Interdisciplinary Project: Portfolio Optimization with Market Data

SID: 510657138

Group 4

FMAT3888: Projects in Financial Mathematics
S2 2024

THE UNIVERSITY OF SYDNEY

Contents

1 Executive Summary

This report examines portfolio optimization strategies using real-world financial data, focusing on both static and dynamic portfolio management approaches to maximize returns and manage risk. We aim to demonstrate the practical applications of optimization models, including utility maximization and variance minimization, within a two-year investment period, reflecting the considerations of modern portfolio managers.

Key concepts from Modern Portfolio Theory and Capital Asset Pricing Model (CAPM) were applied to our data to construct and optimize portfolios for two distinct time periods. We incorporated both utility and variance optimization approaches. The utility maximization approach targets the highest possible returns given an investor's risk preferences, while the variance minimization model prioritizes stability and reduced volatility.

In the static optimization approach, fixed asset allocations were defined at the beginning of the investment period, unable to adapt to changing market conditions. Dynamic optimization, on the other hand, allowed for asset rebalancing between years, improving the portfolio's adaptability and responsiveness to new information. Results showed that while the static approach delivered reasonable returns in stable markets, the dynamic approach offered slightly lower volatility and greater flexibility in reacting to economic shifts.

The study also addresses practical challenges, such as the occurrence of non-positive semidefinite (non-PSD) covariance matrices, which can destabilize optimization results. We provide a method for approximating non-PSD matrices to restore stability in portfolio calculations. Furthermore, we analyzed the Sharpe ratio, a standard measure for risk-adjusted return, to evaluate the efficiency of each portfolio.

Overall, the results indicate that dynamic portfolio optimization, with its ability to adjust allocations based on new market information, yields portfolios that can better adapt to shifting economic conditions while maintaining a stable risk profile. The unconstrained optimization approach, particularly in Sharpe ratio maximization, achieves higher risk-adjusted returns but at the cost of reduced diversification. In contrast, constrained optimizations, which spread allocations across various asset classes, provide a balanced approach, achieving stable returns while adhering to practical risk management principles. This balance between maximizing returns and managing risk underscores the importance of customizing portfolio strategies based on market conditions and investor preferences.

In summary, the findings highlight that while unconstrained portfolios offer higher theoretical returns, diversified, dynamically rebalanced portfolios present a more robust approach to real-world market fluctuations. These results affirm the value of incorporating flexibility and risk management into portfolio strategies, especially in volatile or unpredictable markets.

2 Introduction

Portfolio optimization has become a cornerstone in modern investment strategies due to its ability to enhance returns while managing risk. This involves selecting a combination of assets that maximizes an investor's objectives, such as returns, under specific constraints. The major breakthrough came in 1952, with Markowitz's paper, Portfolio Selection [9]. This theory, now know as Modern Portfolio Theory introduced the idea that an investor can build a portfolio that optimizes expected returns for a given level of risk through diversification [8].

Used in combination with Utility Theory, allowed investors with their own individual preferences — risk aversion, wealth objectives, and investment horizons — impact portfolio construction. The Capital Asset Pricing Model (CAPM) emerged in the 1960s, further advancing the understanding of portfolio risk by introducing the concept of systematic risk, or market risk [13]. CAPM established a relationship between an asset's expected return and its beta, showing that investors are compensated for taking on market risk but not for unsystematic risk, which can be eliminated through diversification.

These theories have transformed the financial decision making process from a classical financial analysis problem into a mathematical optimization problem. As computational tools have advanced, so have the capabilities for portfolio optimization. Today, investors use models that integrate sophisticated risk measures and simulations to optimize portfolios across multiple periods, adjusting to new information and changing market conditions dynamically.

This report is a case-study based on real-world market data for monthly returns for the period of January 2012 to January 2024. We aim to demonstrate the above concepts focusing on static and dynamic portfolio optimization. The analysis involves parameter estimation, optimization of portfolio weights, and comparison of results across different time periods.

3 Parameter Estimation

3.1 Monthly Log-Returns: Statistics

To begin the case study, we first estimate some statistics of our dataset. The parameters of interest are the mean monthly log-returns which we denote as $\boldsymbol{a}$ and the corresponding covariance matrix B. These parameters are calculated for the following assets/asset classes:

1. Australian Listed Equites (AEQ)
2. International Listed Equities (ILE) (Hedged)
3. International Listed Equities (ILE) (Unhedged)
4. Australian Listed Property (ALP)
5. International Listed Property (ILP)
6. International Listed Infrastructure (ILI)
7. Australian Fixed Income (AFI)
8. International Fixed Income (IFI)
9. Cash

Let S_t^i be the price of asset i at month t. The data contains the monthly returns

$$R_t^i \coloneqq \frac{S_t^i}{S_{t-1}^i} - 1$$

Assuming each asset follows a log-normal returns distribution we have the log-return for asset i at month t

$$X_t^i \coloneqq \log\left(\frac{S_t^i}{S_{t-1}^i}\right)$$

We then have the relation

$$X_t^i \coloneqq \log\left(1 + R_t^i\right)$$

With log-returns $\boldsymbol{X}_t = \left(X_t^1, X_t^2, \ldots, X_t^9\right)^\top$ for each month being i.i.d. multivariate normally distributed with mean $\boldsymbol{a} = (a_1, a_2, \ldots, a_9)^\top$ and covariance matrix $B = (b_{ij})$. It is then trivial to perform the computations to find our parameters.

Applying this to our data we obtain $\boldsymbol{a}$ for the time intervals, January 2012 to December 2015 inclusive (Period A) and from January 2016 to December 2019 inclusive (Period B)

$$\mathbf{a}_A = \begin{bmatrix} 0.0091 \\ 0.0118 \\ 0.0152 \\ 0.0151 \\ 0.0115 \\ 0.0105 \\ 0.0044 \\ 0.0051 \\ 0.0024 \end{bmatrix}, \quad \mathbf{a}_B = \begin{bmatrix} 0.0085 \\ 0.0092 \\ 0.0099 \\ 0.0083 \\ 0.0064 \\ 0.0102 \\ 0.0037 \\ 0.0036 \\ 0.0015 \end{bmatrix}$$

And the computed covariance matrix's B_A and B_B can be found in Appendix A.

3.2 Distribution of n-month Returns

Next we want to derive the distribution of n-month return for asset i (from month t to $t+n$). Consider the one-month log return

$$X_t^i \coloneqq \log\left(\frac{S_t^i}{S_{t-1}^i}\right)$$

Then the n-month log-return is

$$Y_t^i = \sum_{k=1}^{n-1} X_{t+k}^i = \log\left(\frac{S_{t+n-1}^i}{S_{t-1}^i}\right)$$

Since the log-returns $\boldsymbol{X}_t$ are multivariate normally distributed with mean $\boldsymbol{a}$ and covariance B, the sum of n log-returns are also normally distributed. Specifically, the sum of X_t's results in $\boldsymbol{Y}_t = \left(Y_t^1, Y_t^2, \ldots, Y_t^9\right)^\top$ with mean $n\boldsymbol{a}$ and covariance nB. Then we can see that the n-month returns given by

$$R_{t,n}^i := \frac{S_{t+n-1}^i}{S_{t-1}^i} - 1$$

can be written as

$$R_{t,n}^i \sim e^{Y_t^i} - 1$$

with $Y_t^i \sim N(na_i, nb_{ii})$, completing the derivation.

3.3 Annual Returns: Statistics

We consider the random vector $\boldsymbol{R}^{(1)} := \left(R_1^{(1)}, \ldots, R_9^{(1)}\right)^\top$ (respectively, $\boldsymbol{R}^{(2)} := \left(R_1^{(2)}, \ldots, R_9^{(2)}\right)^\top$) which models the joint annual (respectively, two-year) returns for the nine assets. Considering the n-month log-return $Y_t^i \sim \mathcal{N}(na_i, nb_{ii})$, the expected value of the n-month return $R_{t,n}^i$ is derived using the properties of the log-normal distribution:

$$\begin{aligned}\mathbb{E}[R_{t,n}^i] &= \mathbb{E}\left[e^{Y_t^i} - 1\right] \\ &= e^{na_i + \frac{1}{2}nb_{ii}} - 1 \\ &= \mu_i^{(k)}\end{aligned}$$

Thus, the expected return over one year (12 months) is

$$\mu_i^{(k)} = e^{12a_i + 6b_{ii}} - 1$$

where k denotes the time horizon (e.g., $k = 1$ for one year).

For the covariance between the n-month returns of assets i and j, we utilize the joint log-normal distribution property:

$$\begin{aligned}\mathrm{Cov}(R_{t,n}^i, R_{t,n}^j) &= \mathbb{E}\left[R_{t,n}^i R_{t,n}^j\right] - \mathbb{E}[R_{t,n}^i]\mathbb{E}[R_{t,n}^j] \\ &= e^{na_i + na_j + \frac{1}{2}n(b_{ii}+b_{jj})}\left(e^{nb_{ij}} - 1\right) \\ &= c_{ij}^{(k)}\end{aligned}$$

where the covariance matrix for the 12-month period is given by

$$c_{ij}^{(k)} = e^{12(a_i+a_j)+6(b_{ii}+b_{jj})}\left(e^{12b_{ij}} - 1\right)$$

for all $i, j = 1, \ldots, 9$.

