Holomorfne funkcije

Vse (holomorfne) zveze, ki veljajo v $\mathbb{R}$, veljajo tudi v $\mathbb{C}$, npr.
$\sin(2z) = 2\cdot\sin z\cdot\cos z$
$\sin(z+w) = \sin z\cdot\cos w + \sin w\cdot\cos z$
(To bomo dokazali na predavanjih)

$$\boxed{e^{iz} = \cos z + i\cdot\sin z = ch(iz) + i\cdot sh(iz)}$$

$\boxed{\cos(iz) = ch z}$
$(\cos(iz) = -ch z = ch z)$
$\boxed{\sin(iz) = i\cdot sh z}$
$\hookrightarrow i\cdot\sin(iz) = sh(-z)$
$i\cdot\sin(iz) = -sh z \;/\cdot i$
$-\sin(iz) = -i\cdot sh z$

(1) Poišči vse kompleksne ničle sin, cos, ch, sh!

a) $\sin z = 0$

$\sin(a+ib) = 0 \;;\; a,b\in\mathbb{R}$

$\sin a\cdot\cos(ib) + \cos a\cdot\sin(ib) = 0$

$\underbrace{\sin a\cdot ch b}_{=0} + \underbrace{\cos a\cdot i\cdot sh b}_{=0} = 0$

Realen del   Imaginarni del

$\Rightarrow \sin a\cdot \underbrace{ch b}_{\text{v } \mathbb{R} \text{ chb nima ničel}} = 0 \Rightarrow \sin a = 0,\; a = k\cdot\pi$

$\underbrace{\cos a}_{\neq 0,\text{ ker je } \sin a = 0}\cdot sh b = 0 \Rightarrow sh b = 0 \Rightarrow b = 0$

$\Big\}\; z = k\cdot\pi,\; k\in\mathbb{Z}$
ničle so tiste, ki jih že poznamo

b) Za ničle $\cos z = 0$ uporabimo zvezo $\sin\left(\frac{\pi}{2} - z\right) = 0$

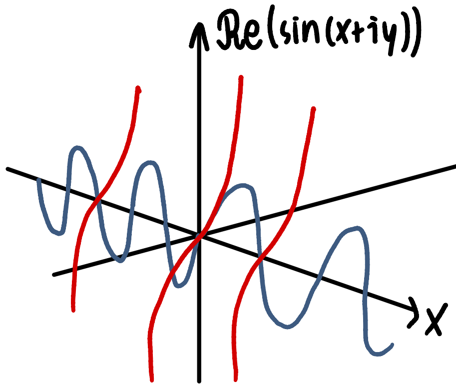
$\Rightarrow \frac{\pi}{2} - z = k\cdot\pi,\; k\in\mathbb{Z} \Rightarrow z = \frac{\pi}{2} + k\cdot\pi,\; k\in\mathbb{Z}$

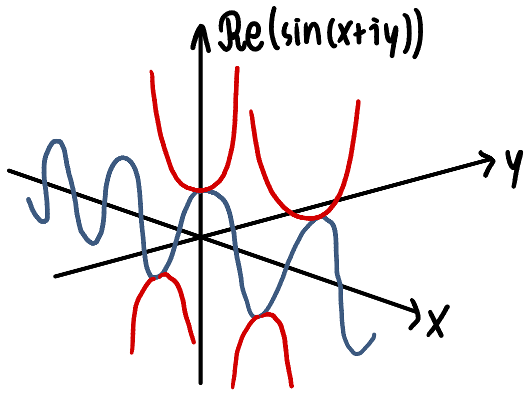
c) $sh z = 0$

$i\cdot\sin(iz) = 0 \Rightarrow iz = k\cdot\pi \;/\cdot(-i) \Rightarrow z = k\pi i,\; k\in\mathbb{Z}$

d) $ch z = 0$

$\cos(iz) = 0 \Rightarrow iz = \frac{\pi}{2} + k\pi \;/\cdot(-i) \Rightarrow z = \frac{\pi}{2}i + k\pi i,\; k\in\mathbb{Z}$





(2) $|\sin(x+iy)|^2 = |\sin x\cdot ch y + i\cdot sh x\cdot\cos y|^2 = \sin^2 x\cdot ch^2 y + sh^2 x\cdot\cos^2 y \overset{DN}{=} \ldots$

