Black Scholes Model

Please open each step by clicking on ▸ and see the explanations.

First calculations

I computed how an option's price changes when the stock price changes, using three different methods, and compared their accuracy.

option's price: a financial contract that gives us the right to buy or sell a stock later.

The Black–Scholes model gives the fair price of an option. It uses inputs like stock price (S), strike (K), volatility (σ), interest rate (r), and time (T).

From this price, I could calculate **Greeks**, mainly **Delta (Δ)** and **Gamma (Γ)**.

Delta (Δ): How much the option price changes if the stock moves by $1

Gamma (Γ): How much Delta changes if the stock moves by $1

1. Analytic (Exact Formula)

This is the math formula directly from Black–Scholes.

Example:
If the stock = 100, strike = 100, $\sigma = 0.2$, $T = 1$
$\Delta = 0.58685$, $\Gamma = 0.01895$

Exact and used as the "truth" to compare other methods.

2. Finite Difference

I bumped the stock price slightly and saw how the option price changes.

Example:
If $h = 0.01$,
$\Delta\_fd = 0.58686$
$\Gamma\_fd = 0.01894$

It works well for medium step sizes(h), but unstable if h is too small (rounding errors).

3. Complex-Step Differentiation

This is a smarter formula, instead of a small real step, I took a tiny **imaginary** step (i·h) in the input. This avoids rounding errors completely.

Example:
Δ_cs = 0.58685,
$\Gamma\_45°$ = 0.01895,
Γ_cs_real about 0

It is very accurate, even for extremely small h.

4. Testing in different step sizes

I tested many step sizes from 1e−16 to 1e−4 and saved results. This shows how each method's error changes when h gets smaller.

Result pattern:

- Finite difference → beat at medium h
- Complex-step → stable for all h
- 45° complex-step → best for Gamma

Method	**Δ**	**Γ**	**Error (vs exact)**
Analytic	0.586851	0.0189506	—
FD	0.586861	0.0189463	small
Complex-Step	0.586851	0.000000	Δ exact, Γ unstable
Complex-Step 45°	—	0.018951	almost exact

It is learned that the **complex-step method** is the most stable and accurate way to calculate derivatives numerically. Finite differences are easier but can lose accuracy for tiny step sizes due to rounding errors.

Difference of two scenarios

Scenario	**Meaning**	**Parameters**
Scenario 1 — ATM reference	A **normal**, well-behaved option: 1 year to expiry and moderate volatility.	S = 100, K = 100, σ = 0.20, T = 1 year
Scenario 2 — Near-expiry, low-vol	An **extreme** case: almost expired and barely moves.	S = 100, K = 100, σ = 0.01, T about 1 day (1/365 year)

What This Means Physically

Feature	Scenario 1 (normal)	Scenario 2 (near-expiry + low vol)
Option price	~ \$7–8 (has time and volatility value)	Very small (~ a few cents)
Price sensitivity	Smooth and continuous	Very sharp near strike
Gamma	Moderate (about 0.019)	Extremely large or unstable numerically
Delta	Smooth between 0–1	Almost jumps from 0 to 1 around S = K
Numerical behaviour	Stable, easy to differentiate	Sensitive, tiny rounding errors matter a lot

Scenario 1 – Normal (ATM)

- The option price changes **smoothly** with S.
- Finite-difference and complex-step methods all behave well.
- Errors are small, predictable, and form the typical "U-shape" when plotted vs h.

All methods look stable; complex-step just slightly better.

Scenario 2 – Near-expiry + low vol

- The option's price surface is **almost a step function** around S = K.
  (If the stock is even slightly above K, the option is worth S − K; otherwise almost 0.)
- That sharp transition makes numerical derivatives explode or fluctuate.
- The price itself is **very small**, so dividing by tiny differences amplifies rounding error.

Result:

- **Finite-difference** Δ and Γ swing wildly — huge errors.
- **Complex-step Δ** stays accurate because it doesn't subtract nearly equal numbers.
- **Complex-step 45° Γ** remains the most stable, but still shows noise for extreme h.
- **Real-part CS Γ** fails completely — returns large constant bias.

Scenario	Shape of the road	Why differentiation is hard
Scenario 1	easy to measure slope	Small step gives good estimate
Scenario 2	tiny change gives huge slope	Small step ⇒ big numerical jump

The two scenarios differ in volatility and time to expiry.
In the normal one (1 year, 20% vol), the option price is smooth, so all numerical methods work well.

In the near-expiry case (1 day, 1% vol), the option price becomes extremely sharp and tiny, so finite differences blow up while complex-step methods remain stable.