The correlation coefficient between the n-month returns of assets i and j is obtained by normalizing the covariance:

$$\rho_{ij}^{(k)} = \frac{c_{ij}^{(k)}}{\sqrt{c_{ii}^{(k)} \cdot c_{jj}^{(k)}}}$$

This results in a correlation matrix $\rho^{(k)}$ where each element represents the linear relationship between the returns of assets i and j over the specified time horizon. The statistics for the two year horizon are computed simply with $n = 24$. The computed statistics can be found in Appendix B.

4 Static Portfolio Optimization

For the static portfolio optimization problem we have the solution

Theorem 1. *Under Markowitz's Portfolio Theory, we assume that all rational risk-averse investors have a utility function $U(x)$ and they pick portfolios to maximize $\mathbb{E}[U(\boldsymbol{x}^\top\boldsymbol{R})]$ (where $\boldsymbol{x}^\top\boldsymbol{R}$ is the portfolio return). Then*

$$\max_{\boldsymbol{x}} \mathbb{E}[U(\boldsymbol{x}^\top\boldsymbol{R})] = \min_{\boldsymbol{x}} Z(\boldsymbol{x}) = -t\boldsymbol{x}^\top\boldsymbol{r} + \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{C}\boldsymbol{x}.$$

A rough sketch of the proof is provided.

Proof. Investors aim to maximize the expected utility of their portfolio returns, specifically $\mathbb{E}[U(\boldsymbol{x}^\top\boldsymbol{R})]$, where $\boldsymbol{x}^\top\boldsymbol{R}$ denotes the portfolio return. Assume that the portfolio return follows a normal distribution:

$$\boldsymbol{x}^\top\boldsymbol{R} \sim \mathcal{N}(\mu_x, \sigma_x^2).$$

Given the utility function satisfies $U'(V) > 0$ and $U''(V) < 0$, it follows that:

$$\frac{\partial\mathbb{E}[U(\boldsymbol{x}^\top\boldsymbol{R})]}{\partial\mu_x} > 0, \quad \frac{\partial\mathbb{E}[U(\boldsymbol{x}^\top\boldsymbol{R})]}{\partial\sigma_x} < 0.$$

This indicates that investors prefer higher expected returns (μ_x) and lower risk (σ_x).

Assume the utility function exhibits constant absolute risk aversion, defined as:

$$U(V) = -e^{-cV},$$

where $c > 0$ is the coefficient of absolute risk aversion. The investor's objective is to maximize:

$$\mathbb{E}[U(\boldsymbol{x}^\top\boldsymbol{R})] = \mathbb{E}\left[-e^{-c\boldsymbol{x}^\top\boldsymbol{R}}\right] = -\mathbb{E}\left[e^{-c\boldsymbol{x}^\top\boldsymbol{R}}\right].$$

Let $\hat{c} := cW_0$, where W_0 is the initial wealth. Since $\boldsymbol{x}^\top \boldsymbol{R} \sim \mathcal{N}(\mu_x, \sigma_x^2)$, the moment-generating function (MGF) of a normal distribution gives

$$\mathbb{E}[U(\boldsymbol{x}^\top \boldsymbol{R})] = -e^{-\hat{c}}\mathbb{E}\left[e^{-\hat{c}\boldsymbol{x}^\top \boldsymbol{R}}\right] = -e^{-\hat{c}-\hat{c}^2\left(\frac{1}{\hat{c}}\mu_x - \frac{1}{2}t^2\sigma_x^2\right)},$$

Since $-e^{-\hat{c}-\hat{c}^2 x}$ is increasing in x, we can maximize $\mathbb{E}[U(\boldsymbol{x}^\top \boldsymbol{R})]$ by maximizing

$$V(\mu_x, \sigma_x) = t\mu_x - \frac{1}{2}\sigma_x^2.$$

where $t = \frac{1}{\hat{c}} = \frac{1}{cW_0}$. And so for a given $t \geq 0$ the equivalent minimization problem is

$$\max_{\boldsymbol{x}} \mathbb{E}[U(\boldsymbol{x}^\top \boldsymbol{R})] = \min_{\boldsymbol{x}} Z(\boldsymbol{x}) = -t\boldsymbol{x}^\top \boldsymbol{r} + \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{C}\boldsymbol{x}.$$

□

4.1 Application to Australian and International Assets: Static Allocation

With the utility maximization properly formulated we can now apply it to our data. We have the following problem

$$\begin{aligned}
&\max_{\boldsymbol{w}} \mathbb{E}[U(\boldsymbol{w}^\top \boldsymbol{R}^{(2)})], \\
&\text{s.t.} \sum_{i=1}^{9} w_i = 1, \\
&\quad w_1 \geq 0.15, \\
&\quad w_2 \geq 0.15, \\
&\quad w_7 \geq 0.05, \\
&\quad w_9 \geq 0.05, \\
&\quad 0.65 \geq \sum_{i=1}^{6} w_i \leq 0.75, \\
&\quad 0 \leq w_i \leq 1, \quad \forall i
\end{aligned}$$

Where w is our vector of weights and $U(x) = -e^{-\gamma x}$ with risk-aversion parameter $\gamma = 1$. The problem consists of maximizing our expected Utility for our two-year return. This is subject to the constraints:

- The sum of our weights equal one,
- A minimum of 15% allocation to Australian Listed Equites,
- A minimum of 15% allocation to International Listed Equities (Hedged),
- A minimum of 5% allocation to Australian Fixed Income,
- A minimum of 5% allocation to Cash,
- A 65% to 75% allocation to Growth Assets,
- All weights must be between 0% to 100%.

Importantly, the combination of the first and last constraints mean that we do not use leverage and do not short-sell. While there exists an analytical closed-form solution for the static portfolio optimization problem [10], this involves a complex calculation involving Karush–Kuhn–Tucker conditions and as a result we have opted for the numerical optimization method. Using scipy.optimize.minimize we obtain

Asset	**Period A**	**Period B**
AEQ	0.15	0.15
ILE (Hedged)	0.15	0.15
ILE (Unhedged)	0.29	0.00
ALP	0.16	0.00
ILP	0.00	0.00
ILI	0.00	0.45
AFI	0.05	0.20
IFI	0.15	0.00
Cash	0.05	0.05
$Z(\boldsymbol{w})$	-0.3170	-0.2182
Expected Return (μ)	0.3244	0.2231
Volatility (σ)	0.1221	0.0993

Table 1: Weights and Statistics for Periods (A) and (B)

In Table 1 we observe significant changes between the two periods. Given our objective function defined as

$$\min_{\boldsymbol{w}} Z(\boldsymbol{w}) = -t\boldsymbol{w}^\top \boldsymbol{r} + \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{C}\boldsymbol{w}$$

We have that Period (A) has a lower $Z(\boldsymbol{w})$ than Period (B). This means that Period A's portfolio optimization results in a better trade-off between maximizing return and minimizing risk.

In both periods we have that both Australian Equities and International Hedged Equities each remained at 15% of the portfolio allocation. This was the minimum value defined in the constraints, suggesting that these assets may not have provided the highest risk-adjusted returns or were not as attractive compared to other options. Similarly, Cash remained at the minimum allocation percentage of 5% for both periods. While their expected returns and volatility's did not justify a larger allocation, relative to other assets, the constraints were set to adequately diversify our portfolio.

A large difference in the weightings come from International Equities (Unhedged) and Australian Properties which went from 45% and 16% in (A) respectively to both 0% in (B). In Period (A), the market showed strong signs of recovery from the financial crisis, attracting investment into international markets [5]. Furthermore, during (A), Australian Property assets performed strongly, benefiting from low interest rates [12], which fueled demand for real estate investments. However, in Period (B), global markets experienced volatility, particularly with Brexit and the U.S.-China trade war leading to high volatility in Unhedged International Equities. In the Australian Property

Market, concerns over overvaluation, particularly in Sydney and Melbourne may have reduced the attractiveness of property investments [1]. This is reflected in Appendix B where the two-year returns for ILE (Unhedged) and ALP dropped significantly, whilst their volatility remained high compared to other assets.

Conversely, this resulted in a large increase of International Infrastructure from 0% to 45%. Infrastructure investments provide steady cash flows and tend to be less sensitive to economic cycles, making them attractive during periods of market uncertainty [4]. Furthermore, global interest rates remained low, further amplifying ILI's attractiveness. This is reflected in Appendix B where ILI had the highest expected return with lowest volatility out of all growth assets. In terms of defensive assets, as volatility remains relatively much lower, it usually remains to pick the one with the highest expected return. This resulted in the optimizer picking International Fixed Income for (A) and Australian Fixed Income for (B), as each had the highest expected return for their respective periods.

