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This paper deals with an economic production quantity inventory model for non-instantaneous deteriorating items under inflationary conditions considering customer returns. We adopt a price- and time-dependent demand function. Also, the customer returns are considered as a function of both price and demand. The effects of time value of money are studied using the Discounted Cash Flow approach. The main objective is to determine the optimal selling price, the optimal replenishment cycles, and the optimal production quantity simultaneously such that the present value of total profit is maximized. An efficient algorithm is presented to find the optimal solution. Finally, numerical examples are provided to solve the presented inventory model using our proposed algorithm, which is further clarified through a sensitivity analysis. The results of analysing customer returns provide important suggestions to financial managers who use price as a control to match the quantity sold to inventory while maximizing revenues. The paper ends with a conclusion and an outlook to future studies.

Keywords: inventory; price- and time-dependent demand; customer returns; non-instantaneous deteriorating items; inflation; efficient algorithm

AMS Subject Classification: 90B05; 78M50; 91B24

1. Introduction

For nearly three decades, the lodging industry has used revenue management practices and theory to extensively enhance inventory optimization and revenue generation. According to Chen and Bell [1], 'Revenue management (*RM*) has its source in the North American airline industry following deregulation in 1979 and has now been applied in many service industries and increasingly for manufactured products'. McGill and VanRyzin [2], Bitran and Caldentey [3], and Elmaghraby and Keskinocak [4] presented reviews of *RM*. Optimal pricing is an important revenue-enhancing business practice within *RM* that is often combined with inventory ordering policy. Federgruen and Heching [5] discussed the simultaneous determination of price and inventory replenishment strategies in a multi-period problem with stochastic demand. They considered both finite and infinite horizon models assuming that prices can either be adjusted arbitrarily (upward or downward) or that they can only be reduced.

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Recently, many researchers have studied the problem of joint pricing and inventory control for deteriorating items. Generally, deterioration is defined as decay, damage, spoilage, evaporation and loss of utility of the product. Most physical goods undergo decay or deterioration over time such as medicines, volatile liquids, blood banks and others.[6] The first attempt to describe optimal ordering policies for deteriorating items was made by Ghare and Schrader [7]. Later, Covert and Philip [8] derived the model with variable deteriorating rate of two-parameter Weibull distribution. Goyal and Giri [9] presented a detailed review of deteriorating inventory literatures. Abad [10,11] considered a pricing and lot-sizing problem for a perishable good under exponential decay and partial backlogging. Dye [12] proposed the joint pricing and ordering policies for a deteriorating inventory with price-dependent demand and partial backlogging. Dye et al. [13] developed an inventory and pricing strategy for deteriorating items with shortages when demand and deterioration rates are continuous and differentiable function of price and time, respectively. Chang et al. [14] introduced a deteriorating inventory model with log-concave demand and partial backlogging. Tsao and Sheen [15] proposed the problem of dynamic pricing and replenishment for deteriorating items under the supplier's trade credit and the retailer's promotional effort. Shi et al. [16] presented an optimal ordering and pricing policy with supplier quantity discounts and price-dependent stochastic demand. Sarkar [17] investigated a production-inventory model with probabilistic deterioration in two-echelon supply chain management. Sarkar and Sarkar [18] developed an inventory model with partial backlogging, time-varying deterioration and stock-dependent demand. Sarkar [19] presented an economic order quantity (EOQ) model with delay in payments, where demand and deterioration rate are both time-dependent. Sarkar [20] studied an EOQ model with delay in payments and stock-dependent demand in an imperfect production system. Sett et al. [21] proposed a two-warehouse inventory model with quadratically increasing demand and time-varying deterioration. Sarkar and Sarkar [22] developed an inventory model with inventory-dependent demand function and time-varying deterioration rate.

In most of the inventory models for deteriorating items in the literature, it is assumed that the deterioration occurs as soon as the commodities arrive in inventory. However, in real life, most goods would have a span of maintaining quality or original condition, i.e. during that period, there is no deterioration occurring. Wu et al. [23] defined the phenomenon as 'non-instantaneous deterioration'. They considered an inventory model for non-instantaneous deteriorating items with stock-dependent demand and permissible delay in payments. This type of phenomena can be commonly observed in food stuffs, fruits, green vegetables and fashionable goods, which have a span of maintaining fresh quality, and during that period there is almost no spoilage. For these kinds of items, the assumption that the deterioration starts from the instant of arrival in stock may cause retailers to make an inappropriate replenishment policy due to overvaluing of the total annual relevant inventory cost. Thus, it is necessary to consider the inventory problems for non-instantaneous deteriorating items. Ouyang et al. [24] proposed an inventory model for non-instantaneous deteriorating items with permissible delay in payments. Chang et al. [25] developed optimal replenishment policies for non-instantaneous deteriorating items with stock-dependent demand. Yang et al. [26] proposed an inventory and pricing strategy for non-instantaneous deteriorating items with price-dependent demand. In their model, shortages are allowed and partially backlogged. Geetha and Uthayakumar [27] proposed EOQ-based model for non-instantaneous deteriorating items with permissible delay in payments. In this model, demand and price are constant and shortages are allowed and partially backlogged. Musa and Sani [28] proposed a mathematical model for inventory control of non-instantaneous deteriorating items with permissible delay in payments. Maihami and Nakhai [29] presented a joint pricing and inventory control for non-instantaneous deteriorating items. In their model, the demand rate is a function of price and time simultaneously and shortages are allowed and partially backlogged. Also, Maihami and Nakhai [30] extended the mentioned model under permissible delay in payments.