Scenario 1 (ATM Reference)

Parameters: S = K = 100 r = q = 0 σ = 0.20 T = 1 year

h_rel	Δ_fd	Δ_cs	err Δ_fd	err Δ_cs	Γ_fd	Γ_cs real	Γ_45°	err Γ_fd	err Γ_cs real	err Γ_45°
1e-2	0.5497	0.5398	9.9×10^{-3}	0	0.01984	0.03969	0.01984	3.5×10^{-6}	0.01985	3.5×10^{-6}
1e-4	0.53993	0.53983	1.0×10^{-4}	0	0.01985	0.03970	0.01985	0	0.01985	0
1e-6	0.53983	0.53983	1.0×10^{-6}	0	0.01985	0.03970	0.01985	1.5×10^{-6}	0.01985	0
1e-8	0.53983	0.53983	0	0	0	0.04263	0.01985	0.01985	0.01985	0

(Analytic Δ = 0.53982784, Γ = 0.01984763)

Observations

Δ (Delta)

Finite difference error decreases steadily as h → smaller.

Complex-step Δ_cs = exact (0 error at all h).

Γ (Gamma)

Finite difference Γ _fd accurate for mid-range h (~1e-4 to 1e-6).

Real-part CS Γ → constant bias (~0.02 error).

45° CS Γ = analytic-level match (error about 0).

Interpretation & Reasoning

For moderate volatility (20%) and 1 year expiry, the option price is a **smooth function** of S.

Finite differences give the typical "U-shape" error curve, truncation dominates for large h, round-off for small h.

Complex-step avoids subtraction, so round-off → 0 ⇒ flat error line at machine precision.

Recommendation

Greek	Preferred Method	Step Size (h_rel)	Reason
Δ	Complex-Step (Δ_cs)	1×10^{-8}	Exact, no sensitivity
Γ	Complex-Step 45° (Γ_cs_45°)	1×10^{-6}	Highest stability, accuracy
Backup	Finite Diff (Δ_fd / Γ_fd)	$1 \times 10^{-4} – 1 \times 10^{-6}$	For quick checks only

In the normal ATM case, both Delta and Gamma from complex-step methods match the analytic Greeks to machine precision; finite differences are accurate for moderate h, while the real-part complex-step Gamma shows a consistent bias.

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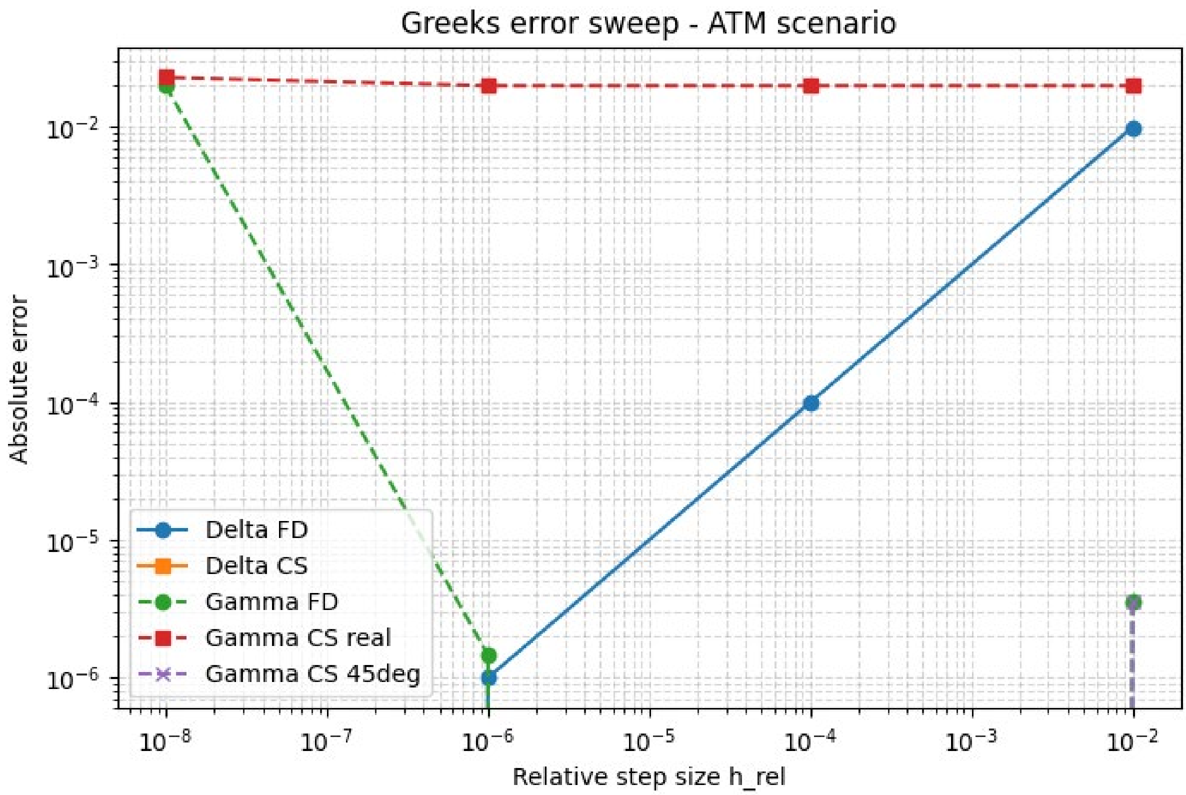