Overall, the shift in asset allocations from Period A to Period B reflects changes in market conditions, investor risk preferences, and the assets' expected returns and volatility's. In Period B, the portfolio becomes more conservative, with a greater emphasis on stable, income-generating assets like infrastructure and fixed income, and a reduced focus on riskier assets like unhedged international equities.

4.2 Comparison with Realised Portfolio Returns (2020-2021)

Using the portfolio weights from Period (B) for the period January 2020 to December 2021, we observe a disparity between the expected and realized returns:

Expected Return (from model):	0.2231
Realized Return (2020-2021):	0.1525

The realized return of 15.25% falls short of the expected return of 22.31% derived from Period (B) model estimates. This difference can be attributed to the unique market conditions during 2020-2021, heavily influenced by the COVID-19 pandemic [2].

The 2020-2021 period saw unprecedented volatility due to the pandemic. Initial market declines in early 2020 were followed by rapid recovery; however, sectors such as international equities and infrastructure faced high variability in returns due to global economic disruptions. This market environment deviated from the more stable growth observed in 2016-2019, impacting the portfolio's performance [2, 3].

With fluctuating global interest rates and shifts in currency valuations, the portfolio's international equity and fixed income assets faced mixed returns. The expected returns in Period (B) assumed lower economic volatility, whereas the actual 2020-2021 period involved significant rate cuts and currency movements, especially in response to central bank interventions. These factors diminished expected gains from both international and domestic investments [3, 6].

While infrastructure assets generally provide stability, they were not immune to 2020's macroeconomic shocks. Investments in infrastructure saw increased volatility, which deviated from the model's assumption of consistent cash flows and low correlation with economic cycles. Additionally, fixed income, although relatively stable, was influenced by inflationary pressures and bond market volatility, leading to lower-than-anticipated returns in this asset class [11].

In summary, the portfolio's realized return was lower than the expected model return due to the unforeseen economic impact of the COVID-19 pandemic, which introduced substantial deviations in asset returns. This outcome showcases the limitations of static optimization based on historical data, particularly in unpredictable macroeconomic scenarios.

4.3 Efficient Frontier and Capital Market Line Analysis

Using our calculated two-year statistics for periods (A) and (B), we plot the Minimum Variance Frontier, Efficient Frontier, Capital Market Line and locate the Market Portfolio as well as the unconstrained optimal portfolio that contains only risky assets and the optimal portfolio which includes the risk-free asset. (So short-selling and leveraging allowed)

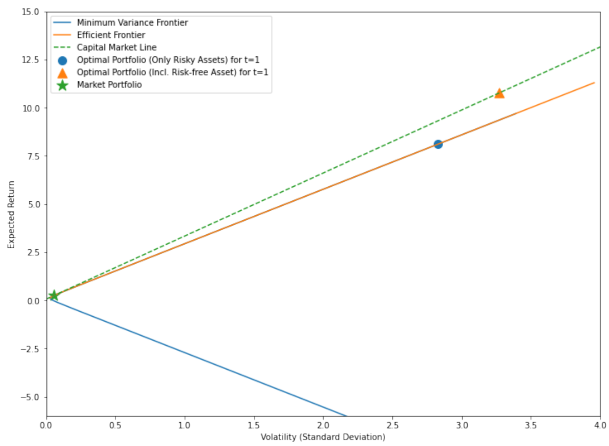

Figure 1: Period (A) Efficient Frontier

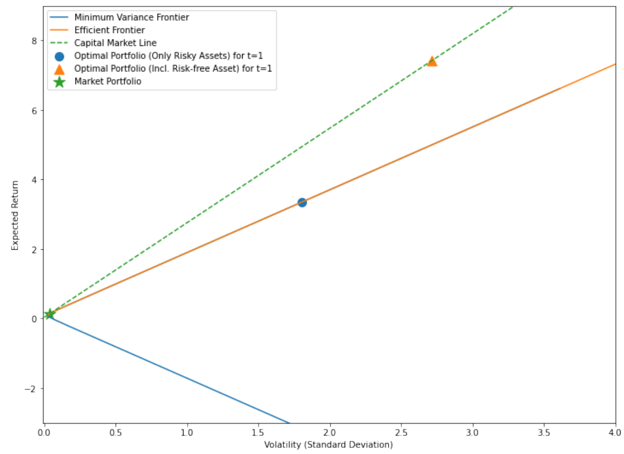

Figure 2: Period (B) Efficient Frontier

The Efficient Frontier (EF) and Capital Market Line (CML) plots for Period A and Period B illustrate the core principles of portfolio optimization. The Minimum Variance Frontier (MVF) shows portfolios with the lowest risk for a given return, curving upwards until it reaches the EF. The EF then represents portfolios that maximize return for each level of risk.

The CML, originating from the risk-free rate and tangent to the EF at the Market Portfolio, displays the highest achievable return for a given risk level when combining the risk-free asset with risky assets. This line allows investors to attain any risk-return combination along its path, adjusting allocations between the risk-free asset and the market portfolio for optimal performance.

The Optimal Portfolio (Only Risky Assets) lies on the EF and comprises purely risky assets, achieving the best return for its risk level among risky portfolios. In contrast, the Optimal Portfolio (Incl. Risk-free Asset), positioned on the CML, includes a mix of the risk-free asset and the market portfolio, offering higher potential returns and risk through leveraging or conservative allocation with the risk-free rate.

The Market Portfolio, marked at the tangency between the EF and CML, represents the optimal blend of risky assets, maximizing the Sharpe ratio. Its positioning shifts slightly between Periods A and B, reflecting variations in asset performance and economic conditions. Period B's Market Portfolio shows a more conservative profile, aligning with the observed shift toward stable assets during this period.

Overall, these plots show the flexibility in constructing tailored risk-return profiles. The Optimal Portfolio with only risky assets sets a benchmark, while the Optimal Portfolio with the risk-free asset offers enhanced trade-offs. The Market Portfolio remains pivotal, representing the optimal point for risk-adjusted returns on the EF and setting a foundation for CAPM-based portfolio construction.

4.4 Portfolio Optimization: Variance

Given the unpredictability of the market, an investor may choose instead to minimize the variance of their portfolio. Consider the variance minimization problem which yields at least 6% expected return. Under the same assumptions as previous we have

$$
\begin{aligned}
\min_{\boldsymbol{w}} \ & \boldsymbol{w}^\top C \boldsymbol{w} \\
\text{s.t.} \ & \sum_{i=1}^{9} w_i = 1, \\
& \sum_{i=1}^{9} w_i \mu_i \geq 0.06 \\
& w_1 \geq 0.15, \\
& w_2 \geq 0.15, \\
& w_7 \geq 0.05, \\
& w_9 \geq 0.05, \\
& 0.65 \geq \sum_{i=1}^{6} w_i \leq 0.75, \\
& 0 \leq w_i \leq 1, \quad \forall i
\end{aligned}
$$

Where the constraints are the same as in the expected utility maximization problem expect for the addition of $\sum_{i=1}^{9} w_i \mu_i \geq 0.06$ which guarantees our expected return to be at least 6%. Similarly, we have used scipy.optimize.minimize to obtain

Asset	Period A	Period B
AEQ	0.1500	0.1500
ILE (Hedged)	0.1500	0.1500
ILE (Unhedged)	0.1315	0.0248
ALP	0.0000	0.0672
ILP	0.0000	0.0000
ILI	0.2185	0.2580
AFI	0.1147	0.0876
IFI	0.0000	0.0000
Cash	0.2353	0.2624
Expected Return (μ)	0.2426	0.1883
Volatility (σ)	0.0997	0.0844

Table 2: Weights and Statistics for Periods (A) and (B)

We observe some key similarities and differences for both periods. In Period A, the portfolio includes a larger allocation to International Listed Equities (Unhedged), with a weight of 13.15%, whereas this is reduced to 2.48% in Period B. This decrease may be attributed to increased currency volatility or a greater perceived risk in international markets as discussed previously.

Similarly, the allocation to Australian Listed Property (ALP) increased from 0% in Period A to 6.72% in Period B, likely due to attractive returns or favorable market conditions within the Australian property sector. The allocation to International Listed Infrastructure (ILI) remains significant across both periods, indicating its continued role as a stable, income-generating asset. However, the weight increases slightly from 21.85% to 25.80% in Period B, suggesting a shift towards more defensive assets in a more volatile market.

The allocation to Cash is higher in Period B (26.24% compared to 23.53% in Period A), which could imply a more conservative stance, reflecting increased risk aversion or uncertainty about market returns. This increase aligns with the lower expected return in Period B (18.83%) compared to Period A (24.26%). The lower volatility in Period B (8.44%) compared to Period A (9.97%) further suggests a reallocation towards less risky assets, likely driven by market conditions that favored stability over high returns.