In all the above-mentioned models, the inflation and the time value of money were disregarded, but most of the countries suffered from large-scale inflation and sharp decline in the purchasing power of money during years. As a result, while determining the optimal inventory policies, the effects of inflation and time value of money cannot be ignored. First, Buzacott [31] presented the EOQ model with inflation. Following Buzacott [31], several researchers (Misra [32], Jolai et al. [33], etc.) extended their approaches to distinguish the inventory models by considering the time value of money, the different inflation rates for the internal and external costs, finite replenishment, shortages, etc. Park [34] derived the EOQ in terms of purchasing credit. Datta and Pal [35] discussed a model with shortages and time-dependent demand rates to study the effects of inflation and time value of money on a finite time horizon. Goal et al. [36] developed the model economic discount value for multiple items with restricted warehouse space and the number of orders under inflationary conditions. Hall [37] presented a new model with the increasing purchasing price over time. Sarker and Pan [38] surveyed the effects of inflation and the time value of money on the optimal ordering quantities and the maximum allowable shortage in a finite replenishment inventory system. Hariga and Ben-Daya [39] presented time-varying lot-sizing models with a time-varying demand pattern and taking into account the effects of inflation and time value of money. Horowitz [40] discussed an EOQ model with a normal distribution for the inflation. Moon and Lee [41] developed an EOQ model under inflation and discounting with a random product life cycle. Mirzazadeh and Sarfaraz [42] presented a multiple-items inventory system with a budget constraint and the uniform distribution function for the external inflation rate. Dey et al. [43] developed the model for deteriorating items with time-dependent demand rate and interval valued lead-time under inflationary conditions. Mirzazadeh et al. [44] considered stochastic inflationary conditions with variable probability density functions (pdfs) over the time horizon and the demand rate is dependent on the inflation rates. Sarkar and Moon [45] developed a production inventory model for stochastic demand with inflation in an imperfect production system. Sarkar et al. [46] presented an economic manufacturing quantity (EMQ) model for time-varying demand with inflation in an imperfect production process. Wee and Law [47] developed a replenishment and pricing policy for deteriorating items taking into account the time value of money. In their model, shortages are allowed and the demand is considered as a function of price. Hsieh and Dye [48] presented pricing and inventory control model for deterioration items taking into account the time value of money. In their model, shortage is allowed and partially backlogged and the demand is considered as a function of price and time. Dye et al. [49] developed inventory and pricing strategies for deteriorating items taking into account time value of money. In their model, demand and deterioration rate are continuous and differentiable function of price and time, respectively, and shortages are allowed. Hou and Lin [50] presented an EOQ model for deteriorating items with price- and stock-dependent selling rates under inflation and time value of money. Hou and Lin [51] developed optimal pricing and ordering policies for deteriorating items under inflation and permissible delay in payments where demand rate is a linear function of price and decreases negative exponentially with time. Ghoreishi et al. [52] studied the problem of joint pricing and inventory control model for deteriorating items taking into account the time-value of money and customer returns. In this model, shortage is allowed and partially backlogged and the demand is deterministic and depends on time and price simultaneously. Sarkar and Sarkar [53] presented an EMQ model with deterioration and exponential demand under the effect of inflation and time value of money, where the production rate is a dynamic variable (varying with time) in a production system. Sarkar [54] proposed an EMQ model with price and advertising demand pattern in an imperfect production process under the effect of inflation, where the development cost, production cost and material costs are dependent on the reliability parameter. Sarkar et al. [55] considered an EMQ model for time-dependent (quadratic) demand pattern in an imperfect production process under the effect of inflation and time value of money. Sarkar and Moon [56] developed a production inventory model for stochastic demand with the effect of inflation. Sarkar et al. [57] discussed a finite replenishment model with time-varying demand under inflation and permissible delay in payments. Taheri-Tolgari et al. [58] proposed an inventory model for imperfect items under inflationary conditions with considering inspection errors. Mirzazadeh [59] presented an optimal inventory control problem with inflation-dependent demand, where the inflation and time horizon, both are random in nature. In this model, shortages are allowed and partially backlogged.

In the traditional economic production quantity (EPQ) model, customer returns are not considered, while in supply chain retailers can return some or all unsold items at the end of the selling season to the manufacturer and receive a full or partial refund. Hess and Mayhew [60] studied the problem of customer return by using regression methods to model the returns for a large direct market. Anderson et al. [61] found that the quantity sold has a strong positive linear relationship with number of returns. The same holds true for Hess and Mayhew; they used regression models to show that as the price increases, both the number of returns and the return rate increase. These empirical investigations provide evidence to support the view that customer returns increase with both the quantity sold and the price set for the product. Chen and Bell [1] considered the customer returns as a function of price and demand simultaneously. Pasternack [62] studied the newsvendor problem framework for a seasonal product where a percentage of the order quantity could be returned from the retailers to the manufacturer. Zhu [63] presented a single-item periodic-review model for the joint pricing and inventory replenishment problem with returns and expediting. Yet, only a few authors investigated the effect of customer returns on joint pricing and inventory control.

