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ORIGINAL PAPER

Testing the robustness of deterministic models of optimal dynamic pricing and lot-sizing for deteriorating items under stochastic conditions

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Abstract Many models within the field of optimal dynamic pricing and lot-sizing models for deteriorating items assume everything is deterministic and develop a differential equation as the core of analysis. Two prominent examples are the papers by Rajan et al. (Manag Sci 38:240–262, 1992) and Abad (Manag Sci 42:1093–1104, 1996). To our knowledge, nobody has ever tested whether the optimal solutions obtained in those papers are valid if the real system is exposed to randomness: with regard to demand process as well as with regard to the deterioration process. The motivation is that although the real world is indeed stochastic, it is often more convenient to work with a deterministic decision model providing a nice closed form solution. The crucial thing is of course whether the results derived in the deterministic setting are robust when tested in a stochastic environment. Therefore, in this paper, we will try to expose the model by Abad (1996) and Rajan et al. (1992) to stochastic inputs; however, designing these stochastic inputs such that they as closely as possible are aligned with the assumptions of those papers. We do our investigation through a numerical test where we test the robustness of the numerical results reported in Rajan et al. (1992) and Abad (1996) in a simulation model. Our numerical results seem to confirm that the results stated in these papers are indeed robust when being imposed to stochastic inputs.

Keywords Inventory control · Optimal dynamic pricing · Deterioration · Simulation

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1 Introduction

There are many researches within the subject of optimal dynamic pricing and lot-sizing models. For a survey, see Elmaghraby and Keskinocak (2003), and for more recent contributions, see Zhao and Zheng (2000), Talluri and van Ryzin (2005), and Besbes and Zeevi (2009). A special field within this research topic is to consider optimal dynamic pricing and lot-sizing models for deteriorating items assume everything is deterministic and develop a differential equation as the core of analysis. For good survey papers, see Goyal and Giri (2001) and Bakker et al. (2012). For recent contributions within this field, see among others Wang et al. (2015), Li et al. (2015) and Zhang et al. (2015). To our knowledge, nobody has ever tested whether the results derived in such a deterministic setting are valid when tested in a stochastic environment. This is the aim for our paper. As indicated above, there are many papers even within this special field. Therefore, we have chosen to focus on a few contributions in our analysis. We have chosen the papers by Rajan et al. (1992) and Abad (1996). This is due to the fact that they are both published in a much-respected journal (Management Science) and are well cited, too.

The rest of the paper is organized as follows. In Sect. 2, we present how the results in Rajan et al. (1992) and Abad (1996) are derived in general. This is done in order to make the paper self-contained. In Sect. 3, we will more critically and in more detail examine the results of those papers, in particular, the paper by Abad (1996). There are three reasons for this. First, there are a lot of misprints in Abad (1996), which is of interest to report to the research community. Second, we think there is a point that is overlooked in Abad (1996), namely that the problem can be transformed into an optimization model with one decision variable enabling more straightforward solution procedures. This will be illustrated by the numerical problem that is used in both Rajan et al. (1992) and Abad (1996). Third, with regard to this numerical problem, there seems to be some numerical inaccuracies in the way Abad (1996) reports his results. In Sect. 4, we will describe how we translate the model of Abad (1996) [the model of Rajan et al. (1992) is contained as a special case of this model] into a simulation model. In Sect. 5, we report the findings we have obtained from our simulation experiments. Finally, Sect. 6 contains some concluding remarks.

2 The models by Rajan et al. (1992) and Abad (1996)

For a list of notation, used throughout the paper, see Table 1.

In order to make the paper self-contained, we will make a short review of the two models. The model by Rajan et al. (1992) considers a monopolistic retailer that at the beginning of each order cycle, when the inventory level goes to zero, orders an amount of Q units at a total cost of $K+c_0Q$, where K is a fixed order cost and c_0 is the unit purchase price. In this model, the delivery lead time is zero. Units on stock are charged a holding cost h per unit per time unit. Denote by t the time elapsed since the last order was placed; in the following just denoted as time t. Denote by $I(t)$ the amount of inventory at time t. There is an inventory deterioration at time t given by

$$W(t) = \sigma(t)I(t) \tag{1}$$

Table 1 Notations

Q	Order quantity
h	Holding cost per unit per unit time
c_0	Purchasing price per unit
σ	Deterioration rate
K	Constant purchasing cost per order
μ	Value drop parameter
θ	Length of time in which there is no inventory shortage (decision variable) in Abad (1996) and the duration of cycle (decision variable) in Rajan et al. (1992)
ψ	Length of time in which there is inventory shortage (decision variable in Abad 1996)
$\delta(t)$	Value drop function
$p(t)$	Price at time t
$TNOW$	Current simulation time
$c(t)$	Total unit cost function
$\lambda(t)$	Arrival rate function
$v(p,t)$	The rate contribution to profit
I_0^*	The optimal initial inventory level
S^*	The optimal maximum demand backlogged
RV	Revenue earned during cycle i
IL	Inventory level
CL	Cycle time for replication i
RN	Random number

At time t, the price $p(t)$ is charged, and at that time, the demand rate is $D(p(t),t)$, where it is assumed that $D(p,t)$ is non-negative, non-increasing and continuous in p and t and $D(0,0)>0$. So the two decision variables are the price function $p(t)$ and the order size Q. However, by letting θ denote the cycle length, we have that the order size Q can be expressed as

$$Q = \int_0^{\theta} \{D\left(p\left(t\right), t\right) + W\left(t\right)\} dt \tag{2}$$

Therefore, it is more convenient to let θ act as a decision variable, and when θ and the price function $p(t)$ have been determined, then the order size Q is derived from (2). The inventory at time t is given by [for details, see Rajan et al. (1992)]

$$I\left(t\right) = \int_t^{\theta} D\left(p\left(r\right), r\right) e^{\int_t^r \sigma(s) ds} dr \tag{3}$$

The objective of the retailer is to determine a cycle time and a price function such that his average profit per time unit $\Pi(\theta,p(t))$ is maximized, that is, to maximize

$$\Pi\left(\theta, p\left(t\right)\right) = \frac{1}{\theta}\left[\int_0^\theta \left\{p\left(t\right) D\left(p\left(t\right), t\right) - c_0\left(D\left(p\left(t\right), t\right) + W\left(t\right)\right) - hI\left(t\right)\right\} dt - K\right]. \tag{4}$$