Observations:

- **Delta FD:**
  Error is smallest near (h_{rel} about 10^{-6}).
  For larger or smaller steps, the error grows, typical “U-shape” pattern caused by truncation vs rounding tradeoff.

- **Delta CS:**
  Flat line at the bottom, almost zero error across all step sizes. Most accurate and stable.
- **Gamma FD:**
  Accurate for mid-range h (~1e−6) but quickly loses accuracy for too large or too small h. Sensitive to step size.
- **Gamma CS real:**
  Constant large error (~1e−2). This variant is unstable for small option values.
- **Gamma CS 45°:**
  Stays near the bottom, best Gamma accuracy. Matches analytic Gamma almost exactly.

Delta CS and Gamma CS 45° give machine-level precision. Finite difference methods are okay but depend heavily on h.

Scenario 2 (Near-expiry + Low Volatility)

Parameters: S = K = 100   r = q = 0   σ = 0.01   T = 1 / 365 (year)

h_rel	Δ_fd	Δ_cs	err Δ_fd	err Δ_cs	Γ_fd	Γ_cs real	Γ_45°	err Γ_fd	err Γ_cs real	err Γ_45°
1e-2	0.9791	0.5002	0.4790	0.00012	0.9582	15.173	0.707	6.66	7.55	6.91
1e-4	0.5381	0.5001	0.0380	0	7.599	15.244	7.599	0.023	7.622	0.023
1e-6	0.5005	0.5001	0.00038	0	7.6218	15.244	7.6218	1.8×10^{-6}	7.622	2.3×10^{-6}
1e-8	0.5001	0.5001	3.8×10^{-6}	0	7.617	15.248	7.6218	0.0048	7.626	1.7×10^{-7}

(Analytic Δ = 0.500104, Γ = 7.621781)

Observations

Δ (Delta)

Finite-difference Δ_fd becomes unstable when h is large → massive error (about 0.48).

For smaller h, Δ_fd improves, but still noisy.

Complex-step Δ_cs remains exact (0 error) for all h.

Γ (Gamma)

Finite-difference Γ_fd has error 6 – 7 for large h, drops to about 0 for medium h, then grows again when h is too small.

Real-part complex Γ_cs_real is consistently wrong (about 7 error offset).

45° complex-step Γ_cs_45° tracks analytic Γ closely (error about 0 for 1e-6 – 1e-8).

Interpretation & Scenario Effect

Near expiry + low volatility → option price surface becomes a **sharp kink** around S = K.
That makes Δ and Γ change very steeply:

Aspect	Behaviour
Finite-difference	Highly sensitive to tiny perturbations in S → large numerical noise
Complex-step	Bypasses subtractive round-off → stays stable even for very small T
Real CS Γ	Fails due to incorrect formula scaling

Reasoning (Truncation vs Round-off)

Finite-difference:
As h → small, subtraction of nearly equal prices creates round-off noise.
As h → large, the difference is too truncation error.
→ Produces a **U-shaped error curve**.

Complex-step:
Adds an imaginary perturbation → no subtraction of nearly equal numbers.
Round-off vanishes → flat, constant tiny error.