Overall, the shift from more aggressive allocations in Period A to more conservative ones in Period B indicates an adaptation to a changing market, with a focus on reducing exposure to higher-volatility assets and increasing allocations to cash and stable income-producing assets. This has resulted in a portfolio with lower expected returns and volatility in Period B.

4.5 Comparison with Realised Portfolio Returns (2020-2021): Utility Maximization and Variance Minimization

Using the portfolio weights from Period (B) for the period January 2020 to December 2021, we observe a disparity between the expected and realized returns:

Expected Return (from model):	0.1884
Realized Return (2020-2021):	0.1457

This lower realised return is likely attributed to the same factors as discussed in Section 4.2. However, we notice that the difference is not as drastic as in the Expected Utility maximization case. As we have minimized variance, this lower volatility portfolio was able to protect the portfolio against the extreme volatility brought about by the COVID-19 pandemic and subsequent economic uncertainty.

While the realized return (14.57%) is lower than the expected return (18.84%) predicted by the model, the discrepancy is relatively moderate. This outcome suggests that the portfolio's defensive allocation strategy favouring lower-risk assets like cash and infrastructure, mitigated the impacts of market downturns observed during this period.

In contrast to the utility maximization case, where a higher variance was accepted in pursuit of greater returns, the variance-minimized portfolio achieved more consistent returns, in-line with its design to withstand volatility. Thus, the reduced discrepancy between expected and realized returns validates the low-volatility focus adopted in Period B, demonstrating the robustness of variance minimization in turbulent market environments.

So, while in stable market conditions, the utility-maximized portfolio provides higher returns with manageable volatility, it becomes more vulnerable during periods of extreme market disruption. The variance-minimized portfolio, on the other hand, was designed specifically to prioritize stability over high returns, making it more resilient in unpredictable environments. This contrast highlights the trade-off between maximizing utility and minimizing risk, underscoring the importance of aligning portfolio strategies with expected market conditions. In volatile periods like 2020-2021, the variance-minimized approach proved to be advantageous, delivering more stable returns in line with its conservative allocation.

5 Dynamic Portfolio Optimization

In this section, we analyze the dynamic portfolio optimization problem over a two-year investment horizon. We demonstrate that the multi-period optimization can be decomposed into a series of single-period utility maximization problems using the principles of Dynamic Programming.

Theorem 2. *Consider a two-year investment period with portfolio allocations $\mathbf{w}$ in Year 1 and $\mathbf{u}$ in Year 2. The portfolio return over the two years is given by*

$$G(\mathbf{w}, \mathbf{u}) = (1 + \mathbf{w}^\top \mathbf{V}_1)(1 + \mathbf{u}^\top \mathbf{V}_2) - 1,$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ are random vectors of asset returns in Year 1 and Year 2, respectively. The investor seeks to maximize the expected utility

$$\max \mathbb{E}\left[U(G(\mathbf{w}, \mathbf{u}))\right],$$

subject to the constraints $\sum_{i=1}^{9} w_i = 1$, $\sum_{i=1}^{9} u_i = 1$, and $U(x) = -e^{-\gamma x}$ with $\gamma = 1$. Additionally, the allocation $\mathbf{u}$ in Year 2 may depend on the realization of $\mathbf{V}_1$. Then, the multi-period optimization problem can be decomposed into a series of single-period utility maximization problems.

A proof (sketch) of this theorem is provided below.

Proof. The objective is to maximize $\mathbb{E}[U(G(\mathbf{w}, \mathbf{u}))]$ where $G(\mathbf{w}, \mathbf{u}) = (1 + \mathbf{w}^\top \mathbf{V}_1)(1 + \mathbf{u}^\top \mathbf{V}_2) - 1$.

Define the value function for Year 2:

$$V_2(\mathbf{w}, \mathbf{V}_1) = \max_{\mathbf{u}} \mathbb{E}\left[U\left((1 + \mathbf{w}^\top \mathbf{V}_1)(1 + \mathbf{u}^\top \mathbf{V}_2) - 1\right) \mid \mathbf{V}_1\right],$$

subject to

$$\sum_{i=1}^{9} u_i = 1, \quad \mathbf{u} \geq 0.$$

The overall value function for Year 1 is:

$$V_1 = \max_{\mathbf{w}} \mathbb{E}\left[V_2(\mathbf{w}, \mathbf{V}_1)\right],$$

subject to

$$\sum_{i=1}^{8} w_i = 1, \quad \mathbf{w} \geq 0.$$

Substituting V_2 into V_1:

$$V_1 = \max_{\mathbf{w}} \mathbb{E}\left[\max_{\mathbf{u}} \mathbb{E}\left[-e^{-\left((1+\mathbf{w}^\top \mathbf{V}_1)(1+\mathbf{u}^\top \mathbf{V}_2)-1\right)} \mid \mathbf{V}_1\right]\right].$$

Applying the properties of the utility function $U(x) = -e^{-x}$, the inner optimization for $\mathbf{u}$ given $\mathbf{V}_1$ is a single-period utility maximization:

$$\mathbf{u}^*(\mathbf{V}_1) = \arg\max_{\mathbf{u}} \mathbb{E}\left[-e^{-\left((1+\mathbf{w}^\top \mathbf{V}_1)(1+\mathbf{u}^\top \mathbf{V}_2)\right)} \mid \mathbf{V}_1\right].$$

Substituting $\mathbf{u}^*(\mathbf{V}_1)$ back into V_1:

$$V_1 = \max_{\mathbf{w}} \mathbb{E}\left[-e^{-\left((1+\mathbf{w}^\top \mathbf{V}_1)(1+\mathbf{u}^*(\mathbf{V}_1)^\top \mathbf{V}_2)\right)}\right].$$

Therefore, the two-period optimization problem is reduced to:

1. Solving the Year 2 problem $V_2(\mathbf{w}, \mathbf{V}_1)$ as a single-period utility maximization given $\mathbf{V}_1$.

2. Solving the Year 1 problem V_1 by maximizing the expected utility considering the optimal Year 2 response.

This decomposition aligns with Bellman's Optimality Principle, confirming that the multi-period problem is composed of a series of single-period utility maximization problems. □

5.1 A Fast Method for Normally Distributed Asset Returns

In practice, solving the dynamic programming recursion using Monte Carlo simulations can be computationally intensive. However, for our specific case where asset returns are multivariate normally distributed, a more efficient method exists [7]. This method leverages the property that, for monotonically increasing concave utility functions, the optimal portfolio allocations lie on the efficient frontier. Consequently, instead of solving the recursive non-linear optimization directly, we can search within a pre-computed set of mean-variance efficient portfolios to identify the one that maximizes the utility function.

To ensure optimality at each stage of the investment horizon, it is crucial that the induced utility function $u_t(W_t)$, defined as

$$u_t(W_t) = \max_{\mathbf{u}} \mathbb{E}\left[u_{t+1}\left((W_t + s_t)\mathbf{u}^\top \mathbf{R}_t\right)\right],$$

remains monotonically increasing and concave, consistent with the terminal utility function $U(W_T) = -e^{-W_T}$.

Let $r_{lk,t}$ denote the return distribution of the k-th allocation point $(\mathbf{a}_k, t)$ on the mean-variance efficient frontier in period t, specifically $r_{lk,t} = \mathcal{N}(\mu_{lk,t}, \sigma^2_{hk,t})$. The dynamic programming recursion can then be expressed as

$$u_t(W_t) = \max_{k} \mathbb{E}\left[u_{t+1}\left((W_t + s_t) r_{lk,t}\right)\right],$$

where $u_T(W_T) = U(W_T)$ and $u_t(W_t)$ for $t < T$ is defined recursively based on the optimal allocations in subsequent periods.

For our problem we have the following implementation of the fast method

1. **Simulate Asset Returns**:

   $$\mathbf{V}_1, \mathbf{V}_2 \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$$

   where $\boldsymbol{\mu}$ is the mean return vector and Σ is the covariance matrix of asset returns.

2. **Pre-compute Mean-Variance Efficient Portfolios**:

   $$\mathcal{W} = \left\{\mathbf{w}_k \;\middle|\; \mathbf{w}_k = \arg\min_{\mathbf{w}} \mathbf{w}^\top \Sigma \mathbf{w}\right\}$$

   subject to:

   $$\sum_{i=1}^{n} w_{k,i} = 1, \quad w_{k,i} \geq 0 \;\forall i$$

3. **Optimize Second-Period Allocation u Given Each $\mathbf{w}_k$**: For each $\mathbf{w}_k \in \mathcal{W}$, determine the optimal $\mathbf{u}_k$ that maximizes expected utility:

   $$\mathbf{u}_k^* = \arg\max_{\mathbf{u}} \mathbb{E}\left[U\left((1 + \mathbf{w}_k^\top \mathbf{V}_1)(1 + \mathbf{u}^\top \mathbf{V}_2) - 1\right)\right]$$

   subject to:

   $$\sum_{i=1}^{n} u_i = 1, \quad u_i \geq 0 \;\forall i,$$

4. **Evaluate Expected Utility for Each Portfolio Pair $(\mathbf{w}_k, \mathbf{u}_k^*)$**:

   $$\mathbb{E}\left[U\left((1 + \mathbf{w}_k^\top \mathbf{V}_1)(1 + \mathbf{u}_k^{*\top} \mathbf{V}_2) - 1\right)\right]$$

   Compute this expectation using the simulated scenarios of $\mathbf{V}_1$ and $\mathbf{V}_2$.