In this paper, we develop an appropriate pricing and inventory model for non-instantaneous deteriorating items under inflation and customer returns. There are a few models on pricing and inventory control with considering non-instantaneous deteriorating items. But, in the real world, the majority deteriorating items would have a span of maintaining quality or original condition, namely, during that period, there is no deterioration occurring. For this type of items, the assumption that the deterioration starts from the instant of arrival in stock may cause retailers to make inappropriate replenishment policies due to overvaluing the total annual relevant inventory cost.

Therefore, in the field of inventory management, it is necessary to consider the inventory problems for non-instantaneous deteriorating items. On the other hand, the coordination of price decisions and inventory control means optimizing the system rather than its individual elements. Thus, the optimal pricing combined with inventory ordering policy can yield considerable revenue increase. Also, in the previous research that considers non-instantaneous deteriorating items on pricing and inventory control, the effect of time value of money is not considered. However, in order to address the realistic circumstances, the effect of time value of money should be considered. In addition, in nearly all papers that consider the impact of customer returns on pricing and inventory control, the return functions are dependent on price or demand, separately. But, the empirical findings of Anderson et al. [61] provide evidence to support the view that customer returns increase with both the quantity sold and the price set for the product. Moreover, in the majority of papers that study pricing and inventory control for non-instantaneously deteriorating items, the demand functions are simple and dependent on price, stock or time, separately. But in the real world, the demand may increase when the price decreases, or it may vary through time. Therefore, in order to incorporate the realistic conditions, the price and the time should be considered simultaneously. Thus, in this work, we develop a finite planning horizon inventory model for non-instantaneous deteriorating items with price- and time-dependent demand rate. In addition, we consider the effects of customer returns and time value of money on replenishment policy and financial performance. We assume that the customer returns increase with both the quantity sold and the product price. An optimization algorithm is presented to derive the optimal length of the production period, selling price and the number of production cycles during the time horizon and then obtain the optimal production quantity when the total present value of profits is maximized. Therefore, the replenishment and price policies are appropriately developed. Numerical examples are provided to illustrate the proposed model, which is further clarified through a sensitivity analysis.

The rest of the paper is organized as follows. In Section 2, assumptions and notations throughout this paper are presented. In Section 3, we establish the mathematical model. Next, in Section 4, an algorithm is presented to find the optimal selling price and inventory control variables. In Section 5, we use numerical examples to illustrate the proposed model. Then a sensitivity analysis over a wide range of problem parameters is performed in Section 6. Finally, we give a summary and some suggestions for future studies in Section 7.

2. Notation and assumptions

The following notation and assumptions are used throughout the paper:

2.1. *Notation*

R	production rate for the item (units/unit time)
p	selling price per unit, where $p > c_2$ (decision variable)
σ	deteriorating rate of the items ($0 < \sigma < 1$)
K	set up cost per set up
c_1	holding cost per unit per unit time
c_2	purchasing price (or the production cost) per unit

T	duration of inventory cycle (decision variable)
t_p	length of the production period in an inventory cycle (decision variable)
t_d	length of time in which the product exhibits no deterioration
Q	production quantity
H	length of planning horizon
N	number of production cycles during the time horizon H
S	salvage value per unit
r	constant representing the difference between the discount (cost of capital) and the inflation rate
T^*	optimal length of the replenishment cycle time
Q^*	optimal production quantity
t_p^*	optimal length of the production period in an inventory cycle
p^*	optimal selling price per unit
I_0	maximum inventory level
$I_1(t)$	inventory level at time $t \in [0,\ t_d]$
$I_2(t)$	inventory level at time $t \in [t_d,\ t_p]$
$I_3(t)$	inventory level at time $t \in [t_p,\ T]$
$f(p,\ t_p, T; N)$	present value of total profit over the time horizon

2.2. *Assumptions*

I The planning horizon is finite.

II The initial and final inventory levels are both zero.

III A single non-instantaneous deteriorating item is assumed.

IV The production rate, which is finite, is higher than the demand rate.

V Delivery lead time is zero.

VI The demand rate, $D(p,t) = (a - bp)e^{\lambda t}$ (where $a, b > 0$) is a linearly decreasing function of the price and decreases (increases) exponentially with time if $\lambda < 0$ ($\lambda > 0$), respectively. [15]

VII Shortages are not allowed.

VIII Following the empirical findings of Anderson et al. [61], we assume that customer returns increase with both the quantity sold and the price. We use the general form: $RC(p,t) = \alpha D(p,t) + \beta p$ $(\beta \geq 0, 0 \leq \alpha < 1)$ that is presented by Chen and Bell [1]. Customers are assumed to return $RC(p,t)$ products during the period for full credit and these units are available for resale in the following period. We assume that the salvage value of the product at the end of the last period is S per unit.

IX The length of the production period is larger than or equal to the length of time in which the product exhibits no deterioration, i.e. $t_p \geq t_d$.

3. The model formulation

Here, we considered a production inventory system for non-instantaneous deteriorating items, which will be described as follows. During the interval [0, t_d], the inventory level increases due to production as the production rate is much greater than demand rate. At time t_d, deterioration starts and, thus, the inventory level increases due to production

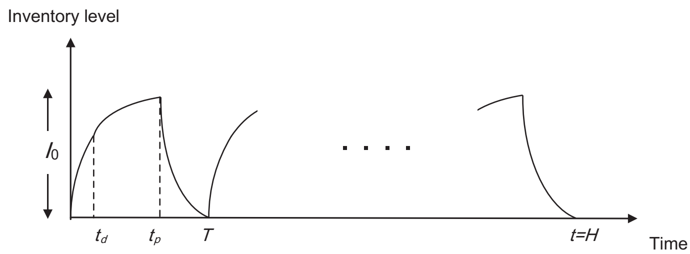

Figure 1. Graphical representation of an inventory system.

rate which is greater than the demand and the deterioration until the maximum inventory level is reached at $t = t_p$. During the interval [t_p, T], there is no production and the inventory level decreases due to demand and deterioration until the inventory level becomes 0 at $t = T$. The graphical representation of the model is shown below in Figure 1. In this illustration, the demand rate increases exponentially with time (i.e. $\lambda > 0$).

