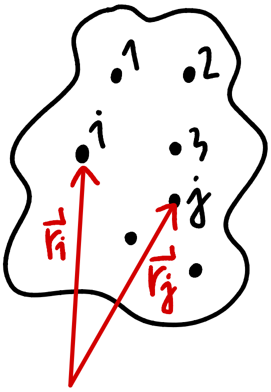
Togo Telo



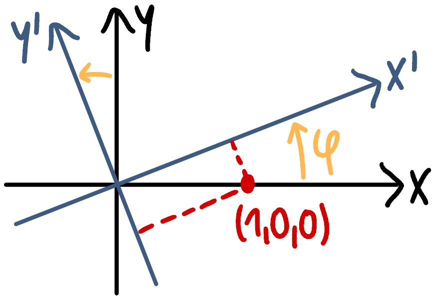
N točk

Pogoj za togo telo: $|\vec r_i - \vec r_j| = c_{ij}$

Imamo 6 prostostnih stopenj:
3 za lego 1 točke v prostoru (npr. težišča),
ostanejo 3 (npr. Eulerjevi koti)

Rotacija koordinatnega sistema

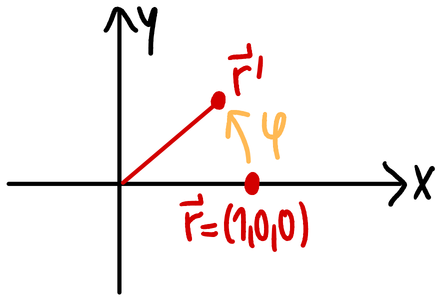
a) Pasivna rotacija: Točke ne vrtiš, vrtiš le koordinatni sistem



$$\begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_3(\varphi)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec r\,' = R_3(\varphi)\vec r$$

b) Aktivna rotacija: Zavrtiš točko, ne pa koordinatnega sistema



$$\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde R_3(\varphi)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{\vec r}\,' = \tilde R_3(\varphi)\vec r$$

Rotacije so ortogonalne transformacije:

$$\vec a\,'\cdot\vec b\,' = R\vec a\cdot R\vec b = \vec a\cdot \underbrace{R^T R}_{I}\,\vec b = \vec a\cdot\vec b$$

$$R^T R = R R^T = I$$

$$R^T = R^{-1}$$

$$R = R_2 R_1 \Rightarrow R^{-1} = R_1^{-1} R_2^{-1}$$

$$R R^{-1} = R^{-1} R = I$$

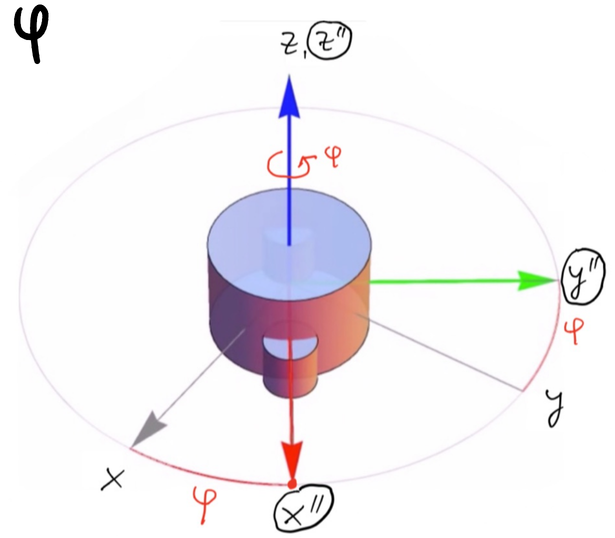
Lastnosti:

$$\det(A\cdot B) = \det A\cdot\det B$$
$$\det A^T = \det A$$
$$\det A^T A = \det A^T\cdot\det A$$

če je $A^T = A^{-1} \Rightarrow 1 = \det A^T A = \det A^T\cdot\det A \Rightarrow \det A^T = \dfrac{1}{\det A} \Rightarrow \det A = \pm 1$