By defining the *total unit cost* function

$$c\left(t\right) = c_0 e^{\int_0^t \sigma(s)ds} + h\int_0^t e^{\int_r^t \sigma(s)ds} dr \tag{5}$$

and the *rate contribution to profit* function

$$v\left(p, t\right) = \left(p - c\left(t\right)\right) D(p, t), \tag{6}$$

the optimization problem can be rewritten to

$$\Pi\left(\theta, p\left(t\right)\right) = \frac{1}{\theta}\left[\int_0^\theta v\left(p, t\right) dt - K\right]. \tag{7}$$

The optimal price function *p*(t)* can now be determined independent of the cycle time θ by

$$p^*\left(t\right) = argmax_p v\left(p, t\right) \tag{8}$$

Thereafter, the optimal cycle time $\theta*$ is determined as

$$\theta^* = argmax_\theta \frac{1}{\theta}\left[\int_0^\theta v\left(p^*\left(t\right), t\right) dt - K\right] \tag{9}$$

The model by Abad (1996) follows all the assumptions of Rajan et al. (1992), but allows for backlogging. It means that the order size Q is given as $I\ (0) + S$, where $I(0)$ is initial inventory at the start of the cycle, and S is the accumulated backlog at the end of a cycle. Now, the cycle time is $\theta + \psi$, where θ, similar to Rajan et al. (1992), is the part of the cycle time with a positive inventory, while ψ is the part of the cycle time with a negative inventory. Obviously, (1) now only holds for $0 \le t \le \theta$. Abad (1996) makes the assumption that during the stock-out period from time $t = \theta$ to time $t = \theta + \psi$, demand occurs at a rate $D\left(p\left(0\right), 0\right)$, that is, the customer response as if a new and fresh batch is offered to the market (though it is not yet arrived). During the stock-out period, at time t, a fraction $B\left(\theta + \psi - t\right)$ is backlogged, where $\tau = \theta + \psi - t$ is the waiting time until the new batch arrives. Since, naturally, customers do not like the wait, the function $B\left(\tau\right)$ is a decreasing function of τ. We have

$$S = D\left(p\left(0\right), 0\right)\int_0^\psi B\left(\tau\right) d\tau \tag{10}$$

It enables Abad (1996) to get to the following expression to maximize

$$\Pi\left(\theta,\psi,p\left(t\right)\right)=\frac{1}{\theta+\psi}\left[\int_{0}^{\theta}v\left(p,t\right)dt-K+v\left(p\left(0\right),0\right)\int_{0}^{\psi}B\left(\tau\right)d\tau\right] \quad (11)$$

Again, the optimization procedure can be decomposed. The optimal price function $p^{*}\left(t\right)$ is found as described in Rajan et al. (1992), that is, from (8). Thereafter, the optimal values for θ and ψ are found from

$$max_{\theta,\psi}\Pi\left(\theta,\psi,p\left(t\right)\right)=\frac{1}{\theta+\psi}\left[\int_{0}^{\theta}v\left(p^{*}\left(t\right),t\right)dt-K+v\left(p^{*}\left(0\right),0\right)\int_{0}^{\psi}B\left(\tau\right)d\tau\right] \quad (12)$$

3 A critical examination of Abad (1996)

With regard to a more detailed and numerical analysis, both Rajan et al. (1992) and Abad (1996) study the following case with constant deterioration, exponential value drop and exponential backordering. That is,

$$D\left(p\left(t\right),t\right)=\left[a-p\left(t\right)e^{\mu t}\right]/b \quad (13)$$

$$\sigma\left(t\right)=\sigma \quad (14)$$

$$B\left(\tau\right)=k_{0}e^{-k_{1}\tau} \quad (15)$$

It holds that

$$c\left(t\right)=\left(c_{0}+\frac{h}{\sigma}\right)e^{\sigma t}-\frac{h}{\sigma} \quad (16)$$

and the optimal price function is

$$p^{*}\left(t\right)=\frac{1}{2}\left[ae^{-\mu t}+\left(c_{0}+\frac{h}{\sigma}\right)e^{\sigma t}-\frac{h}{\sigma}\right] \quad (17)$$

3.1 Misprints in formulas in Abad (1996)

The optimal profit contribution, $v^{*}\left(t\right)=v\left(p^{*}\left(t\right),t\right)$, is not stated in Abad (1996) for this case, but from Rajan et al. (1992) it is

$$v^{*}\left(t\right)=\frac{e^{\mu t}}{4b}\left[ae^{-\mu t}-\left(c_{0}+\frac{h}{\sigma}\right)e^{\sigma t}+\frac{h}{\sigma}\right]^{2} \quad (18)$$

We then get $F(\theta) = \int_0^\theta v^*(t)\,dt$ to be

$$F(\theta) = \frac{1}{4b}\left[\frac{a^2}{\mu}\left(1 - e^{-\mu\theta}\right) + \frac{\left(c_0 + \frac{h}{\sigma}\right)^2}{2\sigma + \mu}\left(e^{(2\sigma+\mu)\theta} - 1\right) + \frac{h^2}{\sigma^2\mu}\left(e^{\mu\theta} - 1\right)\right.$$
$$\left. - \frac{2a\left(c_0 + \frac{h}{\sigma}\right)}{\sigma}\left(e^{\sigma\theta} - 1\right) + \frac{2ah}{\sigma}\theta - \frac{2h\left(c_0 + \frac{h}{\sigma}\right)}{\sigma(\sigma + \mu)}\left(e^{(\sigma+\mu)\theta} - 1\right)\right] \tag{19}$$

This reveals that Eq. (30) in Abad (1996) contains several misprints. The function *F(θ)* is not specified in Rajan et al. (1992).

From Eq. (4) in Abad (1996), we get that

$$I(t) = \frac{1}{b}\int_t^\theta \left[a - p^*(r)\,e^{\mu r}\right] e^{\sigma(r-t)} dr$$
$$= \frac{1}{2b}\left[\frac{a}{\sigma}\left(e^{\sigma(\theta-t)} - 1\right) - \frac{c_0 + \frac{h}{\sigma}}{\mu + 2\sigma}\left(e^{(\mu+2\sigma)\theta} - e^{(\mu+2\sigma)t}\right)e^{-\sigma t}\right.$$
$$\left. + \frac{h}{\sigma(\mu + \sigma)}\left(e^{(\mu+\sigma)\theta} - e^{(\mu+\sigma)t}\right)e^{-\sigma t}\right] \tag{20}$$

and

$$I(0) = \frac{1}{2b}\left[\frac{a}{\sigma}\left(e^{\sigma\theta} - 1\right) - \frac{c_0 + \frac{h}{\sigma}}{\mu + 2\sigma}\left(e^{(\mu+2\sigma)\theta} - 1\right) + \frac{h}{\sigma(\mu + \sigma)}\left(e^{(\mu+\sigma)\theta} - 1\right)\right] \tag{21}$$

When comparing (21) to the first half of Eq. (32) in Abad (1996), it reveals a misprint.