Upoštevaš $\sin^2 x + \cos^2 x = 1$
$ch^2 x - sh^2 x = 1$

(4) $\Omega = \{z;\ \operatorname{Im} z > 0\}$ ... zgornja polravnina $\mathbb{C}$

$f: \Omega \to \mathbb{C}$

$f(x+iy) = \sqrt{\sqrt{x^2+y^2}+x} + i\cdot\sqrt{\sqrt{x^2+y^2}-x}$

a) Preveri, da je f holomorfna.

b) Krajši zapis za f (Namig: f^2)

c) Izračunaj odvod f'

$f(x+iy) = u(x,y) + i\cdot v(x,y)$

f holomorfna $\Leftrightarrow$ u,v parcialno zvezno odvedljivi + veljajo Cauchy-Riemannove enačbe:

$u_x = v_y,\ u_y = -v_x$

$u,v: \mathbb{R}^2 \to \mathbb{R}$

a) u,v zvezno parcialno odvedljivi? Da ✓ $\to$

$$u_x = \frac{1}{2\cdot\sqrt{\sqrt{x^2+y^2}+x}}\cdot\left(\frac{1}{2\cdot\sqrt{x^2+y^2}}\cdot 2x+1\right)$$

$$u_y = \frac{1}{2\cdot\sqrt{\sqrt{x^2+y^2}+x}}\cdot\left(\frac{y}{\sqrt{x^2+y^2}}+0\right)$$

$$v_x = \frac{1}{2\cdot\sqrt{\sqrt{x^2+y^2}-x}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}-1\right)$$

$$v_y = \frac{1}{2\cdot\sqrt{\sqrt{x^2+y^2}-x}}\cdot\left(\frac{y}{\sqrt{x^2+y^2}}-0\right)$$

$u_x \overset{?}{=} v_y$:

$$\frac{1}{2\sqrt{\sqrt{\ldots}+x}}\left(\frac{x}{\sqrt{\ldots}}+1\right) \overset{?}{=} \frac{1}{2\sqrt{\sqrt{\ldots}-x}}\cdot\frac{y}{\sqrt{\ldots}}$$

vedno >0 $\to$

$$\frac{x+\sqrt{\ldots}}{\cancel{2}\sqrt{\sqrt{\ldots}+x}} \overset{?}{=} \frac{y}{\cancel{2}\cdot\sqrt{\sqrt{\ldots}-x}}$$

$$\pm\sqrt{(x^2+y^2)-x^2} = y$$

$$y = y \checkmark$$

$u_y \overset{?}{=} -v_x$:

$$\frac{1}{\cancel{2}\sqrt{\sqrt{\ldots}+x}}\cdot\frac{y}{\cancel{\sqrt{\ldots}}} \overset{?}{=} \frac{-1}{\cancel{2}\sqrt{\sqrt{\ldots}-x}}\cdot\frac{x-\sqrt{\ldots}}{\cancel{\sqrt{\ldots}}}$$

<0

$$\overset{+}{y}\cdot\overset{+}{\sqrt{\sqrt{\ldots}-x}} \overset{?}{=} \overset{+}{\sqrt{\sqrt{\ldots}+x}}\,(\overset{+}{\sqrt{\ldots}}-x) \;\Big|\cdot\sqrt{\sqrt{\ldots}+x}$$

$$y\cdot\sqrt{\cancel{x^2}+y^2-\cancel{x^2}} \overset{?}{=} (\sqrt{\ldots}+x)(\sqrt{\ldots}-x)$$

$$y^2 = y^2 \checkmark$$

b) $$\left(\sqrt{\sqrt{x^2+y^2}+x} + i\cdot\sqrt{\sqrt{x^2+y^2}-x}\right)^2 = \cancel{\sqrt{x^2+y^2}}+x + 2i\cdot\sqrt{(\cancel{x^2}+y^2)-\cancel{x^2}} - \left(\cancel{\sqrt{x^2+y^2}}-x\right) =$$

$$= 2iy + 2x = 2\underbrace{(x+iy)}_{z} = 2z$$

$f(z)^2 = 2z$

$\Rightarrow f(z) = \pm\sqrt{2z}$ Kaj je koren v kompleksnem? $\to$ Def. korena še pride ...

