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# Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric

Przemysław Grzegorzewski*

*Systems Research Institute, Polish Academy of Sciences, 1 ul. Newelska 6, 01-447 Warsaw, Poland*

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**Abstract**

New methods for measuring distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets, based on the Hausdorff metric, are suggested. The proposed new distances are straightforward generalizations of the well known Hamming distance, the Euclidean distance and their normalized counterparts.


*Keywords:* Fuzzy set; Interval-valued fuzzy sets; Intuitionistic fuzzy set; Hamming distance; Euclidean distance; Hausdorff metric

## 1. Introduction

In conventional fuzzy set a membership function assigns to each element of the universe of discourse a number from the unit interval to indicate the degree of belongingness to the set under consideration. The degree of nonbelongingness is just automatically the complement to 1 of the membership degree. However, a human being who expresses the degree of membership of given element in a fuzzy set very often does not express corresponding degree of nonmembership as the complement to 1. This reflects a well known psychological fact that the linguistic negation not always identifies with logical negation.

Thus Atanassov [1] introduced the concept of an intuitionistic fuzzy set which is characterized by two functions expressing the degree of belongingness and the degree of nonbelongingness, respectively. This idea, which is a natural generalization of usual fuzzy set, seems to be useful in modeling many real life situations, like negotiation processes, etc. (see Szmidt and Kacprzyk [18,19,22,23]).

Atanassov [5] and Szmidt and Kacprzyk [20,21] suggested some methods for measuring distances between intuitionistic fuzzy sets that are generalizations of the well known Hamming distance,

* Tel.: +48-22-3735-78; fax: +48-22-3727-72.
*E-mail address:* pgrzeg@ibspan.waw.pl (P. Grzegorzewski).

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Euclidean distance and their normalized counterparts. In the present paper we propose another generalization of those distances based on the Hausdorff metric.

Since Zadeh has introduced fuzzy sets in 1965 [26], many new approaches and theories treating imprecision and uncertainty have been proposed. Some of these theories, like intuitionistic fuzzy sets theory, are extensions of the classical fuzzy set theory. Another, well-known generalization of an ordinary fuzzy set is, the so-called, interval-valued fuzzy set. Generally, the idea of interval-valued fuzzy sets was attributed to Gorzalczany [11] and Turksen [24], but actually they appear earlier in the papers [12,14,17]. Moreover, these approaches are in general not independent and there exist relationships among them. Sometimes they are even mathematically equivalent, however they have arisen on different ground and they have different semantics. For more details we refer the reader to [6,7,9,10,13]. For example, there is a strong connection between intuitionistic fuzzy sets and interval-valued fuzzy sets.

Therefore, one may easily notice that our definitions of the distances between intuitionistic fuzzy sets, based on the Hausdorff metric, could be immediately expressed in terms of interval-valued fuzzy sets.

## 2. Intuitionistic fuzzy sets and interval-valued fuzzy sets

Let $X$ denote a universe of discourse. Then a fuzzy set $A$ in $X$ is defined as a set of ordered pairs

$$A = \{\langle x, \mu_A(x)\rangle : \; x \in X\}, \tag{1}$$

where $\mu_A : X \to [0,1]$ is the membership function of $A$ and $\mu_A(x)$ is the grade of belongingness of $x$ into $A$ (see [26]). Thus automatically the grade of nonbelongingness of $x$ into $A$ is equal to $1-\mu_A(x)$. However, in real life the linguistic negation not always identifies with logical negation (see, e.g., [16]). This situation is very common in natural language processing, computing with words, etc. Therefore Atanassov [1–5] suggested a generalization of classical fuzzy set, called an intuitionistic fuzzy set.

An intuitionistic fuzzy set $A$ in $X$ is given by a set of ordered triples

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle : \; x \in X\}, \tag{2}$$

where $\mu_A, v_A : X \to [0,1]$ are functions such that

$$0 \leqslant \mu_A(x) + v_A(x) \leqslant 1 \quad \forall x \in X. \tag{3}$$

For each $x$ the numbers $\mu_A(x)$ and $v_A(x)$ represent the degree of membership and degree of nonmembership of the element $x \in X$ to $A \subset X$, respectively.

It is easily seen that a $\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle : \; x \in X\}$ is equivalent to (1), i.e. each fuzzy set is a particular case of the intuitionistic fuzzy set. We will denote a family of fuzzy sets in $X$ by $FS(X)$, while $IFS(X)$ stands for the family of all intuitionistic fuzzy sets in $X$.