The maximum backlog at the end of a cycle is

$$S = D\left(p^*(0), 0\right) M_b(\psi) = \frac{1}{2b}(a - c_0)\frac{k_0}{k_1}\left(1 - e^{-k_1\psi}\right) \tag{22}$$

where $M_b(\psi^*(\theta))$ is a multiplier associated with the backlogged demand, $M_b(\psi^*(\theta)) = \int_0^{\psi^*(\theta)} B(\tau)\,d\tau$.

When comparing to Eq. (32) in Abad (1996), there also appears a misprint.

3.2 Simplifying the computational procedure

Nor Rajan et al. (1992) or Abad (1996) address any algorithmic approaches for computing optimal solutions. It may be excused for the first paper, as there is only one variable to optimize, but for the second paper, having an optimization model with two decision variables, it could have been sensible to add some comments about this. We think there is an interesting point which is not addressed explicitly in Abad (1996).

When examining the optimality conditions, we get that, for an optimal solution (if both decision variables are positive), it must hold that

$$v^*(\theta) = v^*(0)\, B(\psi) \tag{23}$$

In the all-exponential case, this transforms to

$$\psi = \frac{1}{k_1} ln \left(\frac{k_0 v^*(0)}{v^*(\theta)} \right) \tag{24}$$

Clearly, in case ψ becomes negative above, we must replace it by *0*. So, we derive that the optimal trajectory is

$$\psi^*(\theta) = \begin{cases} 0 & k_0 v^*(0) \le v^*(\theta) \\ \frac{1}{k_1} ln \left(\frac{k_0 v^*(0)}{v^*(\theta)} \right) & k_0 v^*(0) > v^*(\theta) \end{cases} \tag{25}$$

So, the threshold value of θ, for when backlogging becomes optimal, is independent of the parameter k_1. We can insert this into the objective function (12), so we only need to find the optimal parameter of θ, when solving

$$max \left\{ \frac{F(\theta) - K + v^*(0)\, M_b(\psi^*(\theta))}{\theta + \psi^*(\theta)} : \theta > 0 \right\} \tag{26}$$

3.3 Solving the numerical problem common for both papers

We find the optimal θ and corresponding optimal profit by doing a search using Excel, where the optimal θ is restricted to two decimals. Next, we find the optimal order quantity for this optimal θ using Maple 2015 as well as, Excel in order to verify our results. Here, we report the results of optimal order quantity obtained using Maple 2015.

The numerical problem is presented in Rajan et al. (1992). According to their specifications

$$K = 50,\ c_0 = 1.45,\ h = 0.000822,\ \sigma = 0.03,\ \mu = 0.12 \text{ and } a = 2.55. \tag{27}$$

The parameter b is not directly stated but can be deduced from the specification of the demand function

$$D(p, t) = 576.78 - 226.9 e^{0.12t} p$$

We get that the value of b can be either b =0.004421 (= 2.55/576.78) or b = 0.004407 (= 1/226.9).

When solving the optimization problem of Rajan et al. (1992) where backlogging is not allowed

$$max \left\{ \frac{F(\theta) - K}{\theta} : \theta > 0 \right\} \tag{28}$$

with the parameters as settled in (27) and $b = 0.004421$, we get (when doing a search for the optimal θ restricted to two decimals)

$\theta* = 2.09$, $Q* = 205.66$, and the optimal profit is 14.00.

When redoing the optimization now with b=0.004407, we get

$\theta* = 2.08$, $Q*$=205.65, and the optimal profit is 14.12.

This is almost the same as reported in Rajan et al. (1992).

In Abad (1996), the input specification is

$$K = 50, c_0 = 1.45, h = 0.000822, \sigma = 0.03, \mu = 0.1232, a = 2.542, b = 0.004407. \tag{29}$$

Note that Abad (1996) has added more decimals to the value of μ, his value of a is slightly different to that in Rajan et al. (1992) and a specific value of b is stated (the last of the two deduced above). Despite these inaccuracies, Abad (1996) reports the same numerical values for the demand function and the optimal price function. It is somewhat peculiar that Abad (1996) claims $v^* (2.089) = 12.91$ without commenting on the fact that this value is significantly different to the value of 14.00 reported in Rajan et al. (1992). As we can confirm the results of Rajan et al. (1992), it might reveal some numerical inaccuracies in the numerical study of Abad (1996).

In addition, Abad (1996) also needs to specify the backlog function which has parameters

$$k_0 = 0.8 \quad \text{and} \quad k_1 = 0.05. \tag{30}$$

When solving the optimization problem (26) with the parameters given in (29) and (30), we get

$\theta* = 0.85$, $\psi* = 5.83$, $I_0^* = 96.95$, $S^* = 501.24$, $Q^* = I_0^* + S^* = 598.19$, and optimal profit is 40.29.

The reason why $\psi*$ is so relatively high compared to $\theta*$ is due to the very low value of k_1.

3.4 A new numerical example with zero drop-down value

Here, we will consider the Rajan et al. (1992) example with zero drop-down value. By substituting $\mu = 0$ into Eq. (18), the optimal profit contribution can be obtained as follows

$$v^* (t) = \frac{1}{4b}\left[a - \left(c_0 + \frac{h}{\sigma}\right) e^{\sigma t} + \frac{h}{\sigma}\right]^2 \tag{31}$$

We then get $F(\theta) = \int_0^\theta v^* (t)\, dt$ to be

$$F(\theta) = \frac{1}{4b}\left[a^2\theta + \frac{\left(c_0 + \frac{h}{\sigma}\right)^2}{2\sigma}\left(e^{2\sigma\theta} - 1\right) + \frac{h^2\theta}{\sigma^2}\right.$$

$$- \frac{2a\left(c_0 + \frac{h}{\sigma}\right)}{\sigma}\left(e^{\sigma\theta} - 1\right) + \frac{2ah\theta}{\sigma} - \frac{2h\left(c_0 + \frac{h}{\sigma}\right)}{\sigma^2}\left(e^{\sigma\theta} - 1\right)\Bigg] \quad (32)$$

Now, we consider a new numerical example where the drop down value is zero, and $K = 20$ and $h = 0.5$. This example is based on the following data

$$K = 20, c_0 = 1.45, h = 0.5, \sigma = 0.03, a = 2.542, \; b = 0.004407. \quad (33)$$

When solving the optimization problem (28) with the parameters given in (33), we get

$\theta^* = 0.92$, $Q^* = 88.74$, and the optimal profit is 19.472.

