}$$

Consequently (6.19) can be rewritten in the form

$$(A^T)^{-1}JZA^T = JD, \tag{34.6.21}$$

which reveals that the matrices JZ and JD are related by a *similarity* transformation, and therefore have the same eigenvalues. See Exercise 3.7.16.

What remains is to find the eigenvalues of JD which, according to (3.2.10), (3.2.11), and (6.12) can be written in the block form

$$JD = \begin{pmatrix} \lambda_1 J_2 & & \\ & \lambda_2 J_2 & \\ & & \lambda_3 J_2 \end{pmatrix}. \tag{34.6.22}$$

Let W_2 be the unitary and (complex) symplectic 2×2 matrix

$$W_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{34.6.23}$$

[See (3.9.12).] It has the property

$$W_2^{-1} J_2 W_2 = iK_2 \tag{34.6.24}$$

where K_2 is the matrix

$$K_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34.6.25}$$

From W_2 construct the the 6×6 matrix W given in block form by the rule

$$W = \begin{pmatrix} W_2 & & \\ & W_2 & \\ & & W_2 \end{pmatrix}. \tag{34.6.26}$$

It follows from (6.22) and (6.24) that there is the relation

$$W^{-1}JDW = \begin{pmatrix} i\lambda_1 K_2 & & \\ & i\lambda_2 K_2 & \\ & & i\lambda_3 K_2 \end{pmatrix}. \tag{34.6.27}$$

We see that the eigenvalues of JD, and hence JZ, are pure imaginary and come in the $\pm$ pairs

$$\sigma_j = \pm i\lambda_j. \tag{34.6.28}$$

Conversely, if the eigenvalues $\pm\sigma_j$ of JZ are computed, then the root eigen emittances are given by the relation

$$\lambda_j = |\sigma_j|. \tag{34.6.29}$$

Suppose we combine the relations (6.21) and (6.27) to find the result

$$W^{-1}(A^T)^{-1}JZA^TW = \begin{pmatrix} i\lambda_1 K_2 & & \\ & i\lambda_2 K_2 & \\ & & i\lambda_3 K_2 \end{pmatrix} = \mathrm{diag}\{-i\lambda_1, i\lambda_1, -i\lambda_2, i\lambda_2, -i\lambda_3, i\lambda_3\}. \tag{34.6.30}$$

Let A' be the matrix defined by the rule

$$A' = A^TW, \;\; (A')^{-1} = W^{-1}(A^T)^{-1}. \tag{34.6.31}$$

It will be symplectic since A (and hence A^T) and W are symplectic. With the aid of A' the relation (6.30) takes the form

$$(A')^{-1}JZA' = \mathrm{diag}\{-i\lambda_1, i\lambda_1, -i\lambda_2, i\lambda_2, -i\lambda_3, i\lambda_3\}. \tag{34.6.32}$$

We observe that since Z is positive definite, JZ is a "special" kind of Hamiltonian matrix. As a consequence of Williamson's theorem we have seen that it can be diagonalized by a similarity transformation even if its eigenvalues are not distinct; moreover the diagonalizing matrix A' can be chosen to be real and symplectic. And, as stated earlier, the eigenvalues of JZ are pure imaginary. Note that we have considered the case of 6×6 matrices. But it is evident that analogous results hold in any (even) dimension.

At this point, recall Exercise 3.7.14. There it was found that *one* possibility for Hamiltonian matrices is that they have purely imaginary eigenvalues coming in complex conjugate pairs. Now we have learned that any matrix of the form JZ with Z real and symmetric and *positive definite* (which makes them a special kind of Hamiltonian matrix) *must* have have purely imaginary eigenvalues coming in complex conjugate pairs. In this case there are no other possibilities. Moreover, Hamiltonian matrices of this special kind can always be diagonalized by a real symplectic similarity transformation. Evidently the same result holds if Z is negative definite.

Finally, we note that multiplying both sides of (6.10) by J produces the relation

$$JZ' = JRZR^T = (R^T)^{-1}JZR^T. \tag{34.6.33}$$

Here we have used the fact that R is symplectic. We see that with the use of J the evolution rule (6.10) for Z becomes the *similarity* transformation rule (6.33) for JZ. Since eigenvalues are preserved by similarity transformations, we have found an alternative explanation of why the eigen emittances remain unchanged as a particle distribution evolves.

34.6.5 Classical Uncertainty Principle

Statement

The results of the previous subsection can be used to derive a *classical* uncertainty principle. What we will show is that there is the inequality

$$\langle q_i^2 \rangle \langle p_i^2 \rangle \geq \lambda_{\min}^2,\ i = 1, 2, 3, \tag{34.6.34}$$

where $\lambda_{\min}$ is the minimum of the λ_k. No matter what is done to a beam (ignoring nonlinear and nonsymplectic effects), the products of the mean-square deviations in q_i and p_i for any plane must exceed, or at best equal, $\lambda_{\min}^2$.

Proof

Begin by rewriting (6.13) in the form

$$Z = A^{-1} Z^{\text{norm}} (A^{-1})^T = N^T Z^{\text{norm}} N = N^T D N \tag{34.6.35}$$

where we have made the definition

$$N = (A^{-1})^T. \tag{34.6.36}$$

We note that N will be symplectic if A is symplectic, and conversely.

Let us compute the $\langle q_i^2 \rangle$ and $\langle p_i^2 \rangle$. To compute the $\langle q_i^2 \rangle$ set

$$a = j \text{ with } j = 1, 3, 5 \text{ when } i = 1, 2, 3. \tag{34.6.37}$$

We then find from (6.35) that

$$\begin{aligned} \langle q_i^2 \rangle &= Z_{aa} = (N^T D N)_{aa} = \sum_{cd} (N^T)_{ac} D_{cd} N_{da} \\ &= \sum_c N_{ca} D_{cc} N_{ca} = \sum_c (N_{ca})^2 D_{cc} \\ &\geq \lambda_{\min} \sum_c (N_{ca})^2. \end{aligned} \tag{34.6.38}$$

Similarly, to compute the $\langle p_i^2 \rangle$, upon setting

$$b = j + 1, \tag{34.6.39}$$

we find that

$$\begin{aligned} \langle p_i^2 \rangle &= Z_{bb} = (N^T D N)_{bb} = \sum_{cd} (N^T)_{bc} D_{cd} N_{db} \\ &= \sum_d N_{db} D_{dd} N_{db} = \sum_d (N_{db})^2 D_{dd} \\ &\geq \lambda_{\min} \sum_d (N_{db})^2. \end{aligned} \tag{34.6.40}$$

It follows that

$$\langle q_i^2\rangle\langle p_i^2\rangle \geq \lambda_{\min}^2 \left[\sum_c (N_{ca})^2\right]\left[\sum_d (N_{db})^2\right]. \tag{34.6.41}$$

To proceed further, let u^a and u^b be vectors with the entries

$$u_c^a = N_{ca}, \tag{34.6.42}$$

$$u_d^b = N_{db}. \tag{34.6.43}$$

Evidently u^a and u^b are the $a^{\rm th}$ and $b^{\rm th}$ columns of N. With these definitions we may write (6.41) in the more compact form

$$\langle q_i^2\rangle\langle p_i^2\rangle \geq \lambda_{\min}^2 ||u^a||^2\ ||u^b||^2 \tag{34.6.44}$$

where $|| * ||$ denotes the Euclidean norm. Since N is a symplectic matrix, it follows from the symplectic condition that there is also the relation

$$(u^a, Ju^b) = 1. \tag{34.6.45}$$

See Exercise 3.6.13. It can be shown using the spectral norm for J that (6.45) in turn entails the inequality.

$$||u^a||\ ||u^b|| \geq 1. \tag{34.6.46}$$

See Exercise 3.7.1. Upon combining (6.44) and (6.46) we find the advertised result (6.34).

34.6.6 Minimum Emittance Theorem

Statement

The classical uncertainty principle shows that (in the linear approximation) no matter how a beam is transformed, the product of the spreads in position and the conjugate momentum must satisfy the relation (6.34). There is a related constraint on the mean-square emittances ϵ_i defined by

$$\epsilon_i^2 = \langle q_i^2\rangle\langle p_i^2\rangle - \langle q_i p_i\rangle^2. \tag{34.6.47}$$

What we will show is that (in the linear approximation) no matter how a beam is transformed (symplectically) there is the constraint

$$\epsilon_i^2 \geq \lambda_{\min}^2,\ i = 1, 2, 3. \tag{34.6.48}$$

Together the information provided by the classical uncertainty principle and the minimum emittance theorem is useful when designing a beam line to perform emittance manipulations because it sets lower limits on what one can hope to achieve.

Proof

Suppose, in the 6×6 case under consideration, that we partition Z into nine 2×2 blocks by writing

$$Z = \begin{pmatrix} Z^{11} & Z^{12} & Z^{13} \\ Z^{21} & Z^{22} & Z^{23} \\ Z^{31} & Z^{32} & Z^{33} \end{pmatrix}. \tag{34.6.49}$$

Because Z is symmetric, the blocks will satisfy the relations

$$(Z^{ij})^T = Z^{ji}. \tag{34.6.50}$$

Let R be a 6×6 matrix having the block form

$$R = \begin{pmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{34.6.51}$$

It will be symplectic if A is symplectic. Its use in (6.10) produces a Z' given by

$$Z' = \begin{pmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} Z^{11} & Z^{12} & Z^{13} \\ Z^{21} & Z^{22} & Z^{23} \\ Z^{31} & Z^{32} & Z^{33} \end{pmatrix} \begin{pmatrix} A^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{34.6.52}$$

Carrying out the indicated multiplication gives the result

$$Z' = \begin{pmatrix} AZ^{11}A^T & AZ^{12} & AZ^{13} \\ Z^{21}A^T & Z^{22} & Z^{23} \\ Z^{31}A^T & Z^{32} & Z^{33} \end{pmatrix}. \tag{34.6.53}$$

In particular, we see that

$$(Z')^{11} = AZ^{11}A^T. \tag{34.6.54}$$

We will now seek a symplectic A that brings Z^{11} to Williamson normal form. Define the quantity ϵ_1 by the rules

$$\epsilon_1^2 = Z^{11}_{11}Z^{11}_{22} - (Z^{11}_{12})^2 = \langle q_1^2\rangle\langle p_1^2\rangle - \langle q_1p_1\rangle^2, \tag{34.6.55}$$

$$\epsilon_1 = +\sqrt{\epsilon_1^2}. \tag{34.6.56}$$

It follows from the Schwarz inequality that there is the relation

$$\langle q_1p_1\rangle^2 \leq \langle q_1^2\rangle\langle p_1^2\rangle \tag{34.6.57}$$

and therefore the right side of (6.55) can never be negative. See Exercise 6.4. Consequently ϵ_1 is well defined by (6.55) and (6.56), and is positive. Next define "beam" betatron functions α, β, γ by the rules

$$\alpha = -Z^{11}_{12}/\epsilon_1 = -\langle q_1p_1\rangle/\epsilon_1, \tag{34.6.58}$$

$$\beta = Z^{11}_{11}/\epsilon_1 = \langle q_1^2\rangle/\epsilon_1, \tag{34.6.59}$$

$$\gamma = Z_{22}^{11}/\epsilon_1 = \langle p_1^2 \rangle / \epsilon_1. \tag{34.6.60}$$

In terms of these definitions, Z^{11} takes the form

$$Z^{11} = \epsilon_1 \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}. \tag{34.6.61}$$

From (6.59) and (6.60) there are the inequalities $\beta \geq 0$ and $\gamma \geq 0$. And, from (6.55) through (6.60), there is the relation

$$1 = \beta\gamma - \alpha^2. \tag{34.6.62}$$

Finally, define the matrix A by the rule

$$A = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix}. \tag{34.6.63}$$

Since A is 2×2 and evidently has unit determinant, it is symplectic. Correspondingly, the R given by (6.51) is symplectic. And, from the definitions made and executing the matrix multiplications $AZ^{11}A^T$ indicated in (6.54), we find that

$$(Z')^{11} = AZ^{11}A^T = \text{diag}(\epsilon_1, \epsilon_1). \tag{34.6.64}$$

See Exercise 6.5.

We will now exploit these results. From (6.64) we find that

$$\langle q_1^2 \rangle' = (Z')_{11}^{11} = \epsilon_1, \tag{34.6.65}$$

$$\langle p_1^2 \rangle' = (Z')_{22}^{11} = \epsilon_1. \tag{34.6.66}$$

It follows that

$$\langle q_1^2 \rangle' \, \langle p_1^2 \rangle' = \epsilon_1^2. \tag{34.6.67}$$

But we also have the relation (6.34). We conclude that there is the inequality

$$\epsilon_1^2 \geq \lambda_{\text{min}}^2, \tag{34.6.68}$$

in accord with (6.48). Analogous results hold for the other planes.

Sharpening

We close this subsection by noting that the minimum emittance theorem (6.48) sharpens the classical uncertainty principle (6.34). Indeed, combining (6.47) and (6.48) produces the result

$$\langle q_i^2 \rangle \langle p_i^2 \rangle \geq \lambda_{\text{min}}^2 + \langle q_i p_i \rangle^2, \; i = 1, 2, 3. \tag{34.6.69}$$

We see that to minimize $\langle q_i^2 \rangle \langle p_i^2 \rangle$ we must insure that $\langle q_i p_i \rangle$ vanishes.

34.6.7 Nonexistence of Maximum Emittances

The classical uncertainty principle (6.34) and the minimum emittance theorem (6.48) show that the mean square emittances are bounded from below under the action of linear symplectic maps. We will now see that they are *not* bounded from above if the phase space has 4 or more dimensions.

Consider the 4-dimensional case, and suppose initially a particle distribution has all quadratic moments zero save for the moments $\langle q_1^2\rangle$, $\langle p_1^2\rangle$, $\langle q_2^2\rangle$, and $\langle p_2^2\rangle$. (From the work of Subsection 6.3 we know that there is always a linear symplectic transformation that will bring the quadratic moments to this form.) In this case the mean square emittances ϵ_i^2 are given by the relation

$$\epsilon_i^2 = \langle q_i^2\rangle\langle p_i^2\rangle. \tag{34.6.70}$$

Let $\mathcal{R}$ be the linear symplectic map

$$\mathcal{R} = \exp(\nu : q_1 p_2 :). \tag{34.6.71}$$

Here ν is some some real parameter. It is easily verified that this map has the properties

$$\bar{q}_1 = \mathcal{R} q_1 = q_1, \tag{34.6.72}$$

$$\bar{p}_1 = \mathcal{R} p_1 = p_1 + \nu p_2, \tag{34.6.73}$$

$$\bar{q}_2 = \mathcal{R} q_2 = q_2 - \nu q_1, \tag{34.6.74}$$

$$\bar{p}_2 = \mathcal{R} p_2 = p_2. \tag{34.6.75}$$

It is also easily verified that there are the transformed moment relations

$$\langle \bar{q}_1^2\rangle = \langle q_1^2\rangle, \tag{34.6.76}$$

$$\langle \bar{p}_1^2\rangle = \langle (p_1 + \nu p_2)^2\rangle = \langle p_1^2\rangle + \nu^2\langle p_2^2\rangle, \tag{34.6.77}$$

$$\langle \bar{q}_1 \bar{p}_1\rangle = 0; \tag{34.6.78}$$

$$\langle \bar{q}_2^2\rangle = \langle (q_2 - \nu q_1)^2\rangle = \langle q_2^2\rangle + \nu^2\langle q_1^2\rangle, \tag{34.6.79}$$

$$\langle \bar{p}_2^2\rangle = \langle p_2^2\rangle, \tag{34.6.80}$$

$$\langle \bar{q}_2 \bar{p}_2\rangle = 0. \tag{34.6.81}$$

Correspondingly, the transformed mean square emittance $\bar{\epsilon}_1^2$ satisfies the relation

$$\begin{aligned} \bar{\epsilon}_1^2 &= \langle \bar{q}_1^2\rangle\langle \bar{p}_1^2\rangle - \langle \bar{q}_1\bar{p}_1\rangle^2 \\ &= \langle q_1^2\rangle(\langle p_1^2\rangle + \nu^2\langle p_2^2\rangle) \\ &= \epsilon_1^2 + \nu^2\langle q_1^2\rangle\langle p_2^2\rangle. \end{aligned} \tag{34.6.82}$$

Similarly the transformed mean square emittance $\bar{\epsilon}_2^2$ satisfies the relation

$$\bar{\epsilon}_2^2 = \epsilon_2^2 + \nu^2\langle q_1^2\rangle\langle p_2^2\rangle. \tag{34.6.83}$$

We conclude that both $\bar{\epsilon}_1^2$ and $\bar{\epsilon}_2^2$ can be made arbitrarily large by making $|\nu|$ arbitrarily large.

34.6.8 Second-Order Moments about the Beam Centroid

Definition

Define second-order moments *About* the *Beam Centroid*, denoted as Z_{ab}^{ABC}, by the rule

$$Z_{ab}^{\mathrm{ABC}} = \langle (z_a - \langle z_a \rangle)(z_b - \langle z_b \rangle) \rangle = (1/N) \int d^6z \; h(z)(z_a - \langle z_a \rangle)(z_b - \langle z_b \rangle). \tag{34.6.84}$$

Executing the indicated operations gives the result

$$\begin{aligned} Z_{ab}^{\mathrm{ABC}} &= \langle (z_a - \langle z_a \rangle)(z_b - \langle z_b \rangle) \rangle \\ &= \langle z_a z_b \rangle - \langle z_a \langle z_b \rangle \rangle - \langle \langle z_a \rangle z_b \rangle + \langle \langle z_a \rangle \langle z_b \rangle \rangle \\ &= \langle z_a z_b \rangle - \langle z_a \rangle \langle z_b \rangle \\ &= Z_{ab} - Z_{ab}^{\mathrm{OBC}}. \end{aligned} \tag{34.6.85}$$

Here Z_{ab}^{OBC} denotes the set of second-order moments *Of* the *Beam Centroid* defined by the rule

$$Z_{ab}^{\mathrm{OBC}} = \langle z_a \rangle \langle z_b \rangle. \tag{34.6.86}$$

Intuitively, Z^{OBC} may be viewed as the collection of second-order moments of a beam distribution consisting of a single *macro* particle located at the beam centroid. We also observe, in passing, that (6.85) can be rewritten in the form

$$Z = Z^{\mathrm{ABC}} + Z^{\mathrm{OBC}}, \tag{34.6.87}$$

which is analogous to the fact that the inertia tensor of a rigid body about some specified origin is the sum of its inertia tensor about its center of mass plus the inertia tensor of its center of mass about the specified origin.

Properties under Translations

What are the transformation properties of Z^{ABC} under the action of a translation $\mathcal{T}$? Starting from the definition (6.83) we find the chain of equalities

$$\begin{aligned} (Z_{ab}^{\mathrm{ABC}})' &= (\langle (z_a - \langle z_a \rangle')(z_b - \langle z_b \rangle') \rangle)' \\ &= (1/N) \int d^6z \; h'(z)(z_a - \langle z_a \rangle')(z_b - \langle z_b \rangle') \\ &= (1/N) \int d^6z \; h(\mathcal{T}^{-1}z)(z_a - \langle z_a \rangle')(z_b - \langle z_b \rangle') \\ &= (1/N) \int d^6\bar{z} \; h(\bar{z})(\bar{z}_a + \delta_a - \langle z_a \rangle')(\bar{z}_b + \delta_b - \langle z_b \rangle') \\ &= (1/N) \int d^6\bar{z} \; h(\bar{z})(\bar{z}_a - \langle z_a \rangle)(\bar{z}_b - \langle z_b \rangle) \\ &= Z_{ab}^{\mathrm{ABC}}. \end{aligned} \tag{34.6.88}$$

Here we have used (4.9) and (4.10) to change variables and have used (4.12) to obtain the relation

$$(\bar{z}_a + \delta_a - \langle z_a \rangle')(\bar{z}_b + \delta_b - \langle z_b \rangle') = (\bar{z}_a - \langle z_a \rangle)(\bar{z}_b - \langle z_b \rangle). \tag{34.6.89}$$

We see that Z^{ABC} is *invariant* under translations. (For an alternate proof of this result, see Exercise 6.6.) The quantity Z^{ABC} describes an *intrinsic* property of the beam distribution in that it does not depend on the location of the beam relative to the design orbit. Note that, according to (6.85) and(6.86), each component Z_{ab}^{ABC} of Z^{ABC} depends on first and second moments. Therefore, each component is a moment invariant under the action of translations.

Properties under Linear Symplectic Maps

What are the transformation properties of Z^{ABC} under the action of a linear symplectic map $\mathcal{R}$? Evidently, according to (6.85), we may write the relation

$$(Z^{\mathrm{ABC}})' = Z' - (Z^{\mathrm{OBC}})'. \tag{34.6.90}$$

We see from (4.19) that there is the relation

$$(Z_{ab}^{\mathrm{OBC}})' = \langle z_a\rangle'\langle z_b\rangle' = \sum_{cd} R_{ac}R_{bd}\langle z_c\rangle\langle z_d\rangle, \tag{34.6.91}$$

which can be written in the more compact form

$$(Z^{\mathrm{OBC}})' = RZ^{\mathrm{OBC}}R^T. \tag{34.6.92}$$

Recall also the relation (6.10). It follows that there is the result

$$(Z^{\mathrm{ABC}})' = RZR^T - RZ^{\mathrm{OBC}}R^T = RZ^{\mathrm{ABC}}R^T. \tag{34.6.93}$$

We conclude that Z, Z^{ABC}, and Z^{OBC} all transform in the same manner.

Positive Definiteness

What about positive definiteness? First consider Z^{OBC}. As in Subsection 6.1, u be any real nonzero vector. It follows from (6.86) that there is the result

$$(u, Z^{\mathrm{OBC}}u) = \sum_{ab} u_a Z_{ab}^{\mathrm{OBC}} u_b = \sum_{ab} u_a\langle z_a\rangle\langle z_b\rangle u_b = (u, \langle z\rangle)^2 \geq 0. \tag{34.6.94}$$

We see that $(u, Z^{\mathrm{OBC}}u)$ can never be negative. However if $(u, \langle z\rangle) = 0$, which is certainly possible, then $(u, Z^{\mathrm{OBC}}u) = 0$. Therefore Z^{OBC} is *not* positive *definite*.

Even more can be said. Let R be a symplectic matrix that has the property (4.21). Then we see from (6.91) that in this case

$$(Z^{\mathrm{OBC}})' = D \tag{34.6.95}$$

where D is a diagonal matrix with all entries zero save for D_{11} which has the value $D_{11} = 1$. Correspondingly, DJ has all entries zero save that $(DJ)_{12} = 1$. Consequently, the eigenvalues of DJ and JD, and hence of $Z^{\mathrm{OBC}}J$ and JZ^{OBC}, all vanish. Recall Exercises 3.7.18 and 3.7.16. Moreover, there is the relation

$$(DJ)^2 = 0, \tag{34.6.96}$$

from which it follows that

$$(Z^{\mathrm{OBC}}J)^2 = 0. \tag{34.6.97}$$

We conclude from (5.7) and (6.97) that for Z^{OBC} there is the result

$$I^{(n)}[Z^{\mathrm{OBC}}] = 0. \tag{34.6.98}$$

The second-order moment invariants of a beam distribution consisting of a single macro particle all vanish.

What about second-order moments about the beam centroid, those described by Z^{ABC}? Calculation/insight shows that Z^{ABC} *is* positive definite. See Exercise 6.7. Therefore it has a Williamson normal form and is characterized by eigen emittances. Note that because Z^{ABC} is invariant under translations, these eigen emittances are *invariant* under the action of the *full* group $ISp(2n)$.

We also observe that in general these eigen emittances may differ from those of Z, but they may be of interest if the beam centroid is quite far from the design orbit. See Exercise 6.8. However in practice it is probably desirable, as one of the criteria for beam matching, to arrange to have the beam centroid coincide with the design orbit. This is also natural from an instrumentation perspective since beam position monitors essentially record the spatial coordinates of the beam centroid. Recall Section 3.

34.6.9 Summary of What We Have Learned

The information provided by the classical uncertainty principle and the minimum emittance theorem is useful when designing a beam line to perform emittance manipulations on a beam because it sets lower limits on what one can hope to achieve. It should also be useful in analyzing the results of beam cooling experiments. In this case one can measure all quadratic moments before and after a cooling channel. Next compute the eigen emittances of Z before and Z after. Ideally, one would like to find that all the λ_j^2 have decreased, or at least the minimum of the λ_j^2 has decreased.

We have seen that, in considering what can be achieved under beam transport (in the linear approximation), what counts are the eigen emittances, and these can be viewed as properties of the *initial* particle distribution. Moreover, according to (6.14) through (6.16), the best that can be achieved are the spread relations

$$\langle q_i^2\rangle\langle p_i^2\rangle = \lambda_i^2,\ i = 1, 2, 3 \tag{34.6.99}$$

where the λ_i are the eigen emittances in some order. Thus, in the combined context of both source and beam-line design, the challenge is to produce an initial particle distribution having optimal eigen emittances and to then transform the initial particle distribution in such a way that the optimal spread relations are realized in the desired planes. The next sections will describe various methods for producing initial particle distributions having optimal eigen emittances and how to then transform these distributions in such a way that the optimal spread relations are realized in the desired planes.

Exercises

34.6.1. Verify (5.5). Next suppose that Z' and Z are related by (6.10) and that R is symplectic. Verify that

$$I[Z'] = \text{tr}[(Z'J_2)^2] = \text{tr}[(RZR^TJ_2)^2]. \tag{34.6.100}$$

Next verify that

$$\begin{aligned}
\text{tr}[(RZR^TJ_2)^2] &= \text{tr}[RZR^TJ_2RZR^TJ_2] = \text{tr}[RZJ_2ZR^TJ_2] \\
&= \text{tr}[ZJ_2ZR^TJ_2R] = \text{tr}[ZJ_2ZJ_2] \\
&= \text{tr}[(ZJ_2)^2] = I[Z].
\end{aligned} \tag{34.6.101}$$

Here we have used that assumption that R is symplectic and the trace property (3.6.130). Combining (6.100) and (6.101) shows that I is invariant,

$$I[Z'] = I[Z]. \tag{34.6.102}$$

How might we have known that there should be an invariant? Consider the space of all quadratic polynomial functions of the two-dimensional phase-space variables q, p. For present purposes, a convenient basis for these polynomials is given by the monomials

$$c^1 = q^2, \tag{34.6.103}$$

$$c^2 = qp, \tag{34.6.104}$$

$$c^3 = p^2. \tag{34.6.105}$$

According to Section 24.2, these polynomials carry the $sp(2)$ representation $\Gamma(2)$. Also, comparison of (2.5) and (2.7) shows that polynomials P_α and moments m_α have the *same* transformation properties. Therefore, and in particular, second-order moments of the two-dimensional phase-space variables q, p also carry the representation $\Gamma(2)$.

Next we observe that, for $sp(2)$, there is the Clebsch-Gordan series result

$$\Gamma(2) \otimes \Gamma(2) = \Gamma(0) \oplus \Gamma(2) \oplus \Gamma(4). \tag{34.6.106}$$

This is just the $sp(2)$ analog of the familiar statement that spin 1 and spin 1 combine to make spin 0, spin 1, and spin 2. Even more familiar, it is the analog of the statement that two vectors can be combined to make a scalar by use of the dot product, or can be combined to make another vector by use of the cross product, or can be combined to make a tensor by use of the tensor product. By definition, an entity (if is nonzero) that carries the representation $\Gamma(0)$ will be an invariant. Therefore, according to (6.106), there is at least the hope/possibility of constructing an invariant out of quadratic products of second-order moments. Note that the contents of (5.5) are indeed quadratic products of second-order moments.

How can we construct an entity that carries the representation $\Gamma(0)$? We have already seen some such constructions in Section 24.11, which you should review. You will now have the pleasure of making a similar construction for the problem at hand.

Begin by finding the symmetric matrices S^j associated with the c^j by the rule

$$c^j = (z, S^j z). \tag{34.6.107}$$

Show that these matrices are given by the relations

$$S^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{34.6.108}$$

$$S^2 = (1/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{34.6.109}$$

$$S^3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34.6.110}$$

Next find the associated $sp(2)$ matrices C^j defined by the rule

$$C^j = JS^j. \tag{34.6.111}$$

Show that these matrices are given by the relations

$$C^1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{34.6.112}$$

$$C^2 = (1/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{34.6.113}$$

$$C^3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{34.6.114}$$

From these $sp(2)$ matrices construct the associated down-index metric tensor g by the rule

$$g_{jk} = \mathrm{tr}(C^j C^k). \tag{34.6.115}$$

Show that g has the entries

$$g = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{34.6.116}$$

With g in hand, construct the up-index metric tensor $\hat{g}$ by the rule

$$\hat{g}^{jk} = (g^{-1})_{jk}. \tag{34.6.117}$$

Show that $\hat{g}$ has the entries

$$\hat{g} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{34.6.118}$$

Finally, based on arguments provided in Section 24.11, the quantity I defined by

$$I = \sum_{jk} \langle c^j \rangle \hat{g}^{jk} \langle c^k \rangle \tag{34.6.119}$$

should be invariant. Verify that

$$I = \sum_{jk} \langle c^j \rangle \hat{g}^{jk} \langle c^k \rangle = -2\langle c^1 \rangle \langle c^3 \rangle + 2\langle c^2 \rangle^2 = -2(\langle q^2 \rangle \langle p^2 \rangle - \langle qp \rangle^2), \tag{34.6.120}$$

which agrees with (5.6). If we look at (6.120) from a group-theory perspective, we see that the quantities $\hat{g}^{jk}$ are the $Sp(2)$ Clebsch-Gordan coefficients that couple $\Gamma(2)$ and $\Gamma(2)$ down to $\Gamma(0)$.

There is one further proof of the invariance of I as given by (5.5), actually of ϵ^2 as given by (5.6), that historically probably came first. Begin by verifying, in the case of a two-dimensional phase space, that the second-order moment matrix Z has the form

$$Z = \begin{pmatrix} \langle q^2 \rangle & \langle qp \rangle \\ \langle qp \rangle & \langle p^2 \rangle \end{pmatrix}. \tag{34.6.121}$$

Consequently, the determinant of Z has the value

$$\det Z = \langle q^2 \rangle \langle p^2 \rangle - \langle qp \rangle^2 = \epsilon^2. \tag{34.6.122}$$

Next look at the transformation rule (6.10) in the two-dimensional phase space case. Verify that taking the determinant of both sides of (6.10) and recalling Section 3.3.3 give the result

$$\det Z' = \det RZR^T = (\det R)(\det Z)(\det R^T) = \det Z, \tag{34.6.123}$$

thereby demonstrating the invariance of the mean-square emittance in the two-dimensional phase space case.

34.6.2. The aim of this exercise is to show that $I^{(n)}[Z]$ as given by (5.7) is invariant. First, as warmup steps, verify the relations

$$I^{[n]} = 0 \text{ for odd } n, \tag{34.6.124}$$

$$\mathrm{tr}[(ZJ)^n] = \mathrm{tr}[(JZ)^n]. \tag{34.6.125}$$

Deduce from (6.33) that

$$(JZ')^n = [(R^T)^{-1} JZR^T]^n = (R^T)^{-1} (JZ)^n R^T. \tag{34.6.126}$$

Next verify from (6.126) that

$$\mathrm{tr}[(JZ')^n] = \mathrm{tr}[(JZ)^n]. \tag{34.6.127}$$

You have shown that

$$I^{(n)}[Z'] = \mathrm{tr}[(JZ')^n] = \mathrm{tr}[(Jz)^n] = I^{(n)}[Z]. \tag{34.6.128}$$

34.6.3. The aim of this exercise is to prove (6.18) and to remark on one of its consequences. To do so, begin by verifying the following chain of equalities:

$$I^{(n)}[Z] = I^{(n)}[AZA^T] = I^{(n)}[D] = \mathrm{tr}[(DJ)^n] = \mathrm{tr}[(JD)^n]. \tag{34.6.129}$$

Here we have used the invariance of $I^{(n)}$, the relation (6.11), and the relation (6.125). Next use (6.22) to show that

$$(JD)^2 = -\text{diag}\{\lambda_1^2, \lambda_1^2, \lambda_2^2, \lambda_2^2, \lambda_3^2, \lambda_3^2\}, \tag{34.6.130}$$

from which it follows that

$$(JD)^n = (-1)^{n/2}\text{diag}\{\lambda_1^n, \lambda_1^n, \lambda_2^n, \lambda_2^n, \lambda_3^n, \lambda_3^n\}. \tag{34.6.131}$$

Here we assume n is even since the case of odd n has already been covered in (6.124). Finally, verify that

$$\text{tr}[(JD)^n] = 2(-1)^{n/2}(\lambda_1^n + \lambda_2^n + \lambda_2^n) \tag{34.6.132}$$

thereby proving (6.18).

As a parting comment we remark that, if desired, relations of the form (6.18) can be solved for the λ_j in terms of the $I^{[n]}$, and that the solution is given in terms of radicals for the cases where the phase-space dimension is less than or equal to 8. Find explicit results when the phase-space dimension is 2 or 4.

34.6.4. Let $f(z)$ and $g(z)$ be any two real *polynomial* functions. Such functions form a vector space. By using the phase-space density $h(z)$, which is assumed to fall off sufficiently fast at infinity, define a scalar product (f, g) by the rule

$$(f, g) = (1/N) \int d^6z\, h(z)f(z)g(z). \tag{34.6.133}$$

Verify that (6.133) satisfies all the requirements to be a scalar product including the positive-definite conditions

$$(f, f) \geq 0, \tag{34.6.134}$$

$$(f, f) = 0 \Leftrightarrow f = 0. \tag{34.6.135}$$

Note also that, in terms of moment notation, there is the relation

$$(f, g) = \langle fg \rangle. \tag{34.6.136}$$

Prove the Schwarz inequality in this context, and use it to verity the result (6.57). See Exercise 3.7.1.

34.6.5. Verify (6.62). Let I be the 2×2 identity matrix, and let Z^{11} and A be the matrices (6.61) and (6.63), respectively. Verify the matrix multiplication result

$$AZ^{11}A^T = \epsilon_1 I. \tag{34.6.137}$$

34.6.6. The aim of this exercise is to provide an alternate proof of (6.88).

34.6.7. Suppose Z and Z' are two matrices related by (6.10). Under the assumption that R is nonsingular, but not necessarily symplectic, show that Z' is symmetric and positive definite if the same is true for Z.

A further task is to show that Z^{ABC} is positive definite. First provide a proof along the lines of that in Subsection 6.1. Next $\cdots$.

34.6.8. The purpose of this exercise is to illustrate by a simple example that translation can change an emittance. Consider, for simplicity, the case of a two-dimensional phase space, and suppose a beam initially has the moments

$$\langle q^2 \rangle = \langle p^2 \rangle = \lambda, \tag{34.6.138}$$

$$\langle qp \rangle = \langle q \rangle = \langle p \rangle = 0. \tag{34.6.139}$$

Then the initial mean-square emittance ϵ^2 has the value

$$\epsilon^2 = \lambda^2. \tag{34.6.140}$$

Next consider the effect of a translation that simply augments q by an amount δ. Then, to compute the final emittance, we need the quantities

$$\langle q^2 \rangle' = \langle (q+\delta)^2 \rangle = \langle (q^2 + 2q\delta + \delta^2) \rangle = \langle q^2 \rangle + 2\delta \langle q \rangle + \delta^2 = \lambda + \delta^2, \tag{34.6.141}$$

$$\langle p^2 \rangle' = \langle p^2 \rangle = \lambda \tag{34.6.142}$$

$$\langle qp \rangle' = \langle (q+\delta)p \rangle = \langle qp \rangle + \delta \langle p \rangle = 0. \tag{34.6.143}$$

From these quantities show that the transformed mean-square emittance is given by the relation

$$(\epsilon^2)' = \langle q^2 \rangle' \langle p^2 \rangle' - (\langle qp \rangle')^2 = (\lambda + \delta^2)\lambda = \lambda^2 + \delta^2 \lambda. \tag{34.6.144}$$

Upon comparing (6.140) and (6.144), we see that the mean-square emittance has been *increased* by an amount $\delta^2 \lambda$.

34.7 Construction of Initial Distributions with Small/Optimized Eigen Emittances

34.8 Realization of Eigen Emittances as Mean-Square Emittances

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Chapter 35

Optimal Evaluation of Symplectic Maps

35.1 Overview of Symplectic Map Approximation

Several previous chapters have been devoted to the subject of describing, computing, manipulating, and analyzing symplectic maps. This chapter is devoted to the subject of *applying* symplectic maps to phase-space data. That is, we are given a symplectic map $\mathcal{M}$ in some some form and a general phase-space point z, and we wish to find the phase-space point $\bar{z}$ given by

$$\bar{z} = \mathcal{M}z. \tag{35.1.1}$$

This task is more difficult and more complicated than one might suppose.

In practice, generally the only symplectic maps we can deal with in an explicit way are *truncated* Taylor series of the form (7.5.5) or (7.6.1) or (7.7.13). These Taylor series will obey the symplectic condition to the order through they have been calculated, i.e. they are symplectic jets.[1] But usually they will fail to obey the symplectic condition exactly because of the missing higher-order terms. Of course, we can always make the Lie factorization (7.7.23) and then truncate the infinite product at some order. So doing will still yield a symplectic map. However, its evaluation will generally involve summing infinite series of the form (5.4.1) and (5.42). When working numerically, at best these series can be evaluated to machine precision. But usually this is impractical because of the great effort involved if this is to be done very often. Usually we must truncate the Lie series, in which case we are again left with a symplectic jet which generally does not satisfy the symplectic condition exactly.

Put another way, suppose the map $\mathcal{M}$ is factored in the form

$$\mathcal{M} = \exp(:f_1:)\mathcal{R}\mathcal{N} \tag{35.1.2}$$

where

$$\mathcal{N} = \exp(:f_3:)\exp(:f_4:)\exp(:f_5:)\cdots . \tag{35.1.3}$$

Then we can evaluate $\exp(:f_1:)$ exactly because it simply produces a translation, and we can evaluate the linear part $\mathcal{R}$ to produce a matrix R that is symplectic to machine precision

[1] See Section 7.5.

using the methods of Chapter 4. However, there is generally no easy way to evaluate the action of the nonlinear part $\mathcal{N}$. In particular, approximating its action as a jet generally violates the symplectic condition.

Failure to satisfy the symplectic condition exactly may not produce serious errors when tracking particles through *single-pass* systems such as beam lines or electron microscopes or spot forming systems or linear accelerators or linear colliders. However, failure to satisfy the symplectic condition is very serious when one tries to model the long-term behavior of particles in *circulating* devices such as synchrotrons or damping rings or storage rings.

As a simple example, consider the two-dimensional symplectic map $\mathcal{M}$ given by the relation

$$\mathcal{M} = \mathcal{R}\mathcal{N} \tag{35.1.4}$$

with linear part

$$\mathcal{R} = \exp(-(\theta/2) : p^2 + q^2 :) \tag{35.1.5}$$

and nonlinear part

$$\mathcal{N} = \exp(: qp^2 :). \tag{35.1.6}$$

The map $\mathcal{R}$ can be evaluated exactly, see (1.2.48) and (1.2.49). And, thanks to its simplicity, so can the map $\mathcal{N}$. There is the result

$$\bar{q} = \mathcal{N} q = q(1-p)^2, \tag{35.1.7}$$

$$\bar{p} = \mathcal{N} p = p/(1-p). \tag{35.1.8}$$

See Section 1.4.2. Therefore $\mathcal{M}$ can also be evaluated exactly.

Figure 1.1 shows the result of applying $\mathcal{M}$ repeatedly to seven initial conditions for the case $\theta/2\pi = 0.22$. That is, seven initial conditions have been selected and their orbits have been found under the repeated action of $\mathcal{M}$. One initial condition is near the origin, and its orbit appears to lie on a closed curve that is nearly elliptical. (It would be nearly circular had the horizontal and vertical scales been equal.) This is to be expected because the effect of the nonlinear part $\mathcal{N}$ is small on such orbits so that such orbits are essentially those of $\mathcal{R}$. By contrast, the other initial conditions are successively farther from the origin where the effect of $\mathcal{N}$ becomes ever more significant. Their orbits appear to lie on closed curves that, the farther they are from the origin, are more and more noticeably distorted from circular by nonlinearities.

Now suppose the nonlinear map $\mathcal{N}$ is *truncated*, to become the map $\mathcal{N}^{\rm tr}$, by retaining only the first two terms in its Taylor expansion,

$$\mathcal{N}^{\rm tr} = \mathcal{I} + : qp^2 : . \tag{35.1.9}$$

The truncated map $\mathcal{N}^{\rm tr}$ has the effect

$$\bar{q} = \mathcal{N}^{\rm tr} q = (\mathcal{I} + : qp^2 :)q = q + [qp^2, q] = q - 2qp, \tag{35.1.10}$$

$$\bar{p} = \mathcal{N}^{\rm tr} p = (\mathcal{I} + : qp^2 :)p = p + [qp^2, p] = p + p^2. \tag{35.1.11}$$

Evidently $\mathcal{N}^{\rm tr}$ is a degree-two symplectic jet map.

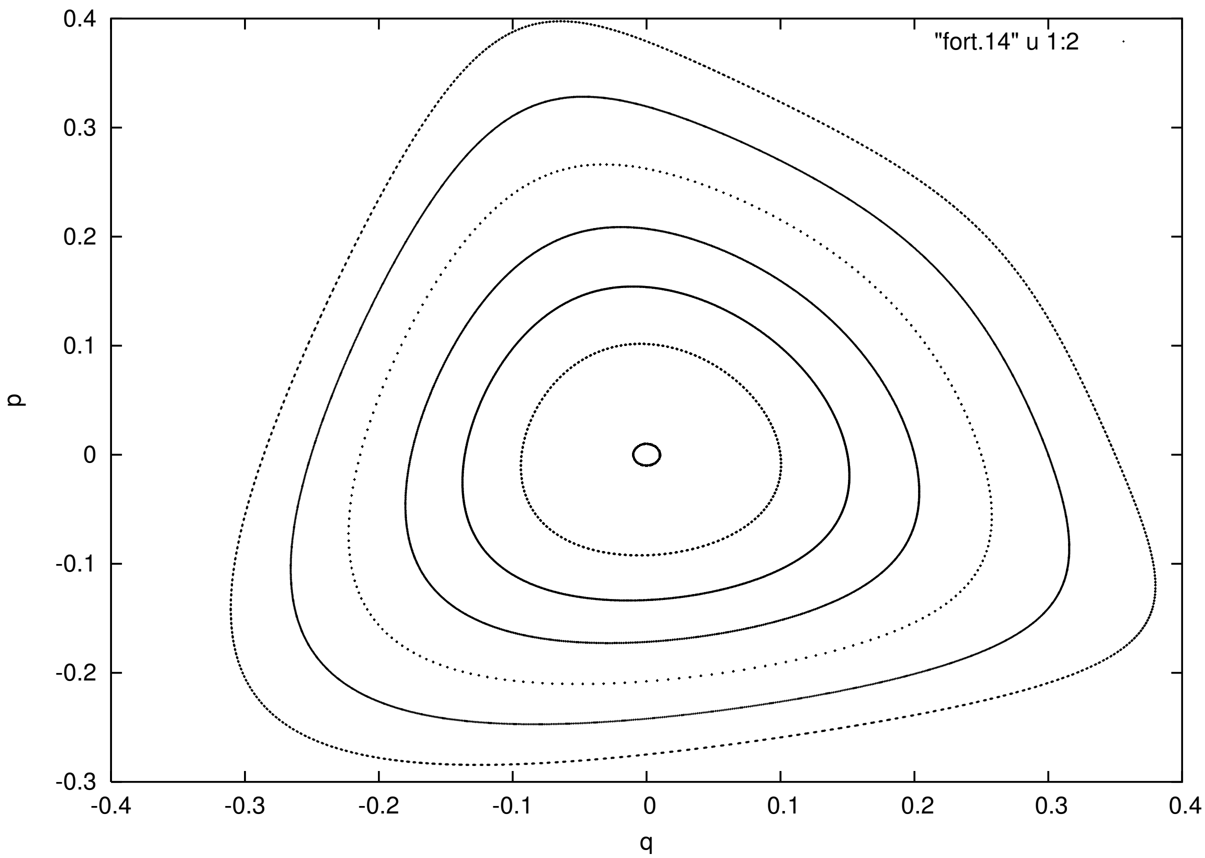

Figure 35.1.1: Phase-space portrait, in the case $\theta/2\pi = 0.22$, resulting from applying the map $\mathcal{M}$ repeatedly (2000 times) to the seven initial conditions $(q, p) = (.01, 0)$, $(.1, 0)$, $(.15, 0)$, $(.2, 0)$, $(.25, 0)$, $(.3, 0)$, and $(.35, 0)$ to find their orbits.

Next define a corresponding map $\mathcal{M}^{\rm tr}$ by writing

$$\mathcal{M}^{\rm tr} = \mathcal{R}\mathcal{N}^{\rm tr}. \tag{35.1.12}$$

Figure 1.2 shows the orbits of $\mathcal{M}^{\rm tr}$ for two initial conditions, one near the origin and one quite far away. Inspection of the figure shows that orbits are no longer distorted circles, but instead appear to spiral into the origin. This motion into the origin occurs because $\mathcal{N}^{\rm tr}$, and consequently $\mathcal{M}^{\rm tr}$, is not symplectic. See Exercise 1.1. Indeed, following the discussion of Section 22.1, the map $\mathcal{M}^{\rm tr}$ must have a factorization of the form

$$\begin{aligned}\mathcal{M}^{\rm tr} = \;& \exp(\mathcal{G}_4)\exp(\mathcal{G}_5)\exp(\mathcal{G}_6)\cdots\times \\ & \exp(-(\theta/2):p^2+q^2:)\exp(:qp^2:)\exp(:f_4:)\exp(:f_5:)\cdots \end{aligned}\tag{35.1.13}$$

with the non-Hamiltonian vector field $\mathcal{G}_4$ being nonzero.

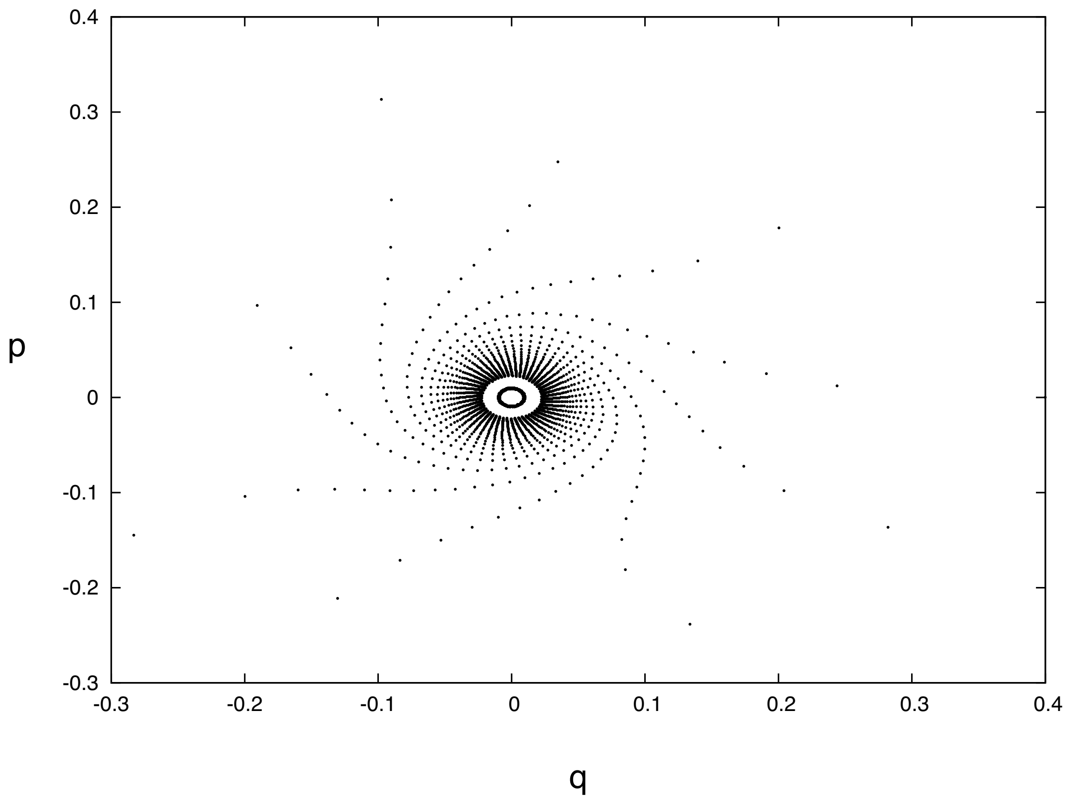

Figure 35.1.2: Phase-space portrait, in the case $\theta/2\pi = 0.22$, resulting from applying the map $\mathcal{M}^{\rm tr}$ repeatedly (2000 times) to the two initial conditions $(q,p) = (.01, 0)$ and $(.4, 0)$ to find their orbits. The orbits appear to spiral into the origin.

Suppose we also retain the next term in the Lie series for $\exp(:qp^2:)$ to form the degree-three symplectic jet map

$$\mathcal{N}^{\rm tr3} = \mathcal{I} + :qp^2: + :qp^2:^2/2. \tag{35.1.14}$$

The truncated map $\mathcal{N}^{\mathrm{tr3}}$ has the effect

$$\begin{aligned} \bar{q} &= \mathcal{N}^{\mathrm{tr3}} q = (\mathcal{I} + :qp^2: + :qp^2:^2/2)q \\ &= q + [qp^2, q] + [qp^2, [qp^2, q]]/2 = q - 2qp + qp^2, \end{aligned} \tag{35.1.15}$$

$$\begin{aligned} \bar{p} &= \mathcal{N}^{\mathrm{tr3}} p = (\mathcal{I} + :qp^2: + :qp^2:^2/2)p \\ &= p + [qp^2, p] + [qp^2, [qp^2, p]]/2 = p + p^2 + p^3. \end{aligned} \tag{35.1.16}$$

Again define a corresponding map $\mathcal{M}^{\mathrm{tr3}}$ by writing

$$\mathcal{M}^{\mathrm{tr3}} = \mathcal{R}\mathcal{N}^{\mathrm{tr3}}. \tag{35.1.17}$$

Figure 1.3 shows the orbits of $\mathcal{M}^{\mathrm{tr3}}$ for four initial conditions relatively near the origin. Now orbits move away from the origin. And the farther they are from the origin, the faster they move further away from the origin. Indeed, the orbits of initial conditions somewhat farther from the origin move very far from the origin under 2000 applications of $\mathcal{M}^{\mathrm{tr3}}$. This motion away from the origin occurs because $\mathcal{N}^{\mathrm{tr3}}$, and consequently $\mathcal{M}^{\mathrm{tr3}}$, is again not symplectic, although more nearly symplectic than $\mathcal{N}^{\mathrm{tr}}$ because $\mathcal{N}^{\mathrm{tr3}}$ is a degree-three symplectic jet whereas $\mathcal{N}^{\mathrm{tr}}$ is a degree-two symplectic jet. Now, following the discussion of Section 22.1, the map $\mathcal{M}^{\mathrm{tr3}}$ must have a factorization of the form

$$\begin{aligned} \mathcal{M}^{\mathrm{tr3}} = &\exp(\mathcal{G}_5)\exp(\mathcal{G}_6)\exp(\mathcal{G}_7)\cdots\times \\ &\exp(-(\theta/2):p^2+q^2:)\exp(:qp^2:)\exp(:f_5:)\exp(:f_6:)\cdots \end{aligned} \tag{35.1.18}$$

with the non-Hamiltonian vector field $\mathcal{G}_5$ being nonzero.

We have learned that violation of the symplectic condition can lead both to spurious damping (motion toward the origin) and spurious growth (motion away from the origin).

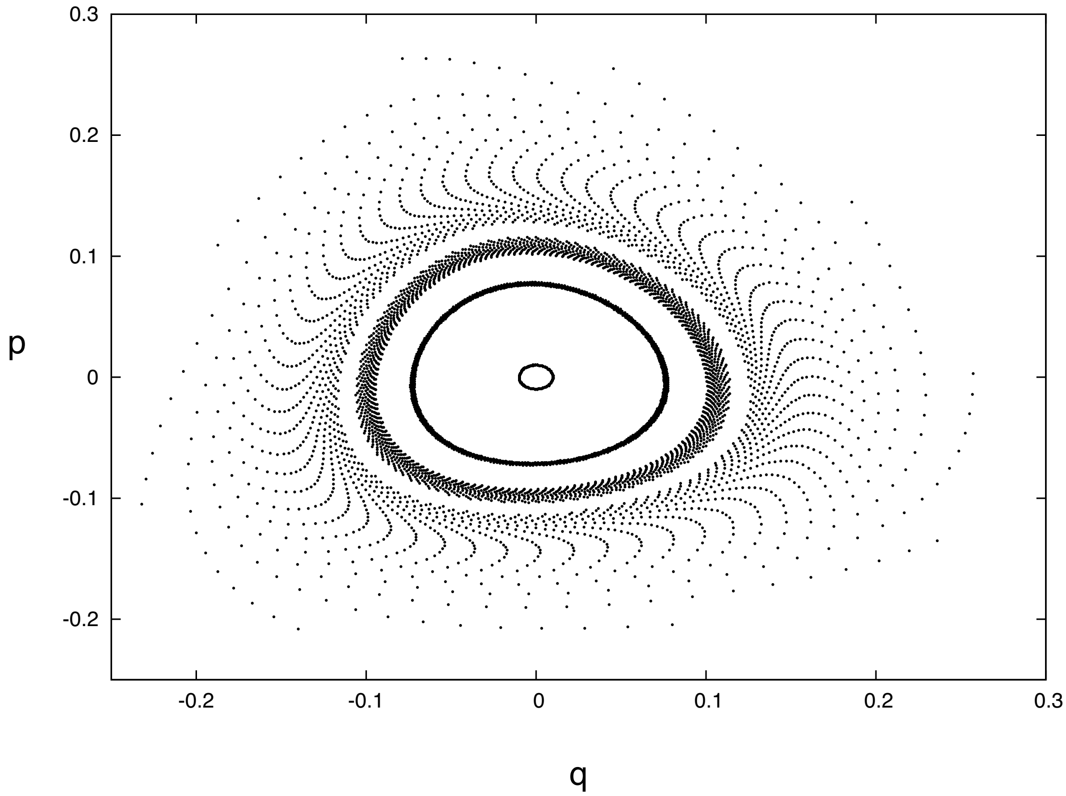

Figure 35.1.3: Phase-space portrait, in the case $\theta/2\pi = 0.22$, resulting from applying the map $\mathcal{M}^{\mathrm{tr}3}$ repeatedly (2000 times) to the four initial conditions $(q, p) = (.01, 0)$, $(.075, 0)$, $(.1, 0)$, and $(.125, 0)$ to find their orbits. The orbits appear to move away from origin.

