

Super sample variance of stacked lensing

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I. INTRODUCTION

[1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15]

II. COVARIANCE OF STACKED LENSING IN A SURVEY WINDOW

A. Halo statistics: number counts and projected power spectrum

First we consider observables of halos. We assume that halos in a survey region can be identified from observables such as optical richness and X-ray observables, and then assume that halo mass and redshift of each halo are available. However, the following discussion can be extended to a more generally case where only proxies of redshift and halo mass for each halo are available.

The number density fluctuation field for halos in the mass range $[M, M + dM]$ is given as

$$n_h(\mathbf{x}; M)dM \simeq \frac{dn}{dM} [1 + \delta_h(\mathbf{x}; M)] \quad (1)$$

where dn/dM is the (ensemble-average) number density of halos with masses $[M, M + dM]$, and $\delta_h(\mathbf{x})$ is the number density fluctuation field of the halos.

Denoting the radial selection function by $f_h(\chi; z_L)$ and the halo mass selection at each redshift $\chi = \chi(z)$ by $S(M; \chi)$, we can define the projected number density field of halos, integrated over ranges of redshift and halo masses, in terms of the 3D density field $n_h(\mathbf{x}; \chi)$ as

$$\begin{aligned} N_h(\mathbf{x}_\perp) &= \int d\chi f_h(\chi; z_L) \int dM S(M; \chi) W(\mathbf{x}_\perp) n_h(\mathbf{x}; M, \chi) \\ &= \int d\chi f_h(\chi; z_L) \int dM S(M; \chi) \frac{dn}{dM} [1 + W(\mathbf{x}_\perp) \delta_h(\chi, \mathbf{x}_\perp)], \end{aligned} \quad (2)$$

where we introduced the notation “ z_L ” in $f_h(\chi)$ to explicitly mean that the radial selection is non-zero around a target redshift z_L , which will be below taken for the redshift slice of lensing halos. The vector \mathbf{x}_\perp denotes the position vector in the two-dimensional plane at distance χ , perpendicular to the line-of-sight direction, which is converted from the angular position $\boldsymbol{\theta}$ via relation $\mathbf{x}_\perp \equiv \chi \boldsymbol{\theta}$. $W(\boldsymbol{\theta})$ is a survey window, which is 1 in the measured region and 0 in the unmeasured region. Hence the survey area is given as

$$\Omega_S \equiv \int d^2\boldsymbol{\theta} W(\boldsymbol{\theta}). \quad (3)$$

Note that N_h is the projected number density field in dimension of $[\text{Mpc}^{-2}]$, not an angular number density, and we throughout this paper employ distant observer approximation as well as flat-sky approximation. The selection functions are generally defined so as to satisfy $0 \leq f_h \leq 1$ and $0 \leq S(M; \chi) \leq 1$. For an ideal, homogeneous complete selection, $f_h = 0$ or 1 and similar for $S(M)$.

The mean surface density estimated from a finite-area survey region is

$$\begin{aligned} \hat{N}_h(z_L) &= \frac{1}{\Omega_S} \int d^2\boldsymbol{\theta} N_h(\mathbf{x}_\perp) \simeq \int d\chi f_h(\chi; z_L) \int dM S(M; \chi) \frac{dn}{dM} \left[1 + \frac{b(M)}{\chi^2 \Omega_S(\chi)} \int d^2\mathbf{x}_\perp W(\mathbf{x}_\perp) \delta_{m, \text{lin}}(\chi, \mathbf{x}_\perp) \right] \\ &= \int d\chi f_h(\chi; z_L) \int dM S(M; \chi) \frac{dn}{dM} [1 + b(M) \delta_b(\chi)] \\ &= \bar{N}_h \left[1 + \int d\chi f_h(\chi) \bar{n}_h^S \bar{b}^S \delta_b \right]. \end{aligned} \quad (4)$$

where we have defined the average halo number density and the average halo bias over the halo mass range at redshift $z = z(\chi)$:

$$\begin{aligned}\bar{n}_h^S(\chi) &\equiv \int dM \frac{dn}{dM} S(M), \\ \bar{b}^S(\chi) &\equiv \frac{1}{\bar{n}_h^S} \int dM \frac{dn}{dM} S(M) b.\end{aligned}\quad (5)$$

Quantities with hat notation, here and hereafter, denote their estimators, and $\chi^2 \Omega_S(\chi)$ is the effective area in the perpendicular plane at distance χ , in units of [Mpc²]. On the r.h.s. of the above equation, we assumed that the halo density field at large scales is related to the underlying mass density field as $\delta_h(\mathbf{x}; M) \simeq b(M) \delta_{m, \text{lin}}(\mathbf{x})$, where $\delta_{m, \text{lin}}(\mathbf{x})$ is the mass density fluctuation field in the linear regime and b is the linear halo bias, assuming that the angular average over a sufficiently wide area smooths out the small-scale fluctuations. The quantity δ_b is the background density contrast of the survey region:

$$\delta_b(\chi) \equiv \frac{1}{\Omega_S} \int d^2\theta W(\theta) \delta_{m, \text{lin}}(\chi, \chi\theta) = \frac{1}{\chi^2 \Omega_S(\chi)} \int d^2\mathbf{x}_\perp W(\mathbf{x}_\perp) \delta_{m, \text{lin}}(\chi, \mathbf{x}_\perp), \quad (6)$$

where we have again used the conversion $\mathbf{x}_\perp = \chi\theta$. Eq. (6) shows that the background mode at each redshift arises from the average of Fourier modes that lie in the two-dimensional plane perpendicular to the line-of-sight direction. If ignoring the effect of radial Fourier mode on clustering observables, which is a good approximation for lensing observables, the background mode can be realized as a shift in the mean mass density in each lens plane. \bar{N}_h is the ensemble average or global mean of the surface halo number density:

$$\bar{N}_h(z_L) \equiv \langle \hat{N}_h(z_L) \rangle = \int d\chi f_h(\chi; z_L) \int dM S(M; \chi) \frac{dn}{dM}, \quad (7)$$

where we have used $\langle \delta_b \rangle = 0$. Here and hereafter quantities with bar notation denote their ensemble average quantities.

The projected number density fluctuation field of halos is

$$\widehat{\delta N}_h(\mathbf{x}_\perp) \equiv \int d\chi f_h(\chi; z_L) \int dM S(M) \frac{dn}{dM} W(\mathbf{x}_\perp) \delta_h(\chi, \mathbf{x}_\perp; M). \quad (8)$$

The Fourier transform is

$$\widehat{\delta N}_h(\mathbf{k}_\perp) \equiv \int d^2\mathbf{x}_\perp \widehat{\delta N}_h(\mathbf{x}_\perp) e^{-i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} = \int d\chi f_h(\chi; z_L) \int dM \frac{dn}{dM} S(M) \int \frac{d\mathbf{k}_\parallel}{2\pi} \tilde{\delta}_h^W(k_\parallel, \mathbf{k}_\perp) e^{ik_\parallel \chi} \quad (9)$$

where

$$\tilde{\delta}_h^W(k_\parallel, \mathbf{k}_\perp) \equiv \int \frac{d^2\mathbf{q}}{(2\pi)^2} \tilde{W}(\mathbf{q}) \tilde{\delta}_h(k_\parallel, \mathbf{k}_\perp - \mathbf{q}), \quad (10)$$

and $\tilde{W}(\mathbf{q}; \chi) \equiv \int d^2\mathbf{x}_\perp W(\mathbf{x}) e^{-i\mathbf{q} \cdot \mathbf{x}_\perp}$.

Hence we can define as estimator of the projected power spectrum of halos as

$$\begin{aligned}\hat{C}_{hh}(k_{\perp, i}) &\equiv \frac{1}{\hat{N}_h(z_L)^2} \int d\chi \int d\chi' f_h(\chi) f_h(\chi') \int dM \int dM' S(M) \frac{dn}{dM} S(M') \frac{dn}{dM'} \\ &\times \frac{1}{\chi^2 \Omega_S} \int_{|\mathbf{k}_\perp| \in k_{\perp, i}} \frac{d^2\mathbf{k}_\perp}{A(k_{\perp, i})} \int \frac{d\mathbf{k}_\parallel}{2\pi} \int \frac{d\mathbf{k}'_\parallel}{2\pi} \tilde{\delta}_h^W(k_\parallel, \mathbf{k}_\perp) \tilde{\delta}_h^W(k'_\parallel, -\mathbf{k}_\perp) e^{ik_\parallel \chi + ik'_\parallel \chi'},\end{aligned}\quad (11)$$

where the integral $\int d^2\mathbf{k}_\perp$ is over an annulus in \mathbf{k}_\perp space of width Δk , and the area $A(k_{\perp, i}) \simeq 2\pi k_{\perp, i} \Delta k$. Using the halo power spectrum definition,

$$\langle \delta_h(k_\parallel, \mathbf{k}_\perp; M) \delta_h(k'_\parallel, \mathbf{k}'_\perp; M') \rangle = (2\pi)^3 \delta_D(k_\parallel + k'_\parallel) \delta_D^2(\mathbf{k}_\perp + \mathbf{k}'_\perp) P_{hh}(k; M, M'), \quad (12)$$

and using the Limber's approximation [16], we can compute the ensemble average of the above power spectrum estimator:

$$\begin{aligned}C_{hh}(k_i) &= \langle \hat{C}_{hh} \rangle(k_{\perp, i}) = \frac{1}{\bar{N}_h(z_L)^2} \int d\chi \int d\chi' f_h(\chi) f_h(\chi') \int dM \int dM' S(M) \frac{dn}{dM} S(M') \frac{dn}{dM'} \\ &\times \frac{1}{\chi^2 \Omega_S} \int \frac{d^2\mathbf{k}_\perp}{A(k_{\perp, i})} \int \frac{d^2\mathbf{q}}{(2\pi)^2} P_{hh}(\sqrt{k_\parallel^2 + |\mathbf{k}_\perp - \mathbf{q}|^2}) |\tilde{W}(\mathbf{q})|^2 e^{ik_\parallel(\chi - \chi')} \\ &\simeq \frac{1}{\bar{N}_h(z_L)^2} \int d\chi f_h(\chi)^2 \int dM \int dM' S(M) \frac{dn}{dM} S(M') \frac{dn}{dM'} P_{hh}(k_{\perp, i}, \chi; M, M') \frac{1}{\chi^2 \Omega_S} \int \frac{d^2\mathbf{q}}{(2\pi)^2} |\tilde{W}(\mathbf{q})|^2 \\ &= \frac{1}{\bar{N}_h(z_L)^2} \int d\chi f_h(\chi)^2 \int dM \int dM' S(M) \frac{dn}{dM} S(M') \frac{dn}{dM'} P_{hh}(k_{\perp, i}, \chi; M, M').\end{aligned}\quad (13)$$

Here we have used that $P_{\text{hh}}(\mathbf{k}_\perp - \mathbf{q}) \simeq P_{\text{hh}}(k_\perp)$ over the integration range of $d^2\mathbf{q}$ which the window function supports and also assumed that $P_{\text{hh}}(k)$ is not a rapidly varying function within the k_\perp -bin. Here and hereafter we will often omit “ \perp ” in k_\perp for notational simplicity. In the third equality on the r.h.s., we used the general identity for the window function:

$$\chi^2 \Omega_S = \int d^2\mathbf{x}_\perp W(\mathbf{x}_\perp)^n = \int \left[\prod_{a=1}^n \frac{d\mathbf{q}_a}{(2\pi)^2} \tilde{W}(\mathbf{q}_a) \right] (2\pi)^2 \delta_D^2(\mathbf{q}_{1\dots n}), \quad (14)$$

where $\mathbf{q}_{1\dots n} \equiv \mathbf{q}_1 + \mathbf{q}_2 + \dots + \mathbf{q}_n$. For $n = 2$, $\chi^2 \Omega_S = \Omega_S \int d^2\mathbf{q} / (2\pi)^2 |\tilde{W}(\mathbf{q})|^2$. If we employ the halo model, where the halo power spectrum is modeled as $P_{\text{hh}}(k; M, M') \simeq b(M)b(M')P_{\text{m}}^{\text{lin}}(k)$, the projected power spectrum is

$$C_{\text{hh}}(k) = \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi f_{\text{h}}(\chi)^2 [\bar{n}_{\text{h}}^S \bar{b}^S]^2 P_{\text{m}}^{\text{lin}}(k; \chi), \quad (15)$$

where $P_{\text{m}}^{\text{lin}}(k)$ is the linear mass power spectrum. The projected power spectrum C_{hh} has a dimension of $[\text{Mpc}^2]$.