Later, we will compare these results with the results obtained where the demand and deterioration are exposed to randomness.

4 Development of simulation model for testing the validity of the models by Abad (1996) and Rajan et al. (1992)

We will try to expose the models by Abad (1996) and Rajan et al. (1992) to stochastic inputs; however, designing these stochastic inputs such that they as closely as possible, are aligned with the assumptions of those papers.

We will assume there is a gross arrival rate which is Poisson process with intensity $\Lambda = a/b$, which, by Abad (1996), is coined "the market potential". As in Abad (1996), we assume a value drop rate function $\delta(t)$ which is increasing in t with $\delta(0) = 1$. We assume that at time t, an arbitrary customer has a reservation price that is perceived by the retailer to be a random variable $R(t)$ that is uniformly distributed on the interval from 0 to $a/\delta(t)$, and the customer will only demand a unit if his reservation price (which is his evaluation of the product) is larger than or equal to the announced price $p(t)$. Therefore, the net arrival process is a time dependent Poisson process with arrival rate (as function of t):

$$\lambda(t) = \Lambda P(R(t) \geq p(t)) = \frac{a}{b}\frac{a - \delta(t)\, p(t)}{a} = \frac{a - \delta(t)\, p(t)}{b} \quad \text{for } 0 \leq t < \theta \quad (34)$$

This corresponds to the demand function on p. 1097 in Abad (1996). This also corresponds to how such demand processes are modelled by others, see for instance Talluri and van Ryzin (2005) and Zhao and Zheng (2000). As mentioned previously, this demand rate is only relevant if there are items on stock. When there are no items on stock, the demand rate is

$$\lambda(t) = \frac{a - p(0)}{b} \quad \text{for } \theta \leq t < \theta + \psi \quad (35)$$

When there are $i(i > 0)$ items on stock, the time interval until the next item will deteriorate is a random variable which is exponentially distributed with mean $1/(\sigma i)$. However, if, in the meantime, a demand has occurred, due to the memoryless characteristic of an exponential distribution, the deterioration process will be regenerated. This corresponds to the assumption of constant deterioration rate of σ.

With regard to simulating the model of Rajan et al. (1992), there are two possible implementations. The first option is to simulate the inventory system with a fixed cycle time (in the numerical example, a cycle time of $\theta*$=2.08 time units), and then, at the end of cycle, bring the inventory level up to level $Q*$ (in the numerical example $Q* = 206$). It has the consequence that the inventory may have gone to zero before the end of the cycle, or that there is still stock on hand at the end of the cycle. In the former case, it means that the retailer could, in reality, have reacted earlier and ordered $Q*$ units as soon as the inventory dropped to zero. In the latter case, it means that a new cycle is started with fresh units and with some that are ordered a cycle (or more) previously. This would probably not make sense in reality either. Therefore, we think that it makes more sense to apply the second option, namely to simulate the inventory system with a stochastic cycle time. This means that Q is the primary decision variable and that a cycle lasts until all $Q*$ units has either been demanded or deteriorated. It also means that the system regenerates itself (that is the start condition is the same, see Tijms (2003; Chapter 2).

With regard to simulating the model by Abad (1996), we need to simulate this model with a fixed cycle time (for a numerical example a cycle time of $\theta* + \psi* = 6.68$ time units). This is due to the way Abad (1996) has constructed his backlog assumption, which requires that the customers arriving in the stock-out period know the exact time of arrival of the next batch. As the numerical example of Abad (1996) is designed in such a way that the stock-out period is relatively long, we will actually not encounter the problem that there might be old items on stock, as the new batch arrives.

5 Simulation set-up

The primary output of the simulation models is to get an estimate of the *profit rate incurred in the long-run.*

With regard to the simulation model for verification of the results by Abad (1996), we use *the method of replication*, see Law and Kelton (1991; Sect. 9.5.1). As the system regenerates itself after the fixed cycle time $\theta* + \psi*$ has elapsed, our run-length (or replication length) only needs to be $\theta* + \psi*$ time units. Then, the simulation is replicated N_A (subscript A stands for Abad—a suitable value for N_A will later be specified) times in order to get an estimate as well as 95% confidence interval.

With regard to the simulation for verification of the results by Rajan et al. (1992), due to the fact that the cycle time is random, we need to have our simulations to run for a longer time in order to collect data on *the profit rate incurred in the long-run*. If we just let the simulation length be the random cycle time, then we would in fact, collect data on *the profit rate incurred per cycle* which is not the same (for an elaboration of why and how much they differ, see later). We could use the straight forward *method of replication* again. For instance, simulate the system over a length of T_R time units where T_R is sufficiently long. Then, after the simulation has run, collect the data on the total profit incurred over these T_R time units. Then, by replicating this experiment N_R times, we can make an estimate of the long-run profit rate as well as a 95% confidence interval. However, as we need to make a simulation over *several cycles* and the system regenerates itself when a new cycle starts we can also use *the regenerative method*, see Law and Kelton (1991; pp 557–559), for our data analysis. This means that we simulate the system for C_R cycles and in each cycle i=1,..., C_R we collect information about the cycle length CL_i and total profit incurred during a cycle PC_i. Then we can estimate the profit rate incurred in the long-run as

$$PR_{Est} = \frac{\frac{1}{C_R}\sum_{i=1}^{C_R} PC_i}{\frac{1}{C_R}\sum_{i=1}^{C_R} CL_i} \tag{36}$$

By constructing a new dataset, $D_i = PC_i - PR_{Est} \cdot CL_i$, $i = 1,\ldots,C_R$ and computing its standard deviation $s(D)$, we can then get a 95% confidence interval using Eq. (9.10) in Law and Kelton (1991). In addition, we could also get information about the *short-term* performance measure of *the profit rate incurred per cycle* by doing statistics on the dataset PC_i/CL_i.