Exercises

35.1.1. Verify that (1.10) and (1.11) are truncated Taylor expansions of (1.7) and (1.8). Show that the map $\mathcal{N}^{\mathrm{tr}}$ given by (1.10) and (1.11) satisfies the relation

$$[\bar{q}, \bar{p}] = 1 - 4p^2, \tag{35.1.19}$$

and is therefore a symplectic jet but not a symplectic map.

35.1.2. Find $\mathcal{G}_4$ and f_4 in the factorization (1.13) for the map $\mathcal{M}^{\mathrm{tr}}$

35.2 Symplectic Completion of Symplectic Jets

35.2.1 Criteria

35.2.2 Monomial Approximation

35.2.3 Generating Function Approximation

35.2.4 Cremona Maps

Kick Approximation

Jolt Approximation

35.3 Connection Between Mixed-Variable Generating Functions and Lie Generators

Sections 6.5 through 6.7 described the parameterization of symplectic maps in terms of mixed-variable generating functions. Chapters 7 through 9 described, among other things, the parameterization of symplectic maps in terms of Lie generators. The purpose of this section is to study the relation between these two parameterizations.

In particular, suppose $\mathcal{N}$ is a *nonlinear* symplectic map of the form

$$\mathcal{N} = \exp(:f_3:)\exp(:f_4:)\exp(:f_5:)\cdots. \tag{35.3.1}$$

Select some Darboux matrix α. Then, generally, there will be some source function $g(u)$ that will produce the same map using (6.7.21). We will see that the source function $g(u)$ has a homogeneous polynomial expansion of the form

$$g = g_2 + g_3 + g_4 + \cdots. \tag{35.3.2}$$

What we wish to do is to find the relation between the f_m and the g_m.

35.3.1 Method of Calculation

The task we have posed is algebraically complicated. As a first step, with the aid of the CBH series, we combine all the exponents appearing in (3.1) into one grand exponent $:e:$. Thus, we may also write

$$\mathcal{N} = \exp(:e:) \tag{35.3.3}$$

with

$$e = e_3 + e_4 + e_5 + \cdots \tag{35.3.4}$$

where

$$\begin{aligned} e_3 &= f_3, \\ e_4 &= f_4, \\ e_5 &= f_5 + [f_3, f_4], \\ e_6 &= f_6 +, \text{ etc.} \end{aligned} \tag{35.3.5}$$

Next define a function h by the rule

$$h = -e = h_3 + h_4 + h_5 + \cdots \tag{35.3.6}$$

so that $\mathcal{N}$ can be written in the form

$$\mathcal{N} = \exp(-:h:). \tag{35.3.7}$$

That is, $\mathcal{N}$ can be viewed as the map generated by integrating from $t = 0$ to $t = 1$ the equations of motion arising from the *time-independent* Hamiltonian h. Our intermediate goal now is to find a relation between the h_m and the g_m.

This goal can be achieved with the aid of the results of Section 6.7.3.2. There we learned that the source function and the Hamiltonian are related by (6.7.131) and (6.7.145). In this instance, we should set $t^i = 0$, $t = 1$, and $H(\zeta, \tau) = h(\zeta)$ to give the result

$$g(u) = g(u, t = 1) = (1/2)(\hat{Z}, \alpha^T S \alpha \hat{Z}) + (1/2)A'(u, t = 1) \tag{35.3.8}$$

with

$$A'(u, t = 1) = \int_0^1 d\tau[(\zeta, J\dot{\zeta}) + 2h(\zeta)]. \tag{35.3.9}$$

It is this result that we will manipulate and evaluate to bring it into usable form.

Let us begin with the first term appearing on the right side of (3.8). For this term we undo (6.7.144) to write

$$(1/2)(\hat{Z}, \alpha^T S \alpha \hat{Z}) = (1/2)(U, u) = (1/2)(u, U) \tag{35.3.10}$$

with

$$U = A^\alpha Z + B^\alpha z. \tag{35.3.11}$$

Now work on $(1/2)A'(u, t = 1)$, the second term on the right side of (3.8). Consider the first term appearing in the integrand of (3.9). We know that $\zeta(\tau)$ is given by the relation

$$\zeta(\tau) = \exp(-\tau :h:)z \tag{35.3.12}$$

and therefore

$$\dot{\zeta} = -\exp(-\tau : h :) : h : z = -\exp(-\tau : h :)[h, z]. \tag{35.3.13}$$

Consequently we find that

$$(\zeta, J\dot{\zeta}) = -(\exp(-\tau : h :)z, J\exp(-\tau : h :)[h, z]) = -\exp(-\tau : h :)(z, J[h, z]). \tag{35.3.14}$$

Let us evaluate $(z, J[h, z])$. In terms of components, and employing the convention of summing over repeated indices, we find that

$$\begin{aligned}(z, J[h, z]) = z_a J_{ab}[h, z_b] &= z_a J_{ab}(\partial h/\partial z_c)J_{cd}(\partial z_b/\partial z_d)\\ &= z_a J_{ab}(\partial h/\partial z_c)J_{cd}\delta_{bd}\\ &= z_a J_{ab}(\partial h/\partial z_c)J_{cb}\\ &= z_a J_{ab}(J^T)_{bc}(\partial h/\partial z_c)\\ &= z_a(JJ^T)_{ac}(\partial h/\partial z_c)\\ &= z_a(\partial h/\partial z_a).\end{aligned} \tag{35.3.15}$$

The right side of (3.15) cries out for Euler's homogeneous function theorem. By this theorem, for each homogeneous component appearing in (3.6), we have the result

$$z_a(\partial h_m/\partial z_a) = mh_m. \tag{35.3.16}$$

Therefore we may also write

$$(z, J[h, z]) = \sum_{m=3}^{\infty} mh_m. \tag{35.3.17}$$

It follows that

$$(\zeta, J\dot{\zeta}) = -(\exp(-\tau : h :)\sum_{m=3}^{\infty} mh_m. \tag{35.3.18}$$

We also note that the second term appearing in the integrand of (3.9) can be rewritten in the form

$$2h(\zeta) = 2h(\exp(-\tau : h :)z) = 2\exp(-\tau : h :)h(z). \tag{35.3.19}$$

Consequently, the full integrand can be rewritten as

$$(\zeta, J\dot{\zeta}) + 2h(\zeta) = \exp(-\tau : h :)\sum_{m=3}^{\infty}(2-m)h_m. \tag{35.3.20}$$

Correspondingly, the integral takes the form

$$A'(u, t = 1) = \int_0^1 d\tau[\exp(-\tau : h :)\sum_{m=3}^{\infty}(2-m)h_m. \tag{35.3.21}$$

Since the τ behavior has been isolated, this integral can be evaluated to give the result

$$A'(u, t = 1) = \mathrm{iex}(- : h :)\sum_{m=3}^{\infty}(2-m)h_m. \tag{35.3.22}$$

We can now put all our results together to obtain the relation

$$g(u) = (1/2)(u, U) + (1/2)\mathrm{iex}(- : h :) \sum_{m=3}^{\infty} (2 - m) h_m. \tag{35.3.23}$$

Recall the relation (6.7.18), which can be rewritten in the form

$$u = C^\alpha Z + D^\alpha z = C^\alpha \mathcal{N} z + D^\alpha z = C^\alpha \exp(- : h :) z + D^\alpha z. \tag{35.3.24}$$

This relation is to solved to give z in terms of u. This $z(u)$ must then be substituted into the right side of (3.23) to yield $g(u)$. Finally, g must be expanded in homogeneous polynomials as in (3.2).

35.3.2 Computing g_2

We begin with the computation g_2. The relation (3.24) has the expansion

$$\begin{aligned} u &= C^\alpha \exp(- : h :) z + D^\alpha z = C^\alpha (\mathcal{I} - : h : + \cdots) z + D^\alpha z \\ &= (C^\alpha + D^\alpha) z + C^\alpha (- : h : + \cdots) z. \end{aligned} \tag{35.3.25}$$

Thus, because the quantity $[(- : h : + \cdots) z]$ consists of terms that are of order 2 and higher, the expansion (3.24) has the inverse expansion

$$z = z^{(1)}(u) + O(u^2) \tag{35.3.26}$$

with

$$z^{(1)}(u) = (C^\alpha + D^\alpha)^{-1} u. \tag{35.3.27}$$

Observe that the second set of terms on the right side of (3.23) is of order 3 and higher in z. It follows that second set of terms contributes only terms of order u^3 and higher. Therefore second degree terms, the ones required for g_2, can only come from the quantity $(1/2)(u, U)$, the first term on the right side of (3.23). We have from (3.11) the expansion

$$\begin{aligned} U &= A^\alpha Z + B^\alpha z = A^\alpha \mathcal{N} z + B^\alpha z \\ &= A^\alpha \exp(- : h :) z + B^\alpha z = A^\alpha (\mathcal{I} - : h : + \cdots) z + B^\alpha z \\ &= (A^\alpha + B^\alpha) z + A^\alpha (- : h : + \cdots) z \\ &= (A^\alpha + B^\alpha) z + O(z^2). \end{aligned} \tag{35.3.28}$$

Next substitute (3.26) and (3.27) into (3.28) to yield the expansion

$$U = (A^\alpha + B^\alpha)(C^\alpha + D^\alpha)^{-1} u + O(u^2). \tag{35.3.29}$$

Also observe that the matrix product appearing in (3.29) can be written in the Möbius transformation form

$$(A^\alpha + B^\alpha)(C^\alpha + D^\alpha)^{-1} = (A^\alpha I + B^\alpha)(C^\alpha I + D^\alpha)^{-1} = T_\alpha(I). \tag{35.3.30}$$

Therefore (3.29) can be rewritten as

$$U = Wu + O(u^2) \tag{35.3.31}$$

with

$$W = T_\alpha(I), \tag{35.3.32}$$

and it follows that

$$(1/2)(u, U) = (1/2)(u, Wu) + O(u^3). \tag{35.3.33}$$

Thus we have the result

$$g_2(u) = (1/2)(u, Wu), \tag{35.3.34}$$

which is what we should have expected. Consult Exercise 6.7.1. We see that W is well defined provided

$$\det(C^\alpha + D^\alpha) \neq 0, \tag{35.3.35}$$

which was also required to write (3.27).

35.3.3 Low Order Results: Computing g_3 and g_4

Let us push on to compute g_3 and g_4. To do so, we will need to retain various higher-order terms in the expressions we have already encountered. We might think that we need to retain higher order terms in (3.25) or (3.26), in (3.28), and in the second term on the right side of (3.23). In fact, this would be one way to proceed. However, at this stage, it is also possible to avoid dealing with (3.28) entirely, thereby achieving a considerable simplification.

Again Euler comes to the rescue. We thought we had to deal with (3.28) because it appeared to be needed to compute the first term on the right side of (3.23), namely $(1/2)(u, U)$. However, using (6.7.14), we may write

$$(u, U) = \sum_a u_a(\partial g/\partial u_a). \tag{35.3.36}$$

Therefore, if we decompose g into homogeneous polynomials as in (3.2) by writing

$$g = \sum_{n=2}^{\infty} g_n, \tag{35.3.37}$$

we find by Euler's relation the result

$$(1/2)(u, U) = (1/2)\sum_a u_a(\partial g/\partial u_a) = (1/2)\sum_{n=2}^{\infty}\sum_a u_a(\partial g_n/\partial u_a) = (1/2)\sum_{n=2}^{\infty} n g_n. \tag{35.3.38}$$

Moreover, we find that

$$g - (1/2)(u, U) = \sum_{n=2}^{\infty}(1 - n/2)g_n. \tag{35.3.39}$$

Consequently, when (6.7.14) is taken into account, the defining relation (3.23) can also be written in the form

$$\sum_{n=3}^{\infty}(2-n)g_n = \text{iex}(-:h:)\sum_{m=3}^{\infty}(2-m)h_m. \tag{35.3.40}$$

It is this form that we will employ to compute g_3 and g_4.

We must still retain higher-order terms in (3.25). Doing so gives the result

$$\begin{aligned} u &= (C^\alpha + D^\alpha)z + C^\alpha(-:h:+:h:^2/2!+\cdots)z \\ &= (C^\alpha + D^\alpha)z + C^\alpha(-:h_3:+:h_3:^2/2!-:h_4:+\cdots)z \\ &= (C^\alpha + D^\alpha)z + C^\alpha(-[h_3,z]+(1/2)[h_3,[h_3,z]]-[h_4,z]+\cdots). \end{aligned} \tag{35.3.41}$$

Let us rewrite this relation in the implicit form

$$z = (C^\alpha + D^\alpha)^{-1}u - (C^\alpha + D^\alpha)^{-1}C^\alpha\{-[h_3,z]+(1/2)[h_3,[h_3,z]]-[h_4,z]+\cdots\}. \tag{35.3.42}$$

We can now invert the relation by iteration. In lowest order we have the results (3.26) and (3.27). In next order, we find

$$z = z^{(2)}(u) + O(u^3). \tag{35.3.43}$$

with

$$z^{(2)}(u) = (C^\alpha + D^\alpha)^{-1}u - (C^\alpha + D^\alpha)^{-1}C^\alpha\{-[h_3,z]\}|_{z=z^{(1)}}. \tag{35.3.44}$$

Also, we need to expand the right side of (3.40). Through terms of degree 4 we have the result

$$\begin{aligned} \text{iex}(-:h:)\sum_{m=3}^{\infty}(2-m)h_m &= (\mathcal{I}-:h:/2+\cdots)(-h_3-2h_4-\cdots) \\ &= -h_3-2h_4+O(z^5). \end{aligned} \tag{35.3.45}$$

Next we need to express both sides of (3.45) as functions of u using (3.26) and (3.43). Doing so we find the result

$$\{\text{iex}(-:h:)\sum_{m=3}^{\infty}(2-m)h_m\}|_{z=z^{(2)}} = -(h_3)|_{z=z^{(2)}} - 2(h_4)|_{z=z^{(1)}} + O(u^5). \tag{35.3.46}$$

Note that because $z^{(2)}(u)$ contains quadratic terms in u, see (3.44), the first quantity on the right of (3.46) will contribute terms that are both of degree 3 and 4 in u. Let us see what they are. Rewrite (3.44) in the form

$$z^{(2)}(u) = z^{(1)}(u) + \Delta \tag{35.3.47}$$

where

$$\Delta = -(C^\alpha + D^\alpha)^{-1}C^\alpha\{-[h_3,z]\}|_{z=z^{(1)}}. \tag{35.3.48}$$

With this notation, and inspired by Taylor, we find the result

$$(h_3)|_{z=z^{(2)}} = h_3(z^{(1)}+\Delta) = h_3(z^{(1)}) + \sum_a \Delta_a(\partial h_3/\partial z_a)|_{z=z^{(1)}} + O(\Delta^2). \tag{35.3.49}$$

Also, the quantity $[h_3, z]$ appearing in Δ can be evaluated,

$$\begin{aligned} [h_3, z_a] &= \sum_{bc} (\partial h_3/\partial z_b) J_{bc} (\partial z_a/\partial z_c) = \sum_{bc} (\partial h_3/\partial z_b) J_{bc} \delta_{ac} \\ &= \sum_{b} (\partial h_3/\partial z_b) J_{ba} = -\sum_{b} J_{ab} (\partial h_3/\partial z_b). \end{aligned} \tag{35.3.50}$$

This result can be written more compactly in vector notation as

$$[h_3, z] = -J(\partial h_3/\partial z). \tag{35.3.51}$$

Correspondingly, Δ takes the more compact form

$$\Delta = K(\partial h_3/\partial z) \tag{35.3.52}$$

where K is the matrix

$$K = -(C^\alpha + D^\alpha)^{-1} C^\alpha J. \tag{35.3.53}$$

Finally, (3.49) takes the compact form

$$(h_3)|_{z=z^{(2)}} = (h_3)|_{z=z^{(1)}} + ((\partial h_3/\partial z), K(\partial h_3/\partial z))|_{z=z^{(1)}} + O(u^5), \tag{35.3.54}$$

and (3.46) becomes

$$\begin{aligned} \{\text{iex}(-:h:) \sum_{m=3}^{\infty} (2-m) h_m\}|_{z=z^{(2)}} &= -(h_3)|_{z=z^{(1)}} - ((\partial h_3/\partial z), K(\partial h_3/\partial z))|_{z=z^{(1)}} \\ &\quad -2(h_4)|_{z=z^{(1)}} + O(u^5). \end{aligned} \tag{35.3.55}$$

Now we are ready to equate terms of like degree in (3.40). Equating terms of degree 3 gives the result

$$-g_3(u) = -h_3(z)|_{z=z^{(1)}}, \tag{35.3.56}$$

or

$$g_3(u) = h_3(z)|_{z=z^{(1)}}. \tag{35.3.57}$$

And equating terms of degree 4 gives the result

$$-2g_4(u) = -((\partial h_3/\partial z), K(\partial h_3/\partial z))|_{z=z^{(1)}} - 2h_4(z)|_{z=z^{(1)}} \tag{35.3.58}$$

or

$$g_4(u) = (1/2)((\partial h_3/\partial z), K(\partial h_3/\partial z))|_{z=z^{(1)}} + h_4(z)|_{z=z^{(1)}} \tag{35.3.59}$$

In terms of the f_m, these relations can be written in the form

$$g_3(u) = -f_3(z)|_{z=z^{(1)}}, \tag{35.3.60}$$

$$g_4(u) = (1/2)((\partial f_3/\partial z), K(\partial f_3/\partial z))|_{z=z^{(1)}} - f_4(z)|_{z=z^{(1)}}. \tag{35.3.61}$$

35.3.4 Two Examples

Eventually we will want to find higher-order results to determine the $g_n(u)$ for, say, $n \leq 8$ and some select Darboux matrices α. Before doing so, let us see what can be said so far for familiar choices of α. If we look at the Darboux α matrices for the generating functions F_1 and F_4, see Table 6.7.1, we observe that the matrices $(C^\alpha + D^\alpha)$ are singular. Hence these Darboux matrices cannot be used for our purposes. By contrast, the matrices $(C^\alpha + D^\alpha)$ for the three Darboux matrices associated with F_2, F_3, and F_+ are invertible. We will study the cases of F_2 and F_+. The case of F_3 is similar to that of F_2.

The Case of F_2

First consider the case of F_2. We find from (6.7.56) that

$$C^\alpha + D^\alpha = I^{2n} \tag{35.3.62}$$

and

$$C^\alpha = \begin{pmatrix} 0 & 0 \\ 0 & I^n \end{pmatrix}. \tag{35.3.63}$$

It follows from (3.27) that

$$z^{(1)}(u) = u. \tag{35.3.64}$$

Also, from (6.7.55) and (3.32), we find that

$$W = \begin{pmatrix} 0 & I^n \\ I^n & 0 \end{pmatrix}. \tag{35.3.65}$$

Finally, from (3.41), (3.47), and (3.63), we find that

$$K = -C^\alpha J = \begin{pmatrix} 0 & 0 \\ I^n & 0 \end{pmatrix}. \tag{35.3.66}$$

We are now ready to compute $g_2(u)$ through $g_4(u)$. As in Exercise 6.7.5, partition u into position-like and momentum-like components by writing

$$u = (v; w). \tag{35.3.67}$$

Then, from (3.42), (3.48), and (3.49) we have for $g_2(u)$ the result

$$g_2(u) = (v, w), \tag{35.3.68}$$

which can also be written in the form

$$g_2(u) = (q, p)|_{z=u}. \tag{35.3.69}$$

Next, from (3.45) and (3.47), we find that

$$g_3(u) = -f_3(u). \tag{35.3.70}$$

Finally, from (3.46), (3.47), and taking into account the form of K given by (3.49), we find the results

$$g_4(u) = -f_4(u) + (1/2)(\partial f_3/\partial q, \partial f_3/\partial p)|_{z=u}. \tag{35.3.71}$$

The Case of F_+

Next consider the case of F_+. We find from (6.7.67) that

$$C^\alpha + D^\alpha = \sqrt{2} I^{2n} \tag{35.3.72}$$

and

$$C^\alpha = (1/\sqrt{2}) I^{2n}. \tag{35.3.73}$$

It follows that in this case

$$z^{(1)}(u) = (1/\sqrt{2})u. \tag{35.3.74}$$

Also, from (6.7.67) and (3.15), we find that

$$W = 0. \tag{35.3.75}$$

Finally, from (3.41), (3.55), and (6.7.67), we find that

$$K = -(1/2)J. \tag{35.3.76}$$

We are again ready to compute $g_2(u)$ through $g_4(u)$. From (3.42) and (3.56) we find that

$$g_2(u) = 0. \tag{35.3.77}$$

Next, from (3.45) and (3.55), we find that

$$g_3(u) = -f_3(u/\sqrt{2}). \tag{35.3.78}$$

Finally we observe from (3.57) that

$$((\partial f_3/\partial z), K(\partial f_3/\partial z)) = -(1/2)((\partial f_3/\partial z), J(\partial f_3/\partial z)) = 0 \tag{35.3.79}$$

since J is an antisymmetric matrix. It follows from (3.46) and (3.60) that in this case g_4 takes the simple form

$$g_4(u) = -f_4(u/\sqrt{2}). \tag{35.3.80}$$

35.3.5 Exploration

Let us explore what maps are produced when the source function consists only of quadratic and cubic terms,

$$g(u) = g_2(u) + g_3(u). \tag{35.3.81}$$

For simplicity, we will explore only the use of F_2 and F_+ generating functions, and work with only a two-dimensional phase space so that $z = (q; p)$ and $Z = (Q; P)$.

Use of F_2

General Discussion

Let us begin with the use of F_2. In that case, we will consider generating functions of the form

$$F_2(q,P) = qP + aq^3 + bq^2P + cqP^2 + dP^3 \tag{35.3.82}$$

with arbitrary coefficients a through d. Then use of (6.7.54) produces the implicit relations

$$p = P + 3aq^2 + 2bqP + cP^2, \tag{35.3.83}$$

$$Q = q + bq^2 + 2cqP + 3dP^2. \tag{35.3.84}$$

Since these equations are quadratic, they can be solved exactly, and we will do so shortly. First, however, let us find the first few terms in the Taylor expansions of $Q(q,p)$ and $P(q,p)$ in powers of q and p. Rewrite (3.64) in the form

$$P = p - 3aq^2 - 2bqP - cP^2. \tag{35.3.85}$$

Now we can expand Q and P in powers of q and p by iteration of (3.65) and (3.66). In lowest approximation, they have the solution

$$Q = q + O(z^2), \tag{35.3.86}$$

$$P = p + O(z^2). \tag{35.3.87}$$

Now substitute (3.67) and (3.68) into (3.65) and (3.66) to get the improved solution

$$Q = q + bq^2 + 2cqp + 3dp^2 + O(z^3), \tag{35.3.88}$$

$$P = p - 3aq^2 - 2bqp - cp^2 + O(z^3) \tag{35.3.89}$$

For our present purposes we will be content with expansions that retain terms through degree three. This can be achieved by substituting (3.69) and (3.70) into (3.65) and (3.66). Doing so gives the results

$$Q = q + bq^2 + 2cqp + 3dp^2 + *q^3 + *q^2p + *qp^2 + *p^3 + O(z^4), \tag{35.3.90}$$

$$P = p - 3aq^2 - 2bqp - cp^2 + *q^3 + *q^2p + *qp^2 + *p^3 + O(z^4). \tag{35.3.91}$$

For comparison, let us evaluate the Taylor series for the transformation

$$Z = \mathcal{N}z \tag{35.3.92}$$

where

$$\mathcal{N} = \exp(:f_3:) \tag{35.3.93}$$

with

$$f_3(z) = -aq^3 - bq^2p - cqp^2 - dp^3. \tag{35.3.94}$$

Then it is easily verified that the Lie transformation

$$Z = \exp(: f_3 :)z = \sum_{m=0}^{\infty}(1/m!) : f_3 :^m z = z+ : f_3 : z + (1/2!) : f_3 :^2 z + O(z^4) \quad (35.3.95)$$

gives the result

$$Q = q + bq^2 + 2cqp + 3dp^2 + *q^3 + *q^2p + *qp^2 + *p^3 + O(z^4), \quad (35.3.96)$$

$$P = p - 3aq^2 - 2bqp - cp^2 + *q^3 + *q^2p + *qp^2 + *p^3 + O(z^4). \quad (35.3.97)$$

We see that the linear and quadratic terms in (3.71) and (3.72) agree with those in (3.77) and (3.78), respectively. However, the cubic terms do not.

These results are to be expected based on the findings of Subsection 27.3.3. There we saw that g_3 and f_3 should be related by (3.53), and that is what has been done in writing (3.63) and (3.76). Therefore the quadratic terms in in (3.71) and (3.72) should agree with those in (3.77) and (3.78). With regard to cubic terms, in writing (3.62) we have implicitly made the requirement

$$g_n(u) = 0 \text{ for } n \geq 4. \quad (35.3.98)$$

And, in writing (3.74), we have implicitly made the requirement

$$f_n(z) = 0 \text{ for } n \geq 4. \quad (35.3.99)$$

But we see from (3.54) that in general these requirements are incompatible. Therefore we expect differences in the cubic (and higher-order) terms.

As promised, let us now solve the relations (3.65) and (3.66) exactly. We must distinguish two cases:

The Case When $c = 0$

If $c = 0$, (3.66) has the immediate solution

$$P = (p - 3aq^2)/(2bq + 1). \quad (35.3.100)$$

And substituting this result into (3.65) gives the complementary result

$$Q = q + bq^2 + 3d(p - 3aq^2)^2/(2bq + 1)^2. \quad (35.3.101)$$

The Case When $c \neq 0$

When $c \neq 0$, the relation (3.66) is quadratic in P and has the solution

$$P = [1/(2c)]\{-(2bq + 1) + [(2bq + 1)^2 + 4c(p - 3aq^2)]^{1/2}\}. \quad (35.3.102)$$

The implicit relation (3.64) can be solved to give the explicit relation

$$P = [1/(2c)]\{-(1 + 2bq) + [1 + 4bq + cp + 3(b^2 - 4ac)q^2]^{1/2}\}. \quad (35.3.103)$$

And (3.81) can then be substituted into (3.65) to give the complementary explicit relation

$$Q = . \tag{35.3.104}$$

We see that, as functions of q and p, Q and P *generically* have branch points.[2] They occur on the surface

$$1 + 4bq + cp + 3(b^2 - 4ac)q^2 = 0. \tag{35.3.105}$$

By contrast, we know from the work of Section 25.3 that the map given by (3.73) and (3.74) generally has poles.

Case When Only $a \neq 0$

Let us consider some special cases. First suppose that only $a \neq 0$. Then the Lie transformation series (3.76) terminates and gives the exact result

$$Q = q, \tag{35.3.106}$$

$$P = p - 3aq^2. \tag{35.3.107}$$

Also, in this case, solution of the implicit relations (3.64) and (3.65) gives identical results. Thus in this case, which is easily verified to be that of a kick map, the use of F_2 and the exact Lie transformation give the same result. It is an easy calculation to show that the same holds true when only $d \neq 0$.

Jolt Case

Next assume that f_3 is of the form

$$f_3 = (\alpha q - \beta p)^3 \tag{35.3.108}$$

which amounts to setting

$$a = -\alpha^3, \tag{35.3.109}$$

$$b = 3\alpha^2\beta, \tag{35.3.110}$$

$$c = -3\alpha\beta^2, \tag{35.3.111}$$

$$d = \beta^3. \tag{35.3.112}$$

Then it is easily verified that the Lie transformation series (3.76) also terminates and gives the exact result

$$Q = q + 3\beta(\alpha q - \beta p)^2, \tag{35.3.113}$$

$$P = p + 3\alpha(\alpha q - \beta p)^2. \tag{35.3.114}$$

See Section 22.3. In fact, $\mathcal{N}$ in this case is a jolt map. See Exercise *. By contrast, solution of the implicit relations (3.64) and (3.65) in this case gives the results

$$Q =, \tag{35.3.115}$$

[2] However.

$$P = . \tag{35.3.116}$$

We see that the Q and P given by (3.91) and (3.92) are entire functions of q and p. By contrast, the relations (3.93) and (3.94) show that the map produced by F_2 in this case has branch points. They are located on the surface

$$= . \tag{35.3.117}$$

Case When Only $c \neq 0$

As another example, suppose f_3 is of the form

$$f_3 = -cqp^2 \tag{35.3.118}$$

as in (1.6). So doing amounts to assuming that only $c \neq 0$ in (3.75). As already seen, setting $c = -1$ in (3.73) and (3.74) leads to the relation

$$Q = \mathcal{N}q = q(1-p)^2, \tag{35.3.119}$$

$$P = \mathcal{N}p = p/(1-p). \tag{35.3.120}$$

By contrast, the implicit equations (3.64) and (3.65) in this case have the explicit solution

$$Q = q(1+4cp)^{1/2}, \tag{35.3.121}$$

$$P = [1/(2c)][(1+4cp)^{1/2} - 1]. \tag{35.3.122}$$

We see that in this case the map produced by F_2 has a branch point (when $c = -1$) on the surface $p = 1/4$ while, according to (3.98), the exact Lie transformation map has a pole on the surface $p = 1$.

The last case to be considered in this vein is that of $b \neq 0$ and all other coefficients in (3.63) or (3.75) set to zero. You, dear reader, will have the pleasure of doing so in Exercise *.

F_2 Symplectic Completion of $\mathcal{N}^{\rm tr}$

Let $\mathcal{N}^{\rm sc}$ be the map given by (3.99) and (3.100) when $c = -1$. We may view $\mathcal{N}^{\rm sc}$ as a *symplectic completion* of the degree-two symplectic jet map $\mathcal{N}^{\rm tr}$ given by (1.9). Corespondingly, we define the associated map $\mathcal{M}^{\rm sc}$ by the relation

$$\mathcal{M}^{\rm sc} = \mathcal{R}\mathcal{N}^{\rm sc}. \tag{35.3.123}$$

Figure 3.1 shows the result of applying $\mathcal{M}^{\rm sc}$ repeatedly to four initial conditions for the case $\theta/2\pi = 0.22$. Note that, unlike the cases of Figures 1.2 and 1.3, points on the orbit no longer spiral into or out of the origin. Moreover, the behavior of the orbits is similar to that shown in Figure 1.1.

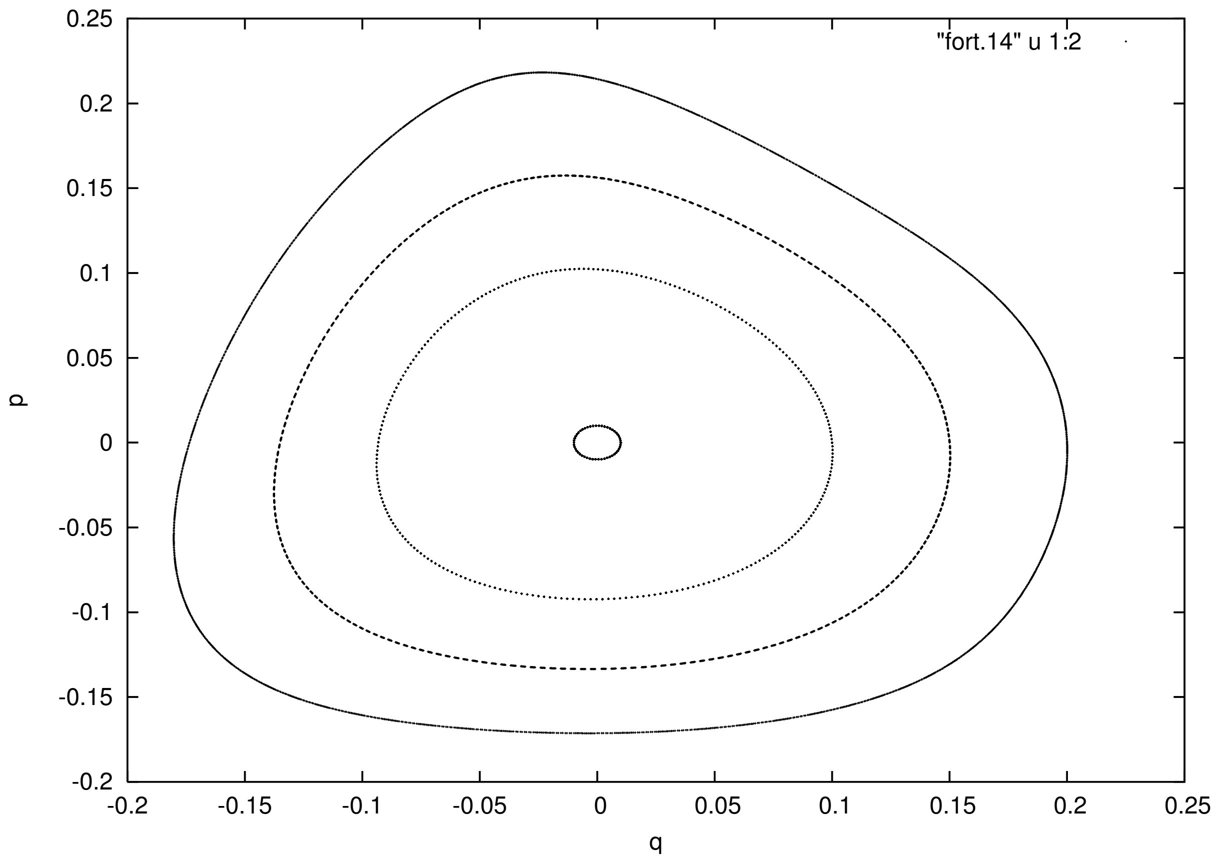

Figure 35.3.1: Phase-space portrait, in the case $\theta/2\pi = 0.22$, resulting from applying the map $\mathcal{M}^{\rm sc}$ repeatedly (2000 times) to the four initial conditions $(q,p) = (.01,0)$, $(.1,0)$, $(.15,0)$, and $(.2,0)$ to find their orbits.

Root Trick

Of course, we know that in this case use of $\mathcal{N}^{\rm sc}$ can at best make sense for $|p| < 1/4$. What can be done for points farther from the origin? For a map of the form (3.3) we can easily extract a square root. In accord with the relation (3.6), we will write

$$\mathcal{N}^{1/2} = \mathcal{N}(1/2). \tag{35.3.124}$$

Also, let $\mathcal{N}^{\rm sc}(1/2)$ denote the map given by (3.99) and (3.100) with $c = -1/2$. We may view $\mathcal{N}^{\rm sc}(1/2)$ as the F_2 symplectification of the degree-two jet map $\mathcal{N}^{\rm tr}(1/2)$. We see, from (3.99) and (3.100) with $c = -1/2$, that the map $\mathcal{N}^{\rm sc}(1/2)$ is well defined for $|p| < 1/2$.

Now watch closely. We know that

$$\mathcal{M} = \mathcal{R}\mathcal{N}^{1/2}\mathcal{N}^{1/2}. \tag{35.3.125}$$

Therefore, it makes sense to consider what we will call the *improved symplectically completed* map $\mathcal{M}^{\rm isc}$ defined by the relation

$$\mathcal{M}^{\rm isc} = \mathcal{R}\mathcal{N}^{\rm sc}(1/2)\mathcal{N}^{\rm sc}(1/2). \tag{35.3.126}$$

This map will be defined over a larger region of phase space and will be a better approximation to $\mathcal{M}$. Figure 3.2 shows the result of applying $\mathcal{M}^{\rm isc}$ repeatedly to seven initial conditions for the case $\theta/2\pi = 0.22$. Again points on the orbits neither spiral into or out of the origin, and the orbits more nearly approximate those of Figure 1.1.

Use of the Poincaré Generating Function F_+

General Discussion

We now repeat much of the work above, but this time for the case where F_+ is used. Now (3.58) holds and, in accord with (3.59), the source function g_3 will have the form

$$g_3(u) = (1/\sqrt{2})^3(av^3 + bv^2w + cvw^2 + dw^3). \tag{35.3.127}$$

In this case use of (6.7.21) with α given by (6.7.67) gives the implicit relations

$$Q = q + (1/4)[b(Q+q)^2 + 2c(Q+q)(P+p) + 3d(P+p)^2], \tag{35.3.128}$$

$$P = p - (1/4)[3a(Q+q)^2 + 2b(Q+q)(P+p) + c(P+p)^2]. \tag{35.3.129}$$

Since these equations are quadratic, they can again be solved exactly. However, the solutions are far too long to record. In section 3.5 we will see they again involve square roots, and therefore the functions $Q(q,p)$ and $P(q,p)$, like the F_2 case, have square root branch-point singularities.

But, again, we can obtain Taylor expansions of Q and P in terms if q and p by iteration. Doing so gives, as first and second passes, the expansions (3.67) through (3.70). And the third pass gives the expansions (3.77) and (3.78). That is, unlike the F_2 case, use of F_+ gives expansions that agree with the exact result through terms of third order.

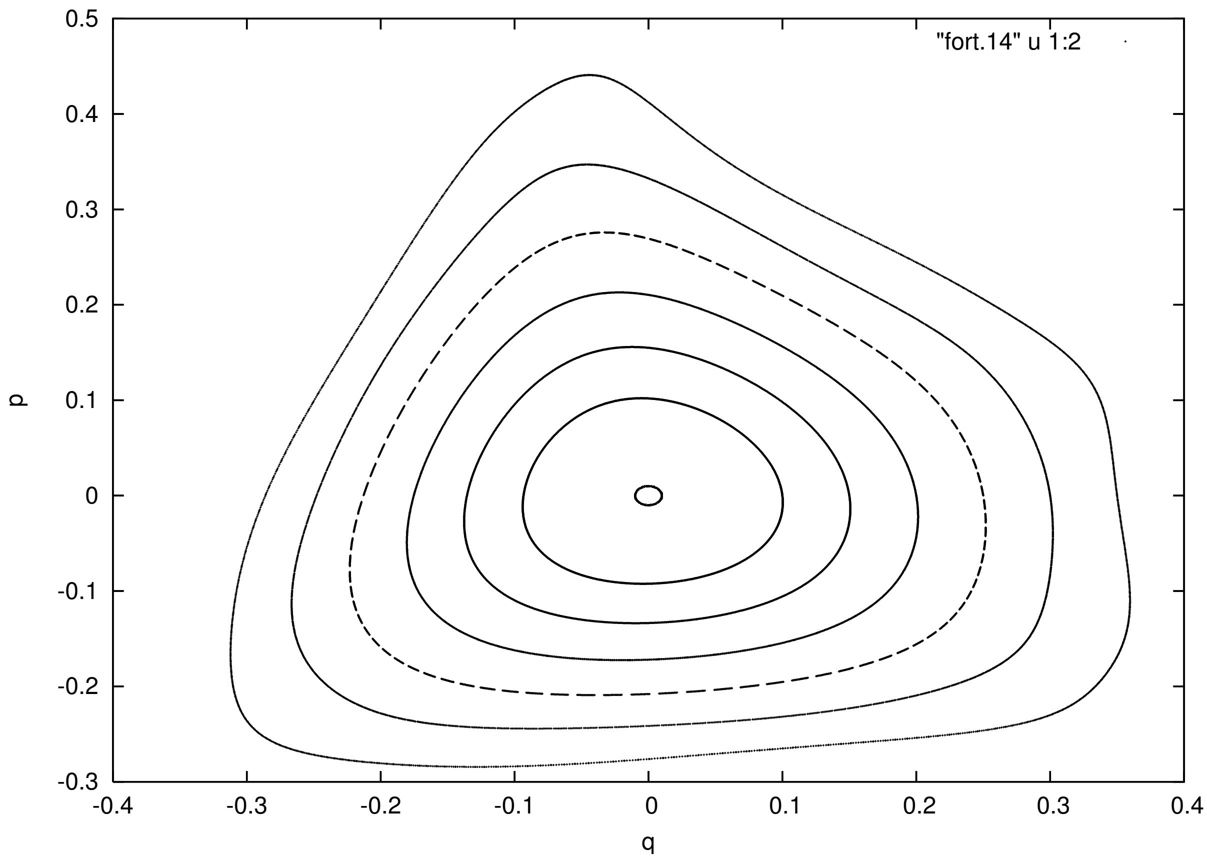

Figure 35.3.2: Phase-space portrait, in the case $\theta/2\pi = 0.22$, resulting from applying the map $\mathcal{M}^{\text{isc}}$ repeatedly (2000 times) to the seven initial conditions $(q,p) = (.01,0)$, $(.1,0)$, $(.15,0)$, $(.2,0)$, $(.25,0)$, $(.3,0)$, and $(.35,0)$ to find their orbits.

This accuracy is again to be expected based on the findings of Subsection 27.3.3. There we saw that g_3 and f_3 should be related by (3.59), and that is what has been done in writing (3.105). Therefore the maps produced by the use of F_+ should agree with those produced by the $\mathcal{N}$ given by (3.73) through quadratic terms. With regard to cubic terms, we see from (3.61) that the implicit assumptions (3.79), which are still in effect, imply that f_4 vanishes. Therefore, the cubic terms must also agree. Later we will see that quartic and higher-order terms need not agree.

Cases When Only a or $d \neq 0$

Let us again consider some special cases. First suppose that only $a \neq 0$. Then (3.106) and (3.107) can be solved immediately to give the results (3.84) and (3.85). And if only $d \neq 0$, (3.106) and (3.107) have the solution

$$Q = q + 3dp^2, \tag{35.3.130}$$

$$P = p. \tag{35.3.131}$$

Thus, like F_2, use of F_+ also gives exact results for kick maps.

Jolt Case

Suppose the values of a through d given by (3.114) through (3.117) are employed in (3.106) and (3.107); and that also the quantities Q and P appearing in (3.106) and (3.107) are replaced by the right sides of equations (3.91) and (3.92). Upon doing so one finds that the resulting two equations (which now involve only the quantities α, β, q, and p) are satisfied *identically* for all values of α, β, q, and p. It follows that, unlike the case of F_2, the use of F_+ gives exact results for jolt maps as well. Why this should be so is explained in a subsequent section.

Case When Only $c \neq 0$

If only $c \neq 0$, (3.106) and (3.107) have the solution

$$Q = q(1 + 2cp)^{1/2}/[2 - (1 + 2cp)^{1/2}], \tag{35.3.132}$$

$$P = -[p + (2/c)] + (2/c)(1 + 2cp)^{1/2}. \tag{35.3.133}$$

We see that in this case use of F_+ produces a map that has a branch point on the surface

$$p = -1/(2c). \tag{35.3.134}$$

By contrast, according to (3.100), use of F_2 in this case produces a map that has a branch point on the surface

$$p = -1/(4c). \tag{35.3.135}$$

Therefore the branch-point surface for F_+ is farther from the origin than that of F_2. In particular, for $c = -1$, it is located at $p = 1/2$. But note that it is still closer to the origin than the pole of the exact map which, we have seen, is on the surface $p = 1$.

Again the last case to be considered in this vein is that of only $b \neq 0$. This case is treated in Exercise *.

F_+ Symplectic Completion of $\mathcal{N}^{\mathrm{tr}}$

Let $\mathcal{N}^{\mathrm{psc}}$ be the map given by (3.113) and (3.114) when $c = -1$. We may view $\mathcal{N}^{\mathrm{psc}}$ as the *Poincaré symplectic completion* of the degree-two symplectic jet map $\mathcal{N}^{\mathrm{tr}}$ given by (1.9). Corespondingly, we define the associated map $\mathcal{M}^{\mathrm{psc}}$ by the relation

$$\mathcal{M}^{\mathrm{psc}} = \mathcal{R}\mathcal{N}^{\mathrm{psc}}. \tag{35.3.136}$$

Figure 3.3 shows the result of applying $\mathcal{M}^{\mathrm{psc}}$ repeatedly to seven initial conditions for the case $\theta/2\pi = 0.22$. Again there is no spurious spiraling into or out of the origin. Note also that we have been able to apply this map to a larger region of phase space than we could for $\mathcal{N}^{\mathrm{sc}}$. Compare Figures 3.1 and 3.3.

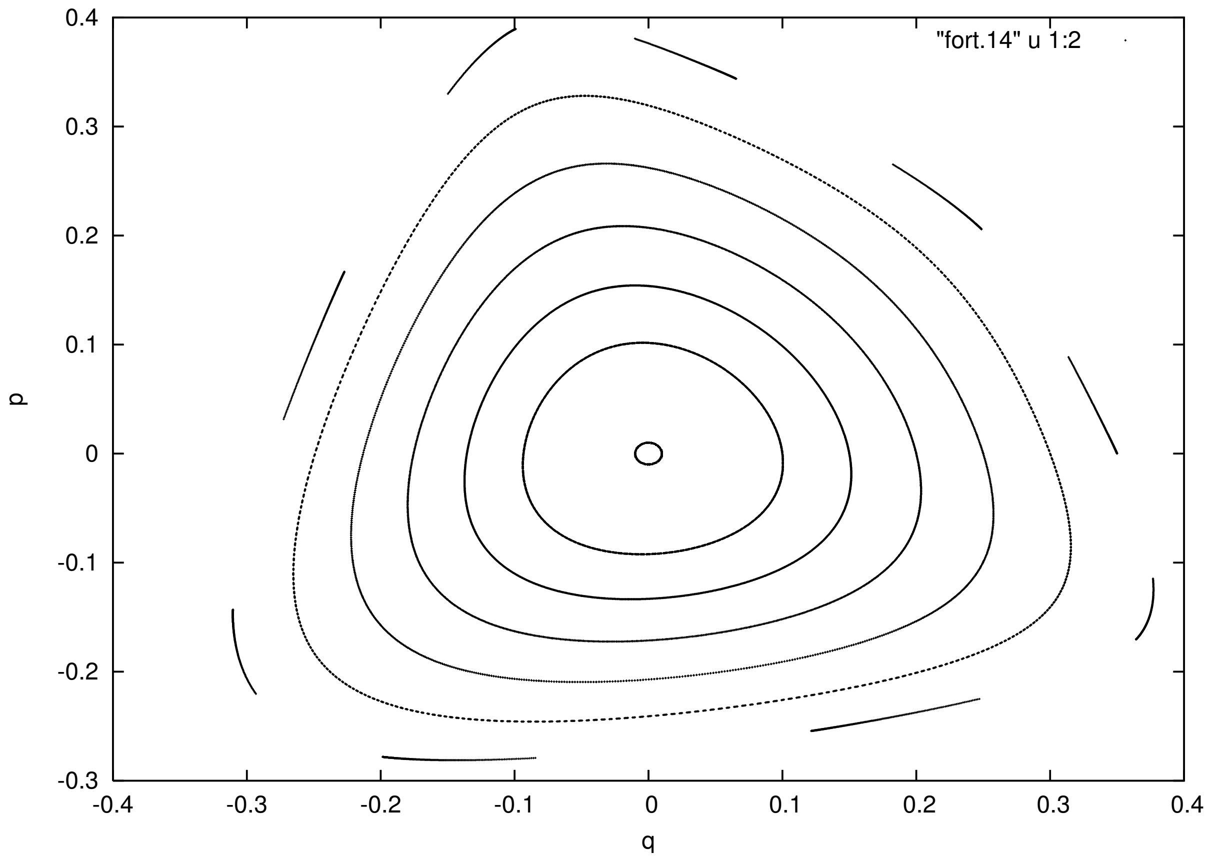

Figure 35.3.3: Phase-space portrait, in the case $\theta/2\pi = 0.22$, resulting from applying the map $\mathcal{M}^{\mathrm{psc}}$ repeatedly (2000 times) to the to the seven initial conditions $(q, p) = (.01, 0)$, $(.1, 0)$, $(.15, 0)$, $(.2, 0)$, $(.25, 0)$, $(.3, 0)$, and $(.35, 0)$ to find their orbits.

Root Trick for F_+

Evidently the root trick can be applied to any symplectification procedure. Here we will explore its use for our example of Poincaré symplectification. Let $\mathcal{N}^{\mathrm{psc}}(1/2)$ denote the map given by (3.113) and (3.114) with $c = -1/2$. We may view $\mathcal{N}^{\mathrm{psc}}(1/2)$ as the F_+

symplectification of the degree-two jet map $\mathcal{N}^{\rm tr}(1/2)$. We see, from (3.115) with $c = -1/2$, that the map $\mathcal{N}^{\rm psc}(1/2)$ has a branch point on the surface $p = 1$, the same surface on which the origin map has a pole. We will now study the behavior of the *improved Poincaré symplectically completed* map $\mathcal{M}^{\rm ipsc}$ defined by the relation

$$\mathcal{M}^{\rm ipsc} = \mathcal{R}\mathcal{N}^{\rm psc}(1/2)\mathcal{N}^{\rm psc}(1/2). \tag{35.3.137}$$

Figure 3.4 shows the result of applying $\mathcal{M}^{\rm ipsc}$ repeatedly to seven initial conditions for the case $\theta/2\pi = 0.22$. We see that now the orbits approximate those of Figure 1.1 remarkably well. Presumably one reason for this improvement is the larger domain of analyticity for $\mathcal{N}^{\rm psc}(1/2)$.

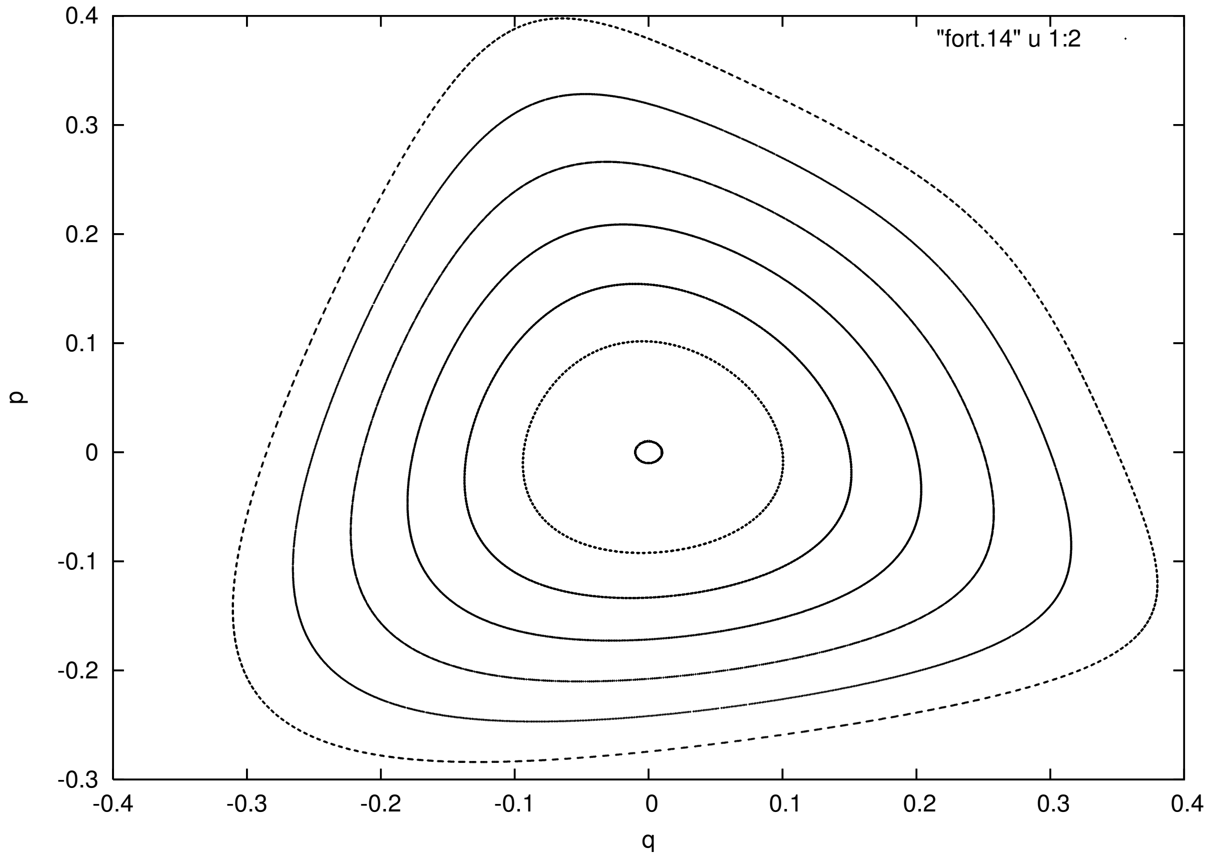

Figure 35.3.4: Phase-space portrait, in the case $\theta/2\pi = 0.22$, resulting from applying the map $\mathcal{M}^{\rm ipsc}$ repeatedly (2000 times) to the seven initial conditions $(q,p) = (.01,0)$, $(.1,0)$, $(.15,0)$, $(.2,0)$, $(.25,0)$, $(.3,0)$, and $(.35,0)$ to find their orbits.