B. Stacked lensing: An estimator of the projected halo-mass cross-correlation

The lensing convergence field at an angular position $\boldsymbol{\theta}$ on the sky arises from a projection of the three-dimensional mass fluctuation field δ_{m} , weighted by the lensing efficiency kernel:

$$\kappa^W(\boldsymbol{\theta}) \equiv W(\boldsymbol{\theta})\kappa(\boldsymbol{\theta}) = \int d\chi f_{\text{h}}(\chi) W(\boldsymbol{\theta}) \delta_{\text{m}}(\chi, \chi\boldsymbol{\theta}; z) = \int d\chi f_{\text{h}}(\chi) \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} \frac{dk_\parallel}{2\pi} \delta_{\text{m}}^W(k_\parallel, \mathbf{k}) e^{ik_\parallel\chi + i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} \quad (16)$$

where $\mathbf{x}_\perp = \chi\boldsymbol{\theta}$, $\delta_{\text{m}}(\mathbf{x})$ is the three-dimensional mass fluctuation field, and the lensing radial kernel $f_{\text{h}}(\chi)$ is given as

$$f_{\text{h}}(\chi) = \bar{\rho}_{\text{m}0} \Sigma_{\text{cr}}^{-1}(\chi). \quad (17)$$

The quantity Σ_{cr} is the critical surface mass density defined for each observer-lens-source system:

$$\Sigma_{\text{cr}}^{-1}(z) \equiv 4\pi G a^{-1} \chi(z) \left[1 - \chi(z) \left\langle \frac{1}{\chi_s} \right\rangle \right], \quad (18)$$

with $\langle 1/\chi_s \rangle = \int_{z_{\text{s,min}}}^{\infty} dz_s p(z_s) 1/\chi(z_s) / \int_{z_{\text{s,min}}}^{\infty} dz_s p(z_s)$, where $p(z_s)$ is the redshift distribution of source galaxies, and $z_{\text{s,min}}$ is the minimum redshift which an observer determines to select source galaxies behind lensing halos at z_L . The factor of $\bar{\rho}_{\text{m}0}$ in Eq. (16), instead of the mean density $\bar{\rho}_{\text{m}}(z)$ at redshift z , is from our use of the comoving coordinates.

For illustrative purpose, let us divide the convergence field into two contributions:

$$\kappa(\boldsymbol{\theta}) \simeq \bar{\rho}_{\text{m}0} \Sigma_{\text{cr}}^{-1}(z_L) \int_{z_L} d\chi \delta_{\text{m}}(\chi_L, \chi_L \boldsymbol{\theta}) + \int_{z \neq z_L} d\chi \bar{\rho}_{\text{m}0} \Sigma_{\text{cr}}^{-1}(z) \delta_{\text{m}}(\chi, \chi \boldsymbol{\theta}). \quad (19)$$

The first term on the r.h.s. denotes the mass distribution at the same redshift as that of lensing halos, and the second term is the lensing contribution from the mass distribution at different redshifts, which we call the cosmic shear contribution.

The stacked lensing profile for lensing halos with known redshifts probes the average mass distribution surrounding the lensing halos. Observationally the stacked lensing profile can be estimated by averaging shapes of background galaxies for all the pairs of halos and source galaxies:

$$\langle \widehat{\Delta\Sigma} \rangle(R) \equiv \frac{1}{N_{\text{pairs}}} \sum_{\text{all pairs}; (l,s)} \Sigma_{\text{cr}}(z_l, z_s) \epsilon_{+(s)} \bigg|_{R=\chi(z_l)\Delta\theta_{ls}}, \quad (20)$$

where the summation runs over all the halo and source galaxy pairs each of which is separated by a particular projected distance R in the lens plane of each halo at z_l , and ϵ_+ is the tangential component of source galaxy ellipticity with respect to the halo center. $\Delta\theta_{ls}$ is the angular separation between lens and source in each pair, and $\chi(z_l)$ is the distance to each halo. Here we assume that redshift information of each source galaxy is also available, e.g., based on the photometric redshift estimation. Thus, thanks to *a priori* knowledge of each lens redshift, we can probe the average projected mass profile as a function of comoving distances in each lens plane, rather than angular radii [8, 17]. For simplicity we do not consider a weighting in the above average, which is often considered in an actual measurement in order to optimize the signal-to-noise ratio.

The ensemble-average expectation, i.e. the cosmological signal of the stacked lensing arises from the cross-correlation between the projected number density fluctuation field of lensing halos and the convergence field:

$$\langle \Delta \Sigma \rangle(R) = \frac{1}{\bar{N}_h(z_L)} \langle \delta N_h(\mathbf{x}_\perp) \Sigma_{\text{cr}}(\chi) \gamma_+(\mathbf{x}'_\perp) \rangle_{R=|\mathbf{x}_\perp - \mathbf{x}'_\perp|; \mathbf{x}'_\perp = \chi \boldsymbol{\theta}}, \quad (21)$$

where $\gamma_+(\mathbf{x}'_\perp)$ is the tangential component of the shear field with respect to halo center, in the angular direction of \mathbf{x}'_\perp via the relation $\mathbf{x}'_\perp = \chi \boldsymbol{\theta}$. If defining the projected cross-power spectrum of halos and the surrounding mass distribution, the stacked lensing profile is expressed as

$$\langle \Delta \Sigma \rangle(R) = \int \frac{k dk}{2\pi} C_{\text{hm}}(k) J_2(kR), \quad (22)$$

where $J_2(x)$ is the 2nd-order Bessel function and the cross-power spectrum is defined as

$$C_{\text{hm}}(k) = \frac{\bar{\rho}_{\text{m}0}}{\bar{N}_h(z_L)} \int d\chi f_h(\chi) \int dM \frac{dn}{dM} S(M) P_{\text{hm}}(k; M, \chi), \quad (23)$$

where $P_{\text{hm}}(k; M, \chi)$ is the 3D cross-power spectrum between the mass distribution and halos of mass M and at redshift $z = z(\chi)$. If we employ the halo model, we can express $C_{\text{hm}}(k)$ by a sum of the 1- and 2-halo terms:

$$C_{\text{hm}}(k) = C_{\text{hm}}^{\text{1h}}(k) + C_{\text{hm}}^{\text{2h}}(k), \quad (24)$$

with

$$\begin{aligned} C_{\text{hm}}^{\text{1h}}(k) &\equiv \frac{\bar{\rho}_{\text{m}0}}{\bar{N}_h(z_L)} \int_{\chi_L} d\chi f_h(\chi) \int dM \frac{dn}{dM} S(M, z_L) M |u_M(k; z_L)| = \frac{\bar{\rho}_{\text{m}0}}{\bar{N}_h(z_L)} \int_{\chi_L} d\chi f_h(\chi) I_1^0(k; \chi), \\ C_{\text{hm}}^{\text{2h}}(k) &= \frac{\bar{\rho}_{\text{m}0}}{\bar{N}_h(z_L)} \int_{\chi_L} d\chi f_h(\chi) \int dM \frac{dn}{dM} S(M) b(M) P_{\text{m}}^{\text{lin}}(k; \chi) = \frac{\bar{\rho}_{\text{m}0}}{\bar{N}_h(z_L)} \int_{\chi_L} d\chi f_h(\chi) \bar{n}_h^S \bar{b}^S P_{\text{m}}^{\text{lin}}(k; \chi), \end{aligned} \quad (25)$$

where we introduced the notations:

$$\begin{aligned} I_\mu^{(S)\beta}(k_1, \dots, k_\mu) &\equiv \int dM \frac{dn}{dM} S(M) b^\beta \prod_{i=1}^\mu u_M(k_i), \\ I_\mu^\beta(k_1, \dots, k_\mu) &\equiv \int dM \frac{dn}{dM} b^\beta \prod_{i=1}^\mu u_M(k_i), \end{aligned} \quad (26)$$

where $u_M(k)$ is the Fourier transform of mass density profile for halos of mass M .

Similarly to Eq. (11), taking into account the survey window, we can define an estimator of the projected cross-power spectrum as

$$\hat{C}_{\text{hm}}(k_i) \equiv \frac{1}{\hat{N}_h(z_L)} \int d\chi \int d\chi' f_h(\chi) \Sigma_{\text{cr}}(\chi) f_\kappa(\chi') \int dM \frac{dn}{dM} S(M) \frac{1}{\chi^2 \Omega_S} \int_{|\mathbf{k}_\perp| \in k_i} \frac{d^2 \mathbf{k}_\perp}{A(k_i)} \int \frac{d\mathbf{k}_\parallel}{2\pi} \frac{d\mathbf{k}'_\parallel}{2\pi} \delta_h^W(k_\parallel, \mathbf{k}_\perp) \tilde{\delta}_m^W(k'_\parallel, -\mathbf{k}_\perp) e^{ik_\parallel \chi + ik'_\parallel \chi'}. \quad (27)$$

Note that the projected number density $\hat{N}_h(z_L)$ in the denominator on the r.h.s. is an estimator of the projected halo number density, and therefore includes a contribution due to the background density δ_b (see Eq. 4). Similarly to Eq. (13), employing the Limber's approximation and assuming that the underlying power spectrum is not a rapidly varying function within the survey window, we can check that the ensemble average of the power spectrum estimator gives the expectation:

$$\begin{aligned} \langle \hat{C}_{\text{hm}}(k_i) \rangle &= \frac{1}{\bar{N}_h(z_L)} \int d\chi f_h(\chi) \Sigma_{\text{cr}}(\chi) f_\kappa(\chi) \int dM \frac{dn}{dM} S(M) \frac{1}{\chi^2 \Omega_S} \int_{|\mathbf{k}_\perp| \in k_i} \frac{d^2 \mathbf{k}_\perp}{A(k_i)} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} P_{\text{hm}}(|\mathbf{k}_\perp - \mathbf{q}|) |tW(\mathbf{q})|^2 \\ &\simeq \frac{\bar{\rho}_{\text{m}0}}{\bar{N}_h(z_L)} \int d\chi f_h(\chi) \int dM \frac{dn}{dM} S(M) P_{\text{hm}}(k_i; M, \chi) = C_{\text{hm}}(k_i), \end{aligned} \quad (28)$$

where we have used $P_{\text{hm}}(|\mathbf{k}_\perp - \mathbf{q}|) \simeq P_{\text{hm}}(k_\perp)$ over the integration range $d^2 \mathbf{q}$ where the survey window supports, and also assumed that $P_{\text{hm}}(k)$ is not a rapidly varying function within the bin width Δk around the k_i -bin.

C. Covariance matrix of the stacked lensing power spectrum

Now we consider the covariance matrix of the projected power spectrum of stacked lensing, which can be defined in terms of the power spectrum estimator (Eq. 27) as

$$C_{ij} \equiv \langle \hat{C}_{\text{hm}}(k_i) \hat{C}_{\text{hm}}(k_j) \rangle - \langle \hat{C}_{\text{hm}}(k_i) \rangle \langle \hat{C}_{\text{hm}}(k_j) \rangle. \quad (29)$$

By inserting the power spectrum estimator (Eq. 27) and then computing the ensemble averages in the above equation, we will below derive an expression of the covariance matrix.