The simulation models have been programmed in Arena, see Kelton et al. (2015). See Figs. 1 and 2, for flowcharts that describes the structure of the simulation programs. As you can see in these figures, we consider two different parts for developing our simulation model: the customer arrival process part and deterioration process part. In order to do the simulation for the customer arrival process part, we use a create module to generate entities according to exponential distribution. Moreover, in order to simulate deterioration process part, we first create one and only one entity. Thereafter, we consider a loop which for any entity entering the deterioration process module, checks the inventory level and if it is positive assigns an exponential delay time for deterioration. During this delay of time, we also need to check if the inventory level has already been changed due to the customer arrival (we check this by defining *MyInv* in our flowchart). If the inventory level has already been changed and it is still positive, *MyInv* will be updated using the new value of the inventory level and as a result, the deterioration process regenerates using the new mean value of exponential deterioration (shown as $\frac{1}{\sigma \cdot MyInv}$ in our flowchart).

For Rajan et al. (1992)'s model, as we have a multiple-cycle inventory system, we also need to check that any entity only affects the cycle it belongs to. Therefore, for any entity belonging to a specific cycle, we assign an attribute of cycle number (*MyCycle* in our flowchart). We use this attribute in order to check that any entity leaving the deterioration process module is in the right cycle.

5.1 The simulation model by Abad (1996)

For simulating this model, we consider, as explained before, a fixed cycle time of CL, which is the cycle length derived by Abad (1996) that is 6.68. Moreover, we consider the same inventory level at the beginning of the cycle as the one obtained by Abad (1996) that is 97 (after rounding). Therefore, we have a fixed value of initial inventory

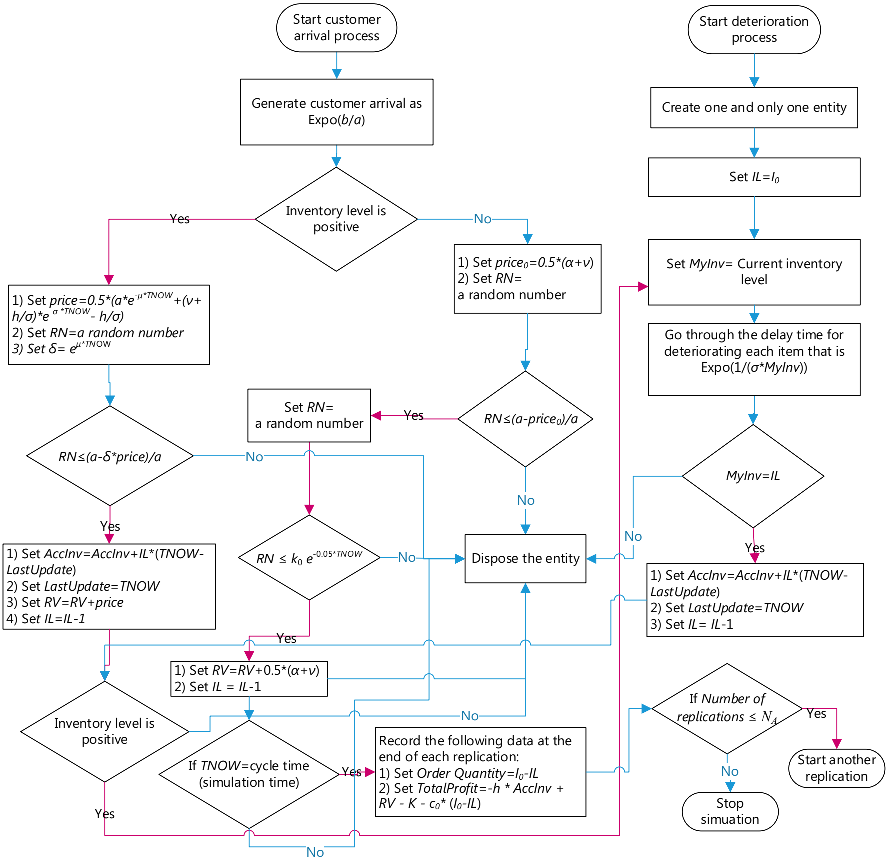

Fig. 1 Flowchart for simulating the flow of an entity through the system of the model by Abad (1996)

level (I_0) and a random value of maximum demand backlogged (S) which leads to have a random order quantity of $Q = I_0 + S$.

In following, we explain the way we collect our data in flowchart given in Fig. 1. In order to find the profit during the cycle, we need to record the changes in revenue and costs over a cycle. In order to find the total revenue during a cycle, we need to keep track of customer arrival process. Therefore, when a customer arrives we need to check if his reservation price is greater than the announced price. If so, we update the revenue (RV in our flowchart). Moreover, according to Fig. 1, we find the order quantity as $Q = I_0 - IL(t = 6.68)$ where $IL(t = 6.68) = -S$. Finally, we can find the inventory holding cost by finding the accumulated inventory level over a cycle. In order to find the accumulated inventory level over a cycle, we can multiply different values of the inventory level by the time interval until one item has either been demanded or deteriorated (we show these values by IL*(*LastUpdate-TNOW*) in our flowchart). Thereafter, by accumulating all values of IL*(*LastUpdate-TNOW*) during a cycle, we can find the accumulated inventory level and as a result, the inventory holding cost over a cycle.