5. **Select the Optimal First-Period Portfolio $\mathbf{w}^*$ and Corresponding $\mathbf{u}^*$**:

   $$(\mathbf{w}^*, \mathbf{u}^*) = \arg\max_{(\mathbf{w}_k, \mathbf{u}_k^*) \in \mathcal{W} \times \mathcal{U}} \mathbb{E}\left[U\left(G(\mathbf{w}_k, \mathbf{u}_k^*)\right)\right]$$

   where $G(\mathbf{w}_k, \mathbf{u}_k^*) = (1 + \mathbf{w}_k^\top \mathbf{V}_1)(1 + \mathbf{u}_k^{*\top} \mathbf{V}_2) - 1$.

5.2 Application to Australian and International Assets: Dynamic Allocation

Now we apply the fast method to our data. For the following problem

$$\begin{aligned} \max_{\boldsymbol{w},\boldsymbol{u}} \ & \mathbb{E}[U(G(\boldsymbol{w},\boldsymbol{u})], \\ \text{s.t. } & \sum_{i=1}^{9} w_i, \sum_{i=1}^{6} u_i = 1, \\ & w_1, u_1 \geq 0.15, \\ & w_2, u_2 \geq 0.15, \\ & w_7, u_7 \geq 0.05, \\ & w_9, u_9 \geq 0.05, \\ & 0.65 \geq \sum_{i=1}^{6} w_i, \sum_{i=1}^{6} u_i \leq 0.75, \\ & 0 \leq w_i, u_i \leq 1, \quad \forall i \end{aligned}$$

Where w is our vector of weights and $U(x) = -e^{-\gamma x}$ with risk-aversion parameter $\gamma = 1$. The constraints are the same as the static optimization case, but applied to both time-steps. Then we obtain the weights

Asset	**1$^{\text{st}}$ Year ($\boldsymbol{w}$)**	**2$^{\text{nd}}$ Year ($\boldsymbol{u}$)**
AEQ	0.1500	0.1500
ILE (Hedged)	0.1500	0.1500
ILE (Unhedged)	0.0134	0.0000
ALP	0.0000	0.0000
ILP	0.0000	0.0000
ILI	0.4366	0.4500
AFI	0.1801	0.2000
IFI	0.0199	0.0000
Cash	0.0500	0.0500
$Z(\boldsymbol{w},\boldsymbol{u})$	-0.8047	
Expected Return (μ)	0.2221	
Volatility (σ)	0.0981	

Table 3: Weights and Statistics for Dynamic Portfolio

In the first year, the dynamic portfolio allocates heavily towards International Listed Infrastructure (ILI) at 43.66%, with notable allocations to Australian Fixed Income (AFI) (18.01%) and Cash (5%). This configuration suggests a balance between growth-oriented assets and conservative holdings. Notably, Australian Equities (AEQ) and International Listed Equities (ILE Hedged) are both held at the minimum constraint of 15%, likely reflecting lower risk-adjusted returns compared to other assets.

Moving into the second year, the portfolio adjustments indicate a shift towards stability. The allocation to ILI increases to 45%, while AFI also rises to 20%. ILE (Unhedged) and IFI are completely removed from the portfolio, reflecting a more cautious stance likely driven by performance observations in the first year. These adjustments align with the dynamic strategy's responsiveness to previous outcomes, aiming for improved stability without substantial reduction in growth potential.

Cash remains steady at 5% across both years, fulfilling the minimal allocation for liquidity. This consistent allocation highlights the importance of maintaining cash reserves even in growth-oriented periods, balancing liquidity needs with the overall investment strategy.

5.3 Comparison to Static Portfolio Allocation

The comparison between the static and dynamic utility maximization portfolios highlights key distinctions in how each approach adapts asset allocations across the two-year investment horizon in response to different market conditions. Keep in mind that the realizations of $\mathbf{V}_1$ and $\mathbf{V}_2$ (Our simulated asset returns for Year 1 and Year 2) were generated from period (B) statistics. Furthermore, the values of the objective functions cannot be directly compared as they are fundamentally different.

The dynamic portfolio achieves a similar expected (average) return (22.21%) to the static portfolio's Period B configuration (22.31%) but with a slight reduction in volatility (9.81% vs. 9.93%). This is to be expected due to the way $\mathbf{V}_1$ and $\mathbf{V}_2$ were generated, resulting in similar performance across the two-year horizon. The key distinction lies in how the dynamic portfolio allocates significantly to infrastructure (ILI) in both years while strategically decreasing exposure to assets like International Listed Equities (Unhedged) and International Fixed Income in the second year. This shift reflects the dynamic model's ability to observe and respond to asset performance after the first year, adjusting allocations to capitalize on lower-risk, income-generating assets while reducing exposure to higher-volatility components.

In a stable market environment, both portfolios are likely to perform similarly due to their shared basis in Period B's statistics. However, the dynamic approach provides a marginal edge in terms of risk control. This advantage may become more pronounced in scenarios where market conditions shift significantly between years, as the dynamic portfolio's rebalancing capability would allow it to respond to emerging risks or opportunities, whereas the static portfolio would remain fixed to its initial allocations for the entire period.

5.4 Dynamic Portfolio Optimization: Variance

Similar to the static optimization problem, an investor may instead to minimise variance to offset the effect of swings in the market. Once again, consider the variance minimization problem which yields at least 6% expected return. Under the same assumptions as previous we have

$$
\begin{aligned}
&\max_{\boldsymbol{w},\boldsymbol{u}} \operatorname{Var}[U(G(\boldsymbol{w},\boldsymbol{u})],\\
&\text{s.t.} \sum_{i=1}^{9} w_i, \sum_{i=1}^{6} u_i = 1,\\
&\quad \mathbb{E}[U(G(\boldsymbol{w},\boldsymbol{u})] \geq 0.06\\
&\quad w_1, u_1 \geq 0.15,\\
&\quad w_2, u_2 \geq 0.15,\\
&\quad w_7, u_7 \geq 0.05,\\
&\quad w_9, u_9 \geq 0.05,\\
&\quad 0.65 \geq \sum_{i=1}^{6} w_i, \sum_{i=1}^{6} u_i \leq 0.75,\\
&\quad 0 \leq w_i, u_i \leq 1, \quad \forall i
\end{aligned}
$$

The solution obtained using the fast-method is

Asset	**1st Year ($\boldsymbol{w}$)**	**2nd Year ($\boldsymbol{u}$)**
AEQ	0.1663	0.1500
ILE (Hedged)	0.1500	0.1500
ILE (Unhedged)	0.0248	0.0239
ALP	0.0486	0.0483
ILP	0.0000	0.0000
ILI	0.2603	0.2778
AFI	0.0500	0.0867
IFI	0.0001	0.0592
Cash	0.2999	0.2041
Expected Return (μ)	0.1873	
Volatility (σ)	0.0827	

Table 4: Weights and Statistics for Dynamic Portfolio

In the first year, the portfolio leans defensively with a high allocation to Cash (29.99%) and International Listed Infrastructure (ILI) (26.03%). This setup reflects an effort to dampen volatility, particularly by holding cash and stable infrastructure assets. Additionally, Australian Listed Equity (AEQ) and International Listed Equity (ILE Hedged) are both constrained to the minimum required allocation of 15%, suggesting limited expected contribution to risk-adjusted returns. Australian Fixed Income (AFI) and Australian Listed Property (ALP) have small allocations at 5% and 4.86%, balancing growth and defensive exposure with the minimum variance objective.

In the second year, the portfolio adapts towards further stability by reducing Cash to 20.41% and reallocating to IFI (5.92%) and AFI (8.67%), diversifing into defensive assets. ILI increases to 27.78%, maintaining its role as a significant stabilizer, given its lower volatility and steady income generation. Notably, allocations to ILE (Unhedged) and ALP remain minimal, each around 2-4%, while AEQ and ILE (Hedged) stay at their minimum 15% allocation, further illustrating the cautious, defensive reallocation approach in the second year.

Overall, this dynamic minimum variance strategy shows careful adjustments, prioritizing low-risk assets like cash, fixed income, and infrastructure. These shifts indicate responsiveness to year-one market conditions, seeking stability without compromising the overall risk-return balance. The resulting portfolio volatility is 8.27%, with an expected return of 18.73%, aligning with the objectives of risk management and income stability across the two-year horizon.