First of all, we have to bear in mind that the overall normalization of the stacked lensing estimator depends on the local mean of the projected halo number density (Eq. 4):

$$\hat{C}_{\text{hm}} \propto \frac{1}{\bar{N}_{\text{h}}} \equiv \frac{1}{\bar{N}_{\text{h}}} (1 - \Delta_{\text{b}}^{\text{h}}) \quad (30)$$

where $\Delta_{\text{b}}^{\text{h}} \equiv (1/\bar{N}_{\text{h}}) \int d\chi \bar{n}_{\text{h}}^{\text{S}} \bar{b}^{\text{S}} \delta_{\text{b}}$. The background density mode δ_{b} is a statistical variable, varies with a survey window, and therefore contributes the covariance. We need to include

If we conceptually express the power spectrum estimator as $\hat{C}_{\text{hm}}(k_i) \propto (1 - \Delta_{\text{b}}^{\text{h}})(\hat{h}\hat{\kappa})_{k_i}$, where let us denote the estimators of the halo field and the lensing field as \hat{h} and $\hat{\kappa}$, the covariance arises from two contributions:

$$\mathbf{C} \rightarrow \langle (1 - \Delta_{\text{b}}^{\text{h}})^2 (\hat{h}\hat{\kappa})_{k_i} (\hat{h}\hat{\kappa})_{k_j} \rangle - \langle (\hat{h}\hat{\kappa})_{k_i} \rangle \langle (\hat{h}\hat{\kappa})_{k_j} \rangle = \langle \hat{h}_{k_i} \hat{\kappa}_{k_j} \rangle \langle \hat{\kappa}_{k_i} \hat{h}_{k_j} \rangle + \langle \hat{h}_{k_i} \hat{h}_{k_j} \rangle \langle \hat{\kappa}_{k_i} \hat{\kappa}_{k_j} \rangle + \langle (1 - \Delta_{\text{b}}^{\text{h}})^2 (\hat{h}\hat{\kappa})_{k_i} (\hat{h}\hat{\kappa})_{k_j} \rangle_c. \quad (31)$$

The first two terms are from products of the power spectra, $\langle \hat{h}\hat{\kappa} \rangle$, $\langle \hat{h}\hat{h} \rangle$ and $\langle \hat{\kappa}\hat{\kappa} \rangle$, which we call the Gaussian covariance term. The third term arises from the 4-point correlation function or the trispectrum in the Fourier space, which we call the non-Gaussian term. If the halo and lensing fields are both Gaussian, e.g. at linear scales, the non-Gaussian term is vanishing. We generalized the non-Gaussian term in that it includes the contribution from the modulation in the projected halo number density.

Hence, following the method in Ref. [4], the covariance matrix can be generally expressed by a sum of the three terms:

$$\mathbf{C} = \mathbf{C}^{\text{Gauss}} + \mathbf{C}^{\text{T}_0} + \mathbf{C}^{\text{SSC}}, \quad (32)$$

where we have consider the modes satisfying $k_i, k_j \gg 1/(\chi \Omega_{\text{S}}^{1/2})$ (χ is in the range of lens redshifts). The first term is the Gaussian term as we stated above, the second term is the non-Gaussian term arising from the trispectrum of sub-survey modes, and the third term is the super-sample covariance arising from super-survey modes expressed in terms of δ_{b} in our formulation.

1. Gaussian covariance

Extending the formulation in Ref. [7] to include the redshift weight $\Sigma_{\text{cr}}(\chi)$ in the estimator (Eq. 27), we can derive the Gaussian covariance term corresponding to the first and second terms in the formal expression (Eq. 31):

$$C^{\text{Gauss}}(k_i, k_j) = \frac{1}{N_{\text{mode}}(k_i)} \delta_{ij}^K \left[C_{\text{hm}}(k_i) + C_{\text{hh}}^{\Sigma_{\text{cr}}}(k_i) C_{\kappa}^{\text{obs}}(k_i) \right], \quad (33)$$

where δ_{ij}^K is the Kronecker delta function; $\delta_{ij}^K = 1$ if $k_i = k_j$ to within the bin width, otherwise $\delta_{ij}^K = 0$. The number of independent k -modes resolved in the shell, by a resolution of Fourier decomposition of a finite-area survey, is given as

$$N_{\text{mode}}(k_i) \simeq \frac{A(k_i) S_{\text{eff}}}{(2\pi)^2} = \frac{A(k_i) \langle \chi_L^2 \rangle \Omega_{\text{S}}}{(2\pi)^2}, \quad (34)$$

where S_{eff} is the effective survey area in dimension of $[\text{Mpc}^2]$, and $\langle \chi_L^2 \rangle$ is the radial distance to lensing halos, averaged over the radial selection function: $\langle \chi_L^2 \rangle \equiv \int d\chi f_{\text{h}}(\chi; z_L) \chi^2 / \int d\chi f_{\text{h}}(\chi; z_L)$. The spectrum $C_{\text{hh}}^{\Sigma_{\text{cr}}}(k)$ is the projected auto-spectrum of halos including the lensing kernel weight $\Sigma_{\text{cr}}(\chi)$ at each lens redshift (see Eq. 21 or 27) as well as the shot noise contamination arising from a finite number of lensing halos used in the stacked lensing measurement, while $C_{\kappa}^{\text{obs}}(k)$ is the cosmic shear power spectrum including the intrinsic shape noise contamination arising from a finite number of source galaxies. These spectra are given as

$$\begin{aligned} C_{\text{hh}}^{\Sigma_{\text{cr}}} &= \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi f_{\text{h}}(\chi)^2 \Sigma_{\text{cr}}(\chi)^2 [\bar{n}_{\text{h}}^{\text{S}} \bar{b}^{\text{S}}]^2 P_{\text{m}}^{\text{lin}}(k; \chi) + \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi f_{\text{h}}(\chi) \Sigma_{\text{cr}}(\chi)^2 \bar{n}_{\text{h}}^{\text{S}}, \\ C_{\kappa}^{\text{obs}}(k) &= \int d\chi f_{\kappa}^2 P_{\text{m}}^{\text{NL}}(k; \chi) + \frac{\sigma_{\epsilon}^2}{\bar{N}_{\text{sg}}}, \end{aligned} \quad (35)$$

where σ_ϵ is the rms intrinsic ellipticity per component [18] and \bar{N}_{sg} is the mean number density of source galaxies, used for the weak lensing measurements, per unit area in dimension of $[\text{Mpc}^{-2}]$, which is given in terms of the angular source density as $\bar{N}_{\text{sg}} = \bar{n}_{\text{gs}} / \langle \chi_L^2 \rangle$, where \bar{n}_{gs} is the mean angular number density of source galaxies per unit solid steradian as often used in the literature. The above formula of the Gaussian term is new in a sense that the covariance is for the stacked lensing in dimension of the surface mass density, $\langle \Delta \Sigma \rangle (R)$, while the literature usually studied the angular power spectrum.

The Kronecker delta function in Eq. (33) ensures that the Gaussian term contributes only the diagonal terms of the covariance matrix. The covariance term scales with the survey area as $C^{\text{Gauss}} \propto 1/\Omega_S$.

2. Non-Gaussian covariance: the trispectrum consistency and super-sample covariance

Now we consider the non-Gaussian term of the stacked lensing covariance. As we stated above, we need to include the contributions of connected 4-point function and the modulation in the projected number density of lensing halos due to super-survey modes, Δ_b^h (Eqs. 30 and 31).

To derive different contributions to the non-Gaussian covariance in an efficient way, we follow the formulation used in Refs. [4, 5]. For that purpose, let us consider the contribution from the connected 4-point function, by inserting the estimator (Eq. 27) into the covariance definition (Eq. 29):

$$\begin{aligned} \langle (\hat{h}\hat{k})_{k_i} (\hat{h}\hat{k})_{k_j} \rangle &\leftarrow \frac{\bar{\rho}_{\text{m}0}^2}{\bar{N}_h(z_L)^2} \int d\chi f_h(\chi)^2 \int dM_1 \int dM_2 \frac{dn}{dM_1} S(M_1) \frac{dn}{dM_2} S(M_2) \\ &\times \frac{1}{(\chi^2 \Omega_S)^2} \int \frac{d^2 \mathbf{q}}{(2\pi)} |\tilde{W}(\mathbf{q})|^2 \int_{|\mathbf{k}| \in k_i} \frac{d^2 \mathbf{k}}{A(k_i)} \int_{|\mathbf{k}'| \in k_j} \frac{d^2 \mathbf{k}'}{A(k_j)} T_{\text{hmhm}}(\mathbf{k}, -\mathbf{k} + \mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}; M_1, M_2, \chi), \end{aligned} \quad (36)$$

where we have used the Limber's approximation and T_{hmhm} is the 3D trispectrum at redshift $z = z(\chi)$ for the two matter fields and the two fields of halos with their masses M_1 and M_2 , defined as

$$\langle \delta_h(\mathbf{k}_1; M_1) \delta_m(\mathbf{k}_2) \delta_h(\mathbf{k}_3; M_2) \delta_m(\mathbf{k}_4) \rangle \equiv (2\pi)^3 \delta_D^3(\mathbf{k}_{1234}) T_{\text{hmhm}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; M_1, M_2) \quad (37)$$

The convolution with the window function in Eq. (36) means that different 4-point configurations separated by less than the Fourier width of the window function and involving contributions from super-survey modes contribute to the non-Gaussian covariances.

We employ the halo model to model the trispectrum T_{hmhm} . For squeezed configurations with $k, k' \gg q$, using the halo model approach, we can express the change in the trispectrum due to the long wavelength q -mode to leading order of q/k as

$$\delta \bar{T}_{\text{hmhm}} \equiv \int dM_1 \int dM_2 \frac{dn}{dM_1} S(M_1) \frac{dn}{dM_2} S(M_2) [T_{\text{hmhm}}(\mathbf{k}, -\mathbf{k} + \mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}) - T_{\text{hmhm}}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}')], \quad (38)$$

with

$$\begin{aligned} \delta \bar{T}_{\text{hmhm}}^{1h} &\approx 0 \\ \delta \bar{T}_{\text{hmhm}}^{2h} &\approx P_m^{\text{lin}}(q) \end{aligned} \quad (39)$$

3. Covariance formula for the stacked lensing

$$\mathbf{C} = \mathbf{C}^{\text{Gauss}} + \mathbf{C}^T \quad (40)$$

$$C_{ij}^{\text{Gauss}} = \frac{2}{N_{\text{mode}}(k_i)} \delta_{ij}^K [C_{\text{hm}}(k_i)^2 + C_{\text{hh}}^{\Sigma_{\text{cr}}}(k_i) C_{\kappa}(k_i)] \quad (41)$$

$$\begin{aligned} N_{\text{mode}}(k) &\simeq \int d\chi f_h(\chi) \frac{2\pi k \Delta k}{(2\pi/\chi \Theta_S)^2} = \Omega_S \int d\chi f_h(\chi) \frac{k \Delta k \chi^2}{2\pi} \simeq \frac{k \Delta k \chi_L^2 \Omega_S}{2\pi} \\ C_{\text{hh}}^{\Sigma_{\text{cr}}}(k) &= \frac{1}{\bar{N}_h(z_L)^2} \int d\chi f_h(\chi)^2 \Sigma_{\text{cr}}(\chi)^2 \left[\int dM \frac{dn}{dM} b(M) \right]^2 P_m^{\text{lin}}(k; \chi) + \frac{\int d\chi f_h(\chi) \Sigma_{\text{cr}}(\chi)^2 \int dM \frac{dn}{dM}}{\bar{N}_h(z_L)^2} \\ C_{\kappa}(k) &= \int d\chi f_h(\chi)^2 P_m^{\text{NL}}(k; \chi) + \frac{\sigma_\epsilon^2}{\bar{n}_{\text{gs}}}, \end{aligned} \quad (42)$$

Using Limber's approximation,

$$C_{ij}^T = \frac{1}{\bar{N}_h(z_L)^2} \int d\chi f_h(\chi)^2 \Sigma_{cr}(\chi)^2 \frac{1}{(\chi^2 \Omega_S)^2} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} |\tilde{W}(\mathbf{q})|^2 \int_{k_i} \frac{d\mathbf{k}}{V(k_i)} \int_{k_j} \frac{d\mathbf{k}}{V(k_j)} T_{hmhm}(\mathbf{k}, -\mathbf{k} + \mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}) \quad (43)$$