c) $f'(z) = \frac{1}{2\sqrt{2z}}\cdot 2 = \frac{\pm 1}{\sqrt{2z}}$

Kompleksne funkcije se odvaja enako kot realne.

holomorfna na $\mathbb{C}$

(5) Poišči vse cele funkcije, katerih realni del je x^3-3xy^2.

$$f(u+iv)=\underbrace{u(x,y)}_{\overset{\parallel}{x^3-3xy^2}}+i\cdot \underbrace{v(x,y)}_{\overset{\parallel}{?}}$$

Upoštevamo C-R: $u_x=v_y,\ v_x=-u_y$ $\Rightarrow \overbrace{\cancel{6xy}-0+C'(x)}^{v_x}=-(0-\cancel{6xy})$

$$C'(x)=0$$
$$C(x)=C$$

$3x^2-3y^2=v_y \quad /\int dy$

$v=\int(3x^2-3y^2)dy=3x^2y-y^3+C(x)$ $\Rightarrow v=3x^2y-y^3+C$

Rezultat: $f(x+iy)=\underbrace{x^3-3xy^2}_{u}+\underbrace{i\cdot 3x^2y-iy^3+Ci}_{iv}=(x+iy)^3+Ci=z^3+Ci,\ C\in\mathbb{R}$

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(2.7) Katere od spodnjih funkcij so holomorfne za vsako celo funkcijo f?

a) $f_1(z)=f(\bar z)$
b) $f_2(z)=\overline{f(z)}$
c) $f_3(z)=\overline{f(\bar z)}$

$f=u+iv=(u,v)$ $\quad z=x+iy=(x,y)$
$f(z)=u(z)+i\cdot v(z)$
$f(x,y)=u(x,y)+i\cdot v(x,y)$

a) $f(z)$ holomorfna, $f(\bar z)$ pa ne. Recimo: $f(z)=z^2$ ali $f(z)=z$
f_1 torej ni holomorfna za $\forall$ celo funkcijo f.

b) Zopet je odgovor NE, iz istega razloga

c) $f(z)=\sin z$
$f_3(z)=\overline{\sin\bar z}=\overline{\bar z-\frac{\bar z^3}{3!}\pm\ldots}=z-\frac{z^3}{3!}\pm\ldots=\sin z$ Konjugiranje se povsod krajša

Rešimo brez razvoja v vrsto (čeprav bi se tudi to dalo)
Naj bo f holomorfna $\Rightarrow f(x+iy)=u(x,y)+i\cdot v(x,y)$. Velja $u_x=v_y,\ u_y=-v_x$

$$f_3(z)=f_3(x+iy)=\overline{f(x-iy)}=\overline{u(x,-y)+i\cdot v(x,-y)}=\underbrace{u(x,-y)}_{\alpha(x,y)}\underbrace{-i\cdot v(x,-y)}_{\beta(x,y)}=\alpha(x,y)+i\cdot\beta(x,y)$$

$\alpha_x(x,y)=u_x(x,-y)\cdot 1+u_y(x,-y)\cdot 0=u_x(x,-y)$
$\alpha_y(x,y)=u_x(x,-y)\cdot 0+u_y(x,-y)\cdot(-1)=-u_y(x,-y)$
$\beta_x(x,y)=-v_x(x,-y)\cdot 1-v_y(x,-y)\cdot 0=-v_x(x,-y)$
$\beta_y(x,y)=-v_x(x,-y)\cdot 0-v_x(x,-y)\cdot(-1)=v_y(x,-y)$

$\alpha_x=\beta_y$
$\alpha_y=-\beta_x$ ✓ Drži.

$$\boxed{\begin{gathered} f(u(x,y),v(x,y)) \\ \frac{d}{dx}f=f_u(u,v)\cdot u_x+f_v(u,v)\cdot v_x \\ \frac{d}{dy}f=f_u(u,v)\cdot u_y+f_v(u,v)\cdot v_y \end{gathered}}$$

(2.8) $f: D \to \mathbb{C}$ holomorfna na območju D (odprta, povezana mn.). $|f(z)| = C \;\forall z \in D$. f konstantna.

