Subject: Elasticity

Assignment no.: 1

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QUESTION 1

For the given matrix/vector Pair, compute the following a_{ii}, $a_{ij}a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, b_ib_i, b_ib_j. For each case, Point out whether the result is scalar vector or matrix.

(a) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & \phi \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Solution:-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & \phi \end{bmatrix}$$

a_{ii}

$$a_{ii} = a_{11} + a_{22} + a_{33}$$
$$= 1 + 4 + 1$$
$$= 6 \text{ (scalar)}$$

$a_{ij}a_{ij}$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$
$$= (1)(1) + (1)(1) + (1)(1) + (0)(0) + (4)(4) + (2)(2) + (0)(0) + (1)(1) + (0)(\phi)$$
$$= 1 + 1 + 1 + 0 + 16 + 4 + 0 + 1 + 0$$
$$= 25 \text{ (scalar)}$$

$a_{ij}a_{jk}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & \phi \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & \phi \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \text{ (matrix)}$$

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$a_{ij}b_j = a_{11}b_1 + a_{12}b_2 + a_{13}b_3$

$i=1, i=2, i=3$

$\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ (vector)

$a_{ij}b_ib_j \quad a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 +$

$a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3$

$+ a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1+0+2+0+0+0+0+0+4$

$= 7$ (scalar)

$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$

(matrix)

b_ib_i

$= b_1b_1 + b_2b_2 + b_3b_3$

$= 1+0+4$

$= 5$ (scalar)

(b) $a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \quad b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Solution:

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$

$a_{ii} \quad a_{11} + a_{12} + a_{13}$

$= 1+2+2$

$= 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21}$

$+ a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32}$

$+ a_{33}a_{33}$

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$= 1+4+0+0+4+1+0+16+4$

$= 30$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix}$$

(matrix)

$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$

$i=1,\ i=2,\ i=3$

$$\begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$$ (Vector)

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_2 + a_{21}b_2b_1$
$a_{22}b_2b_1 + a_{23}b_2b_3 + a_{31}b_3b_1$
$+ a_{32}b_3b_2 + a_{33}b_3b_3$

$= 4+4+0+0+2+1+0+4+2$

$= 17$ (scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$ (matrix)

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3$

$= 4+1+1 \Rightarrow 6$ (scalar)

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(C) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ $b_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Solution:-

$$a_{ii} = a_{11} + a_{12} + a_{13}$$
$$= 1 + 0 + 4$$
$$= 5 \text{ (scalar)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21}$$
$$+ a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31}$$
$$+ a_{32}a_{32} + a_{33}a_{33}$$
$$= 1 + 1 + 1 + 1 + 0 + 4 + 0 + 1 + 16$$
$$= 25 \text{ (scalar)}$$

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 8 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$$
$$i = 1, 2, 3$$
$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ (vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1$$
$$+ a_{22}b_2b_2 + a_{23}b_2b_3 +$$
$$a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$

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$= 1+1+0+1+0+0+0+0+0$

$= 3$ (Scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (matrix)}$$

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3$

$= 1+1+0$

$= 2$ (scalar)

QUESTION 2

Use the decomposition result to express a_{ij} from Exercise 1-1 in terms of the sum of symmetric and antisymmetric matrices. Verify that $a_{(ij)}$ and $a_{[ij]}$ satisfy the conditions given in the last paragraph of Section 1.2.

Solution:-

(a) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate condition.

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(b) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate condition

(c) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate condition.

QUESTION 3

If a_{ij} is Symmetric and b_{ij} is antisymmetric, Prove in General that the Product $a_{ij}b_{ij}$ is zero. Verify this result for the Specific case by using the symmetric and antisymmetric terms from Exercise 2.