For squeezed configurations with $k, k' \gg q$ we can express the change in the trispectrum due to the long wavelength q mode to leading order in q/k as

$$\delta T_{hmhm} \equiv T_{hmhm}(\mathbf{k}, -\mathbf{k} + \mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}) - T_{hmhm}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \quad (44)$$

with

$$\begin{aligned} \delta T_{hmhm}^{1h} &\simeq 0 \\ \delta T_{hmhm}^{2h(22)} &\simeq P_m^{\text{lin}}(q) I_1^{(S)1}(k) I_1^{(S)1}(k') \\ \delta T_{hmhm}^{2h(13)} &\simeq 0 \\ \delta T_{hmhm}^{3h} &\simeq 2 P_m^{\text{lin}}(q) I_1^{(S)1}(k') \mathcal{F}(\mathbf{k}, \mathbf{q}) + 2 P_m^{\text{lin}}(q) I_1^{(S)1}(k) \mathcal{F}(\mathbf{k}', -\mathbf{q}) \\ \delta T_{hmhm}^{4h} &\simeq 4 P_m^{\text{lin}}(q) \mathcal{F}(\mathbf{k}, \mathbf{q}) \mathcal{F}(\mathbf{k}', -\mathbf{q}) \end{aligned} \quad (45)$$

where

$$\mathcal{F}(\mathbf{k}, \mathbf{q}) \equiv \left[P_m^{\text{lin}}(k) F_2(\mathbf{q}, -\mathbf{k}) + P_m^{\text{lin}}(|\mathbf{k} - \mathbf{q}|) F_2(\mathbf{q}, \mathbf{k} - \mathbf{q}) \right] I_0^{(S)0} I_1^{(S)1}(\mathbf{k} - \mathbf{q}) \quad (46)$$

with the mode-coupling kernel F_2 defined as

$$F_2(\mathbf{k}, \mathbf{q}) \equiv \frac{5}{7} + \frac{1}{2} \left(\frac{1}{k^2} + \frac{1}{q^2} \right) (\mathbf{k} \cdot \mathbf{q}) + \frac{2}{7} \frac{(\mathbf{k} \cdot \mathbf{q})^2}{k^2 q^2} \quad (47)$$

The latter must be handled with care since the mode coupling factor F_2 has a pole when one of its arguments goes to zero. Thus we need to consistently expand this expression in q/k (or q/k'). The result of integrating over the direction of \mathbf{k} is

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \mathcal{F}(\mathbf{k}, \mathbf{q}) = \frac{1}{2} \left[\frac{24}{7} - \frac{1}{2} \frac{d \ln P_{hm}^{\text{lin}}(k)}{d \ln k} \right] P_{hm}^{\text{lin}}(k) \quad (48)$$

$$\frac{d \ln P_{hm}(k)}{d \delta_b} = \left[\frac{24}{7} - \frac{1}{2} \frac{d \ln P_{hm}^{\text{lin}}(k)}{d \ln k} \right] P_{hm}^{\text{lin}}(k) + I_1^{(S)1}(k) \quad (49)$$

$$\begin{aligned} \mathbf{C}^{\text{SSC}} &\leftarrow \left\langle \left[1 - 2 \int d\chi f_h(\chi) \int dM S(M) \frac{dn}{dM} b \delta_b \right] \left[\int d\chi f_h(\chi) P_{hm}^S(k) \left\{ 1 + \frac{d \ln P_{hm}^S(k)}{d \delta_b} \delta_b \right\} \right] \right. \\ &\quad \times \left. \left[\int d\chi f_h(\chi) P_{hm}^S(k') \left\{ 1 + \frac{d \ln P_{hm}^S(k')}{d \delta_b} \delta_b \right\} \right] \right\rangle \\ &\simeq \frac{1}{\bar{N}_h^2} \int d\chi f_h(\chi)^2 \left[\frac{d \ln P_{hm}^S(k)}{d \delta_b} \frac{d \ln P_{hm}^S(k')}{d \delta_b} - 2 I_0^{(S)1} \frac{d \ln P_{hm}^S(k)}{d \delta_b} - 2 I_0^{(S)1} \frac{d \ln P_{hm}^S(k')}{d \delta_b} \right] \frac{1}{(\chi^2 \Omega_S)^2} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} |\tilde{W}(\mathbf{q})|^2 P_m^{\text{lin}}(q) \end{aligned} \quad (50)$$

First of all, note that we define the halo number density fluctuation relative to the mean within the survey window, as shown from Eq. (4):

$$\hat{C}_{hm}(k_i) = \frac{C_{hm}(k_i)}{1 + \delta \ln \hat{N}_h|_{\delta_b}} \simeq C_{hm}(k_i) [1 - \delta \ln \hat{N}_h|_{\delta_b}] \quad (51)$$

$$\delta \ln \hat{C}_{hm}(k_i)|_{\delta_b} \simeq \frac{1}{\bar{N}_h(z_L)} \int d\chi f_h(\chi) \int dM \frac{dn}{dM}(M) S(M) b(M) \delta_b(\chi) \quad (52)$$

$$\begin{aligned}
\langle \delta \ln \hat{C}_{\text{hm}}(k_i)|_{\delta_b} \delta \ln \hat{C}_{\text{hm}}(k_j)|_{\delta_b} \rangle &= \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \int d\chi' f_{\text{h}}(\chi) f_{\text{h}}(\chi') \int dM \int dM' S(M) \frac{dn}{dM}(M) S(M') \frac{dn}{dM}(M') b(M) b(M') \langle \delta_b(\chi) \delta_b(\chi') \rangle \\
&= \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \int d\chi' f_{\text{h}}(\chi) f_{\text{h}}(\chi') \int dM \int dM' S(M) \frac{dn}{dM}(M) S(M') \frac{dn}{dM}(M') b(M) b(M') \\
&\quad \times \frac{1}{\chi^2 \Omega_s \chi'^2 \Omega_s} \int \frac{dk_{\parallel}}{2\pi} \int \frac{dk'_{\parallel}}{2\pi} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \int \frac{d^2 \mathbf{q}'}{(2\pi)^2} \tilde{W}(\mathbf{q}) \tilde{W}(\mathbf{q}') \langle \delta_{\text{m,lin}}(k_{\parallel}, -\mathbf{q}) \delta_{\text{m,lin}}(k'_{\parallel}, -\mathbf{q}') \rangle e^{ik_{\parallel}\chi + i\chi' k'_{\parallel}} \\
&= \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \int d\chi' f_{\text{h}}(\chi) f_{\text{h}}(\chi') \int dM \int dM' S(M) \frac{dn}{dM}(M) S(M') \frac{dn}{dM}(M') \\
&\quad \times \frac{1}{\chi^2 \Omega_s \chi'^2 \Omega_s} \int \frac{dk_{\parallel}}{2\pi} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} |\tilde{W}(\mathbf{q})|^2 P_{\text{m}}^{\text{lin}}(\sqrt{k_{\parallel}^2 + \mathbf{q}^2}) e^{ik_{\parallel}(\chi - \chi')} \\
&\simeq \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi f_{\text{h}}(\chi)^2 \left[\int dM S(M) \frac{dn}{dM}(M) b(M) \right]^2 \frac{1}{(\chi^2 \Omega_s)^2} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} |\tilde{W}(\mathbf{q})|^2 P_{\text{m}}^{\text{lin}}(q)
\end{aligned} \tag{53}$$

The covariance matrix describes expected statistical errors in measurements of the stacked lensing profile as well as how the stacked lensing power spectra of different wavenumber bins are correlated with each other. The covariance matrix has three contributions: the Gaussian term, the non-Gaussian term arising from sub-volume modes, and the non-Gaussian, super-sample covariance (SSC) term arising from super-survey modes.

$$\begin{aligned}
\text{Cov}[P_{\Delta\Sigma}(k), P_{\Delta\Sigma}(k')] &= \mathbf{C}^{\text{Gauss}} + \mathbf{C}^{\text{sub-NG}} + \mathbf{C}^{\text{SSC}} \\
&= \frac{\delta_{kk'}}{N_{\text{mode}}(k)} \left[P_{\Delta\Sigma}(k)^2 + P_{\text{hh}}^{\text{obs}}(k) \Sigma_{\text{cr}}(z_L)^2 \chi_L^2 C_{\kappa}^{\text{obs}}(l = \chi_L k) \right] + \frac{1}{\chi_L^2 \Omega_s} \bar{T}_{\text{hhmm}}^{2\text{D}}(k, k'; z_L) + \sigma_b^2 \frac{\partial P_{\Delta\Sigma}(k)}{\partial \delta_b} \frac{\partial P_{\Delta\Sigma}(k')}{\partial \delta_b}
\end{aligned} \tag{54}$$

where $N_{\text{mode}}(k)$ is the number of Fourier modes taken from the survey volume, $P_{\text{hh}}(k)$ is the auto-power spectrum of halos, and $C_{\kappa}(l)$ is the cosmic shear power spectrum.

$$\begin{aligned}
N_{\text{mode}}(k) &\equiv \frac{2\pi k \Delta k}{[2\pi/(\chi_L \Theta_s)]^2} = k \Delta k \chi_L^2 \Omega_s, \\
P_{\text{hh}}^{\text{obs}}(k) &= P_{\text{hh}}(k) + \frac{1}{\bar{N}_{\text{h}}(z_L)} \\
C_{\kappa}^{\text{obs}}(l) &= C_{\kappa} + \frac{\sigma_{\epsilon}^2}{\bar{n}_{\text{sg}}}.
\end{aligned} \tag{55}$$

The halo auto-power spectrum term accounts for the sample variance arising due to a finite number of halos in the survey region. The cosmic shear term accounts for the contribution arising from the mass distribution along the line-of-sight to source redshifts, and can be computed in terms of the nonlinear mass power spectrum:

$$C_{\kappa}(l) = \int_{\chi \neq [\chi_L - \Delta\chi/2, \chi_L + \Delta\chi/2]} d\chi W_{\kappa}(\chi)^2 \chi^{-2} P_{\text{m}}^{\text{NL}}\left(k = \frac{l}{\chi}; \chi\right), \tag{56}$$

where the line-of-sight integration excludes the contribution of lens redshifts, in order to avoid a double counting.

More precisely to estimate the sample variance contribution due to cosmic shear, we can populate exactly the same configurations of lens-source galaxies into ray-tracing simulations, and

$$\begin{aligned}
\frac{\partial \ln P_{\Delta\Sigma}(k)}{\partial \delta_b} &= -\frac{1}{\bar{N}_{\text{h}}(z_L)} \frac{\partial \bar{N}_{\text{h}}(z_L)}{\partial \delta_b} + \frac{1}{\bar{N}_{\text{h}}(z_L)} \int d\chi \chi^2 \frac{\partial}{\partial \delta_b} \left[\int dM \frac{dn}{dM} S(M, \chi; z_L) P_{\text{hm}}(k; M) \right] \\
&= -\bar{b}_1 + \frac{1}{\bar{N}_{\text{h}}(z_L)} \int d\chi \chi^2 \frac{\partial}{\partial \delta_b} \left[\int dM \frac{dn}{dM} S(M, \chi; z_L) P_{\text{hm}}(k; M) \right]
\end{aligned} \tag{57}$$

$$\bar{T}_{\text{hh}\Delta\Sigma\Delta\Sigma}^{2\text{D}}(k, k') = \frac{\bar{\rho}_{\text{m0}}^2}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^4 \int dM dM' \frac{dn}{dM} \frac{dn}{dM'} S(M, \chi; z_L) S(M', \chi; z_L) T_{\text{hhmm}}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \tag{58}$$

$$T = T^{1h} + (T^{2h(22)} + T^{2h(13)}) + T^{3h} + T^{4h} \quad (59)$$

$$T_{\text{hmhm}}^{1h}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') = \frac{\bar{\rho}_{\text{m0}}^2}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^4 \int dM \frac{dn}{dM} S(M, \chi; z_L)^2 \left(\frac{M}{\bar{\rho}_{\text{m0}}} \right)^2 u_M(k) u_M(k') \quad (60)$$

$$T_{\text{hmhm}}^{2h(13)}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') = \frac{\bar{\rho}_{\text{m0}}^2}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^4 \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \times \left[P_{\text{hh}}(k) \left\{ S(M_1) S(M_2) \left(\frac{M_2}{\bar{\rho}_{\text{m0}}} \right)^2 u_{M_2}(k) u_{M_2}(k') + \frac{M_1}{\bar{\rho}_{\text{m0}}} u_{M_1}(k) S(M_2)^2 \frac{M_2}{\bar{\rho}_{\text{m0}}} u_{M_2}(k') \right\} + (k \leftrightarrow k') \right] \quad (61)$$