$f = u + iv$, $u_x = v_y$, $u_y = -v_x$ — uporabimo

$$\sqrt{u^2+v^2} = C \Rightarrow u^2+v^2 = C^2$$

$\frac{\partial}{\partial x}$: $\cancel{2}u \cdot u_x + \cancel{2}v \cdot v_x = 0$

$\frac{\partial}{\partial y}$: $\cancel{2}u \cdot u_y + \cancel{2}v \cdot v_y = 0$

$\Rightarrow u \cdot (-v_x) + v \cdot u_x = 0$

sistem: $u \cdot u_x + v \cdot v_x = 0 \;|\cdot u$
$v \cdot u_x - u \cdot v_x = 0 \;|\cdot v$ $\oplus$

$$\Rightarrow \begin{array}{l} u^2 u_x + uv \cdot v_x = 0 \\ v^2 u_x - uv \cdot v_x = 0 \\ \hline \end{array}$$

$$(u^2+v^2) u_x = 0$$
$$C^2 u_x = 0 \;|:C^2$$
$$u_x = 0 \text{ in } v_y = 0$$
podobno $u_y = 0$ in $v_x = 0$

Če je C že na začetku 0, nimamo kaj za dokazovati, ker je $|f(z)| = 0$. BŠS predpostavimo, da $C \neq 0$.

Vsi parcialni odvodi so enaki 0, zato sta u in v konstantni (ker je D povezana mn.), zato je tudi f konstantna.

(2.10) c) Funkcijo $f(z) = \sin z$ razvij v potenčno vrsto okoli točke i.

$$\sin(z-i) = z-i - \frac{(z-i)^3}{3!} \pm \dots \quad \text{razvoj okrog } i$$

$$f(z) = \sin z = \sin(z-i+i) = \sin(z-i)\cdot\cos i + \cos(z-i)\cdot\sin i =$$

$\cos i = \operatorname{ch} 1$
$\sin i = i \cdot \operatorname{sh} 1$

$$= \left[(z-i) - \frac{(z-i)^3}{3!} \pm\right]\cdot \operatorname{ch} 1 + \left[1 - \frac{(z-i)^2}{2!} \pm \dots\right]\cdot i \cdot \operatorname{sh} 1 =$$

(2.2) $|\sin(x+iy)|^2 = |\sin x \cdot \operatorname{ch} y + i \cdot \cos x \cdot \operatorname{sh} y|^2 = \underbrace{\sin^2 x}_{1-\cos^2 x} \cdot \operatorname{ch}^2 y + \cos^2 x \cdot \operatorname{sh}^2 y =$

$$= (1-\cancel{\cos^2 x}) \cdot \operatorname{ch}^2 y + \cos^2 x (\cancel{\operatorname{ch}^2 y} - 1) =$$
$$= \operatorname{ch}^2 y - \cos^2 x$$

zveze:
$\cos^2 z + \sin^2 z = 1$
$\operatorname{ch}^2 z - \operatorname{sh}^2 z = 1$

(2.11) c) Določi konvergenčno območje potenčne vrste $\sum_{n=0}^{\infty} z^{2^n}$.

$$\sum_{n=0}^{\infty} a_n z^n \Rightarrow R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

Pri nas: $a_n = \begin{cases} 1, & n = 2^k \\ 0, & \text{sicer} \end{cases} \Rightarrow \limsup \sqrt[n]{|a_n|} = 1 \Rightarrow R = 1$

3. Kompleksni integrali

$$\int_\gamma f(z)\,dz = \int_{t_1}^{t_2} f(z(t))\cdot \dot z(t)\,dt$$

$\gamma : z(t)$

brez lukenj

Cauchyjeva formula:

Naj bo f holomorfna na enostavno povezanem območju D, $\alpha \in D$ in γ sklenjena pot v D, ki ne gre skozi α. Potem je:

(1) $\oint_\gamma f(z)dz = 0$ → posledica

(2) $\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\alpha}\,dz = I_\gamma(\alpha)f(\alpha)$ → osnovna Cauchyjeva formula

indeks

(3) $\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz = I_\gamma(\alpha)\cdot\frac{f^{(n)}(\alpha)}{n!}$, $n \in \mathbb{N}_0$ → druga formulacija

(1) Izračunaj po definiciji naslednje integrale. Vse krivulje naj bodo pozitivno orientirane.