35.3.6 Comments and Comparisons

At this point some comments and comparisons are in order. Based on our experience so far, we may draw the following (sometimes tentative) conclusions:

1. The symplectification of symplectic jets overcomes the problem of spurious spiraling into or out of the origin.

2. The use of either F_2 or F_+ (Poincaré) generating functions gives exact results for jets that are kick maps. Kick maps are, of course, symplectic, and their symplectification by use of either F_2 or F_+ (Poincaré) generating functions leaves them unchanged.

3. The use of F_+ (Poincaré) generating functions gives exact results for jets that are jolt maps. Such jet maps are also also exactly symplectic, and their Poincaré symplectification also leaves them unchanged. Such is not the case for F_2 symplectification. Given a jolt map, it generally converts this map into some other map. While the result of this conversion is a symplectic map, it is not generally the original jolt map. Put colloquially, Poincaré symplectification has the good sense to leave a good thing alone, but F_2 symplectification generally does not. In fact, in this case F_2 symplectification replaces a map with no singularities by a map with singularities.

4. The root trick enlarges the applicable domain and improves accuracy. It might also appear to require more work because now a symplectified nonlinear map has to be evaluated twice. However, the iterative method employed to solve numerically the implicit equations associated with the use of $\mathcal{N}^{\text{psc}}(1/2)$ is expected to converge faster than that for $\mathcal{N}^{\text{psc}}(1)$ because $\mathcal{N}^{\text{psc}}(1/2)$ is less nonlinear; and this gain in convergence speed is likely to be greater than the loss associated with evaluating $\mathcal{N}^{\text{psc}}(1/2)$ twice.

5. Compared to a F_2 symplectified map, a Poincaré symplectified map has a larger domain of applicability.

6. Compared to a F_2 symplectified map, a Poincaré symplectified map has higher-order accuracy.

Let us explore item 5 above in some more detail. In Section 26.2 we listed the normal forms for cubic polynomials in two variables, namely those given by (26.2.4) through (26.2.7). It is instructive to compare F_2 and Poincaré symplectification for each. We have already considered the cases (26.2.6) and (26.2.7). We now consider the two remaining cases.

Case When $a = 1$, $b = -3$, $c = d = 0$

We begin with the case (26.2.5), which is the easier of the two. In this case, by suitable rescaling, there is no loss of generality in taking a and b as the the only nonzero coefficients and giving them the values $a = 1$ and $b = -3$.

For these values use of (3.81) and (3.82) gives the map

$$Q = q - 3q^2, \tag{35.3.138}$$

$$P = (p - 3q^2)/(1 - 6q). \tag{35.3.139}$$

We see that for the F_2 case the map is singular on the surface

$$? = . \tag{35.3.140}$$

For these same values of a and b, the implicit relations (3.106) and (3.107), produced by the use of F_+, take the form

$$Q = q - (3/4)(Q + q)^2, \tag{35.3.141}$$

$$P = p - (1/4)[3(Q+q)^2 - 6(Q+q)(P+p)]. \tag{35.3.142}$$

These implicit relations have the solution

$$Q = (2/3)\{-[1 + (3/2)q] + [1 + 6q]^{1/2}\}, \tag{35.3.143}$$

$$P = . \tag{35.3.144}$$

We see that for the F_+ case the map is singular on the surface

$$? = . \tag{35.3.145}$$

Case When $a = d = 1$, $b = c = 0$

For the remaining case (26.2.4) there is no loss of generality in taking a and d as the only nonzero coefficients and giving them the values $a = d = 1$.

For these values use of (3.81) and (3.82) gives the map

$$Q =, \tag{35.3.146}$$

$$P = . \tag{35.3.147}$$

We see that for the F_2 case the map is singular on the surface

$$? = . \tag{35.3.148}$$

For these same values of a and d, the implicit relations (3.106) and (3.107), produced by the use of F_+, take the form

$$Q = q + (3/4)(P+p)^2, \tag{35.3.149}$$

$$P = p - (3/4)(Q+q)^2. \tag{35.3.150}$$

These implicit relations have the solution

$$Q =, \tag{35.3.151}$$

$$P = . \tag{35.3.152}$$

We see that for the F_+ case the map is singular on the surface

$$? = . \tag{35.3.153}$$

Let us also explore item 6 above in some more detail. To do so it is convenient to make some definitions. First, suppose $\mathcal{N}$ is some nonlinear map. We will define $\mathrm{n}(\mathcal{N}, z)$, the local *nonlinearity* of $\mathcal{N}$, by the rule

$$\mathrm{n}(\mathcal{N}, z) = ||(\mathcal{N} - \mathcal{I})z||/||z||. \tag{35.3.154}$$

Here $||\ ||$ denotes the vector norm of a phase-space vector in the usual Euclidean metric. The quantity $n(\mathcal{N}, z)$ measures how much a phase-space point z moves under the action of $\mathcal{N}$ normalized by its distance from the origin.

It may be the case that $\mathcal{N}$ is a symplectic jet. In that case, it is useful to have some measure of the violation of the symplectic condition associated with the action of $\mathcal{N}$. One possibility is to define $\mathrm{sv}(\mathcal{N}, z)$, the local *symplectic violation*, by the rule

$$\mathrm{sv}(\mathcal{N}, z) = ||([\mathcal{N}z_a, \mathcal{N}z_b] - J_{ab})||. \tag{35.3.155}$$

Here $||\ ||$ denotes some matrix norm, say the maximum column sum norm.

Given two nonlinear maps $\mathcal{N}_1$ and $\mathcal{N}_2$, we will also want to have some measure of the *difference* between them. One way to do so is to introduce the quantity $\mathrm{d}(\mathcal{N}_1, \mathcal{N}_2, z)$ by the rule

$$\mathrm{d}(\mathcal{N}_1, \mathcal{N}_2, z) = ||\mathcal{N}_1 z - \mathcal{N}_2 z||. \tag{35.3.156}$$

Some Second Thoughts

We have been comparing the performance of F_2 and F_+. See items 5 and 6 above. However, in retrospect, the comparison has not been entirely fair. In the case of F_+, we assumed an F_+ of the form

$$F_+ = F_+^3, \tag{35.3.157}$$

and found the resulting map $\mathcal{N}$ was of the form

$$\mathcal{N} = \exp(: f_3 :) \exp(: f_5 :) \cdots . \tag{35.3.158}$$

Note that there is *no* f_4 term in (3.158) for any choice of F_+^3; and there is a uniquely determined F_+^3 whose use will produce any desired f_3 in (3.158). By contrast, in the case of F_2, we assumed an F_2 of the form

$$F_2 = F_2^3, \tag{35.3.159}$$

and found the resulting map $\mathcal{N}$ was of the form

$$\mathcal{N} = \exp(: f_3 :) \exp(: f_4 :) \cdots . \tag{35.3.160}$$

Note that in this case there is, *possibly*, an f_4 term in (3.160); and again there is a uniquely determined F_2^3 whose use will produce any desired f_3 in (3.160). Indeed, in this case f_4 is given by

$$f_4 = (1/2)(\partial f_3/\partial q, \partial f_3/\partial p). \tag{35.3.161}$$

In particular, for the case

$$f_3 = pq^2, \tag{35.3.162}$$

there is the result

$$f_4 = . \tag{35.3.163}$$

We see that in this case, in order to achieve $f_4 = 0$ in *, we must employ an F_2 of the form

$$F_2 = F_2^3 + F_2^4 \tag{35.3.164}$$

with

$$F_2^3 = \tag{35.3.165}$$

and

$$F_2^4 = . \tag{35.3.166}$$

Therefore, to be fair, we must compare the use of F_+ as given by * with the use of F_2 as given by *.

$$g_4(u) = -f_4(u) + (1/2)(\partial f_3/\partial q, \partial f_3/\partial p)|_{z=u}. \tag{35.3.167}$$

35.4 Use of Poincaré Generating Function

35.4.1 Determination of Poincaré Generating Function in Terms of H

Suppose we are given a time-independent Hamiltonian H. Use it to generate the symplectic map

$$\mathcal{M}(\tau) = \exp(-\tau : H :) \tag{35.4.1}$$

Let $F_+(\Sigma, \tau)$ be the Poincaré generating function associated with $\mathcal{M}(\tau)$. We want to find a formula for F_+ in terms of H. To do so we will seek a Taylor expansion of $F_+(\Sigma, \tau)$ in powers of τ.

We first note that F_+ is odd in τ. From (2.1) we have the relation

$$\mathcal{M}(-\tau) = \exp(+\tau : H :) = \mathcal{M}^{-1}(\tau). \tag{35.4.2}$$

Next we observe that (6.6.45), which can be written in the form

$$Z = z + J\partial_\Sigma F_+|_{\Sigma=(Z+z)/2}, \tag{35.4.3}$$

can be rewritten in the form

$$z = Z - J\partial_\Sigma F_+|_{\Sigma=(Z+z)/2}, \tag{35.4.4}$$

which reveals that if the Poincaré generating function associated with $\mathcal{M}(\tau)$ is $F_+(\Sigma, \tau)$, then the Poincaré generating function associated with $\mathcal{M}^{-1}(\tau)$ is $-F_+(\Sigma, \tau)$. Consequently, we conclude that

$$F_+(\Sigma, -\tau) = -F_+(\Sigma, \tau). \tag{35.4.5}$$

Since F_+ is odd in τ, only odd powers of τ can occur in its Taylor expansion so that we may write

$$F_+(\Sigma, \tau) = F_+^{(1)}(\Sigma)\tau + F_+^{(3)}(\Sigma)\tau^3 + F_+^{(5)}(\Sigma)\tau^5 + \cdots . \tag{35.4.6}$$

The first term in the expansion is H itself,

$$F_+^{(1)}(\Sigma) = H(\Sigma). \tag{35.4.7}$$

Let $\mathcal{S}(H)$ denote the Hessian of H,

$$\mathcal{S}(H)_{ab} = \partial_a \partial_b H. \tag{35.4.8}$$

Then the next term in the expansion is given by the equation

$$F_+^{(3)}(\Sigma) = (1/24)(\partial H, J\mathcal{S}(H)J\partial H). \tag{35.4.9}$$

The successive terms are ever more complicated to state in explicit form. For $F_+^{(5)}$ there is the intermediate result

$$F_+^{(5)}(\Sigma) = \tag{35.4.10}$$

and the final result

$$F_+^{(5)}(\Sigma) = . \tag{35.4.11}$$

35.4.2 Application to Quadratic Hamiltonian

As a preliminary application of these results, let us first consider the simple case where

$$H(z) = h_2(z) = (1/2)(z, Sz). \tag{35.4.12}$$

Then we have the relations

$$\partial_a H = S_{ab} z_b, \tag{35.4.13}$$

$$\mathcal{S}(H) = S. \tag{35.4.14}$$

Correspondingly, we find the results

$$F_+^{(1)}(\Sigma) = H(\Sigma) = (1/2)(\Sigma, S\Sigma), \tag{35.4.15}$$

$$F_+^{(3)}(\Sigma) = (1/24)(\partial_a H J_{ab} \mathcal{S}(H)_{bc} J_{cd} \partial_d H) = (1/24)(\Sigma, SJSJS\Sigma), \tag{35.4.16}$$

$$F_+^{(5)}(\Sigma) = . \tag{35.4.17}$$

The net result is that $F_+(\Sigma, \tau)$ has the expansion

$$F_+(\Sigma, \tau) = (\Sigma, [(1/2)\tau S + (1/24)\tau^3 S(JS)^2 + ()\tau^5 S(JS)^4 + \cdots]\Sigma) \tag{35.4.18}$$

However, thanks to (), we already know that in this case

$$\begin{aligned} F_+(\Sigma, \tau) &= (1/2)(\Sigma, W'\Sigma) = (\Sigma, -J\tanh[\tau JS/2]\Sigma) \\ &= (\Sigma, [-J\tau JS/2 + J(1/3)(\tau JS/2)^3 + J(2/15)(\tau JS/2)^5/3 + \cdots]\Sigma) \\ &= (\Sigma, [(1/2)\tau S + (1/24)\tau^3 S(JS)^2 + (1/240)\tau^5 S(JS)^4 + \cdots]\Sigma). \end{aligned} \tag{35.4.19}$$

Here we have used the series

$$\tanh x = x - (1/3)x^3 + (2/15)x^5 + \cdots . \tag{35.4.20}$$

Evidently the expansions () and () agree.

35.4.3 Application to Symplectic Approximation

As a second application, suppose H has a homogeneous polynomial expansion of the form

$$H = h_3 + h_4 + h_5 + \cdots . \tag{35.4.21}$$

In this case we wish to obtain a homogeneous polynomial expansion for $F_+(\Sigma, \tau)$ of the form

$$F_+(\Sigma, \tau) = F_+^3(\Sigma, \tau) + F_+^4(\Sigma, \tau) + F_+^5(\Sigma, \tau) + \cdots . \tag{35.4.22}$$

We will now find that each $F_+^m(\Sigma, \tau)$ is also polynomial in the variable τ. Therefore, we may set $\tau = 1$, and drop τ from our variable list. Then we will have the relation

$$\mathcal{M} = \exp(- : H :). \tag{35.4.23}$$

For this symplectic map there will be the Poincaré generating function

$$F_+(\Sigma) = F_+^3(\Sigma) + F_+^4(\Sigma) + F_+^5(\Sigma) + \cdots \tag{35.4.24}$$

with

$$F_+^m(\Sigma) = F_+^m(\Sigma, \tau = 1). \tag{35.4.25}$$

Upon equating like powers of Σ on both sides of () and (), we find, through terms of degree 8, the results

$$F_+^3 = h_3, \tag{35.4.26}$$

$$F_+^4 = h_4, \tag{35.4.27}$$

$$F_+^5 = h_5 + (1/24)(\partial h_3, J\mathcal{S}(h_3)J\partial h_3), \tag{35.4.28}$$

$$F_+^6 = h_6 + (1/24)(\partial h_3, J\mathcal{S}(h_4)J\partial h_3) + (1/12)(\partial h_3, J\mathcal{S}(h_3)J\partial h_4), \tag{35.4.29}$$

$$F_+^7 = h_7 + (1/24)(\partial h_4, J\mathcal{S}(h_3)J\partial h_4) + (1/12)(\partial h_3, J\mathcal{S}(h_4)J\partial h_4), \tag{35.4.30}$$

$$F_+^8 = h_8 + (1/24)(\partial h_4, J\mathcal{S}(h_4)J\partial h_4). \tag{35.4.31}$$

Suppose we know h_3 through h_n and wish to 'evaluate'

$$\mathcal{M}^{[n]} = \exp(- : H^{[n]} :) \tag{35.4.32}$$

where

$$H^{[n]} = h_3 + h_4 + \cdots + h_n. \tag{35.4.33}$$

For this purpose, let us use a corresponding Poincaré generating function also truncated beyond terms of degree n. That is, we use the function $F_+^{[n]}$ defined by the rule

$$F_+^{[n]} = F_+^3 + F_+^4 + \cdots + F_+^n \tag{35.4.34}$$

Let $\mathcal{M}_+^{[n]}$ be the symplectic map produced by the use of $F_+^{[n]}$. By construction it will have the single-exponent Lie representation

$$\mathcal{M}_+^{[n]} = \exp(- : h_3 : - : h_4 : - \cdots - : h_n : + : g_{n+1} : + : g_{n+2} : + \cdots). \tag{35.4.35}$$

That is, the exponent of $\mathcal{M}_+^{[n]}$ will agree with that of $\mathcal{M}^{[n]}$ through terms of degree n, but there will generally be additional terms g_{n+1}, g_{n+2}, $\cdots$ which reflect the fact that the maps $\mathcal{M}_+^{[n]}$ and $\mathcal{M}$ are generally not identical. We will see that these additional terms depend on the given/known h_m, and that this dependence has three desirable properties.

Suppose, for example, that $n = 4$ so that

$$\mathcal{M}^{[4]} = \exp(- : h_3 : - : h_4 :), \tag{35.4.36}$$

$$F_+^{[4]} = F_+^3 + F_+^4, \tag{35.4.37}$$

and

$$\mathcal{M}_+^{[4]} = \exp(- : h_3 : - : h_4 : + : g_5 : + : g_6 : + \cdots). \tag{35.4.38}$$

Evidently truncating the series (2.34) beyond terms of degree 4 is equivalent to including all terms in the series and requiring that

$$F_+^m = 0 \text{ for all } m > 4. \tag{35.4.39}$$

Inspection of (2.28) through (2.31) shows that the requirement (2.39) produces, through terms of degree 8, the relations

$$-h_5 = (1/24)(\partial h_3, J\mathcal{S}(h_3)J\partial h_3), \tag{35.4.40}$$

$$-h_6 = (1/24)(\partial h_3, J\mathcal{S}(h_4)J\partial h_3) + (1/12)(\partial h_3, J\mathcal{S}(h_3)J\partial h_4), \tag{35.4.41}$$

$$-h_7 = (1/24)(\partial h_4, J\mathcal{S}(h_3)J\partial h_4) + (1/12)(\partial h_3, J\mathcal{S}(h_4)J\partial h_4), \tag{35.4.42}$$

$$-h_8 = (1/24)(\partial h_4, J\mathcal{S}(h_4)J\partial h_4), \tag{35.4.43}$$

and we see that

$$g_m = -h_m \text{ for all } m > 4 \tag{35.4.44}$$

with the h_m for $m > 4$ defined by in terms of h_3 and h_4 by the relations (2.40) through (2.43).

Now we are ready to examine in some detail the properties of the dependence of the g_{n+1}, g_{n+2}, $\cdots$ on the h_3, h_4, $\cdots$ h_n. To do so, it is useful to introduce a somewhat more elaborate notation. Let us employ, in place of $\mathcal{M}_+^{[n]}$, the symbols $\mathcal{M}_+^{[n]}\{H^{[n]}\}$ to indicate that the map $\mathcal{M}_+^{[n]}$ depends on h_3, h_4, $\cdots$ h_n. The first property is this: Suppose all the h_m are replaced by $-h_m$. Then we see, from () through () and () through (), that all the F_+^m and all the g_m are replaced by $-F_+^m$ and $-g_m$, respectively. Consequently, there is the relation

$$\mathcal{M}_+^{[n]}\{-H^{[n]}\} = (\mathcal{M}_+^{[n]}\{H^{[n]}\})^{-1}. \tag{35.4.45}$$

In words, if $\mathcal{M}_+^{[n]}$ is the symplectic approximation to $\mathcal{M}^{[n]}$, then $(\mathcal{M}_+^{[n]})^{-1}$ is the symplectic approximation to $(\mathcal{M}^{[n]})^{-1}$. We may invert and then symplectically approximate, or symplectically approximate and then invert. The result of both procedures is the same. We may say that symplectic approximation by the use of a Poincaré generating function is invariant under the operation of map inversion.

The second property is more subtle. Suppose $\mathcal{R}$ is a linear symplectic map with associated symplectic matrix R. Suppose the h_m are transformed under the action of $\mathcal{R}$ to become the homogeneous polynomials $h_m^{\rm tr}$ by the rule

$$h_m^{\rm tr}(z) = \mathcal{R}h_m(z) = h_m(Rz). \tag{35.4.46}$$

Also, let $g_{n+1}^{\rm tr}(z)$, $g_{n+2}^{\rm tr}(z)$, $\cdots$ be the functions obtained by applying the rules defining the $g_{n+1}(z)$, $g_{n+2}(z)$, $\cdots$ to the $h_m^{\rm tr}$. Then there is also the result

$$\begin{aligned} g_{n+1}^{\rm tr}(z) &= \mathcal{R}g_{n+1}, \\ g_{n+2}^{\rm tr}(z) &= \mathcal{R}g_{n+2}, \text{ etc.} \end{aligned} \tag{35.4.47}$$

Suppose we assume, for the moment, that (2.47) is correct. It follows that there is then the relation

$$\mathcal{M}_+^{[n]}\{\mathcal{R}H^{[n]}\} = \mathcal{R}\mathcal{M}_+^{[n]}\{H^{[n]}\}(\mathcal{R})^{-1}. \tag{35.4.48}$$

Of course, we also have the relation

$$\mathcal{R}\mathcal{M}^{[n]}(\mathcal{R})^{-1} = \exp(- : \mathcal{R}H^{[n]} :). \tag{35.4.49}$$

In words, if we conjugate the map $\mathcal{M}^{[n]}$ with $\mathcal{R}$ and then symplectically approximate the result, the outcome is the same as first symplectically approximating $\mathcal{M}^{[n]}$ and then conjugating with $\mathcal{R}$. We may say that symplectic approximation by the use of a Poincaré generating function is invariant under the operation of conjugation with the linear symplectic map $\mathcal{R}$.

35.5 Use of Other Generating Functions

The Poincaré generating function type F_+ had the property that if an F_+ of the form

$$F_+ = F_+^3 \tag{35.5.1}$$

was used to generate an $\mathcal{N}$ with an f_3, then this map was guaranteed to have *vanishing* f_4. Are there any other generating function types with this property? Note that, according to (3.71), F_2 does not have this property.

Use the notation F_γ to denote the generation function type associated with the $4n \times 4n$ syplectic matrix γ. Then we may write, for example,

$$F_+ = F_\gamma \tag{35.5.2}$$

with

$$\gamma = I^{4n}. \tag{35.5.3}$$

See (6.7.69). With this notation, we may rephrase our question: Are there any other γ for which, if an F_γ^3 is used to generate an $\mathcal{N}$ with an f_3, then this map is guaranteed to have vanishing f_4? Actually, we need to be a bit careful here. Since $\mathcal{N}$ has the identity map as its linear part, we may need a generating function of the form

$$F_\gamma = F_\gamma^2 + F_\gamma^3 \tag{35.5.4}$$

to generate an $\mathcal{N}$ with an f_3 and the identity map as its linear part. Again we will want the guarantee that f_4 will vanish.

Here is a perhaps associated question: Suppose $\mathcal{N}$ is a map of the form (3.1). Suppose we wish to represent it using two different generating function types so that we have the two generating functions F_γ and $F_{\gamma'}$. Are they related by a Legendre transformation?

35.6 Cremona Approximation

$$(qc+ps)^3 = q^3c^3 + 3q^2c^2ps + 3qcp^2s^2 + p^3s^3. \tag{35.6.1}$$

$$\begin{aligned} c^3 &= (1/2)^3(e^{i\theta}+e^{-i\theta})^3 = (1/8)(e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta}) \\ &= (1/4)c3 + (3/4)c = (1/4)(c3+3c). \end{aligned} \tag{35.6.2}$$

$$-3c^2s = -(3/4)s3 - (3/4)s \Leftrightarrow c^2s = (1/4)(s3+s), \tag{35.6.3}$$

$$\begin{aligned} s^3 &= [1/(2i)]^3(e^{i\theta}-e^{-i\theta})^3 = -(1/4)[1/(2i)](e^{3i\theta}-3e^{i\theta}+3e^{-i\theta}-e^{-3i\theta}) \\ &= -(1/4)s3 + (3/4)s = (1/4)(-s3+3s), \end{aligned} \tag{35.6.4}$$

$$3cs^2 = -(3/4)c3 + (3/4)c \Leftrightarrow cs^2 = (1/4)(-c3+c). \tag{35.6.5}$$

Note that (6.2) and (6.4) follow from the definitions of the cos and sin functions, and (6.3) and (6.5) follow from differentiating (6.2) and (6.4).

Define vectors/functions by the rules

$$a_c = c, \tag{35.6.6}$$

$$b_c = c3, \tag{35.6.7}$$

$$a_s = s, \tag{35.6.8}$$

$$b_s = s3. \tag{35.6.9}$$

Let $\chi(\phi)$ and $\psi(\phi)$ be any two functions of ϕ. Define their inner product by writing

$$(\chi,\psi) = \int_0^{2\pi} d\phi\, \chi\psi. \tag{35.6.10}$$

By this definition, the functions defined by (6.6) through (6.9) are orthogonal. They also have a squared length π,

$$(a_c, a_c) = (c,c) = (1/2)[(c,c)+(s,s)] = (1/2)(1,1) = \pi, \text{ etc.} \tag{35.6.11}$$

$$u_c = c3 + 3c = b_c + 3a_c, \tag{35.6.12}$$

$$v_c = s3 + s =, \tag{35.6.13}$$

$$u_s = -s3 + 3s, \tag{35.6.14}$$

$$v_s = -c3 + c. \tag{35.6.15}$$

They are obviously linearly independent. Then we find that

$$(u_c, v_c) = (u_s, v_s) = 0, \tag{35.6.16}$$

$$(u_c, u_s) = (v_c, v_s) = 0, \tag{35.6.17}$$

$$(u_c, v_s) = -(c3, c3) + 3(c, c) = 2(c, c) = (c, c) + (s.s) = (1, 1) = 2\pi, \tag{35.6.18}$$

$$(v_c, u_s) = -(s3, s3) + 3(s, s) = 2(s, s) = (c, c) + (s.s) = (1, 1) = 2\pi. \tag{35.6.19}$$

Using, for $\mathcal{L}$, drifts instead of phase advances. Are there more possibilities?
Decomposition of the $sp(6, \mathbb{R})$ representation $\Gamma(\ell, 0, 0)$ into representations of $su(3)$.

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Chapter 36

Orbit Stability, Long-Term Behavior, and Dynamic Aperture

Chapter 37

Reversal Symmetry

The concept of reversibility, and that reversibility has various implications for charged-particle and light optics (and the study of general dynamical systems), are part of the common lore of those working in these fields, and its role and value are generally understood at least on an intuitive level. The purpose of this chapter is to explore reversal symmetry systematically. Reversal symmetry is defined; it is shown that the transfer maps for most common beam-line elements are reversal symmetric including nonlinear effects; and various linear and nonlinear consequences of reversal symmetry are worked out in some detail.

Section 1 defines the operation of reversal and works out some of its properties. Section 2 describes some of the applications of these properties. In particular it defines what is meant for a transfer map to be reversal symmetric, and shows that the transfer maps for many common beam-line elements are reversal symmetric. Section 3 works out some of the general consequences of reversal symmetry for straight and circular machines, and Section 4 treats some special cases. Section 5 studies the consequences of reversal symmetry for closed orbits in a circular machine, and Section 6 studies the consequences for the *Courant-Snyder* functions in a circular machine. A final section treats various nonlinear consequences of reversal symmetry. It seems remarkable that such a simple concept should be so rich in consequences.

37.1 Reversal Operator

We will work with a coordinate system that is particularly useful for charged-particle optics. We write

$$z = (x, p_x; y, p_y; \tau, p_\tau). \tag{37.1.1}$$

The quantities x and y are transverse deviations from a design trajectory, and p_x and p_y are their conjugate momenta. The quantity τ is the difference (time deviation) between the arrival/departure time of a given particle and a particle on the design trajectory. Finally, p_τ is the *negative* of the energy difference between that of the given particle and that of a particle on the design trajectory. Note that this choice of variables presumes that some *coordinate* (it could be Cartesian or angular or path length along some design trajectory) is taken to play the role of the *independent* (time-like) variable. When this is done, its conjugate momentum does not appear in the associated Hamiltonian, and τ and p_τ are both *dependent* variables. Finally, we note that in accelerator physics it is common to scale the transverse coordinates x and y by some convenient scale length ℓ, to scale the transverse momenta p_x and p_y by some "design" momentum p^0, to scale p_t (which is the negative of the energy) by p^0c (where c is the speed of light), and to scale the time t by ℓ/c. In this chapter we do not do so. In particular, for the purpose of this chapter, p_τ is defined in terms of p_t simply by subtracting off the design value of p_t without any scaling, and τ is defined in terms of t simply by subtracting off the time of flight for the design orbit, again without any scaling factor.

Let z be any point in phase space as specified by (1.1). *Define* a "reversal" operator $\mathcal{R}$ acting on phase space by the rule

$$\mathcal{R}z = z^r \tag{37.1.2}$$

with

$$z^r = (x, -p_x; y, -p_y; -\tau, p_\tau). \tag{37.1.3}$$

The reversal operation is analogous to time reversal, but differs from it in two essential ways. First, recall that the definitions (1.1) through (1.3) presume that some coordinate is playing the role of the independent (time-like) variable, its conjugate momentum is absent, and τ and p_τ are dependent variables. Second, the magnetic field does not change sign. It is for these reasons that, as we will see, a transfer map can violate what we will define as reversal symmetry even though the fundamental laws that govern charged-particle motion, the electromagnetic field, and the electromagnetic interaction are all invariant under time reversal as usually defined.

It is easily verified that, when acting on phase space, the effect of $\mathcal{R}$ can be described by a matrix R with

$$\mathcal{R}z_a = (Rz)_a \tag{37.1.4}$$

where R is the matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{37.1.5}$$

We note for future reference that J and R have the properties

$$JR = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{37.1.6}$$

$$J^2 = -I, \tag{37.1.7}$$

$$R^2 = I, \tag{37.1.8}$$

$$(JR)^2 = I, \tag{37.1.9}$$

$$RJR = -J, \tag{37.1.10}$$

$$RJ = -JR. \tag{37.1.11}$$

Here we use the form (3.2.10) for J. We also remark that since R is a symmetric matrix, $R^T = R$, (1.10) can also be written in the form

$$R^T JR = -J. \tag{37.1.12}$$

Therefore, R is an *antisymplectic* matrix. See Exercise 3.12.8. Correspondingly, we will say that $\mathcal{R}$ is an *antisymplectic* map. We have learned that, in our setting of Classical Mechanics, evolution (with some coordinate playing the role of the independent variable) is described by a symplectic map acting on phase space, and reversal is described by an antisymplectic map. This terminology is analogous to that employed in Quantum Mechanics where time evolution is described by a unitary transformation acting on Hilbert space, and time reversal is described by what is called an *antiunitary* transformation.

We have seen that $\mathcal{R}$ is an antisymplectic map. The same is true of the maps $\mathcal{MR}$ and $\mathcal{RM}$ if $\mathcal{M}$ is symplectic. To prove this, let $\mathcal{N}$ be the map given by the product

$$\mathcal{N} = \mathcal{MR}. \tag{37.1.13}$$

By the chain rule its Jacobian matrix N is given by the relation

$$N = MR, \tag{37.1.14}$$

and we find the result

$$N^T JN = RM^T JMR = RJR = -J. \tag{37.1.15}$$

Similarly, if $\mathcal{N}$ is the map given by the product

$$\mathcal{N} = \mathcal{RM}, \tag{37.1.16}$$

we find the results

$$N = RM \tag{37.1.17}$$

and

$$N^T JN = M^T RJRM = M^T(-J)M = -J. \tag{37.1.18}$$

We also note the converse conclusion: If $\mathcal{N}$ is an antisymplectic map, then the maps $\mathcal{RN}$ and $\mathcal{NR}$ are symplectic. Finally, there is an immediate generalization: The product of a symplectic map and an antisymplectic map is antisymplectic, and the product of two antisymplectic maps is symplectic.

Extend $\mathcal{R}$ to phase-space functions $f(z)$ by the rule

$$\mathcal{R}f = f^r \tag{37.1.19}$$

with

$$f^r(z) = f(z^r). \tag{37.1.20}$$

Evidently $\mathcal{R}$ is a linear operator with the property

$$\mathcal{R}^2 = \mathcal{I} \text{ or } \mathcal{R}^{-1} = \mathcal{R}. \tag{37.1.21}$$

For any two functions g and h there is also the property

$$\mathcal{R}(gh) = (\mathcal{R}g)(\mathcal{R}h). \tag{37.1.22}$$

Finally we note that since the application of $\mathcal{R}$ commutes with the operation of scaling variables, everything we will conclude about reversal properties of maps given in terms of unscaled variables will also hold for maps given in terms of scaled variables.

Let us determine the effect of reversal on Lie operators. We claim that $\mathcal{R}$ has the property

$$\mathcal{R} : f : \mathcal{R} = - : \mathcal{R}f := - : f^r : \tag{37.1.23}$$

for any Lie operator $: f :$. To prove this claim, let $\mathcal{R} : f : \mathcal{R}$ act on any function g. The calculation is a bit delicate, and is best done in stages and pieces. To begin, we have the result

$$\begin{aligned} \mathcal{R} : f : \mathcal{R}g &= \mathcal{R}[f, \mathcal{R}g] \\ &= \mathcal{R}\{\sum_j (\partial f/\partial q_j)(\partial(\mathcal{R}g)/\partial p_j) - (\partial f/\partial p_j)(\partial(\mathcal{R}g)/\partial q_j)\} \\ &= \sum_j \{\mathcal{R}(\partial f/\partial q_j)\}\{\mathcal{R}(\partial(\mathcal{R}g)/\partial p_j)\} \\ &- \{\mathcal{R}(\partial f/\partial p_j)\}\{\mathcal{R}(\partial(\mathcal{R}g)/\partial q_j)\}. \end{aligned} \tag{37.1.24}$$

Here we have used (1.22). Next, it follows from (1.2) and (1.3) that there are the operator relations

$$\mathcal{R}(\partial/\partial z_a) = (\partial/\partial z_a)\mathcal{R} \text{ for } a = 1, 3, 6; \tag{37.1.25}$$

$$\mathcal{R}(\partial/\partial z_a) = -(\partial/\partial z_a)\mathcal{R} \text{ for } a = 2, 4, 5. \tag{37.1.26}$$

Therefore we find for the $j = 1$ terms in (1.24) the results

$$\begin{aligned} \{\mathcal{R}(\partial f/\partial x)\}\{\mathcal{R}(\partial(\mathcal{R}g)/\partial p_x)\} &= \{\partial(\mathcal{R}f)/\partial x)\}(-1)\{\mathcal{R}^2(\partial g/\partial p_x)\} \\ &= -(\partial f^r/\partial x)(\partial g/\partial p_x), \end{aligned} \tag{37.1.27}$$

$$\begin{aligned} -\{\mathcal{R}(\partial f/\partial p_x)\}\{\mathcal{R}(\partial(\mathcal{R}g)/\partial x)\} &= +\{\partial(\mathcal{R}f)/\partial p_x\}\{\mathcal{R}^2(\partial g/\partial x)\} \\ &= (\partial f^r/\partial p_x)(\partial g/\partial x). \end{aligned} \tag{37.1.28}$$

Analogous results hold for the $j = 2$ terms, which involve the y, p_y pair. And for the $j = 3$ terms, which involve the τ, p_τ pair, we find the results

$$\begin{aligned} \{\mathcal{R}(\partial f/\partial \tau)\}\{\mathcal{R}(\partial(\mathcal{R}g)/\partial p_\tau)\} &= -\{\partial(\mathcal{R}f)/\partial \tau\}\{\mathcal{R}^2(\partial g/\partial p_\tau)\} \\ &= -(\partial f^r/\partial \tau)(\partial g/\partial p_\tau), \end{aligned} \tag{37.1.29}$$

$$\begin{aligned} -\{\mathcal{R}(\partial f/\partial p_\tau)\}\{\mathcal{R}(\partial(\mathcal{R}g)/\partial \tau)\} &= -\{\partial(\mathcal{R}f/\partial p_\tau)\}(-1)\{\mathcal{R}^2(\partial g/\partial \tau)\} \\ &= (\partial f^r/\partial p_\tau)(\partial g/\partial \tau). \end{aligned} \tag{37.1.30}$$

Now put all these results into (1.24) to obtain the relation

$$\begin{aligned}
\mathcal{R} : f : \mathcal{R}g &= \sum_j \{\mathcal{R}(\partial f/\partial q_j)\}\{\mathcal{R}(\partial(\mathcal{R}g)/\partial p_j)\} \\
&- \{\mathcal{R}(\partial f/\partial p_j)\}\{\mathcal{R}(\partial(\mathcal{R}g)/\partial q_j)\} \\
&= -\sum_j (\partial f^r/\partial q_j)(\partial g/\partial p_j) - (\partial f^r/\partial p_j)(\partial g/\partial q_j) \\
&= -[f^r, g] = - : f^r : g. \qquad (37.1.31)
\end{aligned}$$

Evidently (1.23) is the operator version of (1.31).

Let us next determine the effect of reversal on Lie transformations. From (1.21) and (1.23) we find the additional property

$$\begin{aligned}
\mathcal{R} : f :^n \mathcal{R} &= \mathcal{R} : f :: f :: f : \cdots : f : \mathcal{R} \\
&= \mathcal{R} : f : \mathcal{R}\mathcal{R} : f : \mathcal{R}\mathcal{R} : f : \mathcal{R} \cdots \mathcal{R} : f : \mathcal{R} \\
&= (\mathcal{R} : f : \mathcal{R})^n = (-1)^n : \mathcal{R}f :^n = (-1)^n : f^r :^n . \qquad (37.1.32)
\end{aligned}$$

Suppose $\mathcal{M}$ is a map that, for some f, can be written in the single exponent form

$$\mathcal{M} = \exp(: f :) = \sum_{n=0}^{\infty} : f :^n /n!. \qquad (37.1.33)$$

Then, from (1.32) and (1.33), we find the result

$$\begin{aligned}
\mathcal{R}\mathcal{M}\mathcal{R} &= \sum_{n=0}^{\infty} \mathcal{R} : f :^n \mathcal{R}/n! = \sum_{n=0}^{\infty} (-1)^n : \mathcal{R}f :^n /n! \\
&= \exp(- : \mathcal{R}f :) = \exp(- : f^r :). \qquad (37.1.34)
\end{aligned}$$

The stage is set to define the effect of reversal on maps. Suppose $\mathcal{M}$ is any map that sends initial points z^i to final points z^f,

$$z^f = \mathcal{M}z^i. \qquad (37.1.35)$$

Reverse both z^i and z^f to yield $\mathcal{R}z^i$ and $\mathcal{R}z^f$. We *define* the reversed map $\mathcal{M}^r$ to be that map which sends $\mathcal{R}z^f$ to $\mathcal{R}z^i$,

$$\mathcal{M}^r \mathcal{R}z^f = \mathcal{R}z^i. \qquad (37.1.36)$$

See Figure 1.1. Combining (1.35) and (1.36) gives the result

$$\mathcal{M}^r \mathcal{R}\mathcal{M}z^i = \mathcal{R}z^i. \qquad (37.1.37)$$

Equivalently, we have the operator relation

$$\mathcal{M}^r \mathcal{R}\mathcal{M} = \mathcal{R}. \qquad (37.1.38)$$

This relation can be solved for $\mathcal{M}^r$ to give the intermediate result

$$\mathcal{M}^r = \mathcal{R}(\mathcal{R}\mathcal{M})^{-1} = \mathcal{R}\mathcal{M}^{-1}\mathcal{R}^{-1}, \qquad (37.1.39)$$

and use of (1.21) in (1.39) gives the final equivalent definition for $\mathcal{M}^r$:

$$\mathcal{M}^r = \mathcal{R}\mathcal{M}^{-1}\mathcal{R}. \tag{37.1.40}$$

Note that, in analogy to (1.21), reversing a map twice leaves it unchanged:

$$\begin{aligned}(\mathcal{M}^r)^r &= (\mathcal{R}\mathcal{M}^{-1}\mathcal{R})^r = \mathcal{R}(\mathcal{R}\mathcal{M}^{-1}\mathcal{R})^{-1}\mathcal{R} \\ &= \mathcal{R}\mathcal{R}^{-1}\mathcal{M}\mathcal{R}^{-1}\mathcal{R} = \mathcal{M}.\end{aligned} \tag{37.1.41}$$

From (1.21), (1.38), and (1.40) we deduce the chain of relations

$$\mathcal{M}^r\mathcal{R}\mathcal{M}\mathcal{R} = \mathcal{I}, \tag{37.1.42}$$

$$\mathcal{M}^r\mathcal{R}(\mathcal{M}^{-1})^{-1}\mathcal{R} = \mathcal{I}, \tag{37.1.43}$$

$$\mathcal{M}^r(\mathcal{M}^{-1})^r = \mathcal{I}, \tag{37.1.44}$$

$$(\mathcal{M}^{-1})^r = (\mathcal{M}^r)^{-1}. \tag{37.1.45}$$

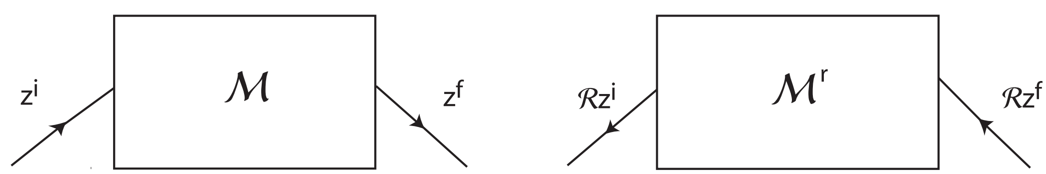

Figure 37.1.1: Actions of a map $\mathcal{M}$ and its reversed counterpart $\mathcal{M}^r$.

We also observe that if $\mathcal{M}$ can be written in the single exponent form (1.33), then use of (1.34) and (1.40) shows that there is the relation

$$\mathcal{M}^r = \mathcal{R}\exp(-:f:)\mathcal{R} = \exp(:f^r:). \tag{37.1.46}$$

If $\mathcal{M}$ is a symplectic map, so is $\mathcal{M}^r$. To prove this, let M^r be the Jacobian matrix of $\mathcal{M}^r$. From (1.40) and the chain rule we find the result

$$M^r = RM^{-1}R. \tag{37.1.47}$$

Let us check whether M^r is a symplectic matrix. We find from (3.1.2), (1.47), and (1.11) the result

$$\begin{aligned}(M^r)^T JM^r &= R(M^{-1})^T RJRM^{-1}R \\ &= -R(M^{-1})^T JM^{-1}R = -RJR = J.\end{aligned} \tag{37.1.48}$$

We see that M^r is a symplectic matrix, and hence $\mathcal{M}^r$ is a symplectic map. The observant reader will have noticed that the same conclusion could have been reached immediately from the discussion surrounding equations (1.13) through (1.18).

Suppose we combine (1.47) with the symplectic condition. From the symplectic condition (3.1.2) we deduce that

$$M^{-1} = -JM^T J, \tag{37.1.49}$$

and hence (1.47) can also be written in the form

$$M^r = JRM^TJR = JRM^T(JR)^{-1}. \tag{37.1.50}$$

Here we have also used (1.11) and (1.9).

From (1.50) it follows that M and M^r have the *same spectrum*. Indeed, let P and P^r be the *characteristic polynomials* of M and M^r,

$$P(\lambda) = \det(M - \lambda I), \tag{37.1.51}$$

$$P^r(\lambda) = \det(M^r - \lambda I). \tag{37.1.52}$$

Then, by use of (1.50), we have the result

$$\begin{aligned} P^r(\lambda) &= \det(M^r - \lambda I) = \det[JRM^T(JR)^{-1} - \lambda I] \\ &= \det[(JR)(M^T - \lambda I)(JR)^{-1}] \\ &= \det(JR)\det[(JR)^{-1}]\det(M^T - \lambda I) \\ &= \det(M - \lambda I) = P(\lambda). \end{aligned} \tag{37.1.53}$$

The last task for this section is to determine the effect of reversal on a relation involving the action of a map on a function. Suppose the function h is the result of the symplectic map $\mathcal{M}$ acting on the function g,

$$h = \mathcal{M}g. \tag{37.1.54}$$

Letting $\mathcal{R}$ act on both sides of (1.54) and using (1.21) and (1.42) give the results

$$\mathcal{R}h = \mathcal{R}\mathcal{M}g = \mathcal{R}\mathcal{M}\mathcal{R}\mathcal{R}g = (\mathcal{M}^r)^{-1}\mathcal{R}g, \tag{37.1.55}$$

or

$$h^r = (\mathcal{M}^r)^{-1}g^r. \tag{37.1.56}$$

37.2 Applications

Suppose $\mathcal{M}$ is a product of several maps $\mathcal{M}_1$ to $\mathcal{M}_n$,

$$\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\cdots\mathcal{M}_n. \tag{37.2.1}$$

Then, from (1.21) and (1.22), there is the result

$$\begin{aligned} \mathcal{M}^r &= \mathcal{R}(\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\cdots\mathcal{M}_n)^{-1}\mathcal{R} \\ &= \mathcal{R}(\mathcal{M}_n^{-1}\cdots\mathcal{M}_3^{-1}\mathcal{M}_2^{-1}\mathcal{M}_1^{-1})\mathcal{R} \\ &= \mathcal{R}\mathcal{M}_n^{-1}\mathcal{R}\cdots\mathcal{R}\mathcal{M}_3^{-1}\mathcal{R}\mathcal{R}\mathcal{M}_2^{-1}\mathcal{R}\mathcal{R}\mathcal{M}_1^{-1}\mathcal{R} \\ &= \mathcal{M}_n^r\cdots\mathcal{M}_3^r\mathcal{M}_2^r\mathcal{M}_1^r. \end{aligned} \tag{37.2.2}$$

Thus, the reverse of a product of maps is the product of the reverses of the individual maps taken in *opposite* order.

Define a map $\mathcal{M}$ to be *reversal symmetric* if it equals its reverse,

$$\mathcal{M}^r = \mathcal{M}. \tag{37.2.3}$$

For a reversal symmetric map (1.38) can be rewritten in the forms

$$\mathcal{RMR} = \mathcal{M}^{-1} \text{ or } \mathcal{MRM} = \mathcal{R}; \tag{37.2.4}$$

$$\mathcal{RMRM} = \mathcal{I}, \tag{37.2.5}$$

$$\mathcal{MRMR} = \mathcal{I}, \text{ or} \tag{37.2.6}$$

$$(\mathcal{RM})^2 = (\mathcal{MR})^2 = \mathcal{I}. \tag{37.2.7}$$

Here we have used (1.21). A map whose square is the identity is called an *involution*. We have seen that $\mathcal{RM}$ and $\mathcal{MR}$ are involutions if $\mathcal{M}$ is reversal symmetric. According to (1.21), $\mathcal{R}$ is also an involution. Finally, since $\mathcal{M}^k$ will be reversal symmetric if $\mathcal{M}$ is reversal symmetric, the maps $\mathcal{RM}^k$ and $\mathcal{M}^k\mathcal{R}$ for any k are also involutions,

$$(\mathcal{RM}^k)(\mathcal{RM}^k) = (\mathcal{RM}^k\mathcal{R})(\mathcal{M}^k) = \mathcal{M}^{-k}\mathcal{M}^k = \mathcal{I}, \tag{37.2.8}$$

$$(\mathcal{M}^k\mathcal{R})(\mathcal{M}^k\mathcal{R}) = \mathcal{M}^k(\mathcal{RM}^k\mathcal{R}) = \mathcal{M}^k\mathcal{M}^{-k} = \mathcal{I}. \tag{37.2.9}$$

Moreover, there are the obvious identities

$$\mathcal{M} = \mathcal{RRM} = (\mathcal{R})(\mathcal{RM}), \tag{37.2.10}$$

$$\mathcal{M} = \mathcal{MRR} = (\mathcal{MR})(\mathcal{R}), \tag{37.2.11}$$

$$\mathcal{M}^k = \mathcal{RRM}^k = (\mathcal{R})(\mathcal{RM}^k), \tag{37.2.12}$$

$$\mathcal{M}^k = \mathcal{M}^k\mathcal{RR} = (\mathcal{M}^k\mathcal{R})(\mathcal{R}). \tag{37.2.13}$$

They show that if $\mathcal{M}$ is reversal symmetric, then $\mathcal{M}$ and $\mathcal{M}^k$ for any k can be written as the product of two involutions. The discovery and classification of the fixed points (closed orbits) of a map are greatly simplified if the map can be written as the product of two involutions. See Section 7.

Suppose $\mathcal{M}$ can be written in the single exponent form (1.33), and is reversal symmetric. Then we see from (1.46) that the generator f must satisfy the relation

$$f^r = f. \tag{37.2.14}$$

Suppose $\mathcal{M}$ can be written as a product of several maps $\mathcal{M}_1$ to $\mathcal{M}_n$ and their reverses,

$$\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_n\mathcal{M}_n^r\cdots\mathcal{M}_2^r\mathcal{M}_1^r. \tag{37.2.15}$$

Then, simple calculation shows that $\mathcal{M}$ is reversal symmetric. Indeed, from (2.2) and (1.40) we find the result

$$\begin{aligned}\mathcal{M}^r &= (\mathcal{M}_1^r)^r(\mathcal{M}_2^r)^r\cdots(\mathcal{M}_n^r)^r\mathcal{M}_n^r\cdots\mathcal{M}_2^r\mathcal{M}_1^r \\ &= \mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_n\mathcal{M}_n^r\cdots\mathcal{M}_2^r\mathcal{M}_1^r = \mathcal{M}.\end{aligned} \tag{37.2.16}$$

We next claim, based on end-to-end symmetry, that the transfer maps $\mathcal{M}$ for many common beamline elements are reversal symmetric. These elements include drifts, bends (including combined-function bends) with equal entry and exit angles, quadrupoles, sextupoles, octupoles, etc. This statement holds even if fringe-field and multipole effects are included provided the element in question has end-to-end symmetry. We will also show that the transfer map for a short on-phase RF cavity (a cavity that maintains bunching, but provides no net acceleration) is reversal symmetric. Finally, we note that the transfer map for a solenoid with end-to-end symmetry is *not* reversal symmetric. Instead, the reversed map for such a solenoid is the map for that solenoid with opposite magnetic field. That is, if $\mathcal{M}[\boldsymbol{B}(\boldsymbol{r})]$ is the map for such a solenoid with magnetic field $\boldsymbol{B}(\boldsymbol{r})$, there is the relation

$$\mathcal{M}^r[\boldsymbol{B}(\boldsymbol{r})] = \mathcal{M}[-\boldsymbol{B}(\boldsymbol{r})]. \tag{37.2.17}$$

Here we have used a square-bracket notation to indicate that the map $\mathcal{M}$ is a *functional* of the magnetic field $\boldsymbol{B}(\boldsymbol{r})$.

Imagine integrating (10.1.8) to find the map $\mathcal{M}$ for some beamline element. Divide the integration interval into $2N$ equal segments each of "duration" h. Label the intervals $1, 2, \cdots N$ followed by $\tilde{N}, \cdots \tilde{2}, \tilde{1}$. Then $\mathcal{M}$ can be written in the product form

$$\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_N\mathcal{M}_{\tilde{N}}\cdots\mathcal{M}_{\tilde{2}}\mathcal{M}_{\tilde{1}} \tag{37.2.18}$$

where $\mathcal{M}_j$ is the map for the j^{th} segment. The segments N and $\tilde{N}$ are on either side of the center of the element, and the segments 1 and $\tilde{1}$ are at the leading and trailing ends, etc. For each map $\mathcal{M}_j$ we have an approximation of the form

$$\mathcal{M}_j = \exp(-h : H_j :) + O(h^2) \tag{37.2.19}$$

where H_j is the Hamiltonian evaluated at the center of the j^{th} segment. Let us compute $(\mathcal{M}_j)^r$. From (1.34), (1.46), and (2.19) we find the result

$$(\mathcal{M}_j)^r = \exp[-h : (H_j)^r :] + O(h^2). \tag{37.2.20}$$

We now make the symmetry assumption

$$(H_{\tilde{j}})^r = H_j \text{ for } j = 1, 2, \cdots N. \tag{37.2.21}$$

It then follows that

$$(\mathcal{M}_{\tilde{j}})^r = \mathcal{M}_j + O(h^2) \tag{37.2.22}$$

and, by (1.41),

$$\mathcal{M}_{\tilde{j}} = (\mathcal{M}_j)^r + O(h^2). \tag{37.2.23}$$

Correspondingly, we may rewrite (2.18) in the form

$$\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_N(\mathcal{M}_N)^r\cdots(\mathcal{M}_2)^r(\mathcal{M}_1)^r + O(Nh^2). \tag{37.2.24}$$

Here, as a worst case estimate, we assume that all the $O(h^2)$ terms in (2.18) add constructively to produce a possible term of order Nh^2 in (2.24). Comparison of (2.15), (2.16), and (2.24) gives the result

$$\mathcal{M}^r = \mathcal{M} + O(Nh^2). \tag{37.2.25}$$

Now let the number N of segments approach infinity and the duration h of each approach zero. Then in this limit, $Nh^2 \to 0$, and we see that $\mathcal{M}^r$ must equal $\mathcal{M}$ exactly, and hence $\mathcal{M}$ is reversal symmetric.

There is a related result that is also of use. Let us write (2.18) in the form

$$\mathcal{M} = \mathcal{M}_\ell \mathcal{M}_t \tag{37.2.26}$$

where $\mathcal{M}_\ell$, the *leading* half of $\mathcal{M}$, is given by the product

$$\mathcal{M}_\ell = \mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_N, \tag{37.2.27}$$

and $\mathcal{M}_t$, the *trailing* half of $\mathcal{M}$, is given by the product

$$\mathcal{M}_t = \mathcal{M}_{\tilde{N}} \cdots \mathcal{M}_{\tilde{2}} \mathcal{M}_{\tilde{1}}. \tag{37.2.28}$$

From (2.27) there is the relation

$$(\mathcal{M}_\ell)^r = (\mathcal{M}_N)^r \cdots (\mathcal{M}_2)^r (\mathcal{M}_1)^r, \tag{37.2.29}$$

and by combining this relation with (2.23) we obtain the estimate

$$(\mathcal{M}_\ell)^r = \mathcal{M}_{\tilde{N}} \cdots \mathcal{M}_{\tilde{2}} \mathcal{M}_{\tilde{1}} + O(Nh^2) = \mathcal{M}_t + O(Nh^2). \tag{37.2.30}$$

Again let the number N of segments approach infinity and the duration h of each approach zero so that $Nh^2 \to 0$. By so doing we conclude that the estimate (2.30) must in fact be the equality

$$(\mathcal{M}_\ell)^r = \mathcal{M}_t. \tag{37.2.31}$$

A few words need to be said about the symmetry assumption (2.21). Consider first *static* elements for which H does not depend on τ. In this case it is only necessary to examine how H depends on p_x and p_y. For a drift H_j is an even function (depends only on p_x^2 and p_y^2) and, of course, independent of the segment j. Therefore (2.21) holds. The same is true for the *body* of any multipole (including dipoles and combined-function dipoles), and therefore (2.21) again holds.