During the time interval [0, t_d], the system is subject to the effect of production and demand. Therefore, the change of the inventory level at time t, $I_1(t)$ is governed by

$$\frac{dI_1}{dt} = R - D(t,p). \tag{1}$$

With the condition $I_1(0) = 0$, solving Equation (1) yields

$$I_1(t) = \frac{(-a + bp)e^{\lambda t} - bp + Rt\lambda + a}{\lambda} \quad (0 \le t \le t_d). \tag{2}$$

In the time interval [t_d, t_p], the system is affected by the combination of the production, demand and deterioration. Hence, the change of the inventory level at time t, $I_2(t)$, is governed with

$$\frac{dI_2}{dt} + \theta I_2(t) = R - D(t,p). \tag{3}$$

With the condition $I_2(t_p) = I_0$, Equation (3) yields

$$I_2(t) = \frac{(\theta(a - bp)(e^{t_p(\theta+\lambda)} - e^{t(\theta+\lambda)}) + ((I_0\theta - R)e^{t_p\theta} + Re^{\theta t})(\theta + \lambda))e^{-\theta t}}{\theta(\theta + \lambda)} \quad (t_d \le t \le t_p). \tag{4}$$

In the third interval [t_p, T], the inventory level decreases due to demand and deterioration. Thus, the differential equation below represents the inventory status:

$$\frac{dI_3}{dt} + \theta I_3(t) = -D(t,p). \tag{5}$$

By the condition $I_3(T) = 0$, the solution of Equation (5) is

$$I_3(t) = \frac{(a - bp)(-e^{t(\theta+\lambda)} + e^{T(\theta+\lambda)})e^{-\theta t}}{\theta + \lambda} \quad (t_p \le t \le T). \tag{6}$$

Furthermore, in this interval with the condition $I_3(t_p) = I_0$, the maximum inventory level (I_0) yields the following value:

$$I_0 = \frac{(a - bp)(-e^{t_p(\theta+\lambda)} + e^{T(\theta+\lambda)})e^{-\theta t_p}}{\theta + \lambda}. \tag{7}$$

Note that the production occurs in continuous time-spans $[0,\ t_p]$. Hence, the lot size in this problem is given by

$$Q = R \cdot t_p. \tag{8}$$

Now, we can obtain the present-value inventory costs and sales revenue for the first cycle, which consists of the following elements:

(a) *SR*: the present value of the sales revenue for the first cycle:

$$SR = p\left(\int_0^T D(p, t) \cdot e^{-r \cdot t} dt\right). \tag{9}$$

(b) *PC*: The present value of production cost for the first cycle:

$$PC = c_2(R \cdot t_p). \tag{10}$$

(c) *K*: Since production set-up in each cycle is done at the beginning of each cycle, the present value of set-up cost for the first cycle is *K*, which is a constant value.

(d) *HC*: The present-value of inventory carrying cost for the first cycle:

$$HC = c_1\left(\int_0^{t_d} I_1(t) \cdot e^{-r \cdot t} dt + e^{-r \cdot t_d} \int_{t_d}^{t_p} I_2(t) \cdot e^{-r \cdot t} dt + e^{-r \cdot t_p} \int_{t_p}^{T} I_3(t) \cdot e^{-r \cdot t} dt\right). \tag{11}$$

(e) We assume that returns from period $i - 1$ are available for resale at the beginning of period i (except the first period in which there is no cycle previous to it). Also, it is assumed that the salvage value of product at the end of the last period ($i = N$) is S. Therefore, the present value of return cost and resale revenue for each cycle is obtained as follows:

$$PRC_i = \begin{cases} p\int_0^T (\alpha D(p,t) + \beta p)e^{-r.t} dt, & \text{for } i = 1, \\ PRC = p\int_0^T (\alpha D(p,t) + \beta p)e^{-r.t} dt - c_2\int_0^T (\alpha D(p,t) + \beta p) dt, & \text{for } i = 2, \ldots, N-1, \\ p\int_0^T (\alpha D(p,t) + \beta p)e^{-r.t} dt - c_2\int_0^T (\alpha D(p,t) + \beta p) dt & \\ \quad - Se^{-r.T}\int_0^T (\alpha D(p,t) + \beta p) dt, & \text{for } i = N. \end{cases} \tag{12}$$