Solution:

$$a_{ij}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij}$$

$$\Rightarrow 2a_{ij}b_{ij} = 0$$

$$\Rightarrow a_{ij}b_{ij} = 0$$

From Exercise 1-2(a): $a_{(ij)}\, a_{[ij]}$

$$= \frac{1}{4} tr \left[\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T \right] = 0$$

From Exercise 1-2(b):

$$a_{(ij)} a_{[ij]} = -\frac{1}{4} tr \left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix} \right) = 0$$

From Exercise 1-2(c);

$$a_{(ij)} a_{[ij]} = \frac{1}{4} tr \left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T \right) = 0$$

QUESTION 4

Explicity verify the following Properties of the kronecker delta

$$\delta_{ij} a_j = a_i$$

$$\delta_{ij} a_{jk} = a_{ik}$$

Solution:-

$$\delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3$$

$$= \begin{bmatrix} \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

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$$\delta_{ij} a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} \cdots \\ \vdots & \vdots \\ & \delta_{11}a_{13} + \delta_{12}a_{23} + \delta_{13}a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

QUESTION 5

Formally the expand the expression for the determinant and justify that either index notation form yields a result that matches the traditional form for $\det[a_{ij}]$.

Solution:-

$$\det(a_{ij}) = \varepsilon_{ijk}\, a_{1i} a_{2j} a_{3k}$$

$$= \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{231} a_{12} a_{23} a_{31} + \varepsilon_{312} a_{13} a_{21} a_{32} + \varepsilon_{321} a_{13} a_{22} a_{31} + \varepsilon_{132} a_{11} a_{23} a_{32} + \varepsilon_{213} a_{12} a_{21} a_{33}$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{12} a_{33}$$

$$= a_{11}(a_{22} a_{33} - a_{23} a_{23}) - a_{12}(a_{21} a_{33} - a_{23} a_{31}) + a_{13}(a_{21} a_{32} - a_{22} a_{31})$$

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$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

QUESTION 6

Determine the Components of the vector b_i and matrix a_{ij} given in Exercise 1 in a new coordinate system found through a rotation of $45°$ ($\pi/4$ radian) about the x_1-axis. The rotation direction follows the Positive sense Presented in Example.

Solution:-

$45°$ rotation about x_1-axis $\Rightarrow$ $Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

From Exercise 1-1(a) ; $b_i' = Q_{ij} b_j$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

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$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

From Exercise 1-1(b): $b_i' = Q_{ij} b_j$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jp} Q_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

From Exercise 1-1(c): $b_i' = Q_{ij} b_j$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

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QUESTION 7

Consider the 2-D Coordinate transformation through the Counterclockwise rotation θ, a new Polar Coordinate system is created. Show that the transformation matrix for this case is given by

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Solution:-

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \cos(90^\circ - \theta) \\ \cos(90^\circ + \theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12} + a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta \\ a_{21}\cos^2\theta - (a_{\ldots} \ldots \end{bmatrix}$$

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QUESTION 8

Show that the second order tensor $a\delta_{ij}$. where a is an arbitrary constant, retains its form under any transformation Q_{ij}. This form is then an isotropic second order tensor.

Solution:-

$$a'\delta'_{ij} = Q_{ip} Q_{jq} a\delta_{pq}$$

$$= aQ_{ip}Q_{jp}$$

$$= a\delta_{ij}$$

QUESTION 9

The most general form of a fourth-order isotrophic tensor can be expressed by

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

Where α, β and γ are arbitrary constants. Verify that this forms remains the same under the general transformation.