$$T_{\text{hmhm}}^{2h(22)}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') = \frac{\bar{\rho}_{\text{m0}}^2}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^4 \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \times \left[P_{\text{hh}}(|\mathbf{k} + \mathbf{k}'|) S(M_1)^2 \left(\frac{M_2}{\bar{\rho}_{\text{m0}}} \right)^2 u_{M_2}(k) u_{M_2}(k') + P_{\text{hh}}(|\mathbf{k} - \mathbf{k}'|) S(M_1) S(M_2) \frac{M_1}{\bar{\rho}_{\text{m0}}} \frac{M_1}{\bar{\rho}_{\text{m0}}} u_{M_1}(k') u_{M_2}(k) \right] \quad (62)$$

$$T_{\text{hmhm}}^{3h}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') = \frac{\bar{\rho}_{\text{m0}}^2}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^4 \int dM_1 dM_2 dM_3 \frac{dn}{dM_1} \frac{dn}{dM_2} \frac{dn}{dM_3} \times \left[B_{\text{hhh}}(\mathbf{k} + \mathbf{k}', -\mathbf{k}, -\mathbf{k}') S(M_1)^2 \frac{M_2}{\bar{\rho}_{\text{m0}}} u_{M_2}(k) \frac{M_3}{\bar{\rho}_{\text{m0}}} u_{M_3}(k') + B_{\text{hhh}}(\mathbf{k} - \mathbf{k}', -\mathbf{k}, \mathbf{k}') S(M_1) \frac{M_1}{\bar{\rho}_{\text{m0}}} u_{M_1}(k') \frac{M_2}{\bar{\rho}_{\text{m0}}} u_{M_2}(k) S(M_3) + B_{\text{hhh}}(-\mathbf{k} + \mathbf{k}', \mathbf{k}, -\mathbf{k}') S(M_1) \frac{M_1}{\bar{\rho}_{\text{m0}}} u_{M_1}(k) S(M_2) \frac{M_3}{\bar{\rho}_{\text{m0}}} u_{M_3}(k') + B_{\text{hhh}}(-\mathbf{k} - \mathbf{k}', \mathbf{k}, \mathbf{k}') \left(\frac{M_1}{\bar{\rho}_{\text{m0}}} \right)^2 u_{M_1}(k) u_{M_1}(k') S(M_2) S(M_3) \right] \quad (63)$$

$$T_{\text{hmhm}}^{4h}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') = \frac{\bar{\rho}_{\text{m0}}^2}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^4 \times \int dM_1 dM_2 dM_3 dM_4 \frac{dn}{dM_1} \frac{dn}{dM_2} \frac{dn}{dM_3} \frac{dn}{dM_4} S(M_1) S(M_3) T_{\text{hhhh}}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \quad (64)$$

$$T_{\text{hmhm}}^{1h}(\mathbf{k}, -\mathbf{k} + \mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}) \simeq T_{\text{hmhm}}^{1h}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \quad (65)$$

$$T_{\text{hmhm}}^{2h(13)}(\mathbf{k}, -\mathbf{k} + \mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}) \simeq T_{\text{hmhm}}^{2h(22)}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') + P_{\text{hh}}(q) S(M_1) \frac{M_1}{\bar{\rho}_{\text{m0}}} u_{M_1}(k) S(M_2) \frac{M_2}{\bar{\rho}_{\text{m0}}} u_{M_2}(k') \quad (66)$$

$$T_{\text{hmhm}}^{3h}(\mathbf{k}, -\mathbf{k} + \mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}) \simeq T_{\text{hmhm}}^{3h}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') + B_{\text{hhh}}(\mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}) S(M_1) \frac{M_1}{\bar{\rho}_{\text{m0}}} u_{M_1}(k) S(M_2) \frac{M_3}{\bar{\rho}_{\text{m0}}} u_{M_3}(k') + B_{\text{hhh}}(-\mathbf{q}, \mathbf{k}, -\mathbf{k} + \mathbf{q}) S(M_1) \frac{M_1}{\bar{\rho}_{\text{m0}}} u_{M_1}(k') S(M_2) \frac{M_3}{\bar{\rho}_{\text{m0}}} u_{M_3}(k) \quad (67)$$

$$T_{\text{hmhm}}^{4h}(\mathbf{k}, -\mathbf{k} + \mathbf{q}, \mathbf{k}', -\mathbf{k}' - \mathbf{q}) \simeq T_{\text{hhhh}}^{4h}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') + \dots \quad (68)$$

$$B_{\text{m}}^{\text{PT}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 2F_2(\mathbf{k}_1, \mathbf{k}_2) P_{\text{m}}^{\text{lin}}(k_1) P_{\text{m}}^{\text{lin}}(k_2) + 2 \text{ perm.} \\ T_{\text{m}}^{\text{PT}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = 4 \left[F_2(\mathbf{k}_{13}, -\mathbf{k}_1) F_2(\mathbf{k}_{13}, \mathbf{k}_2) P_{\text{m}}^{\text{lin}}(k_{13}) P_{\text{m}}^{\text{lin}}(k_1) P_{\text{m}}^{\text{lin}}(k_2) + 11 \text{ perm.} \right] \\ + 6 \left[F_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) P_{\text{m}}^{\text{lin}}(k_1) P_{\text{m}}^{\text{lin}}(k_2) P_{\text{m}}^{\text{lin}}(k_3) + 3 \text{ perm.} \right] \quad (69)$$

$$F_2(\mathbf{k}_1, \mathbf{k}_2) \equiv \frac{5}{7} + \frac{1}{2} \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} \right) (\mathbf{k}_1 \cdot \mathbf{k}_2) + \frac{2}{7} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2}, \\ F_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \equiv \frac{7}{18} \frac{\mathbf{k}_{12} \cdot \mathbf{k}_1}{k_1^2} [F_2(\mathbf{k}_2, \mathbf{k}_3) + G_2(\mathbf{k}_1, \mathbf{k}_2)] + \frac{1}{18} \frac{k_{12}^2 (\mathbf{k}_1 \cdot \mathbf{k}_2)}{k_1^2 k_2^2} [G_2(\mathbf{k}_2, \mathbf{k}_3) + G_2(\mathbf{k}_1, \mathbf{k}_2)], \\ G_2(\mathbf{k}_1, \mathbf{k}_2) \equiv \frac{3}{7} + \frac{1}{2} \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} \right) (\mathbf{k}_1 \cdot \mathbf{k}_2) + \frac{4}{7} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} \quad (70)$$

The lensing convergence field, more generally any two-dimensional field projected along the line-of-sight, can be expressed as

$$\kappa(\boldsymbol{\theta}) = \int d\chi W(\chi) \delta_m(\chi, \chi\boldsymbol{\theta}) \simeq \sum_i \Delta\chi W(\chi_i) \delta_m(\chi_i, \chi\boldsymbol{\theta}), \quad (71)$$

where $W(\chi)$ is the radial weight function. To measure a cross-correlation of the projected field with the distribution of halos at a particular redshift z_L [8, 17], we can estimate the three-dimensional distribution of the underlying mass density field at the halo redshift:

$$\hat{\delta}_m^*(\mathbf{k}_\perp; z_L) \equiv \int_S d^2\mathbf{x}_\perp W^{-1}(z_L) \kappa(\boldsymbol{\theta}) e^{-i\mathbf{k}_\perp \cdot \chi_L \boldsymbol{\theta}} \simeq \delta_m(\mathbf{k}_\perp, k_\parallel; \chi_L) + \chi_L^{-2} W^{-1}(z_L) \kappa_1 \quad (72)$$

where $\chi_L \equiv \chi(z_L)$ and \mathbf{x}_\perp and \mathbf{k}_\perp are the two-dimensional spatial vector and wavevector in the plane perpendicular to the line-of-sight direction.

$$\langle \widehat{\Delta\Sigma}(k) (2\pi)^2 \delta_D^2(\mathbf{k} + \mathbf{k}') \rangle \equiv \langle \hat{\delta}_h^{2D}(\mathbf{k}, \chi) \Sigma_{\text{cr}}(\chi) \rangle \quad (73)$$

The two-dimensional Fourier mode in the lens plane at redshift z can be estimated as

$$\widehat{\Delta\Sigma}(\mathbf{k}_\perp; \chi) \equiv \Sigma_{\text{cr}}(\chi) \int_{S=\chi\Omega_S} d^2\mathbf{x}_\perp \kappa(\boldsymbol{\theta}) e^{-i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}, \quad (74)$$

where \mathbf{x}_\perp is the two-dimensional position vector in the plane at z , $\mathbf{x}_\perp = \chi\boldsymbol{\theta}$.

$$\widehat{\Delta\Sigma}(\mathbf{k}_\perp) = \chi_L^2 \kappa(\mathbf{l} = \chi_L \mathbf{k}_\perp) \quad (75)$$

$$\begin{aligned} \langle \widehat{\Delta\Sigma}(\mathbf{k}_\perp) \widehat{\Delta\Sigma}(\mathbf{k}'_\perp) \rangle &\equiv P_{\Delta\Sigma}(k_\perp) (2\pi)^2 \delta_D^2(\mathbf{k}_\perp + \mathbf{k}'_\perp) \\ &\leftarrow \chi_L^4 C_\kappa(l = \chi_L k_\perp) (2\pi)^2 \delta_D^2(\mathbf{l} + \mathbf{l}')|_{\mathbf{l}=\chi_L \mathbf{k}_\perp} \\ &= \chi_L^2 C_\kappa(l = \chi_L k_\perp) (2\pi)^2 \delta_D^2(\mathbf{k}_\perp + \mathbf{k}'_\perp) \end{aligned} \quad (76)$$

Cross-correlating the convergence field on the sky with the distribution of halos in the particular redshift slice around z_L allows us to probe the mass power spectrum at the redshift:

$$\Delta\Sigma(R; z_L) \equiv \langle \delta_h^{2D}(\mathbf{x}_\perp; z_L) \Sigma_{\text{cr}}(z_L) \kappa(\boldsymbol{\theta}) \rangle_{R=|\mathbf{x}_\perp - \chi_L \boldsymbol{\theta}|}, \quad (77)$$

where the average is done over all the pairs of source galaxy and lensing halo that are separate by the projected distance R to within the bin width, in the plane perpendicular to the line-of-sight. The stacked lensing profiles can be expressed in terms of the power spectrum as

$$\begin{aligned} \Sigma(R; z_L) &= \int \frac{k dk}{2\pi} P_{\Delta\Sigma}(k; z_L) J_0(kR), \\ \Delta\Sigma(R; z_L) &= \int \frac{k dk}{2\pi} P_{\Delta\Sigma}(k; z_L) J_2(kR), \end{aligned} \quad (78)$$

where $J_0(x)$ and $J_2(x)$ are the zero-th and 2nd order Bessel functions, and the power spectrum $P_{\Delta\Sigma}(k)$ is given as

$$P_{\Delta\Sigma}(k) = \frac{\bar{\rho}_{m0}}{\bar{N}_h(z_L)} \int d\chi \chi^2 \int dM \frac{dn}{dM} S(M, z_L) P_{\text{hm}}(k; M). \quad (79)$$

Hence an estimator of the matter power spectrum at z_L can be constructed from the cross-correlation function:

$$\hat{P}_{\text{hm}}(k; z_L) = \langle \delta_h^*(\mathbf{k}; z_L) \hat{\delta}_m^*(\mathbf{k}) \rangle \quad (80)$$