5.5 Comparison to Static Portfolio Allocation and Utility Maximization

Comparing the dynamic variance-minimization portfolio with the static variance-minimization and dynamic utility maximization portfolios reveals notable distinctions in risk-reward profiles and asset allocation strategies over the two-year horizon. Both variance-minimization strategies target stability, yet the dynamic approach shows flexibility in adapting to year-one outcomes, while the static allocation remains constant, unable to respond to intermediate performance. This responsiveness in the dynamic portfolio allows it to marginally reduce overall volatility (8.27% vs. 8.44% for static) while achieving an expected return (18.73%) close to the static allocation for Period B (18.83%).

The dynamic variance-minimization portfolio stands out in its allocation to defensive assets, especially in the second year. By reducing Cash from 29.99% to 20.41% and shifting to International Fixed Income (IFI) and Australian Fixed Income (AFI), it seeks diversification among low-volatility assets while maintaining stability through a high allocation to International Listed Infrastructure (ILI) (27.78%). In contrast, the static variance-minimization portfolio for Period B maintains higher cash holdings (26.24%) and reduces exposure to ILI. This indicates a conservative approach to risk without the dynamic strategy's ability to adjust based on performance, potentially limiting the portfolio's adaptability to market shifts.

Overall, similar to the previous dynamic and static utility maximization problem, due to the way we generated $\mathbf{V}_1$ and $\mathbf{V}_2$, and in combination with a relatively stable market in Period (B), the results should be similar especially for a high number of simulations.

The dynamic utility maximization portfolio pursues higher returns at a controlled but slightly elevated risk compared to the variance-minimization approaches. It allocates more heavily to growth-oriented assets like ILI, with 43.66% in the first year and a slight increase to 45% in the second, focusing on income-generating assets with favorable risk-adjusted returns. The lack of high cash allocations in the utility maximization portfolio contrasts with the variance-focused strategies, which prioritize cash and fixed income for stability. Consequently, the utility maximization portfolio achieves a higher expected return

(22.21%) at a slightly increased volatility (9.81%), positioning it for investors with a greater tolerance for risk in pursuit of higher returns.

6 Extensions

6.1 Handling Non-Positive Semidefinite Covariance Matrices

In real-world portfolio optimization, it is possible for the covariance matrix C of asset returns to become non-positive semidefinite (non-PSD) due to sudden market disruptions or estimation errors. A non-PSD covariance matrix can lead to optimization algorithms failing or producing unreliable portfolio allocations. This section explores how to handle such scenarios by:

1. Introducing a change that renders C non-PSD.
2. Approximating the altered C with a PSD matrix.
3. Analyzing the impact of this approximation on portfolio optimization.

To simulate a market disruption, we modify an element of the covariance matrix C such that it violates the PSD property. Specifically, we alter the covariance between Australian Fixed Income and International Listed Equity (Unhedged).

Given the original covariance matrix C, we introduce a market disruption by significantly increasing the negative covariance between AFI (Asset Index 7) and ILE (Unhedged) (Asset Index 3):

$$C'_{7,3} = C'_{3,7} = -0.0050$$

To confirm that C' is no longer PSD, we compute its eigenvalues. The presence of any negative eigenvalues indicates that C' is indeed non-PSD. For our data the eigenvalues of C' are

$$\lambda_{C'} = [-0.0009, 0, \ldots, 0.106]$$

At least one eigenvalue of C' is negative, confirming that C' is not positive semidefinite.

To restore the PSD property, we approximate C' with a PSD matrix $\hat{C}$. One common method involves Eigenvalue Decomposition:

1. **Eigenvalue Decomposition:**

   Decompose C' into its eigenvalues and eigenvectors:

   $$C' = Q\Lambda Q^\top$$

   where Q is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues.

2. **Modify Eigenvalues:**

   Set any negative eigenvalues in Λ to zero:

   $$\Lambda^+ = \max(\Lambda, 0)$$

3. **Reconstruct the PSD Matrix:**

   Reconstruct $\hat{C}$ using the modified eigenvalues:

   $$\hat{C} = Q\Lambda^+ Q^\top$$

4. **Ensure Symmetry:**

   Due to numerical precision, ensure $\hat{C}$ is symmetric:

   $$\hat{C} = \frac{1}{2}\left(\hat{C} + \hat{C}^\top\right)$$

Applying the PSD approximation method to C' results in $\hat{C}$ which for our data, can be found in Appendix C. Then using our approximated $\hat{C}$ we obtain the following results for the variance minimization problem based on period (B)

Asset	C	$\hat{C}$
AEQ	0.1500	0.1500
ILE (Hedged)	0.1500	0.1500
ILE (Unhedged)	0.0248	0.0526
ALP	0.0672	0.0420
ILP	0.0000	0.0000
ILI	0.2580	0.2554
AFI	0.0876	0.1481
IFI	0.0000	0.0577
Cash	0.2624	0.1442
Expected Return (μ)	0.1883	0.1961
Volatility (σ)	0.0844	0.0846

Table 5: Weights and Statistics for C and $\hat{C}$ on (B)

The optimized portfolios under the original covariance matrix C and the approximated PSD matrix $\hat{C}$ have some key differences in asset allocations, expected return, and volatility. Notably, allocations to International Listed Equities (Unhedged) and Australian Fixed Income increased when using $\hat{C}$, with ILE (Unhedged) rising from 2.48% to 5.26% and AFI from 8.76% to 14.81%. This indicates that, under the adjusted risk structure in $\hat{C}$, these assets appear more effective for variance reduction. Conversely, the allocation to Cash decreased significantly, from 26.24% in the C-optimized portfolio to 14.42% with $\hat{C}$. This change suggests that the adjusted covariance structure increases the relative attractiveness of fixed-income assets, thereby reducing the portfolio's reliance on cash for risk mitigation.

The expected return also increased slightly, from 18.83% under C to 19.61% under $\hat{C}$, while volatility remained stable at 8.44% and 8.46%, respectively. This stability in risk suggests that the PSD adjustment in $\hat{C}$ allowed for a feasible solution with marginally higher returns, without compromising the low-volatility objective. Overall, the PSD approximation produced a portfolio that reallocated assets towards fixed income while achieving a marginally higher return at nearly identical risk levels.

6.2 Static Portfolio Optimization: Sharpe Ratio

In real-world scenarios, portfolio managers often seek to maximize the Sharpe ratio, as it measures the portfolio's risk-adjusted return. The Sharpe ratio, introduced by William F. Sharpe, is defined as:

$$\text{Sharpe Ratio} = \frac{\mathbb{E}[\boldsymbol{w}^\top \boldsymbol{R}] - r_f}{\sqrt{\boldsymbol{w}^\top C \boldsymbol{w}}},$$

where $\mathbb{E}[\boldsymbol{w}^\top \boldsymbol{R}]$ is the portfolio's expected return, r_f represents the risk-free rate, and C is the covariance matrix of asset returns. This metric provides a framework to assess the excess return achieved for each unit of risk undertaken, guiding managers toward portfolios that maximize efficiency. A higher Sharpe ratio indicates a more attractive risk-return profile, where returns are enhanced without a proportional increase in volatility.

The optimization objective then becomes to maximize the Sharpe ratio, as follows

$$\max_{\boldsymbol{w}} \frac{\boldsymbol{w}^\top (\boldsymbol{r} - r_f \mathbf{1})}{\sqrt{\boldsymbol{w}^\top C \boldsymbol{w}}},$$

where $\boldsymbol{w}$ is the vector of portfolio weights, $\boldsymbol{r}$ is the vector of expected returns, and $r_f\mathbf{1}$ adjusts each return relative to the risk-free rate.

Optimizing for the Sharpe ratio is especially relevant for investors aiming to enhance risk-adjusted performance, as it identifies portfolios that achieve the best possible trade-off between return and volatility. This approach highlights efficient allocations that not only seek high returns but also manage risk exposure effectively.

Applying the Sharpe ratio maximization problem to our data we have

$$\begin{aligned}
&\max_{\boldsymbol{w}} \frac{\boldsymbol{w}^\top (\boldsymbol{r} - r_f \mathbf{1})}{\sqrt{\boldsymbol{w}^\top C \boldsymbol{w}}}, \\
&\text{s.t.} \sum_{i=1}^{8} w_i = 1, \\
&\qquad w_1 \geq 0.15, \\
&\qquad w_2 \geq 0.15, \\
&\qquad w_7 \geq 0.05, \\
&\qquad 0.65 \geq \sum_{i=1}^{6} w_i \leq 0.75, \\
&\qquad 0 \leq w_i \leq 1, \quad \forall i
\end{aligned}$$

Which are the same constraints described in the Utility Maximization Problem (Section 4.21) except for the constraint on Cash. Note that we have treated Cash as the risk-free asset, using its return (0.0365) as r_f in the Sharpe ratio calculation. This means that we exclude Cash from the risky assets resulting in an 8-asset expected returns vector $\boldsymbol{r}$ and an 8×8 covariance matrix C.