Solution:-

$$\alpha'\delta'_{ij}\delta'_{kl} + \beta'\delta'_{ik}\delta'_{jl} + \gamma'\delta'_{il}\delta'_{jk}$$

$$= Q_{im}Q_{jn}Q_{kp}Q_{lq}(\alpha\,\delta_{mn}\delta_{pq} + \beta\delta_{mp}\delta_{nq} + \gamma\delta_{mq}\delta_{np})$$

$$= \alpha Q_{im}Q_{jm}Q_{kp}Q_{lp} + \beta Q_{im}Q_{jn}Q_{km}Q_{ln}$$

$$+ \gamma Q_{im}Q_{jn}Q_{kn}Q_{lm}$$

$$= \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

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Q10:-
For the fourth-order isotropic tensor given in Exercise 1-9, show that if $B = \gamma$, then the tensor will have the following symmetry $C_{ijkl} \cdot C_{klij}$ -

Soln:-

$$C_{ijkl} = \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}$$
$$= \alpha \delta_{ij}\delta_{kl} + B(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$
$$= \alpha \delta_{kl}\delta_{ij} + \beta(\delta_{ki}\cdot\delta_{lj} + \delta_{kj}\delta_{li})$$
$$= C_{klij}.$$

Q1-11:-
Show that the fundamental invariants can be expressed in terms of principal values as given by relations (1.6.5)-

Soln.:-

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix}$$

$$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix}$$

$$= \lambda_1\lambda_2\lambda_3$$

Q1-12:-
Determine the invariants and principal values and directions of the following matrices. Use the determined principal directions to establish a principal co-ordinate system, and following the procedure in Example 1.3, formally transform (rotate)

the given matrix into the principal system to arrive at the appropriate diagonal form

(a) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = a_{ij}$

$I_a = a_{ij} = a_{11} + a_{22} + a_{33}$

$= -1 - 1 + 1$

$= -1$

$II_a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

$= \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$

$= (\cancel{1} - \cancel{1}) + (-1-0) + (-1+0)$

$= 0 - 1 - 1$

$= -2$

$III_a = \det [a_{ij}]$

$= \begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= (-1) \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} - (1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + (0) \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$

$= (-1)(-1-0) - (1)(1-0) + 0$

$= \cancel{1} - \cancel{1} + 0$

$= 0$

Characteristic Equation is:-

$0 = -\lambda^3 + I_a \lambda^2 - II_a \lambda + III_a$

$0 = -\lambda^3 + (-1)\lambda^2 - (-2)\lambda + (0)$

$0 = -\lambda^3 - \lambda^2 + 2\lambda$

$\lambda(\lambda^2 + \lambda - 2) = 0$

$\lambda(\lambda+2)(\lambda-1) = 0$

Roots:- $\lambda_1 = 0, \ \lambda_2 = -2, \ \lambda_3 = 1$

Case-I :-

$\lambda_2 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$-n_1^{(2)} + n_2^{(2)} = 0 \Rightarrow n_1 = n_2 = \pm\sqrt{2}/2$$

$$n_3^{(2)} = 0$$

$$n_1^{(2)^2} + n_2^{(2)^2} + n_3^{(2)^2} = 1$$

$$\Rightarrow n^{(2)} = \pm \frac{\sqrt{2}}{2}(1, 1, 0)$$

Case-II :-

$\lambda_1 = -2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$n_1^{(1)} + n_2^{(1)} = 0$$

$$n_3^{(1)} = 0 \qquad \Rightarrow n_1 = -n_2 = \pm\sqrt{2}/2.$$

$$n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1$$

$$\Rightarrow n^{(1)} = \pm(\sqrt{2}/2)(-1, 1, 0).$$

Case-III :-

$\lambda_3 = 1$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0.$$

$$-2n_1^{(3)} + n_2^{(3)} = 0$$

$$n_1^{(3)} - 2n_2^{(3)} = 0 \Rightarrow n_1 = n_2 = 0, n_3 = 1.$$

$$n_1^{(3)^2} + n_2^{(3)^2} + n_3^{(3)^2} = 1$$

$$\Rightarrow \pm(0, 0, 1) = n^{(3)}$$

The rotation matrix is given by $Q_{ij} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$

and

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$I_a = a_{11} + a_{22} + a_{33} = -2-2+0 = -4$.

$II_a = 3$, $III_a = 0$.

Characteristic equation is:-

$-\lambda^3 + I_a \lambda^2 - II_a \lambda + III_a = 0$.