At the ends of a multipole H can have odd terms in p_x and p_y. For example, for a quadrupole, the Hamiltonian is of the form

$$H = -[(p_t/c)^2 - m^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2]^{1/2} - qA_z. \tag{37.2.32}$$

Here we have abandoned the notation (1.1). Instead, z is now a Cartesian coordinate in the longitudinal direction, and we take it to be the *independent* variable. Also, p_t is the negative of the total energy. The vector potential $\boldsymbol{A}$ for a quadruple has an expansion (shown through fourth order) of the form

$$A_x = \frac{g'(z)}{4}(x^3 - xy^2) + \cdots\,, \tag{37.2.33}$$

$$A_y = -\frac{g'(z)}{4}(y^3 - x^2y) + \cdots\,, \tag{37.2.34}$$

$$A_z = -\frac{g(z)}{2}(x^2 - y^2) + \frac{g''(z)}{12}(x^4 - y^4) + \cdots\,. \tag{37.2.35}$$

Here $g(z)$ is the on-axis field gradient, and the quantities $g'(z)$ and $g''(z)$ are derivatives of g with respect to z. We note that once $g(z)$ is specified (and quadrupole symmetry is imposed), then all other terms are determined by the Maxwell equations. Inspection of (2.32) shows that H is unchanged by the substitution $(p_x, p_y) \rightarrow (-p_x, -p_y)$ provided there is also the substitution $(A_x, A_y) \rightarrow (-A_x, -A_y)$. Suppose, for convenience, we choose the z coordinate so that $z = 0$ is at the center of the quadrupole. Then, for what we would intuitively call a symmetric quadrupole in the sense of having end-to-end symmetry, $g(z)$ should be an *even* function of z,

$$g(-z) = g(z). \tag{37.2.36}$$

From (2.36) we deduce that $g''(z)$, $g^{iv}(z)$, etc. are then also *even* functions of z; and $g'(z)$, $g'''(z)$, etc. are *odd* functions of z. It follows from (2.33) through (2.35) that, for a quadrupole with end-to-end symmetry, A_x and A_y are odd functions of z,

$$A_x(x, y, -z) = -A_x(x, y, z), \tag{37.2.37}$$

$$A_y(x, y, -z) = -A_y(x, y, z), \tag{37.2.38}$$

and A_z is an even function,

$$A_z(x, y, -z) = A_z(x, y, z). \tag{37.2.39}$$

[Note that the conditions (2.37) through (2.39) imply for the magnetic field the symmetry relations $B_{x,y}(x, y, -z) = B_{x,y}(x, y, z)$ and $B_z(x, y, -z) = -B_z(x, y, z)$]. From (2.32) and (2.37) through (2.39) we conclude that

$$H^r(-z) = H(z), \tag{37.2.40}$$

and therefore (2.21) is again satisfied. Thus, our intuitive sense of symmetry for a quadrupole coincides with the precise definition (2.21) for the Hamiltonian, which in turn implies the reversal symmetry condition (2.3) for the associated transfer map.

The same can be shown to be true for any multipole, including skew multipoles, with end-to-end symmetry. Finally, the same can be shown to be true for any dipole, with or without additional multipoles intended or otherwise, provided the magnet (including all multipole and fringe fields) has end-to-end symmetry. [Note that the Hamiltonian (2.32) can also be used for curved elements providing the bending angle is less than π. And for larger bend angles an analogous treatment can be formulated using cylindrical coordinates.] That is, in all these cases the relations (2.37) through (2.39) hold, and they imply the relation (2.40). By contrast, the transfer map for a dipole with unequal entry and exit angles, or for a combined function dipole with excessive quadrupole field at one end, will not be reversal symmetric.

Consider next the case of a short RF cavity phased to act as a buncher. Such an idealized cavity can be described by a map $\mathcal{M}$ of the form (1.32) with f given by the relation

$$f = (V/\omega) \cos \omega\tau. \tag{37.2.41}$$

Here V and ω are the voltage and frequency of the cavity. We see that f is *even* in τ, and therefore $f^r = f$. It follows that $\mathcal{M}$ is reversal symmetric. The case of a finite length RF cavity with realistic electromagnetic fields awaits investigation. Finally, it is evident that the transfer map for a short RF cavity phased to operate as an accelerating element rather than a bunching element [$\cos\omega\tau$ replaced by $\cos(\omega\tau+\phi)$] is not reversal symmetric.

We have seen that the transfer maps for many common beam-line elements are reversal symmetric. Suppose that $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are reversal symmetric maps. Then, by (2.1) and (2.2), the maps $\mathcal{M}$ given by products of the form

$$\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_1, \tag{37.2.42}$$

$$\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\mathcal{M}_2\mathcal{M}_1 \tag{37.2.43}$$

will be reversal symmetric. For example, $\mathcal{M}_1$ could be the map for a drift and $\mathcal{M}_2$ could be the map for a quadrupole. It follows that the map for a quadrupole sandwiched between two equal length drifts, given by (2.42), is reversal symmetric. Or $\mathcal{M}_1$ and $\mathcal{M}_3$ could be maps for quadrupoles and $\mathcal{M}_2$ could be the map for a drift. Then (2.43) would be the map for a quadrupole triplet, and we conclude that such maps are reversal symmetric. In the case of solenoids with end-to-end symmetry we could consider maps of the form

$$\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3 \tag{37.2.44}$$

where $\mathcal{M}_1$ is a solenoid map, $\mathcal{M}_2$ is a drift or quadrupole, and $\mathcal{M}_3$ is a map for an identical solenoid except for a reversed field. These maps (2.44) would also be reversal symmetric.

It frequently happens that a given map $\mathcal{M}$ is not reversal symmetric, but is *conjugate* to a map $\mathcal{N}$ that is reversal symmetric. That is, given $\mathcal{M}$, there exists a conjugating map $\mathcal{A}$ such that $\mathcal{M}$ can be written in the form

$$\mathcal{M} = \mathcal{A}^{-1}\mathcal{N}\mathcal{A} \tag{37.2.45}$$

where $\mathcal{N}$ is reversal symmetric. Consider, for example, the case of a FODO cell. Its map $\mathcal{M}$ is given by the product

$$\mathcal{M} = \mathcal{F}\mathcal{O}\mathcal{D}\mathcal{O} \tag{37.2.46}$$

where $\mathcal{F}$ and $\mathcal{D}$ are the maps for (horizontally) *focusing* and *defocussing* quadrupoles and $\mathcal{O}$ is the map for a *drift* (or a reversal symmetric dipole). Evidently $\mathcal{M}$ is not reversal symmetric although, according to our previous discussion, its factors are. However, we know that $\mathcal{F}$ (including all multipole and fringe-field effects) can be written as the product

$$\mathcal{F} = \mathcal{F}_\ell\mathcal{F}_t \tag{37.2.47}$$

where $\mathcal{F}_\ell$ and $\mathcal{F}_t$ are the maps for the leading and trailing halves of the focusing quadrupole. Moreover, there is the relation

$$\mathcal{F}_\ell^r = \mathcal{F}_t. \tag{37.2.48}$$

As a result of (2.47) $\mathcal{M}$ can be written in the form

$$\mathcal{M} = \mathcal{F}_\ell\mathcal{F}_t\mathcal{O}\mathcal{D}\mathcal{O} = \mathcal{F}_\ell\mathcal{F}_t\mathcal{O}\mathcal{D}\mathcal{O}\mathcal{F}_\ell\mathcal{F}_\ell^{-1} = \mathcal{A}^{-1}\mathcal{N}\mathcal{A} \tag{37.2.49}$$

with

$$\mathcal{A}^{-1} = \mathcal{F}_\ell \tag{37.2.50}$$

and

$$\mathcal{N} = \mathcal{F}_t \mathcal{O}\mathcal{D}\mathcal{O}\mathcal{F}_\ell. \tag{37.2.51}$$

Let us compute the reverse of $\mathcal{N}$. We find, using (2.1) and (2.48), the result

$$\mathcal{N}^r = \mathcal{F}_\ell^r \mathcal{O}^r \mathcal{D}^r \mathcal{O}^r \mathcal{F}_t^r = \mathcal{F}_t \mathcal{O}\mathcal{D}\mathcal{O}\mathcal{F}_\ell = \mathcal{N}. \tag{37.2.52}$$

Therefore, $\mathcal{N}$ is reversal symmetric. This example illustrates that the one-turn map for a ring is often reversal symmetric providing the surface of section (location at which the one-turn map is computed) is properly chosen.

As a second illustration, suppose that (through some order) $\mathcal{M}$ can be brought to normal form. (For example, assume that the eigenvalues of the linear part of $\mathcal{M}$ lie on the unit circle and that the corresponding tunes are not resonant through some order.) Then we may take $\mathcal{A}$ to be the normalizing map, and $\mathcal{N}$ to be the normal form of $\mathcal{M}$. The map $\mathcal{N}$ can be written in terms of a single exponent,

$$\mathcal{N} = \exp(:h:). \tag{37.2.53}$$

In the case of a static ring (no RF) h takes the form

$$\begin{aligned} h \;=\; & -(\phi_x/2)(p_x^2 + x^2) - (\phi_y/2)(p_y^2 + y^2) + b^1 p_\tau^2 + b^2 p_\tau^3 + b^3 p_\tau^4 \\ & + a_{xx}(p_x^2 + x^2)^2 + a_{xy}(p_x^2 + x^2)(p_y^2 + y^2) + a_{yy}(p_y^2 + y^2)^2 \\ & + c_x^1 (p_x^2 + x^2) p_\tau + c_x^2 (p_x^2 + x^2) p_\tau^2 \\ & + c_y^1 (p_y^2 + y^2) p_\tau + c_y^2 (p_y^2 + y^2) p_\tau^2 + \cdots . \end{aligned} \tag{37.2.54}$$

We see that h is a power series in the quantities K_x, K_y, and p_τ where

$$K_x = (p_x^2 + x^2), \tag{37.2.55}$$

$$K_y = (p_y^2 + y^2). \tag{37.2.56}$$

The quantities $[\phi/(2\pi)]$ are (fractional) tunes, the quantities b are related to phase slip (momentum compaction), the quantities a are related to anharmonicities, and the quantities c are related to chromaticities. In the dynamic case h takes the form

$$\begin{aligned} h \;=\; & -(\phi_x/2)K_x - (\phi_y/2)K_y - (\phi_\tau/2)K_\tau \\ & + a_{xx}K_x^2 + a_{yy}K_y^2 + a_{\tau\tau}K_\tau^2 + a_{xy}K_xK_y \\ & + a_{x\tau}K_xK_\tau + a_{y\tau}K_yK_\tau + \cdots . \end{aligned} \tag{37.2.57}$$

Now h is a power series in K_x, K_y, and K_τ with K_x, K_y defined as above and

$$K_\tau = (p_\tau^2 + \tau^2). \tag{37.2.58}$$

We see that in both cases $h^r = h$, and therefore $\mathcal{N}$ is reversal symmetric. Of course, the normal form procedure usually leads to divergent series and therefore these normal form results are only approximate in the formal sense of holding to any order, but usually not exactly.

We have defined a map $\mathcal{M}$ to be reversal symmetric if it satisfies the condition (2.3). We now define a map to be reversal *antisymmetric* if it satisfies the condition

$$\mathcal{M}^r = \mathcal{M}^{-1}. \tag{37.2.59}$$

Suppose $\mathcal{M}$ can be written in the single exponent form (1.33), and is reversal antisymmetric. Then we see from (1.46) that the generator f must satisfy the relation

$$f^r = -f. \tag{37.2.60}$$

Note that reversal antisymmetric maps form a subgroup. In particular, if two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are reversal antisymmetric, so is their product $\mathcal{M}_1\mathcal{M}_2$:

$$(\mathcal{M}_1\mathcal{M}_2)^r = \mathcal{M}_2^r\mathcal{M}_1^r = (\mathcal{M}_2^{-1})(\mathcal{M}_1^{-1}) = (\mathcal{M}_1\mathcal{M}_2)^{-1}. \tag{37.2.61}$$

Correspondingly, functions that satisfy (2.60) form a Lie algebra under Poisson bracketing. Indeed, if f and g are *any* two functions, there is the general relation

$$\mathcal{R}[f,g] = \mathcal{R} : f : g = \mathcal{R} : f : \mathcal{R}\mathcal{R}g = - : f^r : g^r = -[f^r, g^r]. \tag{37.2.62}$$

And, if f and g satisfy (2.60), there is the result

$$\mathcal{R}[f,g] = -[f,g]. \tag{37.2.63}$$

Thus, if f and g change sign under reversal, so does their Poisson bracket.

We close this section by showing that under certain circumstances a symplectic map $\mathcal{M}$ can be written uniquely as the product of a reversal symmetric symplectic map and a reversal antisymmetric symplectic map. Given any symplectic map $\mathcal{M}$, define an associated symplectic map $\mathcal{B}$ by the relation

$$\mathcal{B} = \mathcal{M}\mathcal{M}^r. \tag{37.2.64}$$

Then, by (2.1) and (2.2), $\mathcal{B}$ is reversal symmetric. Assume that it is possible to find a *square root* $\mathcal{S}$ for $\mathcal{B}$, and that this square root is also reversal symmetric. That is, assume there is an $\mathcal{S}$ satisfying

$$\mathcal{S}^r = \mathcal{S} \tag{37.2.65}$$

such that

$$\mathcal{B} = \mathcal{S}^2. \tag{37.2.66}$$

For example, suppose that $\mathcal{B}$ can be written in the single exponent form

$$\mathcal{B} = \exp(: f :) \tag{37.2.67}$$

and that f has the property (2.14). Then we may define $\mathcal{S}$ by the relation

$$\mathcal{S} = \exp(: f : /2). \tag{37.2.68}$$

Next define a symplectic map $\mathcal{A}$ by the relation

$$\mathcal{A} = \mathcal{S}^{-1}\mathcal{M}. \tag{37.2.69}$$

Then $\mathcal{A}$ will be reversal antisymmetric. Indeed, we have the results

$$\mathcal{A}^r = \mathcal{M}^r\mathcal{S}^{-1}, \tag{37.2.70}$$

$$\mathcal{A}\mathcal{A}^r = \mathcal{S}^{-1}\mathcal{M}\mathcal{M}^r\mathcal{S}^{-1} = \mathcal{S}^{-1}\mathcal{B}\mathcal{S}^{-1} = \mathcal{S}^{-1}\mathcal{S}^2\mathcal{S}^{-1} = \mathcal{I}, \tag{37.2.71}$$

and therefore $\mathcal{A}$ is reversal antisymmetric. Finally, (2.69) can be rewritten in the form

$$\mathcal{M} = \mathcal{S}\mathcal{A}. \tag{37.2.72}$$

We see that, under the assumptions made, $\mathcal{M}$ can be written as the product of a reversal symmetric map $\mathcal{S}$ and a reversal antisymmetric map $\mathcal{A}$, and we have found expressions for each.

Exercises

37.2.1. Review the first part of Exercise 6.2.9. Let $\mathcal{S}_3$ be the set of all reversal antisymmetric symplectic maps. Show that $\mathcal{S}_3$ forms a group, and that there is the inclusion relation

$$\mathcal{S}_0 \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3. \tag{37.2.73}$$

37.3 General Consequences for Straight and Circular Machines

Suppose $\mathcal{M}$ has a factorization of the form

$$\mathcal{M} = \mathcal{L}\ \exp : f_3 : \exp : f_4 : \cdots . \tag{37.3.1}$$

Here $\mathcal{L}$ is the linear part of $\mathcal{M}$. Then, we find for $\mathcal{M}^r$ the factorization

$$\begin{aligned} \mathcal{M}^r &= \cdots [\exp(: f_4 :)]^r[\exp(: f_3 :)]^r\mathcal{L}^r \\ &= \cdots \exp(: f_4^r :) \exp(: f_3^r :)\mathcal{L}^r. \end{aligned} \tag{37.3.2}$$

Here we have used (2.2) and (1.46). Also, if the action of $\mathcal{L}^r$ is described by the matrix L^r, then the operator relation

$$\mathcal{L}^r = \mathcal{R}\mathcal{L}^{-1}\mathcal{R} \tag{37.3.3}$$

yields the matrix relation

$$L^r = RL^{-1}R \tag{37.3.4}$$

consistent with (1.47). Also, as done before for (1.49), we deduce from the symplectic condition (3.1.2) that (3.3) can also be written in the form

$$L^r = JRL^TJR. \tag{37.3.5}$$

Let us (in the 6×6 case) write L using the standard matrix notation,

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} & L_{16} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} & L_{26} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} & L_{36} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} & L_{46} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} & L_{56} \\ L_{61} & L_{62} & L_{63} & L_{64} & L_{65} & L_{66} \end{pmatrix}. \tag{37.3.6}$$

Then, upon evaluating (3.5), we find that L^r is the matrix

$$L^r = \begin{pmatrix} L_{22} & L_{12} & L_{42} & L_{32} & -L_{62} & -L_{52} \\ L_{21} & L_{11} & L_{41} & L_{31} & -L_{61} & -L_{51} \\ L_{24} & L_{14} & L_{44} & L_{34} & -L_{64} & -L_{54} \\ L_{23} & L_{13} & L_{43} & L_{33} & -L_{63} & -L_{53} \\ -L_{26} & -L_{16} & -L_{46} & -L_{36} & L_{66} & L_{56} \\ -L_{25} & -L_{15} & -L_{45} & -L_{35} & L_{65} & L_{55} \end{pmatrix}. \tag{37.3.7}$$

Now suppose that $\mathcal{M}$ is reversal symmetric so that (2.3) holds. Taken together (2.3) and (3.2) give the result

$$\cdots \exp(:f_4^r:) \exp(:f_3^r:) \mathcal{L}^r = \mathcal{L} \exp(:f_3:) \exp(:f_4:) \cdots . \tag{37.3.8}$$

The right side of (3.8) can be rewritten in the form

$$\begin{aligned} \mathcal{L} \exp(:f_3:) \exp(:f_4:) \cdots &= \mathcal{L} \exp(:f_3:) \exp(:f_4:) \cdots \mathcal{L}^{-1} \mathcal{L} \\ &= \cdots \exp(:g_4:) \exp(:g_3:) \mathcal{L}, \end{aligned} \tag{37.3.9}$$

where the quantities $g_3, g_4, \cdots$ are yet to be determined. Upon comparing (3.8) and (3.9) we see that reversal symmetry requires the relations

$$\mathcal{L}^r = \mathcal{L}, \tag{37.3.10}$$

$$f_m^r = g_m. \tag{37.3.11}$$

The linear part $\mathcal{L}$ of $\mathcal{M}$ must be reversal symmetric, and the Lie generators of the nonlinear part must satisfy (3.11).

To work out the implications of (3.11) for the nonlinear part in more detail, we must find the g_m in terms of the f_m. From (3.9) we conclude that there is the relation

$$\exp(:\mathcal{L} f_3:) \exp(:\mathcal{L} f_4:) \cdots = \cdots \exp(:g_4:) \exp(:g_3:). \tag{37.3.12}$$

Now use the Baker-Campbell-Hausdorff series (8.2.28) and (8.2.29) to combine the exponents on each side of (3.12) and equate terms of like degree. Doing so yields the relations

$$g_3 = \mathcal{L} f_3, \tag{37.3.13}$$

$$g_4 = \mathcal{L} f_4, \tag{37.3.14}$$

$$g_5 = \mathcal{L}f_5 + [\mathcal{L}f_3, \mathcal{L}f_4], \tag{37.3.15}$$

$$g_6 = \mathcal{L}f_6 + [\mathcal{L}f_3, \mathcal{L}f_5] + (1/2)[\mathcal{L}f_3, [\mathcal{L}f_3, \mathcal{L}f_4]], \text{ etc.} \tag{37.3.16}$$

Finally, employ (3.11) in the relations (3.13) through (3.16) to find the results

$$f_3^r = \mathcal{L}f_3, \tag{37.3.17}$$

$$f_4^r = \mathcal{L}f_4, \tag{37.3.18}$$

$$f_5^r = \mathcal{L}f_5 + [\mathcal{L}f_3, \mathcal{L}f_4], \tag{37.3.19}$$

$$f_6^r = \mathcal{L}f_6 + [\mathcal{L}f_3, \mathcal{L}f_5] + (1/2)[\mathcal{L}f_3, [\mathcal{L}f_3, \mathcal{L}f_4]], \text{ etc.} \tag{37.3.20}$$

Let us explore the consequences of reversal symmetry for $\mathcal{L}$. The relation (3.15) implies the associated matrix relation

$$L^r = L. \tag{37.3.21}$$

In view of (3.6) and (3.7), and in the 6×6 case, simple enumeration shows that reversal symmetry places on the entries in L the 15 restrictions listed below:

$$L_{11} = L_{22}, \tag{37.3.22}$$

$$L_{13} = L_{42}, \tag{37.3.23}$$

$$L_{14} = L_{32}, \tag{37.3.24}$$

$$L_{15} = -L_{62}, \tag{37.3.25}$$

$$L_{16} = -L_{52}, \tag{37.3.26}$$

$$L_{23} = L_{41}, \tag{37.3.27}$$

$$L_{24} = L_{31}, \tag{37.3.28}$$

$$L_{25} = -L_{61}, \tag{37.3.29}$$

$$L_{26} = -L_{51}, \tag{37.3.30}$$

$$L_{33} = L_{44}, \tag{37.3.31}$$

$$L_{35} = -L_{64}, \tag{37.3.32}$$

$$L_{36} = -L_{54}, \tag{37.3.33}$$

$$L_{45} = -L_{63}, \tag{37.3.34}$$

$$L_{46} = -L_{53}, \tag{37.3.35}$$

$$L_{55} = L_{66}. \tag{37.3.36}$$

Of course, there are also restrictions on L that follow from the symplectic condition. Suppose that $\mathcal{M}$ is a static (time independent) map. From the work of Section 19.4 we know that in the static case L must have the form

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} & 0 & (\check{L}\delta)_1 \\ L_{21} & L_{22} & L_{23} & L_{24} & 0 & (\check{L}\delta)_2 \\ L_{31} & L_{32} & L_{33} & L_{34} & 0 & (\check{L}\delta)_3 \\ L_{41} & L_{42} & L_{43} & L_{44} & 0 & (\check{L}\delta)_4 \\ -\delta_2 & \delta_1 & -\delta_4 & \delta_3 & 1 & L_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{37.3.37}$$

Recall that we have previously defined a matrix $\hat{L}$ by the rule

$$\hat{L} = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} & 0 & 0 \\ L_{21} & L_{22} & L_{23} & L_{24} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & L_{34} & 0 & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{37.3.38}$$

and that we have used the notation

$$\hat{L} = \begin{pmatrix} \check{L} & 0 \\ 0 & I \end{pmatrix} \tag{37.3.39}$$

where $\check{L}$ is the 4×4 matrix

$$\check{L} = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}. \tag{37.3.40}$$

See (19.*.*). Moreover, we know that the linear map $\hat{\mathcal{L}}$ associated with the matrix $\hat{L}$ is a symplectic map, and hence $\hat{L}$ and $\check{L}$ are symplectic matrices.

Suppose, now, that the static linear map described by (3.37) is also reversal symmetric so that the conditions (3.22) through (3.36) also hold. Then, from the form (3.37) for L, it is evident that the five conditions (3.25), (3.29), (3.32), (3.34), and (3.36) on the matrix elements L_{15} through L_{55} are automatically satisfied. By contrast, the four conditions (3.26), (3.30), (3.33), and (3.35) on the matrix elements L_{16} through L_{46} yield the relations

$$(\check{L}\delta)_1 = -\delta_1, \tag{37.3.41}$$

$$(\check{L}\delta)_2 = \delta_2, \tag{37.3.42}$$

$$(\check{L}\delta)_3 = -\delta_3, \tag{37.3.43}$$

$$(\check{L}\delta)_4 = \delta_4. \tag{37.3.44}$$

In the spirit of (3.40), let us use J and R as given by (3.2.10) and (1.5) to define associated 4×4 matrices $\check{J}$ and $\check{R}$,

$$\check{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{37.3.45}$$

$$\check{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{37.3.46}$$

With the aid of $\check{R}$ the relations (3.41) through (3.44) can be written in the more compact forms

$$\check{L}\delta = -\check{R}\delta \tag{37.3.47}$$

or

$$\check{R}\check{L}\delta = -\delta. \tag{37.3.48}$$

Moreover, the remaining six reversal symmetry relations (3.22), (3.23), (3.24), (3.27), (3.28), and (3.31) for the entries of $\check{L}$ can be written in the compact form

$$\check{L} = \check{J}\check{R}\check{L}^T\check{J}\check{R}. \tag{37.3.49}$$

Finally, the entries of $\check{L}$ must also satisfy the symplectic condition

$$\check{L}^T\check{J}\check{L} = \check{J}. \tag{37.3.50}$$

The imposition of reversal symmetry on $\mathcal{L}$ also implies an associated relation that must be satisfied by the f_1 in (19.*.*). This relation can be obtained by matrix and vector manipulation using (19.*.*) and (3.48). It is also instructive to obtain the condition on f_1 by Lie manipulation starting with (19.*.*). Rewrite (19.*.*) in the explicit form

$$\mathcal{L} = \exp(:\xi p_\tau^2:)\exp(:p_\tau f_1:)\hat{\mathcal{L}}, \tag{37.3.51}$$

and apply reversal to find the equivalent relation

$$\begin{aligned}
\mathcal{L}^r &= \hat{\mathcal{L}}^r\exp(:p_\tau f_1^r:)\exp(:\xi p_\tau^2:) \\
&= \exp(:\xi p_\tau^2:)\hat{\mathcal{L}}^r\exp(:p_\tau f_1^r:)(\hat{\mathcal{L}}^r)^{-1}\hat{\mathcal{L}}^r \\
&= \exp(:\xi p_\tau^2:)\exp(:p_\tau\hat{\mathcal{L}}^r f_1^r:)\hat{\mathcal{L}}^r.
\end{aligned} \tag{37.3.52}$$

Here we have also used (8.2.27). Now require that $\mathcal{L}$ satisfy the reversal symmetry condition (3.10). Comparison of (3.51) and (3.52) then gives the relation

$$\hat{\mathcal{L}}^r = \hat{\mathcal{L}}, \tag{37.3.53}$$

which is equivalent to (3.49), and the relation

$$\hat{\mathcal{L}}^r f_1^r = f_1. \tag{37.3.54}$$

In view of (3.53), the relation (3.54) can also be written in the form

$$f_1^r = \hat{\mathcal{L}}^{-1}f_1, \tag{37.3.55}$$

which is equivalent to (3.47) or (3.48).

37.4 Consequences for some Special Cases

Let us explore the implication of reversal symmetry for some special cases. Again assume that $\mathcal{M}$ is static, and suppose further that $\check{L}$ does not couple the x, p_x, and y, p_y degrees of freedom. In this case L as given by (3.37) takes the form

$$L = \begin{pmatrix} a & b & 0 & 0 & 0 & \check{\delta}_1 \\ c & d & 0 & 0 & 0 & \check{\delta}_2 \\ 0 & 0 & e & f & 0 & \check{\delta}_3 \\ 0 & 0 & g & h & 0 & \check{\delta}_4 \\ -\delta_2 & \delta_1 & -\delta_4 & \delta_3 & 1 & L_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{37.4.1}$$

where

$$\check{\delta}_a = (\check{L}\delta)_a. \tag{37.4.2}$$

And, by combining (3.6), (3.7), and (3.48), L^r takes the form

$$L^r = \begin{pmatrix} d & b & 0 & 0 & 0 & -\delta_1 \\ c & a & 0 & 0 & 0 & \delta_2 \\ 0 & 0 & h & f & 0 & -\delta_3 \\ 0 & 0 & g & e & 0 & \delta_4 \\ -\check{\delta}_2 & -\check{\delta}_1 & -\check{\delta}_4 & -\check{\delta}_3 & 1 & L_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{37.4.3}$$

Imposing the reversal symmetry condition (3.21) yields the relations

$$\check{\delta}_1 = -\delta_1, \tag{37.4.4}$$

$$\check{\delta}_2 = \delta_2, \tag{37.4.5}$$

$$\check{\delta}_3 = -\delta_3, \tag{37.4.6}$$

$$\check{\delta}_4 = \delta_4, \tag{37.4.7}$$

which echo (3.41) through (3.44), and the additional relations

$$a = d, \tag{37.4.8}$$

$$e = h. \tag{37.4.9}$$

Moreover, the symplectic condition (3.50) provides the relations

$$ad - bc = 1, \tag{37.4.10}$$

$$eh - fg = 1. \tag{37.4.11}$$

Suppose we further assume that, say in the x, p_x plane, the system is *imaging* (and therefore $b = 0$) or *telescopic* (and therefore $c = 0$). Then it follows from (4.8) and (4.10) that

$$a = d = \pm 1. \tag{37.4.12}$$

Similar conclusions hold for the y, p_y plane. We have learned that if a system is reversal symmetric and imaging or telescopic, then (up to a sign) the system must have unit magnification or unit "parallelization".

Suppose instead that the eigenvalues of $\check{L}$ lie on the unit circle (but differ from ± 1) and that the x, p_x and y, p_y degrees of freedom are uncoupled. Then $\hat{\mathcal{L}}$ can be written in the form

$$\hat{\mathcal{L}} = \exp(:f_2:) \tag{37.4.13}$$

where

$$\begin{aligned} f_2 = \; & - \; (\phi_x/2)(\beta_x p_x^2 + 2\alpha_x x p_x + \gamma_x x^2) \\ & - \; (\phi_y/2)(\beta_y p_y^2 + 2\alpha_y y p_y + \gamma_y y^2). \end{aligned} \tag{37.4.14}$$

Here $[\phi_x/(2\pi)]$ and $[\phi_y/2\pi]$ are the horizontal and vertical (fractional) tunes, and $\alpha_x, \beta_x, \gamma_x$ and $\alpha_y, \beta_y, \gamma_y$ are the horizontal and vertical Courant-Snyder (Twiss) functions. Then reversal symmetry requires the relation (2.14) from which it follows that

$$\alpha_x = \alpha_y = 0. \tag{37.4.15}$$

Thus, the Courant-Snyder ellipses for both the x and y degrees of freedom are *upright* if $\mathcal{M}$ is reversal symmetric.

37.5 Consequences for Closed Orbit in a Circular Machine

There is a further factorization of a static linear sympletic map $\mathcal{L}$ as given by (19.*.*) that is of use. Let $\mathcal{A}_d$ be the map given by the relation

$$\mathcal{A}_d = \exp(:p_\tau g_1:) \tag{37.5.1}$$

where g_1 is yet to be determined (but does not depend on τ and p_τ). Here we use the subscript d to indicate that $\mathcal{A}_d$ is analogous to $\mathcal{D}$ as given by (19.*.*). Consider the map $\mathcal{N}_d$ given by

$$\mathcal{N}_d = \mathcal{A}_d \mathcal{L} \mathcal{A}_d^{-1}. \tag{37.5.2}$$

From (19.*.*) we have the result

$$\mathcal{N}_d = \mathcal{A}_d \mathcal{C}\mathcal{D}\hat{\mathcal{L}}\mathcal{A}_d^{-1} = \mathcal{C}\mathcal{A}_d\mathcal{D}\hat{\mathcal{L}}\mathcal{A}_d^{-1} = \mathcal{C}\mathcal{A}_d\mathcal{D}\hat{\mathcal{L}}\mathcal{A}_d^{-1}\hat{\mathcal{L}}^{-1}\hat{\mathcal{L}}. \tag{37.5.3}$$

At this point we ask if there is a choice of $\mathcal{A}_d$ such that

$$\mathcal{A}_d\mathcal{D}\hat{\mathcal{L}}\mathcal{A}_d^{-1}\hat{\mathcal{L}}^{-1} = \mathcal{C}(\xi') \tag{37.5.4}$$

where $\mathcal{C}(\xi')$ is a map of the form (19.*.*). With the aid of (5.13.14), (8.2.27) and the Baker-Campbell-Hausdorff series (3.7.34) we find the result

$$\begin{aligned} & \mathcal{A}_d\mathcal{D}\hat{\mathcal{L}}\mathcal{A}_d^{-1}\hat{\mathcal{L}}^{-1} = \\ & \exp(:p_\tau g_1:)\exp(:p_\tau f_1:)\hat{\mathcal{L}}\exp(-:p_\tau g_1:)\hat{\mathcal{L}}^{-1} = \\ & \exp(:p_\tau g_1:)\exp(:p_\tau f_1:)\exp(-:p_\tau \hat{\mathcal{L}} g_1) \\ & = \exp\{:p_\tau(g_1 + f_1 - \hat{\mathcal{L}}g_1):\} \times \\ & \exp\{:(p_\tau^2/2)(-[f_1, \hat{\mathcal{L}}g_1] + [g_1, f_1] - [g_1, \hat{\mathcal{L}}g_1]):\}. \end{aligned} \tag{37.5.5}$$

We see that (5.4) can be achieved providing a g_1 can be found such that

$$(\hat{\mathcal{L}} - \mathcal{I})g_1 = f_1. \tag{37.5.6}$$

Inspection of (3.39) and (3.40) shows that (5.6) can be solved (uniquely) to obtain g_1 in terms of f_1 providing $\check{L}$ as given by (3.40) does not have +1 as an eigenvalue. That is, neither transverse tune is integer. For the record, we also note that when g_1 is specified by (5.6), the map $\mathcal{C}(\xi')$ is given by the relation

$$\xi' = [g_1, f_1]/2. \tag{37.5.7}$$

By combining (5.3) and (5.4), we find the relation

$$\mathcal{N}_d = \mathcal{C}(\eta)\hat{\mathcal{L}} \tag{37.5.8}$$

with $\mathcal{C}(\eta)$ given by the product

$$\mathcal{C}(\eta) = \mathcal{C}(\xi)\mathcal{C}(\xi') = \mathcal{C}(\xi + \xi'). \tag{37.5.9}$$

Finally, solving (5.2) for $\mathcal{L}$ and use of (5.8) yields the factorization

$$\mathcal{L} = \mathcal{C}(\eta)\mathcal{A}_d^{-1}\hat{\mathcal{L}}\mathcal{A}_d. \tag{37.5.10}$$

The quantities η and g_1 appearing in $\mathcal{C}(\eta)$ and $\mathcal{A}_d$ have a physical interpretation. In analogy with (19.*.*), write g_1 in the form

$$g_1 = \Delta_2 x - \Delta_1 p_x + \Delta_4 y - \Delta_3 p_y. \tag{37.5.11}$$

The operator relation (5.10) is equivalent to the matrix relation

$$L = A_d\hat{L}A_d^{-1}C(\eta) = A_dC(\eta)\hat{L}A_d^{-1} \tag{37.5.12}$$

where $C(\eta)$ and A_d are the matrices associated with $\mathcal{C}(\eta)$ and $\mathcal{A}_d$. In analogy with (19.*.*) and (19.*.*) they are given by the equations

$$C(\eta) = C(\xi + \xi'), \tag{37.5.13}$$

$$A_d = D(\Delta), \tag{37.5.14}$$

$$A_d^{-1} = D(-\Delta). \tag{37.5.15}$$

[Note that in writing (5.12) we have employed the fact that $C(\eta)$ commutes with A_d^{-1}, $\hat{L}$, and A_d.] Now let $z^0(p_\tau)$ be the vector with all zero entries save for p_τ in the last entry,

$$z^0(p_\tau) = (0, 0, 0, 0, 0, p_\tau). \tag{37.5.16}$$

It evidently has the properties

$$\hat{L}z^0(p_\tau) = z^0(p_\tau), \tag{37.5.17}$$

$$C(\eta)z^0(p_\tau) = \bar{z}^0(p_\tau) \tag{37.5.18}$$

where $\bar{z}^0(p_\tau)$ has the entries

$$\bar{z}^0(p_\tau) = (0, 0, 0, 0, \eta p_\tau, p_\tau). \tag{37.5.19}$$

Next let $z^c(p_\tau)$ be the vector defined by the relation

$$z^c(p_\tau) = A_d z^c(p_\tau). \tag{37.5.20}$$

Carrying out the indicated multiplication shows that it has the entries

$$z^c(p_\tau) = (\Delta_1 p_\tau, \Delta_2 p_\tau, \Delta_3 p_\tau, \Delta_4 p_\tau, 0, p_\tau). \tag{37.5.21}$$

Also, by construction, it has the property

$$z^0(p_\tau) = A_d^{-1} z^c(p_\tau). \tag{37.5.22}$$

Finally, let us apply L to $z^c(p_\tau)$. From (5.12), (5.17) through (5.20), and (5.22) we find the result

$$\begin{aligned} Lz^c(p_\tau) &= A_d C'' \hat{L} A_d^{-1} z^c(p_\tau) = A_d C'' \hat{L} z^0(p_\tau) \\ &= A_d C'' z^0(p_\tau) = A_d z^0(p_\tau) = \bar{z}^c(p_\tau) \end{aligned} \tag{37.5.23}$$

where $\bar{z}^c(p_\tau)$ is the vector

$$\bar{z}^c(p_\tau) = A_d \bar{z}^0(p_\tau). \tag{37.5.24}$$

From (19.*.*), (5.14), (5.19), and (5.24) we find that $\bar{z}^c(p_\tau)$ has the entries

$$\bar{z}^c(p_\tau) = (\Delta_1 p_\tau, \Delta_2 p_\tau, \Delta_3 p_\tau, \Delta_4 p_\tau, \eta p_\tau, p_\tau). \tag{37.5.25}$$

Comparison of (5.21) and (5.25) shows that $z^c(p_\tau)$ and $\bar{z}^c(p_\tau)$ *agree* in all entries except for the fifth. We conclude that the quantities $\Delta_a p_\tau$, for $a = 1$ to 4, are the transverse phase-space coordinates for the *closed* orbit (in the linear approximation to $\mathcal{M}$ described by $\mathcal{L}$). Thus, the quantities Δ_1 and Δ_3 are dispersions, and Δ_2 and Δ_4 are their momentum counterparts. And, as is evident from the fifth components of $z^c(p_\tau)$ and $\bar{z}^c(p_\tau)$, the quantity ηp_τ is the differential time-of-flight on the closed orbit. Correspondingly, η is the *phase slip* factor.

What can be said about g_1 and $\mathcal{A}_d$ if $\mathcal{L}$ is reversal symmetric? Rewrite (5.6) in the form

$$f_1 = \hat{\mathcal{L}} g_1 - g_1. \tag{37.5.26}$$

Applying reversal produces the equivalent relation [see (1.56)]

$$f_1^r = (\hat{\mathcal{L}}^r)^{-1} g_1^r - g_1^r, \tag{37.5.27}$$

and multiplying both sides of this relation by $\hat{\mathcal{L}}^r$ yields the result

$$\hat{\mathcal{L}}^r f_1^r = g_1^r - \hat{\mathcal{L}}^r g_1^r. \tag{37.5.28}$$

Now suppose that $\mathcal{L}$ is reversal symmetric so that (3.53) and (3.54) hold. Then (5.28) becomes

$$f_1 = g_1^r - \hat{\mathcal{L}} g_1^r, \tag{37.5.29}$$

and this result when combined with (5.26) yields the relation

$$(\hat{\mathcal{L}} - \mathcal{I})g_1 = -(\hat{\mathcal{L}} - \mathcal{I})g_1^r. \tag{37.5.30}$$

We have already assumed that $(\hat{\mathcal{L}} - \mathcal{I})$ is invertible, and therefore (5.30) implies the relation

$$g_1^r = -g_1. \tag{37.5.31}$$

This is the condition for $\mathcal{A}_d$ to be reversal antisymmetric. We have learned that if $\mathcal{L}$ is reversal symmetric, it can be written in the form (5.10) with $\mathcal{A}_d$ being reversal antisymmetric,

$$\mathcal{A}_d^r = \mathcal{A}_d^{-1}. \tag{37.5.32}$$

Moreover, comparison of (5.11) and (5.31) shows that there is the relation

$$\Delta_2 = \Delta_4 = 0 \tag{37.5.33}$$

if $\mathcal{L}$ is reversal symmetric. In view of (5.21), this relation can also be written in the form

$$Rz^c(p_\tau) = z^c(p_\tau). \tag{37.5.34}$$

If $\mathcal{L}$ is reversal symmetric the off-energy closed orbit has no transverse momentum components (at the ring location for which $\mathcal{L}$ is computed).

How do the quantities Δ_a and η vary from place to place around a ring? We will see shortly that η is constant. First let us consider the Δ_a. Suppose that in the linear approximation the one-turn map $\mathcal{L}$ for a ring can be written as the product of $2n$ maps in the form

$$\mathcal{L} = \mathcal{L}_1\mathcal{L}_2 \cdots \mathcal{L}_n\mathcal{L}_{\tilde{n}} \cdots \mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}} \tag{37.5.35}$$

with

$$\mathcal{L}_{\tilde{j}} = \mathcal{L}_j^r. \tag{37.5.36}$$

Figure 5.1 illustrates such a ring. In view of (5.36), we may say that the ring has location 0 as a symmetry point. From (2.15) and (2.16) we see that $\mathcal{L}$ is reversal symmetric, i.e. (3.10) holds. Our task is to compute the closed-orbit quantities Δ_a at various other locations (Poincaré surfaces of section) j and $\tilde{j}$ around the ring.

Introduce the maps $\mathcal{S}^j$ and $\mathcal{S}^{\tilde{j}}$ defined by the rules

$$\mathcal{S}^j = \mathcal{L}_1\mathcal{L}_2 \cdots \mathcal{L}_j, \tag{37.5.37}$$

$$\mathcal{S}^{\tilde{j}} = \mathcal{L}_{\tilde{j}} \cdots \mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}}, \tag{37.5.38}$$

and let S^j and $S^{\tilde{j}}$ be their corresponding matrices. Let z^{cj} be the closed-orbit vector at location j. In analogy to (5.21) we write it in the form

$$z^{cj}(p_\tau) = (\Delta_1^j p_\tau, \Delta_2^j p_\tau, \Delta_3^j p_\tau, \Delta_4^j p_\tau, *, p_\tau) \tag{37.5.39}$$

where the entry labeled $*$ need not concern us. Also, from (5.21) and (5.34), the closed-orbit vector at location 0, denote it by $z^{c0}(p_\tau)$, is given by the relation

$$z^{c0}(p_\tau) = z^c(p_\tau) = (\Delta_1^0 p_\tau, 0, \Delta_3^0 p_\tau, 0, 0, p_\tau) \tag{37.5.40}$$

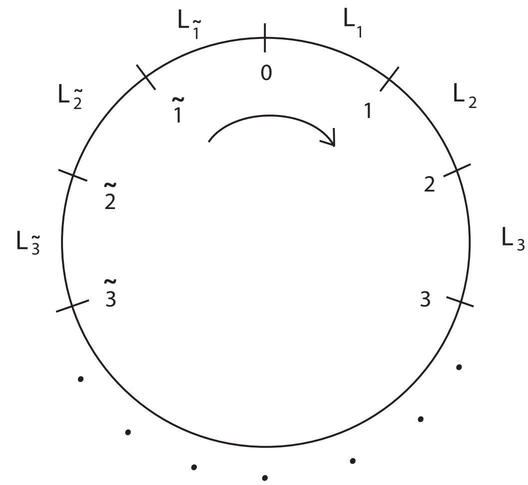

Figure 37.5.1: Schematic drawing of a ring showing the locations (Poincaré surfaces of section) $0; 1, \tilde{1}; 2, \tilde{2}; 3, \tilde{3}$; etc.

with $\Delta_1^0 = \Delta_1$ and $\Delta_3^0 = \Delta_3$. Then, inspection of Figure 5.1 shows that the closed-orbit vector at location j must be the result of propagating the closed-orbit initial conditions from location 0 to location j. That is, there is the vector and matrix relation

$$z^{cj}(p_\tau) = S^j z^{c0}. \tag{37.5.41}$$

Similarly, let $z^{c\tilde{j}}$ be the closed-orbit vector at location $\tilde{j}$, and write it in the form

$$z^{c\tilde{j}}(p_\tau) = (\Delta_1^{\tilde{j}} p_\tau, \Delta_2^{\tilde{j}} p_\tau, \Delta_3^{\tilde{j}} p_\tau, \Delta_4^{\tilde{j}} p_\tau, *, p_\tau). \tag{37.5.42}$$

Then, by following the closed orbit *backwards* from location 0 to location $\tilde{j}$, we see that there is the relation

$$z^{c\tilde{j}} = (S^{\tilde{j}})^{-1} z^{c0}. \tag{37.5.43}$$

We are now prepared to compare z^{cj} and $z^{c\tilde{j}}$. From (5.36) through (5.38) it is easily checked that there is the operator relation

$$(\mathcal{S}^j)^r = \mathcal{S}^{\tilde{j}} \tag{37.5.44}$$

and hence also the corresponding matrix relation

$$(S^j)^r = S^{\tilde{j}}. \tag{37.5.45}$$

It follows that (5.43) can be written in the form

$$\begin{aligned} z^{c\tilde{j}} &= ((S^j)^r)^{-1} z^{c0} = (R(S^j)^{-1}R)^{-1} z^{c0} \\ &= RS^jRz^{c0} = RS^jz^{c0} = Rz^{cj}. \end{aligned} \tag{37.5.46}$$

Here we have also used (1.8), (3.4), and (5.34). From (5.46) we conclude that there are the relations

$$\Delta_1^{\tilde{j}} = \Delta_1^j, \tag{37.5.47}$$

$$\Delta_2^{\tilde{j}} = -\Delta_2^j, \tag{37.5.48}$$

$$\Delta_3^{\tilde{j}} = \Delta_3^j, \tag{37.5.49}$$

$$\Delta_4^{\tilde{j}} = -\Delta_4^j. \tag{37.5.50}$$

We have learned that Δ_1 and Δ_3 are "even" functions of position about the symmetry location 0, and Δ_2 and Δ_4 are "odd" functions.

37.6 Consequences for Courant-Snyder Functions in a Circular Machine

We next turn to the task of determining the behavior of the Courant-Snyder lattice functions (as well as the phase slip η) for a ring with a reversal symmetric one-turn map, and for a ring that also has a symmetry point. Some preparatory work is required. The relation (5.10) can be rewritten in the form

$$\mathcal{L} = \mathcal{A}_d^{-1}[\mathcal{C}(\eta)\hat{\mathcal{L}}]\mathcal{A}_d. \tag{37.6.1}$$

In this form we see that a general $\mathcal{L}$ has been expressed as the similarity transform of the simpler map $\mathcal{C}(\eta)\hat{\mathcal{L}}$. Let us now further simplify $\hat{\mathcal{L}}$ itself. With reference to (3.39) and (3.40), suppose that the eigenvalues of $\check{L}$ lie on the unit circle and are distinct. Then, according to normal form theory, there is a symplectic matrix $\check{A}_b$ such that

$$\check{A}_b^{-1}\check{L}\check{A}_b = \check{N}_b \tag{37.6.2}$$

where $\check{N}_b$ takes the simple (*normal*) form

$$\check{N}_b(\phi_1, \phi_2) = \begin{pmatrix} \cos\phi_1 & \sin\phi_1 & 0 & 0 \\ -\sin\phi_1 & \cos\phi_1 & 0 & 0 \\ 0 & 0 & \cos\phi_2 & \sin\phi_2 \\ 0 & 0 & -\sin\phi_2 & \cos\phi_2 \end{pmatrix}. \tag{37.6.3}$$

Here the quantities ϕ are the (eigen) phase advances of $\check{L}$ and the $[\phi/(2\pi)]$ are the (eigen) tunes. The subscript b indicates the connection that $\check{A}_b$ and $\check{N}_b$ have with *betatron* oscillations.

In the spirit of (3.39), let $\hat{A}_b$ and $\hat{N}_b$ denote the matrices

$$\hat{A}_b = \begin{pmatrix} \check{A}_b & 0 \\ 0 & I \end{pmatrix}, \tag{37.6.4}$$

$$\hat{N}_b = \begin{pmatrix} \check{N}_b & 0 \\ 0 & I \end{pmatrix}, \tag{37.6.5}$$

and let $\mathcal{A}_b$ and $\mathcal{N}_b$ denote the corresponding symplectic maps. Then the matrix relation (6.2) is equivalent to the map relation

$$\mathcal{A}_b\hat{\mathcal{L}}\mathcal{A}_b^{-1} = \mathcal{N}_b. \tag{37.6.6}$$

We note that $\mathcal{N}_b$ can be written in the Lie form

$$\mathcal{N}_b = \exp(: h_2 :) \tag{37.6.7}$$

with

$$h_2 = -(\phi_1/2)(p_x^2 + x^2) - (\phi_2/2)(p_y^2 + y^2). \tag{37.6.8}$$

The relation (6.6) can also be written in the form

$$\hat{\mathcal{L}} = \mathcal{A}_b^{-1}\mathcal{N}_b\mathcal{A}_b, \tag{37.6.9}$$

and then inserted into (6.1) to give the relation

$$\mathcal{L} = \mathcal{A}_d^{-1}\mathcal{C}(\eta)\mathcal{A}_b^{-1}\mathcal{N}_b\mathcal{A}_b\mathcal{A}_d. \tag{37.6.10}$$

From (19.*.*) and (6.4) it is evident that $\mathcal{C}(\eta)$ and $\mathcal{A}_b^{-1}$ commute. Therefore (6.10) can be cast in the still simpler form

$$\mathcal{L} = \mathcal{A}^{-1}\mathcal{N}\mathcal{A} \tag{37.6.11}$$

where

$$\mathcal{A} = \mathcal{A}_b\mathcal{A}_d, \tag{37.6.12}$$

$$\mathcal{N} = \mathcal{C}(\eta)\mathcal{N}_b. \tag{37.6.13}$$

Inspection of (19.*.*), (5.13), (6.7), and (6.8) shows that $\mathcal{N}$ can be written in the form

$$\mathcal{N} = \exp(: h_2 :) \tag{37.6.14}$$

with

$$h_2 = -(\phi_1/2)(p_x^2 + x^2) - (\phi_2/2)(p_y^2 + y^2) - (\eta/2)p_\tau^2. \tag{37.6.15}$$

Let us again explore the effect of reversal. Applying reversal to (6.6) produces the equivalent result

$$(\mathcal{A}_b^{-1})^r\hat{\mathcal{L}}^r\mathcal{A}_b^r = \mathcal{N}_b^r, \tag{37.6.16}$$

which can be rewritten in the form

$$(\mathcal{A}_b^r)^{-1}\hat{\mathcal{L}}^r\mathcal{A}_b^r = \mathcal{N}_b. \tag{37.6.17}$$

Here we have used (2.2) and the fact that $\hat{\mathcal{N}_b}$ is manifestly reversal symmetric. Let us solve (6.16) and (6.17) for $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}^r$ to yield the relations

$$\hat{\mathcal{L}} = \mathcal{A}_b^{-1}\mathcal{N}_b\mathcal{A}_b, \tag{37.6.18}$$

$$\hat{\mathcal{L}}^r = \mathcal{A}_b^r\mathcal{N}_b(\mathcal{A}_b^r)^{-1}. \tag{37.6.19}$$

Now suppose that $\mathcal{L}$ is reversal symmetric, in which case $\hat{\mathcal{L}}$ is also reversal symmetric. See (3.53). Then comparison of (6.18) and (6.19) gives the relation

$$\mathcal{A}_b^{-1}\mathcal{N}_b\mathcal{A}_b = \mathcal{A}_b^r\mathcal{N}_b(\mathcal{A}_b^r)^{-1}, \tag{37.6.20}$$

which can be rewritten in the forms

$$(\mathcal{A}_b^r)^{-1}\mathcal{A}_b^{-1}\mathcal{N}_b\mathcal{A}_b\mathcal{A}_b^r = \mathcal{N}_b, \tag{37.6.21}$$

$$(\mathcal{A}_b\mathcal{A}_b^r)^{-1}\mathcal{N}_b(\mathcal{A}_b\mathcal{A}_b^r) = \mathcal{N}_b, \tag{37.6.22}$$

$$\mathcal{N}_b(\phi_1,\phi_2)(\mathcal{A}_b\mathcal{A}_b^r) = (\mathcal{A}_b\mathcal{A}_b^r)\mathcal{N}_b(\phi_1,\phi_2). \tag{37.6.23}$$

In the last form we have made explicit the dependence of $\mathcal{N}_b$ on ϕ_1 and ϕ_2.