Eulerjevi koti

φ



Rotacija okoli osi z za kot φ

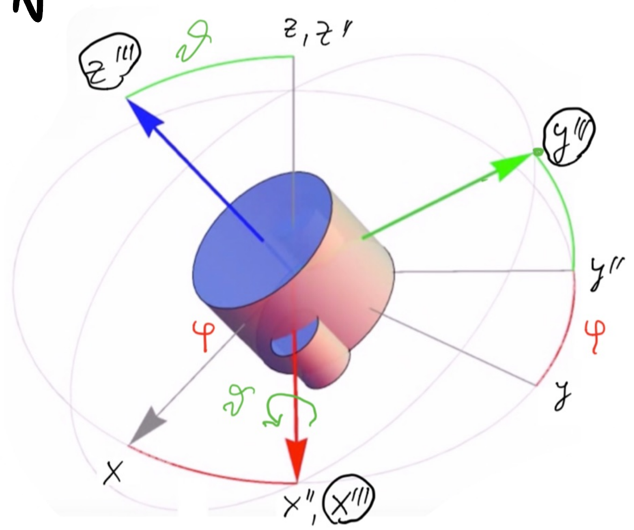
$\vec{r} \rightarrow \vec{r}\,''$: precesija

$$T(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T(\varphi) = R_3(\varphi) = T^{-1}(-\varphi)$$

$$T^{-1} = T^T$$

ϑ



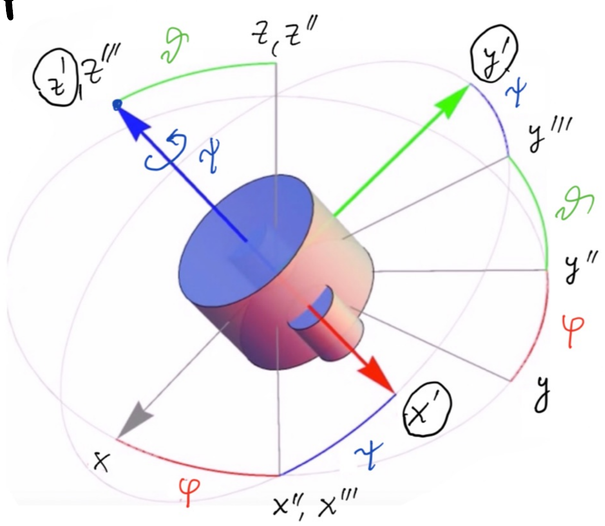
Rotacija okoli osi x'' za kot ϑ

$\vec{r}\,'' \rightarrow \vec{r}\,'''$: nutacija

$$U(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$U(\vartheta) = R_1(\vartheta)\,;\; U^{-1} = U^T$$

ψ



Rotacija okoli osi z''' za kot ψ

$\vec{r}\,''' \rightarrow \vec{r}\,'$: rotacija

$$V(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$V(\psi) = R_3(\psi)\,;\; V^{-1} = V^T$$

skupaj: $\vec{r} \rightarrow \vec{r}\,'$ $\quad \boxed{R = V(\psi)\cdot U(\vartheta)\cdot T(\varphi)}$ $\;;\; R^T = T^T U^T V^T = R^{-1}$

Eulerjev izrek o rotacijah (1775)

Naj bo R rotacijska matrika, ki transformira $\vec{r} \to \vec{r}\,' : R\vec{r} = \vec{r}\,'$.
Za $\forall R \ \exists \vec{n} \neq 0 \ni : R\vec{n} = \vec{n}$, torej obstaja lastni vektor z lastno vrednostjo 1.

↳ To pomeni, da so 3 zaporedne rotacije v resnici samo ena rotacija.

Dokaz: $R^{-1} = R^T : RR^T = R^T R = I$

$\det(RR^T) = \det R \cdot \det R^T = (\det R)^2 = 1$

$\Rightarrow \det R = \pm 1$ → rotacija (+), zrcaljenje (−)

$R\vec{n} = \vec{n} = I\vec{n}$

$(R - I)\vec{n} = 0$ → problem lastnih vrednosti

$\det(R - I) = 0$

Spomnimo:
$\det(-A) = (-1)^3 \det A$
$\det(A^{-1}) = (\det A)^{-1}$

$$\det(R-I) = \det(R-I)^T = \det(R^T - I) = \det(R^{-1} - R^{-1}R) = \det[R^{-1}(I-R)] =$$
$$= \underbrace{\det R^{-1}}_{1} \cdot \det(I-R) = -\det(R-I) = 0$$

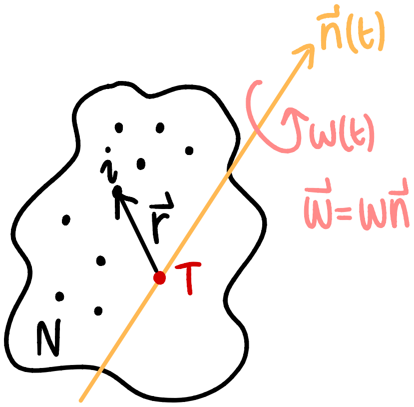
$\Rightarrow \exists \vec{n} : R\vec{n} = \vec{n} = \lambda \vec{n} \ ; \ \lambda = 1$ $\square$