$-\lambda^3 - 4\lambda^2 - 3\lambda = 0$.

$\lambda(\lambda^2 + 4\lambda + 3) = 0$.

$\lambda(\lambda+3)(\lambda+1) = 0$.

Roots:- $\lambda_1 = -3$, $\lambda_2 = -1$, $\lambda_3 = 0$.

Case-I:-

$\lambda_1 = -3$

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$

$n_1^{(1)} + n_2^{(1)} = 0 \Rightarrow n_1^{(1)} = -n_2^{(1)} = +\sqrt{2}/2$

$n_3^{(1)} = 0$

${n_1^{(1)}}^2 + {n_2^{(1)}}^2 + {n_3^{(1)}}^2 = 1$

$\Rightarrow n^{(1)} = \pm \frac{\sqrt{2}}{2}(-1, 1, 0)$

Case-II:-

$\lambda_2 = -1$.

$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$.

$-n_1^{(2)} + n_2^{(2)} = 0$

$n_3^{(2)} = 0 \qquad \Rightarrow n_1 = n_2 = \pm \frac{\sqrt{2}}{2}$.

${n_1^{(2)}}^2 + {n_2^{(2)}}^2 + {n_3^{(2)}}^2 = 1$.

$\Rightarrow n^{(2)} = \pm \frac{\sqrt{2}}{2}(1, 1, 0)$

Case-III:- $\lambda_3 = 0$.

$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$

$-2n_1^{(3)}+n_2^{(3)}=0.$

$n_1^{(3)}-2n_2^{(3)}=0 \Rightarrow n_1=n_2=0, n_3^{(3)}=1$

$n_1^{(3)^2}+n_2^{(3)^2}+n_3^{(3)^2}=1$

$\Rightarrow n^{(3)}=\pm(0,0,1)$

The rotation matrix is given by $Q_{ij}=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}$

$$a_{ij}'=Q_{ip}Q_{jp}a_{pq}=\frac{1}{2}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & \frac{2}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}^T$$

$$=\begin{bmatrix}-3 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

(c) $\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}.$

$I_a=-2,\ II_a=0,\ III_a=0.$

Characteristic equation is:

$-\lambda^3-2\lambda^2=0.$

$\lambda^2(\lambda+2)=0.$

Roots:-

$\lambda_1=-2,\ \lambda_2=\lambda_3=0$

Case-I:-

$\lambda_1=-2$

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}=0.$$

$\Rightarrow n_1^{(1)}+n_2^{(2)}=0$

$n_3^{(1)}=0 \Rightarrow n_1^{(1)}=-n_2^{(1)}=\pm\frac{\sqrt{2}}{2}$

$n_1^{(1)^2}+n_2^{(1)^2}+n_3^{(1)^2}=1$

$\Rightarrow n^{(1)}=\pm\frac{\sqrt{2}}{2}(-1,1,0)$

Case II, III :-

$\lambda_2 = \lambda_3 = 0$.

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$-n_1 + n_2 = 0 \Rightarrow n_1 = n_2, n_3^2 = 1 - 2n_1^2$

$n_1^2 + n_2^2 + n_2^2 = 1$

$\Rightarrow n = \pm (K, K\sqrt{1-2K^2})$.

For arbitrary K, and thus directions are not uniquely determined. For convenience we may choose

$K = \frac{\sqrt{2}}{2}$ and 0

To get $n^{(2)} = \pm \sqrt{2}/2 \, (1,1,0)$ & $n^{(3)} = \pm (0,0,1)$

The rotation matrix is given by $Q_{ij} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