a) $\int_{|z|=1} z^n dz$; $n \in \mathbb{Z}$

$z = e^{it}$, $t \in [0, 2\pi]$ parametrizacija krožnice

$dz = i\cdot e^{it}dt$

e je periodična fun. s periodo $2\pi i$

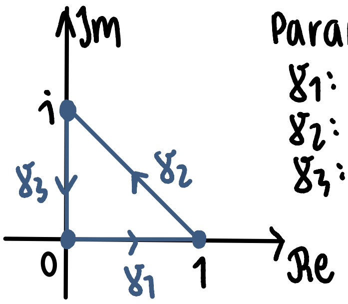
$$\Rightarrow \int_{|z|=1} z^n dz = \int_0^{2\pi} e^{nit}\cdot i e^{it}dt = i\cdot\int_0^{2\pi} e^{(n+1)it}dt = \cancel{i}\cdot\frac{1}{(n+1)\cancel{i}}\cdot e^{(n+1)it}\Big|_0^{2\pi} = \frac{1}{(n+1)}\Big[\underbrace{e^{(n+1)i\cdot 2\pi}}_{1} - 1\Big] =$$

$$= \frac{1}{(n+1)}(1-1) = 0 \ ; \ n \neq -1$$

$$n=-1 \Rightarrow \int_0^{2\pi} i\cdot e^0 dt = 2\pi i$$

b) $\int_\gamma \bar z\, dz$; γ rob trikotnika $\Delta(0,1,i)$

ni holomorfna funkcija



Parametrizacija γ:

γ_1: $z(t) = t$; $t \in [0,1]$

γ_2: $z(t) = 1 + t\cdot(i-1)$; $t \in [0,1]$

γ_3: $z(t) = i + t(0-i)$; $t \in [0,1]$

$$\int_\gamma \bar z\,dz = \int_{\gamma_1}\ldots + \int_{\gamma_2}\ldots + \int_{\gamma_3}\ldots = \int_0^1 t\,\underbrace{dt}_{dz} + \int_0^1 (1-it-t)(i-1)\,dt + \int_0^1 (-i+it)\underbrace{(-i)dt}_{dz} =$$

$$= \frac{1}{2} + \int_0^1 (i-1+2t)\,dt + \int_0^1 (-1+t)\,dt = \cancel{\tfrac{1}{2}} + i - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{\tfrac{1}{2}} = i$$

c) DN

(2) Izračunaj s pomočjo Cauchyjeve formule:

$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\alpha}dz = f(\alpha)\cdot \mathrm{Ind}_\gamma(\alpha)$

a) $\int_{|z|=2} \frac{\sin z}{z+i}dz = 2\pi i \cdot \sin(-i) = 2\pi \cdot \mathrm{sh}1$

b) $\int_{|z|=3} \frac{z^2}{z-2i}dz = 2\pi i \cdot f(2i) = 2\pi i \cdot (2i)^2 = -8\pi i$

c) $\int_{|z|=1} \bar{z}dz$ Problem: $\bar{z}$ ni holomorfna in nima pola. $\bar{z}$ želimo nadomestiti z nečim, kar se na krožnici ujema z $\bar{z}$.

Na $|z|=1$ velja: $\bar{z}\cdot z = 1$

$$\Rightarrow \int_{|z|=1} \bar{z}dz = \int_{|z|=1} \frac{1}{z}dz = 2\pi i \cdot f(0) = 2\pi i \qquad f(z)=1$$

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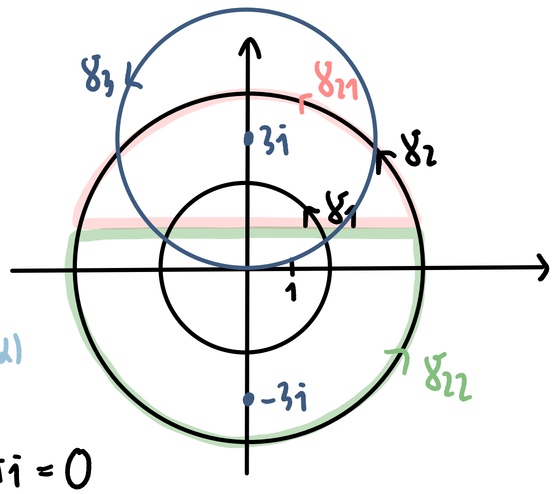
(3.2) d) $\oint_\gamma \frac{dz}{9+z^2} = \oint_\gamma \frac{dz}{(z+3i)(z-3i)}$