For each element $x \in X$ we can compute, so called, the intuitionistic fuzzy index of $x$ in $A$ defined as follows

$$\pi_A(x) = 1 - \mu_A(x) - v_A(x). \tag{4}$$

It is seen immediately that $\pi_A(x) \in [0,1]$ $\forall x \in X$. If $A \in FS(X)$ then $\pi_A(x) = 0$ $\forall x \in X$.

Now let us briefly recall some notation used in the interval-valued fuzzy set theory (see [25]). Let $[I]=[a^-,a^+]$: $a^- \leqslant a^+, a^-, a^+ \in [0,1]$. Then a mapping $A$: $X \to [I]$ is called an interval-valued fuzzy set on $X$.

Let $IVF(X)$ denote all interval-valued fuzzy sets on $X$. For each $A \in IVF(X)$ let $A(x)=[A^-(x), A^+(x)]$, where $A^-(x) \leqslant A^+(x)$ and $x \in X$. Then fuzzy sets $A^-: X \to [I]$ and $A^+: X \to [I]$ are called a lower fuzzy set of $A$ and a upper fuzzy sets of $A$, respectively.

We have to stress that our ideas developed in this paper might be expressed equivalently using either intuitionistic fuzzy set or interval-value fuzzy set notation. However, since our motivations go back to intuitionistic fuzzy sets, we use them in the presentation of the paper.

## 3. Measuring distances

In many theoretical and practical problems we want to express numerically the difference between two objects (notions, etc.) by means of a distance between corresponding fuzzy sets. From here we will assume that the universe of discourse under study is finite, i.e. $X=\{x_1,\ldots,x_n\}$. One can define and use different metrics in a family of fuzzy subsets of given universe of discourse $X$. Especially, the Hamming metric and the Euclidean metric are most often used. For any two fuzzy subsets $A$ and $B$ of $X$ with membership functions $\mu_A$ and $\mu_B$, respectively, we have (see [15]):

- the Hamming distance $d(A,B)$

$$d(A,B)=\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|, \tag{5}$$

- the normalized Hamming distance $l(A,B)$

$$l(A,B)=\frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|, \tag{6}$$

- the Euclidean distance $e(A,B)$

$$e(A,B)=\sqrt{\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2}, \tag{7}$$

- the normalized Euclidean distance $q(A,B)$

$$q(A,B)=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2}. \tag{8}$$

These formulas are straightforward generalizations of distances used in classical set theory obtained by replacing the characteristic functions of two sets with the membership functions. Atanassov [5] suggested a direct generalization of distances (5)–(8) for intuitionistic fuzzy sets. For two

$A,B \in IFS(X)$ we get:

- the Hamming distance $d'(A,B)$

$$d'(A,B) = \frac{1}{2}\sum_{i=1}^{n}\left[|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|\right], \tag{9}$$

- the normalized Hamming distance $l'(A,B)$

$$l'(A,B) = \frac{1}{2n}\sum_{i=1}^{n}\left[|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|\right], \tag{10}$$

- the Euclidean distance $e'(A,B)$

$$e'(A,B) = \sqrt{\frac{1}{2}\sum_{i=1}^{n}\left[(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2\right]}, \tag{11}$$

- the normalized Euclidean distance $q'(A,B)$

$$q'(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2\right]}. \tag{12}$$

Then starting from the geometrical interpretation of intuitionistic fuzzy sets Szmidt and Kacprzyk [20,21] modified these distances. They proposed to take into account the three parameter characterization of intuitionistic fuzzy sets: the degree of membership $\mu(x)$, the degree of nonmembership $\nu(x)$ and the intuitionistic fuzzy index $\pi(x)$. Here are the definitions of the distances given by Szmidt and Kacprzyk:

- the Hamming distance $d''(A,B)$

$$d''(A,B) = \frac{1}{2}\sum_{i=1}^{n}\left[|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|\right], \tag{13}$$

- the normalized Hamming distance $l''(A,B)$

$$l''(A,B) = \frac{1}{2n}\sum_{i=1}^{n}\left[|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|\right], \tag{14}$$

- the Euclidean distance $e''(A,B)$

$$e''(A,B) = \sqrt{\frac{1}{2}\sum_{i=1}^{n}\left[(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2\right]} \tag{15}$$

