2.3.2021

RAZVOJ FUNKCIJE $E_1(x)$ (Integralska eksponentna funkcija)

$$E_1(x)=\int_x^\infty \frac{e^{-t}}{t}\,dt\,,\quad E_i=\int_{-\infty}^{x}\frac{e^{t}}{t}\,dt$$

← alternativni zapis

ASIMPTOTIČNA VRSTA: $x\sim\infty$

→ Ni nujno, da sploh kje konvergirajo

Asimptotika bo $\sim e^{-x}$ (ker v integralu e^t prevlada)

Najprej se moramo znebiti x v meji:

Če gre $x\to\infty \Rightarrow \frac{1}{x}\sim\emptyset$

$u=t-x$, $du=dt$:

$$E_1(x)=\int_0^\infty \frac{e^{-(u+x)}}{u+x}\,du = e^{-x}\int_0^\infty \frac{e^{-u}}{u+x}\,du =$$

← Tole zdaj spominja na Γ: $u! = \int_0^\infty x^n e^{-x}\,dx$

po tem integriramo

$$= \frac{e^{-x}}{x}\cdot\int_0^\infty \frac{e^{-u}}{1+\frac{x}{u}}\,du = \frac{e^{-x}}{x}\int_0^\infty \left(1-\frac{u}{x}+\frac{u^2}{x^2}-\frac{u^3}{x^3}+O(x^{-4})\right)e^{-u}\,du$$

→ x in u sta razklopljena

x je tu le parameter

majhno. Je res?

zaradi tega ne bo konvergence... Vedno bo $\exists$ nek $u>x$...

$$\Rightarrow \ldots = \frac{e^{-x}}{x}\left[0!-\frac{1!}{x}+\frac{2!}{x^2}-\frac{3!}{x^3}+\ldots\right]$$

Divergentna vrsta, ampak UPORABNA!

Drugi način: per partes originala, ponavljamo per partes, odrivamo nerešljiv integral naprej...

$$\int \frac{e^{-t}}{t}\,dt = \underbrace{-\frac{e^{-t}}{t}\Big|_x^\infty}_{=\frac{e^{-x}}{x}\text{ 1. člen}} - \int_x^\infty \frac{e^{-t}}{t^2}\,dt = \ldots$$

$\frac{1}{t}=u$, $du=-\frac{1}{t^2}dt$

$dv=e^{-t}dt$, $v=-e^{-t}$

Kodre 27/2

Debye: $C_V=\frac{3R}{M}D\left(\frac{T}{\Theta}\right)$, $D(x)=12x^3\int_0^{1/x}\frac{u^3\,du}{e^u-1}-\frac{3}{x(e^{1/x}-1)}$

ε, ko gre $x\to\infty$

Limiti: $x\to\infty$, $x\to 0$ } Potrebujemo oboje, da vemo, kako se funkcija obnaša

$\frac{1}{x}$ majhen, u ne preseže $\frac{1}{x}$ ⇒ integrand smemo razviti po u:

$$\frac{1}{e^u-1}=\frac{1}{\cancel{1}+u+\frac{u^2}{2!}+\frac{u^3}{3!}+O(u^4)-\cancel{1}}=\frac{1}{u}\cdot\frac{1}{1+\underbrace{\frac{u}{2!}+\frac{u^2}{3!}+O(u^3)}_{\varepsilon}}=$$

$\frac{1}{1+\varepsilon}=1-\varepsilon+\varepsilon^2-\varepsilon^3\pm\ldots$

$$=\frac{1}{u}\left(1-\left(\frac{u}{2!}+\frac{u^2}{3!}+O(u^3)\right)+\left(\frac{u}{2!}+\frac{u^2}{3!}+O(u^3)\right)^2+O(u^3)\right)=$$

$$=\frac{1}{u}\left(1-\frac{u}{2}-\frac{u^2}{6}+\frac{u^2}{4}+O(u^3)\right)=\frac{1}{u}\left(1-\frac{u}{2}+\frac{u^2}{12}+O(u^3)\right)$$

$$D(x) = 12x^3 \cdot \int_0^{1/x} \left(u^2 - \frac{u^3}{2} + \frac{u^4}{12} + \mathcal{O}(u^5)\right) du - \frac{3}{x}\left(\frac{1}{1/x} - \frac{1}{2} + \frac{\left(\frac{1}{x}\right)}{12} + \mathcal{O}(x^{-2})\right) =$$