Consequently, the *present value of total profit*, denoted by $f(p, t_p, T;\, N)$, is given by:

$$f(p, t_p, T;\, N) = \sum_{i=0}^{N-1}(SR - PC - K - HC - PRC)e^{-r\cdot i\cdot T} + S\cdot e^{-r\cdot H}\int_0^T (\alpha D(p,t) + \beta p)dt$$
$$- c_2\int_0^T (\alpha D(p,t) + \beta p)dt, \tag{13}$$

which we want to maximize subject to the following constraints:

$$p > 0,\ 0 < t_p < T, N \in \mathbb{N}.$$

The value of the variable T can be replaced by the Equation $T = H/N$, for some constant $H > 0$, and we will use Maclaurin's approximation for $\sum_{i=0}^{N-1} e^{-r\cdot i\cdot T} \cong (1 - e^{-r\cdot N\cdot T})/(1 - e^{-r\cdot T})$. Thus, the objective of this paper is to determine the values of t_p, p and N that maximize $f(p, t_p, T; N)$ subject to $p > 0$ and $0 < t_p < T$, where N is a discrete variable and p and t_p are continuous variables, can be reduced to maximizing $f(p, t_p, H/N; N)$. However, since $f(p, t_p, T; N)$, and still $f(p, t_p, H/N; N)$, is a very complicated function due to high-power expressions in the exponential function, it is difficult to show analytically the validity of the sufficient conditions. Hence, if more than one local maximum value exists, we would attain the largest of the local maximum values over the regions subject to $p > 0$ and $0 < t_p < T$. The largest value is referred to as the global maximum value of $f(p, t_p, T; N)$. So far, the procedure is to locate the optimal values of p and t_p when N is fixed. Since N is a discrete variable, the following algorithm can be used to determine the optimal values of p, t_p and N for the proposed model. We may refer to $f(p, t_p, H/N; N)$ and, for the sake of convenience, just denote it by $f(p, t_p, N)$.

4. The optimal solution procedure

The objective function has three variables. The number of replenishments (N) is a discrete variable and the production period in an inventory cycle (t_p) and the selling price per unit (p) are continuous variables. The following algorithm is used to obtain the optimal amount of t_p, p and N:

Step 1 let $N = 1$.

Step 2 Take the partial derivatives of $f(p, t_p, N)$ with respect to p and t_p, and equate the results to zero, the necessary conditions for optimality are

$$\frac{\partial}{\partial p} f(p, t_p, N) = 0 \tag{14}$$

and

$$\frac{\partial}{\partial t_p} f(p, t_p, N) = 0. \tag{15}$$

In Appendix A, we use the formula of $f(p, t_p, T\,; N)$ from Equation (13), inserted into Equations (14) and (15).

Step 3 For different integer N values, derive t_p^* and p^* from Equations (14) and (15). Substitute (p^*, t_p^*, N^*) to Equation (13) to derive $f(p^*, t_p^*, N^*)$.

Step 4 Add one unit to N and repeat Steps 2 and 3 for the new N. If there is no increasing in the last value of $f(p, t_p, N)$, then show the previous one which has the maximum value.

The point (p^*, t_p^*, N^*) and the value $f(p^*, t_p^*, N^*)$ constitute the optimal solution and satisfy the following conditions:

$$\Delta f(p^*, t_p^*, N^*) < 0 < \Delta f(p^*, t_p^*, N^* - 1), \tag{16}$$

where

$$\Delta f(p^*, t_p^*, N^*) = f(p^*, t_p^*, N^* + 1) - f(p^*, t_p^*, N^*). \tag{17}$$

We substitute (p^*, t_p^*, N^*) into Equation (8) to derive the Nth production lot size.

If the objective function was strictly concave, the following *sufficient* conditions must be satisfied:

$$\left(\frac{\partial^2 f}{\partial p \partial t_p}\right)^2 - \left(\frac{\partial^2 f}{\partial t_p^2}\right)\left(\frac{\partial^2 f}{\partial p^2}\right) < 0, \tag{18}$$

and any one of the following conditions:

$$\frac{\partial^2 f}{\partial t_p^2} < 0, \quad \frac{\partial^2 f}{\partial p^2} < 0. \tag{19}$$

Since f is a very complicated function due to high-power expression of the exponential function, it is unlikely to show analytically the validity of the above sufficient conditions. Our optimization problem is even more complex by that one of the variables, N, is an integer. However, it can be assessed numerically in the following illustrative examples.

5. Numerical examples

To illustrate the solution procedure and the results, let us apply the proposed algorithm to solve the following numerical examples. The results can be found by applying the Maple 13.

Example 1 This example is based on the following parameters and functions. R = 500 units/per unit time, c_1 = \$8/per unit/per unite time, c_2 = \$10/per unit, t_d = 0.04 unit time,

Table 1. Optimal solution of Example 1.

		Time interval			
N	p	t_p	T	Q	f
21	55.97656	0.94569	1.90476	472.84520	9358.184253
22*	55.92422*	0.94555*	1.81818*	472.77838*	9435.394724*
23	55.87688	0.94533	1.73913	472.66768	9434.405265

*Optimal solution.

K = \$250/per production run, σ = 0.08, r = 0.08, a = 200, b = 0.5, λ = −0.02, H = 40 unit time, α = 0.5, β = 0.7, S = \$3/per unit.

Using the solution procedure described above, the related results are shown in Table 1 and all the given conditions in Equations (18) and (19) are satisfied. In this example, the maximum present value of the total profit is found when the number of cycle (N) is 22. With 22 replenishments, the optimal solution is as follows:

$$p^* = 55.92422,\ t_p^* = 0.94555,\ T^* = 1.81818,\ f^* = 9435.394724,\ Q^* = 472.77838$$

Example 2 This example is based on the following data. R = 700 units/per unit time, c_1 = \$5/per unit/per unit time, c_2 = \$3/per unit, t_d = 1/12 unit time, K = \$250/per production run, σ = 0.08, r = 0.08, a = 200, b = 0.5, λ = −0.02, H = 40, α = 0.2, β = 0.875, S = \$1/per unit.