III. RESULTS

IV. DISCUSSION

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$$\begin{aligned}
w_{\text{hh}}(\mathbf{R}; z_L) &= \left\langle \delta_{\text{h}}^{2\text{D}}(\mathbf{x}_{\perp}; z_L) \delta_{\text{h}}^{2\text{D}}(\mathbf{x}'_{\perp}; z_L) \right\rangle_{R=|\mathbf{x}_{\perp}-\mathbf{x}'_{\perp}|} \\
&= \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^2 \int d\chi' \chi'^2 \int dM \frac{dn}{dM} S(M, \chi; z_L) \int dM' \frac{dn}{dM'} S(M', \chi'; z_L) \\
&\quad \times \int \frac{d\mathbf{k}_{\parallel} d^2 \mathbf{k}_{\perp}}{(2\pi)^3} \int \frac{d\mathbf{k}'_{\parallel} d^2 \mathbf{k}'_{\perp}}{(2\pi)^3} \langle \delta_{\text{h}}(\mathbf{k}; \chi, M) \delta_{\text{h}}(\mathbf{k}'; \chi', M') \rangle e^{i(\chi k_{\parallel} + \mathbf{x}_{\perp} \cdot \mathbf{k}_{\perp}) + i(\chi' k'_{\parallel} + \mathbf{x}'_{\perp} \cdot \mathbf{k}'_{\perp})} \\
&= \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^2 \int d\chi' \chi'^2 \int dM \frac{dn}{dM} S(M, \chi; z_L) \int dM' \frac{dn}{dM'} S(M', \chi'; z_L) \\
&\quad \times \int \frac{d\mathbf{k}_{\parallel} d^2 \mathbf{k}_{\perp}}{(2\pi)^3} P_{\text{hh}}(k; M, M') e^{ik_{\parallel}(\chi - \chi') + i\mathbf{k}_{\perp} \cdot \mathbf{R}} \\
&\simeq \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \int d\chi \chi^4 \int dM \int dM' \frac{dn}{dM} S(M, \chi; z_L) \frac{dn}{dM'} S(M', \chi'; z_L) \int \frac{d^2 \mathbf{k}_{\perp}}{(2\pi)^2} P_{\text{hh}}(k_{\perp}; M, M', z) e^{i\mathbf{k}_{\perp} \cdot \mathbf{R}} \\
&\simeq \frac{1}{\bar{N}_{\text{h}}(z_L)^2} \chi_L^4 \Delta\chi \int dM \int dM' \frac{dn}{dM} S(M, \chi; z_L) \frac{dn}{dM'} S(M', \chi'; z_L) \int \frac{d^2 \mathbf{k}_{\perp}}{(2\pi)^2} P_{\text{hh}}(k_{\perp}; M, M', z) e^{i\mathbf{k}_{\perp} \cdot \mathbf{R}} \\
&= \frac{1}{\frac{dn}{dM}(z_L)^2 \Delta\chi} \int dM \int dM' \frac{dn}{dM} S(M, \chi; z_L) \frac{dn}{dM'} S(M', \chi'; z_L) \int \frac{d^2 \mathbf{k}_{\perp}}{(2\pi)^2} P_{\text{hh}}(k_{\perp}; M, M', z) e^{i\mathbf{k}_{\perp} \cdot \mathbf{R}}
\end{aligned} \tag{81}$$

$$\begin{aligned}
\Delta\Sigma(R; z_L) &\equiv \langle \delta_h(\mathbf{x}_\perp; z_L) \Sigma_{\text{cr}}(z_L) \kappa(\boldsymbol{\theta}) \rangle_{|\mathbf{x}_\perp - \chi_L \boldsymbol{\theta}|=R} \\
&= \frac{1}{\bar{N}_h(z_L)} \Sigma_{\text{cr}}(z_L) \int d\chi \chi^2 \int d\chi' f_\kappa(z) \int dM \frac{dn}{dM} S(M, \chi; z_L) \langle \delta_h(\chi, \mathbf{x}_\perp; M) \delta_m(\chi', \chi' \boldsymbol{\theta}) \rangle \\
&= \frac{1}{\bar{N}_h(z_L)} \Sigma_{\text{cr}}(z_L) \int d\chi \chi^2 \int d\chi' f_\kappa(z) \int dM \frac{dn}{dM} S(M, \chi; z_L) \\
&\quad \times \int \frac{d^2 \mathbf{k}_\perp d k_\parallel}{(2\pi)^3} \int \frac{d^2 \mathbf{k}'_\perp d k'_\parallel}{(2\pi)^3} \langle \delta_h(\mathbf{k}; M) \delta_m(\mathbf{k}') \rangle e^{i(\chi k_\parallel + \mathbf{k}_\perp \cdot \mathbf{x}_\perp) + i(\chi' k'_\parallel + \mathbf{k}'_\perp \cdot \chi' \boldsymbol{\theta})} \\
&= \frac{1}{\bar{N}_h(z_L)} \Sigma_{\text{cr}}(z_L) \int d\chi \chi^2 \int d\chi' f_\kappa(z) \int dM \frac{dn}{dM} S(M, \chi; z_L) \int \frac{d^2 \mathbf{k}_\perp d k_\parallel}{(2\pi)^3} P_{\text{hm}}(k; M) e^{ik_\parallel(\chi - \chi') + i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \chi' \boldsymbol{\theta})} \\
&\simeq \frac{1}{\bar{N}_h(z_L)} \Sigma_{\text{cr}}(z_L) \int d\chi \chi^2 f_\kappa(\chi) \int dM \frac{dn}{dM} S(M, \chi; z_L) \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} P_{\text{hm}}(k_\perp; M) e^{i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \chi \boldsymbol{\theta})} \\
&\simeq \frac{1}{\bar{N}_h(z_L)} \Sigma_{\text{cr}}(z_L) \Delta\chi \chi_L^2 f_\kappa(\chi_L) \int dM \frac{dn}{dM} S(M; z_L) \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} P_{\text{hm}}(k_\perp; M) e^{i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \chi_L \boldsymbol{\theta})} \\
&= \frac{1}{\frac{dn}{dM}(z_L)} \bar{\rho}_{\text{m}0} \int dM \frac{dn}{dM} S(M; z_L) \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} P_{\text{hm}}(k_\perp; M) e^{i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \chi_L \boldsymbol{\theta})} \tag{82}
\end{aligned}$$

$$P_{\Delta\Sigma}(k; z_L) = \frac{1}{\frac{dn}{dM}(z_L)} \bar{\rho}_{\text{m}0} \int dM \frac{dn}{dM} S(M; z_L) P_{\text{hm}}(k; M). \tag{83}$$

$$\Delta\Sigma(R; z_L) = \int \frac{k dk}{2\pi} P_{\Delta\Sigma}(k; z_L) J_2(kR). \tag{84}$$

$$f(\mathbf{x}_\perp) = \frac{1}{\chi_L^2 \Omega_S} \sum_{\mathbf{k}} f(\mathbf{k}_\perp) e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} \simeq \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} f(\mathbf{k}_\perp) e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} \tag{85}$$

$$\hat{P}_{\text{hm}}(k_i) = \frac{1}{\chi_L^2 \Omega_S N_{\text{mode}}(k)} \sum_{\mathbf{k}; k \in k_i} \hat{\delta}_h(\mathbf{k}) \widehat{\Delta\Sigma}(-\mathbf{k}) \tag{86}$$

$$\begin{aligned}
\text{Cov}[P_{\text{hm}}(k_i), P_{\text{hm}}(k_j)]^{\text{NG}} &= \frac{1}{\chi_L^4 \Omega_S^2 N_{\text{mode}}(k) N_{\text{mode}}(k')} \sum_{\mathbf{k}; k \in k_i} \sum_{\mathbf{k}'; k' \in k_j} \langle \hat{\delta}_h(\mathbf{k}) \widehat{\Delta\Sigma}(-\mathbf{k}) \hat{\delta}_h(\mathbf{k}') \widehat{\Delta\Sigma}(-\mathbf{k}') \rangle_c \\
&= \frac{1}{\chi_L^4 \Omega_S^2 N_{\text{mode}}(k) N_{\text{mode}}(k')} \sum_{\mathbf{k}; k \in k_i} \sum_{\mathbf{k}'; k' \in k_j} (\chi_L^2 \Omega_S) \delta_{\mathbf{k}-\mathbf{k}+\mathbf{k}'-\mathbf{k}'}^K T_{\text{hh}\Delta\Sigma\Delta\Sigma}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \\
&= \frac{1}{\chi_L^2 \Omega_S N_{\text{mode}}(k) N_{\text{mode}}(k')} \sum_{\mathbf{k}; k \in k_i} \sum_{\mathbf{k}'; k' \in k_j} T_{\text{hh}\Delta\Sigma\Delta\Sigma}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \\
&= \frac{\chi_L^2 \Omega_S}{N_{\text{mode}}(k) N_{\text{mode}}(k')} \int_{k \in k_i} \frac{d^2 \mathbf{k}}{(2\pi)^2} \int_{k' \in k_j} \frac{d^2 \mathbf{k}'}{(2\pi)^2} T_{\text{hh}\Delta\Sigma\Delta\Sigma}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \\
&= \frac{1}{\chi_L^2 \Omega_S} \int_{k \in k_i} \frac{d^2 \mathbf{k}}{2\pi k_i \Delta k} \int_{k' \in k_j} \frac{d^2 \mathbf{k}'}{2\pi k_j \Delta k} T_{\text{hh}\Delta\Sigma\Delta\Sigma}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \\
&= \frac{1}{4\pi f_{\text{sky}} \chi_L^2} \int_{k \in k_i} \frac{d^2 \mathbf{k}}{2\pi k_i \Delta k} \int_{k' \in k_j} \frac{d^2 \mathbf{k}'}{2\pi k_j \Delta k} T_{\text{hh}\Delta\Sigma\Delta\Sigma}(\mathbf{k}, -\mathbf{k}, \mathbf{k}', -\mathbf{k}') \tag{87}
\end{aligned}$$

$$\begin{aligned}
& \langle \hat{\delta}_h^{2D}(\mathbf{k}_1) \hat{\delta}_h^{2D}(\mathbf{k}_2) \widehat{\Delta \Sigma}(\mathbf{k}_3) \widehat{\Delta \Sigma}(\mathbf{k}_4) \rangle = \frac{\Sigma_{\text{cr}}(z_L)^2}{\bar{N}_h(z_L)^2} \int d\chi_1 d\chi_2 d\chi_3 d\chi_4 \chi_1^2 \chi_2^2 W_\kappa(\chi_3) W_\kappa(\chi_4) \\
& \quad \times \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} S(M_1, \chi_1; z_L) S(M_2, \chi_2; z_L) \\
& \quad \times \int \frac{dk_{1\parallel}}{2\pi} \dots \frac{dk_{4\parallel}}{2\pi} \langle \delta_h(\mathbf{k}_1) \delta_h(\mathbf{k}_2) \delta_m(\mathbf{k}_3) \delta_m(\mathbf{k}_4) \rangle e^{ik_{1\parallel}\chi_1 + \dots + ik_{4\parallel}\chi_4} \\
& = (2\pi)^2 \delta_D^2(\mathbf{k}_{1\perp} + \dots + \mathbf{k}_{4\perp}) \frac{\Sigma_{\text{cr}}(z_L)^2}{\bar{N}_h(z_L)^2} \int d\chi_1 d\chi_2 d\chi_3 d\chi_4 \chi_1^2 \chi_2^2 W_\kappa(\chi_3) W_\kappa(\chi_4) \\
& \quad \times \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} S(M_1, \chi_1; z_L) S(M_2, \chi_2; z_L) \\
& \quad \times \int \frac{dk_{1\parallel}}{2\pi} \dots \frac{dk_{3\parallel}}{2\pi} T_{\text{hhmm}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) e^{ik_{1\parallel}(\chi_1 - \chi_4) + ik_{2\parallel}(\chi_2 - \chi_4) + ik_{3\parallel}(\chi_3 - \chi_4)} \\
& \simeq (2\pi)^2 \delta_D^2(\mathbf{k}_{1\perp} + \dots + \mathbf{k}_{4\perp}) \frac{\Sigma_{\text{cr}}(z_L)^2}{\bar{N}_h(z_L)^2} \int d\chi \chi^4 W_\kappa(\chi)^2 \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} S(M_1, \chi_1; z_L) S(M_2, \chi_2; z_L) T_{\text{hhmm}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; M_1, M_2, \chi_L) \\
& \simeq (2\pi)^2 \delta_D^2(\mathbf{k}_{1\perp} + \dots + \mathbf{k}_{4\perp}) \frac{\bar{\rho}_{\text{m}0}^2}{\bar{N}_h(z_L)^2} \chi_L^4 \Delta\chi \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} S(M_1, \chi_1; z_L) S(M_2, \chi_2; z_L) T_{\text{hhmm}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; \chi_L) \\
& = (2\pi)^2 \delta_D^2(\mathbf{k}_{1\perp} + \dots + \mathbf{k}_{4\perp}) \frac{\bar{\rho}_{\text{m}0}^2}{\frac{dn}{dM}(z_L)^2 \Delta\chi} \int dM_1 dM_2 \frac{dn}{dM} \frac{dn}{dM} S(M, z_L) S(M', z_L) T_{\text{hhmm}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; M, M', \chi_L) \tag{88}
\end{aligned}$$