- the normalized Euclidean distance $q''(A,B)$

$$q''(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[(\mu_A(x_i)-\mu_B(x_i))^2+(\nu_A(x_i)-\nu_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2\right]}. \quad (16)$$

Szmidt and Kacprzyk claim that their approach ensures that the distances for fuzzy sets and intuitionistic fuzzy sets can be easily compared since it reflects distances in three dimensional space, while distances due to Atanassov are the orthogonal projections of the real distances. Nevertheless this reasoning seems to be somehow strange and nonconvincing since their modification reduces to add the parameter that is a linear combination of two other parameters used in the Atanassov definitions. Also examples shown in papers [20,21] do not motivate the conclusion that their distances are better. Of course, we can consider their distances just as the Atanassov distances but we see no reasons sufficient to assert that these or those are better.

## 4. Distances based on the Hausdorff metric

Beside distances described in the previous section, some distances based on the Hausdorff metric are also used in the fuzzy sets theory. For any two subsets $U$ and $W$ of a Banach space $Z$ the Hausdorff metric is

$$d_H(U,W) = \max\left\{\sup_{u\in U}\inf_{w\in W}|u-w|, \sup_{w\in W}\inf_{u\in U}|u-w|\right\}. \quad (17)$$

If $Z=\mathbb{R}$ and $U=[u_1,u_2]$ and $W=[w_1,w_2]$ are intervals then (17) reduces to

$$d_H(U,V) = \max\{|u_1-w_1|, |u_2-w_2|\}. \quad (18)$$

It seems that this metric applied for intervals could be successfully used in the case of intuitionistic fuzzy sets too. Let us start from the following example connected with a decision making problem.

**Example 1.** Suppose that $X$ is a set of $n$ experts who vote for/against given decision (voting for/against given localization of the motorway, voting for/against given candidate or his opponent, etc.). Every expert $x_i$ is for—to the extent $0\leqslant\mu(x_i)\leqslant 1$ and is against—to the extent $0\leqslant v(x_i)\leqslant 1-\mu(x_i)$. Unless $v(x_i)=1-\mu(x_i)$ he is hesitated to the extent $\pi(x_i)=1-\mu(x_i)-v(x_i)$. Thus the situation at the beginning of negotiations might be described both by the intuitionistic fuzzy set of the form (2), i.e. $\{\langle x_i,\mu_A(x_i),v_A(x_i)\rangle\colon\ x_i\in X\}$, as well as by a set of ordered pairs $\{\langle x_i,[\mu_A(x_i),\mu_A(x_i)+\pi_A(x_i)]\rangle\colon\ x_i\in X\}$, where interval $[\mu_A(x_i),\mu_A(x_i)+\pi_A(x_i)]=[\mu_A(x_i),1-v_A(x_i)]$ shows all possible outcomes of the negotiations. For example, assume the expert $x_i$ at the beginning of negotiations is for to the extent $\mu(x_i)=0.4$ and against to the extent $v(x_i)=0.3$. It means he hesitates to the extent $\pi(x_i)=1-0.4-0.3=0.3$. If we persuade that he should vote for, the best result we can achieve is $\mu_{\text{final}}(x_i)=\mu(x_i)+\pi(x_i)=0.7$ with $v_{\text{final}}(x_i)=0.3$. On the other hand, the best result the opponents can achieve is $v_{\text{final}}(x_i)=v(x_i)+\pi(x_i)=0.6$ with $\mu_{\text{final}}(x_i)=0.4$. It may also happen that neither our nor the opponents' arguments would be strong enough to achieve the best result and then $\mu_{\mathrm{final}}(x_i)$ could assume any number from the interval $[0.4, 0.7]$.

As it was shown in the example there is one-to-one correspondence between definition (2) and the intuitionistic fuzzy set description based on intervals of possible values that the degree of membership and degree of nonmembership of each element could assume if the hesitation were removed. And hence, one can look on the distance between two intuitionistic fuzzy sets $A$ and $B$ in a degenerated universe of discourse $X = \{x\}$, where $A = \{\langle x, \mu_A(x), v_A(x)\rangle\}$ and $B = \{\langle x, \mu_B(x), v_B(x)\rangle\}$, respectively, as on the distance between corresponding intervals $[\mu_A(x), 1 - v_A(x)]$ and $[\mu_B(x), 1 - v_B(x)]$. Using the Hausdorff metric (18) we get

$$\begin{aligned} d(A,B) &= \max\{|\mu_A(x) - \mu_B(x)|, |1 - v_A(x) - (1 - v_B(x))|\} \\ &= \max\{|\mu_A(x) - \mu_B(x)|, |v_A(x) - v_B(x)|\}. \end{aligned} \tag{19}$$