The relation (6.23) states that the product $(\mathcal{A}_b\mathcal{A}_b^r)$ *commutes* with $\mathcal{N}_b$. Inspection of (6.7) and (6.8) shows that (6.23) can hold only if the product $(\mathcal{A}_b\mathcal{A}_b^r)$ is of the form

$$\mathcal{A}_b\mathcal{A}_b^r = \mathcal{N}_b(\phi_1',\phi_2') \tag{37.6.24}$$

for some values of ϕ_1' and ϕ_2'. We now show that, without loss of the desired relations (6.6) through (6.8), we can require the condition

$$\mathcal{A}_b\mathcal{A}_b^r = \mathcal{I} \text{ or } \mathcal{A}_b^r = \mathcal{A}_b^{-1}. \tag{37.6.25}$$

That is, if $\mathcal{L}$ is reversal symmetric, there is a reversal antisymmetric $\mathcal{A}_b$ that accomplishes the desired goals (6.6) through (6.8). Indeed, given an $\mathcal{A}_b$ that satisfies (6.24), define an associated map $\bar{\mathcal{A}}_b$ by the rule

$$\bar{\mathcal{A}}_b = \mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)\mathcal{A}_b. \tag{37.6.26}$$

From (6.6) we see that it satisfies the desired normalizing relation,

$$\begin{aligned}\bar{\mathcal{A}}_b\hat{\mathcal{L}}\bar{\mathcal{A}}_b^{-1} &= \mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)\mathcal{A}_b\hat{\mathcal{L}}\mathcal{A}_b^{-1}[\mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)]^{-1}\\ &= \mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)\mathcal{N}_b(\phi_1,\phi_2)[\mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)]^{-1}\\ &= \mathcal{N}_b(\phi_1,\phi_2).\end{aligned} \tag{37.6.27}$$

Also, it has the desired product relation,

$$\begin{aligned}\bar{\mathcal{A}}_b\bar{\mathcal{A}}_b^r &= \mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)\mathcal{A}_b\mathcal{A}_b^r\mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)\\ &= \mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)\mathcal{N}_b(\phi_1',\phi_2')\mathcal{N}_b(-\phi_1'/2,-\phi_2'/2)\\ &= \mathcal{I}.\end{aligned} \tag{37.6.28}$$

The generalized Courant-Snyder quadratic invariants I_x and I_y (which include the possibility of coupling between the x, p_x and y, p_y degrees of freedom) are defined by the rules

$$I_x = \mathcal{A}_b^{-1}(p_x^2 + x^2), \tag{37.6.29}$$

$$I_y = \mathcal{A}_b^{-1}(p_y^2 + y^2). \tag{37.6.30}$$

In the no-coupling case they take the form

$$I_x = \beta_x p_x^2 + 2\alpha_x x p_x + \gamma_x x^2, \text{ etc.} \tag{37.6.31}$$

The relations (6.29) and (6.30) have the reversed counterparts

$$I_x^r = \mathcal{A}_b^r(p_x^2 + x^2), \text{ etc.} \tag{37.6.32}$$

Here we have used (1.56). Now assume that $\mathcal{L}$ is reversal symmetric so that $\mathcal{A}_b$ can be taken to satisfy (6.25). In this case we find the relations

$$I_x^r = \mathcal{A}_b^r(p_x^2 + x^2) = \mathcal{A}_b^{-1}(p_x^2 + x^2) = I_x, \text{ etc.} \tag{37.6.33}$$

Thus, if $\mathcal{L}$ is reversal symmetric, the generalized Courant-Snyder invariants (at the location for which $\mathcal{L}$ is computed) are free of terms that are *odd* in the momenta.

We now explore how I_x, I_y, and η vary about a ring that has a symmetry point. Refer again to Figure 5.1. Let $\mathcal{L}^0$ be the one-turn map starting at location 0,

$$\mathcal{L}^0 = \mathcal{L}_1\mathcal{L}_2 \cdots \mathcal{L}_n\mathcal{L}_{\tilde{n}} \cdots \mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}} = \mathcal{L}. \tag{37.6.34}$$

Similarly, let $\mathcal{L}^1$ and $\mathcal{L}^{\tilde{1}}$ be the maps starting at the locations 1 and $\tilde{1}$. For these maps we find the results

$$\begin{aligned}
\mathcal{L}^1 &= \mathcal{L}_2\mathcal{L}_3 \cdots \mathcal{L}_n\mathcal{L}_{\tilde{n}} \cdots \mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}}\mathcal{L}_1 \\
&= \mathcal{L}_1^{-1}\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 \cdots \mathcal{L}_n\mathcal{L}_{\tilde{n}} \cdots \mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}}\mathcal{L}_1 \\
&= \mathcal{L}_1^{-1}\mathcal{L}\mathcal{L}_1,
\end{aligned} \tag{37.6.35}$$

$$\begin{aligned}
\mathcal{L}^{\tilde{1}} &= \mathcal{L}_{\tilde{1}}\mathcal{L}_1\mathcal{L}_2 \cdots \mathcal{L}_n\mathcal{L}_{\tilde{n}} \cdots \mathcal{L}_{\tilde{3}}\mathcal{L}_{\tilde{2}} \\
&= \mathcal{L}_{\tilde{1}}\mathcal{L}_1\mathcal{L}_2 \cdots \mathcal{L}_n\mathcal{L}_{\tilde{n}} \cdots \mathcal{L}_{\tilde{1}}\mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}}\mathcal{L}_{\tilde{1}}^{-1} \\
&= \mathcal{L}_{\tilde{1}}\mathcal{L}\mathcal{L}_{\tilde{1}}^{-1}.
\end{aligned} \tag{37.6.36}$$

In the same way we find for $\mathcal{L}^2$ and $\mathcal{L}^{\tilde{2}}$ the results

$$\mathcal{L}^2 = \mathcal{L}_2^{-1}\mathcal{L}_1^{-1}\mathcal{L}\mathcal{L}_1\mathcal{L}_2 = (\mathcal{L}_1\mathcal{L}_2)^{-1}\mathcal{L}(\mathcal{L}_1\mathcal{L}_2), \tag{37.6.37}$$

$$\mathcal{L}^{\tilde{2}} = \mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}}\mathcal{L}\mathcal{L}_{\tilde{1}}^{-1}\mathcal{L}_{\tilde{2}}^{-1} = (\mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}})\mathcal{L}(\mathcal{L}_{\tilde{2}}\mathcal{L}_{\tilde{1}})^{-1}. \tag{37.6.38}$$

Finally, in terms of the maps $\mathcal{S}^j$ and $\mathcal{S}^{\tilde{j}}$ defined by (5.37) and (5.38), the relations (6.34) through (6.38) etc. take the general form

$$\mathcal{L}^j = (\mathcal{S}^j)^{-1}\mathcal{L}\mathcal{S}^j, \tag{37.6.39}$$

$$\mathcal{L}^{\tilde{j}} = (\mathcal{S}^{\tilde{j}})\mathcal{L}(\mathcal{S}^{\tilde{j}})^{-1}. \tag{37.6.40}$$

Insert the representation (6.11) into (6.39) and (6.40) to obtain the relations

$$\mathcal{L}^j = (\mathcal{S}^j)^{-1}\mathcal{A}^{-1}\mathcal{N}\mathcal{A}(\mathcal{S}^j), \tag{37.6.41}$$

$$\mathcal{L}^{\tilde{j}} = (\mathcal{S}^{\tilde{j}})\mathcal{A}^{-1}\mathcal{N}\mathcal{A}(\mathcal{S}^{\tilde{j}})^{-1}. \tag{37.6.42}$$

These relations can be rewritten in the form

$$\mathcal{L}^j = (\mathcal{A}^j)^{-1}\mathcal{N}^j\mathcal{A}^j, \tag{37.6.43}$$

$$\mathcal{L}^{\tilde{j}} = (\mathcal{A}^{\tilde{j}})^{-1}\mathcal{N}^{\tilde{j}}\mathcal{A}^{\tilde{j}}, \tag{37.6.44}$$

where

$$\mathcal{A}^j = \mathcal{A}\mathcal{S}^j, \tag{37.6.45}$$

$$\mathcal{N}^j = \mathcal{N}, \tag{37.6.46}$$

$$\mathcal{A}^{\tilde{j}} = \mathcal{A}(\mathcal{S}^{\tilde{j}})^{-1}, \tag{37.6.47}$$

$$\mathcal{N}^{\tilde{j}} = \mathcal{N}. \tag{37.6.48}$$

Comparison of (6.46) and (6.48) immediately gives the result

$$\mathcal{N}^{\tilde{j}} = \mathcal{N}^j, \tag{37.6.49}$$

from which we conclude that ϕ_1, ϕ_2 and η are *global* properties of a ring. [See (6.15).] Their values are independent of the choice of the surface of section. Note that in arriving at this conclusion no assumptions were required about reversal symmetry.

Next, in analogy to (6.12), make the factorizations

$$\mathcal{A}^j = \mathcal{A}_b^j \mathcal{A}_d^j, \tag{37.6.50}$$

$$\mathcal{A}^{\tilde{j}} = \mathcal{A}_b^{\tilde{j}} \mathcal{A}_d^{\tilde{j}}. \tag{37.6.51}$$

Here the maps $\mathcal{A}_d^j$ and $\mathcal{A}_d^{\tilde{j}}$ are defined by the analogs of (5.1) and (5.11):

$$\mathcal{A}_d^j = \exp(: p_\tau g_1^j :) \tag{37.6.52}$$

with

$$g_1^j = \Delta_2^j x - \Delta_1^j p_x + \Delta_4^j y - \Delta_3^j p_y; \tag{37.6.53}$$

$$\mathcal{A}_d^{\tilde{j}} = \exp(: p_\tau g_1^{\tilde{j}} :) \tag{37.6.54}$$

with

$$g^{\tilde{j}} = \Delta_2^{\tilde{j}} x - \Delta_1^{\tilde{j}} p_x + \Delta_4^{\tilde{j}} y - \Delta_3^{\tilde{j}} p_y. \tag{37.6.55}$$

So far we have not necessarily assumed that $\mathcal{L}$ is reversal symmetric and that the ring has 0 as a symmetry point. Do so now. Then (5.47) through (5.50) can be used to rewrite (6.53) in the form

$$g_1^j = -\Delta_2^{\tilde{j}} x - \Delta_1^{\tilde{j}} p_x - \Delta_4^{\tilde{j}} y - \Delta_3^{\tilde{j}} p_y. \tag{37.6.56}$$

Now we see that there is the relation

$$(g_1^j)^r = -g_1^{\tilde{j}}, \tag{37.6.57}$$

from which it follows that

$$(\mathcal{A}_d^j)^r = (\mathcal{A}_d^{\tilde{j}})^{-1}. \tag{37.6.58}$$

We also know that, since $\mathcal{L}$ is reversal symmetric, the maps $\mathcal{A}_d$ and $\mathcal{A}_b$ are reversal antisymmetric. See (5.32) and (6.25). It follows from the group property for reversal antisymmetric maps that $\mathcal{A}$ as given by (6.12) is also reversal antisymmetric,

$$\mathcal{A}^r = \mathcal{A}^{-1}. \tag{37.6.59}$$

We are now prepared to study the relation between $\mathcal{A}_b^j$ and $\mathcal{A}_b^{\tilde{j}}$. Solving (6.50) and (6.51) for $\mathcal{A}_b^j$ and $\mathcal{A}_b^{\tilde{j}}$ and using (6.45) and (6.47) give the results

$$\mathcal{A}_b^j = \mathcal{A}^j(\mathcal{A}_d^j)^{-1} = \mathcal{A}\mathcal{S}^j(\mathcal{A}_d^j)^{-1}, \tag{37.6.60}$$

$$\mathcal{A}_b^{\tilde{j}} = \mathcal{A}^{\tilde{j}}(\mathcal{A}_d^{\tilde{j}})^{-1} = \mathcal{A}(\mathcal{S}^{\tilde{j}})^{-1}(\mathcal{A}_d^{\tilde{j}})^{-1}. \tag{37.6.61}$$

Let us compute the map $((\mathcal{A}_b^j)^r)^{-1}$. From (6.60) we find the results

$$(\mathcal{A}_b^j)^r = ((\mathcal{A}_d^j)^{-1})^r(\mathcal{S}^j)^r\mathcal{A}^r, \tag{37.6.62}$$

$$\begin{aligned}((\mathcal{A}_b^j)^r)^{-1} &= (\mathcal{A}^r)^{-1}((\mathcal{S}^j)^r)^{-1}(\mathcal{A}_d^j)^r \\ &= \mathcal{A}(\mathcal{S}^{\tilde{j}})^{-1}(\mathcal{A}_d^{\tilde{j}})^{-1} \\ &= \mathcal{A}_b^{\tilde{j}}. \end{aligned}\tag{37.6.63}$$

Here we have used (5.44), (6.58), and (6.59). Note that (6.63) can also be written in the form

$$(\mathcal{A}_b^j)^r = (\mathcal{A}_b^{\tilde{j}})^{-1}, \tag{37.6.64}$$

which is analogous to the relation (6.58) for $(\mathcal{A}_d^j)^r$.

We are ready for the coup de maître. The horizontal Courant-Snyder (eigen) invariants at the locations j and $\tilde{j}$ are given by the expressions

$$I_x^j = (\mathcal{A}_b^j)^{-1}(p_x^2 + x^2), \tag{37.6.65}$$

$$I_x^{\tilde{j}} = (\mathcal{A}_b^{\tilde{j}})^{-1}(p_x^2 + x^2); \tag{37.6.66}$$

and there are analogous expressions for their vertical counterparts. Now apply the reversal operation to (6.65). Doing so gives the result

$$(I_x^j)^r = (\mathcal{A}_b^j)^r(p_x^2 + x^2) = (\mathcal{A}_b^{\tilde{j}})^{-1}(p_x^2 + x^2) = I_x^{\tilde{j}}. \tag{37.6.67}$$

Similarly, there is the relation

$$(I_y^j)^r = I_y^{\tilde{j}}. \tag{37.6.68}$$

Consider the coefficients of the monomials in I_x^j or I_y^j that are *even* under $\mathcal{R}$. The relations (6.67) and (6.68) show that these coefficients are also *even* functions of position about the symmetry location 0. For example, in the no-coupling case (6.31), there are the relations

$$\beta_x^{\tilde{j}} = \beta_x^j, \tag{37.6.69}$$

$$\gamma_x^{\tilde{j}} = \gamma_x^j. \tag{37.6.70}$$

Next consider the coefficients of the monomials in I_x^j or I_y^j that are *odd* under $\mathcal{R}$. The relations (6.67) and (6.68) show that these coefficients are also *odd* functions of position about the symmetry location 0. For example, in the non-coupling case (6.31) there is the relation

$$\alpha_x^{\tilde{j}} = -\alpha_x^j. \tag{37.6.71}$$

37.7 Some Nonlinear Consequences

So far we have mostly explored the consequences of reversal symmetry for $\mathcal{L}$, the linear part of $\mathcal{M}$. We now explore some of the consequences of reversal symmetry for the full map $\mathcal{M}$. One line of inquiry would be to generalize the results of the previous Sections 5 and 6 to include nonlinear terms. Equations (5.25) and (5.39) give the linear terms of a power series (in p_τ) expansion of $z^{cj}(p_\tau)$. The higher order terms in this expansion could be found and their dependence on the location j could be explored. Similarly a full normalization of $\mathcal{M}$ [analogous to that given in (6.11) for its linear part $\mathcal{L}$] could be found. The *betatron* part of the associated normalizing map could then be used in (6.65), etc., to find the generalized Courant-Snyder invariants that take into account all nonlinear effects through any desired order. These invariants contain, in addition to quadratic monomials, monomials of degree 3 and higher. When considered as functions of location j, the coefficients of these monomials yield *nonlinear* lattice functions. Just how these coefficients depend on j could also be explored.

We will leave these generalizations to the reader. Instead, we will devote this section to the exploration of how reversal symmetry affects the dynamic aperture of a ring including the location of fixed points, and how it limits the kind of nonlinearities that can occur in $\mathcal{M}$.

Consider the map $\mathcal{M}$ on two-dimensional phase space given by the product

$$\mathcal{M}(\theta) = \exp[-(\phi/4) :p^2 + q^2:] \exp(:q^3:) \exp[-(\phi/4) :p^2 + q^2 :]. \tag{37.7.1}$$

As described in Section 1.2.3, this map consists of a $\phi/2$ phase advance, followed by a sextupole kick, followed again by a $\phi/2$ phase advance. It may be viewed as describing horizontal betatron motion in an idealized storage ring with a single thin *sextupole* insertion S, and an *observation* point O (Poincaré surface of section) located diametrically across the ring from the sextupole insertion. Recall Figure 1.2.8. We verified in Section 18.8.4 that this map is a variant of the usual Hénon map, and differs from it only by a linear change of variables. Now we note that $\mathcal{M}$ as given by (7.1) is reversal symmetric.

Figure 7.1, which is a replication of Figure 1.2.9, shows the dynamic aperture for our variant of the Hénon map for the case $\phi/(2\pi) = .22$. Points in the black area of the q, p (mapping) plane remain there under repeated application of the map. [Actually, the points shown remain there for at least 10,000 iterations ($\mathcal{M}^n$ with $n \leq 10,000$).] By contrast, any point launched in the white area eventually iterates away to infinity. Inspection of the figure suggests symmetry about the q axis. That is, all features of the figure are invariant under reversal.

By construction, the map sends the origin into itself (the origin is a fixed point of $\mathcal{M}$), and the origin is unchanged under reversal. Not shown, because it is unstable and also outside the viewing window, there is also a fixed point of $\mathcal{M}$ at $p = 0$ and $q = .7158$, and this point is also unchanged under reversal. Next observe that there are 5 islands. They surround 5 fixed points of the map $\mathcal{M}^5$. [Note that the tune $\phi/(2\pi) = .22$ is close to 1/5.] These fixed points appear to be located symmetrically about the q axis. Also, each island is surrounded by 6 smaller islands. These smaller islands surround fixed points of $\mathcal{M}^{30}$, and these fixed points appear to be located symmetrically about the q axis. Finally, the whole

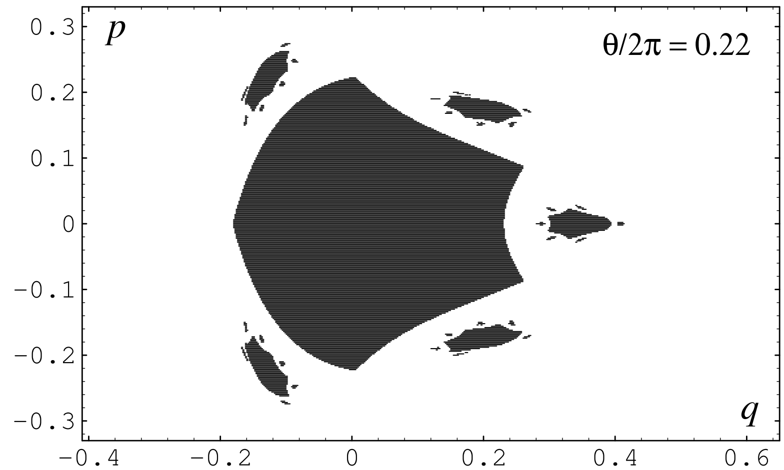

Figure 37.7.1: The dynamic aperture of the Hénon map for the case $\phi/(2\pi) = 0.22$.

dynamic aperture for our variant of the Hénon map appears to be symmetrical about the q axis. That is, the dynamic aperture appears to be invariant under reversal.

To explore these conjectures, let us first think about fixed points of $\mathcal{M}$ and its powers. Suppose $\mathcal{M}$ is a general map in any number of phase-space dimensions, and suppose z^f is a *fixed* point,

$$\mathcal{M}z^f = z^f. \tag{37.7.2}$$

Suppose also that $\mathcal{M}$ is reversal symmetric so that (2.5) holds. Applying $\mathcal{RMR}$ to both sides of (7.2) yields the result

$$\mathcal{RMRM}z^f = \mathcal{RMR}z^f. \tag{37.7.3}$$

Use of (2.5) in (7.3) gives the relation

$$z^f = \mathcal{RMR}z^f, \tag{37.7.4}$$

which, in view of (1.21), can be rewritten as

$$\mathcal{M}(\mathcal{R}z^f) = (\mathcal{R}z^f). \tag{37.7.5}$$

We see that if z^f is a fixed point of $\mathcal{M}$, so is the point $\mathcal{R}z^f$. An analogous result holds for powers of $\mathcal{M}$ because $\mathcal{M}^n$ will be reversal symmetric when $\mathcal{M}$ is reversal symmetric: If z^f is a fixed point of $\mathcal{M}^n$, so is the point $\mathcal{R}z^f$.

We define a fixed point z^f to be *symmetric* if it satisfies the relation

$$\mathcal{R}z^f = z^f. \tag{37.7.6}$$

For example, for the Hénon map of Figure 7.1, the origin, and the point $(q,p) = (.7158, 0)$ not shown, are symmetric fixed points of $\mathcal{M}$; and $(q,p) = (.3458, 0)$ is a symmetric fixed point of $\mathcal{M}^5$. From appearances there are two symmetric fixed points of $\mathcal{M}^{30}$: those on the q axis and surrounding the fixed point of $\mathcal{M}^5$ also on the q axis. The discovery (or ruling out the existence) of symmetric fixed points is easier than the discovery of general fixed points. Let Fix $(\mathcal{R})$ be the set of points z satisfying

$$\mathcal{R}z = z. \tag{37.7.7}$$

From the definition (1.2) and (1.3) of $\mathcal{R}$ it is evident that (for a 6-dimensional phase space) the set Fix $(\mathcal{R})$ is 3 dimensional. [In general, if the dimension of phase space is $2m$, the dimension of Fix $(\mathcal{R})$ is m.] Consequently, the dimension of the space to be searched to find a symmetric fixed point is only half as large as the dimension of the full phase space that must be searched to find a general fixed point.

We next remark that we have found that $\mathcal{R}$ and $\mathcal{R}\mathcal{M}^k$ and $\mathcal{M}^k\mathcal{R}$ (for any k) are all involutions. See (2.8) and (2.9). Moreover, they are all antisymplectic. We have also seen that Fix $(\mathcal{R})$ is m dimensional. Based on a theorem of *Bochner* and *Montgomery*, it can be shown that the sets Fix $(\mathcal{R}\mathcal{M}^k)$ and Fix $(\mathcal{M}^k\mathcal{R})$ are also m dimensional if phase space is $2m$ dimensional. That is, the set of points obeying

$$\mathcal{R}\mathcal{M}^k z = z \tag{37.7.8}$$

is m dimensional for any value of k, and so is the set of points obeying

$$\mathcal{M}^k\mathcal{R}z = z. \tag{37.7.9}$$

See References 19, 22, and 23 in the Bibliography at the end this chapter.

Suppose that z^f is the fixed point of some power $\mathcal{M}$, say $\mathcal{M}^n$, but not of some lower power. Suppose that z^f is also symmetric. Then more can be said. For example, suppose $n = 2$. Then we have the relations

$$\mathcal{M}^2 z^f = z^f, \tag{37.7.10}$$

$$\mathcal{M}^2\mathcal{M}z^f = \mathcal{M}\mathcal{M}^2 z^f = \mathcal{M}z^f. \tag{37.7.11}$$

Thus z^f and $\mathcal{M}z^f$ are *two distinct* fixed points of $\mathcal{M}^2$. By assumption z^f is symmetric. What about $\mathcal{M}z^f$? We have the relations

$$\mathcal{R}\mathcal{M}z^f = \mathcal{R}\mathcal{M}\mathcal{R}\mathcal{R}z^f = \mathcal{M}^{-1}z^f = \mathcal{M}^{-1}\mathcal{M}^2 z^f = \mathcal{M}z^f. \tag{37.7.12}$$

Here we have used (1.21), (1.40), (7.6), and (7.10). We see that $\mathcal{M}z^f$ is also a symmetric fixed point. It is easy to show that this result can be generalized to the case of any even n: If z^f is a symmetric fixed point of $\mathcal{M}^n$, then $\mathcal{M}^{n/2}z^f$ is also a symmetric fixed point of $\mathcal{M}^n$. Thus, symmetric fixed points of $\mathcal{M}^n$ occur in pairs when n is even. As an example, we have now proved that the two fixed points of $\mathcal{M}^{30}$ that appear to lie on the q axis in Figure 5.1 are indeed symmetric.

The case of odd n is a bit more complicated. Suppose for example that $n = 3$. Then, if z^f is a fixed point of $\mathcal{M}^3$, so are the two other points $\mathcal{M}z^f$ and $\mathcal{M}^2 z^f$. By assumption z^f is symmetric. For $\mathcal{M}z^f$ and $\mathcal{M}^2 z^f$ we find the following results:

$$\mathcal{R}\mathcal{M}z^f = \mathcal{R}\mathcal{M}\mathcal{R}\mathcal{R}z^f = \mathcal{M}^{-1}z^f \tag{37.7.13}$$

from which it follows that

$$(\mathcal{M}^2\mathcal{R})(\mathcal{M}z^f) = (\mathcal{M}z^f); \tag{37.7.14}$$

$$\mathcal{RM}^2 z^f = \mathcal{RM}^2\mathcal{RR}z^f = \mathcal{M}^{-2}z^f \tag{37.7.15}$$

from which it follows that

$$(\mathcal{MR})(\mathcal{M}^2 z^f) = \mathcal{M}^{-1}z^f = \mathcal{M}^{-1}\mathcal{M}^3 z^f = (\mathcal{M}^2 z^f). \tag{37.7.16}$$

We know that $\mathcal{R}$ and $\mathcal{MR}$ and $\mathcal{M}^2\mathcal{R}$ are involutions. Thus, the relations (7.6), (7.14), and (7.16) are all analogous: Some particular fixed point is invariant under some particular involution.

Finally, suppose the set of points that satisfies (7.9) for some $k = k_1$ *intersects* the set of points that satisfies (7.9) for some other $k = k_2$. That is, suppose there is a common point w such that

$$\mathcal{M}^{k_1}\mathcal{R}w = w, \tag{37.7.17}$$

$$\mathcal{M}^{k_2}\mathcal{R}w = w. \tag{37.7.18}$$

Then we have the relation

$$\mathcal{M}^{k_2}w = \mathcal{M}^{k_2}\mathcal{M}^{k_1}\mathcal{R}w = \mathcal{M}^{k_1}\mathcal{M}^{k_2}\mathcal{R}w = \mathcal{M}^{k_1}w, \tag{37.7.19}$$

and therefore

$$\mathcal{M}^{k_1-k_2}w = w. \tag{37.7.20}$$

We have found a fixed point, namely w, of the map $\mathcal{M}^{k_1-k_2}$! Note that this fixed point need not be symmetric. For some problems it is possible to construct the m-dimensional manifolds Fix $(\mathcal{M}^k\mathcal{R})$ that satisfy (7.9) for various values of k, and then determine their intersections to find fixed points of $\mathcal{M}^n$. See References 12 and 19 in the Bibliography at the end of this chapter.

So far we have studied fixed points. Next consider general points. What can be said about symmetry for them? We can think about this question for a general map $\mathcal{M}$ in any number of phase-space dimensions as follows: Suppose $\mathcal{M}$ sends the origin into itself. Let N be some large integer, and let Σ be some set in phase space such that all the points $\mathcal{M}^n z$ with $z \in \Sigma$ and $n \in [1, 2N]$ have some desirable property such as being near the origin. Let Γ be the set $\mathcal{M}^N\Sigma$. It is the set of all points obtained by letting $\mathcal{M}^N$ act on all points in Σ. Evidently Γ is a set such that all the points $\mathcal{M}^n z$ with $z \in \Gamma$ and $n \in [-N, N]$ have the same desirable property. Now suppose that $\mathcal{M}$ is reversal symmetric. Then Γ is also reversal symmetric,

$$\Gamma^r = \Gamma. \tag{37.7.21}$$

That is, if the phase-space point z is in Γ, so is the point z^r.

To see the truth of this assertion, suppose $z \in \Gamma$. Consider the set of points $\mathcal{M}^n z^r$ for $n \in [1, N]$. From (1.21) and (1.39) we have the result

$$\begin{aligned}\mathcal{M}^n z^r &= \mathcal{M}^n\mathcal{R}z = \mathcal{RRM}^n\mathcal{R}z \\ &= \mathcal{R}(\mathcal{RMR})^n z = \mathcal{R}(\mathcal{M}^r)^{-n}z \\ &= \mathcal{RM}^{-n}z.\end{aligned} \tag{37.7.22}$$

Here, in the last step, we have used the assumption that $\mathcal{M}$ is reversal symmetric. Now we know that the sequence of points $\mathcal{M}^{-n}z$ is well behaved since $z \in \Gamma$. Therefore the sequence of points $\mathcal{R}\mathcal{M}^{-n}z$ is well behaved. (Note that, for any point $\bar{z}$, the points $\bar{z}$ and $\bar{z}^r$ are equidistant from the origin.) It follows from (7.22) that the sequence of points $\mathcal{M}^n z^r$ is well behaved.

Next consider the set of points $\mathcal{M}^{-n}z^r$. In this case we have the result

$$\begin{aligned}\mathcal{M}^{-n}z^r &= \mathcal{M}^{-n}\mathcal{R}z = \mathcal{R}\mathcal{R}\mathcal{M}^{-n}\mathcal{R}z \\ &= \mathcal{R}(\mathcal{R}\mathcal{M}^{-1}\mathcal{R})^n z = \mathcal{R}(\mathcal{M}^r)^n z \\ &= \mathcal{R}\mathcal{M}^n z. \end{aligned} \tag{37.7.23}$$

By hypothesis the sequence of points $\mathcal{M}^n z$ is well behaved, and therefore by (7.23) the sequence $\mathcal{M}^{-n}z^r$ is well behaved.

We have learned that the points $\mathcal{M}^n z^r$ are well behaved for $n \in [-N, N]$. It follows that $z^r \in \Gamma$, and hence (7.21) is correct.

We end this section with an exploration of what restrictions reversal invariance places on the nonlinear part of $\mathcal{M}$. Technically, we have already found these restrictions. They are given by the relations (3.17) through (3.20). What we want to do here is to explore these restrictions in more detail using some of the results of previous sections. For brevity we will treat only the case of static maps, which is actually somewhat more complicated than the dynamic case.

Rather than working with $\mathcal{M}$, it is convenient to work with the related map $\mathcal{M}'$ defined by the relation

$$\mathcal{M}' = \mathcal{A}\mathcal{M}\mathcal{A}^{-1} \tag{37.7.24}$$

with $\mathcal{A}$ given by (6.12). From the representation (3.1) and (6.11) it follows that $\mathcal{M}'$ has the representation

$$\begin{aligned}\mathcal{M}' &= \mathcal{A}\mathcal{L}\exp(:f_3:)\exp(:f_4:)\cdots\mathcal{A}^{-1} \\ &= [\mathcal{A}\mathcal{L}\mathcal{A}^{-1}][\mathcal{A}\exp(:f_3:)\exp(:f_4:)\cdots\mathcal{A}^{-1}] \\ &= \mathcal{N}\exp(:g_3:)\exp(:g_4:)\cdots \end{aligned} \tag{37.7.25}$$

where the g_m are given by the relation

$$g_m = \mathcal{A}f_m. \tag{37.7.26}$$

Next use the Baker-Campbell-Hausdorff series to combine the exponents of the nonlinear terms on the right side of (7.25),

$$\exp(:g_3:)\exp(:g_4:)\exp(:g_5:)\cdots = \exp(:h:) \tag{37.7.27}$$

where

$$h = h_3 + h_4 + h_5 + h_6 + \cdots. \tag{37.7.28}$$

with

$$h_3 = g_3, \tag{37.7.29}$$

$$h_4 = g_4, \tag{37.7.30}$$

$$h_5 = g_5 + (1/2)[g_3, g_4], \tag{37.7.31}$$

$$h_6 = g_6 + (1/2)[g_3, g_5] + (1/12)[g_3, [g_3, g_4]], \text{ etc.} \tag{37.7.32}$$

The net result is that $\mathcal{M}'$ can be written in the form form

$$\mathcal{M}' = \mathcal{N} \exp(: h :). \tag{37.7.33}$$

If $\mathcal{M}$ is reversal symmetric, $\mathcal{M}'$ will also be reversal symmetric,

$$(\mathcal{M}')^r = (\mathcal{A}^{-1})^r \mathcal{M}^r \mathcal{A}^r = \mathcal{A}\mathcal{M}\mathcal{A}^{-1} = \mathcal{M}'. \tag{37.7.34}$$

Here we have used (6.59), which holds if $\mathcal{M}$ is reversal symmetric. Now employ the representation (7.33) in (7.34) to obtain the condition

$$\exp(: h^r :)\mathcal{N} = \mathcal{N} \exp(: h :) \tag{37.7.35}$$

or

$$\exp(: h^r :) = \mathcal{N} \exp(: h :)\mathcal{N}^{-1} \tag{37.7.36}$$

from which it follows that

$$h^r = \mathcal{N} h. \tag{37.7.37}$$

Here we have used (8.2.27) and the fact that $\mathcal{N}$ is reversal symmetric as is evident from (6.14) and (6.15).

To explore the implications of (7.37) it is convenient to expand h in a *static resonance* basis. Introduce polynomials $R_{abcde}(z)$ and $I_{abcde}(z)$ defined by the relations

$$R_{abcde}(z) = \text{ Re } [(x + ip_x)^a (x - ip_x)^b (y + ip_y)^c (y - ip_y)^d p_\tau^e], \tag{37.7.38}$$

$$I_{abcde}(z) = \text{ Im } [(x + ip_x)^a (x - ip_x)^b (y + ip_y)^c (y - ip_y)^d p_\tau^e], \tag{37.7.39}$$

Here the quantities (exponents) a through e are integers. The first few such polynomials of interest are given by the relations

$$R_{11001} = (x^2 + p_x^2)p_\tau, \tag{37.7.40}$$

$$R_{00111} = (y^2 + p_y^2)p_\tau, \tag{37.7.41}$$

$$I_{11001} = I_{00111} = 0, \tag{37.7.42}$$

$$R_{30000} = x^3 - 3xp_x^2, \tag{37.7.43}$$

$$I_{30000} = 3x^2 p_x - p_x^3, \tag{37.7.44}$$

$$R_{20100} = x^2 y - p_x^2 y - 2xp_x p_y, \tag{37.7.45}$$

$$I_{20100} = x^2 p_y - p_x^2 p_y + 2xp_y y. \tag{37.7.46}$$

It is evident from (7.38) and (7.39) that the R_{abcde} and I_{abcde} form a basis for all static polynomials; and it is also evident that they have the simple reversal properties

$$R_{abcde}^r = R_{abcde}, \tag{37.7.47}$$

$$I_{abcde}^r = -I_{abcde}. \tag{37.7.48}$$

They also have simple properties under the action of $\mathcal{N}$, which will allow us to exploit (7.37). From (6.15) we have the results

$$\mathcal{N}(x \pm ip_x) = [\exp(\pm i\phi_1)](x \pm ip_x), \tag{37.7.49}$$

$$\mathcal{N}(y \pm ip_y) = [\exp(\pm i\phi_2)](y \pm ip_y), \tag{37.7.50}$$

$$\mathcal{N}p_\tau = p_\tau. \tag{37.7.51}$$

Introduce the *resonance* phase advances ψ_{abcd} defined by the rules

$$\psi_{abcd} = (a - b)\phi_1 + (c - d)\phi_2. \tag{37.7.52}$$

It follows from (8.3.52) and (7.49) through (7.52) that there are the relations

$$\mathcal{N}R_{abcde} = (\cos\psi_{abcd})R_{abcde} - (\sin\psi_{abcd})I_{abcde}, \tag{37.7.53}$$

$$\mathcal{N}I_{abcde} = (\sin\psi_{abcd})R_{abcde} + (\cos\psi_{abcd})I_{abcde}. \tag{37.7.54}$$

We are now ready to work out the implications of (7.37). Expand h in terms of the static resonance basis by writing

$$h = \sum_{abcde} A_{abcde}R_{abcde} + B_{abcde}I_{abcde}. \tag{37.7.55}$$

Then, by (7.47) and (7.48), we have the relation

$$h^r = \sum_{abcde} A_{abcde}R_{abcde} - B_{abcde}I_{abcde}; \tag{37.7.56}$$

and by (7.53) and (7.54) we have the relation

$$\begin{aligned}\mathcal{N}h &= \sum_{abcde}[A_{abcde}\cos\psi_{abcd} + B_{abcde}\sin\psi_{abcd}]R_{abcde} \\ &+ [-A_{abcde}\sin\psi_{abcd} + B_{abcde}\cos\psi_{abcd}]I_{abcde}.\end{aligned} \tag{37.7.57}$$

Upon comparing (7.56) and (7.57) we see that (7.37) implies the restrictions

$$A_{abcde}(-1 + \cos\psi_{abcd}) + B_{abcde}\sin\psi_{abcd} = 0, \tag{37.7.58}$$

$$-A_{abcde}\sin\psi_{abcd} + B_{abcde}(1 + \cos\psi_{abcd}) = 0. \tag{37.7.59}$$

These restrictions are equivalent, and yield the final result

$$B_{abcde} = [\tan(\psi_{abcd}/2)]A_{abcde}. \tag{37.7.60}$$

What we have learned is that if $\mathcal{M}$ is reversal symmetric, then the net Lie generator of its nonlinear part has only about half the number of linearly independent nonlinear basis generators that it otherwise might have. For example, suppose we were to employ, in a reversal symmetric system, various nonlinear correctors (sextupoles, octupoles, etc.) to drive various A_{abcde} to zero in the representation (7.55). Then, according to (7.60), the corresponding B_{abcde} would also automatically be driven to zero. Applications of this principal include the construction of high-order achromats.

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Chapter 38

Standard First- and Higher-Order Optical Modules

Chapter 39

Analyticity and Convergence

We have learned from Poincaré's Theorem 1.3.3 that trajectories will be analytic functions of the initial conditions in some domain if the right sides of the equations of motion (1.3.4) are analytic. Appendix F shows that for particle motion in electric and magnetic fields this desired analyticity is realized under very general circumstances. Correspondingly, the Taylor map (7.5.5) will converge in some domain about the origin.

The purpose of this chapter is to describe in some detail what is meant by analyticity and to apply the results of Analytic Function Theory to various problems of interest. Section 38.1 reviews briefly what is needed for the case of one complex variable, and much of its material should already be familiar to the reader. Sections 38.2 and 38.3 treat the case of several complex variables, and may be less familiar. The remaining sections discuss various applications.

39.1 Analyticity in One Complex Variable

In the theory of analytic functions of one complex variable, there are two common ways to *define* analyticity due to Riemann and Weierstrass, respectively. Riemann's approach assumes that the function in question, call it $f(z)$, is defined in some domain (open set) $\mathcal{D}$ of the complex z plane. It then says that f is analytic at the point z if it is *complex* (or *totally*) differentiable there. Complex or totally differentiable means that the limit

$$f'(z) = \lim_{\zeta \to 0}[f(z+\zeta) - f(z)]/\zeta \tag{39.1.1}$$

exists and is the same for all possible ways (e.g. directions) in which the complex variable ζ can approach zero. This definition leads directly to the Cauchy-Riemann equations. Further, f is said to be analytic in $\mathcal{D}$ if f is analytic at each point in $\mathcal{D}$.

In comparison to Weierstrass' definition of analyticity, which will be presented shortly, Riemann's definition is remarkably succinct. It also has some other advantages. For example, it immediately follows from the chain rule for differentiation that the composition of two analytic functions produces a function that is again analytic. That is, if f and g are analytic functions, then $h(z)$ defined by the rule

$$h(z) = g(f(z)) \tag{39.1.2}$$

is also an analytic function. (Of course, for z values of interest, the range of f must also be in the domain of g.) We have the mantra "an analytic function of an analytic function is analytic".

By contrast, Weierstrass defines analyticity in terms of representations by convergent Taylor series. His definition is more involved and, in order to discuss Taylor series, we first must remind ourselves of some properties of infinite sequences and infinite series. We will do this by recalling definitions and stating theorems without proof. Further information may be found in the references listed at the end of this chapter.

Definition 1.1: Consider an infinite sequence c_m of real or complex numbers. We say that this sequence *converges* to a number c,

$$\lim_{m\to\infty} c_m = c,$$

if, given any $\epsilon > 0$, there exists a positive integer N such that

$$|c_m - c| < \epsilon \text{ if } m > N.$$

Definition 1.2: Suppose the sequence c_m has the property that, given any $\epsilon > 0$, there exists an N such that

$$|c_\ell - c_m| < \epsilon \text{ if } \ell, m > N.$$

Such a sequence is called *Cauchy* or *fundamental*.

Theorem 1.1: A sequence is convergent if and only if it is Cauchy.

Definition 1.3: Consider an infinite series $\sum_{n=0}^{\infty} b_n$. We say that this series converges to the (finite) value s if the sequence of partial sums s_m, with

$$s_m = \sum_{n=0}^{m} b_n, \tag{39.1.3}$$

converges to the value s.

Definition 1.4: If an infinite series does not converge, we say that it is *divergent*. By this definition, divergence includes the possibility that the sequence of partial sums increases (in absolute value) without bound, or has more than one limit point. For example, the sequence of partial sums for the series 1, 1/2, 1/3, 1/4 $\cdots$ grows without bound, and the sequence of partial sums for the series 1, -1, 1, -1, $\cdots$ has the limit points 1 and 0. Thus, in the context of this definition, *divergent* simply means *not* convergent. The sequence of partial sums need not grow without bound for a series to be classified as divergent.

Theorem 1.2: If the infinite series $\sum b_n$ converges, then its terms b_n must tend to zero,

$$\lim_{n\to\infty} b_n = 0.$$

Theorem 1.3: Suppose the terms b_n are all real and alternate in sign. Suppose also that

$$\lim_{n\to\infty} b_n = 0.$$

Then the series $\sum b_n$ converges.

Definition 1.5: We say that the infinite series $\sum b_n$ converges *absolutely* if the series $\sum |b_n|$ converges.
Theorem 1.4: If the infinite series $\sum b_n$ converges absolutely, then it also converges in the sense of Definition 1.3.
Definition 1.6: Suppose $g(m)$ is a function that provides an invertible mapping of the (non-negative) integers onto themselves (a bijection). Then we say that the series

$$\sum_{m=0}^{\infty} b_{g(m)} \tag{39.1.4}$$

is a *rearrangement* of the series $\sum b_n$.
Theorem 1.5: If the infinite series $\sum b_n$ converges absolutely, then all *rearrangements* of the series $\sum b_n$ converge (in the sense of Definition 1.3 and also absolutely), and they all converge to the same value s.
Definition 1.7: Supposes that the infinite series $\sum b_n$ converges, but not absolutely. That is, the series $\sum |b_n|$ diverges. Then we say that the series $\sum b_n$ converges *conditionally.*
Theorem 1.6 (Riemann): Suppose that the b_n are all real and the series $\sum b_n$ converges conditionally. Then, there are rearrangments of the series that diverge. Moreover, given any (real) number s', there are rearrangements of the series that converge to s'.
Theorem 1.7 (Levy): Suppose the (possibly complex) series $\sum b_n$ is conditionally convergent. Then there is at least one line in the complex plane such that, upon selecting any point w' on that line, there is a rearrangement of the series $\sum b_n$ that converges to w'.
Theorem 1.8 (Steinitz): All the possible sums of a conditionally convergent complex series obtained by rearrangement lie on a straight line or else cover the whole complex plane.

Definitions 1.5 through 1.7 and Theorems 1.4 through 1.8 illustrate the importance of absolute convergence and the need to handle conditionally convergent series with care.

We are now prepared to discuss Taylor series. Without loss of generality, we will for the most part restrict our attention to Taylor series about the origin. Suppose the terms b_n in an infinite series are of the form $a_n z^n$ where z is a complex variable and (a_n) is a sequence of complex numbers. In this case we have the result
Theorem 1.9 (Abel): Suppose that for some (possibly complex) nonzero value $z = z'$ the series $\sum a_n z^n$ converges. Then the series converges absolutely for $|z| < |z'|$.
Moreover, suppose a *radius of convergence* R is defined in terms of the coefficients a_n by the rule

$$1/R = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{39.1.5}$$

Then we have the more specific result
Theorem 1.10 (Cauchy-Hadamard): The series $\sum a_n z^n$ converges absolutely for every z with $|z| < R$, and the convergence is uniform in every closed disk $|z| \le \rho < R$. (*Uniform* means that given any closed disk and any ϵ as in Definition 1.1, there is an associated N that suffices for all points z in the disk.) If $|z| > R$, the terms in the series are unbounded, and (by Theorem 1.2) the series is divergent. Suppose the series is differentiated term by term any number of times. Then the resulting series is also absolutely convergent for $|z| < R$, and uniformly convergent in every closed disk $|z| \le \rho < R$.

At this point we observe that if the radius of convergence R of a Taylor series is nonzero, then (by Theorem 1.10) we obtain an infinitely differentiable function f in some neighborhood of the origin by writing

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{39.1.6}$$

Moreover, we have the relations

$$a_n = (1/n!) f^{(n)}(0). \tag{39.1.7}$$

We are ready for Weierstrass' definition of analyticity. According to his definition, a function f is analytic in the domain $\mathcal{D}$ if, for any point $z^0 \in \mathcal{D}$, the function $f(z)$ has a convergent Taylor expansion about z^0,

$$f(z) = \sum_{n=0}^{\infty} a_n(z^0)(z - z^0)^n. \tag{39.1.8}$$

[Here we have used the notation $a_n(z^0)$ to indicate that the expansion coefficients in general depend on the expansion point z^0.] The definition of Weierstrass has the advantage that initially f need not be defined for complex values of z. For example, when working around the origin, we may at first require only that f have an expansion of the form (1.6) with z real and near zero. Then the series (1.6) automatically *extends* the definition of f to complex values of z. Moreover, it can be verified that the expansion point z^0, which was initially the origin in (1.6), can subsequently be moved to any point in the open disk $|z| < R$ and (26.1.8) will hold. Thus, with Weierstrass' definition, a function that is represented by a Taylor series about the origin is automatically analytic within its disk of convergence. Conversely, if a function is analytic in the disc $|z| < R$, then it will have a Taylor expansion about the origin that converges within that disc.

Contour integration is a fundamental tool in the Theory of Functions of a Single Complex Variable. Applications include the integral formula specified by
Theorem 1.11 (Cauchy): An analytic function f has the representation

$$f(z) = [1/(2\pi i)] \oint dz' f(z')/(z' - z). \tag{39.1.9}$$

Also, its Taylor coefficients can be found by contour integration. Suppose f is analytic about the origin in a disc of radius R. Let $R' = R - \epsilon$ where ϵ is any small positive number. Then the Taylor coefficients of f about the origin are given by the integrals

$$a_n = [1/(2\pi i)] \oint_{|z|=R'} dz f(z)/z^{n+1}. \tag{39.1.10}$$

Use of this representation provides the bound

$$|a_n| \leq K(R')^{-n} \tag{39.1.11}$$

where the constant K is given by the relation

$$K = \max |f(z)| \text{ over the points } |z| = R'. \tag{39.1.12}$$

Suppose f is some given analytic function, and we wish to find the radius of convergence for its Taylor expansion about some point, say the origin. One procedure would be to find its Taylor coefficients and then form the limit (1.5). However, there is an easier way. Simply find the location of the singularity of f that is closest to the origin. Consistent with Theorems 1.10 and 1.11, its distance from the origin will be R.

Finally, we remark that Cauchy's integral formula can be obtained from Riemann's definition of analyticity; and this formula, in turn, can be used to prove the existence of convergent Taylor expansions. Thus we have the key result that Riemann's and Weierstrass' definitions of analyticity are mathematically equivalent.

Historically, the adjective *holomorphic* was used to describe the case where a function was analytic as defined by Riemann, and the adjective *analytic* was used for analyticity as defined by Weierstrass. Now, because these two definitions are mathematically equivalent, these two adjective are commonly used interchangeably.

39.2 Analyticity in Several Complex Variables

The Theory of Functions of Several Complex Variables proceeds in a way that is somewhat analogous to the case of one complex variable, but it is much richer and much less explored. Again there are two ways to define analyticity. According to Riemann, a function $f(z) = f(z_1, z_2, \cdots z_m)$ of m complex variables is analytic if it is analytic (complex differentiable) in each variable *separately*.

As before, Riemann's definition immediately shows that the composition of two analytic functions produces an analytic function. For example, let $f(z_1, z_2, \cdots z_m)$ be an analytic function of m complex variables, and let $g(w)$ be an analytic function of the single complex variable w. Then, by the chain rule, $h(z_1, z_2, \cdots z_m)$ defined by

$$h(z_1, z_2, \cdots z_m) = g(f(z_1, z_2, \cdots z_m)) \tag{39.2.1}$$

is also an analytic function of the m complex variables $z_1, z_2, \cdots z_m$ providing the range of f is in the domain of g for the z values of interest.

According to Weierstrass, a function of m complex variables is analytic in the neighborhood of a point $z^0 = (z_1^0, z_2^0, \cdots z_m^0)$ if it has a convergent multiple Taylor expansion of the form

$$f(z) = f(z_1, z_2, \cdots z_m) = \sum [a_{n_1, n_2, \cdots n_m}(z^0)][(z_1 - z_1^0)^{n_1}(z_2 - z_2^0)^{n_2} \cdots (z_m - z_m^0)^{n_m}]. \tag{39.2.2}$$

The definition of Weierstrass again has the advantage that initially f need not be defined for complex values of the z_ℓ. For example, when working around the origin, we may at first require only that f have an expansion of the form (2.2) with $z^0 = 0$ and the z_ℓ real and near zero. Then, as will become evident in our subsequent discussion, the series expansion automatically extends the definition of f to complex values of the z_ℓ. Moreover, we also get analyticity in an open set.

The handling of *multiple* Taylor series requires some care. For notational simplicity, we mostly limit our discussion to the case of two complex variables and to expansions about the origin. The extension to more variables and general expansion points will generally be obvious.

In the case of double Taylor series about the origin, we have expressions of the form

$$f(z) = f(z_1, z_2) = \sum_{jk} a_{jk} z_1^j z_2^k. \tag{39.2.3}$$

Since this is a double Taylor series, there are many ways of listing its terms sequentially, and the meaning of the infinite sum of these terms is not well defined as it stands. By an *ordering* or *arrangement* of these terms, we mean some procedure for putting the pairs of integers j, k into one-to-one correspondence (a bijection) with the integers $n = 0, 1, 2, \cdots$. Since the set of pairs of integers and the set of integers both have the same cardinality, such a correspondence is always possible. We will denote this correspondence by writing

$$j = j(n), \tag{39.2.4}$$

$$k = k(n).$$

In particular, for any correspondence and given any pair j', k', there is always a unique finite number m such that

$$j' = j(m),$$

$$k' = k(m). \tag{39.2.5}$$

With these ideas in mind we see that, to be precise and without serious loss of generality, we may consider in place of the double series (2.3) a *simple* series of the form

$$f(z) = f(z_1, z_2) = \sum_{n=0}^{\infty} a_{j(n),k(n)} z_1^{j(n)} z_2^{k(n)} = \sum_{n=0}^{\infty} b_n(z) \tag{39.2.6}$$

with

$$b_n(z) = a_{j(n),k(n)} z_1^{j(n)} z_2^{k(n)}. \tag{39.2.7}$$

The treatment of such series is amenable to the methods of Definition 1.3. Figure 2.1 illustrates two of many possible orderings for the case of a double series. In ordering a, successive terms are taken sequentially from the two sides of ever larger squares. In ordering b, successive terms are taken sequentially from the long sides of ever larger triangles.

We can now discuss the convergence properties of multiple Taylor series. First, we define the *convergence set* $\mathcal{T}$ of a Taylor series to be the set of all points for which the series (2.6) converges for some ordering. We have already seen that, according to Theorem 1.10, the convergence set for a Taylor series in one complex variable is a disc. What can be said about the "shape" of the convergence set $\mathcal{T}$ for a Taylor series in several complex variables? We begin with a several complex variable analog of Theorem 1.9.

<u>Theorem 2.1</u>: Suppose that for some pair of (possibly complex) nonzero values z_1', z_2' and some ordering the series (2.6) converges. Define real positive numbers R_1', R_2' by the relations

$$R_1' = |z_1'|, \; R_2' = |z_2'|. \tag{39.2.8}$$

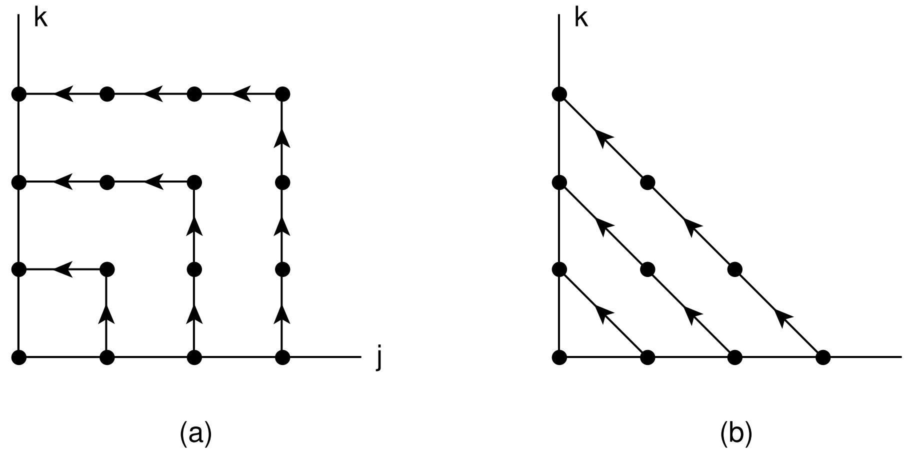

Figure 39.2.1: Two of many possible orderings for the terms in a double series.

Let $\mathcal{P}$ denote the domain (called a *polydisc* or *polycylinder* or *polycircle* about the origin with radii R'_ℓ)

$$|z_1| < R'_1,\ |z_2| < R'_2. \tag{39.2.9}$$

Then, the series (2.6) converges absolutely for all $z_1, z_2 \in \mathcal{P}$, and (by Theorem 1.5) for any ordering. Moreover, by a result of *Fubini*, the iterated series

$$f(z) = f(z_1, z_2) = \sum_{j=0}^{\infty} z_1^j \sum_{k=0}^{\infty} a_{jk} z_2^k = \sum_{k=0}^{\infty} z_2^k \sum_{j=0}^{\infty} a_{jk} z_1^j \tag{39.2.10}$$

also converge (including absolutely) for $z_1, z_2 \in \mathcal{P}$. Finally, all the various series and orderings converge to the same value.

The theorem just quoted, like Theorem 1.9, can be extended. To do so requires the introduction of some notation and further definitions. With regard to notation, let $\boldsymbol{\tau} = (\tau_1, \tau_2 \cdots \tau_m)$ be a collection of m complex numbers. Then by the symbol $\boldsymbol{\tau} z$ we mean the collection of variables $(\tau_1 z_1, \tau_2 z_2, \cdots \tau_m z_m)$. We are ready to give

Definition 2.1: Suppose $\mathcal{R}$ is a set having the property that if z is in $\mathcal{R}$, then so is $\boldsymbol{\tau} z$ providing the τ_ℓ satisfy $|\tau_\ell| \leq 1$. Such a set is called a *complete Reinhardt* set with center at the origin.