γ krožnica, $|z|=2$, $|z|=4$ oz. $|z-2i|=2$

$$\Rightarrow \oint_{\gamma_1} \frac{dz}{(z+3i)(z-3i)} = 0$$

Holomorfna (nima polov znotraj)

parc. ulomki

$\oint \frac{f(z)}{z-\alpha}dz = f(\alpha)\cdot 2\pi i \cdot \mathrm{ind}_\gamma(\alpha)$

$$\oint_{\gamma_2} \frac{dz}{(z+3i)(z-3i)} = \oint_{\gamma_2}\left(\frac{-1/6i}{(z+3i)} + \frac{1/6i}{(z-3i)}\right)dz = \left(-\frac{1}{6i}\right)\cdot 2\pi i + \frac{1}{6i}\cdot 2\pi i = 0$$

2. način: brez parcialnih ulomkov, γ_2 razdelimo na dve polkrožnici, ki vsaka obkroži le en pol:

nima polov znotraj

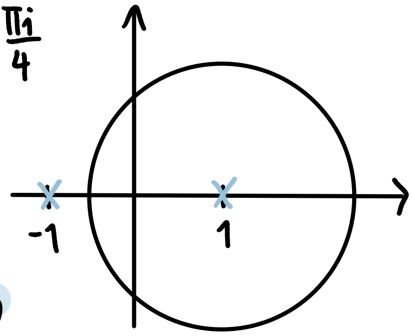
$$\oint_{\gamma_2} \frac{dz}{9+z^2} = \oint_{\gamma_{21}} \frac{dz}{9+z^2} + \oint_{\gamma_{22}} \frac{dz}{9+z^2} = \oint_{\gamma_{21}} \frac{\overbrace{\frac{1}{z+3i}}^{f_1(z)}}{z-3i}dz + \oint_{\gamma_{22}} \frac{\overbrace{\frac{1}{z-3i}}^{f_2(z)}}{z+3i}dz = f_1(3i)\cdot 2\pi i + f_2(-3i)\cdot 2\pi i =$$

$$= \frac{1}{6i}\cdot 2\pi i - \frac{1}{6i}\cdot 2\pi i = 0$$

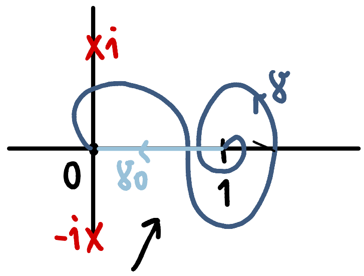
$$\oint_{\gamma_3} \frac{dz}{z^2+9} = \oint_{\gamma_3} \frac{\frac{1}{z+3i}}{z-3i}dz = 2\pi i \frac{1}{3i+3i} = \frac{\pi}{3}$$

e) $\oint\limits_{|z-1|=\frac{3}{2}} \frac{dz}{(z+1)(z-1)^3} = \oint\limits_\gamma \frac{\frac{1}{z+1}}{(z-1)^3}\,dz = 2\pi i \cdot \frac{(-1)(-2)\cdot(z+1)^{-3}}{2!}\Big|_{z=1} = \pi i\cdot 2\cdot 2^{-3} = \frac{\pi i}{4}$

$$\boxed{\oint \frac{f(z)}{(z-\alpha)^{n+1}}\,dz = 2\pi i\,\frac{f^{(n)}(\alpha)}{n!}\cdot \mathrm{ind}_\gamma(\alpha)}$$



(3.3) Katere vrednosti lahko zavzame $\int_0^1 \frac{dz}{z^2+1}$, če integriramo po vseh možnih poteh od 0 do 1?