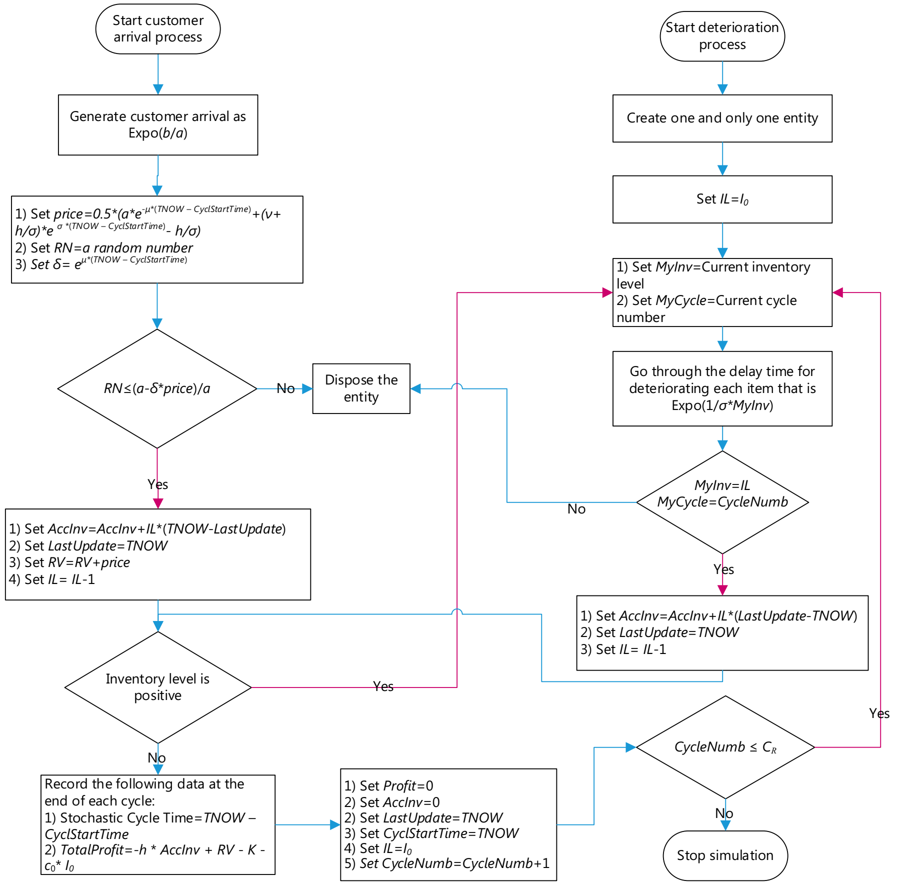

Fig. 2 Flowchart for simulating the flow of an entity through the system of the model by Rajan et al. (1992)

Here, we use *the method of replication* with the duration of CL for each replication. Therefore, for each replication, we record the *Total Profit During a Cycle* (*TPR*) as follows:

$$TPR = RV - (K + c_0 \cdot Q) - h \int_0^{CL} Max\,(I\,(t)\,,0)\,dt \tag{37}$$

The RV and Q are the total revenue and the order size, respectively, that are collected during the replication i of the simulation model, see Fig. 1. Denote by CL and PR the cycle length and the profit rate, respectively. Note that for simulating Abad (1996)'s model, we consider the cycle length (CL) to be constant.

Finally, we can find the average value of the above statistics over all replications as follows:

$$Expected\ TPR = \text{Average } TPR \text{ over all replications} \tag{38}$$

$$Expected\ Q = \text{Average } Q \text{ over all replications} \tag{39}$$

$$Expected\ PR = \frac{Expected\ TPR}{CL} \tag{40}$$

5.2 The simulation model by Rajan et al. (1992)

As explained previously, for the simulation model by Rajan et al. (1992), we consider a fixed order quantity of Q, which is the order quantity derived by Rajan et al. (1992) that is 206 (after rounding). Moreover, we use the same initial inventory level as the one obtained by Rajan et al. (1992) that is 206 (after rounding).

In Fig. 2, we present our flowchart in order to simulate a flow of an entity through the system of the model by Rajan et al. (1992). In this flowchart, we find the revenue and the holding cost in the same way as it is described for Abad (1996)'s simulation model. But in order to find the order quantity during a cycle, we collect the number of times one item has either been demanded or deteriorated.

Here, we use the *regenerative method* to simulate our model where the system regenerates itself as soon as the inventory level reaches zero. Therefore, for each cycle, we record the *Total Profit During a Cycle* (*TPR*) as follows:

$$TPR = RV - (K + c_0 \cdot Q) - h \int_0^{CL} I(t)\, dt \tag{41}$$

The *RV* and *CL* are the total revenue and the cycle time, respectively, that are collected during the cycle i of the simulation model, see Fig. 2. Moreover, recall that CL is a random variable here which varies from one cycle to another.

Finally, we can find the average value of the above statistics over all cycles as follows:

$$Expected\ CL = \text{Average } CL \text{ over all cycles} \tag{42}$$

$$Expected\ TPR = \text{Average } TPR \text{ over all cycles} \tag{43}$$

$$Expected\ PR = \frac{Expected\ TPR}{Expected\ CL} \tag{44}$$

6 Results from the simulation analysis

We subdivide this section into two parts. First, we report our findings when using the parameter values from Sect. 3. In the second part, we use the simulation model developed for the verification of Abad (1996) to explore what is the optimal cycle time in case the starting inventory and price function is unchanged. Similarly, we use the simulation model developed for the verification of Rajan et al. (1992) to explore what is the optimal batch size in case price function is unchanged. Note that the number of cycles are chosen in a way, so that the half width of the 95% confidence interval is reasonably small.

Table 2 Results obtained for the simulation model by Abad (1996)

Expected order quantity	600.74
Expected total profit per cycle	273.3783
Expected profit rate	40.9249
Lower confidence interval	40.8849
Upper confidence interval	40.9649
Difference between the results obtained using the simulation model and the results obtained using deterministic model	
Difference in profit rate	1.55%
Difference in order quantity	0.61%

For Abad (1996), we report all results using the *method of replications*. We run our simulation model for 9000 replications each with a duration of 6.68 time units. Moreover, for Rajan et al. (1992), we use the *regenerative method* and run our multiple-cycle simulation models for 5000 cycles and one replication. For the first part where we report our findings when using the given parameter values, we report our results using both *long-run* and *short-term* measure, but for the second part of our analysis where we do optimization, we apply the *long-run* measure. In addition, for Rajan et al. (1992) with zero drop-down value, we also we use the *regenerative method* and run our multiple-cycle simulation models for 5000 cycles and one replication.

For the first part, we present the simulation outputs for the parameter values from Sect. 3 as well as, the results of comparing these simulation results with the results obtained for deterministic model.

In following, we summarize the results obtained using a deterministic model in Sects. 3.3 and 3.4. Recall that as there were some inaccuracies in model by Abad (1996), we redid the optimization and got the profit per time unit and order quantity to be 40.29 and 598.19, respectively. In addition, after recalculating the results obtained by Rajan et al. (1992), we obtained almost the same optimal profit per time unit and cycle time as it is reported in the paper that are 14.12 and 2.08, respectively. Moreover, recall that for the model by Rajan et al. (1992) with zero drop-down value, we obtained the optimal profit per time unit and the cycle time as 19.472 and 0.92, respectively.