We also consider the following case, where we do not impose specific limits on the weights of the assets. This problem is described as

$$\begin{aligned}
&\max_{\boldsymbol{w}} \frac{\boldsymbol{w}^\top (\boldsymbol{r} - r_f \mathbf{1})}{\sqrt{\boldsymbol{w}^\top C \boldsymbol{w}}}, \\
&\text{s.t.} \sum_{i=1}^{8} w_i = 1, \\
&\qquad 0 \leq w_i \leq 1, \quad \forall i
\end{aligned}$$

Using the period (B) two-year returns $\boldsymbol{R}_B^{(2)}$ and covariance matrix $C_B^{(2)}$ (with Cash excluded) scipy.optimize.minimize we obtain

Asset	**Constrained**	**Unconstrained**
AEQ	0.1500	0.0000
ILE (Hedged)	0.1500	0.0000
ILE (Unhedged)	0.1003	0.1999
ALP	0.0000	0.0000
ILP	0.0000	0.0000
ILI	0.2497	0.0000
AFI	0.3500	0.8001
IFI	0.0000	0.0000
Expected Return (μ)	0.2064	0.1313
Volatility (σ)	0.0842	0.0279
Sharpe Ratio	2.0183	3.4010

Table 6: Weights and Statistics for Constrained and Unconstrained Sharpe Ratio Optimization

Our results provide an interesting insight. Our attempts to diversify have resulted in our constrained portfolio having a much lower Sharpe ratio of 2.0183 with an expected return of 20.64% with a volatility of 8.42%. In contrast, the unconstrained portfolio allocates heavily to Australian Fixed Income (80.01%) and International Listed Equities (Unhedged) (19.99%), with no allocation to other assets. Whilst this results in a higher Sharpe ratio of 3.4010, this lack of diversification results in a portfolio that may be more sensitive to shocks.

In summary, while our attempts to diversify in the constrained case align with practical risk management, they produce a portfolio with a lower Sharpe ratio and higher volatility. The unconstrained optimization provides a theoretical maximum Sharpe ratio by sacrificing diversification, achieving higher risk-adjusted performance but at the expense of robustness. This trade-off underscores the importance of balancing diversification and risk-adjusted return in real-world portfolio construction.

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A Question 1: Parameter Calculation

```
Period A Mean Vector (a_A):
Australian Listed Equity [G]          0.0091
Int'l Listed Equity (Hedged) [G]      0.0118
Int'l Listed Equity (Unhedged) [G]    0.0152
Australian Listed Property [G]        0.0151
Int'l Listed Property [G]             0.0115
Int'l Listed Infrastructure [G]       0.0105
Australian Fixed Income [D]           0.0044
Int'l Fixed Income (Hedged) [D]       0.0051
Cash [D]                              0.0024
dtype: float64

Period A Covariance Matrix (B_A):
[[ 0.0012  0.0007  0.0003  0.0009  0.0007  0.0005  0.      0.0001  0.    ]
 [ 0.0007  0.0009  0.0005  0.0003  0.0005  0.0005 -0.      0.      0.    ]
 [ 0.0003  0.0005  0.0008  0.0001  0.0001  0.0002  0.     -0.      0.    ]
 [ 0.0009  0.0003  0.0001  0.0011  0.0008  0.0005  0.0001  0.0001  0.    ]
 [ 0.0007  0.0005  0.0001  0.0008  0.0011  0.0007  0.0001  0.0002  0.    ]
 [ 0.0005  0.0005  0.0002  0.0005  0.0007  0.0007  0.      0.0001  0.    ]
 [ 0.     -0.      0.      0.0001  0.0001  0.      0.0001  0.      0.    ]
 [ 0.0001  0.     -0.      0.0001  0.0002  0.0001  0.      0.0001  0.    ]
 [ 0.      0.      0.      0.      0.      0.      0.      0.      0.    ]
```

```
Period B Mean Vector (a_B):
Australian Listed Equity [G]          0.0085
Int'l Listed Equity (Hedged) [G]      0.0092
Int'l Listed Equity (Unhedged) [G]    0.0099
Australian Listed Property [G]        0.0083
Int'l Listed Property [G]             0.0064
Int'l Listed Infrastructure [G]       0.0102
Australian Fixed Income [D]           0.0037
Int'l Fixed Income (Hedged) [D]       0.0036
Cash [D]                              0.0015
dtype: float64

Period B Covariance Matrix (B_B):
[[ 0.0008  0.0005  0.0005  0.0005  0.0003  0.0002 -0.     -0.      0.    ]
 [ 0.0005  0.0009  0.0006  0.0002  0.0005  0.0004 -0.0001 -0.0001 -0.    ]
 [ 0.0005  0.0006  0.0008  0.0003  0.0002  0.0002 -0.     -0.     -0.    ]
 [ 0.0005  0.0002  0.0003  0.001   0.0006  0.0003  0.0001  0.0001  0.    ]
 [ 0.0003  0.0005  0.0002  0.0006  0.001   0.0006  0.      0.0001  0.    ]
 [ 0.0002  0.0004  0.0002  0.0003  0.0006  0.0006  0.      0.0001  0.    ]
 [-0.     -0.0001 -0.      0.0001  0.      0.      0.0001  0.      0.    ]
 [-0.     -0.0001 -0.      0.0001  0.0001  0.0001  0.      0.0001  0.    ]
 [ 0.     -0.     -0.      0.      0.      0.      0.      0.      0.    ]]
```