A complete Reinhardt set can be descripted pictorially by a *Reinhardt diagram*. Figure 2.2 shows a Reinhardt diagram for the case of two complex variables. Note that the diagram displays the real numbers $|z_1|$ and $|z_2|$. The complete Reinhardt set corresponds to the shaded area. It may also include portions of the axes, called *thorns*, that correspond to the dark lines that extend from the shaded area. A thorn is a polydisc all of whose radii are zero save one.

We next need to define a *logarithmically convex* complete Reinhardt set. Suppose we replot Figure 2.2 in terms of the variables $w_\ell = \log |z_\ell|$. Figure 2.3 shows the resulting *logarithmic image* of the shaded area in Figure 2.2. This image is *convex* if any two points in it can be joined by a straight line that is also in the image. With these concepts in mind we may state

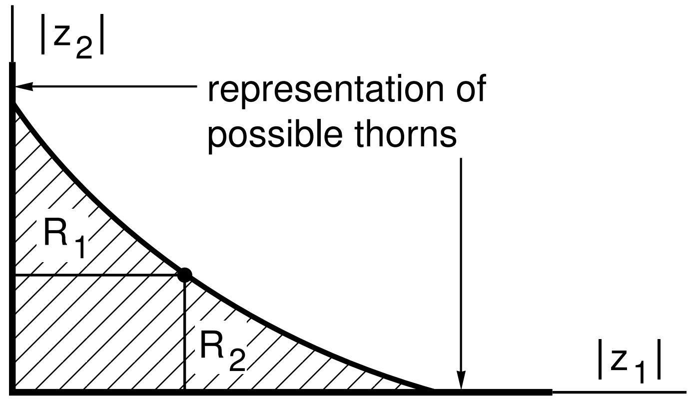

Figure 39.2.2: Possible Reinhardt diagram for the case of two complex variables. The complete Reinhardt set consists of those points $z = (z_1, z_2)$ for which the pairs $|z_1|$, $|z_2|$ lie in the shaded area, or the darkened portions of the axes representing possible thorns. The quantities R_1, R_2 on the boundary of the shaded area are conjugate radii.

Definition 2.2: A complete Reinhardt set is logarithmically convex if the logarithmic image of the "shaded" portion of its corresponding Reinhardt diagram (ignoring possible thorns) is convex.

We are now prepared to describe the characteristic features of the convergence set $\mathcal{T}$ of a Taylor series in several complex variables. The result will be given for a Taylor series about the origin,

$$f(z) = f(z_1, z_2, \cdots z_m) = \sum a_{n_1,n_2,\cdots n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}, \tag{39.2.11}$$

but can be easily extended to any expansion point z^0 by simple translation.

Theorem 2.2: The (absolute) convergence set of a multiple Taylor series of the form (2.11) is the *interior* of some logarithmically convex complete Reinhardt set about the origin. (This interior is called a logarithmically convex complete Reinhardt *domain*.) The series also converges absolutely in possible thorns of the set save perhaps at their "tips". Suppose the series is differentiated term by term any number of times with respect to any number of its variables. Then the resulting series is also absolutely convergent. Moreover, let R_ℓ be the value of $|z_\ell|$ at a point on the boundary of the set excluding points for which some $|z_\ell| = 0$. (Note that this requirement excludes thorns.) See Figure 2.2. Then the R_ℓ, called *conjugate radii*, satisfy the condition [a generalization of (1.5)]

$$\limsup_{|n|\to\infty}[|a_{n_1,n_2,\cdots n_m}|R_1^{n_1} R_2^{n_2} \cdots R_m^{n_m}]^{1/|n|} = 1. \tag{39.2.12}$$

Here we have introduced the notation

$$|n| = n_1 + n_2 + \cdots + n_m. \tag{39.2.13}$$

For points on the boundary of the logarithmically convex complete Reinhardt set the series may or may not converge, either absolutely or conditionally. Finally, for any point z outside the set, the terms in the series are unbounded, and hence the series is divergent for any ordering.

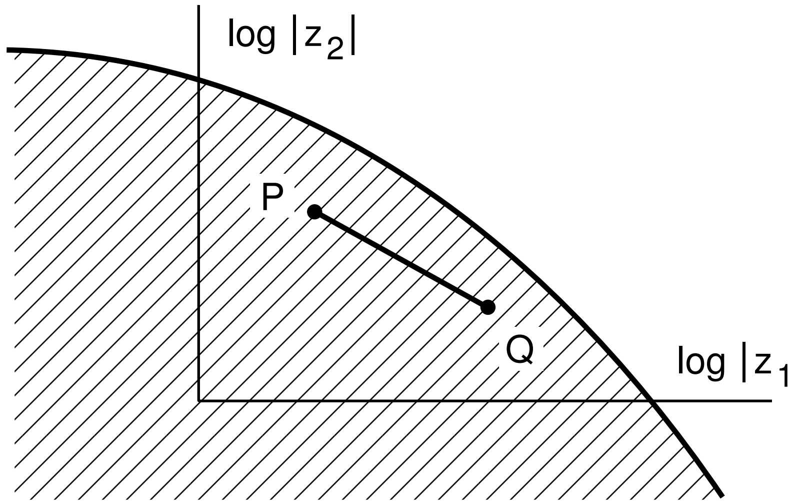

Figure 39.2.3: The logarithmic image of the shaded area in Figure 2.2. This image is convex if any two points P, Q in the image can be joined by a straight line that is also in the image.

According to this theorem, a convergent multiple Taylor series about the origin defines an infinitely differentiable function (with respect to all variables) in a logarthmically convex complete Reinhardt domain about the origin. Moreover, we have the relations

$$a_{n_1,n_2,\cdots n_m} = (1/n_1!)(1/n_2!)\cdots(1/n_m!)\partial_1^{n_1}\partial_2^{n_2}\cdots\partial_m^{n_m} f(z)|_{z=0}\,. \tag{39.2.14}$$

Finally, it can be shown that the expansion point z^0, which was initially the origin in (2.11), can subsequently be moved to any interior point in the logarithmically convex complete Reinhardt domain and (2.2) will hold for z sufficiently near z^0. Thus, a function of several complex variables that is represented by a multiple Taylor series about the origin is automatically analytic within its logarithmically convex complete Reinhardt domain of convergence. Conversely, if a function is analytic in a logarithmically convex complete Reinhardt domain about the origin, then it will have a Taylor expansion about the origin that converges within this logarithmically convex complete Reinhardt domain.

Contour integration also plays an important role in the Theory of Several Complex Variables. For example, there is a generalized Cauchy's Theorem which shows that functions of several complex variables have integral representations of the form

$$\begin{aligned} &f(z) = f(z_1, z_2, \cdots z_m) = \\ &[1/(2\pi i)]^m \oint\oint\cdots\oint dz_1' dz_2' \cdots dz_m' \times \\ &f(z')/(z_1' - z_1)(z_2' - z_2)\cdots(z_m' - z_m). \end{aligned} \tag{39.2.15}$$

Also, the Taylor coefficients (2.14) can be found by contour integration. Let ϵ be any small positive number. Define quantities R_ℓ' by the relation

$$R_\ell' = R_\ell - \epsilon \tag{39.2.16}$$

where the R_ℓ are some set of conjugate radii. Then the Taylor coefficients of f about the origin are given by the integrals

$$a_{n_1,n_2,\cdots n_m} = [1/(2\pi i)]^m \oint_{|z_1|=R'_1} \oint_{|z_2|=R'_2} \cdots \oint_{|z_m|=R'_m} dz_1 dz_2 \cdots dz_m f(z)/(z_1^{n_1+1} z_2^{n_1+1} \cdots z_m^{n_m+1}). \tag{39.2.17}$$

From this representation we deduce the Cauchy bound

$$|a_{n_1,n_2,\cdots n_m}| \le K(R'_1)^{-n_1}(R'_2)^{-n_2} \cdots (R'_m)^{-n_m} \tag{39.2.18}$$

where the constant K is given by the relation

$$K = \max|f(z)| \text{ over the points } |z_1| = R'_1, |z_2| = R'_2, \cdots |z_m| = R'_m. \tag{39.2.19}$$

Finally, starting from Riemann's definition of analyticity, Hartogs showed that analyticity in each variable separately implies continuity as well with respect to the set of all variables. The generalized Cauchy's Theorem (2.15) then follows; and from it follows the existence of convergent multiple Taylor expansions. Thus, Hartogs obtained the celebrated result that Riemann's and Weierstrass' definitions of analyticity are also equivalent in the much more difficult case of several complex variables.

We have seen at the end of Section 31.1 that, in the case of a function of one complex variable, it is possible to determine the radius of convergence of its Taylor series simply from a knowledge of the locations of the singularities of the function. The same is true for a function of several complex variables: the logarithmically convex complete Reinhardt domain of convergence of its multiple Taylor series expansion can be found from a knowledge of the locations of its singularities. However, the calculation is considerably more involved.

Consider, for simplicity, the case of a function $f(z_1, z_2)$ of two complex variables and its Taylor expansion about the origin. First we examine the two functions $f_1(z_1) = f(z_1, 0)$ and $f_2(z_2) = f(0, z_2)$. They are analytic functions of a single complex variable, and the radii of convergence of their Taylor expansions can be found using the method described at the end of Section 31.1. These two radii determine the tips of the two possible thorns represented in Figure 2.2.

Next we determine the conjugate radii R_1 and R_2. To do so we carry out the following steps:

1. Write z_1 in the form
$$z_1 = \hat{z}_1(\phi_1) = R_1 \exp(i\phi_1). \tag{39.2.20}$$

2. Hold R_1 fixed and positive, and initially small.

3. Examine the function
$$g_2(z_2, \phi_1) = f(\hat{z}_1, z_2). \tag{39.2.21}$$

4. Find the locations of the singularities of g_2 (viewed as a function of z_2) in the complex z_2 plane for each value of $\phi_1 \in [0, 2\pi)$.

5. Follow these singularities as ϕ_1 varies from 0 to 2π.

6. Define R_2 by the relation
$$R_2 = \min_{\phi_1} |z_2^c(\phi_1)| \tag{39.2.22}$$
where $|z_2^c(\phi_1)|$ is the distance from the origin of the singularity of g_2 that is *closest* to the origin for each value of ϕ_1.

7. Repeat steps 1 through 6 above, while slowly increasing the value of R_1, until R_2 becomes zero.

The result of this process is the set of R_1, R_2 values that describe the boundary of the logarithmically convex complete Reinhardt domain. See Figure 2.4. Alternatively, we may carry out a similar process, but interchange the roles of z_1 and z_2. See Figure 2.5. It can be shown that both processes yield identical sets of R_1, R_2 values.

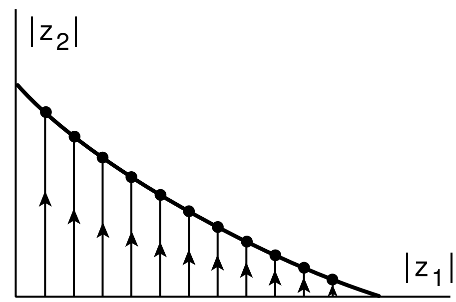

Figure 39.2.4: Determining the conjugate radii by fixing R_1 and searching for the closest singularity in z_2 as ϕ_1 varies to yield R_2.

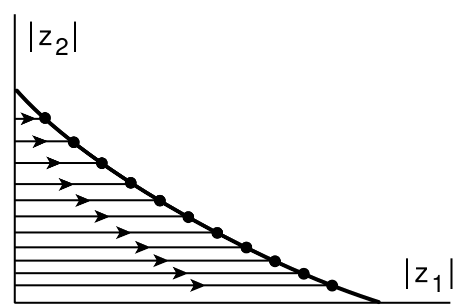

Figure 39.2.5: Determining the conjugate radii by fixing R_2 and searching for the closest singularity in z_1 as ϕ_2 varies to yield R_1.

Example 2.1: Consider the function of three variables defined by the equation

$$\psi(\boldsymbol{r}) = [x_1^2 + (x_2 - 1)^2 + x_3^2]^{-1/2}. \tag{39.2.23}$$

Up to a normalization, ψ is the electrostatic potential due to a point charge located at unit distance from the origin along the x_2 axis. See Figure 2.6.

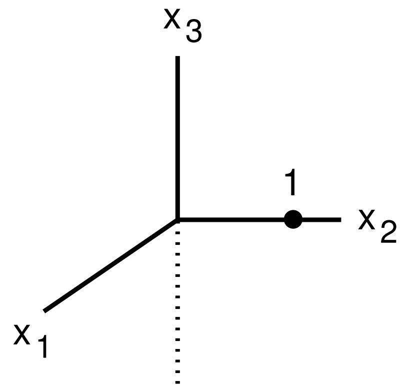

Figure 39.2.6: A point charge located at unit distance from the origin along the x_2 axis.

Suppose we expand $\psi(\boldsymbol{r})$ in a triple Taylor series about the origin,

$$\psi(\boldsymbol{r}) = \sum_{jk\ell} a_{jk\ell} x_1^j x_2^k x_3^\ell. \tag{39.2.24}$$

To examine the convergence of this series, we consider the function

$$f(z) = [z_1^2 + (z_2 - 1)^2 + z_3^2]^{-1/2}. \tag{39.2.25}$$

It is singular when the argument of the square root vanishes,

$$z_1^2 + (z_2 - 1)^2 + z_3^2 = 0. \tag{39.2.26}$$

The Reinhardt diagram in this case is 3-dimensional. An extension of the method just described shows that its boundary is given by the relation

$$R_1 = \min_{\phi_2,\phi_3} |\{-[R_2 \exp(i\phi_2) - 1]^2 - [R_3 \exp(i\phi_3)]^2\}^{1/2}| \text{ with } R_2, R_3 \in [0, 1], \tag{39.2.27}$$

or equivalently,

$$R_1^2 = \min_{\phi_2,\phi_3} |[R_2 \exp(i\phi_2) - 1]^2 + [R_3 \exp(i\phi_3)]^2| \text{ with } R_2, R_3 \in [0, 1]. \tag{39.2.28}$$

This result follows, esssentially, from solving (2.26) for z_1. Now it is easily verified from geometric considerations in the complex plane that the minimum sought in (2.28) occurs when $\phi_2 = 0$ and $\phi_3 = \pi/2$. See Exercise 2.4. Consequently, we have the result

$$R_1^2 = (R_2 - 1)^2 - R_3^2 \text{ or } R_1^2 + R_3^2 = (R_2 - 1)^2. \tag{39.2.29}$$

This is just the equation for a cone. See Figure 2.7.

Example 2.2: Suppose instead we hold x_3 fixed (and real) and simply expand $\psi(\boldsymbol{r})$ in a double Taylor series in x_1 and x_2 about the origin,

$$\psi(x_1, x_2; x_3) = \sum_{jk} a_{jk}(x_3) x_1^j x_2^k. \tag{39.2.30}$$

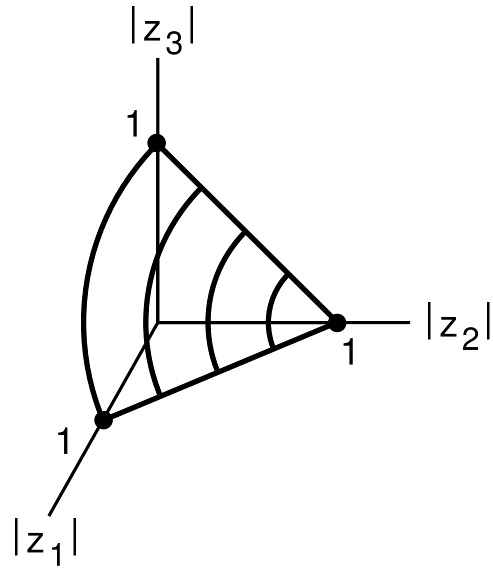

Figure 39.2.7: The Reinhardt diagram for the series (2.24), which represents the function $f(z)$ given by (2.25), is a cone.

To study the convergence of this series we must consider the function

$$f(z;x_3) = [z_1^2 + (z_2-1)^2 + x_3^2]^{-1/2}, \tag{39.2.31}$$

which is singular when

$$z_1^2 + (z_2-1)^2 + x_3^2 = 0. \tag{39.2.32}$$

For this simpler 2-dimensional example direct application of (2.20) through (2.22) gives the result

$$R_2 = \min_{\phi_1} |1 \pm [-R_1^2 \exp(2i\phi_1) - x_3^2]^{1/2}|. \tag{39.2.33}$$

In general the evaluation of (2.33) requires numerical computation. However for the special case $x_3 = 0$, there is the simple result

$$R_1 + R_2 = 1 \text{ when } x_3 = 0. \tag{39.2.34}$$

Figure 2.8 shows, for three values of x_3, the Reinhardt diagrams in the $|z_1|$, $|z_2|$ plane of the series (2.30) that represents $f(z;x_3)$. It is easily verified that the boundaries of these diagrams in the $|z_1| = 0$ and $|z_2| = 0$ planes are hyperbolas,

$$R_2^2 = 1 + x_3^2 \text{ when } |z_1| = 0, \tag{39.2.35}$$

$$R_1^2 = 1 + x_3^2 \text{ when } |z_2| = 0. \tag{39.2.36}$$

Some observations are in order. Note that the double series (2.30) is an iterated version of the triple series (2.24),

$$\psi = \sum_{jk\ell} a_{jk\ell} x_1^j x_2^k x_3^\ell = \sum_{jk} x_1^j x_2^k \sum_\ell a_{jk\ell} x_3^\ell = \sum_{jk} a_{jk}(x_3) x_1^j x_2^k, \tag{39.2.37}$$

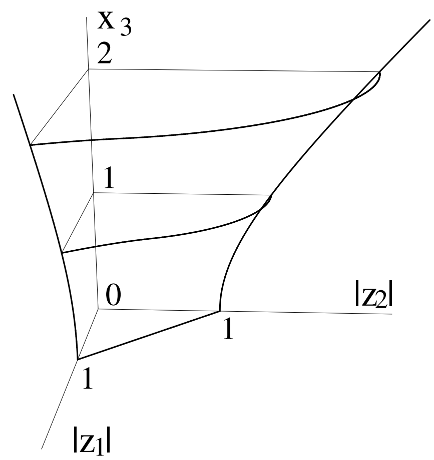

Figure 39.2.8: Reinhardt diagrams, for three values of x_3, of the series (2.30) that represents $f(z; x_3)$. Diagrams for negative values of x_3 are not shown since the diagrams for $\pm x_3$ are identical.

with

$$a_{jk}(x_3) = \sum_{\ell} a_{jk\ell} x_3^{\ell}. \tag{39.2.38}$$

Next we observe from Figure 2.8 that, unlike the previous example of Figure 2.7, the size of the convergence set *increases* with increasing $|x_3|$. We see that summation over some of the variables in a multiple series can yield a series with an extended convergence domain.

We close this section by describing briefly a remarkable feature of analytic functions of *one* complex variable, and a further remarkable feature of functions of *many* complex variables that distinguishes them from functions of a single complex variable. These features have to do with *analytic continuation*.

We begin with functions of a single complex variable, which we call w. Suppose $g(w)$ is such a function that is defined and analytic in some arbitrarily shaped simply-connected domain $\mathcal{D}$. Then we may attempt to extend g beyond $\mathcal{D}$ by analytic continuation. In its simplest description, analytic continuation consists of two steps:

1. Choose some path that leads out of $\mathcal{D}$.

2. Make successive related Taylor expansions of g in overlapping discs along the path. In this process, the Taylor coefficients for the expansion in an adjacent subsequent disc are found from those for the previous disc by multiply differentiating the Taylor expansion for the previous disc and evaluating the result at the new expansion point. That is, two successive Taylor series must give identical values for g in the region where their expansion discs overlap.

See Figure 2.9. However, this process may fail. (The convergence radii of successive Taylor expansions may shrink to zero.) It can be shown that there are functions that are anaytic in $\mathcal{D}$, but cannot be extended in an analytic way beyond the boundary of $\mathcal{D}$. Such functions are said to have a *natural boundary*, which, in this case, is the boundary of $\mathcal{D}$. [Basically, a function (of a single complex variable) with a natural boundary has a dense set of singularities on the boundary that prevent its analytic extension beyond the boundary.] Thus, given any arbitrarily shaped boundary in the complex plane, it is possible to find a function that has this boundary as a natural boundary.

A function that is specified and analytic on a small domain is called a *germ*. What we have learned is the remarkable fact that a germ specifies the full function. If a function of a single variable is defined and analytic is some domain ever so small (it could even be a small line segment), then the function is uniquely extendable over the whole domain to which it can be continued until it is defined everywhere (perhaps in a multiple sheeted way) or a natural boundary is encountered. Moreover, any boundary in the complex plane is a potential natural boundary for some analytic function.

The case of analytic functions of many complex variables is very different. Again, one can begin with a function defined in a small domain, a germ, and then attempt to extend the domain by analytic continuation. But, remarkably, now there are restrictions to what could possibly be natural boundaries.

Specifically, suppose again that $\mathcal{D}$ is some arbitrary domain, but now in C^m, the space of m complex variables. Suppose also that $f(z_1, z_2, \cdots z_m)$ is some function that is analytic in $\mathcal{D}$, and we seek to analytically continue f beyond $\mathcal{D}$. (To carry out analytic continuation in C^m, one may use overlapping polydiscs.) What can be said about this challenge?

Consider, for example, a function $f(z_1, z_2)$ of two complex variables that is analytic within the complete Reinhardt domain whose Reinhardt diagram is displayed in Figure 2.10. Figure 2.11, which shows the logarithmic image of Figure 2.10, illustrates that this domain is not logarithmically convex. However, it becomes logarithmically convex if the region corrresponding to the area below the dashed curved line segment (which is the inverse image of the dashed straight line segment in Figure 2.11) is added to the region corresponding to the shaded area. Indeed, this is the minimum territory that must be added to make the full domain logarithmically convex. A domain augmented in this way is called the *logarithmically convex hull* of the original domain.

It can be shown that *any* function $f(z_1, z_2)$ that is analytic in the complete Reinhardt domain described by the Reinhardt diagram of Figure 2.10 must have an analytic continuation into the minimal logarithmically convex extension obtained by adding to the Reinhardt diagram the area below the dashed curved line. Of course, there may be some functions that can be analytically continued even further. However, there will be other functions that cannot.

Suppose a domain has the property that there exists a function that is analytic in the domain, but this function cannot be analytically continued beyond the domain. Such a domain is called a *domain of holomorphy*. The determination of domains of holomorphy is a major topic in the Theory of Analytic Functions of Several Complex Variables. An important result, for our purposes, is that any logarithmically convex complete Reinhardt domain is a domain of holomorphy.

Often one begins with some particular domain, and would like to know the smallest domain of holomorphy that contains this domain. Such a domain is said to be an *envelope of holomorphy* for the original domain. This question is important, because *any* function that is analytic in the original domain must automatically be analytic in its envelope of holomorphy.

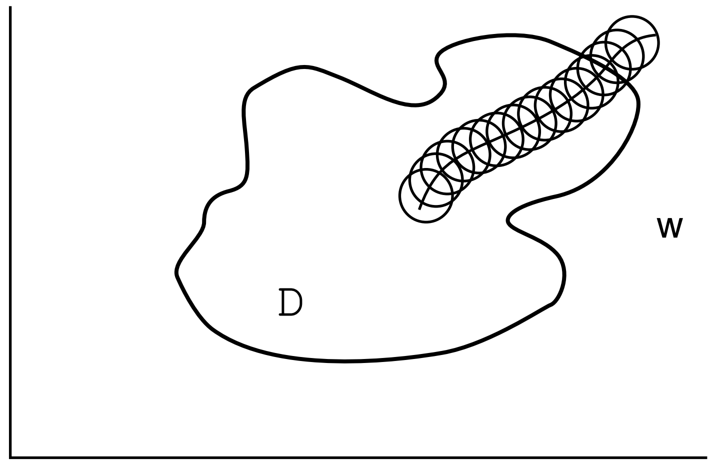

Figure 39.2.9: Analytic continuation along a path out of a domain $\mathcal{D}$ in the complex w plane by making successive related Taylor expansions in overlapping disks along the path.

Roughly speaking, for functions of many complex variables, the known existence of some analyticity often implies the existence of more analyticity.

We conclude that the complete Reinhardt domain described by the Reinhardt diagram of Figure 2.10 is not a domain of holomorphy. By contrast, the minimal logarithmically convex extension of this domain, which is a logarithmically convex complete Reinhardt domain, is a domain of holomorphy. Moreover, it is the envelope of holomorphy for the original domain.

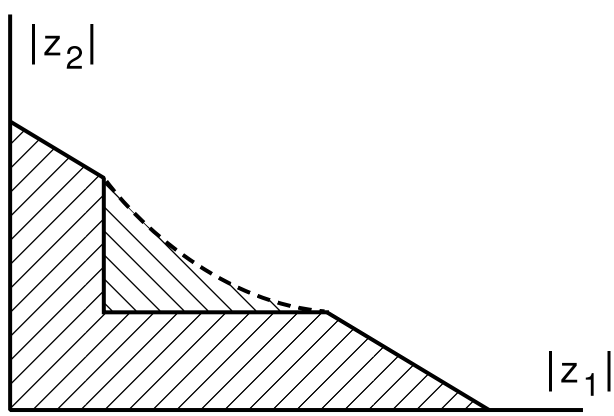

Figure 39.2.10: Reinhardt diagram for a complete Reinhardt domain that is not logarithmically convex. The dashed *curved* line segment is the inverse image of the dashed *straight* line segment in Figure 2.11. The domain becomes logarithmically convex if the region corresponding to the area below the dashed line is annexed to that corresponding to the shaded area.

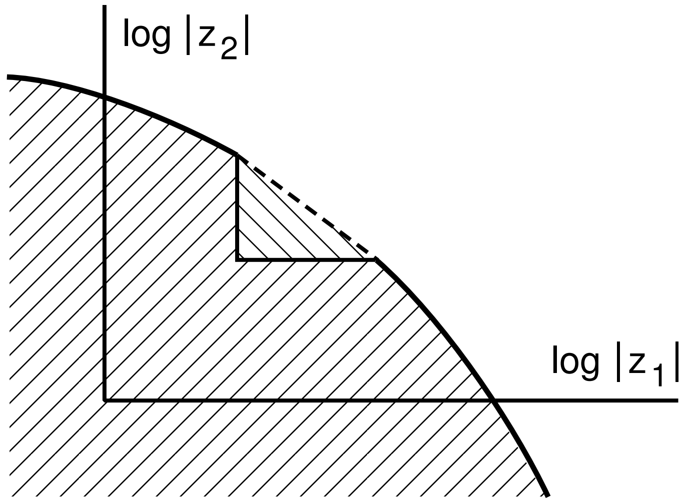

Figure 39.2.11: The logarithmic image of the shaded region of Figure 2.10. Augmenting the image by adding the area below the dashed *straight* line segment makes the image convex.

Exercises

39.2.1. Consider the function

$$f(z) = 1/[1 - (z_1 + z_2)]. \tag{39.2.39}$$

Expand f in a double Taylor series about the origin, and determine the convergence set of this series. Verify that the set is logarithmically convex.

39.2.2. Repeat Exercise 2.1 above for the function

$$f(z) = 1/(1 - z_1 z_2). \tag{39.2.40}$$

39.2.3. Repeat Exercise 2.1 above for the function

$$f(z) = 1/[1 - (z_1 + z_2 + z_1 z_2)]. \tag{39.2.41}$$

39.2.4. Find the ϕ_2, ϕ_3 values that minimize (2.28) by drawing suitable pictures (in the complex plane) of the sets

$$[R_2 \exp(i\phi_2) - 1]^2, \tag{39.2.42}$$

$$[R_3 \exp(i\phi_3)]^2, \tag{39.2.43}$$

$$[R_2 \exp(i\phi_2) - 1]^2 + [R_3 \exp(i\phi_3)]^2. \tag{39.2.44}$$

Verify (2.29).

39.2.5. Verify (2.33) through (2.36). Write a computer program to evaluate R_2 as given by (2.33).

39.2.6. Find the first few coefficients $a_{jk}(x_3)$ as given by (2.30) and (2.38). Replace x_3 by the complex variable z_3, and determine the domain of analyticity for the $a_{jk}(z_3)$.

39.2.7. Consider the map, described by (1.4.21) and (1.4.22), that arises from the monomial Hamiltonian (1.4.20). Suppose this map is expanded in a double Taylor series about the origin. Determine the convergence set for this series, and verify that it is logarithmically convex. Discuss the analytic properties of the Hénon map given by (1.2.23).

39.2.8. Exercise about *real environment*.

39.3 Convergence of Homogeneous Polynomial Series

We have seen in Section 31.2 that, to work with multiple Taylor series in a precise way, it is necessary to order their terms. In this section we will explore the effect of ordering and then *grouping* various terms together during the process of summation. In particular, we will study the effect of grouping terms of like degree together to form homogeneous polynomials, and then summing the resulting series of homogeneous polynomials. For us such series are of particular interest because, as we have seen in earlier chapters, they arise naturally in the Lie algebraic treatment of maps.

As an illustration of the concept of grouping terms, consider the simple series (2.6). Let $n_0,\ n_1,\ n_2,\ n_3\cdots$ be a set of increasing integers with $n_0 = 0$. Suppose we write the series (2.6) in the form

$$f(z) = \sum_{n=0}^{\infty} b_n(z) = \sum_{n=0}^{n_1-1} b_n(z) + \sum_{n=n_1}^{n_2-1} b_n(z) + \sum_{n=n_2}^{n_3-1} b_n(z) + \cdots = \sum_{\ell=0}^{\infty} \sum_{n=n_\ell}^{n_{\ell+1}-1} b_n(z) = \sum_{\ell=0}^{\infty} c_\ell(z)$$

with

$$c_\ell(z) = \sum_{n=n_\ell}^{n_{\ell+1}-1} b_n(z). \tag{39.3.1}$$

We have carried out group sums of the kind $(b_{n_\ell} + b_{n_\ell+1} + b_{n_\ell+2} + \cdots + b_{n_{\ell+1}-1})$ to produce the quantities c_ℓ, and then formed an infinite sum over the c_ℓ.

What can be said about the convergence of the sum $\sum c_\ell$? According to Definition 1.3, to answer this question we must examine the sequence of partial sums. However inspection reveals that, by construction, the sequence of partial sums of $\sum c_\ell$ is a *subsequence* of the sequence of partial sums of $\sum b_n$. Now we know that every subsequence of a convergent sequence is also convergent; indeed, it must converge to the same limit that the full sequence converges to. Moreover, a subsequence may be convergent even if the full sequence is not. We conclude that the grouping of terms can never spoil the convergence of a series, and it may even help in the sense that it may yield a finite result for what would otherwise have been a divergent series. In the case that it helps, and in the context of multiple Taylor series, we would like to show that the result obtained by arranging and grouping is equivalent to that obtained by analytic continuation out of the original domain of convergence of the series.

Let us apply grouping to the ordering of Figure 2.1b. Suppose the successive groups consist of the terms corresponding to the long sides of the successive triangles. Evidently, the terms corresponding to the long side of any given triangle all have degree $(j + k)$, and their sum is a homogeneous polynomial of degree $(j+k)$. Thus, with this ordering and grouping, the sum (2.6) can be rewritten in the form

$$f(z) = \sum_{n=0}^{\infty} a_{j(n),k(n)} z_1^{j(n)} z_2^{k(n)} = \sum_{\ell=0}^{\infty} P_\ell(z) \tag{39.3.2}$$

where $P_\ell(z)$ is the homogeneous polynomial of total degree ℓ

$$P_\ell(z) = \sum_{j+k=\ell} a_{j,k} z_1^j z_2^k. \tag{39.3.3}$$

We have ordered and grouped a double Taylor series into what we will call a *homogeneous polynomial* series. Evidently, an analogous ordering and grouping can be carried out for any multiple Taylor series. Moreover, we know from the results of Sections 31.1 and 31.2, and the arguments just made, that the homogeneous polynomial series must converge in the same set as the original Taylor series, and must converge to the same result. Indeed, we have from (3.3) the inequality

$$|P_\ell(z)| \le \sum_{j+k=\ell} |a_{j,k} z_1^j z_2^k|. \tag{39.3.4}$$

Consequently, the homogeneous polynomial series $\sum P_\ell$ converges absolutely whenever the Taylor series converges absolutely. Finally, the homogeneous polynomial series may even converge *outside* the convergence set of the underlying Taylor series.

How can we find the convergence set of the homogenous polynomial series? Remarkably, we will see that this task is *easier* than finding the convergence set of the underlying Taylor series.

Suppose λ is some complex number. By the symbol λz we mean the collection of variables $(\lambda z_1, \lambda z_2, \cdots \lambda z_m)$. Also, $|z|$ will denote the magnitude of z defined by the rule

$$|z|^2 = \sum_{\ell=1}^{m} |z_\ell|^2 = \sum_{\ell=1}^{m} (x_\ell^2 + y_\ell^2) \tag{39.3.5}$$

with

$$z_\ell = x_\ell + i y_\ell. \tag{39.3.6}$$

[However, for exponents we continue to use (2.13).] Consider the function $g(\lambda, z)$ defined by the relation

$$g(\lambda, z) = f(\lambda z) \tag{39.3.7}$$

where $f(z)$ is analytic about the origin and therefore has a convergent expansion of the form (2.11). Combining (2.11) and (3.7) gives the expansion

$$\begin{aligned} g(\lambda, z) = f(\lambda z) &= \sum a_{n_1, n_2, \cdots n_m} (\lambda z_1)^{n_1} (\lambda z_2)^{n_2} \cdots (\lambda z_m)^{n_m} \\ &= \sum \lambda^{|n|} a_{n_1, n_2, \cdots n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}. \end{aligned} \tag{39.3.8}$$

For any fixed z, this series will converge for sufficiently small λ. To see this, make some simple estimates. Let $R_1, R_2, \cdots R_m$ be some set of (nonzero) conjugate radii as described in Theorem 2.2, and let R be the smallest of them,

$$R = \min R_\ell. \tag{39.3.9}$$

Next define R' by

$$R' = R - \epsilon \tag{39.3.10}$$

where ϵ is a small positive number. Then from (2.18) we get the bound

$$|a_{n_1,n_2,\cdots n_m}| \leq K(R')^{-|n|}. \tag{39.3.11}$$

Also, from (3.5) we have the bounds

$$|z_\ell| \leq |z|, \tag{39.3.12}$$

$$|z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}| \leq |z|^{|n|}. \tag{39.3.13}$$

It follows that the series (3.8) has the comparison series

$$\text{comparison series } = \sum K(|\lambda||z|/R')^{|n|} = K\sum_{\ell=0}^{\infty}(|\lambda||z|/R')^\ell \sum_{|n|=\ell} 1. \tag{39.3.14}$$

A moment's reflection shows that the second sum in (3.14) is $N(\ell, m)$, the number of monomials of degree ℓ in m variables,

$$\sum_{|n|=\ell} 1 = N(\ell, m) = (\ell + m - 1)!/[\ell!(m-1)!]. \tag{39.3.15}$$

See (7.3.36). Moreover, for fixed m the quantity $N(\ell, m)$ is a *polynomial* in ℓ of degree $(m-1)$, and for large ℓ has the behavior

$$N(\ell, m) \sim \ell^{m-1}/(m-1)!. \tag{39.3.16}$$

It follows that the comparison series (3.14), and therefore also the series (3.8), converge provided λ is small enough to satisfy the inequality

$$|\lambda| < R'/|z|. \tag{39.3.17}$$

We have seen that the series (3.8) for $g(\lambda, z)$ converges absolutely for sufficiently small λ and z. It follows from the discussion of Section 31.2 that g is an analytic function, in the vicinity of the origin, of all the $(m+1)$ complex variables $z_1, z_2, \cdots z_m$ *and* λ. Moreover, the terms in the series for g can be arranged and grouped to take the form

$$g(\lambda, z) = \sum_{\ell=0}^{\infty} \lambda^\ell \sum_{|n|=\ell} a_{n_1,n_2,\cdots n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(z) \tag{39.3.18}$$

where $P_\ell(z)$ is the homogeneous polynomial

$$P_\ell(z) = \sum_{|n|=\ell} a_{n_1,n_2,\cdots n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}. \tag{39.3.19}$$

Correspondingly, for f itself we have the homogeneous polynomial expansion

$$f(z) = g(1,z) = \sum_{\ell=0}^{\infty} P_\ell(z). \tag{39.3.20}$$

For fixed z, regard (3.18) as a Taylor series in λ with Taylor coefficients $P_\ell(z)$. By essentially the same arguments we have just endured, these coefficients have the bound

$$|P_\ell(z)| \leq KN(\ell,m)(|z|/R')^\ell. \tag{39.3.21}$$

Consequently, as expected, this series will converge and produce an analytic function of λ at least within the disc (3.17). It follows from Theorem 1.11 that the $P_\ell(z)$ are given by the integrals

$$P_\ell(z) = [1/(2\pi i)] \oint d\lambda g(\lambda,z)/\lambda^{\ell+1} = [1/(2\pi i)] \oint d\lambda f(\lambda z)/\lambda^{\ell+1} \tag{39.3.22}$$

for any contour about the origin in the λ plane for which (3.17) holds. Continue to hold z fixed. Let $\lambda_c(z)$ be the singularity of $g(\lambda,z) = f(\lambda z)$ in the λ plane that is *closest* to the origin. Since $f(\lambda z)$ is always analytic in the disc (3.17), we know that $\lambda_c(z)$ is always nonzero. Define a radius R'' by the rule

$$R'' = \rho|\lambda_c(z)| \tag{39.3.23}$$

where ρ is any number slightly less than but arbitrarily near 1,

$$\rho \simeq 1 \text{ but } \rho < 1. \tag{39.3.24}$$

Then the λ contour in (3.22) can be expanded to give the relation

$$P_\ell(z) = [1/(2\pi i)] \oint_{|\lambda|=R''} d\lambda f(\lambda z)/\lambda^{\ell+1} \tag{39.3.25}$$

and the bound

$$|P_\ell(z)| \leq M/(R'')^\ell \tag{39.3.26}$$

where

$$M = \max|f(\lambda z)| \text{ for } |\lambda| = R''. \tag{39.3.27}$$

We conclude that the homogeneous polynomial expansion (3.20) converges absolutely and uniformly provided $\lambda_c(z)$ satisfies the relation

$$|\lambda_c(z)| \geq \rho' > 1. \tag{39.3.28}$$

Conversely, the series (3.18) must diverge for any λ that satisfies

$$|\lambda| > |\lambda_c(z)|. \tag{39.3.29}$$

For if it converged, then (by Theorem 1.9 and the discussion surrounding it) $g(\lambda, z)$ would be analytic in a disc having a radius larger than $|\lambda_c|$, and could not be singular at $\lambda = \lambda_c$. Finally we note that the calculations involved in the relations (3.22) through (3.27), and the divergence criterion associated with (3.29), hold for general z.

We now have the ingredients for determining the domain of convergence of a homogeneous polynomial expansion. The recipe is this:

1. Let $\mathcal{S}^{2m-1}$ be the unit sphere in C^m defined by the condition

$$|z| = 1. \tag{39.3.30}$$

2. For any point $\hat{z} \in \mathcal{S}^{2m-1}$ let $\lambda_c(\hat{z})$ be the singularity of $f(\lambda\hat{z})$ in the λ plane that is closest to the origin.

3. Define the positive number $\sigma(\hat{z})$ by the rule

$$\sigma(\hat{z}) = |\lambda_c(\hat{z})|. \tag{39.3.31}$$

4. Let $\zeta(\hat{z})$ be the *ray* that goes from the origin to the point $\sigma(\hat{z})\hat{z}$,

$$\zeta(\hat{z}) = \text{ set of all points } r\sigma\hat{z} \text{ with } r \in [0, 1]. \tag{39.3.32}$$

5. Let $\mathcal{H}$ be the union of all rays $\zeta(\hat{z})$ for all points $\hat{z} \in \mathcal{S}^{2m-1}$,

$$\mathcal{H} = \bigcup_{\hat{z}\in\mathcal{S}^{2m-1}} \zeta(\hat{z}). \tag{39.3.33}$$

Then the convergence set of the homogeneous polynomial expansion (3.20) is $\mathcal{H}$. Specifically, the homogeneous polynomial expansion converges absolutely for all z in the interior of $\mathcal{H}$. Moreover, if z is any point in the exterior of $\mathcal{H}$, the terms $P_\ell(z)$ are unbounded for increasing ℓ, and hence the homogeneous polynomial expansion diverges at all exterior points.

We note that this recipe is considerably simpler, particularly in the case of many complex variables, than the analogous recipe given in Section 31.2 for finding conjugate radii of Taylor series. Moreover, it has the advantage that it can be applied, if desired, using only *real* points $\hat{x}$ in $\mathcal{S}^{2m-1}$. This feature is of interest because, in the context of maps, we often need to know only about the convergence of series when all variables are real. Finally, we observe that by construction the *interior* of the convergence set $\mathcal{H}$ has the property that if z is in $\mathcal{H}$, then so is τz where τ is any complex number satisfying $|\tau| \leq 1$. See Exercise 3.4. A domain having this property is called a *complete circular* domain. Thus, the natural domain of convergence for a homogeneous polynomial series is a complete circular domain.

The statements just made about convergence and divergence are easily proved. First, given any $z \neq 0$, there is a unique ray that goes from the origin to z, and this ray (extended if necessary) intersects $\mathcal{S}^{2m-1}$ in the point $\hat{z}$ given by

$$\hat{z} = z/|z|. \tag{39.3.34}$$

Now suppose that z is in the *interior* of $\mathcal{H}$. Then z can be written in the form

$$z = r\sigma(\hat{z})\hat{z} \text{ with } 0 < r \leq \rho'' < \rho < 1. \tag{39.3.35}$$

Here ρ is the same number that appears in (3.23). For $P_\ell(z)$ we find, by using (3.23), (3.26), (3.31), and (3.35), the result

$$\begin{aligned} |P_\ell(z)| &= |P_\ell(r\sigma\hat{z})| = |(r\sigma)^\ell P_\ell(\hat{z})| \\ &\leq |(\rho''\sigma)^\ell P_\ell(\hat{z})| = (\rho''/\rho)^\ell |(\rho\sigma)^\ell P_\ell(\hat{z})| \\ &= (\rho''/\rho)^\ell |(\rho|\lambda_c(\hat{z})|)^\ell P_\ell(\hat{z})| \\ &= (\rho''/\rho)^\ell |(R'')^\ell P_\ell(\hat{z})| \leq (\rho''/\rho)^\ell M. \end{aligned} \tag{39.3.36}$$

However, from (3.35) we have the inequality

$$(\rho''/\rho) < 1. \tag{39.3.37}$$

It follows that (3.20) has a geometric series as a comparison series, and hence it converges absolutely when z is an interior point in $\mathcal{H}$.

To complete the proof, suppose that z is in the *exterior* of $\mathcal{H}$. Then z can be written in the form

$$z = r\sigma(\hat{z})\hat{z} \text{ with } r > 1. \tag{39.3.38}$$

For $P_\ell(z)$ we find the result

$$P_\ell(z) = P_\ell(r\sigma\hat{z}) = (r\sigma)^\ell P_\ell(\hat{z}) \tag{39.3.39}$$

and (3.20) becomes

$$\sum_{\ell=0}^{\infty} P_\ell(z) = \sum_{\ell=0}^{\infty} (r\sigma)^\ell P_\ell(\hat{z}), \tag{39.3.40}$$

which is a series of the form (3.18) with

$$\lambda = r\sigma. \tag{39.3.41}$$

Since $r > 1$, we see from (3.31) and (3.41) that

$$|\lambda| = |r\sigma| = r|\lambda_c| > |\lambda_c|, \tag{39.3.42}$$

and conclude from (3.29) that the series (3.40) is divergent. Because (3.40) is a divergent Taylor series, the terms $P_\ell(z)$ that comprise it must be unbounded. See (3.39) and Theorem 1.9.

It remains to be shown that if the homogeneous polynomial series converges outside the convergence set of the underlying Taylor series, then it provides an analytic continuation of the function specified by the Taylor series. The reader has the pleasure of working out a proof in Exercise 3.5.

Example 3.1: Consider again the function $\psi(\boldsymbol{r})$ given by (2.23). Suppose the series (2.24) is grouped into homogeneous polynomials,

$$\psi(x_1, x_2, x_3) = \sum_{jk\ell} a_{jk\ell} x_1^j x_2^k x_3^\ell = \sum_{m=0}^{\infty} \sum_{j+k+\ell=m} a_{jk\ell} x_1^j x_2^k x_3^\ell = \sum_{m=0}^{\infty} P_m(x_1, x_2, x_3) \tag{39.3.43}$$

where

$$P_m(x_1, x_2, x_3) = \sum_{j+k+\ell=m} a_{jk\ell} x_1^j x_2^k x_3^\ell. \tag{39.3.44}$$

Let us determine the domain of convergence of this homogeneous polynomial series in the (real) x_1,x_2,x_3 3-dimensional space. We parameterize points $\hat{x}_1, \hat{x}_2, \hat{x}_3 \in \mathcal{S}^3$ by writing

$$\hat{x}_2 = \sin\theta, \tag{39.3.45}$$

$$\hat{x}_1 = \cos\theta\cos\phi, \tag{39.3.46}$$

$$\hat{x}_3 = \cos\theta\sin\phi. \tag{39.3.47}$$

From (2.26) we see that singularities in λ satisfy the equation

$$(\lambda\cos\theta\cos\phi)^2 + (\lambda\sin\theta - 1)^2 + (\lambda\cos\theta\sin\phi)^2 = 0.$$

This equation has the solutions

$$\lambda = \sin\theta \pm i\cos\theta \tag{39.3.48}$$

from which we find that

$$|\lambda_c| = 1. \tag{39.3.49}$$

It follows that the homogeneous polynomial series (3.43) for $\psi(x_1, x_2, x_3)$ converges in the *unit ball* about the origin,

$$0 \le x_1^2 + x_2^2 + x_3^2 \le 1. \tag{39.3.50}$$

Note that this set includes points that lie outside the convergence set described by the Reinhardt diagram of Figure 2.7.

Example 3.2: Suppose the series (2.30) is grouped into homogeneous polynomials,

$$\psi(x_1, x_2; x_3) = \sum_{jk} a_{jk}(x_3) x_1^j x_2^k = \sum_{\ell=0}^{\infty} \sum_{j+k=\ell} a_{jk}(x_3) x_1^j x_2^k = \sum_{\ell=0}^{\infty} P_\ell(x_1, x_2; x_3) \tag{39.3.51}$$

where

$$P_\ell(x_1, x_2; x_3) = \sum_{j+k=\ell} a_{jk}(x_3) x_1^j x_2^k. \tag{39.3.52}$$

For fixed (real) x_3, let us determine the convergence set of this homogeneous polynomial series in the (real) x_1, x_2 plane. We parameterize points $\hat{x}_1, \hat{x}_2 \in \mathcal{S}^2$ by writing the relations

$$\hat{x}_1 = \cos\phi, \tag{39.3.53}$$

$$\hat{x}_2 = \sin\phi. \tag{39.3.54}$$

Then, from(2.32), we see that singularities in λ satisfy the equation

$$(\lambda\cos\phi)^2 + (\lambda\sin\phi - 1)^2 + x_3^2 = 0. \tag{39.3.55}$$

This equation has the solutions

$$\lambda = \sin\phi \pm i[\cos^2\phi + x_3^2]^{1/2} \tag{39.3.56}$$

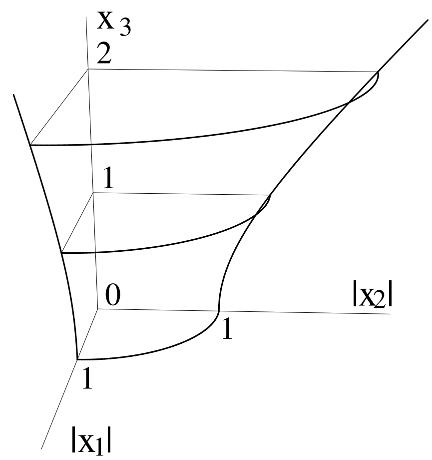

Figure 39.3.1: Real x_1, x_2 convergence sets for the homogeneous polynomial series (3.51) for various values of x_3. Together they form a hyperbola of revolution. Sets are not shown for negative values of x_3 since the sets for $\pm x_3$ are identical.

from which we find that

$$|\lambda_c| = (1 + x_3^2)^{1/2}. \tag{39.3.57}$$

It follows that the homogeneous polynomial series (3.51) for $\psi(x_1, x_2; x_3)$ converges in a *disc* about the origin in the x_1,x_2 plane with radius $(1+x_3^2)^{1/2}$. Figure 3.1 shows several of these discs for various values of x_3. Together they form the hyperbola of revolution

$$x_1^2 + x_2^2 = 1 + x_3^2. \tag{39.3.58}$$

Note that these sets include points that lie outside the convergence sets described by the Reinhardt diagrams of Figure 2.8.

Exercises

39.3.1. Find the convergence set in the real x_1, x_2 plane for the homogeneous polynomial expansion about the origin of the function f given by (2.39).

39.3.2. Repeat Exercise 3.1 for the f given by (2.40).

39.3.3. Repeat Exercise 3.1 for the f given by (2.41).

39.3.4. Verify Equations (3.45) through (3.50).

39.3.5. Verify Equations (3.53) through (3.58).

39.3.6. Verify that the interior of the convergence set $\mathcal{H}$ constructed following the steps (3.30) through (3.33) is a complete circular domain.

39.3.7. Show that if the homogeneous polynomial series converges outside the convergence set of the underlying Taylor series, then it provides an analytic continuation of the function specified by the Taylor series.

39.3.8. Consider the map, described by (1.4.21) and (1.4.22), that arises from the monomial Hamiltonian (1.4.20). See Exercise 2.7. Suppose this map is expanded in a homogeneous polynomial series about the origin. Find the convergence set in the real q, p plane for this expansion.

39.3.9. Using the methods of this section, determine the domain of analyticity and the convergence set for the monopole doublet $\psi(x, y, z)$ given by (13.11.3).

39.4 Application to Potentials and Fields

39.5 Application to Taylor Maps: The Anharmonic Oscillator

39.6 Application to Taylor Maps: The Pendulum

39.7 Convergence of the BCH Series

39.8 Convergence of Lie Transformations and the Factored Product Representation

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The Anharmonic Oscillator and the Pendulum

Chapter 40

Truncated Power Series Algebra

40.1 Introduction

With regard to the simplest transcendental function, the exponential function, it has been said that

God created e^x but never heard of a polynomial.

In this chapter we will be studying polynomials in multiple variables. For mortals, polynomials are a major portal to the transcendent.

Anyone familiar with the rudiments of Computer Science recognizes that the proper choice of algorithm can make an enormous difference in both computational speed and storage requirements. Thus, of several different algorithms which execute the same computational task, some may perform much better than others. Recall, for example, the problem of sorting n items: the number of operations required for the simple *bubble sort* method scales as n^2, while that for the more sophisticated *quick sort* or *heap sort* scales as only $n \log_2 n$. As a second example, the operation count for the ordinary discrete Fourier transform scales as n^2 (where n is the number of data points) while that for the celebrated fast (discrete) Fourier transform again scales as $n \log_2 n$.

The purpose of this chapter is to describe how truncated power series (finite sums of monomials in several variables) can be manipulated by computer. We will explore various methods for labeling and storing monomials together with their relation to efficient algorithms for executing various polynomial operations. These operations, which we refer to as Truncated Power Series Algebra (TPSA), include addition, multiplication, differentiation, Poisson bracketing, the commutation of vector fields, and the composition of functions. The emphasis here will be on computational speed, storage requirements, and program flexibility. Section 39.2 describes how monomials may be labeled and stored. Subsequent sections describe how truncated power series composed of these monomials can be added, multiplied, and otherwise manipulated.

Since TPSA has important applications to the study of dynamical systems, our discussion will often be couched in those terms — sometimes with a particular emphasis on applications to accelerator physics. However, we stress that much of the material presented here applies generically to any use of TPSA.

Before discussing possible schemes for storing and manipulating polynomials, we introduce some useful terminology and definitions: A typical monomial in d variables may be written in the form

$$z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d}, \tag{40.1.1}$$

where the exponents j_k are a set of non-negative integers. We shall sometimes write the d-tuple of exponents $(j_1, \cdots, j_d)$ simply as a vector j, and similarly abbreviate the corresponding monomial by writing

$$z_1^{j_1} \cdots z_d^{j_d} = z^j. \tag{40.1.2}$$

Indeed, for the sake of brevity we shall sometimes refer to the exponent vector j as "the monomial j". For the degree of this monomial, we introduce the notation

$$|j| = j_1 + j_2 + \cdots + j_d. \tag{40.1.3}$$

Recall from Section 7.3 that $N(m, d)$, the number of monomials of degree m in d variables, is given by the binomial coefficient

$$N(m, d) = \begin{pmatrix} m + d - 1 \\ m \end{pmatrix} = \frac{(m + d - 1)!}{m!(d - 1)!}. \tag{40.1.4}$$

Various values of $N(m, d)$ are listed in Table 7.3.1. Also $S(m, d)$, the number of monomials of degrees 1 through m in d variables, is given by the relation

$$S(m, d) = \begin{pmatrix} m + d \\ m \end{pmatrix} - 1 = \frac{(m + d)!}{m!d!} - 1. \tag{40.1.5}$$

See Section 7.9. Various values of $S(m, d)$ are listed in Table 7.9.1. Finally, for some calculations it is also useful to employ the quantity $S_0(m, d)$, the number of monomials of degrees 0 through m in d variables. It is given by the relation

$$S_0(m, d) = S(m, d) + 1 = \begin{pmatrix} m + d \\ m \end{pmatrix} = \begin{pmatrix} m + d \\ d \end{pmatrix} = \frac{(m + d)!}{m!d!}. \tag{40.1.6}$$

40.2 Monomial Indexing

Any program that manipulates polynomials must have a scheme for labeling and storing the coefficients of the basis monomials. In this section we will describe some ways in which this can be done.