According to the computational results shown in Table 2, the optimal solution is as follows:

$$p^* = 63.59471,\ t_p^* = 0.69464,\ T^* = 1.42857,\ f^* = 45629.96427,\ Q^* = 486.24675.$$

6. Sensitivity analysis

First, we obtain the results of Example 1 for analysing the impact of customer returns on the optimal solutions and financial performance (Tables 3). The results illustrate that when returns are proportional to the quantity sold only (i.e. β = 0), the firm should raise the price and reduce the production quantity but if returns are proportional to price only (i.e. α = 0) the firm should decrease the price and increase the production quantity. The results confirm that when returns increase with the product price (when purchase costs are constant), the firm should set a lower price to the no-returns case (in order to discourage returns). Increasing α and/or β reduces the firm's profit.

Table 2. Optimal solution of Example 2.

		Time interval			
N	p	t_p	T	Q	F
27	63.59674	0.69455	1.48148	486.18623	45615.15453
28*	63.59471*	0.69464*	1.42857*	486.24675*	45629.96427*
29	63.59293	0.69468	1.37931	486.274309	45621.16343

*Optimal solution.

Table 3. The impact of customer returns on the optimal solutions of Example 1.

α, β	p^*	t_p^*	T^*	Q^*	f^*
$\alpha = 0.5$, $\beta = 0.7$	55.92422	0.94555	1.81818	472.77838	9435.394724
$\alpha = 0$, $\beta = 0.7$	86.52641	0.94486	1.42857	472.43026	70677.88368
$\alpha = 0.5$, $\beta = 0$	204.82234	0.94195	2.22222	470.97935	176561.56649
$\alpha = 0$, $\beta = 0$	204.25977	0.94218	2.10526	471.09078	208358.46520

Table 4. Sensitivity analysis results.

	Parameter change (%)	−%50	−%20	−%10	+%10	+%20	+%50
R							
	N	36	31	29	27	26	21
	t_p	0.69727	0.69553	0.69508	0.69424	0.69386	0.69166
	p	63.58495	63.58997	63.59294	63.59674	63.59906	63.61749
	f^*	49721.39324	47156.64293	46379.01178	44906.73240	44206.51149	39233.09699
	f^* change (%)	+8.9	+3.35	+1.64	−1.58	−3.12	−14.1
K							
	N	30	29	28	28	27	27
	t_p	0.69468	0.69468	0.69464	0.69464	0.69455	0.69455
	p	63.59136	63.59293	63.59472	63.59472	63.59674	63.59674
	f^*	46776.29252	46080.23991	45852.01643	45407.91227	45186.02154	44542.32208
	f^* change (%)	+2.51	+0.98	+0.48	−0.48	−0.97	−2.38
σ							
	N	28	28	28	28	28	29
	t_p	0.70239	0.69771	0.69616	0.69311	0.69161	0.68718
	p	63.58329	63.59011	63.59240	63.59703	63.59937	63.60377
	f^*	45699.90390	45658.47074	45644.30614	45615.44460	45600.74681	45557.54999
	f^* change (%)	+0.15	0	0	0	0	−0.15
r							
	N	28	28	28	28	28	28
	t_p	0.68167	0.68937	0.69199	0.69731	0.70001	0.70827
	p	63.63239	63.61048	63.60267	63.58664	63.57852	63.55412
	f^*	76694.21429	55090.68515	49990.89089	41876.57263	38625.48769	31116.99342
	f^* change (%)	+68.08	+20.73	+9.56	−8.23	−15.35	−31.81
a							
	N	21	25	26	30	31	36
	t_p	0.69176	0.69357	0.69407	0.69503	0.69536	0.69574
	p	38.83705	51.18680	57.38908	69.80974	76.03027	94.72272
	f^*	9997.744804	25312.86010	34847.98728	57643.07881	70888.00974	117967.57681
	f^* change (%)	−78.09	−44.53	−23.63	+26.32	+55.35	+158.53

Table 4. (*Continued*).

Parameter change (%)	−%50	−%20	−%10	+%10	+%20	+%50
b						
N	30	29	28	28	28	27
t_p	0.69495	0.69477	0.69468	0.69459	0.69455	0.69435
p	75.01590	67.70194	65.58007	61.72902	59.97248	55.28591
f^*	56381.45661	49496.39081	47499.49394	43873.19570	42219.29192	37803.14853
f^* change (%)	+23.56	+8.47	+4.09	−3.85	−7.47	−17.15
λ						
N	29	28	28	28	28	27
t_p	0.69464	0.69462	0.69463	0.69464	0.69464	0.69457
p	63.30295	63.47445	63.53455	63.65493	63.71522	63.90955
f^*	44994.44404	45370.66792	45500.14077	45760.13904	45890.66633	46290.21543
f^* change (%)	−1.39	−0.57	−0.28	+0.29	+0.57	+1.47
c_1						
N	24	26	27	29	30	32
t_p	0.84845	0.76297	0.73389	0.65983	0.63876	0.48997
p	63.45631	63.53133	63.55784	63.62874	63.64872	63.80809
f^*	45711.89023	45690.27201	45670.21366	45579.27676	45541.98259	45433.42647
f^* change (%)	+0.17	+0.13	0	−0.11	−0.12	−0.43
c_2						
N	40	32	30	26	25	21
t_p	0.38744	0.57171	0.63317	0.75613	0.81767	1.00241
p	63.15891	63.42239	63.50871	63.68042	63.76372	64.01287
f^*	50099.98208	47581.02306	46627.30443	44593.23085	43527.01351	40155.76549
f^* change (%)	+9.79	+4.28	+2.19	−2.27	−4.61	−11.99
t_d						
N	30	29	29	28	28	27
t_p	0.64893	0.67343	0.68222	0.69977	0.70857	0.73491
p	63.63672	63.61401	63.60527	63.58962	63.58090	63.55669
f^*	46082.22962	45841.66587	45752.00185	45578.51265	45488.60382	45221.15257
f^* change (%)	+0.99	+0.46	+0.27	−0.13	−0.31	−0.89

Table 4. (*Continued*).