Razvili $\frac{1}{e^{1/x}-1}$

Pomnožili z u^3 v integralu

int.

$$= 12x^3\left[\frac{\left(\frac{1}{x}\right)^3}{3} - \frac{\left(\frac{1}{x}\right)^4}{2\cdot 4} + \frac{\left(\frac{1}{x}\right)^5}{12\cdot 5} + \mathcal{O}(x^{-6})\right] - 3 + \frac{3}{2x} - \frac{1}{4x^2} + \mathcal{O}(x^{-3}) =$$

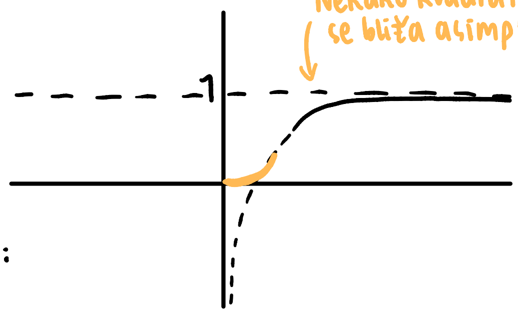
$$= 4 - \cancel{\frac{3}{2x}} + \frac{1}{5x^2} + \mathcal{O}(x^{-3}) - 3 + \cancel{\frac{3}{2x}} - \frac{1}{4x^2} =$$

$$= 1 - \frac{1}{20x^2} + \mathcal{O}(x^{-3})$$

$$\frac{1}{\cancel{1} + 1/x + \frac{1}{2}\left(\frac{1}{x}\right)^2 + \ldots - \cancel{1}} =$$

$$= 1/x\left(1 + \tfrac{1}{2}\left(\tfrac{1}{x}\right) + \tfrac{1}{6}\left(\tfrac{1}{x}\right)^2\right) + \ldots$$

Nekako kvadratično se bliža asimptoti 1



Ko gre $x \to 0$ je integral le neka št., neodvisna od x:

$$\int_0^\infty \frac{u^3 du}{e^u - 1} = \zeta(4)\cdot\Gamma(4) = \frac{\pi^4}{90} 3! = \frac{\pi^4}{15}$$

$$\boxed{\text{Riemann-zeta: } \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}, \quad \zeta(s) = \frac{1}{\Gamma(s)} \cdot \int_0^\infty \frac{u^{s-1} du}{e^u - 1}}$$

Drugače pa uporabimo trik:

$$\int_0^\infty \frac{e^{-u} u^3 du}{1 - \underbrace{e^{-u}}_{\leq 1}} = \int_0^\infty (1 + e^{-u} + e^{-2u} + e^{-3u} + \ldots) e^{-u} u^3 du = \overbrace{\int_0^\infty e^{-u} u^3 du}^{= 3!} + \overbrace{\int_0^\infty e^{-2u} u^3 du}^{= \frac{1}{2^4}\cdot 3!} + \int_0^\infty e^{-3u} u^3 du + \ldots =$$

$2u = t$

$$= 3!\left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right] = 3! \cdot \zeta(4)$$

$$D(x) \sim 12x^3 \cdot \frac{\pi^4}{15}$$

Taylorjev razvoj $e^{-1/x}$ ne $\exists$, ker je $e^{-1/x}$ patološka fun. (vsi njeni odvodi so $= 0$).

Ostanek: $D(x) = 12x^3 \left(\int_0^\infty - \int_{1/x}^\infty\right) \frac{u^3 du}{e^u - 1} - \frac{3}{x(e^{1/x} - 1)}$

ostanek → Pogledati moramo, katerega reda je to

RAZVOJ SFERIČNE BESSLOVE FUNKCIJE OKROG 0.

Besslove f. rešijo DE: $x^2 y'' + 2xy' + (x^2 - n(n+1))y = 0$

→ Z nastavkom + ujemanje koeficientov

nesemo v DE

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

↳ s tem ignoriramo singularno rešitev.

$$\Rightarrow x^2(2a_2 + 3a_3 x + 4\cdot 3 a_4 x^2 + 5\cdot 4 a_5 x^3 + \ldots) + 2x(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots) +$$

$$+ (x^2 - n(n+1))(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) = 0$$

$-n(n+1)a_0 = 0$ rekurzija

$(2-n(n+1))a_1 = 0$

$(2+2\cdot 2-n(n+1))a_2 = -a_0$

$(3\cdot 2+2\cdot 3-n(n+1))a_3 = -a_1$

$(4\cdot 3+2\cdot 4-n(n+1))a_4 = -a_2$

$\vdots$

$(l(l-1)+2l-n(n+1))a_l = -a_{l-2}$

$\boxed{(l(l+1)-n(n+1))a_l = -a_{l-2}}$ splošni člen → skače po 2: lihe in sode izmenično