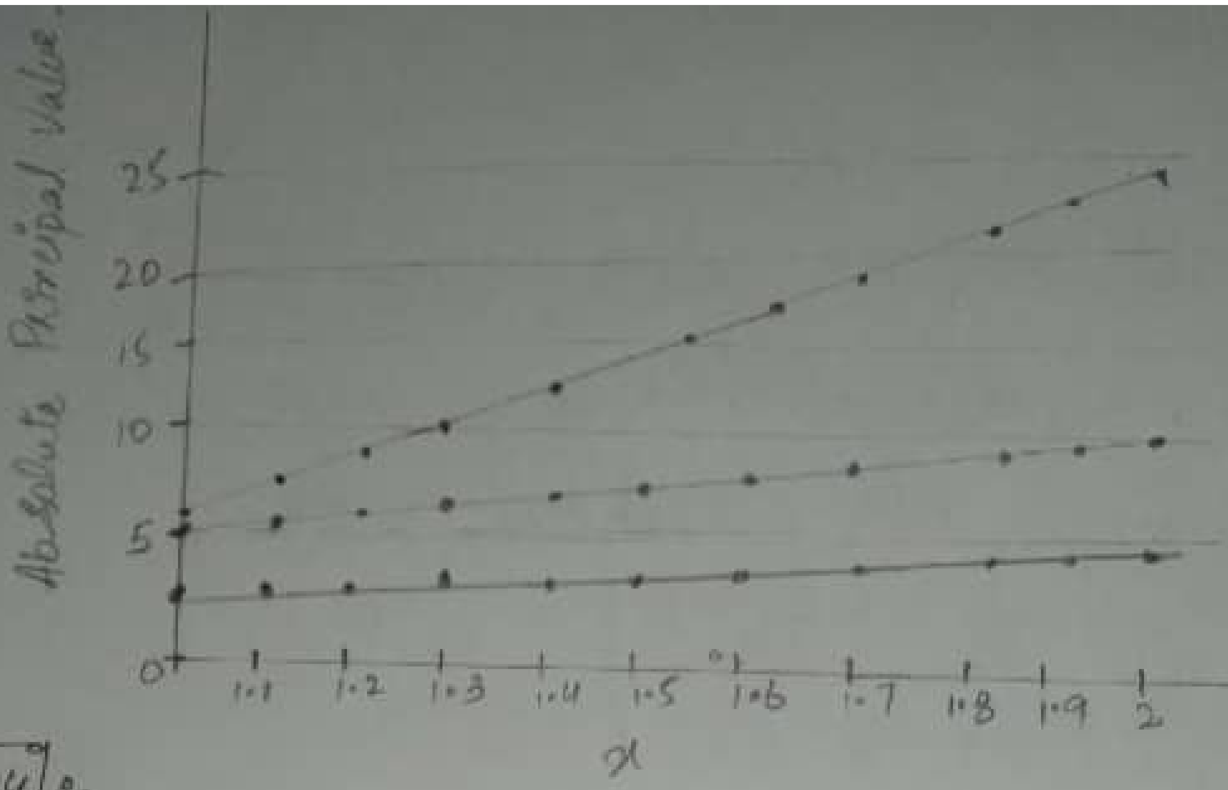
$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\boxed{Q1-13}$:-

A second order symmetric tensor field is given by:

$$a_{ij} = \begin{bmatrix} 2x_1 & x_1 & 0 \\ x_1 & -6x_1^2 & 0 \\ 0 & 0 & 5x_1 \end{bmatrix}$$

Use MATLAB (or similar software) - investigate the nature of the variation of the principal values and directions over the interval $1 \le x_1 \le 2$. Formally plot the variation of the absolute value of each principal value over the range $1 \le x_1 \le 2$



Q 1-14:

Calculate the quantities $\nabla\cdot U$, $\nabla\times U$, $\nabla^2 U$, ∇U, $tr(\nabla U)$ for the following cartesian.