Da γ sklenemo, dodamo še daljico

→ γ daljica $0\to 1$: $z(t)=t,\ t\in[0,1]$: $\int\limits_\gamma \frac{dz}{z^2+1} = \int_0^1 \frac{dt}{t^2+1} = \mathrm{arctg}\Big|_0^1 = \frac{\pi}{4}$

→ γ ni daljica: $\delta = \gamma * \gamma_0$

$$\oint\limits_\delta \frac{dz}{(z+i)(z-i)} = \oint\limits_\delta \frac{dz}{z^2+1} = \int\limits_\gamma \frac{dz}{z^2+1} + \int\limits_{\gamma_0} \frac{dz}{z^2+1} = \int\limits_\gamma \frac{dz}{z^2+1} - \frac{\pi}{4}$$

δ razdelimo na 2 krivulji, kjer vsaka obkroži le eno izmed singularnosti

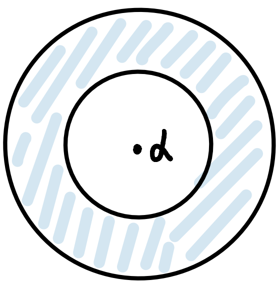
$$\oint\limits_{\delta_1} \frac{\frac{1}{z+i}}{z-i}\,dz + \oint\limits_{\delta_2} \frac{\frac{1}{z-i}}{(z+i)}\,dz = 2\pi i\,\frac{1}{i+i}\cdot I_{\delta_1}(i) + 2\pi i\cdot\frac{1}{-i-i}\cdot I_{\delta_2}(-i) =$$

$$= \pi\cdot(\underbrace{I_{\delta_1}(i)-I_{\delta_2}(-i)}_{K})$$

Več kot to se ne da narediti, ker je rezultat odvisen od ovojnega št.

$\Rightarrow \int\limits_\gamma \frac{dz}{z^2+1} = \frac{\pi}{4} + \pi\cdot K,\ k\in\mathbb{Z}$

4. Laurentova vrsta



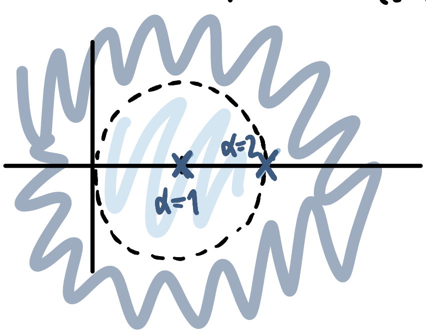
Laurentov razvoj holomorfne funkcije f na kolobarju $r < |z-\alpha| < R$ je

$$f(z) = \sum_{n=0}^{\infty} c_n(z-\alpha)^n + \sum_{n=1}^{\infty} c_{-n}(z-\alpha)^{-n} = \sum_{n=-\infty}^{\infty} c_n(z-\alpha)^n, \qquad c_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz,$$

kjer je γ poljubna sklenjena pot na danem kolobarju z ovojnim številom $I_\gamma(\alpha) = 1$. Prva vrsta v formuli konvergira vsaj na območju $|z-\alpha| < R$, druga pa vsaj na območju $|z-\alpha| > r$.

(1) Razvij v Laurentovo vrsto funkcijo $f(z) = \frac{1}{(z-1)(z-2)}$, tako, da bo vrsta konvergirala na

a) $0 < |z-1| < 1$
b) $|z-1| > 1$



razvijemo v geometrijsko vrsto

$$a)\ \frac{1}{(z-1)(z-2)} = \frac{1}{z-1}\cdot\frac{1}{(z-1)-1} = \frac{-1}{z-1}\cdot\frac{1}{1-\underbrace{(z-1)}_{}} = \underbrace{-\frac{1}{z-1}}_{\to\ |z-1|\neq 0}\cdot\left(1+(z-1)+(z-1)^2+\dots\right) =$$