Vodilni člen je x^n ($l=n$ prvi neničeln člen)

$$j(x) = x^n + a_{n+2}x^{n+2} + \dots, \quad \boxed{a_l = -\frac{a_{l+2}}{l(l+1)-n(n+1)}}$$

$j_0 = \frac{\sin x}{x}$

$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$

$\vdots$

$n=0$

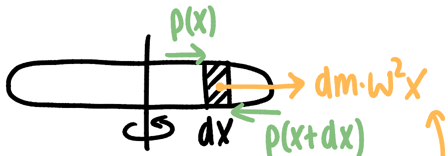
$a_0 = 1$

$a_2 = \frac{-1}{2\cdot 3}$

$a_4 = \frac{1}{2\cdot 3\cdot 4\cdot 5}$

KODRE 30/11

Zataljeno cevko polno plina centrifugiramo



diferencial centrifugalne sile

$$dm\cdot w^2 x = S(p(x+dx)-p(x))$$

$$\rho \cancel{S} w^2 x\,dx = \cancel{S}\,dp$$

$$\rho w^2 x = \frac{dp}{dx}$$

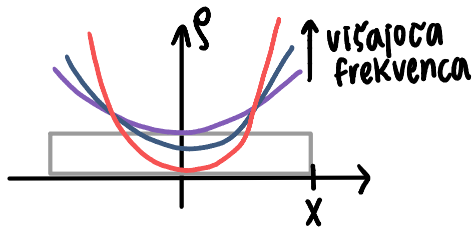
Predpostavimo izotermno situacijo

$p = \rho\frac{R}{M}T$

$dp = d\rho\frac{R}{M}T$

$$\Rightarrow \int_0^x w^2 x\,dx = \frac{R}{M}T\int_{\rho_0}^{\rho}\frac{d\rho}{\rho}$$

~> Iščemo gostotni profil $\rho(x)$



Problem: ρ_0 ni znan, določen je z GLOBALNIM POGOJEM (ohranitev mase)

$$\Rightarrow \frac{w^2x^2}{2} = \frac{R}{M}T\cdot\ln\frac{\rho}{\rho_0}$$

$u^2 = \frac{Mw^2x^2}{2RT}$

↳ $u_{max} = \sqrt{\frac{M}{2RT}}\,wR$

$$\Rightarrow \rho = \rho_0 e^{\frac{M}{RT}\cdot\frac{w^2x^2}{2}} = \rho_0\cdot e^{u^2}$$

ohranitev mase je integral gostote po dolžini ⇒ robni pogoj je določen z integralom

$C = \sqrt{\frac{M}{2RT}}\,w$

$du = \sqrt{\frac{M}{2RT}}\,w$

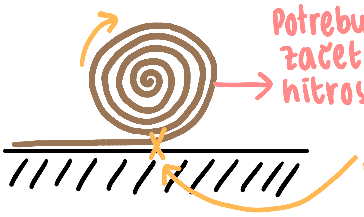
Pogoj: $m = S\cdot\int_{-R}^{R}\rho(x)dx = S\cdot\rho_0 C\int_{-u_{max}}^{u_{max}} e^{u^2}du$

Dawsonov integral: $D(x) = e^{-x^2}\int_0^x e^{u^2}du$

Če R ni prevelik, $\int_0^x e^{u^2}du \sim \int 1+u^2+\frac{u^4}{2!}+\dots du = x+\frac{x^3}{3!}+\frac{x^5}{5\cdot 2!}+\dots$

(gledaš, kdaj členi dovolj padejo)

ODVIJANJE TEPIHA x(t)



1. ideja: Ohranitev Γ, G

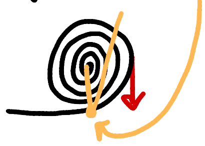
$J \sim \frac{3}{2} m r^2 = \frac{3}{2}\sigma\pi r^4$; $\sigma = \frac{dm}{dS}$ ploskovna gostota

se spreminja

$\Gamma = J\omega \sim r^3 \cdot v \Rightarrow v \sim r^{-3}$ oz. $\omega \sim r^{-4}$

$W_K \sim J\omega^2 \sim r^4 \cdot r^{-8} \sim r^{-4}$

Če se Γ ohranja, to pomeni, da W_K divergira ?!
→ Problem: kdaj se Γ ohrani? $M=0$
To očitno ne bo OK.