(a) $U = x_1 e_1 + x_1 x_2 e_2 + 2x_1 x_2 x_3 e_3$

$$\nabla\cdot U = U_{1,1} + U_{2,2} + U_{3,3}$$
$$= 1 + x_1 + 2x_1 x_2$$

$$\nabla\times U = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1 x_2 & 2x_1 x_2 x_3 \end{vmatrix}$$

$$= 2x_1 x_3 e_1 - 2x_2 x_3 e_2 + x_2 e_3$$

$$\nabla^2 U = 0e_1 + 0e_2 + 0e_3 = 0$$

$$\nabla U = \begin{vmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2 x_3 & 2x_1 x_3 & 2x_1 x_2 \end{vmatrix}$$

$$tr(\nabla U) = 1 + x_1 + 2x_1 x_2$$

(b) $U = x_1^2 e_1 + 2x_1 x_2 e_2 + x_3^3 e_3$

$$\nabla\cdot U = U_{1,1} + U_{2,2} + U_{3,3} = 2x_1 + 2x_1 + 3x_3^2$$

$$\nabla\times U = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1 x_2 & x_3^3 \end{vmatrix}$$

$$= 0e_1 - 0e_2 + 2x_2 e_3$$

$\nabla^2 u = 2e_1 + 0e_2 + 6x_3 e_3 = 0$.

$$\nabla u = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}.$$

$tr(\nabla u) = 4x_1 + 3x_3^2$.

(c) $u = x_2^2 e_1 + 2x_2 x_3 e_2 + 4x_1^2 e_3$

$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0 = 2x_3$.

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2 x_3 & 4x_1^2 \end{vmatrix}$$

$= -2x_2 e_1 - 8x_1 e_2 - 2x_2 e_3$.

$\nabla^2 u = 2e_1 + 0e_2 + 8e_3 = 0$.

$$\nabla u = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}$$

$tr(\nabla u) = 3x_3$.

$\boxed{Q1-15}$:-

The dual vector a_i of an anti symmetric second-order tensor a_{ij} is defined by $a_i = -1/2\, \varepsilon_{ijk} a_{jk}$. Show that this expression can be inverted to get $a_{jk} = -\varepsilon_{ijk} a_i$.

Soln.:- $a_i = -\frac{1}{2} \varepsilon_{ijk} a_{jk}$

$$\varepsilon_{imn} a_i = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{imn} a_{jk}$$

$$= -\frac{1}{2} \begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} a_{jk}$$

$$= -\frac{1}{2} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_{jk}$$

$$= -\frac{1}{2} (a_{mn} - a_{nm}) = -\frac{1}{2}(a_{mn} + a_{mn}) = -a_{mn}$$

$\therefore a_{jk} = -\varepsilon_{ijk}\, a_i$.

Q1-16 :-
Using index notation, explicitly verify the vector identities (5

(a) $(1.8.5)_{1,2,3}$.

$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$.

$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k}$

$= \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$.

$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$.

$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$.

$\nabla\cdot(\phi U) = (\phi U_k)_{,k} = \phi U_{k,k} + \phi_{,k}U_k$

$= \nabla\phi\cdot U + \phi(\nabla\cdot U)$.

(b) $(1.8.5)_{4,5,6,7}$.

$\nabla\times(\phi U) = \varepsilon_{ijk}(\phi U_k)_{,j} = \varepsilon_{ijk}(\phi U_{k,j} + \phi_{,j}U_k)$

$= \varepsilon_{ijk}\phi_{,j}U_k + \phi\varepsilon_{ijk}U_k = \nabla\phi\times U + \phi(\nabla\times U)$.

$\nabla\cdot(U\times V) = (\varepsilon_{ijk}U_jV_k)_{,i} = \varepsilon_{ijk}(U_jV_{k,i} + U_{j,i}V_k)$

$= V_k\varepsilon_{ijk}U_{j,i} + U_j\varepsilon_{ijk}V_{k,i} = V\cdot(\nabla\times U) - U\cdot(\nabla\times U)$

$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0$
because of symmetry and antisymmetry in jk

$\nabla\cdot\nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$.