$$\begin{aligned}
\delta_h(\mathbf{x}_1) &= \frac{1}{\bar{n}_h} n_h(\mathbf{x}_1) - 1 = \frac{1}{\frac{dn}{dM}} \sum_i \delta_D^3(\mathbf{x}_i - \mathbf{x}_1) - 1 \\
\delta_m(\mathbf{x}_1) &= \frac{1}{\bar{\rho}_{\text{m}0}} \sum_i m_i u_{m_i}(\mathbf{x}_1 - \mathbf{x}_i) - 1 \tag{89}
\end{aligned}$$

$$\xi_{4,\text{hhmm}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \langle \delta_h(\mathbf{x}_1) \delta_h(\mathbf{x}_2) \delta_m(\mathbf{x}_3) \delta_m(\mathbf{x}_4) \rangle \tag{90}$$

$$\begin{aligned}
\frac{dn}{dM} \rho_{\text{m}0}^2 \xi_{4,\text{hhmm}}^{\text{1h}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \left\langle \sum_i \delta_D^3(\mathbf{x}_1 - \mathbf{x}_i) \delta_D^3(\mathbf{x}_2 - \mathbf{x}_i) M_i u_{M_i}(\mathbf{x}_3 - \mathbf{x}_i) M_i u_{M_i}(\mathbf{x}_4 - \mathbf{x}_i) \right\rangle \\
&= \int dM \frac{dn}{dM} M^2 \int d\mathbf{x} \delta_D^3(\mathbf{x}_1 - \mathbf{x}) \delta_D^3(\mathbf{x}_2 - \mathbf{x}) u_M(\mathbf{x}_3 - \mathbf{x}) u_M(\mathbf{x}_4 - \mathbf{x}) \\
&= \int dM \frac{dn}{dM} M^2 \int d\mathbf{x} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} e^{-i\mathbf{x} \cdot (\mathbf{k}_1 + \dots + \mathbf{k}_4)} u_M(k_3) u_M(k_4) \\
&= \int dM \frac{dn}{dM} M^2 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} u_M(k_3) u_M(k_4) (2\pi)^3 \delta_D^3(\mathbf{k}_1 + \dots + \mathbf{k}_4) \\
&\rightarrow T_{\text{hhmm}}^{\text{1h}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \frac{1}{\frac{dn}{dM}^2} \int dM \frac{dn}{dM} \left(\frac{M}{\bar{\rho}_{\text{m}0}} \right)^2 u_M(k_3) u_M(k_4) \tag{91}
\end{aligned}$$

$$\begin{aligned}
\frac{dn}{dM} \rho_{m0}^2 \xi_{4,hmmm}^{2h-(13)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \left\langle \sum_{i,j;i \neq j} \delta_D^3(\mathbf{x}_1 - \mathbf{x}_i) \delta_D^3(\mathbf{x}_2 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_3 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_4 - \mathbf{x}_j) \right\rangle \\
&\quad + \left\langle \sum_{i,j;i \neq j} M_i u_{M_i}(\mathbf{x}_3 - \mathbf{x}_i) \delta_D^3(\mathbf{x}_1 - \mathbf{x}_j) \delta_D^3(\mathbf{x}_2 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_4 - \mathbf{x}_j) \right\rangle + (\text{perm.}) \\
&= \int d^3 \mathbf{y}_1 d^3 \mathbf{y}_2 \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \xi_{hh}(\mathbf{y}_1 - \mathbf{y}_2; M_1, M_2) \left[\delta_D^3(\mathbf{x}_1 - \mathbf{y}_1) \delta_D^3(\mathbf{x}_2 - \mathbf{y}_2) M_2^2 u_{M_2}(\mathbf{x}_3 - \mathbf{y}_2) u_{M_2}(\mathbf{x}_4 - \mathbf{y}_2) \right. \\
&\quad \left. + M_1 u_{M_1}(\mathbf{x}_3 - \mathbf{y}_1) \delta_D^3(\mathbf{x}_1 - \mathbf{y}_2) \delta_D^3(\mathbf{x}_2 - \mathbf{y}_2) M_2 u_{M_2}(\mathbf{x}_4 - \mathbf{y}_2) + (\text{perm.}) \right] \\
&= \int d^3 \mathbf{y}_1 d^3 \mathbf{y}_2 \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} P_{hh}(q; M_1, M_2) e^{i\mathbf{q} \cdot (\mathbf{y}_1 - \mathbf{y}_2)} \\
&\quad \times \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \left[e^{-i\mathbf{k}_1 \cdot \mathbf{y}_1} e^{-i\mathbf{y}_2 \cdot (\mathbf{k}_2 + \mathbf{x}_3 + \mathbf{k}_4)} M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \right. \\
&\quad \left. + e^{-i\mathbf{y}_1 \cdot \mathbf{k}_3} e^{-i\mathbf{y}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_4)} M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + (\text{perm.}) \right] \\
&= \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} P_{hh}(q; M_1, M_2) \\
&\quad \times \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \left[(2\pi)^3 \delta_D^3(\mathbf{q} - \mathbf{k}_1) (2\pi)^3 \delta_D^3(\mathbf{q} + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \right. \\
&\quad \left. + (2\pi)^3 \delta_D^3(\mathbf{q} - \mathbf{k}_3) (2\pi)^3 \delta_D^3(\mathbf{q} + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_4) M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + (\text{perm.}) \right] \\
&= \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} (2\pi)^3 \delta_D^3(\mathbf{k}_{1234}) e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \\
&\quad \times \left[P_{hh}(k_1) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) + P_{hh}(k_3) M_1 M_2 u_{M_1}(k_3) u_{M_2}(k_4) + (\text{perm.}) \right] \\
&\rightarrow T_{4,hmmm}^{2h(13)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \left[\{P_{hh}(k_1) + P_{hh}(k_2)\} M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \right. \\
&\quad \left. + P_{hh}(k_3) M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + P_{hh}(k_4) M_1 u_{M_1}(k_4) M_2 u_{M_2}(k_4) \right]
\end{aligned} \tag{92}$$

$$\begin{aligned}
\frac{dn}{dM} \rho_{m0}^2 \xi_{4,hmmm}^{2h-(22)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \left\langle \sum_{i,j;i \neq j} \delta_D^3(\mathbf{x}_1 - \mathbf{x}_i) \delta_D^3(\mathbf{x}_2 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_3 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_4 - \mathbf{x}_j) \right\rangle \\
&\quad + \left\langle \sum_{i,j;i \neq j} \delta_D^3(\mathbf{x}_1 - \mathbf{x}_i) \delta_D^3(\mathbf{x}_2 - \mathbf{x}_i) M_i u_{M_j}(\mathbf{x}_3 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_4 - \mathbf{x}_j) \right\rangle + (\text{perm.}) \\
&= \int d^3 \mathbf{y}_1 d^3 \mathbf{y}_2 \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \xi_{hh}(\mathbf{y}_1 - \mathbf{y}_2; M_1, M_2) \left[\delta_D^3(\mathbf{x}_1 - \mathbf{y}_1) \delta_D^3(\mathbf{x}_2 - \mathbf{y}_1) M_2^2 u_{M_2}(\mathbf{x}_3 - \mathbf{y}_2) u_{M_2}(\mathbf{x}_4 - \mathbf{y}_2) \right. \\
&\quad \left. + \delta_D^3(\mathbf{x}_1 - \mathbf{y}_1) M_1 u_{M_1}(\mathbf{x}_3 - \mathbf{y}_1) \delta_D^3(\mathbf{x}_2 - \mathbf{y}_2) M_2 u_{M_2}(\mathbf{x}_4 - \mathbf{y}_2) + (\text{perm.}) \right] \\
&= \int d^3 \mathbf{y}_1 d^3 \mathbf{y}_2 \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} P_{hh}(q; M_1, M_2) e^{i\mathbf{q} \cdot (\mathbf{y}_1 - \mathbf{y}_2)} \\
&\quad \times \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \left[e^{-i\mathbf{y}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2)} e^{-i\mathbf{y}_2 \cdot (\mathbf{k}_3 + \mathbf{k}_4)} M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \right. \\
&\quad \left. + e^{-i\mathbf{y}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_3)} e^{-i\mathbf{y}_2 \cdot (\mathbf{k}_2 + \mathbf{k}_4)} M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + (\text{perm.}) \right] \\
&= \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} P_{hh}(q; M_1, M_2) \\
&\quad \times \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \left[(2\pi)^3 \delta_D^3(\mathbf{q} - \mathbf{k}_1 - \mathbf{k}_2) (2\pi)^3 \delta_D^3(\mathbf{q} + \mathbf{k}_3 + \mathbf{k}_4) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \right. \\
&\quad \left. + (2\pi)^3 \delta_D^3(\mathbf{q} - \mathbf{k}_1 - \mathbf{k}_3) (2\pi)^3 \delta_D^3(\mathbf{q} + \mathbf{k}_2 + \mathbf{k}_4) M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + (\text{perm.}) \right] \\
&= \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} (2\pi)^3 \delta_D^3(\mathbf{k}_{1234}) e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \\
&\quad \times \left[P_{hh}(k_{12}) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) + P_{hh}(k_{13}) M_1 M_2 u_{M_1}(k_3) u_{M_2}(k_4) + (\text{perm.}) \right] \\
&\rightarrow T_{4,hmmm}^{2h(22)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \int dM_1 dM_2 \frac{dn}{dM_1} \frac{dn}{dM_2} \left[P_{hh}(k_{12}) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \right. \\
&\quad \left. + \{P_{hh}(k_{13}) + P_{hh}(k_{14})\} M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) \right]
\end{aligned} \tag{93}$$