40.2.1 An Obvious but Memory Intensive Method

One very obvious such scheme uses a multi-dimensional array indexed by the monomial exponents. For purposes of illustration, consider the six-variable case. Then, using a phase-space notation, we have monomials of the form

$$z^j = X^{j_1} P_x^{j_2} Y^{j_3} P_y^{j_4} \tau^{j_5} P_\tau^{j_6}. \tag{40.2.1}$$

Suppose, for example, we are interested in the case of homogeneous polynomials of degrees 0 through 12. Then each j_k lies in the interval $[0, 12]$, we have a $13 \times 13 \times \cdots$ (six factors) array, and such an array requires $13^6 \simeq 4.8 \times 10^6$ storage locations. By contrast, the entry for $S(12, 6)$ in Tablel 7.9.1 shows that in principle only 18,564 $(= 18, 563 + 1)$ locations should be required. In general, if we use an $(m + 1) \times (m + 1) \cdots$ (d factors) array to store the coefficients of monomials of degree 0 through m in d variables, we shall need $(m+1)^d$ storage locations. This number is much larger than $S_0(m, d)$. Thus this obvious method requires dedicating but never using a very large amount of memory. We do not want to allocate but not use large amounts of memory. Ideally, we do not want to have any allocated but unused gaps in memory at all. As a consequence, it is desirable — even essential — to consider other possible schemes for labeling and storing monomials.

40.2.2 Polynomial Grading

Implicit in our discussion so far is the assumption that the objects of interest really are polynomials of degrees 0 through m. More explicitly, we assume that we are interested in *grading* polynomials according to their *total* degree. See Section 8.9. For phase-space variables this assumption seems natural, because we expect that the possible excursions from a design trajectory (in suitable scaled coordinates) could be of comparable size in any direction. It may be less natural (and perhaps a different treatment is called for) if we wish simultaneously to make expansions in various parameter variables. Indeed, in this latter case it might be better to have a scheme where the orders of the phase-space variables and the parameter variables could be set independently. One would then have polynomials in the phase-space variables whose coefficients are either numbers or polynomials in the parameter variables.

A remark about nomenclature: Consider the set of all (zero or positive) integers $j_1 \cdots j_d$ that obey the condition

$$|j| \leq m \tag{40.2.2}$$

for a fixed value of m. They form a collection of $S_0(m, d)$ points in d-dimensional space. Some authors refer to this set of points as a *pyramid*. The reader is invited to sketch these points in the cases $d = 2$ and $d = 3$ to see why the name is apt. A set of values assigned to points on a pyramid is referred to as a pyramidal data structure. Finally, some authors refer to sets of points of the kind described in Subsection 32.2.1 as (possible high dimensional) *boxes* or *cubes*. See Appendix S.

40.2.3 Monomial Ordering

Because a well-defined ordering facilitates the systematic implementation of polynomial algebra on a computer, we will continue this section by describing the concept of monomial orderings.

A *monomial ordering* is a relation $>$ on the set of all monomials (exponents) $\alpha = (\alpha_1, \alpha_2 \cdots)$ that satisfies the following three conditions:

1. The relation $>$ is a *total ordering*, meaning that for any two monomials α and β,

exactly one of the following statements holds true:

$$\alpha > \beta\ ,\quad \alpha = \beta\ ,\quad \beta > \alpha. \tag{40.2.3}$$

This condition allows us to arrange the terms of a polynomial in an unambiguous way.

2. If $\alpha > \beta$, then $(\alpha + \gamma) > (\beta + \gamma)$. To see the value of this condition, note that multiplying both z^α and z^β by z^γ yields the results $z^{\alpha+\gamma}$ and $z^{\beta+\gamma}$. Now consider a polynomial whose terms are ordered using the relation $>$. If the above condition holds, then multiplying this polynomial (term-by-term) by z^γ will not alter the arrangement of the terms.

3. The relation $>$ is a *well-ordering*, meaning that every strictly decreasing sequence $\alpha > \beta > \gamma > \cdots$ must eventually terminate. This condition facilitiates proofs that various polynomial algorithms terminate after a finite number of steps.

Suppose we take the $S_0(m, d)$ monomials of degrees 0 through m in d variables, and list them sequentially in some fashion. Next we try to declare that this list constitutes a monomial ordering. For example, as we go down the list, we might declare that each successive monomial is less than all its predecessors. This declaration is consistent with the requirements 1 and 3. However, it generally violates requirement 2. Consequently, we will distinguish between *orderings* and *arrangements*. By an arrangement we will mean any sequential listing, whereas an ordering will mean a monomial ordering as defined above.

The following examples illustrate some of the commonly used monomial orderings. Here we will somes write $z^\alpha > z^\beta$ if $\alpha > \beta$.

Example 1. Lexicographic Order (lex). Let $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\beta = (\beta_1, \cdots, \beta_d)$. We say $\alpha >_{\text{lex}} \beta$ if in the vector difference $\alpha - \beta$ the *left-most* nonzero entry is *positive*. (Note that this construction is identical to that used to order weights in Section 5.8.)

For lexicographic ordering we have the relations

$$(1, 0, \cdots, 0) >_{\text{lex}} (0, 1, \cdots, 0) >_{\text{lex}} \cdots >_{\text{lex}} (0, 0, \cdots, 1), \tag{40.2.4}$$

or, to use a more familiar notation,

$$z_1 >_{\text{lex}} z_2 >_{\text{lex}} \cdots >_{\text{lex}} z_d. \tag{40.2.5}$$

Therefore the variables z_i themselves are arranged in descending order as their subscript label increases. Moreover, it is easily checked that

$$(z_i)^m >_{\text{lex}} (z_j)^n \text{ whenever } i < j, \tag{40.2.6}$$

independent of the powers m, n (but assuming $m > 0$).

Unfortunately, this latter property can be inconvenient for problems where the total degree of monomials is important. (A common such problem in accelerator physics is the construction of a factorized Lie map.) However, there is a simple remedy to this situation: first order monomials by total degree, then apply lex ordering only to monomials of the same total degree. A formal definition of this scheme is given in the next example.

Example 2. Graded Lexicographic Order (glex, sometimes called grlex). We say $\alpha >_{\rm glex} \beta$ if either $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and $\alpha >_{\rm lex} \beta$.

For monomials of degree one,

$$z_1 >_{\rm glex} z_2 >_{\rm glex} \cdots >_{\rm glex} z_d, \tag{40.2.7}$$

just as for lex order. For quadratic monomials,

$$z_1^2 >_{\rm glex} z_1 z_2 >_{\rm glex} z_1 z_3 >_{\rm glex} \cdots >_{\rm glex} z_2^2 >_{\rm glex} z_2 z_3 >_{\rm glex} \cdots >_{\rm glex} z_d^2. \tag{40.2.8}$$

Example 3. Graded Reverse Lexicographic Order (grevlex). In this ordering we say $\alpha >_{\rm grevlex} \beta$ if either $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and the *right-most* nonzero entry in the vector difference $\alpha - \beta$ is *negative*.

The grevlex order may seem less intuitive than glex, but it does have certain advantages. As with glex, the variables themselves are ordered:

$$z_1 >_{\rm grevlex} z_2 >_{\rm grevlex} \cdots >_{\rm grevlex} z_d. \tag{40.2.9}$$

For quadratic monomials, by contrast,

$$z_1^2 >_{\rm grevlex} z_1 z_2 >_{\rm grevlex} z_2^2 >_{\rm grevlex} z_1 z_3 >_{\rm grevlex} \cdots >_{\rm grevlex} z_d^2. \tag{40.2.10}$$

Thus as one scans monomials of a fixed degree in descending grevlex order, one encounters later variables only after earlier ones have been "exhausted".

40.2.4 Labeling Based on Ordering

We will now describe how monomial ordering can be used to assign a label or index to each monomial.[1] Basically, the idea is to list the monomials in some sequence, and then label the monomials by where they occur in the list.

Consider, as a case of special interest, monomials in 6 variables as in (2.1). We have found it useful to list them in the arrangement shown in Table 2.1 below. Note that the monomials are graded: First the monomial of degree 0 appears, next those of degree 1, then those of degree 2, etc. Now let us, for the moment, regard each exponent $j = (j_1 \cdots j_6)$ as some six-digit number having the digits (reading from left to right) j_1 through j_6. Then observe that within each group of monomials of fixed degree these six-digit numbers appear in descending order as one reads down a column. For example, the monomials of degree 1 have the ordering $100000 > 010000 > 001000 > 000100 > 000010 > 000001$. Similarly, the monomials of degree 2 have the ordering $200000 > 110000 > 101000 > 100100 > \cdots$, etc. That is, for a given degree, the monomials appear in descending lex order. We will refer to this arrangement of monomials as *modified glex sequencing*.

Finally, each monomial has been given an *index*, starting with 0 for the monomial of degree 0, that increases by 1 for each successive monomial.[2] Thus, as the index increases,

[1] Equivalently, we will assign a label to each point in a pyramid.

[2] This is the indexing scheme used in the program *MaryLie*. MaryLie is a program for charged-particle beam transport based on Lie algebraic methods. It is named in honor of Queen Henrietta Maria, patron of the English colony that was to become the state of Maryland, and Sophus Lie.

we first encounter the monomial of degree 0, then those of degree 1, then those of degree 2, etc. Furthermore, as the index *increases* within each set of monomials of a given degree, we encounter monomials in *descending* lex order. Note, as is easily verified by comparison of Tables 7.9.1 and 2.1, that there is the relation

$$\text{index }(00000m) = S(m, 6). \tag{40.2.11}$$

We remark that one might consider a true glex ordering in which the monomials of a given degree would occur in *ascending* lexicographic order as the index increased. According to (2.6) and (2.7), however, such an indexing scheme would list temporal variables first, and this order is not convenient for applications to accelerator physics.

There is, though, another arrangement that is satisfactory, and perhaps even superior: One could list the monomials by degree as before, and within each degree arrange them in descending reverse lexicographical (revlex) order. This could be called *modified grevlex sequencing*. Table 2.2 illustrates, for the case of 6 variables, an indexing scheme based on this procedure. Note that monomials with temporal variables now occur at the end of each monomial set of a given degree.

40.2.5 Formulas for Lowest and Highest Indices

For future reference, we note that any indexing scheme based on a graded ordering has the property that all the indices associated with monomials of a fixed degree are contiguous and lie within a fixed range. For example, reference to Tables 2.1 or 2.2 shows that (in the case of 6 variables) monomials of degree 2 begin at index 7 and end at index 27, and those of degree 3 begin at index 28 and end at index 83. Let *itop(ideg)* be the highest (top) index for monomials of degree *ideg*, and let *ibot(ideg)* be the lowest (bottom) index. Then we have the relations

$$itop(ideg) = S(ideg, d), \tag{40.2.12}$$

$$ibot(ideg) = itop(ideg - 1) + 1 = S[(ideg - 1), d] + 1. \tag{40.2.13}$$

Here d is the number of variables. Table 2.3 lists values of *ibot* and *itop* for the case of 6 variables ($d = 6$).

Table 40.2.1: Modified glex sequencing, a possible glex related indexing scheme for monomials in 6 variables.

Index	Exponents of		
	$X\ P_x$	$Y\ P_y$	$\tau\ P_\tau$
0	0 0	0 0	0 0
1	1 0	0 0	0 0
2	0 1	0 0	0 0
3	0 0	1 0	0 0
4	0 0	0 1	0 0
5	0 0	0 0	1 0
6	0 0	0 0	0 1
7	2 0	0 0	0 0
8	1 1	0 0	0 0
9	1 0	1 0	0 0
10	1 0	0 1	0 0
11	1 0	0 0	1 0
12	1 0	0 0	0 1
13	0 2	0 0	0 0
14	0 1	1 0	0 0
15	0 1	0 1	0 0
16	0 1	0 0	1 0
17	0 1	0 0	0 1
18	0 0	2 0	0 0
19	0 0	1 1	0 0
20	0 0	1 0	1 0
21	0 0	1 0	0 1
22	0 0	0 2	0 0
23	0 0	0 1	1 0

Index	Exponents of		
	$X\ P_x$	$Y\ P_y$	$\tau\ P_\tau$
24	0 0	0 1	0 1
25	0 0	0 0	2 0
26	0 0	0 0	1 1
27	0 0	0 0	0 2
28	3 0	0 0	0 0
29	2 1	0 0	0 0
30	2 0	1 0	0 0
31	2 0	0 1	0 0
32	2 0	0 0	1 0
33	2 0	0 0	0 1
34	1 2	0 0	0 0
35	1 1	1 0	0 0
36	1 1	0 1	0 0
37	1 1	0 0	1 0
38	1 1	0 0	0 1
⋮			
77	0 0	0 1	2 0
78	0 0	0 1	1 1
79	0 0	0 1	0 2
80	0 0	0 0	3 0
81	0 0	0 0	2 1
82	0 0	0 0	1 2
83	0 0	0 0	0 3
⋮			

Table 40.2.2: A possible grevlex related indexing scheme for monomials in 6 variables.

Index	Exponents					
0	0	0	0	0	0	0
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	0	0	0
4	0	0	0	1	0	0
5	0	0	0	0	1	0
6	0	0	0	0	0	1
7	2	0	0	0	0	0
8	1	1	0	0	0	0
9	0	2	0	0	0	0
10	1	0	1	0	0	0
11	0	1	1	0	0	0
12	0	0	2	0	0	0
13	1	0	0	1	0	0
14	0	1	0	1	0	0
15	0	0	1	1	0	0
16	0	0	0	2	0	0
17	1	0	0	0	1	0
18	0	1	0	0	1	0
19	0	0	1	0	1	0
20	0	0	0	1	1	0
21	0	0	0	0	2	0
22	1	0	0	0	0	1
23	0	1	0	0	0	1
24	0	0	1	0	0	1
25	0	0	0	1	0	1
26	0	0	0	0	1	1
27	0	0	0	0	0	2
28	3	0	0	0	0	0
29	2	1	0	0	0	0
30	1	2	0	0	0	0
31	0	3	0	0	0	0
32	2	0	1	0	0	0
33	1	1	1	0	0	0
34	0	2	1	0	0	0
35	1	0	2	0	0	0
36	0	1	2	0	0	0
37	0	0	3	0	0	0
38	2	0	0	1	0	0
⋮						
77	0	0	0	0	2	1
78	1	0	0	0	0	2
79	0	1	0	0	0	2
80	0	0	1	0	0	2
81	0	0	0	1	0	2
82	0	0	0	0	1	2
83	0	0	0	0	0	3
⋮						

40.2.6 The Giorgilli Formula

By construction, use of Table 2.1 (or Table 2.2) assigns a unique index i to each monomial exponent j. Moreover, i takes on every possible (non-negative) integer value. Therefore, there is a invertible function $i(j)$ that provides a 1-to-1 mapping between the integers and the exponent vectors j. To proceed further, it would be very useful to have an explicit formula for $i(j)$. Such a formula, which we will call the *Giorgilli* formula, exists. We will illustrate it for the case of 6 variables with monomials indexed as in Table 2.1. In this case the exponent vectors have the form $j = (j_1, \cdots, j_6)$. We begin by defining the integers

$$n(\ell; j_1, \cdots, j_6) = \ell - 1 + \sum_{k=0}^{\ell-1} j_{6-k} \tag{40.2.14}$$

Table 40.2.3: Lowest and highest indices for monomials of degree *ideg* in 6 variables.

ideg	*ibot*	*itop*
0	0	0
1	1	6
2	7	27
3	28	83
4	84	209
5	210	461
6	462	923
7	924	1715
8	1716	3002
9	3003	5004
10	5005	8007
11	8008	12375
12	12736	18563

for $\ell \in \{1, 2, \cdots 6\}$. Then to the general monomial z^j we assign the index

$$i(j) = i(j_1, \cdots j_6) = \sum_{\ell=1}^{6} \text{ Binomial } [n(\ell; j_1, \cdots j_6), \ell]. \tag{40.2.15}$$

Here the quantities

$$\text{Binomial } [n, \ell] = \left(\begin{array}{c} n \\ \ell \end{array} \right) = \left\{ \begin{array}{ccc} \frac{n!}{\ell!(n-\ell)!} & , & 0 \le \ell \le n \\ 0 & , & \text{otherwise} \end{array} \right\} \tag{40.2.16}$$

denote the usual binomial coefficients.

40.2.7 Finding the Required Binomial Coefficients

At this point something needs to be said about what binomial coefficients are actually required and how they can be computed. Note that the formula (2.15) can be written in the form

$$i(j_1, \cdots, j_6) = \text{ Binomial } [n(1; j_1, \cdots, j_6), 1] + \sum_{\ell=2}^{6} \text{ Binomial } [n(\ell; j_1, \cdots j_6), \ell]. \tag{40.2.17}$$

From (2.14) we obtain

$$n(1; j_1, \cdots j_6) = j_6, \tag{40.2.18}$$

and we know that

$$\text{Binomial } [j_6, 1] = j_6. \tag{40.2.19}$$

Thus we may write (2.15) as

$$i(j_1, \cdots, j_6) = j_6 + \sum_{\ell=2}^{6} \text{ Binomial } [n(\ell; j_1, \cdots j_6), \ell]. \tag{40.2.20}$$

Now let *maxdeg* be the maximum degree of the polynomials being stored. Then we have the inequality

$$\sum_{k=0}^{\ell-1} j_{6-k} \leq maxdeg. \tag{40.2.21}$$

Consequently, according to (2.14), $n(\ell; j_1, \cdots, j_6)$ must lie in the range

$$n \in [\ell - 1, \ell - 1 + maxdeg]. \tag{40.2.22}$$

We therefore need only those binomial coefficients Binomial $[n, \ell]$ with $\ell \in [2, 6]$ and, for each ℓ, values of n lying in the range (2.22).

As an example, Exhibit 2.1 shows a program that computes and stores the required binomial coefficients for the case $maxdeg = 6$. It uses the recursion relation (7.3.56), and needs to be executed only once.

Exhibit 32.2.1: A program to compute and store binomial coefficients.

```
      subroutine binom5
c
c     computes a table of the binomial coefficients
c
      implicit double precision (a-h,o-z)
      integer bin5(24,20)
      common /bin5/ bin5
      save/bin5/
      do 1 i=1,20
      bin5(i,1)=i
      do 1 k=2,20
      if (i-k) 2,3,4
    2 bin5(i,k)=0
      go to 1
    3 bin5(i,k)=1
      go to 1
    4 ip=i-1
      kp=k-1
      bin5(i,k)=bin5(ip,kp)+bin5(ip,k)
    1 continue
      return
      end
```

40.2.8 Computation of the Index i Given the Exponent Array j

We will prove eventually that the formula (2.15) does indeed produce the indexing scheme of Table 2.1. Before doing so we will exhibit a computer program that computes $i(j)$. As an example, Exhibit 2.2 shows a computer program that computes $i(j)$ using (2.20) and stored binomial coefficients, and assuming $maxdeg = 6$. For efficiency, the required binomial coefficients have been hard-wired in, and arranged in a convenient order, with the use of a *data* statement. Alternatively, they could have been computed and rearranged in advance, using a variant of the program shown in Exhibit 2.1, and then stored in a common block for use in the program of Exhibit 2.2.

Note that the program requires various binomial lookups and 10 integer adds to compute an index for the case of 6 variables. In the general case of d variables, computing an index requires $2(d-1)$ such adds. The program is therefore reasonably fast.

Exhibit 32.2.2: A program to compute the index $i(j)$ using (2.20) and stored binomial coefficients.

```
      subroutine ndex(j,ind)
c
c  This subroutine calculates the MaryLie index ind, given j1 through j6
c  which are stored in the array j, based on the Giorgilli formula.
c  The use of rearrangement in this algorithm is due to Liam Healy.
c  AJD 6/20/95
c
      integer j(6)
c  cord = cumulative order: sum of exponents from ib-th to 6th.
      integer cord
c  obin = binomial coefficients rearranged to speed up calculation
c         obin(m,i)=Binomial[(m+6-i),(7-i)]
      integer obin(0:6,5)
      save obin
      data obin
     & /0,1,7,28,84,210,462,
     &  0,1,6,21,56,126,252,
     &  0,1,5,15,35, 70,126,
     &  0,1,4,10,20, 35, 56,
     &  0,1,3, 6,10, 15, 21/
c
c  calculate the index
      ind=j(6)
      cord=ind
      do 100 ib=5,1,-1
        cord=cord+j(ib)
  100   ind=ind+obin(cord,ib)
c
      return
      end
```

40.2.9 Preparing a Look-Up Table for the Exponent Array j Given the Index i

Given any exponent array j, we have seen how to compute a corresponding index $i(j)$. There is also the inverse problem: Given an index i, find the exponent array

$$j(i) = \{j_1(i), j_2(i), \cdots j_d(i)\} \tag{40.2.23}$$

that corresponds to this index. From a computational perspective, the most efficient procedure is to prepare a look-up table (a rectangular array) that contains this information.

One way to construct an appropriate look-up table involves finding an algorithm for generating—in the general case of d variables—a modified glex sequence of exponents. With such an algorithm we can produce a look-up table of exponents simply by storing successive d-tuples of exponents as they are generated. In particular, we can initialize the index by setting $i = 0$ for the first exponent vector, $(0, 0, \cdots, 0)$, and then increment i by 1 with each successive exponent vector in the sequence. We now outline a method for generating a modified glex sequence.

As described earlier, one may view the exponents j_1 through j_d that define a particular monomial as the components of a vector $j = (j_1, \cdots j_d)$. Let us refer to any given sequence of such vectors simply as a *list*. Now look at Table 2.1 or recall the definition of glex ordering to see that the list of monomials of a given degree m always begins with the vector

$$j = (m, 0, \cdots 0) \tag{40.2.24}$$

and ends with the vector

$$j = (0, \cdots 0, m). \tag{40.2.25}$$

In addition, the very next element in the list after (2.25) is the vector

$$j = (m + 1, 0, \cdots 0), \tag{40.2.26}$$

which begins the list of monomials of degree $m + 1$. We conclude that it is easy to specify the monomials that begin and end the degree m portion of a modified glex sequence, and to make the transition from the last monomial of degree m to the first one of degree $m + 1$.

We now seek a rule that converts any given j vector, for some monomial of degree m, into the j vector for the next monomial in the list. If the j vector has the form (2.25), then obviously the next j vector has the form (2.26). Otherwise, it is evident that some of the entries in j must be increased and some decreased in such a way as to keep the total degree constant. Moreover, to achieve a modified glex sequence, the entries on the "left end" of j (those j_k with smaller k) should be decreased as little as possible, and the entries on the "right end" of j (those j_k with larger k) should be increased as much as possible. A careful examination of the entries in Table 2.1 shows that one may convert from any j vector not of the form (2.25) to the next j vector via the following sequence of steps:

- Store the value of j_d as $icarry$, and then set $j_d = 0$.
- Test the j_k from right to left to find the right-most non-zero j_k. Let ℓnzj be the subscript for this j_k. (Here ℓnzj is a mnemonic for "last non-zero j".) Thus j_ℓ, with $\ell = \ell nzj$, is the last non-zero component of j.
- Decrease j_ℓ by 1, set $j_{\ell+1} = (1 + icarry)$, and leave intact all other entries of j.

The result of these steps is the next j in the list.

For example, consider the j in Table 2.1 having index 33, $j = (200001)$. In this case we have $icarry = 1$, and setting $j_6 = 0$ yields the vector (200000). We then find that the right-most non-zero entry is j_1; hence $\ell nzj = 1$ and (with $\ell = \ell nzj = 1$) $j_\ell = j_1 = 2$. Decreasing j_1 by 1 and replacing $j_{\ell+1} = j_2$ by $(1+icarry)$, we find the new vector (120000). Examination of Table 2.1 shows that this vector has index 34, as desired. Readers are invited to check other cases in Table 2.1 to satisfy themselves that this procedure works in general.

Exhibit 2.3 shows a routine that carries out the algorithm just described in the case of 6 variables and assuming $maxdeg = 4$. This routine has the further feature [resulting from the initialization of ℓ (ℓnzj) and the use of suitable 'if' statements] that it also automatically makes the transition from the last monomial of degree m to the first one of degree $m + 1$; and it does so by use of the same algorithm just described for generating successive j vectors elsewhere in the list. That is, the same algorithm also produces the transition from (2.25) to (2.26). Readers are also invited to check that this procedure works as claimed.

Exhibit 32.2.3: A program to produce a look-up table for the exponents $j(i)$.

```
      subroutine jtable
c
c  This program creates the look-up table jtbl based on a method of Liam Healy:
c        ind = monomial index and imax = maximum value of ind.
c        ipsv = phase space variable and id = number of phase space variables.
c        For example when id=6, ipsv= 1...id corresponds to X...P_t.
c        jtbl(ipsv,ind) is the exponent of phase space variable
c        'ipsv' (1 to id)  for monomial index 'ind' (1 to imax).
c        For example, when id=6, monomial number 109 is X*P_X*P_X*P_t.
c        Consequently, jtbl(1 to 6,109)=1,2,0,0,0,1.
c
      parameter (imax = 209, id=6)
      dimension jtbl(id,imax)
c  j = array of exponents
      dimension j(id)
c  initialize exponents
      data j/id*0/
c  icarry = temporarily stored value of j(id).
c  lnzj = last non-zero j
c
c  Sequentially create exponent table jtbl
c
      do 150 ind=1,imax
c  set quantities
        icarry=j(id)
        j(id)=0
        lnzj=0
c  search for last nonzero j
        do 100 ipsv=1,id-1
          if (j(ipsv).gt.0) lnzj=ipsv
  100   continue
c  find next set of exponents
        if (lnzj.gt.0) j(lnzj)=j(lnzj)-1
        j(lnzj+1)=1+icarry
c  store exponents in jtbl
        do 120 ipsv=1,id
          jtbl(ipsv,ind)=j(ipsv)
  120   continue
  150 continue
c
c write out table
      do 70 i=1,imax
      write(6,500) i,
     & jtbl(1,i),jtbl(2,i),jtbl(3,i),
     & jtbl(4,i),jtbl(5,i),jtbl(6,i)
  500 format (1h ,i4,2x,3(i2,1x,i2,2x))
   70 continue
c
      end
```

40.2.10 Verification of the Giorgilli Formula

The last task for this section, as promised, is to show that the formula (2.15) does indeed produce modified glex indexing. For this purpose, the reader is invited to examine Table 2.4 which displays a modified glex sequence for the simple case of 3 variables through terms of degree 4.

Consider the $N(m, d)$ monomials of degree m in d variables. We may view each of these monomials $(z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d})$ as a product of two constituent monomials:

$$z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d} = z_1^{j_1} \times z_2^{j_2} \cdots z_d^{j_d}. \tag{40.2.27}$$

Thus, for example, monomials of degree two in three variables comprise three distinct groups (see Table 2.4):

1. z_1^2 times a monomial of degree zero in the two variables z_2 and z_3;
2. z_1^1 times monomials of degree one in the two other variables;
3. z_1^0 times monomials of degree two in the two other variables.

In general, of course, one may write all the monomials of order m in d variables as products between the monomial $z_1^{j_1}$—with $j_1 \in \{0, 1, \ldots, m\}$—and the monomials of order $m - j_1$ in the remaining $d - 1$ variables. The reader may observe that listing the monomials in a modified glex sequence—as in Table 2.4—makes clear the structure just described.

Consider now only those monomials of fixed degree m, and examine their exponents as listed in the modified glex sequence. The reader should note that because the exponents (of fixed degree) are listed in *descending* lexicographic order, eliminating the left-most column—the exponents j_1—will leave behind $d - 1$ columns which contain exactly the modified glex sequence for the monomials in $d - 1$ variables of degree m and smaller. Thus, for example, the transformation displayed in Figure 2.1 shows explicitly how this happens for degree-three monomials in three variables: removing the left column leaves behind the listing, in a modified glex sequence, of all monomials in two variables of degree zero through three.

As a consequence of the observation just described, the index of a given monomial can be determined by using a simple counting procedure, which we illustrate with the following example:

- Look at Table 2.4 and select an exponent, say $j = (1, 0, 2)$, whose index is to be determined. This exponent j represents a monomial of *degree* 3 ($|j| = j_1 + j_2 + j_3 = 1 + 0 + 2 = 3$) in 3 variables.

Table 40.2.4: Modified glex sequence for j in 3 variables.

Index	j_1	j_2	j_3
1	1	0	0
2	0	1	0
3	0	0	1
4	2	0	0
5	1	1	0
6	1	0	1
7	0	2	0
8	0	1	1
9	0	0	2
10	3	0	0
11	2	1	0
12	2	0	1
13	1	2	0
14	1	1	1
15	1	0	2
16	0	3	0
17	0	2	1
18	0	1	2
19	0	0	3
20	4	0	0
21	3	1	0
22	3	0	1
23	2	2	0
24	2	1	1
25	2	0	2
26	1	3	0
27	1	2	1
28	1	1	2
29	1	0	3
30	0	4	0
31	0	3	1
32	0	2	2
33	0	1	3
34	0	0	4

Figure 40.2.1: Sample extraction of a two-column array from a three-column array.

$$\begin{array}{ccc} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{array} \quad \mapsto \quad \begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 1 & 1 \\ 0 & 2 \\ 3 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{array}$$

- Dropping the first entry from j yields a reduced exponent $j' = (0, 2)$, which represents a monomial of *degree* 2 ($|j'| = j_2 + j_3 = 0 + 2 = 2$) in 2 variables.

- Dropping the first two entries from j yields $j'' = (2)$, which represents a monomial of *degree* 2 ($|j''| = j_3 = 2$) in 1 variable.

- Record the *degrees* of j, j', and j'', as described in the previous steps. In the present case we obtain the numbers $|j| = 3$, $|j'| = 2$, and $|j''| = 2$, respectively. Based on these degrees, construct a "path" through Table 2.4, as illustrated in Figure 2.2: Begin the path at the top and proceed down the j_1 column until you reach exponents of degree $|j| = 3$. Then shift over one column and proceed down the j_2's until you reach exponents of degree $|j'| = 2$. Finally, shift over to the last column and proceed down the j_3's until you reach exponents of degree $|j''| = 2$.

- Now determine the index by the evident procedure of simply adding together the "lengths" of the vertical portions of the path just constructed, and then adding 1. The lengths are given by the relations

$$\begin{aligned} \text{length along } j_1 \text{ column} &= S(2,3), & (40.2.28)\\ \text{length along } j_2 \text{ column} &= S_0(1,2), & (40.2.29)\\ \text{length along } j_3 \text{ column} &= S_0(1,1), & (40.2.30) \end{aligned}$$

  and hence the index of the monomial $j = (1, 0, 2)$ is given by

$$i(j) = i(1,0,2) = S(2,3) + S_0(1,2) + S_0(1,1) + 1. \tag{40.2.31}$$

  Using (1.6), we may write (2.31) in the more pleasing form

$$i(1,0,2) = S_0(2,3) + S_0(1,2) + S_0(1,1) = 10 + 3 + 2 = 15, \tag{40.2.32}$$

  in agreement with the index given in Table 2.4 (or Figure 2.2).

Figure 40.2.2: Path to the exponent $j = (1, 0, 2)$ down the modified glex sequence in 3 variables.

Index	j_1	j_2	j_3	
1	1	0	0	
2	0	1	0	
3	0	0	1	
4	2	0	0	
5	1	1	0	$S(2,3) = 9 = S_0(2,3) - 1$
6	1	0	1	
7	0	2	0	
8	0	1	1	
9	0	0	2	
10	3	0	0	
11	2	1	0	$S_0(1,2) = 3$
12	2	0	1	
13	1	2	0	$S_0(1,1) = 2$
14	1	1	1	
15	1	0	2	1
16	0	3	0	
17	0	2	1	
18	0	1	2	
19	0	0	3	
20	4	0	0	
21	3	1	0	
22	3	0	1	
23	2	2	0	
24	2	1	1	
25	2	0	2	
26	1	3	0	
27	1	2	1	
28	1	1	2	
29	1	0	3	
30	0	4	0	
31	0	3	1	
32	0	2	2	
33	0	1	3	
34	0	0	4	

By generalizing the procedure of the example just described, we can now state a procedure for determining the index $i(j)$ for any monomial whose exponent list is given by $j = (j_1, j_2, \ldots, j_d)$.

1. For each $\nu \in \{1, \ldots, d\}$ define m_ν as the degree of the monomial obtained by dropping from the exponent list j the first $(\nu - 1)$ entries:
$$m_\nu = \sum_{k=\nu}^{d} j_k. \tag{40.2.33}$$

2. Then define i_ν as the total number of monomials in $d - (\nu - 1)$ variables which have degree *less* than m_ν:
$$i_\nu = S_0(m_\nu - 1, d - \nu + 1). \tag{40.2.34}$$

3. The index $i(j)$ is then given by the formula
$$i(j_1, \ldots, j_d) = \sum_{\nu=1}^{d} i_\nu. \tag{40.2.35}$$

Finally, we must demonstrate that the prescription just given for determining the index is indeed equivalent to the formula (2.15). To see this, note first that
$$m_\nu = \sum_{k=\nu}^{d} j_k = \sum_{k=0}^{d-\nu} j_{d-k}. \tag{40.2.36}$$

Then recall the definition (2.14) and simply compute:
$$\begin{aligned} i(j_1, \ldots, j_d) &= \sum_{\nu=1}^{d} i_\nu = \sum_{\nu=1}^{d} S_0(m_\nu - 1, d - \nu + 1) = \sum_{\nu=1}^{d} \begin{pmatrix} m_\nu + d - \nu \\ d - \nu + 1 \end{pmatrix} \\ &= \sum_{\nu=1}^{d} \begin{pmatrix} (\sum_{k=0}^{d-\nu} j_{d-k}) + d - \nu \\ d - \nu + 1 \end{pmatrix} = \sum_{\ell=1}^{d} \begin{pmatrix} (\sum_{k=0}^{\ell-1} j_{d-k}) + \ell - 1 \\ \ell \end{pmatrix} \\ &= \sum_{\ell=1}^{d} \begin{pmatrix} n(\ell; j) \\ \ell \end{pmatrix}. \end{aligned} \tag{40.2.37}$$

This result confirms that the formula (2.15) does indeed return the desired index.

Exercises

40.2.1. Verify that one can easily convert between the glex and grevlex orderings by a "double reversal" procedure: among each set of monomials of a given degree first reverse the order of the variables, then reverse the order of the monomials. For example, consider the monomials of degree 2 in 3 variables. Under glex ordering we have
$$z_1^2 > z_1 z_2 > z_1 z_3 > z_2^2 > z_2 z_3 > z_3^2. \tag{40.2.38}$$

Now reverse the order of the variables by making the replacement $z_1,\ z_2,\ z_3 \to z_3,\ z_2,\ z_1$, and replace $>$ by $<$. Then (2.38) becomes

$$z_3^2 < z_2 z_3 < z_1 z_3 < z_2^2 < z_1 z_2 < z_1^2, \tag{40.2.39}$$

or, equivalently,

$$z_1^2 > z_1 z_2 > z_2^2 > z_1 z_3 > z_2 z_3 > z_3^2. \tag{40.2.40}$$

Upon comparing (2.40) and (2.10), we see that (2.40) is in grevlex order.

40.3 Scalar Multiplication and Polynomial Addition

The use of any indexing scheme optimizes the operations of scalar multiplication and polynomial addition. Let $M_i(z)$ denote the generic monomial

$$M_i(z) = z^{j(i)}, \tag{40.3.1}$$

where it is assumed that the exponent j vectors (arrays) are indexed by an index i. Suppose f is any truncated power series. Then, since the M_i form a basis, there is a unique decomposition of the form

$$f = \sum_i f^i M_i \tag{40.3.2}$$

where the f^i are known coefficients. Indeed, the function f is stored by storing each coefficient f^i at the i^{th} location in some array.

Now suppose h is some other function that is related to f by scalar multiplication:

$$h = af \tag{40.3.3}$$

where a is some scalar. Then h has the decomposition

$$h = \sum_i h^i M_i \tag{40.3.4}$$

with the coefficients h^i given by the relation

$$h^i = af^i. \tag{40.3.5}$$

Thus, when any indexing scheme is employed, multiplication of a function by a scalar is equivalent to scalar multiplication of a vector.

Next suppose that f and g are any two polynomials, and we wish to compute the sum

$$h = f + g. \tag{40.3.6}$$

Then we have unique decompositions

$$f = \sum_i f^i M_i, \tag{40.3.7}$$

$$g = \sum_i g^i M_i, \tag{40.3.8}$$

$$h = \sum_i h^i M_i, \tag{40.3.9}$$

where the f^i and g^i are known coefficients, and the h^i are to be determined by (3.6). It follows, again from the fact that the M_i form a basis, that we have the relation

$$h^i = f^i + g^i. \tag{40.3.10}$$

Thus, when any indexing scheme is employed, polynomial addition is equivalent to the simple process of vector addition.

40.4 Polynomial Multiplication

Multiplication of polynomials is more complicated than addition. As before, define basis monomials $M_i(z)$ by (3.1). Also, suppose the polynomials f and g have the decompositions

$$f = \sum_i f^i M_i, \tag{40.4.1}$$

$$g = \sum_k g^k M_k, \tag{40.4.2}$$

and we wish to compute the product

$$h = fg. \tag{40.4.3}$$

The polynomial h will have the decomposition

$$h = \sum_\ell h^\ell M_\ell, \tag{40.4.4}$$

and the problem is two determine the h^ℓ from the relation

$$\begin{aligned} h &= \sum_\ell h^\ell M_\ell = fg = \sum_i f^i M_i \sum_k g^k M_k \\ &= \sum_{i,k} f^i g^k M_i M_k. \end{aligned} \tag{40.4.5}$$

We see that the basic problem consists of computing the products $M_i M_k$.

There are at least 3 ways to solve this problem in the context of indexing:

1. Given the indices i and k, find (say by table look-up) the corresponding exponents $j(i)$ and $j(k)$. Add these exponents as vectors to get the resultant "sum" exponent vector j^s,

   $$j^s = j(i) + j(k). \tag{40.4.6}$$

   Here we have reckoned with the obvious relation

   $$z^{j(i)} z^{j(k)} = z^{j(i)+j(k)} = z^{j^s}. \tag{40.4.7}$$

Next find the index ℓ corresponding to j^s. If the monomials have been indexed using modified glex sequencing, one can evaluate the Giorgilli formula (2.15) for this purpose to find

$$\ell = i(j^s). \tag{40.4.8}$$

Finally, increment h^ℓ by the quantity $f^i g^k$. (Here we have assumed that all the h^ℓ were initially set to zero.)

2. Given the indices i and k, find directly from a specially prepared look-up table the corresponding value of ℓ. Then increment h^ℓ by the quantity $f^i g^k$.

3. Given the index ℓ, use specially prepared *look-back* tables to find all indices i and k such that

$$M_i M_k = M_\ell. \tag{40.4.9}$$

Then increment h^ℓ by all the products $f^i g^k$.

All these methods will be discussed and compared in subsequent sections. At this point we simply remark that the computation of the products $M_i M_k$ is facilitated by the use of a *graded* indexing scheme. Since we are limiting our computations and their results to polynomials of degree less than or equal to m, we know that many of the products $M_i M_k$ need not be computed because their results are monomials of degree larger than m. The identification of such unneeded products is easy in any graded indexing scheme, because the degree of a product is the sum of the degrees of the factors.

40.5 Look-Up Tables

Let f and g be two polynomials. They can be decomposed into sums of homogeneous polynomials, and hence can be written in the form

$$f = f_0 + f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + \cdots , \tag{40.5.1}$$

$$g = g_0 + g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + \cdots . \tag{40.5.2}$$

Here f_m and g_m denote homogeneous polynomials of degree m. Correspondingly, the product fg of any two polynomials can also be organized into terms of common degree. Doing so gives the result

$$\begin{aligned} fg &= (f_0g_0)_0 + (f_0g_1 + g_0f_1)_1 + (f_0g_2 + g_0f_2 + f_1g_1)_2 \\ &+ (f_0g_3 + g_0f_3 + f_1g_2 + g_1f_2)_3 + (f_0g_4 + g_0f_4 + f_1g_3 + g_1f_3 + f_2g_2)_4 \\ &+ (f_0g_5 + g_0f_5 + f_1g_4 + g_1f_4 + f_2g_3 + g_2f_3)_5 \\ &+ (f_0g_6 + g_0f_6 + f_1g_5 + g_1f_5 + f_2g_4 + g_2f_4 + f_3g_3)_6 + \cdots . \end{aligned} \tag{40.5.3}$$

Here the terms appearing in fg have been collected according to their degrees using parentheses, and the parentheses have been given a subscript indicating the degree of the terms enclosed.

There is no particular problem in carrying out multiplications of the form $f_0 g_m$ and $g_0 f_m$ (since this operation is equivalent to scalar multiplication if the entries in g_m and f_m are viewed as the components of a vector). Multiplications of the form $f_m g_n$ and $g_m f_n$ with $m, n > 0$ are more complicated to execute.

Suppose we rearrange the computationally intensive terms (those not of the form $f_0 g_m$ and $g_0 f_m$) in (5.3) as shown below:

$$\begin{aligned}
&\text{computationally intensive terms } = \\
&(f_1g_1) + (f_1g_2 + f_1g_3 + f_1g_4 + f_1g_5 + \cdots) + (g_1f_2 + g_1f_3 + g_1f_4 + g_1f_5 + \cdots) \\
&+(f_2g_2) + (f_2g_3 + f_2g_4 + \cdots) + (g_2f_3 + g_2f_4 + \cdots) \\
&+(f_3g_3) + \cdots .
\end{aligned} \tag{40.5.4}$$

If these multiplications were to be performed with the aid of a look-up table, what would this table look like, and how big would it be?

As a simple but instructive example, consider the case of polynomials through degree 4 in 2 variables ($m = 4$ and $d = 2$). Table 5.1 below shows, using phase-space notation, the monomials for this case listed in a modified glex sequence. We wish to multiply polynomials composed of these monomials, but only retain terms through degree 4.

Table 40.5.1: Modified glex sequence when $m = 4$ and $d = 2$.

i	j_1	j_2	$\vert j\vert$	monomial
1	1	0	1	X
2	0	1		P_x
3	2	0	2	X^2
4	1	1		XP_x
5	0	2		P_x^2
6	3	0	3	X^3
7	2	1		X^2P_x
8	1	2		XP_x^2
9	0	3		P_x^3
10	4	0	4	X^4
11	3	1		X^3P_x
12	2	2		$X^2P_x^2$
13	1	3		XP_x^3
14	0	4		P_x^4

Now consider the result of multiplying a monomial in f with index if and a monomial in g with index ig. The result will be some monomial in h with index ih. (Here, to simplify notation, we denote the relevant indices by if, ig, and ih rather than the i, k, ℓ of the previous Section.) Table 5.2 shows a multiplication table for this process. It gives, for each value of if and ig, the value of ih corresponding to the product monomial. Entries with an asterisk "$*$′′ correspond to monomials having degree greater than 4. See Table 5.1. These entries fall outside our interest. Consider, for example, the relation

$$X^2 \times XP_x = X^3P_x. \tag{40.5.5}$$

Inspection of Table 5.1 shows that the monomials X^2 and XP_x have the indices $if = 3$ and $ig = 4$, respectively; and their product X^3P_x has the index $ih = 11$. Correspondingly, the entry in Table 5.2 for $if = 3$ and $ig = 4$ has the value $ih = 11$.

Table 40.5.2: Multiplication table (when $m = 4$ and $d = 2$) giving values of ih.

$if\backslash ig$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	3	4	6	7	8	10	11	12	13	*	*	*	*	*
2	4	5	7	8	9	11	12	13	14	*	*	*	*	*
3	6	7	10	11	12	*	*	*	*	*	*	*	*	*
4	7	8	11	12	13	*	*	*	*	*	*	*	*	*
5	8	9	12	13	14	*	*	*	*	*	*	*	*	*
6	10	11	*	*	*	*	*	*	*	*	*	*	*	*
7	11	12	*	*	*	*	*	*	*	*	*	*	*	*
8	12	13	*	*	*	*	*	*	*	*	*	*	*	*
9	13	14	*	*	*	*	*	*	*	*	*	*	*	*
10	*	*	*	*	*	*	*	*	*	*	*	*	*	*
11	*	*	*	*	*	*	*	*	*	*	*	*	*	*
12	*	*	*	*	*	*	*	*	*	*	*	*	*	*
13	*	*	*	*	*	*	*	*	*	*	*	*	*	*
14	*	*	*	*	*	*	*	*	*	*	*	*	*	*

Note that Table 5.2 has the form of a *symmetric* matrix with 5 diagonal entries and 18 entries above the diagonal. This symmetry results from the commutativity of (ordinary) multiplication. Let $\{m, d\}$ denote the number of locations (look-up table size) required to store a multiplication table when this symmetry is taken into account. Then we have the result $\{4, 2\} = (5 + 18) = 23$. Table 5.3 lists values of $\{m, d\}$ for various values of m and d.

As will be seen in later sections, it may sometimes be advantageous to sacrifice the savings in storage associated with symmetry in order to gain speed. We should therefore also calculate the storage required when symmetry is ignored. Let $\{m, d\}^{ns}$ denote the number of locations required to store the *full* multiplication table when symmetry is not taken into account. (Here the superscript ns denotes *no symmetry.*) Then we have the result $\{4, 2\}^{ns} = (5 + 18 + 18) = 41$. Table 5.4 lists values of $\{m, d\}^{ns}$ for various values of m and d.

We close this section with a description of how the dimensions $\{m, d\}$ and $\{m, d\}^{ns}$ can be computed in a general case. Suppose, for example, we wish to know the dimensionality of the look-up table associated with the terms displayed in (5.4). (This is equivalent to retaining terms through degree $m = 6$.) Let us also consider the case of 6 variables, $d = 6$. Thus, we need to find $\{6, 6\}$ and $\{6, 6\}^{ns}$.

We begin with the case where symmetry is taken into account. Since the portion of the look-up table associated with (f_1g_1) is 6×6 and symmetric, we have the result

$$\{f_1g_1\} = [(6 \times 6) - 6)]/2 + 6 = (6 \times 7)/2 = 21. \tag{40.5.6}$$

Here we use the notation { } to denote the dimensionality of the associated portion of the look-up table. Similarly, we find the results

Table 40.5.3: Multiplication look-up table size for polynomials of degree 1 through m in various numbers of variables.

m	$\{m,2\}$	$\{m,3\}$	$\{m,4\}$	$\{m,5\}$	$\{m,6\}$	$\{m,7\}$	$\{m,8\}$	$\{m,9\}$
2	3	6	10	15	21	28	36	45
3	9	24	50	90	147	224	324	450
4	23	75	185	385	714	1218	1950	2970
5	45	180	525	1260	2646	5040	8910	14850
6	82	388	1309	3570	8400	17724	34386	62403
7	134	748	2905	8960	23520	54768	116226	229020
8	210	1354	5975	20655	60087	153615	355113	757185
9	310	2300	11475	44250	142065	397320	997425	2295150
10	445	3746	20941	89501	315546	961576	2612753	6470178
11	615	5852	36489	172116	663894	2197272	6444009	17131686
12	833	8869	61270	317366	1333976	4779320	15086409	42955185

m	$\{m,10\}$	$\{m,11\}$	$\{m,12\}$	$\{m,13\}$	$\{m,14\}$	$\{m,15\}$
2	55	66	78	91	105	120
3	605	792	1014	1274	1575	1920
4	4345	6149	8463	11375	14980	19380
5	23595	36036	53235	76440	107100	146880
6	107250	176176	278551	426244	634032	920040
7	424710	748748	1264627	2058784	3246320	4977600
8	1510795	2851563	5134090	8875802	14811930	23963370
9	4915625	9912760	18990530	34807500	61386150	104652000

m	$\{m,16\}$	$\{m,17\}$	$\{m,18\}$	$\{m,19\}$	$\{m,20\}$	$\{m,21\}$
2	136	153	171	190	210	231
3	2312	2754	3249	3800	4410	5082
4	24684	31008	38475	47215	57365	69069
5	197676	261630	341145	438900	557865	701316
6	1306212	1818813	2488962	3353196	4454065	5840758

$$\{f_2 g_2\} = (21 \times 22)/2 = 231, \tag{40.5.7}$$

$$\{f_3 g_3\} = (56 \times 57)/2 = 1596. \tag{40.5.8}$$

See Table 7.3.1. Next consider the portion of the look-up table required for $(f_1g_2 + f_1g_3 + f_1g_4 + f_1g_5)$. We write its dimensionality in the form

$$\begin{aligned}\{f_1g_2 + f_1g_3 + f_1g_4 + f_1g_5\} &= \{f_1\} \times \{g_2 + g_3 + g_4 + g_5\} \\ &= 6 \times (21 + 56 + 126 + 252) = 2730. \end{aligned} \tag{40.5.9}$$

Table 40.5.4: Multiplication look-up table size for polynomials of degree 1 through m in various numbers of variables when symmetry is not exploited.

m	$\{m,2\}^{ns}$	$\{m,3\}^{ns}$	$\{m,4\}^{ns}$	$\{m,5\}^{ns}$	$\{m,6\}^{ns}$	$\{m,7\}^{ns}$	$\{m,8\}^{ns}$	$\{m,9\}^{ns}$
2	4	9	16	25	36	49	64	81
3	16	45	96	175	288	441	640	891
4	41	141	356	750	1401	2401	3856	5886
5	85	351	1036	2500	5265	10045	17776	29646
6	155	757	2584	7085	16717	35329	68608	124587
7	259	1477	5776	17865	46957	109417	232288	457821
8	406	2674	11881	41185	119965	306901	709732	1513656
9	606	4566	22881	88375	283921	794311	1994356	4589586
10	870	7437	41757	178751	630631	1922361	5224220	12938355
11	1210	11649	72853	343981	1327327	4393753	12886732	34261371
12	1639	17655	122331	634271	2667029	9556925	30169816	85905366

m	$\{m,10\}^{ns}$	$\{m,11\}^{ns}$	$\{m,12\}^{ns}$	$\{m,13\}^{ns}$	$\{m,14\}^{ns}$	$\{m,15\}^{ns}$
2	100	121	144	169	196	225
3	1200	1573	2016	2535	3136	3825
4	8625	12221	16836	22646	29841	38625
5	47125	71995	106380	152776	214081	293625
6	214215	351989	556648	851929	1267385	1839265
7	849135	1497133	2528800	4117009	6491961	9954385
8	3020590	5701762	10266361	17749225	29620801	47922865
9	9830250	19824156	37979241	69612621	122769241	209300125

m	$\{m,16\}^{ns}$	$\{m,17\}^{ns}$	$\{m,18\}^{ns}$	$\{m,19\}^{ns}$	$\{m,20\}^{ns}$	$\{m,21\}^{ns}$
2	256	289	324	361	400	441
3	4608	5491	6480	7581	8800	10143
4	49216	61846	76761	94221	114500	137886
5	395200	523090	682101	877591	1115500	1402380
6	2611456	3636487	4976595	6704853	8906360	11679493

Here we use the notation $\{f_m\}$ and $\{g_m + \cdots + g_n\}$ to denote the dimensionality of the spaces of polynomials of degree m and degrees m through n, respectively. Similarly, we find the results

$$\begin{aligned}\{f_2g_3 + f_2g_4\} &= \{f_2\} \times \{g_3 + g_4\} \\ &= 21 \times (56 + 126) = 3822. \end{aligned} \tag{40.5.10}$$

Finally we note that, by symmetry, the evaluation of $(f_1g_2 + f_1g_3 + f_1g_4 + f_1g_5)$ and $(g_1f_2 + g_1f_3 + g_1f_4 + g_1f_5)$ can be done with the same look-up information, and similarly for $(f_2g_3 + f_2g_4)$ and $(g_2f_3 + g_2f_4)$, etc. Thus, we have covered all possibilities for polynomials of degree 1 through 6. We find that the total dimension of the look-up table for such polynomials (in 6 variables) is given by the sum

$$\{6, 6\} = 21 + 231 + 1596 + 2730 + 3822 = 8400. \tag{40.5.11}$$

Consider next the case where symmetry is ignored. Then (for $d = 6$) we have the results

$$\{f_1g_1\}^{ns} = 6 \times 6 = 36, \tag{40.5.12}$$

$$\{f_2g_2\}^{ns} = 21 \times 21 = 441, \tag{40.5.13}$$

$$\{f_3g_3\}^{ns} = 56 \times 56 = 3136. \tag{40.5.14}$$

The quantities (5.9) and (5.10) remain unchanged, but equal numbers of storage locations must now be allocated for their reversed counterparts. Consequently, we find that the total number of storage locations required through degree 6 (in 6 variables) when symmetry is not exploited is given by the relation

$$\{6, 6\}^{ns} = 36 + 441 + 3136 + 2 \times 2730 + 2 \times 3822 = 16,717. \tag{40.5.15}$$

We see that this number is slightly less than twice (5.11).

It is remarkable how much the decision to retain only terms through degree m (e.g. ignore the * terms in Table 5.2) affects the size of the multiplication look-up table. Consider the case of polynomials of degree 1 through 12 in 6 variables. According to Table 7.9.1 there are

$$S(12, 6) = 18,563 \tag{40.5.16}$$

basis monomials in this case. Thus one might naively expect that

$$18,563 \times 18,563 \simeq 3.4 \times 10^8 \tag{40.5.17}$$

storage locations would be required for a multiplication look-up table. This number is prohibitively large. However, examination of Table 5.4 gives the much smaller result

$$\{12, 6\}^{ns} = 2,667,029. \tag{40.5.18}$$

According to (2.10) and (5.16) the largest index in the case $m = 12$ and $d = 6$ is 18,563. We note that 2^{15} = 32,768. Thus, the indices could all be stored in 2 byte entries, and the storage required by (5.18) would be approximately 5.3 megabytes.