(c) $(1.8.5)_{8,9,10}$

$\nabla\cdot(\nabla\times U) = (\varepsilon_{ijk}U_{k,j})_{,i} = \varepsilon_{ijk}U_{k,ji} = 0$
because of symmetry and antisymmetry in ij.

$\nabla\times(\nabla\times U) = \varepsilon_{mni}(\varepsilon_{ijk}U_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}U_{k,jn}$

$= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})U_{k,jn} = U_{n,nm} - U_{m,nn}$

$= \nabla(\nabla\cdot U) - \nabla^2 U$

$U\times(\nabla\times U) = \varepsilon_{ijk}U_j(\varepsilon_{kmn}U_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}U_jU_{n,m}$

$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})U_jU_{n,m} = U_nU_{n,i} - U_mU_{i,m} = \frac{1}{2}\nabla(U\cdot U) - U\cdot\nabla U$

Q1-17 :-

Extend the results found in Example 1-5, and determine the forms of ∇f, $\nabla\cdot u$, $\nabla^2 f$ and $\nabla\times u$ for a three-dimensional cylindrical co-ordinate system.

Cylindrical co-ordinates :

$$\xi^1 = r,\ \xi^2 = \theta,\ \xi^3 = z.$$

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2$$

$$\Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1$$

$$\hat{e}_r = \cos\theta e_1 + \sin\theta e_2 ,$$

$$\hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2 ,$$

$$\hat{e}_z = e_3 .$$

$$\frac{\partial \hat{e}_r}{\partial\theta} = \hat{e}_\theta ,\quad \frac{\partial \hat{e}_\theta}{\partial\theta} = -\hat{e}_r .$$

$$\frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial\theta} = \frac{\partial \hat{e}_z}{\partial z} = 0 .$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_z \frac{\partial f}{\partial z}$$

$$\nabla\cdot u = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} .$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla\times u = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right)\hat{e}_z$$

Q1-18:

For the spherical co-ordinate system (R, ϕ, θ) show that
$h_1 = 1,\ h_2 = R,\ h_3 = R\sin\phi$.

Soln:

Spherical coordinate: $\xi^1 = R,\ \xi^2 = \phi,\ \xi^3 = \theta$.

$x^1 = \xi^1 \sin\xi^2 \cos\xi^3,\ x^2 = \xi^1 \sin\xi^2 \sin\xi^3,\ x^3 = \xi^1 \cos\xi^2$.

Scale factors:

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1} \cdot \frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1$$
$$\Rightarrow h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2} \frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

$$(h_3)^3 = \frac{\partial x^k}{\partial \xi^3} \frac{\partial x^k}{\partial \xi^3} = R^2 \sin^2\phi \Rightarrow h_3 = R\sin\phi.$$

Unit Vectors

$\hat{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3$.

$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3$.

$\hat{e}_\theta = -\sin\phi\, e_1 + \cos\theta\, e_2$

$$\frac{\partial \hat{e}_R}{\partial R} = 0,\ \frac{\partial \hat{e}}{\partial \phi} = \hat{e}_\phi,\ \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi\, \hat{e}_\theta.$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0,\ \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_r,\ \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta.$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0,\ \frac{\partial \hat{e}_\theta}{\partial \phi} = 0,\ \frac{\partial \hat{e}_\theta}{\partial \theta} = \cos\phi\, \hat{e}_\phi$$

$$\nabla = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}$$

$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\nabla \cdot U = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2\sin\phi\, U_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi\, U_\phi)$$

$$+ \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}(R U_\theta)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 U_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, U_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}(U_\theta$$