$\vec{n}$ je os vrtenja

$$\det R = 1 = \overset{1}{\lambda_1} \cdot \lambda_2 \cdot \lambda_3 \Rightarrow \lambda_2 \cdot \lambda_3 = 1 \Rightarrow \lambda_{2,3} = e^{\pm i\varphi}, \varphi \in \mathbb{R}$$

Obstaja natanko 1 os vrtenja

Togo telo z eno nepremično točko (prvič)



Telo se vrti okoli osi $\vec{n}$, ki gre skozi težišče.
Koordinatni sistem postavimo v težišče.

$$\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)_{nein.} + \vec{\omega} \times \vec{r}$$

$$\vec{v}(t) = \vec{\omega} \times \vec{r}$$

Vrtilna količina telesa:

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i = \sum_{i=1}^{N} m_i(\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{N} m_i(\vec{r}_i \times (\vec{\omega}_i \times \vec{r}_i)) = \underline{\underline{J}}\vec{\omega}$$

↳ Tenzor vztrajnostnega momenta

$$\vec{L} = \sum_{i=1}^{N} m_i[(\vec{r}_i \cdot \vec{r}_i)\vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\vec{r}_i] = \underline{\underline{J}}\vec{\omega}$$

Definiramo gostoto: $\rho(r) = \lim_{\Delta V \to 0} \frac{\Delta m_i}{\Delta V_i} \longrightarrow \frac{dm}{dV} \Rightarrow M = \sum_{i=1}^{N} m_i \longrightarrow \int \rho \, d^3\vec{r}$

Izberemo si koordinatni sistem: $\vec{\omega}^T = (\omega_1, \omega_2, \omega_3)$

Sledi:
$$\boxed{\underline{\underline{J}} = \int \rho(\vec{r}) \cdot \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} d^3 r}$$
Tenzor vztrajnostnega momenta

$$J_{\alpha\beta} = \int \rho(\vec{r}) [r^2 \delta_{\alpha\beta} - x_\alpha x_\beta] \, d^3 r$$

$\vec{L}' = R\vec{L}$, $\vec{\omega}' = R\vec{\omega}$ ... ista količina, zapisana v novih koordinatah

$$\vec{L}' = R\vec{L} = R\underline{\underline{J}}\vec{\omega} = R\underline{\underline{J}}R^{-1}\underbrace{R\vec{\omega}} = \underbrace{R J R^{-1}}_{\underline{\underline{J}}'} \vec{\omega}'$$
$\rightarrow$ Če se matrika tako obnaša, ji pravimo tenzor

Energija telesa:

$$\boxed{T = \frac{1}{2}\sum_{i=1}^{N} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2}\sum_{i=1}^{N} m_i (\vec{\omega} \times \vec{r}_i) \cdot \vec{v}_i = \frac{1}{2}\sum_{i=1}^{N} m_i \vec{\omega} (\vec{r}_i \times \vec{v}_i) = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}\vec{\omega}\underline{\underline{J}}\vec{\omega}}$$

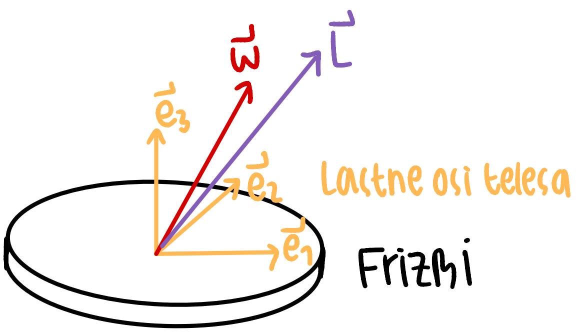
Za osi izberemo lastne osi telesa = lastni sistem:

$$\underline{\underline{J}} = \begin{pmatrix} J_1 & & \emptyset \\ & J_2 & \\ \emptyset & & J_3 \end{pmatrix}, \quad \vec{\omega} = \sum_{\alpha=1}^{3} \omega_\alpha \vec{e}_\alpha$$

$$\underline{\underline{J}}\vec{\omega} = \sum_{\alpha=1}^{3} J_{\alpha\alpha} \omega_\alpha \vec{e}_\alpha$$

$$\boxed{T = \frac{1}{2}\sum_{\alpha=1}^{3} J_{\alpha\alpha} \omega_\alpha^2}$$

Enačbe gibanja:



$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{L}}{dt}\right)_{nein.} + \vec{\omega} \times \vec{L} = \vec{M}$$

$$\vec{L} = \underline{\underline{J}}\vec{\omega} = \sum_{\alpha=1}^{3} J_{\alpha\alpha} \omega_\alpha \vec{e}_\alpha = \begin{pmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{pmatrix}$$

$\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ v lastnem sist.

$$\left(\frac{d\vec{L}}{dt}\right)_{nein.} = \sum_{\alpha=1}^{3} J_{\alpha\alpha} \dot{\omega}_\alpha \vec{e}_\alpha$$

$$\vec{\omega} \times \vec{L} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} J_1\omega_1 \\ J_2\omega_2 \\ J_3\omega_3 \end{bmatrix} = \begin{bmatrix} (J_3 - J_2)\omega_2\omega_3 \\ (J_1 - J_3)\omega_3\omega_1 \\ (J_2 - J_1)\omega_1\omega_2 \end{bmatrix}$$

To moramo sešteti, da dobimo navor

Sledi: $$\boxed{\begin{aligned} J_1\dot\omega_1-(J_2-J_3)\omega_2\omega_3&=M_1\\ J_2\dot\omega_2-(J_3-J_1)\omega_3\omega_1&=M_2\\ J_3\dot\omega_3-(J_1-J_2)\omega_1\omega_2&=M_3\end{aligned}}$$ Eulerjeve enačbe

Primeri:

Nevpeto telo (frizbi, ni upora in teže)

Najenostavnejši primer: $\vec M=0\rightarrow\vec L=\text{konst.}=\vec L_0$

$\vec\omega$ se vrti okoli $\vec L$, ki je konstanten, sam frizbi pa se tudi vrti. Več kot to ne gre, saj so enačbe zaradi nelinearnosti v splošnem nerešljive.

→ Rešitev bi bila pojav Dzanibekovega

a) Osno simetrično telo: $J_1=J_2$ (npr. frizbi)

(v primeru $J_1=J_2=J_3=J$ ležita $\vec\omega$ in $\vec L$ na isti nosilki in nimamo tresenja)

$$J_3\dot\omega_3=0\ \Rightarrow\ \omega_3=\text{konst.}=\omega_0$$
$$J_1\dot\omega_1-(J_1-J_3)\,\omega_2\omega_0=0$$
$$J_2\dot\omega_2-(J_3-J_1)\,\omega_1\omega_0=0$$

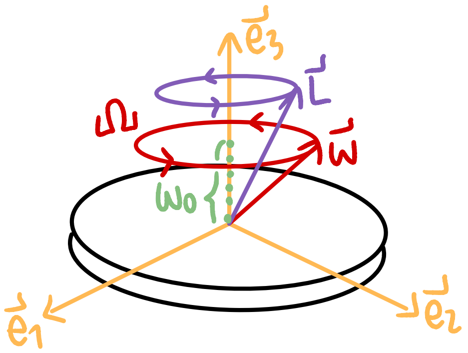
$$\dot\omega_1=\frac{J_1-J_3}{J_1}\omega_0\omega_2 \qquad \dot\omega_2=\underbrace{\frac{J_3-J_1}{J_1}}_{\Omega}\omega_0\omega_1 \quad\Rightarrow\quad \dot\omega_1=-\Omega\omega_2,\quad \dot\omega_2=\Omega\omega_1 \quad\Big| : \quad \frac{\dot\omega_1}{\dot\omega_2}=-\frac{\omega_2}{\omega_1}$$

vst.

$$\Rightarrow\ \ddot\omega_1=-\Omega^2\omega_1\ \Rightarrow\ \begin{aligned}\omega_1(t)&=A\cdot\cos(\Omega t+\delta)\\ \omega_2(t)&=A\cdot\sin(\Omega t+\delta)\end{aligned}\ ;\ \omega_1^2+\omega_2^2=A^2=\text{konst.}$$

odvisna od zač. pog.

Sledi: $$\boxed{\vec\omega(t)=\begin{bmatrix}\omega_1(t)\\ \omega_2(t)\\ \omega_0\end{bmatrix}}\ ;\ |\omega|^2=\omega_0^2+A^2$$

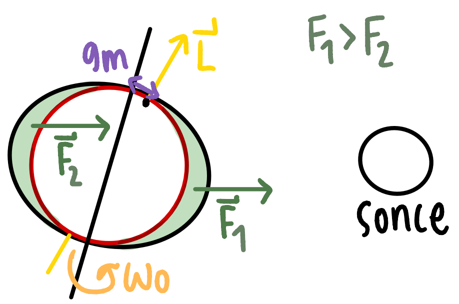


b) Feynman: $J_3=\frac{1}{2}MR^2$, $J_1=J_2=\frac{1}{4}MR^2$

$$\Omega=\frac{J_3-J_1}{J_1}\omega_0=\frac{\frac12-\frac14}{\frac14}\omega_0=\omega_0$$