**Example 2.** Let us consider following intuitionistic fuzzy sets $A, B, D, G, E \in X = \{x\}$:

$$A = \{\langle x, 1, 0\rangle\}, \quad B = \{\langle x, 0, 1\rangle\}, \quad D = \{\langle x, 0, 0\rangle\},$$

$$G = \left\{\left\langle x, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle\right\}, \quad E = \left\{\left\langle x, \tfrac{1}{4}, \tfrac{1}{4}\right\rangle\right\}.$$

Going back to the decision making problem discussed above we can say that expert $A$ is completely convinced to vote for, expert $B$ is completely convinced to vote against, expert $D$ is absolutely hesitant (i.e. undecided whether vote for or against). According to (19) we get

$$\begin{aligned} d(A,B) &= \max\{|1-0|, |0-1|\} = 1, \\ d(A,D) &= \max\{|1-0|, |0-0|\} = 1, \\ d(B,D) &= \max\{|0-0|, |1-0|\} = 1, \\ d(A,G) &= \max\left\{\left|1-\tfrac{1}{2}\right|, \left|0-\tfrac{1}{2}\right|\right\} = \tfrac{1}{2}, \\ d(A,E) &= \max\left\{\left|1-\tfrac{1}{4}\right|, \left|0-\tfrac{1}{4}\right|\right\} = \tfrac{3}{4}, \\ d(B,G) &= \max\left\{\left|0-\tfrac{1}{2}\right|, \left|1-\tfrac{1}{2}\right|\right\} = \tfrac{1}{2}, \\ d(B,E) &= \max\left\{\left|0-\tfrac{1}{4}\right|, \left|1-\tfrac{1}{4}\right|\right\} = \tfrac{3}{4}, \\ d(D,G) &= \max\left\{\left|0-\tfrac{1}{2}\right|, \left|0-\tfrac{1}{2}\right|\right\} = \tfrac{1}{2}, \\ d(D,E) &= \max\left\{\left|0-\tfrac{1}{4}\right|, \left|0-\tfrac{1}{4}\right|\right\} = \tfrac{1}{4}, \\ d(G,E) &= \max\left\{\left|\tfrac{1}{2}-\tfrac{1}{4}\right|, \left|\tfrac{1}{2}-\tfrac{1}{4}\right|\right\} = \tfrac{1}{4}. \end{aligned}$$

Now we are able to suggest how to measure the distance between fuzzy sets on arbitrary finite universe of discourse utilizing the Hausdorff metric. Our definitions are natural counterparts of the Hamming distance, the Euclidean distance and their normalized versions.

**Definition 1.** For any two intuitionistic fuzzy subsets $A = \{\langle x_i, \mu_A(x_i), v_A(x_i)\rangle : x_i \in X\}$ and $B = \{\langle x_i, \mu_B(x_i), v_B(x_i)\rangle : x_i \in X\}$ of the universe of discourse $X = \{x_1, \ldots, x_n\}$, we have:

- the Hamming distance $d_h(A, B)$

$$d_h(A,B) = \sum_{i=1}^{n} \max\{|\mu_A(x_i) - \mu_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}, \tag{20}$$

- the normalized Hamming distance $l_h(A, B)$

$$l_h(A,B) = \frac{1}{n} \sum_{i=1}^{n} \max\{|\mu_A(x_i) - \mu_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}, \tag{21}$$

- the Euclidean distance $e_h(A, B)$

$$e_h(A,B) = \sqrt{\sum_{i=1}^{n} \max\{(\mu_A(x_i) - \mu_B(x_i)^2), (v_A(x_i) - v_B(x_i)^2)\}}, \tag{22}$$

- the normalized Euclidean distance $q_h(A, B)$

$$q_h(A,B) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \max\{(\mu_A(x_i) - \mu_B(x_i)^2), (v_A(x_i) - v_B(x_i)^2)\}}. \tag{23}$$

Here are some lemmas on elementary properties of these concepts.