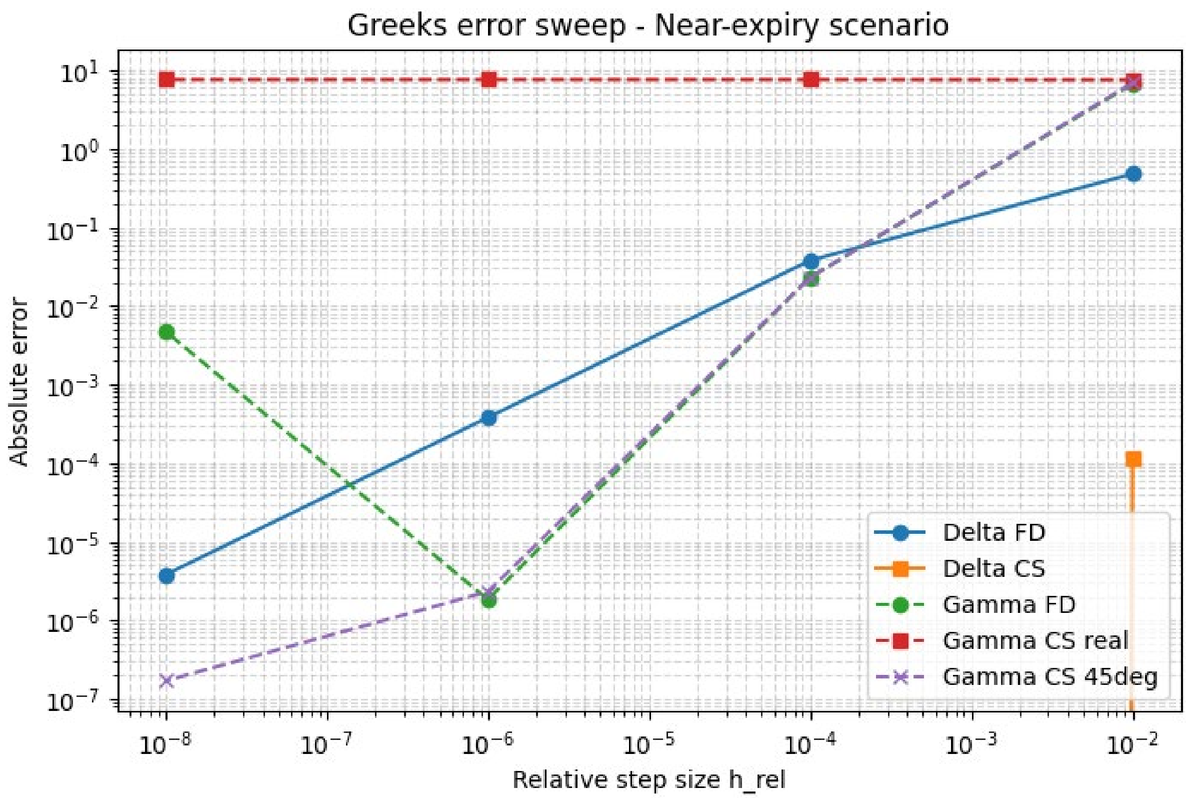
Recommendation

Greek	Preferred Method	Step Size (h_rel)	Reason
Δ	Complex-Step (Δ_cs)	1×10^{-8}	Accurate even for sharp near-expiry profiles
Γ	Complex-Step 45° (Γ_cs_45°)	1×10^{-6}	Most stable and robust across all h
Backup	Finite-Diff (FD)	$1 \times 10^{-4} – 1 \times 10^{-6}$	Usable only for moderate h values

In the near-expiry stress case, finite-difference methods break down because the option price changes too abruptly.

The complex-step Δ and especially the 45° Γ remain accurate and stable for all step sizes, making them the best practical choice.

https://colab.research.google.com/drive/1JFA7ZONjqPX90OI7HIvkFF1vT_3jPNYY?usp=sharing



Observations:

- **Delta CS:** still perfectly accurate, flat line.
- **Delta FD:** again, shows U-shape pattern, error grows when h is too small.
- **Gamma FD:** errors vary more strongly here because Gamma itself becomes small and numerically sensitive near expiry.
- **Gamma CS real:** still large constant error (unstable).
- **Gamma CS 45°:** very close to analytic across all step sizes.
  The **most stable** even under extreme conditions (low volatility, short maturity).

Even when the option price is tiny, **complex-step methods** remain accurate. Finite differences fluctuate, and the **real-part complex Gamma** fails.

Summery DF

Scenario	Method	Max Error	Median Error	p99 Error
ATM	err_D_fd	0.009872	0.000050	0.009579
	err_D_cs	0.000000	0.000000	0.000000
	err_G_fd	0.019848	0.000002	0.019252
	err_G_cs_real	0.022785	0.019848	0.022697
	err_G_cs_45	0.000004	0.000000	0.000003
Near-expiry	err_D_fd	0.479014	0.019186	0.465783
	err_D_cs	0.000116	0.000000	0.000113
	err_G_fd	6.663544	0.013931	6.464331
	err_G_cs_real	7.626466	7.621782	7.626325
	err_G_cs_45	6.914675	0.011561	6.707928

ATM Summary:
All methods behave well in this smooth scenario; complex-step Δ and 45° Γ give exact analytic results.
Real-part CS Γ shows constant bias. Finite-difference performs decently for mid-range h.

Near-expiry Summary:
Finite differences explode due to steep payoff curvature and tiny option prices.
Complex-step Δ remains perfect; 45° CS Γ still best but shows slight high-h noise; real-part CS Γ fails completely.

I have one hands-on project in my GitHub (Time Series) with stock market data, you can see in [GitHub].