Solution of Non-Linear Equations

1- Graphical method

Find the graphical solution (+ve root) for equation
$x^3 - 6x - 13 = 0$.

Solution

Given that,

$f(x) = x^3 - 6x - 13 = 0$ — (1)

$f(0) = 0 - 0 - 13 = -13$,

$f(1) = 1 - 6 - 13 = -18$

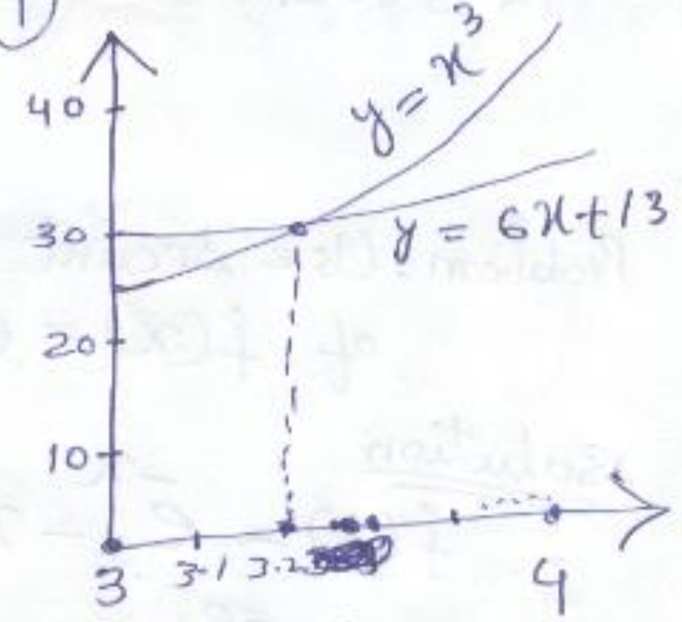
$f(-1) = -1 + 6 - 13 = -8$

$f(2) = 8 - 12 - 13 = -17$

$f(3) = 27 - 18 - 13 = -3$

$f(-2) = -8 + 12 - 13 = -9$

$f(4) = 64 - 24 - 13 = 27$



$f(3) = -3$ & $f(4) = 27$ Let origin be $(3, 0)$.

$f(4) = 64 - 24$ $\quad$ $x = 3.2$

Since $f(3) = -3$ & $f(4) = 27$ $\quad$ let origin be $(3, 0)$.

It means root lies b/w 3 & 4.

Eqn (1) can be written as

$$x^3 = 6x - 13$$

$$\Rightarrow y = x^3 \text{ \& } y = 6x - 13$$

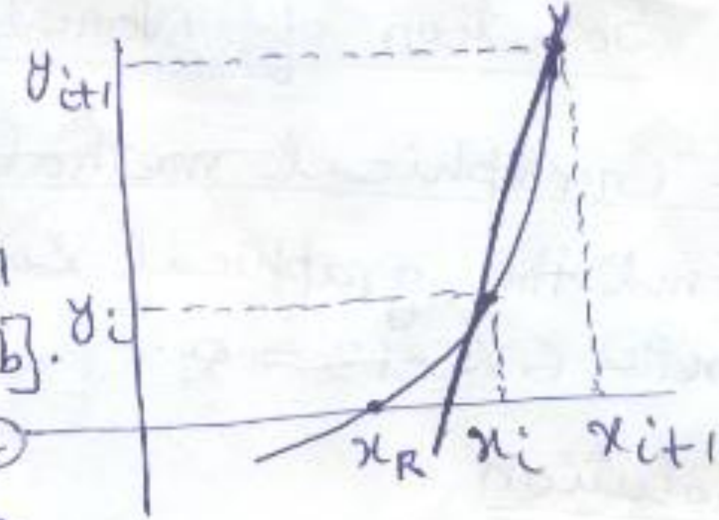
Now drawing the curves for both functions.

For $y = x^3$

x	3	3.2	3.4	3.6	3.8	4.0
$x^3 = y$	27	32.8	39.3	46.7	54.9	64
$y = 6x - 13$	31	32.2	33.4	34.6	35.8	37

The graphs of $y = x^3$ and $y = 6x - 13$ intersects at 3.2. $\therefore$ The root of given equation is "3.2".

Secant method



1- Given $y = f(x)$

2- If $f(x_0) * f(x_1) < 0$ then the reqd root lies $[a, b]$.