Navori okoli te točke so vedno prisotni: pospešujemo težišče dol + sila podlage ne gre skozi sredino role

2. ideja: Ohranimo energijo: $W = \frac{1}{2}J\omega^2 + mgr$ → ali se to ohrani ali disipira?

Torej ali se padanje tepiha pretvori v W_K ali gre v stiskanje tepiha?

$$W = \frac{3}{4}\sigma\pi r^4\omega^2 + gr\sigma\pi r^2 = \frac{3}{4}\sigma\pi r^2 v^2 + \sigma\pi g r^3 \Rightarrow \boxed{v = \sqrt{\frac{W - \sigma\pi g r^3}{3/4\,\sigma\pi r^2}}}$$

Kako do x(t)? Rabimo r(t) oz. r(x). Ohranitev ploščine:

$$\frac{d}{dt}\quad S = \pi r^2 = h(\ell - x)$$

višina tepiha
dolžina tepiha
preostali tepih

$$2\pi r \dot r = -h\dot x = -hv$$

$$\Rightarrow -\frac{2\pi r}{h}\dot r = \sqrt{\frac{W - \sigma\pi g r^3}{3/4\,\sigma\pi r^2}}$$ → grdo! Zato imamo radi brezdimenzijske spremenljivke

Imamo pa poseben primer, ko je $\dot r = 0 \Leftrightarrow W = \sigma\pi g r_x^3$ → referenčni r; to je začetni r, ki ne bi rabil zaleta, tepih bi se odvijal pod lastno težo

$$\Rightarrow r_x = \sqrt[3]{\frac{W}{\sigma\pi g}}$$ → Uvedemo $u = \frac{r}{r_x} \le 1$

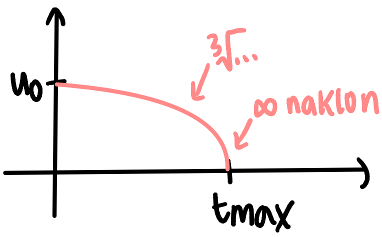
$$\underbrace{\dot u r_x}_{\dot r} = -\frac{h}{2\pi u^2 r_x}\sqrt{\frac{W(1-u^3)}{3/4\,\sigma\pi r_x^2}} \Rightarrow u^2\dot u = -\underbrace{\frac{h}{2\pi r_x^3}\sqrt{\frac{4W}{3\sigma\pi}}}_{\lambda}\cdot\sqrt{1-u^3} \Rightarrow \int_{r_0/r_x}^{0 \text{ ali } r/r_x}\frac{u^2\,du}{\sqrt{1-u^3}} = -\int_0^t \lambda\,dt$$

$W = 1-u^3$

(Po tem modelu ∃ le en možen način odvijanja tepiha)
r_x = pri $v=0$, r_0 = pri začetku

$$\Rightarrow \frac{1}{3}\int_{W_0}^{W}\frac{dW}{\sqrt{W}} = \lambda t = \frac{2}{3}\left(\sqrt{W} - \sqrt{W_0}\right) = \frac{2}{3}\left(\sqrt{1-u^3} - \sqrt{1-u_0^3}\right)$$

...



$$\Rightarrow u = \sqrt[3]{1 - \left(\sqrt{1-u_0^3} + \frac{3}{2}\lambda t\right)^2}$$ ⇒ Čas odvijanja: $\boxed{t = \frac{2}{3}\lambda^{-1}\left(1 - \sqrt{1-u_0^3}\right)}$

Kaj pa če $g \to 0$? (Potencialna en. ne prispeva k rolanju). V tem primeru $r_x \to \infty$ izgubi smisel. Namesto limite začnemo znova:

$$W = \frac{3}{4}\sigma\pi r^2 v^2\ ;\ v = -\frac{2\pi r\dot r}{h} = \sqrt{\frac{4W}{3\sigma\pi}}\cdot\frac{1}{r} \Rightarrow r^2\,dr = -\underbrace{\frac{h}{2\pi}\sqrt{\frac{4W}{3\sigma\pi}}}_{\Lambda\,[m^3/s]}\,dt$$