$|q|<1$ da bo geom. vrsta konvergirala, torej $|z-1|<1$

$$= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \dots$$

Produkt bo konv. na $0<|z-1|<1$

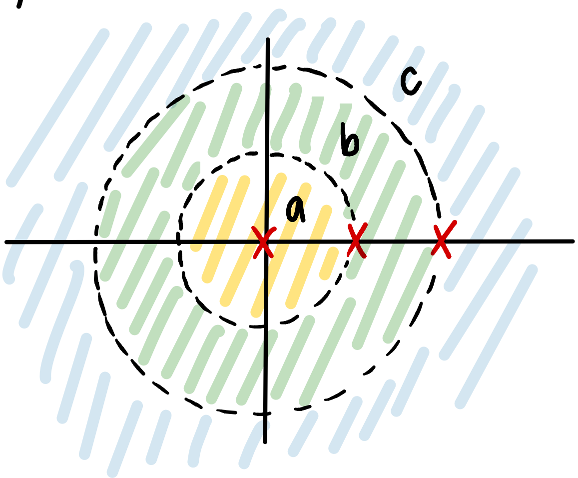
$$b)\ \frac{1}{(z-1)(z-2)} = \frac{1}{z-1}\cdot\frac{1}{z-2} = \frac{1}{z-1}\cdot\frac{1}{(z-1)-1}\ \begin{matrix}|:(z-1)\\ |:(z-1)\end{matrix} = \frac{1}{z-1}\cdot\frac{\frac{1}{z-1}}{1-\underbrace{\frac{1}{z-1}}_{q}} = \frac{1}{(z-1)^2}\left(1+\frac{1}{z-1}+\left(\frac{1}{z-1}\right)^2+\dots\right) =$$

$q \to$ zdaj bomo to razvili: $\left|\frac{1}{z-1}\right|<1 \Rightarrow |z-1|>1$ ✓✓

$$= \frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\dots$$

(2) $f(z) = \frac{1}{z(z-1)(z-2)}$ → Laurentova vrsta, da bo konvergirala na

a) $0<|z|<1$
b) $1<|z|<2$
c) $|z|>2$



nima singularnosti v $z=0$ ⇒ razvijemo v Taylorjevo vrsto

$$a)\ f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{z}\cdot\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(\underbrace{\frac{-1}{z-1}+\frac{1}{z-2}}_{}\right) = \frac{1}{z}\cdot\left(\frac{1}{1-z}+\frac{-1}{2(1-\frac{z}{2})}\right) =$$

vsakega od teh dveh členov zdaj razvijemo v geometrijsko vrsto

$|z|>0$ ✓ $\quad |z|<1$ ✓ $\quad \left|\frac{z}{2}\right|<1$ oz $|z|<2$ ✓

$$= \frac{1}{z}\left(1+z+z^2+\dots - \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)\right) = \frac{1}{2z}+\frac{3}{4}+\frac{7}{8}z+\frac{15}{16}z^2+\dots = \sum_{n\geq -1}\left(1-\frac{1}{2^{n+2}}\right)z^n$$

$$b)\ f(z) = \frac{1}{z}\cdot\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(\frac{-1}{z-1}\begin{matrix}|:z\\|:z\end{matrix}+\frac{1}{z-2}\right) = \frac{1}{z}\left(\frac{-1/z}{1-\underbrace{\frac{1}{z}}_{q}}+\frac{1}{z-2}\right) = \frac{1}{z}\cdot\left(-\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\dots\right)-\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right)\right) =$$

$|z|>0$ ✓ $\quad \left|\frac{1}{z}\right|<1 \Rightarrow |z|>1$ ✓ $\quad \left|\frac{z}{2}\right|<1 \Rightarrow |z|<2$ ✓

$$= \dots -\frac{1}{z^4}-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{2z}-\frac{1}{4}-\frac{1}{8}z-\frac{1}{16}z^2-\dots$$

$\left|\frac{1}{z}\right|<1 \Rightarrow |z|>1$ ✓ $\quad \left|\frac{2}{z}\right|<1 \Rightarrow |z|>2$ ✓

$$c)\ f(z) = \frac{1}{z}\left(\frac{-1}{z-1}\begin{matrix}|:z\\|:z\end{matrix}+\frac{1}{z-2}\begin{matrix}|:z\\|:z\end{matrix}\right) = \frac{1}{z}\left(\frac{-1/z}{1-\underbrace{\frac{1}{z}}_{q_1}}+\frac{1/z}{1-\underbrace{\frac{2}{z}}_{q_2}}\right) = \frac{1}{z}\left(-\frac{1}{z}\left(\cancel{1}+\frac{1}{z}+\frac{1}{z^2}+\dots\right)+\frac{1}{z}\left(\cancel{1}+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\dots\right)\right) =$$

$$= (2-1)\cdot z^{-3}+(2^2-1)z^{-4}+(2^3-1)z^{-5}+\dots = \sum_{n=1}(2^n-1)\cdot z^{-(n+2)}$$