$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right)$$

$$+ \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

$$\nabla \times U = \left(\frac{1}{R^2\sin\phi}\left[\frac{\partial}{\partial \phi}(R\sin\phi\, U_\theta) - \frac{\partial}{\partial \theta}(R U_\phi)\right]\right)\hat{e}_R$$

$$+ \left(\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial \theta}(U_R) - \frac{\partial}{\partial R}(R\sin\phi\, U_\theta)\right]\right)\hat{e}_\phi$$

$$+ \left(\frac{1}{R}\frac{\partial}{\partial R}[(R U_\phi) - \frac{\partial}{\partial \phi}(U_R)\right)\hat{e}_\theta$$

$$= \left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial \phi}(\sin\phi\, U_\theta) - \frac{\partial U}{\partial \theta}\right)\right]\hat{e}_R +$$

$$\left[\frac{1}{R\sin\phi}\frac{\partial U_R}{\partial \theta} - \frac{1}{R}\frac{\partial}{\partial R}(R U_\theta)\right]\hat{e}_\phi$$

$$+ \left[\frac{1}{R}\left(\frac{\partial}{\partial R}(R U_\phi) - \frac{\partial U_R}{\partial \phi}\right)\right]\hat{e}_\theta$$

Question:-
Transform Strain-Displacement relation from cartesian to cylindrical and spherical co-ordinates

Answer:-

1) Cylindrical Co-ordinates.

$$U_x = U_r \cos\theta - U_\theta \sin\theta$$

$$U_y = U_r \sin\theta + U_\theta \cos\theta$$

$$U_z = U_z$$

2) Derivatives of $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ where $r = \sqrt{x^2+y^2}$, $\theta = \arctan\left(\frac{y}{x}\right)$, is given by.

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

It follows that

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \left(\cos\theta\frac{\partial}{\partial r}\right)\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$- \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial}{\partial r}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} - \cos\theta\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} - \cos\theta\sin\theta\left[-\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial \theta}\right]$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial \theta\partial r}$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2} \sin\theta\cos\theta \frac{\partial}{\partial \theta} - \frac{1}{r} \sin\theta\cos\theta \frac{\partial^2}{\partial r \partial \theta}$$

$$- \frac{\sin\theta\cos\theta}{r} \frac{\partial^2}{\partial \theta \partial r} + \frac{\sin^2\theta}{r} \frac{\partial}{\partial r}$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \sin^2\theta \left[\frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] +$$

$$2\sin\theta\cos\theta \left(\frac{1}{r^2} \frac{\partial}{\partial \theta} - \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} \right)$$

Likewise

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \frac{\partial^2}{\partial r^2} + \cos^2\theta \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial r^2}{\partial \theta^2} \right)$$

$$- 2\sin\theta\cos\theta \left(\frac{1}{r^2} \frac{\partial}{\partial \theta} - \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} \right)$$

$$e_{xx} = \frac{\partial v_x}{\partial x} = \cos\theta \frac{\partial}{\partial r} (v_r\cos\theta - v_\theta \sin\theta)$$

$$- \frac{\sin\theta}{r} \left(\frac{\partial}{\partial \theta} (v_r\cos\theta - v_\theta \sin\theta) \right)$$

$$= \frac{\partial v_r}{\partial r} \cos^2\theta - \frac{\partial v_\theta}{\partial r} \sin\theta\cos\theta - \frac{\partial v_r}{\partial \theta} \frac{\sin\theta\cos\theta}{r}$$

$$+ \frac{v_r}{r} \sin^2\theta + \frac{\partial v_\theta}{\partial \theta} \frac{\sin^2\theta}{r} + \frac{v_\theta}{r} \sin\theta\cos\theta$$

$$= \frac{\partial v_r}{\partial r} \cos^2\theta + \left(\frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \sin\theta\cos\theta$$

$$+ \left(\frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \right) \sin^2\theta$$

$$e_{yy} = \frac{\partial v_y}{\partial y} = \sin\theta \frac{\partial}{\partial r} (v_r \sin\theta + v_\theta \cos\theta) + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} (v_r \sin\theta + v_\theta\cos\theta$$

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$$e_{xy} = 2\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\right)$$

thus,

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

and

$$e_{zz} = \frac{\partial u_z}{\partial z}$$