↳ v lastnem sistemu, $2\omega_0$ v mirujočem sistemu

c) Tresenje Zemlje (wobbling)



Zemlja ima obliko geoida: $J_3\neq J_1=J_2$, zato vektor $\vec L$ ne kaže v isto smer kot $\vec\omega$

$$\frac{J_3-J_1}{J_1}=0.00327,\ \omega_0=\frac{2\pi}{\text{dan}}\ \Rightarrow\ \Omega=433\text{ dni}$$

Ker je $F_1 > F_2$ deluje Sonce z navorom na Zemljo tako, da skuša njeno os zavrteti pokonci. Zaradi tega Zemljina os precesira po stožcu s frekvenco ~25.000 let. Poleg tega se spreminja tudi ekscentričnost orbite in ravnina v kateri kroži. Vsi ti efekti so zbrani v Milankovićevih ciklih, ki opisujejo periodično spreminjanje Zemljinega podnebja in pojasnijo ledene dobe:

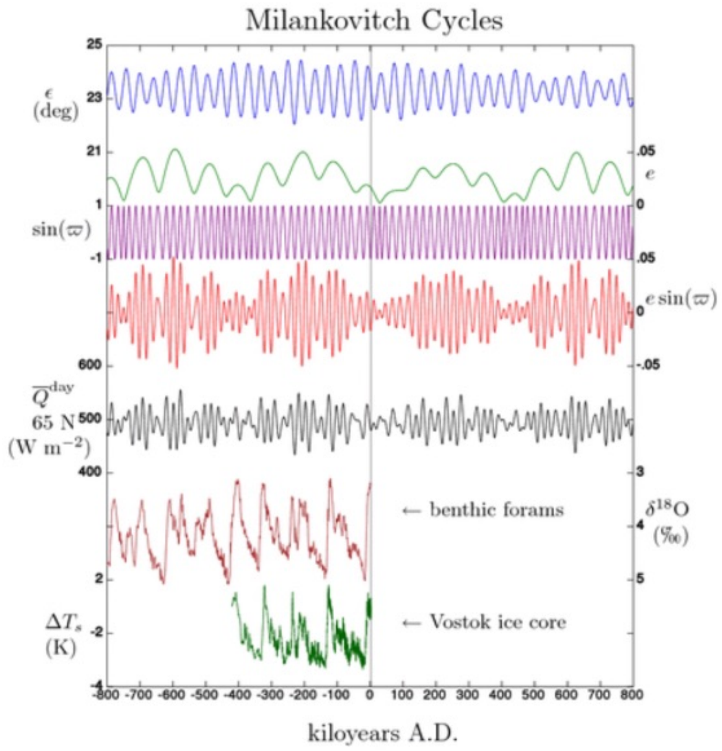
Milankovitch cycles

Article Talk

Milankovitch cycles describe the collective effects of changes in the Earth's movements on its climate over thousands of years. The term was coined and named after Serbian geophysicist and astronomer Milutin Milanković. In the 1920s, he hypothesized that variations in eccentricity, axial tilt, and precession combined to result in cyclical variations in the intra-annual and latitudinal distribution of solar radiation at the Earth's surface, and that this orbital forcing strongly influenced the Earth's climatic patterns.[citation needed]

Similar astronomical hypotheses had been advanced in the 19th century by Joseph Adhemar, James Croll, and others, but verification was difficult because there was no reliably dated evidence, and because it was unclear which periods were important.[citation needed]

In recent years, materials on Earth that have been unchanged for millennia (obtained via ice, rock, and deep ocean sediment cores) have been studied to indicate the history of Earth's climate. Although broadly consistent with the Milankovitch hypothesis, a number of specific observations are not explained by the hypothesis.[citation needed]



Past and future Milankovitch cycles via VSOP model
- Graphic shows variations in five orbital elements:
  - Axial tilt or obliquity (ε).
  - Eccentricity (*e*).
  - Longitude of perihelion (sin(ϖ)).
  - Precession index (*e* sin(ϖ))
- Precession index and obliquity control insolation at each latitude:
  - Daily-average insolation at top of atmosphere on summer solstice ($\overline{Q}^{\text{day}}$) at 65° N
- Ocean sediment and Antarctic ice strata record ancient sea levels and temperatures:
  - Benthic forams (57 widespread locations)
  - Vostok ice core (Antarctica)
- Vertical gray line shows present (2000 CE)