

# Local Volatility Model

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## 1 Setup

We assume the following asset process:

$$dS_t = \sigma(S_t, t, \dots) S_t dW_t . \quad (1)$$

Here,  $\sigma(S_t, t, \dots)$  is an adapted function that can have other adapted processes as arguments in addition to the asset price  $S_t$ . Call option and its derivatives in terms of the strike are

$$C(K) = E \left[ (S_T - K)^+ \right] , \quad (2)$$

$$\frac{\partial C(K)}{\partial K} = -E \left[ H(S_T - K) \right] , \quad (3)$$

$$\frac{\partial^2 C(K)}{\partial K^2} = E \left[ \delta(S_T - K) \right] . \quad (4)$$

## 2 Derivations of the Dupire equation

### 2.1 The original derivation by using Fokker-Planck equation

Dupire [3, 2] and Derman and Kani [1] gave a simple derivation.

### 2.2 Sophisticated Derivation by using SDE

This derivation was given by Gatheral [4]. The stochastic difference of call is

$$dC = dE \left[ (S_T - K)^+ \right] \quad (5)$$

$$= E \left[ d(S_T - K)^+ \right] . \quad (6)$$

$$f(S_T) = (S_T - K)^+ = (S_T - K)H(S_T - K) , \quad (7)$$

$$f'(S_T) = H(S_T - K) + \underbrace{(S_T - K)\delta(S_T - K)}_{=0} , \quad (8)$$

$$f''(S_T) = \delta(S_T - K)^+ , \quad (9)$$

$$d(S_T - K)^+ = f'(S_T)dS_T + \frac{1}{2}f''(S_T)dS_T^2 \quad (10)$$

$$= H(S_T - K)\sigma(S_T, T, \dots)S_T dW_t + \frac{1}{2}\delta(S_T - K)\sigma^2(S_T, T, \dots)S_T^2 dt. \quad (11)$$

$$E \left[ d(S_T - K)^+ \right] = \frac{1}{2}E \left[ \delta(S_T - K)\sigma^2(S_T, T, \dots)S_T^2 \right] dT \quad (12)$$

$$= \frac{1}{2}E \left[ \delta(S_T - K) \right] \cdot E \left[ \sigma^2(S_T, T, \dots) | S_T = K \right] K^2 dT. \quad (13)$$

We get

$$\frac{dC}{dT} = \frac{1}{2} \frac{\partial^2 C}{\partial K^2} E \left[ \sigma^2(S_T, T, \dots) | S_T = K \right] K^2. \quad (14)$$

## A Fokker-Planck equation

We derive the Fokker-Planck equation.

## References

- [1] Emanuel Derman and Iraj Kani, *Riding on a smile*, Risk **7** (1994), no. 2, 32–39.
- [2] Bruno Dupire, *Pricing and hedging with smiles*, Mathematics of derivative securities **1** (1997), no. 1, 103–111.
- [3] Bruno Dupire et al., *Pricing with a smile*, Risk **7** (1994), no. 1, 18–20.
- [4] Jim Gatheral, *The volatility surface: a practitioner's guide*, vol. 357, John Wiley & Sons, 2011.