Exercises

40.5.1.

40.6 Scripts

Suppose we wish to carry out some calculation that potentially involves a great deal of index manipulation, and suppose we have some simple-minded method for doing so. What we can then do is carry out this simple-minded method, while at the same time making and keeping a *record* of the *results* of the *index manipulation*. We will call this record a *script*. Then, any time we wish to repeat the calculation, we can bypass the need for index manipulation by use of the script.

As an example of this approach, consider the problem of multiplying two polynomials f and g to find the product

$$h = fg. \tag{40.6.1}$$

For simplicity, we consider only the computationally intensive terms (5.4). Exhibit 6.1 shows a simple-minded procedure for doing so.

Exhibit 32.6.1: Simple-minded program for polynomial multiplication.

```
c  Loop over degree ifdeg and indices if and ig.
      do 10 ifdeg=1,maxdeg-1
      do 20 if=ibot(ifdeg),itop(ifdeg)
      do 30 ig=1,itop(maxdeg-ifdeg)
c  Look up exponent vectors corresponding to if and ig and add them.
      do l=1,6
      jf(l)=jtbl(l,if)
      jg(l)=jtbl(l,ig)
      jsum(l)=jf(l)+jg(l)
      end do
c  Find the index of vector sum.
      call ndex(jsum,ih)
c  Carry out multiplication, and store result in h(ih).
      h(ih)=h(ih)+f(if)*g(ig)
  30  continue
  20  continue
  10  continue
c
      return
      end
```

What is shown is a fragment of a FORTRAN program with various parameter and dimension statements omitted. The procedure employs a triple loop that goes through all relevant degrees in f and all relevant index pairs if and ig. Suppose *maxdeg* is the maximum degree of the monomials we wish to retain. Then, in the factor f, we only need work with monomials whose degree *ifdeg* lies in the range $1 \le ifdeg \le maxdeg - 1$. Note that exponents (and hence degrees) add under the operation of multiplication; and recall that we are only considering computationally intensive terms so that in both the factors f, g only terms of degree 1 and higher appear. Correspondingly, when working with terms of degree *ifdeg* in the factor f, the only terms in g that are required are those whose degree *igdeg* lies in the range $1 \leq igdeg \leq maxdeg - ifdeg$. All these considerations are implemented conveniently, as shown, with the use of the arrays *ibot* and *itop*. See Section 32.2.5.

For each *if*, *ig* pair the following operations are performed:

1. Look up the corresponding exponents jf and jg from a previously prepared and stored table $jtb\ell$.

2. Add these exponents to find the resulting exponent.

3. Compute the index ih corresponding to this exponent.

4. Carry out the multiplication of the monomial coefficients from f and g, and increment the monomial coefficient of h corresponding to the index ih found in step 3 by the resulting product.

Evidently this procedure requires numerous look-ups (step 1), numerous additions (step 2), and numerous calls to an index computation routine (in this case the subroutine *ndex*, see Exhibit 2.2).

Exhibit 6.2 shows the same routine except that step 4 above is replaced by storage of the relevant index triplets if, ig, and ih. They are stored sequentially in terms of a "counting" index ic. The result of running this routine is the arrays (look-up tables) $iftb\ell$, $igtb\ell$, and $ihtb\ell$. The size of each of these arrays is $\{m, d\}^{ns}$ with $m = maxdeg$ and (in this example) $d = 6$. The array *icmin*, whose purpose will be described later, is also filled.

```
Exhibit 32.6.2: Program for preparation of script for polynomial multiplication.
This program prepares the tables iftbl, igtbl, ihtbl, and records their size,
icmax.  It also fills the array icmin.

c  Initialize counter.
      ic=0
c  Loop over degree ifdeg and indices if and ig.
      do 10 ifdeg=1,maxdeg-1
      do 20 if=ibot(ifdeg),itop(ifdeg)
      do 30 ig=1,itop(maxdeg-ifdeg)
c  Look up exponent vectors corresponding to if and ig and add them.
      do l=1,6
      jf(l)=jtbl(l,if)
      jg(l)=jtbl(l,ig)
      jsum(l)=jf(l)+jg(l)
      end do
c  Find the index of vector sum.
      call ndex(jsum,ih)
c  Update counter and fill tables.
      ic=ic+1
      iftbl(ic)=if
      igtbl(ic)=ig
      ihtbl(ic)=ih
c  Fill icmin.
      if(ig .eq. 1) icmin(if)=ic
   30 continue
```

```
  20  continue
  10  continue
c  Set icmax
      icmax=ic
c
      return
      end
```

The arrays $iftb\ell$, $igtb\ell$, and $ihtb\ell$ can now be used as a script to carry out multiplication. Exhibit 6.3 shows how. We see, by comparing Exhibits 6.1 and 6.3, that the triple loop has been replaced by a single loop, and all index and exponent manipulations and calculations are replaced by table look up. Indeed, the routine consists entirely of the actual calculation to be performed (step 4 above) and table look up. Evidently this program should run far faster than that of Exhibit 6.1. The price paid for this speed is the storage required for the tables $iftb\ell$, $igtb\ell$, and $ihtb\ell$.

Exhibit 32.6.3: Program for polynomial multiplication using a script.

```
      do 10 ic=1,icmax
      if=iftbl(ic)
      ig=igtbl(ic)
      ih=ihtbl(ic)
  10  h(ih)=h(ih)+f(if)*g(ig)
      end
```

An even more compact program for polynomial multiplication using a script.

```
      do 10 ic=1,icmax
  10  h(ihtbl(ic))=h(ihtbl(ic))+f(iftbl(ic))*g(igtbl(ic))
      end
```

Now that we understand the basic idea of how a script works, we should seek to optimize the procedure. In particular, the tables $iftb\ell$ and $igtb\ell$ are not actually necessary. This is good because their size, $\{m, d\}^{ns}$, can be quite large.

Here the arrays *ibot* and *itop* can again be utilized. Consider the program fragment shown in Exhibit 6.4. Evidently, by construction, this code will produce the *same* set of if and ig values, and in the *same order*, as those stored in the routine of Exhibit 6.2 and used in the routine of Exhibit 6.3. Thus, the need for the tables $iftb\ell$ and $igtb\ell$ has been eliminated.

Exhibit 32.6.4: Program for producing if and ig pairs using less storage and fewer look ups.

```
      do 10 ifdeg=1,maxdeg-1
      do 20 if=ibot(ifdeg),itop(ifdeg)
      do 30 ig=1,itop(maxdeg-ifdeg)
      .
      .
      .
   30 continue
   20 continue
```

```
10 continue
   end
```

We are now ready to reap the fruit of our deliberations. We know that the routine of Exhibit 6.4 produces the same if and ig values, and in the same order, as those stored in the routine of Exhibit 6.2 and used in the multiplication routine of Exhibit 6.3. Also, we see from Exhibit 6.2 that the pointer ic is incremented by 1 each time an if, ig pair is produced. As a result of these facts, we may write a multiplication routine equivalent to that of Exhibit 6.3 using the loop structure of Exhibit 6.4. This multiplication routine is shown in Exhibit 6.5 below. We see that this routine uses the table $ihtb\ell$, as does the routine of Exhibit 6.3, but does not use the tables $iftb\ell$ and $igtb\ell$. As noted earlier, this is a considerable savings in storage since these tables may be large.

Exhibit 32.6.5: Program for polynomial multiplication using only one index look-up table.

```
   ic=1
   do 10 ifdeg=1,maxdeg-1
   do 20 if=ibot(ifdeg),itop(ifdeg)
   do 30 ig=1,itop(maxdeg-ifdeg)
   h(ihtbl(ic))=h(ihtbl(ic))+f(if)*g(ig)
   ic=ic+1
30 continue
20 continue
10 continue
   end
```

There is one last possible improvement to be considered. Suppose one of the factors in (3.1), say f, is sparse. This is often the case in problems of practical interest. In this case, the program shown in Exhibit 6.5 can be improved to exploit sparseness in f. To see how this may be done, consider the contents of the arrays $iftb\ell$, $igtb\ell$, and $ihtb\ell$. Table 6.1, for example, shows the contents of these arrays in the case $m = maxdeg = 4$ and $d = 2$. Inspection of Table 6.1 shows that for each value of if there is a minimum value of ic for which $iftb\ell(ic) = if$. Evidently, for a given value of if, the minimum value of ic occurs when $ig = 1$. Suppose these values are put in an array which we will call $icmin(if)$. For example, Table 6.2 shows such an array in the case $m = 4$ and $d = 2$. Finally, inspection of the code in Exhibit 6.2 shows how the array $icmin$ can be constructed in general, and in particular for the case $d = 6$.

Table 40.6.1: Contents of the arrays $iftb\ell$, $igtb\ell$, and $ihtb\ell$ in the case $m = 4$ and $d = 2$.

ic	$iftb\ell$	$igtb\ell$	$ihtb\ell$
1	1	1	3
2		2	4
3		3	6
4		4	7
5		5	8
6		6	10
7		7	11
8		8	12
9		9	13
10	2	1	4
11		2	5
12		3	7
13		4	8
14		5	9
15		6	11
16		7	12
17		8	13
18		9	14
19	3	1	6
20		2	7
21		3	10
22		4	11
23		5	12
24	4	1	7
25		2	8
26		3	11
27		4	12
28		5	13
29	5	1	8
30		2	9
31		3	12
32		4	13
33		5	14
34	6	1	10
35		2	11

ic	$iftb\ell$	$igtb\ell$	$ihtb\ell$
36	7	1	11
37		2	12
38	8	1	12
39		2	13
40	9	1	13
41		2	14

Table 40.6.2: Contents of the array $icmin$ in the case $m = 4$ and $d = 2$.

if	$icmin$
1	1
2	10
3	19
4	24
5	29
6	34
7	36
8	38
9	40

Now consider the program shown in Exhibit 6.6 below. It tests the factor $f(if)$ before going into the do loop over ig. This whole loop is skipped if $f(if)$ is zero. If this loop were ever skipped in the program of Exhibit 6.5, the value of ic would be upset because it is incremented within this loop. However, if ic is properly set each time the program goes into this loop, which is what use of the array $icmin$ does, then ic always has the proper value even if this loop has possibly been skipped for some earlier values of if.

Exhibit 32.6.6: Program for polynomial multiplication using only one index look-up table and designed to exploit possible sparseness in the factor f.

```
      do 10 ifdeg=1,maxdeg-1
      do 20 if=ibot(ifdeg),itop(ifdeg)
      if(f(if) .ne. 0.d0) then
      ic=icmin(if)
      do 30 ig=1,itop(maxdeg-ifdeg)
      h(ihtbl(ic))=h(ihtbl(ic))+f(if)*g(ig)
      ic=ic+1
   30 continue
      endif
   20 continue
   10 continue
      end
```

Let us briefly compare the theoretical speeds of the programs shown in Exhibits 6.5 and 6.6. We see that the program in Exhibit 6.6 makes $ifmax = itop(maxdeg - 1) =$

$S(maxdeg - 1, d)$ "if" tests. See (2.20). When an if test is passed, there is the additional burden of an $icmin(if)$ look up. However, when an if test fails, there is a savings of $igmax = itop(maxdeg - ifdeg)$ additions and multiplies as well as the overhead and other operations associated with the ig loop. Since $igmax$ can often be large, and multiplications are slow, we conclude (providing the if test burden is not too large) that the program of Exhibit 6.6 should be significantly faster than that of Exhibit 6.5 if f is sparse, and only slightly slower if f is dense.

In many applications both f and g are known to be homogeneous. Homogeneity may be viewed as a kind of sparseness, and this sparseness can also be exploited to produce a still faster multiplication routine. Exhibit 6.7 shows such a routine. Note that it is now necessary to offset the counter ic by the amount *icoff* to take into account the fact that the ig loop now begins at $ig = ibot(igdeg)$ rather than $ig = 1$.

Exhibit 32.6.7: Program for polynomial multiplication using only one index look-up table and designed to exploit the fact that f and g are homogeneous of degrees ifdeg and igdeg, respectively. It is also designed to exploit possible additional sparseness in the factor f.

```
      icoff=itop(igdeg-1)
      do 10 if=ibot(ifdeg),itop(ifdeg)
      if(f(if) . ne. 0.d0) then
      ic=icmin(if)+icoff
      do 20 ig=ibot(igdeg),itop(igdeg)
      h(ihtbl(ic))=h(ihtbl(ic))+f(if)*g(ig)
      ic=ic+1
   20 continue
      endif
   10 continue
      end
```

Suppose it is known in advance which entries in f are nonzero. Then it is possible to eliminate the if statements in programs like those in Exhibits 6.6 and 6.7 with a possible improvement in computational speed—particularly in the case of vector or pipe-lined computer architecture for which the if test burden is relatively large. If (exactly) k entries in f are known to be *nonzero*, then we can set up an array $nzf(i)$ such that the values $nzf(i)$ with $i \in [1, k]$ are the indices for the nonzero entries in f. With this array in hand we can, for example, reformulate the routine of Exhibit 6.7 as shown in Exhibit 6.8. Here we assume, as in Exhibit 6.7, that f is homogeneous of degree $ifdeg$. We remark that by the introduction of still further arrays it is possible to exploit known sparseness in both f and g.

Exhibit 32.6.8: Program for polynomial multiplication using only one look-up table, known sparseness in f, and homogeneity in f and g.

```
      icoff=itop(igdeg-1)
      do 10 i=1,k
      if=nzf(i)
      ic=icmin(if)+icoff
      do 20 ig=ibot(igdeg),itop(igdeg)
      h(ihtbl(ic))=h(ihtbl(ic))+f(if)*g(ig)
```

```
   ic=ic+1
20 continue
10 continue
   end
```

Note that the look-up table *ihtbℓ* in Exhibits 6.5 through 6.8 has size $\{m,d\}^{ns}$. We close this section by commenting that it is also possible to write multiplication routines that employ a look-up table having the minimum size $\{m,d\}$. That is, it is possible to write a routine that exploits the commutative symmetry of multiplication. However, we have not found a way of doing so that simultaneously exploits sparseness.

Exercises

40.6.1.

40.7 Look-Back Tables

Consider the array $ihtb\ell(ic)$ of ih values described in Section 6. See, for example, the right column of Table 6.1. For each value of ic ranging from 1 through $icmax = \{m,d\}^{ns}$, there is a corresponding value of ih, and this value lies in the range $(d+1)$ through $S(m,d)$. (Note that there are no zero or first order monomials under consideration since we are only worried about computationally intensive terms.) Since $\{m,d\}^{ns} > S(m,d)$, there are generally many ic values that yield a given value of ih. Indeed, in (6.1) there are many factors $f(if)$ and $g(ig)$ that contribute to a given $h(ih)$.

Next look at the program in Exhibit 6.3. We see that ic runs *successively* through the values $1,2,\cdots icmax$. However, we also see that the same result would be achieved if ic ran through the values $1,2,\cdots icmax$ in *any* order. That is, the outcome of following the script is *independent* of the order in which its instructions are executed. Suppose the array $ihtb\ell(ic)$ is rearranged so that the entries are listed in order of increasing ih. At the same time we rearrange the arrays $iftb\ell$ and $igtb\ell$. Finally, we set up a new ic index that again runs successively through the values $1,2,\cdots icmax$. For example, Table 7.1 below shows the result of this rearrangement applied to Table 6.1.

Examine Table 7.1. We see that for each value of ih in the column $ihtb\ell$ there are corresponding values of if and ig in the columns $iftb\ell$ and $igtb\ell$, respectively. These if, ig pairs are the indices for the monomials that are *factors* of the monomial labelled by ih. Thus, the arrays $iftb\ell$ and $igtb\ell$ (after the rearrangement just described) provide what we have called look-back tables. That is, given some ih, we can look back using these tables to find the if, ig pairs that produced this ih. We note that each table has $\{m,d\}^{ns}$ entries.

These look-back tables can be used to construct a program for multiplication. To do this, we make some observations. We see that for each value of ih in the column $ihtb\ell$ there is a minimum (bottom) value of the variable new ic, call it $icbot$, and a maximum (top) value, call it $ictop$. Use these observations to construct two arrays: $icbot(ih)$ and $ictop(ih)$. For example, Table 7.2 shows the contents of these arrays in the case $m = 4$ and $d = 2$.

Also, we see that ih has the minimum value $ihmin$ given by the relation

$$ihmin = d + 1, \tag{40.7.1}$$

and a maximum value $ihmax$ given by

$$ihmax = S(m, d) \tag{40.7.2}$$

with, in this case, $m = maxdeg = 4$ and $d = 2$.

Table 40.7.1: The result of rearranging Table 6.1 in order of increasing ih.

new ic	old ic	$iftb\ell$	$igtb\ell$	$ihtb\ell$
1	1	1	1	3
2	2	1	2	4
3	10	2	1	
4	11	2	2	5
5	3	1	3	6
6	19	3	1	
7	4	1	4	7
8	12	2	3	
9	20	3	2	
10	24	4	1	
11	5	1	5	8
12	13	2	4	
13	25	4	2	
14	29	5	1	
15	14	2	5	9
16	30	5	2	
17	6	1	6	10
18	21	3	3	
19	34	6	1	
20	7	1	7	11
21	15	2	6	
22	22	3	4	
23	26	4	3	
24	35	6	2	
25	36	7	1	
26	8	1	8	12
27	16	2	7	
28	23	3	5	
29	27	4	4	
30	31	5	3	
31	37	7	2	
32	38	8	1	
33	9	1	9	13
34	17	2	8	
35	28	4	5	

new ic	old ic	$iftb\ell$	$igtb\ell$	$ihtb\ell$
36	32	5	4	
37	39	8	2	
38	40	9	1	
39	18	2	9	14
40	33	5	5	
41	41	9	2	

Table 40.7.2: The arrays *icbot* and *ictop* in the case $m = 4$ and $d = 2$.

ih	$icbot$	$ictop$
3	1	1
4	2	3
5	4	4
6	5	6
7	7	10
8	11	14
9	15	16
10	17	19
11	20	25
12	26	32
13	33	38
14	39	41

Now look at Exhibit 7.1. It shows a program for polynomial multiplication using the arrays $iftb\ell$ and $igtb\ell$ as look-back tables. (Note that, although we have used the same notation, here the tables $iftb\ell$ and $igtb\ell$ are *rearranged* versions of their original counterparts.) We see that, with the use of the arrays *icbot* and *ictop*, the program ranges over all the proper values of *ic*, and therefore from our previous discussion must give the same result as the program of Exhibit 6.3.

`Exhibit 32.7.1: Program for polynomial multiplication using look-back tables.`

```
      do 10 ih=ihmin,ihmax
      do 20 ic=icbot(ih),ictop(ih)
      h(ih)=h(ih)+f(iftbl(ic))*g(igtbl(ic))
   20 continue
   10 continue
      end
```

How do the programs shown in Exhibits 6.5 and 7.1 compare? Here are some reasons to believe that (for multiplication) the use of look-up tables is preferable to the use of look-back tables:

1. Examine Table 6.1. Evidently successive values of $iftb\ell(ic)$ and $igtb\ell(ic)$ are *contiguous* as one goes down the list (increments ic). Therefore successive values of $f[iftb\ell(ic)]$ and $g[igtb\ell(ic)]$ are adjacent in memory. Since modern computers are often designed to fetch many adjacent items from memory at once and place them in fast-access cache memory or in on-chip registers in anticipation that they may be needed shortly, one expects there may be relatively few separate calls to slow access memory to find the coefficients in f and g when look-up tables are used. Note, by contrast, that successive values of $ihtb\ell(ic)$ are not contiguous. Therefore there is the penalty that the results of multiplication have to be scattered into slow access memory. They are, however, not too widely dispersed since if f and g are of given degrees (as in Exhibit 6.7), then all the entries in h will at least be of the same fixed degree, and therefore (in a graded indexing scheme) stored fairly close together. Moreover, if monomial ordering is used, then for each fixed if and all successive ig, the relevant ih values also occur in increasing order. We conclude that, with the use of look-up tables, access to the coefficients in both f and g may be fast, and access to the coefficients in h may be somewhat slow. Now examine Table 7.1. Here the successive values of $ihtb\ell(ic)$ are contiguous. However, those of $iftb\ell(ic)$ and $igtb\ell(ic)$ are not. Indeed, the entries in f and g that need to be accessed are not even of fixed degree. We conclude that, with the use of look-back tables, access to the coefficients in *both* f and g may be quite slow, and *only* access to the coefficients in h may be fast. Therefore, the use of look-up tables is likely to yield faster code.

2. It appears that the look-back method requires the storage of two tables of dimension $\{m, d\}^{ns}$, while the look-up method involves the storage of only one. As we have seen, these tables may be large. However, this objection may not be as serious as it sounds. Examination of Table 7.1 shows that, for each fixed value of $ihtb\ell$, there is a close relation between the contents of $iftb\ell$ and $igtb\ell$. Indeed, one list is the reverse of the other. Therefore, at the expense of some slightly more complicated logic and a few additional look ups, it may be possible to work with a single table of dimension $\{m, d\}^{ns}$.

3. The look-back method also requires storage of the arrays *icbot* and *ictop* which are each roughly of size $S(m, d)$. The look-up method requires the additional arrays *ibot* and *itop* which are only of size $(maxdeg + 1)$ with $maxdeg = m$.

4. Finally, the look-up method can be modified to exploit possible sparseness and homogeneity as shown in Exhibits 6.6 through 6.8. This does not seem to be possible for the look-back method.

Strictly speaking, some of the consideration listed above apply only to the case of computation with one processor. Suppose one has several processors available in some form of large-scale parallel architecture. Then one might assign the computation of various $h(ih)$ values to various processors, all to be computed in parallel. In this case, each computation would use look-back tables, and the use of look-back tables might be preferable to the use of look-up tables.

Let us pause to reflect. Look back over our discussion so far in this section, and the content of the previous two sections. After some thought, we see that what we have learned is that the idea of a script is a unifying concept, and that the use of a look-up table and the use of look-back tables are simply alternate ways of going through the script. Indeed, if we look at Table 6.1 and regard the indices if and ig as entries in a two-component number (if, ig) with most significant digit if and least significant digit ig, then we see that these numbers are arranged in increasing size as we go down the list. Alternatively, if we look at Table 7.1, we see that the list has been "graded" according to the value of ih; and the items having a given ih are again arranged in increasing size based on the two-component numbers (if, ig).

We close this section by observing that look-back tables can sometimes be employed to optimal advantage for other calculations. Their use in the calculation of Poisson brackets is described in Section 8. Here we describe their use in the evaluation of polynomials. Suppose $f(z)$ is a polynomial written in the form

$$f(z) = \sum_i f_i M_i(z), \tag{40.7.3}$$

where the f_i are a given set of coefficients, and we wish to know the value of f at the point $z = w$. Here, as before in Section 32.3, we have used the notation

$$M_i(z) = z^{j(i)}. \tag{40.7.4}$$

The coefficients f_i may be viewed as the entries in a *vector* of dimension $S_0(m, d)$. Define another such vector with entries γ_i given by the relation

$$\gamma_i = M_i(w). \tag{40.7.5}$$

Then the value $f(w)$ can be written in the form

$$f(w) = \sum_i f_i \gamma_i. \tag{40.7.6}$$

We see that (7.6) can be viewed as a *vector dot product* and, providing we know the entries γ_i, this dot product can be computed very efficiently by computers having vector or pipe-lined architecture.

But how can be find the γ_i in an efficient manner? If we use indexing based on a modified *glex* sequence, we have the results

$$\gamma_0 = 1, \tag{40.7.7}$$

$$\gamma_i = w_i \text{ for } i = 1, 2, \cdots d. \tag{40.7.8}$$

Moreover, we claim there are two look-back tables $i1(i)$ and $i2(i)$ such that the remaining γ_i can found from a recursion relation of the form

$$\gamma_i = [\gamma_{i1(i)}][\gamma_{i2(i)}]\ ,\ i = d + 1, \cdots S(m, d). \tag{40.7.9}$$

Thus, it is possible to evaluate the γ_i by carrying out only $[S(m, d) - d]$ multiplications.

To see how to construct the tables $i1(i)$ and $i2(i)$, we observe that each $M_i(z)$ of degree $n = |j(i)|$, and assuming $n \geq 2$, can be factored as the product of a *first* degree monomial with index $i1(i)$ and another monomial of degree $n - 1$ having index $i2(i)$:

$$M_i(z) = [M_{i1(i)}(z)][M_{i2(i)}(z)]. \tag{40.7.10}$$

Exhibit 7.2 shows a routine written to find two tables $i1(i)$ and $i2(i)$ having this property. Inspection of the routine shows that it works as follows: Find and examine the exponent $j(i)$. Proceeding from the left, let $j_k(i)$ be the first nonzero entry in $j(i)$. By construction k must lie in the range $1 \leq k \leq d$, and we have the result

$$z^{j(i)} = z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d} = z_k z^{j'(i)} \tag{40.7.11}$$

where the exponent array $j'(i)$ has the entries

$$j'_\ell(i) = j_\ell(i) - 1 \text{ when } \ell = k, \tag{40.7.12}$$

$$j'_\ell(i) = j_\ell(i) \text{ when } \ell \neq k. \tag{40.7.13}$$

Now define the table entries $i1(i)$ and $i2(i)$ by the rules

$$i1(i) = \text{ index for } z_k = k, \tag{40.7.14}$$

$$i2(i) = \text{ index for the exponent } j'(i). \tag{40.7.15}$$

Table 7.3 displays the result of running this routine for the case $m = 3$ and $d = 6$. The entries in this table should be compared with those in Table 2.1.

Exhibit 7.3 shows a routine for computing the γ_i based on the relations (7.7) through (7.9). Look at Table 7.3. We see that the results of the algorithm of Exhibit 7.2 have the pleasing feature that, for the most part, the values of $i2(i)$ are contiguous for successive values of i. Inspection of the routine of Exhibit 7.3 shows that it requires the values $\gamma_{i1(i)}$ and $\gamma_{i2(i)}$ for successive values of i. Since the addresses are usually contiguous, it is very likely that the values of the required $\gamma_{i1(i)}$ and $\gamma_{i2(i)}$ will either be in fast-access cache or in registers when needed, and consequently this routine should be very fast.

Exhibit 32.7.2: Program for factoring monomials. It produces arrays i1(i) and i2(i) such that monomial(i)=monomial(i1(i))*monomial(i2(i)) and the monomials monomial(i1(i)) in the first factor are all of degree one.

```
      do 10 i=1,imax
      do 20 k=1,6
         j(k)=jtbl(k,i)
   20 continue
      do 30 k=1,6
         if (j(k) .ne. 0) then
             j(k)=j(k)-1
             il(i)=k
             call ndex (j,ij)
             i2(i)=ij
             go to 40
         endif
   30 continue
   40 continue
   10 continue
      end
```

Exhibit 32.7.3: Program for building vector of monomial values using look-back tables i1(i) and i2(i).

```
       gam(0)=1.d0
      do 10 i=1,id
         gam(i)=w(i)
   10 continue
      do 20 i=id+1,imax
         gam(i)=gam(i1(i))*gam(i2(i))
   20 continue
      end
```

Table 40.7.3: The arrays $i1(i)$ and $i2(i)$ in the case $m = 3$ and $d = 6$.

i	$i1$	$i2$	i	$i1$	$i2$
7	1	1	28	1	7
8	1	2	29	1	8
9	1	3	30	1	9
10	1	4	31	1	10
11	1	5	32	1	11
12	1	6	33	1	12
13	2	2	34	1	13
14	2	3	35	1	14
15	2	4	36	1	15
16	2	5	37	1	16
17	2	6	38	1	17
18	3	3	$\vdots$		
19	3	4	77	4	25
20	3	5	78	4	26
21	3	6	79	4	27
22	4	4	80	5	25
23	4	5	81	5	26
24	4	6	82	5	27
25	5	5	83	6	27
26	5	6	$\vdots$		
27	6	6			

Exercises

40.7.1.

40.8 Poisson Bracketing

When computing Poisson brackets of homogeneous polynomials, there are three natural cases to consider. First, there are brackets of the form $[f_1, g_1]$. They are trivial to compute in view of (1.7.10). Next in order of complexity are brackets of the form $[f_m, z_a]$ with $m \geq 2$. They will be referrred to as *single-variable* Poisson brackets, and are an essential ingredient in the computation of $\mathcal{M}z_a$ as in (7.1.1). With the aid of (7.6.10), they can be evaluated by the formula

$$: f_m : z_a = [f_m, z_a] = -\partial f_m / \partial z_a^*. \tag{40.8.1}$$

Finally, there are brackets of the form $[f_m, g_n]$ with $m, n \geq 2$. They will be called *general* Poisson brackets.

The purpose of this section is to describe efficient algorithms for the computation of single-variable and general Poisson brackets.

We begin with the case of single-variable Poisson brackets. That is, we wish to compute

$$g =: f : z_a. \tag{40.8.2}$$

For the discussion of Poisson brackets it is convenient to order the q's and p's in conjugate pairs and to define z^j (for the case of a 6-dimensional phase-space) in analogy with (2.1),

$$z^j = q_1^{j_1} p_1^{j_2} q_2^{j_3} p_2^{j_4} q_3^{j_5} p_3^{j_6}. \tag{40.8.3}$$

Then, for example and in agreement with (8.1), we find the result

$$: z^j : q_1 = [z^j, q_1] = -\partial z^j/\partial p_1 = -j_2 q_1^{j_1} p_1^{j_2-1} q_2^{j_3} p_2^{j_4} q_3^{j_5} p_3^{j_6}. \tag{40.8.4}$$

Evidently every monomial single-variable Poisson bracket $[z^j, z_a]$ results in at most one term, and many produce none. This circumstance can be exploited using look-back tables. Suppose the monomial z^k with

$$z^k = q_1^{k_1} p_1^{k_2} q_2^{k_3} p_2^{k_4} q_3^{k_5} p_3^{k_6} \tag{40.8.5}$$

produces the monomial z^j as a result of the multiplication

$$z^k z_b = z^j. \tag{40.8.6}$$

Then we have the result

$$\partial z^j/\partial z_b = j_b z^k \tag{40.8.7}$$

with j_b guaranteed to be nonzero. Let $i = i(k)$ be the index of the monomial with exponent k. Then, corresponding to the relation (8.6), we can define a single-variable multiplication table $msv(i, b)$ by the rule

$$msv(i, b) = \text{ index of monomial with exponent } j. \tag{40.8.8}$$

Exhibit 8.1 below shows a program that prepares a script for single-variable Poisson bracketing. It makes the single-variable multiplication table $msv(i, b)$ as well as the tables $coef(i, b)$ and $scoef(i, b)$, which contain various coefficients such as the $(-j_2)$ that appears in (8.4). From their construction it is evident that all these tables have the modest dimension $itop(maxdeg-1) \times 6$. [Actually the table $coef(i, b)$ is not needed for single-variable Poisson bracketing, but is useful for multplication-based general Poisson bracketing. See Exhibit 8.5.]

```
Exhibit 32.8.1: Program to produce script for single-variable
Poisson bracket routine.

      subroutine svpbs
c
      data icon /2,1,4,3,6,5/
      data sign /1.d0,-1.d0,1.d0,-1.d0,1.d0,-1.d0/
c
```

```
c Loop over phase-space variables z_k
c
      do 10 k=1,6
      izc=icon(k)
c
c Loop over ic
c
      do 20 ic=ibot(1), itop(maxdeg-1)
c
c find and store exponents
c
      do m=1,6
      jl(m)=jtbl(m,ic)
      end do
      jltizc=jl(izc)
c
c fill tables
c
      jl(izc)=jltizc+1
      ifac=jl(izc)
      coef(ic,k)=dfloat(ifac)
      scoef(ic,k)=sign(k)*coef(ic,k)
      if=ndex(jl)
      msv(ic,k)=if
c
c restore exponent
c

      jl(izc)=jltizc
c
  20  continue
  10  continue
c
      return
      end
```

Exhibit 8.2 shows the actual single-variable Poisson bracketing routine that uses the script prepared by the program of Exhibit 8.1. It works directly with the index ig for each monomial in the result g, and uses the table msv to look back to find the index $if = msv(ig, k)$ of the monomial in f that produced this result. Because each monomial in f contributes to at most one term in g, no attempt has been made to exploit possible sparseness in f. To test in advance of their use the various $f(if)$ to see if they vanished would result in significant computational overhead with little associated reward. Finally, we note that, due to the use of monomial ordering, successive if values appear in an increasing sequence. The required $f(if)$ values are therefore likely to be in cache or in on-chip registers.

Exhibit 32.8.2: Program for single variable Poisson bracket.

```
      subroutine svpb(f,ideg,k,g)
c
c This subroutine finds the single variable Poisson
```

```
c bracket g=:f:z_k.  Here f is homogeneous of degree ideg.
c
      do ig=ibot(ideg-1),itop(ideg-1)
      g(ig)=scoef(ig,k)*f(msv(ig,k))
      end do
c
      return
      end
```

We next turn to the general Poisson bracket case. We wish to compute $h = [f, g]$ when both f and g are of degree two and higher. For a 6-dimensional phase space, the Poisson bracket of any two monomials is given by the standard rule

$$[z^j, z^k] = \sum_{i=1}^{3} (\partial z^j/\partial q_i)(\partial z^k/\partial p_i) - (\partial z^j/\partial p_i)(\partial z^k/\partial q_i). \tag{40.8.9}$$

At first count it might appear that a typical monomial Poisson bracket could contain 6 distinct terms. In fact, there are at most 3 distinct terms. To verify this assertion, consider the $i = 1$ terms on the right side of (8.9). They give the result

$$\begin{aligned}
&(\partial z^j/\partial q_1)(\partial z^k/\partial p_1) - (\partial z^j/\partial p_1)(\partial z^k/\partial q_1)\\
&= (j_1 z_1^{j_1-1} z_2^{j_2} z_3^{j_3} z_4^{j_4} \cdots)(k_2 z_1^{k_1} z_2^{k_2-1} z_3^{k_3} z_4^{k_4} \cdots)\\
&\quad -(j_2 z_1^{j_1} z_2^{j_2-1} z_3^{j_3} z_4^{j_4} \cdots)(k_1 z_1^{k_1-1} z_2^{k_2} z_3^{k_3} z_4^{k_4} \cdots)\\
&= (j_1 k_2 - j_2 k_1)(z_1^{j_1+k_1-1} z_2^{j_2+k_2-1} z_3^{j_3+k_3} z_4^{j_4+k_4} \cdots).
\end{aligned} \tag{40.8.10}$$

Evidently in the Poisson bracket result there is at most one monomial term for each value of i in the sum (8.9), and there is none whenever (for odd ℓ) the coefficient $(j_\ell k_{\ell+1} - j_{\ell+1} k_\ell)$ vanishes.

Exhibit 8.3 below shows a program that prepares a script for general Poisson bracketing.

Exhibit 32.8.3: Program to produce script for general Poisson bracket routine.

```
      subroutine pbsc
c
c
c set counters
c
      ic1=0
      ic2=0
c
c loop over degrees
c
      do 10 ifdeg=2,maxdeg
      maxgdeg=maxdeg+2-ifdeg
      do 20 igdeg=2,maxgdeg
c
c loop over if and ig
c
      do 30 if=ibot(ifdeg),itop(ifdeg)
      do 40 ig=ibot(igdeg),itop(igdeg)
```

```
c
c find and store exponents and their sums
c
      do k=1,6
      jf(k)=jtbl(k,if)
      jg(k)=jtbl(k,ig)
      js(k)=jf(k)+jg(k)
      end do
c
c increment ic2 counter
c
      ic2=ic2+1
c
c find and count possible ih indices and coefficients,
c and set up tables
c
      it=0
      do 50 k=5,1,-2
      iz=k
      izc=k+1
      ival=jf(iz)*jg(izc)-jf(izc)*jg(iz)
      if(ival .eq. 0) go to 50
c
c compute exponents from exponent sums,
c and compute index
c
      jsizt=js(iz)
      jsizct=js(izc)
      js(iz)=jsizt-1
      js(izc)=jsizct-1
      index=ndex(js)
c
c restore exponent sums
c
      js(iz)=jsizt
      js(izc)=jsizct
c
c increment it and ic1 counters,
c and store coefficients and indices
c
      it=it+1
      ic1=ic1+1
      pbcoef(ic1)=dfloat(ival)
      ih(ic1)=index
c
   50 continue
c
c store the number of terms
c
      nt(ic2)=it
c
   40 continue
c
c set reset tables just after leaving ig loop
```

```
c
      irst1(if,igdeg)=ic1
      irst2(if,igdeg)=ic2
c

   30 continue
   20 continue
   10 continue
c
c record maximum required storage
c
      maxic1=ic1
      maxic2=ic2
c
      return
      end
```

The sizes of the arrays produced by this script can be quite large. The array nt is indexed by the integer variable *ic2*, and has dimension *maxic2*. The arrays $pbcoef$ and ih are indexed by the integer variable *ic1*, and have dimension *maxic1*. The values of *maxic1* and *maxic2* are listed in Table 8.1 for various values of $maxdeg$, the maximum degree of the polynomials one is considering.

The computation of *maxic2* is elementary. The variable *ic2* labels all possible if, ig pairs that can potentially occur in a Poisson bracket calculation, and *maxic2* is the total number of such pairs. For example, if $maxdeg = 3$, we have to consider Poisson bracket terms of the form $[f_2, f_2]$, $[f_2, f_3]$, and $[f_3, f_2]$. Inspection of Table 7.3.1 shows that (for a 6-dimensional phase space) there are 21 f_2 basis monomials and 56 f_3 basis monomials. Thus, in this case we expect the result

$$\mathit{maxic2} = 21 \times 21 + 21 \times 56 + 56 \times 21 = 441 + 1176 + 1176 = 2793,$$

in agreement with the corresponding entry in Table 8.1. We note that the quantites $\mathit{maxic2}(maxdeg)$ for Lie (Poisson bracket) multiplication are much larger than their counterparts $\{maxdeg, 6\}^{ns}$ for ordinary multiplication. Compare Tables 5.4 and 8.1. This increased size results from the (-2) term in (7.6.16), which does not occur for ordinary multiplication. As a simple example, many of the absent terms denoted by an asterisk "*" in Table 5.2 for ordinary multiplication would not be absent for Poisson bracket multiplication.

The computation of *maxic1* is more complicated, and is most easily done simply by counting as in Exhibit 8.3. As described earlier, each possible if, ig monomial pair is labeled by a value of *ic2*. The quantity *nt(ic2)* is the number of monomial terms that result from Poisson bracketing the monomial pair with label *ic2*. We have already seen that this number (including the possibility of a vanishing bracket) ranges from 0 to 3. The quantity *maxic1* is the sum over nt,

$$\mathit{maxic1} = \sum_{\mathit{ic2}=1}^{\mathit{maxic2}} \mathit{nt(ic2)}. \tag{40.8.11}$$

Thus, for a fixed value of $maxdeg$, the quantity *maxic1* is the number of nonzero monomials that can occur in a Poisson bracket calculation including repetitions.

Inspection of Table 8.1 shows that *maxic1*, for *maxdeg* ≤ 6, is smaller than or comparable to *maxic2*. Evidently, although the Poisson bracket of each monomial pair could potentially produce as many as 3 monomial terms, most Poisson brackets produce fewer or are in fact zero. By contrast, *maxic1* exceeds *maxic2* for *maxdeg* > 6.

Table 40.8.1: Array sizes *maxic1* and *maxic2* (in the case of 6 phase-space variables) for various values of *maxdeg*.

maxdeg	*maxic1*	*maxic2*	*maxdeg*	*maxic1*	*maxic2*
2	210	441	8	648342	570619
3	1662	2793	9	1514382	1231279
4	7986	11221	10	3320694	2518565
5	29622	35917	11	6902358	4923317
6	92802	99421	12	13701822	9254645
7	257190	247933			

Exhibit 8.4 shows the actual general Poisson bracket routine that uses the script prepared by the program of Exhibit 8.3. It is designed to exploit possible sparseness in f and known homogeneity in both f and g. The array *pbcoef(ic1)* contains precomputed (and nonzero) coefficients of the form $(j_\ell k_{\ell+1} - j_{\ell+1} k_\ell)$, and the look-up table *ih(ict1)* specifies where contributions to the various terms in h are to be placed. For each *if*,*ig* pair, the *ih* values for successive *ict1* values are arranged in increasing order in the hope of rapid memory access. Thanks to the contents of the array *nt(ic2)*, only potentially nonzero terms are computed. The arrays *irst1* and *irst2* set the counters *ic1* and *ic2* to take into account skipped terms due to sparseness and homogeneity.

`Exhibit 32.8.4: Script-driven program for general Poisson bracket.`

```
      subroutine pb(f,ifdeg,g,igdeg,h)
c
c This subroutine computes the Poisson bracket
c h=[f,g].  The input polynomials f and g are
c homogeneous of degrees ifdeg and igdeg, respectively.
c
c clear h
c
      do i=1,imax
      h(i)=0.d0
      end do
c
c find offsets
c
      ic1=irst1(itop(ifdeg-1),igdeg)
      ic2=irst2(itop(ifdeg-1),igdeg)
c
c loop over if and ig
c
      do 10 if=ibot(ifdeg),itop(ifdeg)
```

```
      if(f(if) .eq. 0.d0) then
      ic1=irst1(if,igdeg)
      ic2=irst2(if,igdeg)
      go to 10
      end if
      do 20 ig=ibot(igdeg),itop(igdeg)
      if(g(ig) .eq. 0.d0) then
      ic1=ic1+nt(ic2)
      ic2=ic2+1
      go to 20
      end if
      if(nt(ic2) .ne. 0) then
      prod=f(if)*g(ig)
      do 30 it=1,nt(ic2)
      h(ih(ic1)) = h(ih(ic1)) + pbcoef(ic1)*prod
      ic1=ic1+1
   30 continue
      end if
      ic2=ic2+1
   20 continue
   10 continue
c
      return
      end
```

As remarked earlier, Table 8.1 shows that the script arrays required to drive the routine of Exhibit 8.4 can be quite large for vaues of *maxdeg* beyond 9. This may not be an issue as memory becomes ever more plentiful. However, it is worth remarking that there is an alternate approach that requires much less memory but, of course, is computationally slower. What one can do, as in the case of single-variable Poisson brackets, is use look-back tables to find the various terms in $\partial f/\partial z_a$ and $\partial g/\partial z_b$, and then use look-up tables, as in Exhibit 6.8, to carry out the multiplications $(\partial f/\partial z_a)(\partial g/\partial z_b)$. Exhibit 8.5 below shows a general Poisson bracket routine that uses this procedure.

Exhibit 32.8.5: General Poisson bracket program based on multiplication.

```
      subroutine pb(f,ifdeg,g,igdeg,h)
c
      data icon /2,1,4,3,6,5/
c
      icoff=ibot(igdeg-1)
      do 10 iz=1,6
      izc=icon(iz)
c
      do 20 ifl=ibot(ifdeg-1),itop(ifdeg-1)
      if=msv(ifl,iz)
      if(f(if) .ne. 0.d0) then
      ic=icmin(ifl)+icoff
      do 30 igl=ibot(igdeg-1),itop(igdeg-1)
      ig=msv(igl,izc)
      if(g(ig).ne. 0.d0) then
      fac=scoef(ifl,iz)*coef(igl,izc)
```

```
      h(ihtbl(ic))=h(ihtbl(ic))+fac*f(if)*g(ig)
      end if
      ic=ic+1
   30 continue
      end if
   20 continue
   10 continue
c
      return
      end
```

40.9 Linear Map Action

Let f be any function of the $2n$ phase-space variables z, and let M be any $2n \times 2n$ matrix. Then we define a *transformed* function g by the rule

$$g(z) = f(Mz). \tag{40.9.1}$$

One may view (9.1) as the action of the linear (usually but not necessarily symplectic) map respresented by the matrix M on the function f. See (8.4.23) and (10.4.36) where actions of this type arise in the concatenation and computation of maps. The purpose of this section is to describe an efficient algorithm for carrying out the operation (9.1).

To achieve efficiency, it is useful to employ a precomputed list of variables $jvblist(iv, ind)$. Here ind is a monomial index, and for an nth-order monomial there will be n non-zero variable numbers. For example, for the indexing scheme of Table 2.1, the monomial with index 7 is $X^2 = XX$. Correspondingly, the variable list (array) $jvblist$ will have the entries

$$jvblist(iv = 1 \text{ to } 2, 7) = 1, 1. \tag{40.9.2}$$

As a second example, the monomial with index 19 is YP_y. Correspondingly $jvblist$ will have the entries

$$jvblist(iv = 1 \text{ to } 2, 19) = 3, 4. \tag{40.9.3}$$

For a third example, the monomial with index 77 is $P_y\tau^2 = P_y\tau\tau$. Correspondingly $jvblist$ will have the entries

$$jvblist(iv = 1 \text{ to } 3, 77) = 4, 5, 5. \tag{40.9.4}$$

Exhibit 9.1 below shows a program that produces $jvblist$. It is a modification of the program in Exhibit 2.3 so that both $jtbl$ and $jvblist$ are created simultaneously.

```
Exhibit 32.9.1:  A program to produce both jtbl and jvblist based on  a
method of Liam Healy.

       subroutine tables

c       ind = monomial index and imax = maximum value of ind.
c       ipsv = phase space variable and id = number of phase space
variables.
c
       parameter (imax = 209, id=6)
```

```
      dimension jtbl(id,imax), jvblist(id,imax)
c  j = array of exponents
      dimension j(id)
c  initialize exponents
      data j/id*0/
c  icarry = temporarily stored value of j(id).
c  lnzj = last non-zero j
c
c  sequentially create exponent table jtbl and the array jvblist
c
      do ind=1,imax
c
c  set quantities
c
        icarry=j(id)
        j(id)=0
        lnzj=0
c
c  search for last nonzero j
c
        do ipsv=1,id-1
          if (j(ipsv).gt.0) lnzj=ipsv
        enddo
c
c  find next set of exponents
c
        if (lnzj.gt.0) j(lnzj)=j(lnzj)-1
        j(lnzj+1)=1+icarry
c
c  store exponents in jtbl
c
        do ipsv=1,id
          jtbl(ipsv,ind)=j(ipsv)
        enddo
c
c create jvblist
c
      iv=1
      do ipsv=1,id
       do k=1,j(ipsv)
        jvblist(iv,ind)=ipsv
        iv=iv+1
       enddo
      enddo
c
      enddo
c
      return
      end
```

With this background information in mind, we are ready to present the algorithm that carries out the operation (9.1). It is shown in Exhibit 9.2 below.

Exhibit 32.9.2: Program for linear map action.

```
      subroutine xform(f,ideg,em,g)
c
c     Transforms a polynomial f of degree ideg by the linear
c     map whose matrix representation is em.  The coefficients
c     of the resultant polynomial f(em*z) are stored in the
c     array g.  Thus g=f(em*z).
c
c initialise g array
c
      do k=1,imax
      g(k) = 0.d0
      end do
c
c loop over monomials of degree ideg
c
      do 100 n=ibot(ideg),itop(ideg)
      if(f(n) .eq. 0.d0) goto 100
c
c clear the array ta
c
      do k=7,itop(ideg)
      ta(k) = 0.d0
      end do
c
c work on the ideg variables in the monomial
c
c treatment of first variable
c
c transform first variable in monomial and place result in ta1
c and ta
c
      jiv = jvblist(1,n)
      do k=1,6
      ta1(k) = em(jiv,k)
      ta(k) = ta1(k)
      end do
c
c loop over remaining variables
c
      do 110 iv=2,ideg
c
c find next variable
c
      jivn = jvblist(iv,n)
c
c if next variable is the same as the previous one,
c build up product in ta
c
      if(jivn .eq. jiv) go to 120
c
c otherwise, if the next variable is different from
c the previous one, transform that variable, place
c result in ta1, and build up product in ta
```

```
c
      jiv = jivn
      do k=1,6
      ta1(k) = em(jiv,k)
      end do
c
  120 continue
c
c build up product
c
      icdeg = iv-1
      do 130 i1=1,6
      if(ta1(i1) .ne. 0.d0) then
      do 140 i2=ibot(icdeg),itop(icdeg)
  140 ta(iprodex(i2,i1)) = ta1(i1)*ta(i2)
      endif
  130 continue
c
  110 continue
c
c accumulate sum in g
c
      do nn=ibot(ideg),itop(ideg)
      g(nn) = g(nn) + f(n)*ta(nn)
      end do
c
  100 continue
c
      return
      end
```

40.10 General Vector Fields

Let $\boldsymbol{f} = (f_1, f_2, \cdots f_d)$ be a collection of d functions of z, and suppose each function f_a is a truncated power series in z. Let $\mathcal{L}_{\boldsymbol{f}}$ be the vector field associated with $\boldsymbol{f}$. See Section 5.3. Using the tools already developed, it is easy to envision how to construct programs that would represent, multiply by scalars, and form linear combinations of such vector fields. We simply store and manipulate the underlying collections of functions in the obvious way.

Beyond these operations, we would also like to apply $\mathcal{L}_{\boldsymbol{f}}$ to a function g where it assumed that g is also a truncated power series. That is, we wish to find the truncated power series for h defined by the equation

$$h = \mathcal{L}_{\boldsymbol{f}} g = \sum_a f_a (\partial g / \partial z_a). \tag{40.10.1}$$

Moreover, suppose $\boldsymbol{f}$ and $\boldsymbol{g}$ are two collections of truncated power series. Let $\mathcal{L}_{\boldsymbol{f}}$ and $\mathcal{L}_{\boldsymbol{g}}$ be their associated vector fields. It is easily verified that the commutator of two vector fields is again a vector field. That is, there is a collection of functions $\boldsymbol{h}$ such that

$$\#\mathcal{L}_{\boldsymbol{f}} \#\mathcal{L}_{\boldsymbol{g}} = \{\mathcal{L}_{\boldsymbol{f}}, \mathcal{L}_{\boldsymbol{g}}\} = \mathcal{L}_{\boldsymbol{f}} \mathcal{L}_{\boldsymbol{g}} - \mathcal{L}_{\boldsymbol{g}} \mathcal{L}_{\boldsymbol{f}} = \mathcal{L}_{\boldsymbol{h}}. \tag{40.10.2}$$

Given the collections of truncated power series for $\boldsymbol{f}$ and $\boldsymbol{g}$, we would like to find the collection of truncated power series for $\boldsymbol{h}$.

The purpose of this section is to describe programs for computing h in (10.1) and $\boldsymbol{h}$ in (10.2).

It is evident from (10.1) that the computation of h involves partial differentiation and function multiplication. The same is true for the computation of $\boldsymbol{h}$. From the representation

$$\mathcal{L}_{\boldsymbol{h}} = \sum_b h_b(\partial/\partial z_b) \tag{40.10.3}$$

and (10.2) we find the result

$$h_a = \mathcal{L}_{\boldsymbol{h}} z_a = \mathcal{L}_{\boldsymbol{f}}\mathcal{L}_{\boldsymbol{g}} z_a - \mathcal{L}_{\boldsymbol{g}}\mathcal{L}_{\boldsymbol{f}} z_a = \mathcal{L}_{\boldsymbol{f}} g_a - \mathcal{L}_{\boldsymbol{g}} f_a. \tag{40.10.4}$$

Therefore, the computation of commutators involves the action (10.1) of vector fields on functions which, in turn, again involves partial differentiation and function multiplication.

In principle, it is possible to write scripted programs that would perform all the operations in (10.1) and (10.3) in an optimal way. However, for simplicity, we will only describe a scripted program for partial differentiation. This program can then be used in conjunction with those for multplication and addition to carry out all the required operations.

The operation of partial differentiation is similar to, and in fact simpler than, single-variable Poisson bracketing. See (8.1). Therefore, it is conviently done with the aid of a script and look-back tables. Exhibit 10.1 shows a program that generates a script for partial differentiation. It makes the single-variable multiplication table $msv(ic,k)$ and the table $coef(ic,k)$ that contains the coefficients j_b that occur in (8.7). Finally, Exhibit 10.2 shows the partial differentiation routine that uses the script prepared by the program of Exhibit 10.1.

```
Exhibit 32.10.1: Program to produce script for partial
differentiation routine.

      subroutine pds
c
c Loop over phase-space variables z_k
c
      do 10 k=1,6
      iz=k
c
c Loop over ic
c
      do 20 ic=ibot(1), itop(maxdeg-1)
c
c Find and store exponents
c
      do m=1,6
      jl(m)=jtbl(m,ic)
      end do
      jltiz=jl(iz)
c
c Fill tables
```

```
c
      jl(iz)=jlizt+1
      ifac=jl(iz)
      coef(ic,k)=dfloat(ifac)
      if=ndex(jl)
      msv(ic,k)=if
c
c Restore exponent
c
      jl(iz)=jltiz
c
  20  continue
  10  continue
c
      return
      end
```

Exhibit 32.10.2: Program for partial differentiation.

```
      subroutine pd(f,ideg,k,g)
c
c This subroutine finds the partial derivative
c g=df/dz_k.  Here f is homogeneous of degree ideg.
c
      do ig=ibot(ideg-1), itop(ideg-1)
      g(ig)=coef(ig,k)*f(msv(ig,k))
      end do
c
      return
      end
```

40.11 Expanding Functions of Polynomials

40.12 Automatic Differentiation/Differential Algebra

40.13 Other Methods

Bibliography

Computer Science, Indexing Schemes, Polynomial Manipulation, Etc.

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[2] A. Haro, M. Canadell, J-L. Figueras, A. Luque, and J-M. Mondelo, *The Parameterization Method for Invariant Manifolds: From Rigorous Results to Effective Computations*, Applied Mathematical Sciences Volume 195, Springer (2016).

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Combinatorics

[4] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Springer-Verlag (1992).

Programs

[5] The program *xform* presented in Exhibit 9.2 was written by F. Neri.

Web Sites

[6] http://www.autodiff.org/