B Question 3: Parameter Calculation (Annual and 2-Year)

```
Period A, Return 1 Year(s):

Expected Returns:
Australian Listed Equity [G]          0.124108
Int'l Listed Equity (Hedged) [G]      0.157963
Int'l Listed Equity (Unhedged) [G]    0.206616
Australian Listed Property [G]        0.207008
Int'l Listed Property [G]             0.155966
Int'l Listed Infrastructure [G]       0.138659
Australian Fixed Income [D]           0.055088
Int'l Fixed Income (Hedged) [D]       0.063849
Cash [D]                              0.029630
dtype: float64

Covariance Matrix:
[[ 0.0184  0.0103  0.0052  0.0143  0.0108  0.0076  0.0002  0.0011  0.    ]
 [ 0.0103  0.0141  0.0088  0.0055  0.0074  0.0074 -0.0006  0.0002  0.    ]
 [ 0.0052  0.0088  0.0142  0.0015  0.0024  0.0036  0.0003 -0.0003  0.    ]
 [ 0.0143  0.0055  0.0015  0.0199  0.0132  0.0083  0.0014  0.0018  0.0001]
 [ 0.0108  0.0074  0.0024  0.0132  0.0184  0.0113  0.0014  0.0024  0.    ]
 [ 0.0076  0.0074  0.0036  0.0083  0.0113  0.0107  0.0007  0.0013  0.    ]
 [ 0.0002 -0.0006  0.0003  0.0014  0.0014  0.0007  0.0008  0.0005  0.    ]
 [ 0.0011  0.0002 -0.0003  0.0018  0.0024  0.0013  0.0005  0.0007  0.    ]
 [ 0.      0.      0.      0.0001  0.      0.      0.      0.      0.    ]]

Correlation Matrix:
[[ 1.      0.6424  0.3191  0.7464  0.5887  0.5387  0.059   0.2832  0.1415]
 [ 0.6424  1.      0.6243  0.3302  0.4582  0.6021 -0.1841  0.0708  0.0703]
 [ 0.3191  0.6243  1.      0.088   0.1477  0.2887  0.0791 -0.0946  0.0957]
 [ 0.7464  0.3302  0.088   1.      0.6886  0.5697  0.3403  0.4755  0.2198]
 [ 0.5887  0.4582  0.1477  0.6886  1.      0.8033  0.3653  0.6406  0.1137]
 [ 0.5387  0.6021  0.2887  0.5697  0.8033  1.      0.2392  0.4668  0.11  ]
 [ 0.059  -0.1841  0.0791  0.3403  0.3653  0.2392  1.      0.6698  0.3411]
 [ 0.2832  0.0708 -0.0946  0.4755  0.6406  0.4668  0.6698  1.      0.2701]
 [ 0.1415  0.0703  0.0957  0.2198  0.1137  0.11    0.3411  0.2701  1.    ]]
```

```
Period A, Return 2 Year(s):

Expected Returns:
Australian Listed Equity [G]          0.263618
Int'l Listed Equity (Hedged) [G]      0.340879
Int'l Listed Equity (Unhedged) [G]    0.455921
Australian Listed Property [G]        0.456869
Int'l Listed Property [G]             0.336257
Int'l Listed Infrastructure [G]       0.296544
Australian Fixed Income [D]           0.113210
Int'l Fixed Income (Hedged) [D]       0.131775
Cash [D]                              0.060137
dtype: float64

Covariance Matrix:
[[ 0.0468  0.027   0.014   0.0389  0.0282  0.0194  0.0005  0.0025  0.0001]
 [ 0.027   0.0379  0.0247  0.0155  0.0198  0.0196 -0.0015  0.0006  0.    ]
 [ 0.014   0.0247  0.0415  0.0043  0.0067  0.0098  0.0007 -0.0008  0.0001]
 [ 0.0389  0.0155  0.0043  0.0583  0.0369  0.023   0.0035  0.0047  0.0002]
 [ 0.0282  0.0198  0.0067  0.0369  0.0495  0.0299  0.0035  0.0059  0.0001]
 [ 0.0194  0.0196  0.0098  0.023   0.0299  0.028   0.0017  0.0032  0.0001]
 [ 0.0005 -0.0015  0.0007  0.0035  0.0035  0.0017  0.0019  0.0012  0.    ]
 [ 0.0025  0.0006 -0.0008  0.0047  0.0059  0.0032  0.0012  0.0017  0.    ]
 [ 0.0001  0.      0.0001  0.0002  0.0001  0.0001  0.      0.      0.    ]]

Correlation Matrix:
[[ 1.      0.6409  0.3178  0.7451  0.587   0.5372  0.0588  0.2822  0.141 ]
 [ 0.6409  1.      0.6231  0.3288  0.4567  0.601  -0.1835  0.0706  0.0701]
 [ 0.3178  0.6231  1.      0.0876  0.1469  0.2878  0.079  -0.0944  0.0955]
 [ 0.7451  0.3288  0.0876  1.      0.6872  0.5683  0.3393  0.4742  0.219 ]
 [ 0.587   0.4567  0.1469  0.6872  1.      0.8023  0.3642  0.6389  0.1133]
 [ 0.5372  0.601   0.2878  0.5683  0.8023  1.      0.2387  0.466   0.1098]
 [ 0.0588 -0.1835  0.079   0.3393  0.3642  0.2387  1.      0.6697  0.341 ]
 [ 0.2822  0.0706 -0.0944  0.4742  0.6389  0.466   0.6697  1.      0.2701]
 [ 0.141   0.0701  0.0955  0.219   0.1133  0.1098  0.341   0.2701  1.    ]]
```

```
Period B, Return 1 Year(s):

Expected Returns:
Australian Listed Equity [G]          0.112020
Int'l Listed Equity (Hedged) [G]      0.122220
Int'l Listed Equity (Unhedged) [G]    0.130618
Australian Listed Property [G]        0.111277
Int'l Listed Property [G]             0.085837
Int'l Listed Infrastructure [G]       0.133333
Australian Fixed Income [D]           0.046199
Int'l Fixed Income (Hedged) [D]       0.044571
Cash [D]                              0.018108
dtype: float64

Covariance Matrix:
[[ 0.0114  0.0075  0.0075  0.0067  0.0045  0.0027 -0.     -0.0003  0.    ]
 [ 0.0075  0.0139  0.0092  0.0032  0.0074  0.0061 -0.001  -0.0008 -0.    ]
 [ 0.0075  0.0092  0.0116  0.0052  0.0032  0.0033 -0.     -0.0006 -0.    ]
 [ 0.0067  0.0032  0.0052  0.0155  0.0085  0.005   0.0018  0.0018  0.    ]
 [ 0.0045  0.0074  0.0032  0.0085  0.0139  0.0085  0.0004  0.0011  0.    ]
 [ 0.0027  0.0061  0.0033  0.005   0.0085  0.0086  0.0003  0.0008  0.    ]
 [-0.     -0.001  -0.      0.0018  0.0004  0.0003  0.0008  0.0006  0.    ]
 [-0.0003 -0.0008 -0.0006  0.0018  0.0011  0.0008  0.0006  0.0008  0.    ]
 [ 0.     -0.     -0.      0.      0.      0.      0.      0.      0.    ]]

Correlation Matrix:
[[ 1.      0.5936  0.6559  0.5074  0.3594  0.2698 -0.0108 -0.1005  0.0901]
 [ 0.5936  1.      0.7259  0.2161  0.531   0.5601 -0.2959 -0.2439 -0.0277]
 [ 0.6559  0.7259  1.      0.3867  0.2524  0.3303 -0.0091 -0.2123 -0.0365]
 [ 0.5074  0.2161  0.3867  1.      0.5812  0.4304  0.5061  0.5306  0.2112]
 [ 0.3594  0.531   0.2524  0.5812  1.      0.7797  0.1258  0.3363  0.0741]
 [ 0.2698  0.5601  0.3303  0.4304  0.7797  1.      0.1078  0.3104  0.1169]
 [-0.0108 -0.2959 -0.0091  0.5061  0.1258  0.1078  1.      0.763   0.2628]
 [-0.1005 -0.2439 -0.2123  0.5306  0.3363  0.3104  0.763   1.      0.1761]
 [ 0.0901 -0.0277 -0.0365  0.2112  0.0741  0.1169  0.2628  0.1761  1.    ]]
```

```
Period B, Return 2 Year(s):

Expected Returns:
Australian Listed Equity [G]          0.236589
Int'l Listed Equity (Hedged) [G]      0.259378
Int'l Listed Equity (Unhedged) [G]    0.278298
Australian Listed Property [G]        0.234936
Int'l Listed Property [G]             0.179042
Int'l Listed Infrastructure [G]       0.284443
Australian Fixed Income [D]           0.094532
Int'l Fixed Income (Hedged) [D]       0.091129
Cash [D]                              0.036544
dtype: float64

Covariance Matrix:
[[ 0.0282  0.0187  0.019   0.0167  0.0109  0.0067 -0.0001 -0.0007  0.    ]
 [ 0.0187  0.0353  0.0236  0.0079  0.0181  0.0157 -0.0024 -0.0019 -0.    ]
 [ 0.019   0.0236  0.0299  0.0131  0.0079  0.0085 -0.0001 -0.0015 -0.    ]
 [ 0.0167  0.0079  0.0131  0.0385  0.0206  0.0126  0.0043  0.0042  0.0001]
 [ 0.0109  0.0181  0.0079  0.0206  0.0329  0.0211  0.001   0.0025  0.    ]
 [ 0.0067  0.0157  0.0085  0.0126  0.0211  0.0223  0.0007  0.0019  0.    ]
 [-0.0001 -0.0024 -0.0001  0.0043  0.001   0.0007  0.0018  0.0013  0.    ]
 [-0.0007 -0.0019 -0.0015  0.0042  0.0025  0.0019  0.0013  0.0017  0.    ]
 [ 0.     -0.     -0.      0.0001  0.      0.      0.      0.      0.    ]]

Correlation Matrix:
[[ 1.      0.5924  0.6549  0.506   0.3582  0.269  -0.0108 -0.1002  0.0899]
 [ 0.5924  1.      0.7249  0.2151  0.5296  0.559  -0.2949 -0.2431 -0.0276]
 [ 0.6549  0.7249  1.      0.3854  0.2515  0.3295 -0.009  -0.2117 -0.0364]
 [ 0.506   0.2151  0.3854  1.      0.5798  0.4292  0.5048  0.5293  0.2105]
 [ 0.3582  0.5296  0.2515  0.5798  1.      0.7788  0.1254  0.3354  0.0739]
 [ 0.269   0.559   0.3295  0.4292  0.7788  1.      0.1076  0.3099  0.1167]
 [-0.0108 -0.2949 -0.009   0.5048  0.1254  0.1076  1.      0.7629  0.2627]
 [-0.1002 -0.2431 -0.2117  0.5293  0.3354  0.3099  0.7629  1.      0.1761]
 [ 0.0899 -0.0276 -0.0364  0.2105  0.0739  0.1167  0.2627  0.1761  1.    ]]
```

C Extension:Resulting PSD Covariance Matrix

$$\hat{C} = \begin{bmatrix} 0.0282 & 0.0187 & 0.0190 & 0.0167 & 0.0109 & 0.0067 & -0.0001 & -0.0007 & 0.0000 \\ 0.0187 & 0.0353 & 0.0235 & 0.0080 & 0.0180 & 0.0157 & -0.0025 & -0.0018 & 0.0000 \\ 0.0190 & 0.0235 & 0.0300 & 0.0130 & 0.0079 & 0.0085 & -0.0047 & -0.0015 & 0.0000 \\ 0.0167 & 0.0080 & 0.0130 & 0.0385 & 0.0206 & 0.0126 & 0.0041 & 0.0043 & 0.0001 \\ 0.0109 & 0.0180 & 0.0079 & 0.0206 & 0.0330 & 0.0211 & 0.0011 & 0.0025 & 0.0000 \\ 0.0067 & 0.0157 & 0.0085 & 0.0126 & 0.0211 & 0.0223 & 0.0006 & 0.0019 & 0.0000 \\ -0.0001 & -0.0025 & -0.0047 & 0.0041 & 0.0011 & 0.0006 & 0.0025 & 0.0012 & 0.0000 \\ -0.0007 & -0.0018 & -0.0015 & 0.0043 & 0.0025 & 0.0019 & 0.0012 & 0.0017 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

D Supplementary Code

https://github.com/Mas0nFeng/fmat3888