**Lemma 2.** *Let $X$ denote a finite universe of discourse. Then functions $d_h, l_h, e_h, q_h : IFS(X) \to \mathbb{R}^+ \cup \{0\}$ given by* (20)–(23), *respectively, are metrics.*

**Proof.** We give the proof only for $d_h$; the other cases are left to the reader.

For any two intuitionistic fuzzy subsets $A$ and $B$ of the finite universe of discourse $X = \{x_1, \ldots, x_n\}$, $d_h(A, B)$ given by (20) is positive definite, i.e. $d_h(A, B) \geqslant 0$, because of the absolute value properties.

If $A = B$ then $\mu_A(x_i) = \mu_B(x_i)$ and $v_A(x_i) = v_B(x_i)$ for each $x_i \in X$ and hence $d_h(A, B) = 0$. Conversely, if $d_h(A, B) = 0$ then for each $x_i \in X$ we have $\max\{|\mu_A(x_i) - \mu_B(x_i)|, |v_A(x_i) - v_B(x_i)|\} = 0$. Thus both $\mu_A(x_i) - \mu_B(x_i) = 0$ and $v_A(x_i) - v_B(x_i) = 0$ and we conclude that $A = B$.

The symmetry property $d_h(A, B) = d_h(B, A)$ holds because $|\mu_A(x_i) - \mu_B(x_i)| = |\mu_B(x_i) - \mu_A(x_i)|$ and $|v_A(x_i) - v_B(x_i)| = |v_B(x_i) - v_A(x_i)|$ for each $x_i \in X$.

To prove the triangle inequality we have to show that for any $A, B, C \in IFS(X)$ with membership functions $\mu_A, \mu_B, \mu_C$ and nonmembership functions $v_A, v_B, v_C$, respectively, a following inequality holds:

$$\sum_{i=1}^{n} \max\{|\mu_A(x_i) - \mu_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}$$

$$+\sum_{i=1}^{n}\max\{|\mu_B(x_i)-\mu_C(x_i)|,|v_B(x_i)-v_C(x_i)|\}$$

$$\geqslant \sum_{i=1}^{n}\max\{|\mu_A(x_i)-\mu_C(x_i)|,|v_A(x_i)-v_C(x_i)|\}.$$

We get it immediately since for any nonnegative numbers $a_1,a_2,a_3,b_1,b_2,b_3$ such that $a_1+a_2\geqslant a_3$ and $b_1+b_2\geqslant b_3$ we have $\max\{a_1,b_1\}+\max\{a_2,b_2\}\geqslant \max\{a_3,b_3\}$. This finishes the proof. □

**Lemma 3.** *For any two intuitionistic fuzzy subsets* $A=\{\langle x_i,\mu_A(x_i),v_A(x_i)\rangle\colon\ x_i\in X\}$ *and* $B=\{\langle x_i,\mu_B(x_i),v_B(x_i)\rangle\colon\ x_i\in X\}$ *of the universe of discourse* $X=\{x_1,\dots,x_n\}$, *the following inequalities hold*:

$$d_h(A,B)\ \leqslant\ n, \tag{24}$$

$$l_h(A,B)\ \leqslant\ 1, \tag{25}$$

$$e_h(A,B)\ \leqslant\ \sqrt{n}, \tag{26}$$

$$q_h(A,B)\ \leqslant\ 1. \tag{27}$$

**Proof.** By (3) we have $|\mu_A(x_i)-\mu_B(x_i)|\leqslant 1$ and $|v_A(x_i)-v_B(x_i)|\leqslant 1$ for each $x_i\in X$, and hence $d_h(A,B)\leqslant \sum_{i=1}^{n}1=n$, $l_h(A,B)\leqslant 1/n\sum_{i=1}^{n}1=1$, $e_h(A,B)\leqslant\sqrt{\sum_{i=1}^{n}1}=\sqrt{n}$ and $q_h(A,B)\leqslant\sqrt{1/n\sum_{i=1}^{n}1}=1$. □

**Lemma 4.** *For any two intuitionistic fuzzy subsets* $A=\{\langle x_i,\mu_A(x_i),v_A(x_i)\rangle\colon\ x_i\in X\}$ *and* $B=\{\langle x_i,\mu_B(x_i),v_B(x_i)\rangle\colon\ x_i\in X\}$ *of the universe of discourse* $X=\{x_1,\dots,x_n\}$, *the following inequalities are valid*:

$$d'(A,B)\ \leqslant\ d_h(A,B)\ \leqslant\ d''(A,B), \tag{28}$$

$$l'(A,B)\ \leqslant\ l_h(A,B)\ \leqslant\ l''(A,B), \tag{29}$$

$$e'(A,B)\ \leqslant\ e_h(A,B)\ \leqslant\ e''(A,B), \tag{30}$$

$$q'(A,B)\ \leqslant\ q_h(A,B)\ \leqslant\ q''(A,B). \tag{31}$$

**Proof.** Now we give the proof only for $e_h$; the other cases are left to the reader.

Since for any two numbers $a$ and $b$ we have $\frac{1}{2}(a+b)\leqslant\max\{a,b\}$, hence immediately $e'(A,B)\leqslant e_h(A,B)$. Moreover, since $\pi_A(x_i)=1-\mu_A(x_i)-v_A(x_i)$ and $\pi_B(x_i)=1-\mu_B(x_i)-v_B(x_i)$ for each $x_i\in X$, hence

$$\begin{aligned}&\tfrac{1}{2}[(\mu_A(x_i)-\mu_B(x_i))^2+(v_A(x_i)-v_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2]\\&\quad=\tfrac{1}{2}[(\mu_A(x_i)-\mu_B(x_i))^2+(v_A(x_i)-v_B(x_i))^2\\&\qquad+(\mu_B(x_i)-\mu_A(x_i))+(v_B(x_i)-v_A(x_i))^2]\end{aligned}$$

$$= \tfrac{1}{2}[(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\mu_B(x_i) - \mu_A(x_i))^2$$
$$+ 2(\mu_B(x_i) - \mu_A(x_i))(\nu_B(x_i) - \nu_A(x_i)) + (\nu_B(x_i) - \nu_A(x_i))^2]$$
$$= (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\mu_B(x_i) - \mu_A(x_i))(\nu_B(x_i) - \nu_A(x_i))$$
$$\geqslant \max\{(\mu_A(x_i) - \mu_B(x_i)^2), (v_A(x_i) - v_B(x_i)^2)\}$$

for each $x_i \in X$. It follows easily that $e''(A,B) \geqslant e_h(A,B)$ which completes the proof. □

When generalizing any notion it is desirable that the new object should be consistent with the primary one and it should reduce to that primary one in some particular cases. As it was mentioned above each fuzzy set is an intuitionistic fuzzy set with the intuitionistic fuzzy index equal to zero. Thus it would be desirable that our definitions (20)–(23) should reduce to the Hamming distance (5), the Euclidean distance (6) and their normalized versions (7) and (8) for ordinary fuzzy sets. One can check easily that

**Lemma 5.** *For any two intuitionistic fuzzy subsets* $A, B \in X = \{x_1, \ldots, x_n\}$ *such that* $A = \{\langle x_i, \mu_A(x_i), 1 - \mu_A(x_i)\rangle\colon\ x_i \in X\}$ *and* $B = \{\langle x_i, \mu_B(x_i), 1 - \mu_B(x_i)\rangle\colon\ x_i \in X\}$, *respectively, following equalities hold*:

$$d_h(A,B) = d(A,B), \tag{32}$$

$$l_h(A,B) = l(A,B), \tag{33}$$

$$e_h(A,B) = e(A,B), \tag{34}$$

$$q_h(A,B) = q(A,B). \tag{35}$$

**Remark.** As we have mentioned in Section 2, one can also express the suggested distances using interval-valued fuzzy set notation. Thus, for example, if $A(x) = [A^-(x), A^+(x)]$ and $B(x) = [B^-(x), B^+(x)]$ are two interval-valued fuzzy sets in $X$ then the Hamming distance between them is given by

$$d_h(A,B) = \sum_{i=1}^{n} \max\{|A^-(x_i) - B^-(x_i)|, |A^+(x_i) - B^+(x_i)|\}. \tag{36}$$

Similarly, we can write down counterparts of (21)–(23).

## 5. Conclusions

In the present paper we have suggested new definitions of distances between intuitionistic fuzzy sets that are counterparts of the well known Hamming distance and Euclidean distance. Our definitions are based on the Hausdorff metric seems to be natural and easy for applications.

In the paper we have considered distances between intuitionistic fuzzy sets on finite universe of discourses only which are usually used in a computing environment. However, our method of utilizing the Hausdorff metric in the distance measure construction could be also used in a more general case, i.e. for larger universe of discourses. Moreover, it seems that there are some connections between the metric proposed in this paper and distances based on the Hausdorff metric considered by Diamond and Kloeden [8] for usual fuzzy sets. However, this would be a topic of our future work.

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