Parameter change (%)	−%50	−%20	−%10	+%10	+%20	+%50
S						
N	28	28	28	28	28	28
t_p	0.69464	0.69464	0.69463	0.69463	0.69463	0.69464
p	63.59082	63.59316	63.59393	63.59549	63.59627	63.59862
f^*	45616.39510	45624.53655	45627.25033	45632.67810	45635.39201	45643.53389
f^* change (%)	0	0	0	0	0	0

Second, we carry out the sensitivity analysis of various parameters based on Example 2. The change in the values of parameters can take place due to uncertainties in any decision-making situation. In order to examine the implications of these changes, the sensitivity analysis will be of great help in a decision-making process. The values of the system which are considered here are N, t_p, p and f. Results of the sensitivity analysis are shown in Table 4. The main conclusions which one can draw from the sensitivity analysis are as follows:

(1) There is an increase (decrease) in the f value when a is increased (decreased).
(2) There is an increase (decrease) in the f value when R, K, r, b or c_2 are decreased (increased).
(3) t_p is less sensitive but p and N are moderately sensitive.
(4) All other changes in parameters do not affect the f significantly.

7. Conclusion and outlook

In real life, most goods would have a span of maintaining quality or original condition, i.e. during that period, there is no deterioration occurring. For this type of items, the assumption that the deterioration starts from the instant of arrival in stock may cause retailers to adopt inappropriate replenishment policies due to overvaluing the total annual relevant inventory cost. Therefore, in the field of inventory management, it is necessary to consider the inventory problems for non-instantaneous deteriorating items. On the other hand, the coordination of price decisions and inventory control means optimizing the system rather than its individual elements. Thus, the optimal pricing combined with inventory ordering policy can yield considerable revenue increase. Also, if we ignored inflation and time value of money the optimal present value of total profit is overstated. The overstatement of profits will lead to the wrong management decision. As a result, it is important to consider the effects of inflation and the time value of money on inventory policy. Moreover, the empirical findings of Anderson et al. [61] provide evidence to support the view that customer returns increase with both the quantity sold and the price set for the product. Consequently, in order to address the realistic circumstances, the customer returns should be considered as a function of both the quantity sold and the price. Moreover, in the majority of papers that study pricing and inventory control for non-instantaneously deteriorating items, the demand functions are simple and dependent on price, stock or time, separately. But in the real world, the demand may increase when the price decreases, or it may vary through time. Therefore, in order to incorporate the realistic conditions, the price and the time should be considered simultaneously.

In this work, we addressed the problem of joint pricing and inventory control model for non-instantaneous deteriorating items taking into account the time value of money and customer returns. The demand is deterministic and depends on time and price simultaneously. Also, the customer returns assumed as a function of both the quantity sold and the price. An algorithm is presented for deriving the optimal replenishment and pricing policy that wants to maximize the present value of total profit. Finally, numerical examples are solved and the sensitivity of the solution to changes in the values of different parameters is discussed. The results show that the present value of total profit is sensitive to changes in c_2, r, a, b and R. Hence, it is important to consider the effects of inflation and the time value of money on inventory policy and financial performance.

To the best of our knowledge, this is the first work that focuses on the optimal pricing and inventory control policy for non-instantaneously deteriorating items with the finite replenishment rate considering time- and price-dependent demand, customer returns and time value of money.

Our results suggest the following managerial insights:

- Since the customer returns have an effect on the income statement, balance sheet and cash flow statement, it will ultimately have an effect on the ratios used by financial managers to measure and compare a company's profitability, liquidity, activity and solvency. Therefore, the results of analysing customer returns provide important findings to financial managers who use the price as a control to match the quantity sold to inventory while maximizing revenues. A cost of raising the price will be an increase in returns and this cost needs to be taken into account when optimizing prices. Therefore, a firm facing customer returns that depend on the quantity of product sold should increase price and decrease production quantity to mitigate the loss in profit resulting from the customer returns. On the other hand, when customer returns increase with price, the firm should reduce price and increase the production quantity leading to a fewer returns. If the quantity of returns depends on both price and quantity sold, the firm may set a higher or lower price depending on which returns form is dominant.
- The result is intuitive, easy to implement and provides managerial insights of the effect of the change in the values of parameters shown in the sensitivity analysis table. These changes can take place due to uncertainties in any decision-making situation. Thus, in order to examine the implications of these changes, the sensitivity analysis will be of great help in a decision-making process.
- It can be seen that the present value of total profit in the instantaneous deterioration items case decrease. This implies the insight that the present value of total profit could be increased by changing the instantaneously to non-instantaneously items using the improved stock condition.
- Our model provides a decision support system fostered by Operational Research that could be implemented in management sciences, business administration and economics.