3- Find $f(x_0), f(x_1)$ for ② if true then find x_2 & $f(x_3)$

Problem: Use secant method to estimate the root of $f(x) = e^{-x} - x$ start with $x_0 = 0, x_1 = 1$

Solution

$$f(x) = e^{-x} - x$$

$$f(x_0) = f(0) = e^{-0} - 0 = 1$$

$$f(x_1) = e^{-1} - 1 = \frac{1}{e} - 1 = -0.63212$$

$f(x_0) = f(0) = e^{-0} - 0 = 1$

$f(x_1) = f(1) = e^{-1} - 1 = \frac{1}{e} - 1 = -0.63212$

Since $f(0) * f(1) < 0$ the root lies b/w 0 & 1.

Now using Secant formula,

$$x_{i+2} = x_{i+1} - \frac{f(x_{i+1})(x_i - x_{i+1})}{f(x_i) - f(x_{i+1})}$$

Firstiteration

$x_0 = 0 \Rightarrow f(x_0) = 1$

$x_1 = 1 \Rightarrow f(x_1) = -0.63212$

$$x_2 = x_1 - \frac{f(x_1)(x_0 - x_1)}{f(x_0) - f(x_1)}$$

$$x_2 = 1 - \frac{-0.63212(0-1)}{1-(-0.63212)} = 0.61270$$

Second iteration

$$x_3 = x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)}$$

---- so on ----

Newton-Raphson method

Formula: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ — (1)

1- $y=f(x)$ given 2- Start with one point "x_0"

3- Find $f'(x)$, $f(x_0)$ & $f'(x_0)$ then calculate x_2 and so on

Prob: Start with $x_0 = 3$, find the root of $x^3 - 3x - 5 = 0$, correct to 3 decimal places by using NR method.

Solution let $f(x) = x^3 - 3x - 5$

$$f'(x) = 3x^2 - 3$$

$f(x_0) = f(3) = 27 - 9 - 5 = 13$

$f'(x_0) = 27 - 3 = 24$

First iteration

$i=0$ (1) $\Rightarrow x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3 - \dfrac{13}{24} = 2.45833$

Second iteration

$i=1$ $\quad f(x_1) =$

$$① \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.4583 - \frac{2.4812}{15.12971} = 2.2943$$

Third iteration

$i=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.2943 - \frac{0.1939}{12.7914} = 2.2791$$

$f(2.279) = 0.0010$, Hence the reqd root is 2.2791.

Prob 2: Find the root of eqn $e^x = 3x$ b/w 0 and 1 by NR-method.

Sol. $f(x) = e^x - 3x = 0$

start with $x_0 = \frac{0+1}{2} = 0.5$

Using NR-formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{--- } (1)$$

$$i=0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{--- } (2)$$

$f(x) = e^x - 3x$, $f'(x) = e^x - 3$

$f(x_0) = e^{0.5} - 3(0.5)$ $\quad f'(x_0) = e^{0.5} - 3$

$f(0.5) = 0.149$ $\quad f'(0.5) = -1.351$

$$(2) \Rightarrow x_1 = 0.5 - \frac{0.149}{-0.1351} = 0.61$$

$f(x_1) = f(0.61) = 0.01$

$f'(x_1) = f'(0.61) = -1.16$

$$\underline{i=1} \quad (2) \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.61 - \frac{0.01}{-1.16} = 0.6186$$

$f(0.6186) = 0.0005$, So, $\boxed{x = 0.6186}$ Ans

<u>Prob 3</u> $f(x) = x^4 - x - 10 = 0$, near $x = 2$, 3 dp.

$x_0 = 2$, $f'(x) = 4x^3 - 1$

$f(x_0) = f(2) = 4$

$f'(x_0) = f'(2) = 31$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.871$$

$f(x_1) = 0.379$

$f'(x_1) = 25.1988$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.856$$

$f(x_2) = 0.0102$

$f'(x_2) = 24.5736$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.8556$$

$f(x_3) = 0.00038$

$f'(x_3) = 24.5572$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.85558$$

correct to 3 dp say $\simeq 1.8556$ Ans

Chebyshev method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} - \frac{[f(x_i)]^2 f''(x_i)}{2[f'(x_i)]^3} \quad (1)$$

Prob Solve the equation $x^3 - 29 = 0$; {3 dp accuracy

Solution
let $f(x) = x^3 - 29$, $f'(x) = 3x^2$, $f''(x) = 6x$

$f(0) = -29$

$f(1) = -28$

$f(2) = -21$

$f(3) = -2$

$f(4) = 35$

Since the root lies b/w 3 & 4.

let $x_0 = 4$

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$f(x_0) = f(4) = 35$

$f'(x_0) = f'(4) = 48$

$f''(x_0) = f''(4) = 24$

$$(1) \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{(f(x_0))^2 f''(x_0)}{2(f'(x_0))^3} = 3.137912$$

