Local Volatility Model

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1 Setup

We assume the following asset process:

$$dS_t = \sigma(S_t, t, \cdots) S_t dW_t \ . \tag{1}$$

Here, $\sigma(S_t, t, \cdots)$ is an adapted function that can have other adapted processes as arguments in addition to the asset price S_t. Call option and it defivatives in terms of the strike are

$$\begin{aligned} C(K) &= E\left[(S_T - K)^+\right] \ , &(2)\\ \frac{\partial C(K)}{\partial K} &= -E\left[H\left(S_T - K\right)\right] \ , &(3)\\ \frac{\partial^2 C(K)}{\partial K^2} &= E\left[\delta\left(S_T - K\right)\right] \ . &(4)\end{aligned}$$

2 Derivations of the Dupire equation

2.1 Simple Derivation

Derman and Kani gave a simple derivation [1].

2.2 Sophisticated Derivation by using SDE

This derivation was given by Gatheral [2]. The stochastic difference of call is

$$\begin{aligned} dC &= dE\left[(S_T - K)^+\right] &(5)\\ &= E\left[d\left(S_T - K\right)^+\right]. &(6)\end{aligned}$$

$$\begin{aligned} f(S_T) &= (S_T - K)^+ = (S_T - K)H(S_T - K) \ , &(7)\\ f'(S_T) &= H(S_T - K) + \underbrace{(S_T - K)\delta(S_T - K)}_{=0} \ , &(8)\\ f''(S_T) &= \delta(S_T - K)^+ \ , &(9)\end{aligned}$$

$$\begin{aligned} d\left(S_T - K\right)^+ &= f'(S_T)dS_T + \frac{1}{2}f''(S_T)dS_T^2 &(10)\\ &= H(S_T - K)\sigma(S_T, T, \cdots)S_T dW_t + \frac{1}{2}\delta(S_T - K)\sigma^2(S_T, T, \cdots)S_T^2 dt. &(11)\end{aligned}$$

$$\begin{aligned} E\left[d\left(S_T - K\right)^+\right] &= \frac{1}{2}E\left[\delta(S_T - K)\sigma^2(S_T, T, \cdots)S_T^2\right] dT &(12)\\ &= \frac{1}{2}E\left[\delta(S_T - K)\right] \cdot E\left[\sigma^2(S_T, T, \cdots)|S_T = K\right] K^2 dT. &(13)\end{aligned}$$

We get

$$\frac{dC}{dT} = \frac{1}{2}\frac{\partial^2 C}{\partial K^2} E\left[\sigma^2(S_T, T, \cdots)|S_T = K\right] K^2. \tag{14}$$

2.3 Using Fokker-Planck equation

Using Fokker-Planck equation.

References

[1] Emanuel Derman and Iraj Kani, *Riding on a smile*, Risk **7** (1994), no. 2, 32–39.

[2] Jim Gatheral, *The volatility surface: a practitioner's guide*, vol. 357, John Wiley & Sons, 2011.