This paper can be extended in several ways. For instance, the constant deterioration rate could be extended to a time-dependent function. Furthermore, the deterministic deterioration may be extended to the stochastic deterioration. Also, the deterministic demand function could be extended to the stochastic demand function. Finally, we plan to extend the model to incorporate some more realistic features such as quantity discounts, two warehouse, allowable shortage and permissible delay in payments.

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Appendix A

For a given value of N, the necessary conditions for finding the optimal values of p^* and t_p^* are given as follows:

$$\begin{aligned}
\frac{\partial}{\partial p} f(p,t_p,T) = &-\frac{1}{(-\lambda+r)(\sigma+r)(\sigma+\lambda)r\lambda N(-1+e^{-\frac{rH}{N}})}\\
&\times (Nbr\lambda c_1 e^{-rt_d} e^{(-\sigma-r)t_p-rt_d}(e^{-rH}-1)(-\lambda+r)e^{\frac{((t_p-t_d)\sigma+rt_p)N+(\sigma+\lambda)H}{N}}\\
&+ Nbrc_1 e^{-rt_p} e^{-\frac{rH}{N}}\lambda(e^{-rH}-1)(-\lambda+r)e^{\frac{(\sigma+\lambda+r)H-Nt_p(\sigma+r)}{N}}\\
&- Nbrc_1 e^{-rt_p} e^{-\frac{rH}{N}}\lambda(e^{-rH}-1)(\sigma+r)e^{\frac{-t_p(-\lambda+r)N+rH}{N}}\\
&- Nbr\lambda c_1 e^{-rt_d} e^{(-\sigma-r)t_p-rt_d}(e^{-rH}-1)(-\lambda+r)e^{\frac{(\sigma+\lambda)H+rt_dN}{N}}\\
&+ Nbr\lambda c_1 e^{-rt_d}(-e^{t_p(\sigma+r)+\lambda rt_d}+e^{t_p(\sigma+r)+rt_d})(e^{-rH}-1)\\
&\times(\sigma+r)e^{(-\sigma-r)t_p-rt_d}+(\sigma+\lambda)(-Nr\lambda(e^{-rH}-1)(\sigma+r)(-2bp+a)e^{\frac{-H(-\lambda+r)}{N}}\\
&+(-Nr((-c_1be^{-rt_p}\lambda+\alpha(\sigma+r)(Sbr-((-2p+S)b+a)\lambda))e^{-rH}\\
&+c_1be^{-rt_p}\lambda+(-bc_2r+((c_2-2p)b+a)\lambda)\alpha(\sigma+r)e^{\frac{\lambda H}{N}}\\
&+(-N((\alpha a-2\alpha bp+2\beta p)r-2\beta p\lambda)\lambda(e^{-rH}-1)e^{\frac{rH}{N}}\\
&+(-\lambda+r)((S(\beta H\lambda+\alpha Nb)r+2\lambda N\beta p)e^{-rH}-c_2(\beta H\lambda+\alpha Nb)r\\
&-2\lambda N\beta p))(\sigma+r))e^{-\frac{rH}{N}}-(-Nbr\alpha e^{-rH}(-\lambda+r)(S-c_2)e^{\frac{\lambda H}{N}}\\
&+(-Nbc_1(-r-e^{rt_d}\lambda+re^{\lambda t_d}+\lambda)e^{-rt_d}+r((S-c_2)(\beta H\lambda+\alpha Nb)r\\
&-(H\beta(S-c_2)\lambda+N(((S-c_2)\alpha-2p)b+a))\lambda))e^{-rH}\\
&+N(bc_1(-r-e^{rt_d}\lambda+re^{\lambda t_d}+\lambda)e^{-rt_d}+\lambda r(-2pb+a))(\sigma+r))=0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t_p} f(p,t_p,T) = &-\frac{1}{(-\lambda+r)(\sigma+r)(\sigma+\lambda)(-1+e^{-\frac{rH}{N}})}\\
&\times((-2e^{-\frac{rH}{N}}(-\lambda+r)c_1e^{-rt_p}\left(r+\frac{1}{2}\sigma\right)(a-bp)e^{\frac{(\sigma+\lambda+r)H-Nt_p(\sigma+r)}{N}}\\
&+2e^{-\frac{rH}{N}}c_1(\sigma+r)e^{-rt_p}(a-bp)\left(r-\frac{1}{2}\lambda\right)e^{\frac{t_p(-\lambda+r)N+rH}{N}}\\
&+c_1e^{-rt_d}e^{(-\sigma-r)t_p-rt_d}(-\lambda+r)(\sigma+r)(a-bp)e^{\frac{(\sigma+\lambda)H+rt_dN}{N}}+(-R(\sigma+\lambda)e^{(2\sigma+r)t_p-\sigma t_d}\\
&-(\sigma+r)(a-bp)e^{t_p(\sigma+\lambda)+rt_d}+Re^{rt_d+\sigma t_p}(\sigma+\lambda))(-\lambda+r)e^{-rt_d}c_1e^{(-\sigma-r)t_p-rt_d}\\
&+(-rc_1e^{-rt_p}e^{\frac{\lambda H}{N}}(a-bp)e^{-\frac{rH}{N}}+Rc_2(-\lambda+r)(\sigma+r))(\sigma+\lambda))(e^{-rH}-1))=0.
\end{aligned}$$