Now, $f(x_1) = 1.897434$, $f'(x_1) = 29.539$, $f''(x_1) = 18.8274$

$$(1) \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} - \frac{(f(x_1))^2 f''(x_1)}{2(f'(x_1))^3} = 3.072363$$

Now, $f(x_2) = 0.001324$, $f'(x_2) = 28.3182$

$f''(x_2) = 18.4342$

$$(1) \Rightarrow x_3 = x_2 - \frac{[f'(x_2)]^2 f''(x_2)}{2[f'(x_2)]^3} - \frac{f(x_2)}{f'(x_2)} = 3.072316$$

Ans

(6) Graeffe's Root Square method

Q Solve the eqn $x^3 - x^2 - 17x - 15$ by using Graeffe's root square method squaring 4 times?

1 - Write the coefficients in ascending order.

2 - Take sq. of all

3 - Perform the 1st operation (1st * 3rd * -2)

(2nd * 4th * -2)

4 - Add then obtain new coefficients, cont. upto desire. roots.

No. of Sq.	Roots	Coefficients			
m=0	x_i	1	-1	-17	-15
		1	1	289	225
	operation			1st * 3rd * -2 (1 * -17 * -2) 34	2nd * 4th * -2 -1 * -15 * -2 -30
	Add				
m=1	x_i^2	1	35	259	225

	Add				
$m=1$	x^2	1	35	259	225
		1	1225	67081	50625
			(1st × 3rd × −2)	(2nd × 4th × −2)	
			−518	−15750	
$m=2$	x^4	1	707	51331	50625
		1	499849	2634871 56	25628906
			−102662	−71583750	
$m=3$	x^{16}	1	397187	2563287811	25628906
		1	1.57757×10^{11}	6.570444×10^{18}	6.56840×10^{18}
			−512675622	-2.03589×10^{15}	
		1	1.5263234×10^{11}	6.568408×10^{18}	6.56840×10^{18}

$$x_1^{16} = \frac{a_2}{a_1} = \frac{1.526309374 \times 10^{11}}{1} = 1.526309374 \times 10^{11}$$

$$x_1 = \left(1.526309374 \times 10^{11}\right)^{1/16} = \pm 5.000088148 \cong \pm 5$$

$$x_2^{16} = \frac{a_3}{a_2} = \frac{6.568408508 \times 10^{18}}{1.526309374 \times 10^{11}} = 43034587814 5$$

$$x_2 = \left(\quad \right)^{1/16} = \pm 2.99994746 \cong \pm 3$$

$$x_3^{16} = \frac{a_4}{a_3} = 0.999 \cong 1$$

Varification

$$= (1)^3 - (1)^2 - 17(1) - 15 = -1 - 1 + 17 - 15$$

$$x_3 = -1$$

Graeffe's Root Square method

1- Direct squaring method
2- Tabulation method

Prob Solve $y^3-30y^2+129y-100=0$ by GRSM by squaring 3 times.

Sol

$$y^3-30y^2+129y-100=0 \quad -(1)$$

$$y^3-129y = 30y^2+100$$ Separating even & odd power terms then squaring b/s

$$y(y^2-129)=10(3y^2+10)$$

squaring on b/s

$$\Rightarrow y^2(y^2-129)^2 = 100(3y^2+10)^2$$

Let $y^2 = Z$

$$\Rightarrow Z(Z-129)^2 = 100(3Z+10)^2$$

$$z(z^2+258z+1664) = 100(9z^2+60z+100)$$

$$z^3-642z^2+10641z-10000=0$$

$$\Rightarrow z^3-10641z = 642z^2+10000$$

$$z(z^2-10641) = (642z^2+10000) \quad \text{Squaring on b/s}$$

$$z^2(z^2-10641)^2 = (642z^2+10000)^2$$

Let $z^2 = u$

$$\Rightarrow u(u-10641)^2 = (642u+10000)^2$$

$$\underset{a_0}{u^3} - \underset{a_1}{390882}u^2 + \underset{a_2}{100390881}u - \underset{a_3}{100000000} = 0$$

$$|x_1| = \left[\left(\frac{a_1}{a_0}\right)\right]^{1/8} = \left[(-390882)\right]^{1/8} = 5.004$$

$$|x_2| = \left[\left(\frac{a_2}{a_1}\right)\right]^{1/8} = \left|\frac{100390881}{390882}\right|^{1/8} = 2.00081$$

$$|x_3| = \left[\frac{a_3}{\ldots}\right]^{1/8} \quad \left|\frac{100000000}{\ldots}\right|^{1/8}$$