The results of simulation as well as, comparison of these results with the results obtained using a deterministic model are summarized in Tables 2, 3 and 4 for Abad (1996), Rajan et al. (1992), and Rajan et al. (1992) with zero drop-down value, respectively. Moreover, the results of Rajan et al. (1992) simulation model with the *short-term* performance measure is given in Table 5.

From Tables 2 and 3, we can see that the optimal profit per time unit obtained using a deterministic model does not fall within the confidence interval of 95% obtained using a simulation model. However, the difference between the results obtained using a simulation model and the optimal profit per time unit obtained using a deterministic model seems to be negligible. From Table 4, we can see that the optimal profit per time unit obtained using a deterministic model fall within the confidence interval of 95% obtained using a simulation model. However, the confidence interval of 95% for this case which is the case of Rajan et al. (1992) with zero drop-down value is rather wide.

Table 3 Results obtained for the simulation model by Rajan et al. (1992)

Expected cycle time	2.095
Expected total profit per cycle	29.318
Expected profit rate	13.9939
Standard deviation of observations	6.8606
Lower confidence interval	13.9031
Upper confidence interval	14.084
Difference between the results obtained using the simulation model and the results obtained using deterministic model	
Difference in profit rate	0.9011%
Difference in cycle length	0.716%

Table 4 Results obtained for the simulation model by Rajan et al. (1992) with zero drop-down value

Expected cycle time	0.9528
Expected total profit per cycle	17.8647
Expected profit rate	18.75
Standard deviation of observations	31.4461
Lower confidence interval	17.8351
Upper confidence interval	19.6648
Difference between the optimal results obtained using the simulation model and the results obtained using deterministic model	
Difference in profit rate	3.8507%
Difference in cycle length	3.4425%

Table 5 Results obtained for the simulation model by Rajan et al. (1992) using the *short-term* performance measure

Expected profit rate	14.2452
Standard deviation of observations	3.2966
Lower confidence interval	14.1538
Upper confidence interval	14.3366

From Table 5, we can see that the expected profit rate obtained using the *short-term* measure is greater than the expected profit rate obtained using the *long-run* measure, i.e.,

$E\left(\frac{TPR}{CL}\right) > \frac{E(TPR)}{E(CL)}$. Bellow we argue why this holds.

$$E\left(\frac{TPR}{CL}\right) = E\left(TPR \cdot \frac{1}{CL}\right) = E\left(TPR\right) \cdot E\left(\frac{1}{CL}\right) + Cov\left(TPR, \frac{1}{CL}\right) \quad (45)$$

We expect that TPR and CL to be negatively correlated. This is due to the fact that when the cycle time becomes shorter (due to the randomness in demand and deterioration) the more items would be sold at higher prices, the inventory holding

Table 6 Optimal solution obtained for the simulation model by Abad (1996)

Cycle time	Expected profit per time unit
6.67	40.9242
6.68	40.9249
6.69	40.9272
6.70	40.9278
6.71	40.9286
6.72	40.9278
6.73	40.9292
6.74	40.9294
6.75	40.9294
6.76	40.9289
6.77	40.9292
6.78	40.9249
6.79	**40.9322**
6.80	40.9319
6.81	40.9312
6.82	40.9305
6.83	40.9307
6.84	40.9308
6.85	40.9312
6.86	40.9305
6.87	40.9307
6.88	40.9304
6.89	40.9309
6.90	40.9313
6.91	40.9295
6.92	40.9291
6.93	40.9270
6.94	40.9272
6.95	40.9256
6.96	40.9257
6.97	40.9239
Difference between the optimal results obtained using the simulation model and the results obtained using deterministic model	
Difference in profit rate	1.56%
Difference in cycle length	1.62%

Optimal solution is shown in bold

cost would become less, and less items would be deteriorated. Therefore, the total profit earned during the cycle (*TPR*) in this case, will be higher compared to the case where the cycle time is longer. Thus, we would expect to have a negative covariance of $Cov\,(TPR, CL) < 0$ and consequently, a positive covariance of $Cov\left(TPR, \frac{1}{CL}\right) > 0$.

Moreover, as the function of $f(x) = \frac{1}{x}$ is a convex function on $x > 0$, from the Jensen's inequality, we get that $E\left(\frac{1}{CL}\right) \geq \frac{1}{E(CL)}$. Therefore, by inserting the value of $\frac{1}{E(CL)}$ instead of $E\left(\frac{1}{CL}\right)$ into Eq. (45) we get the following inequality

$$E\left(\frac{TPR}{CL}\right) \geq E\,(TPR) \cdot \frac{1}{E\,(CL)} + Cov\left(TPR, \frac{1}{CL}\right). \tag{46}$$

Moreover, as we argued that $Cov\left(TPR, \frac{1}{CL}\right) > 0$, we can get the following from inequality (46)

$$E\left(\frac{TPR}{CL}\right) > E\,(TPR) \cdot \frac{1}{E\,(CL)} \tag{47}$$

Therefore, we justified that the expected profit rate obtained using the *short-term* measure is greater than the one obtained using the *long-run* measure. This can also be seen by examining the simulation outputs.

Next, we do the analysis related to the second part where we do some numerical experiments in order to find the optimal solutions of the common example by Abad (1996) and Rajan et al. (1992) using the simulation models. Here, we consider that the optimal price function is given.

For the model by Abad (1996), as we need to know the duration of cycle for estimating the fraction of shortage backordered, we find the profit per time unit for a range of given cycle times. In this way, by comparing the profit per time unit over different values of cycle times, we find the optimal cycle time. From Table 6, we can see that the most of the solutions are equally good.

The possible concavity of the profit function for the simulation model by Abad (1996) is illustrated numerically in Fig. 3. This figure is drawn using the linear interpolation from points that are shown in Table 6, as well as some other optional points that give an illustration of profit per time unit over a wide range of cycle times.

For the simulation model by Rajan et al. (1992), we find the optimal order quantity by finding the maximum profit per time unit obtained over different values of order quantities. From Table 7, we can again see that the most of the solutions are equally good, as well.

The possible concavity of the profit function for the simulation model by Rajan et al. (1992) is illustrated numerically (see Fig. 4). This figure is drawn using the linear interpolation from the points that are shown in Table 7 and some other optional points that give an illustration of profit per time unit over a wide range of order quantities.

For the simulation model by Rajan et al. (1992) with zero drop-down value, we find the optimal order quantity by finding the maximum profit per time unit obtained over different values of order quantities. From Table 8, we can again see that the most of the solutions are equally good, as well.