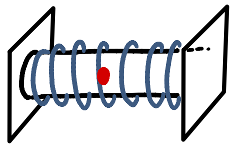
$$\Rightarrow \frac{r^3}{3} = \frac{r_0^3}{3} - \Lambda t \Rightarrow r = \sqrt[3]{r_0^3 - 3\Lambda t}\ ;\ t_{max} = \frac{r_0^3}{3\Lambda}$$

9.3.2021

2x vpeta vzmet

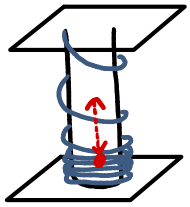
$m = 1\,kg$
$K = 100\,\frac{N}{m}$

naravna dolžina vzmeti je dolžina nosilca



vzmet navijemo na nosilec

$\Rightarrow$



potem nosilec obrnemo pokonci. Zanima nas, za koliko se premakne središče vzmeti.

Hookov zakon:

$$\frac{F}{S} = E \cdot \frac{dy}{dx}$$

$\hookrightarrow \frac{\Delta e}{e}$

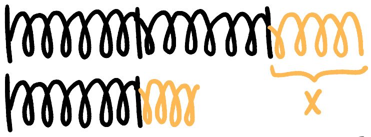
y je kumulativni raztezek, = koliko raztezka se nabere od začetka koord. sistema

x je štetje navojev oz. položaj na neraztegnjeni vzmeti

→ Mi imamo Hookov zakon v tej obliki:

$$F = k \cdot y$$

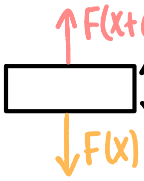
Seštevanje vzmeti:



$\hookrightarrow$ Polovična vzmet: $F = \left(K \cdot \frac{e}{e/2}\right) \cdot y$

oz. $F = K \cdot e \cdot \frac{dy}{dx}$ (*)

→ Zapišimo sile na košček vzmeti:



Teža: gdm

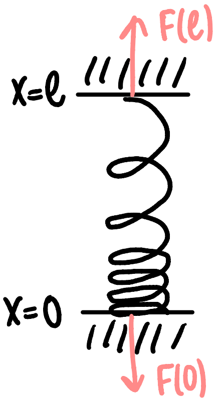
$F(x+dx)$, $F(x)$, dx

$x+dx+y(x+dx)$
$x+y(x)$

sta že povešena, zato sta v bistvu na drugih položajih

$\left(dm = \frac{m}{e} dx\right)$

→ Ravnovesje sil: $F(x+dx) = F(x) + gm\frac{dx}{e} \Rightarrow \boxed{\frac{dF}{dx} = \frac{gm}{e}}$ Kot pri visečem traku ✓



$F(e)$, $x=e$, $x=0$, $F(0)$

Robni pogoj je na $y(0)=0$ in $y(e)=0$

$F(0), F(e)$ sta neznanki! To je robni pogoj 1. vrste.

$\frac{dF}{dx} = \frac{gm}{e}$ Lahko integriramo, a ne vemo, kakšen je $F(0)$:

$$F = F(0) + \frac{mgx}{e}$$

nesemo v (*)

F večji → večji raztezek

*Pričakujemo, da bo $F(0)<0$, saj je tam vzmet skrčena

$$F = Ke \cdot \frac{dy}{dx} = F(0) + \frac{mgx}{e}$$

$$\Rightarrow \int_0^y dy = \int_0^x \left[\frac{F(0)}{Ke} + \frac{mgx}{Ke^2}\right] dx \Rightarrow y = \frac{F(0)}{Ke} x + \frac{mg}{2Ke^2} x^2$$

$y(0)=0$

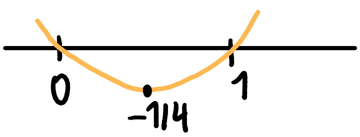
$F(0)$ dobimo iz pogoja $y(e)=0 \Rightarrow 0 = \frac{F(0)}{K\cancel{e}} \cdot \cancel{e} + \frac{mg}{2K\cancel{e^2}} \cdot \cancel{e^2} \Rightarrow F(0) = -\frac{mg}{2}$