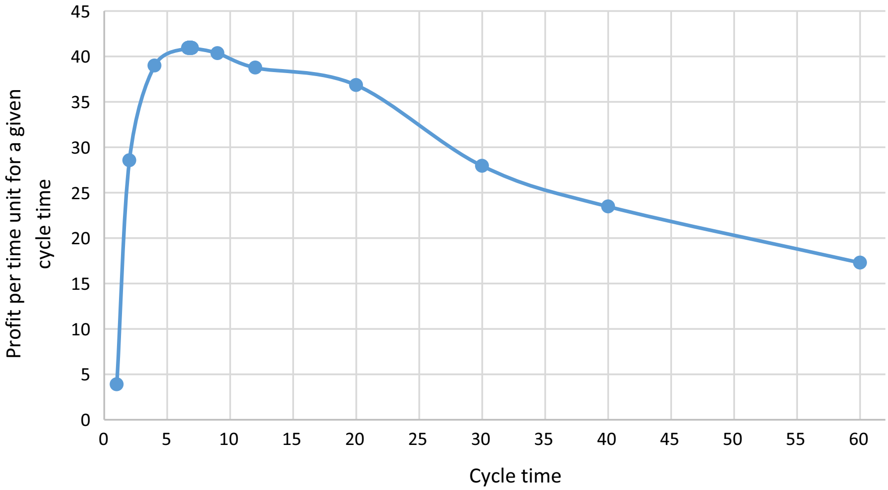

Fig. 3 The graphical illustration of the possible concavity of the profit function for the simulation model by Abad (1996)

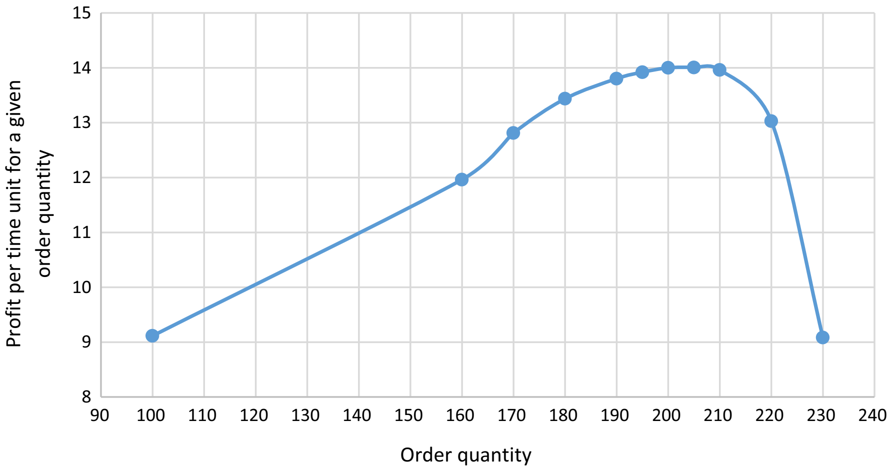

Fig. 4 The graphical illustration of the possible concavity of the profit function for the simulation model by Rajan et al. (1992)

7 Conclusions

In this paper, we investigate the effect of randomness with regard to deterioration and demand on two well-known deterministic models within the subject of optimal dynamic pricing and lot-sizing models for deteriorating items, namely the model by Abad (1996) and Rajan et al. (1992). The aim is to investigate whether the optimal results found in these deterministic models are indeed robust under stochastic conditions. In this regard, we develop simulation models in order to test the numerical

Table 7 Optimal solution obtained for the simulation model by Rajan et al. (1992)

Order quantity	Expected profit per time unit
220	13.0293
210	13.9609
209	13.9360
208	13.6738
207	13.8468
206	13.9939
205	14.0061
204	**14.0082**
203	13.9762
202	13.9799
201	13.9894
200	14.0005
199	13.9964
198	13.9666
197	13.9522
196	13.8928
195	13.9215
194	13.9169
193	13.8548
192	13.8629
191	13.8704
190	13.8031
180	13.4368
Difference between the optimal results obtained using the simulation model and the results obtained using deterministic model	
Difference in profit rate	0.7981%
Difference in order quantity	0.9804%

Optimal solution is shown in bold

results obtained by these authors. We do our experiment in two levels. In the first level, for the model Abad (1996), we examine whether we obtain the same profit per time unit and order quantity if we insert the optimal price function and cycle time obtained by Abad (1996) into our simulation model. In addition, for the model by Rajan et al. (1992), we examine whether we get the same profit per time unit and cycle time if we insert the optimal price function and order quantity obtained by Rajan et al. (1992) into our simulation model. In the second level of the experiment, for the model by Abad (1996), we find the optimal cycle time and profit per time unit numerically using simulation model. Moreover, for Rajan et al. (1992)'s model, we find the optimal order quantity and profit per time unit numerically through simulation model. We also test the robustness of the optimal solutions obtained for model by Rajan

Table 8 Optimal solution obtained for the simulation model by Rajan et al. (1992) with zero drop-down value

Order quantity	Expected profit per time unit
70	17.4483
80	19.0380
81	19.1267
82	19.1004
83	19.1985
84	19.2130
85	**19.2399**
86	19.1774
87	18.5280
88	19.7539
89	18.7499
90	18.2262
100	9.9282
Difference between the optimal results obtained using the simulation model and the results obtained using deterministic model	
Difference in profit rate	1.2063%
Difference in order quantity	4.7059%

Optimal solution is shown in bold

et al. (1992) with zero-drop down value under stochastic conditions. Here, we do the same experiments as those for model by Rajan et al. (1992) with non-zero-drop-down value.

From our first set of examinations, we observe that the optimal profit per time unit given in deterministic models do not exactly fall within the 95% confidence interval of profit per time unit obtained using a simulation model. However, the difference between the results obtained through simulation model and the optimal solutions derived using a deterministic model seems to be negligible. Therefore, it seems that the results are still robust under stochastic conditions. A well-known case, where this robustness of results is in fact true, is with regard to the Economic Order Quantity (*EOQ*).

From our second set of examinations, we can see that there still seems to be a negligible difference between the optimal solutions obtained using a simulation model and the optimal solutions obtained using a deterministic model. As a result, it seems that the optimal solutions obtained using a deterministic model still hold under stochastic conditions.

For further studies, it could be interesting to apply the proposed investigations on other deterministic models existing in literature. In addition, more research could be done in order to study the robustness of deterministic models under a condition where the inventory system faces more stochastic inputs.

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