$\leadsto$ Strop in tla si delita vsak polovico bremena $\Rightarrow F(e) = +\frac{mg}{2}$

$$\boxed{y(x) = -\frac{mg}{2Ke}x + \frac{mg}{2Ke^2}x^2 = \frac{mg}{2K} \cdot \frac{x}{e}\left(\frac{x}{e} - 1\right)}$$

$\hookrightarrow$ med 0 in 1

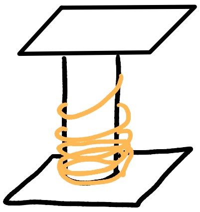
$$y\left(\frac{e}{2}\right) = -\frac{mg}{8K} \sim 1{,}25\,cm$$



0, 1, $-1/4$

središčna točka se najbolj povesi

Če vzmet ni vpeta:



$\rightsquigarrow F(0)=-mg$

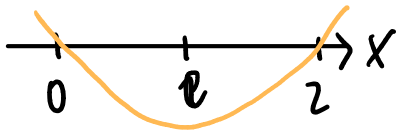
$F(\ell)=0$

$F(\ell)=F(0)+\frac{mg\ell}{K}=0$

$\frac{dF}{dx}=\frac{gm}{\ell}$ bi lahko določeno int.: $\int_F^0 dF=\int_x^\ell \frac{gm}{\ell}dx$

$F(x)=-mg+mg\frac{x}{\ell}=mg\left(\frac{x}{\ell}-1\right)$

$y(x)=-\frac{mg}{k\ell}\cdot x+\frac{mg}{k\ell^2}x^2=\frac{mg}{2k}\cdot\frac{x}{\ell}\left(\frac{x}{\ell}-2\right)$

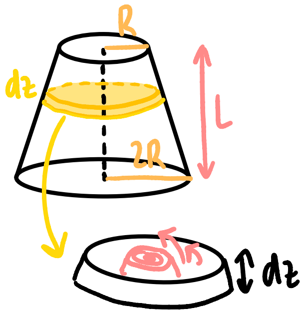


$y\left(\frac{\ell}{2}\right)=\frac{mg}{2k}\cdot\frac{1}{2}\left(-\frac{3}{2}\right)=-\frac{3}{8}\frac{mg}{k}$

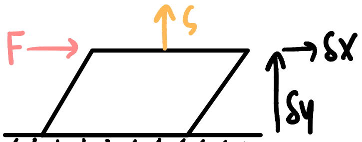
Nevpeta vzmet se bo posedla 3x bolj kot vpeta

Vrtež stožca

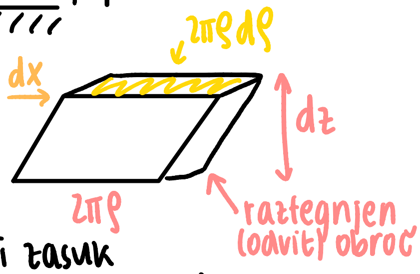
Prisekan polstožec iz gume, polmera R in 2R, višina L. Kolikšen je zasuk, če delujemo z navorom M in je strižni modul G.

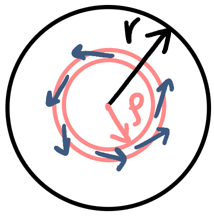


$r=2R-R\frac{z}{\ell}$



$\frac{F}{S}=G\cdot\frac{\delta x}{\delta y}$





zaradi navora zgornje in sp. plasti

$\frac{dF}{dS}=G\frac{dx}{dz}$ $= \rho d\varphi$

φ = absolutni zasuk

$d\varphi$ = razlika zasukov sosednjih plasti dz narazen

$\rho d\varphi$ = strig kolobarja plasti

$dF=G\cdot 2\pi\rho d\rho\cdot\rho\cdot\frac{d\varphi}{dz}$

$dM=\rho dF=G\cdot 2\pi\rho^3 d\rho\frac{d\varphi}{dz}\Rightarrow M=2\pi G\cdot\int_0^r\rho^3 d\rho\cdot\frac{d\varphi}{dz}=\underbrace{\frac{1}{2}\pi G r^4}_{\text{Torzijski koeficient}}\cdot\frac{d\varphi}{dz}$

Konst. (3NZ, navor se prenaša med plastmi)

$d\varphi=\frac{M}{\frac{1}{2}\pi G r^4}dz$; $r=2R-R\frac{z}{\ell}$

$dr=-\frac{R}{\ell}dz$

$d\varphi=-\frac{2M}{\pi G}\frac{\ell}{R}\cdot\frac{dr}{r^4}$