$$\begin{aligned}
\frac{dn}{dM} \rho_{m0}^2 \epsilon_{4,hmm}^{3h}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \left\langle \sum_{i,j,k} \delta_D^3(\mathbf{x}_1 - \mathbf{x}_i) \delta_D^3(\mathbf{x}_2 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_3 - \mathbf{x}_j) M_k u_{M_k}(\mathbf{x}_4 - \mathbf{x}_k) \right\rangle \\
&\quad + \left\langle \sum_{i,j,k} \delta_D^3(\mathbf{x}_1 - \mathbf{x}_i) M_i u_{M_i}(\mathbf{x}_3 - \mathbf{x}_i) \delta_D^3(\mathbf{x}_2 - \mathbf{x}_j) M_k u_{M_k}(\mathbf{x}_4 - \mathbf{x}_k) \right\rangle + (\text{perm.}) \\
&= \int d^3 \mathbf{y}_1 d^3 \mathbf{y}_2 d^3 \mathbf{y}_3 \int dM_1 dM_2 dM_3 \frac{dn}{dM_1} \frac{dn}{dM_2} \frac{dn}{dM_3} \int \frac{d^3 \mathbf{q}_1}{(2\pi)^3} \frac{d^3 \mathbf{q}_2}{(2\pi)^3} B_{hhh}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_{12}) e^{i\mathbf{q}_1 \cdot (\mathbf{y}_1 - \mathbf{y}_3) + i\mathbf{q}_2 \cdot (\mathbf{y}_2 - \mathbf{y}_3)} \\
&\quad \times \left[\delta_D^3(\mathbf{x}_1 - \mathbf{y}_1) \delta_D^3(\mathbf{x}_2 - \mathbf{y}_1) M_2 u_{M_2}(\mathbf{x}_3 - \mathbf{y}_2) M_3 u_{M_3}(\mathbf{x}_4 - \mathbf{y}_3) \right. \\
&\quad \left. + \delta_D^3(\mathbf{x}_1 - \mathbf{y}_1) M_1 u_{M_1}(\mathbf{x}_3 - \mathbf{y}_1) \delta_D^3(\mathbf{x}_2 - \mathbf{y}_2) M_2 u_{M_2}(\mathbf{x}_4 - \mathbf{y}_3) \right. \\
&\quad \left. + M_1^2 u_{M_1}(\mathbf{x}_3 - \mathbf{y}_1) u_{M_1}(\mathbf{x}_4 - \mathbf{y}_1) \delta_D^3(\mathbf{x}_1 - \mathbf{y}_2) \delta_D^3(\mathbf{x}_1 - \mathbf{y}_3) \right] \\
&= \int d^3 \mathbf{y}_1 d^3 \mathbf{y}_2 d^3 \mathbf{y}_3 \int dM_1 dM_2 dM_3 \frac{dn}{dM_1} \frac{dn}{dM_2} \frac{dn}{dM_3} \int \frac{d^3 \mathbf{q}_1}{(2\pi)^3} \frac{d^3 \mathbf{q}_2}{(2\pi)^3} B_{hhh}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_{12}) e^{i\mathbf{q}_1 \cdot (\mathbf{y}_1 - \mathbf{y}_3) + i\mathbf{q}_2 \cdot (\mathbf{y}_2 - \mathbf{y}_3)} \\
&\quad \times \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \left[e^{-i\mathbf{y}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2)} e^{-i\mathbf{y}_2 \cdot \mathbf{k}_3} e^{-i\mathbf{y}_3 \cdot \mathbf{k}_4} M_2 u_{M_2}(k_3) M_3 u_{M_3}(k_4) \right. \\
&\quad \left. + e^{-i\mathbf{y}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_3)} e^{-i\mathbf{y}_2 \cdot \mathbf{k}_2} e^{-i\mathbf{y}_3 \cdot \mathbf{k}_4} M_1 u_{M_1}(k_3) M_3 u_{M_3}(k_4) \right. \\
&\quad \left. + e^{-i\mathbf{y}_1 \cdot (\mathbf{k}_3 + \mathbf{k}_4)} e^{-i\mathbf{y}_2 \cdot \mathbf{k}_1} e^{-i\mathbf{y}_3 \cdot \mathbf{k}_2} M_1^2 u_{M_1}(k_3) u_{M_1}(k_4) \right] \\
&= \int dM_1 dM_2 dM_3 \frac{dn}{dM_1} \frac{dn}{dM_2} \frac{dn}{dM_3} \int \frac{d^3 \mathbf{q}_1}{(2\pi)^3} \frac{d^3 \mathbf{q}_2}{(2\pi)^3} B_{hhh}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_{12}) \\
&\quad \times \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \left[(2\pi)^9 \delta_D^3(\mathbf{q}_1 - \mathbf{k}_1 - \mathbf{k}_2) \delta_D^3(\mathbf{q}_2 - \mathbf{k}_3) \delta_D^3(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{k}_4) M_2 u_{M_2}(k_3) M_3 u_{M_3}(k_4) \right. \\
&\quad \left. + (2\pi)^9 \delta_D^3(\mathbf{q}_1 - \mathbf{k}_1 - \mathbf{k}_3) \delta_D^3(\mathbf{q}_2 - \mathbf{k}_2) \delta_D^3(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{k}_4) M_1 u_{M_1}(k_3) M_3 u_{M_3}(k_4) \right. \\
&\quad \left. + (2\pi)^9 \delta_D^3(\mathbf{q}_1 - \mathbf{k}_3 - \mathbf{k}_4) \delta_D^3(\mathbf{q}_2 - \mathbf{k}_1) \delta_D^3(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{k}_2) M_1^2 u_{M_1}(k_3) u_{M_1}(k_4) \right] \\
&= \int dM_1 dM_2 dM_3 \frac{dn}{dM_1} \frac{dn}{dM_2} \frac{dn}{dM_3} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \dots \frac{d^3 \mathbf{k}_4}{(2\pi)^3} (2\pi)^3 \delta_D^3(\mathbf{k}_{1234}) e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \dots + \mathbf{k}_4 \cdot \mathbf{x}_4)} \\
&\quad \times [B_{hhh}(\mathbf{k}_{12}, \mathbf{k}_3, \mathbf{k}_4) M_2 u_{M_2}(k_3) M_3 u_{M_3}(k_4) + B_{hhh}(\mathbf{k}_{13}, \mathbf{k}_2, \mathbf{k}_4) M_1 u_{M_1}(k_3) M_3 u_{M_3}(k_4) \\
&\quad + B_{hhh}(\mathbf{k}_{34}, \mathbf{k}_1, \mathbf{k}_2) M_1^2 u_{M_1}(k_3) u_{M_1}(k_4)] \\
\rightarrow T_{hhh}^{3h}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) &= \int dM_1 dM_2 dM_3 \frac{dn}{dM_1} \frac{dn}{dM_2} \frac{dn}{dM_3} [B_{hhh}(\mathbf{k}_{12}, \mathbf{k}_3, \mathbf{k}_4) M_2 u_{M_2}(k_3) M_3 u_{M_3}(k_4) \\
&\quad + B_{hhh}(\mathbf{k}_{13}, \mathbf{k}_2, \mathbf{k}_4) M_1 u_{M_1}(k_3) M_3 u_{M_3}(k_4) + B_{hhh}(\mathbf{k}_{34}, \mathbf{k}_1, \mathbf{k}_2) M_1^2 u_{M_1}(k_3) u_{M_1}(k_4) + (\text{perm.})]
\end{aligned} \tag{94}$$

$$\Omega_S = \int \left[\prod_i^n \frac{d\mathbf{q}_i}{(2\pi)^2} \right] \tilde{W}(\mathbf{q}_1) \tilde{W}(\mathbf{q}_2) \dots \tilde{W}(\mathbf{q}_n) = \int \left[\prod_i^{(n-1)} \frac{d\mathbf{q}_i}{(2\pi)^2} \right] \tilde{W}(\mathbf{q}_1) \tilde{W}(\mathbf{q}_2) \dots \tilde{W}(-\mathbf{q}_1 - \mathbf{q}_2 - \dots - \mathbf{q}_{n-1}) \tag{95}$$

$$\begin{aligned}
\hat{N}_h(z_L) &\equiv \int d^2 \boldsymbol{\theta} W(\boldsymbol{\theta}) \int d\chi \frac{d^2 V}{d\chi d\Omega} \int dM S(M) f_h(\chi; z_L) n_h(\mathbf{x}; M) \\
&= \int d^2 \boldsymbol{\theta} W(\boldsymbol{\theta}) \int d\chi \chi^2 f_h(\chi; z_L) \int dM S(M) \frac{dn}{dM}(M; \chi) [1 + b(M) \delta_m(\chi, \chi \boldsymbol{\theta})] \\
&= \Omega_S \int d\chi \chi^2 f_h(\chi; z_L) \int dM S(M) \frac{dn}{dM}(M; \chi) [1 + b(M) \delta_b(\chi)],
\end{aligned} \tag{96}$$

where quantities with hat notation $\hat{}$, here and hereafter, denote their estimator, the quantity with bar notation denotes the ensemble average expectation, and the background mode is defined as

$$\delta_b(\chi) \equiv \frac{1}{\Omega_S} \int d^2 \boldsymbol{\theta} W(\boldsymbol{\theta}) \delta_m(\chi, \chi \boldsymbol{\theta}) = \frac{1}{A_S(\chi)} \int d^2 \mathbf{x}_\perp W(\mathbf{x}_\perp) \delta_m(\chi, \mathbf{x}_\perp), \tag{97}$$

$\mathbf{x}_\perp \equiv \chi \boldsymbol{\theta}$, and $A_S(\chi) \equiv \chi^2 \Omega_S$ is the effective survey area of the plane at distance χ , in units of $[\text{Mpc}^2]$. Eq. (6) shows that the background mode at each redshift is from the average of Fourier modes perpendicular to the line-of-sight direction. If ignoring the effect of radial Fourier mode on clustering observables, which is a good approximation for lensing observables, the background mode can be realized to cause a shift in the mean mass density in each redshift plane. Note that, throughout this paper, we employ a flat-geometry universe, where the radial and angular-diameter distances are the same, and we employ the flat sky approximation. The ensemble average expectation can be computed as

$$\begin{aligned} \bar{N}_h(z_L) &\equiv \langle \hat{N}_h(z_L) \rangle = \int d^2\boldsymbol{\theta} W(\boldsymbol{\theta}) \int d\chi \chi^2 f_h(\chi; z_L) \int dM S(M) \frac{dn}{dM}(M; \chi) [1 + b(M) \langle \delta_m(\chi, \chi \boldsymbol{\theta}) \rangle] \\ &= \Omega_S \int d\chi \chi^2 f_h(\chi; z_L) \int dM S(M) \frac{dn}{dM}(M; \chi), \end{aligned} \quad (98)$$

where we have used the fact $\langle \delta_b \rangle = 0$.

Similarly, taking into account the survey window, we define an estimator of the number density fluctuation field of halos of mass M and at redshift χ as

$$\delta n_h^W(\mathbf{x}_\perp; M, \chi) = \frac{dn}{dM}(M; \chi) b(M) W(\boldsymbol{\theta}) \delta_m(\chi, \chi \boldsymbol{\theta}). \quad (99)$$

Since we later consider the projected field, we focus on the field in the two-dimensional plane perpendicular to the line-of-sight direction; hence, a mapping between the angular position and the comoving, perpendicular position vector for each halo is given as $\mathbf{x}_\perp = \chi \boldsymbol{\theta}$. This is feasible as redshift of each is assumed to be known. Integrating the above field in the light cone, weighted by the selection function yields an estimator of the projected number density fluctuation field:

$$\Sigma_h^W(\boldsymbol{\theta}) \equiv \int d\chi \chi^2 f_h(\chi; z_L) \int dM S(M) \delta n_h^W(M; \chi). \quad (100)$$

The Fourier transform of δn_h is

$$\tilde{\delta n}_h^W(\mathbf{k}_\perp, \chi) \equiv \int d^2\mathbf{x}_\perp \delta n_h^W(\mathbf{x}_\perp) e^{-i\mathbf{x}_\perp \cdot \mathbf{k}_\perp} = \frac{dn}{dM}(M; \chi) b(M) \int \frac{d\mathbf{k}_\parallel}{2\pi} \tilde{\delta}_m^W(\mathbf{k}_\perp, k_\parallel; \chi) e^{i\mathbf{k}_\parallel \cdot \chi \boldsymbol{\theta}}, \quad (101)$$

where

$$\tilde{\delta}_m^W(\mathbf{k}, k_\parallel; \chi) \equiv \int \frac{d^2\mathbf{q}}{(2\pi)^2} \tilde{W}(\mathbf{l} = \chi \mathbf{q}) \tilde{\delta}_m(k_\parallel, \mathbf{k}_\perp - \mathbf{q}), \quad (102)$$

and $\tilde{W}(\mathbf{q}; \chi) \equiv \int d^2\mathbf{x}_\perp W(\mathbf{x}) e^{-i\mathbf{q} \cdot \mathbf{x}_\perp}$ with $\mathbf{x}_\perp = \chi \boldsymbol{\theta}$.

$$\hat{P}_{hh}(k_{\perp,i}) \equiv \frac{1}{\hat{N}_h(z_L)^2} \int d\chi \int d\chi' f_h(\chi) f_h(\chi') \frac{1}{\chi^2 \Omega_S} \int \frac{d^2\mathbf{k}_\perp}{A(k_{\perp,i})} \tilde{\delta n}_h^W(\mathbf{k}_\perp, \chi) \tilde{\delta n}_h^W(-\mathbf{k}_\perp, \chi') \quad (103)$$

Let us define an estimator of the projected power spectrum of halos in the light cone as

$$\hat{P}_{hh}(k_{\perp,i}) = \frac{1}{\Omega_S \hat{N}_h(z_L)^2} \int d\chi \int d\chi' \chi^2 \chi'^2 f_h(\chi; z_L) f_h(\chi'; z_L) \int dM \int dM' S(M) S(M') \int_{k_{\perp,i}} \frac{d^2\mathbf{k}_\perp}{V(k_{\perp,i})} \tilde{\delta n}_h^W(\mathbf{k}_\perp, \chi) \tilde{\delta n}_h^W(-\mathbf{k}_\perp, \chi') \quad (104)$$

$$\begin{aligned} \langle \hat{P}_{hh}(k_{\perp,i}) \rangle &= \frac{1}{\Omega_S \hat{N}_h(z_L)^2} \int d\chi \int d\chi' \chi^2 \chi'^2 f_h(\chi; z_L) f_h(\chi'; z_L) \int dM \int dM' S(M) S(M') \chi^2 \chi'^2 b(M) \frac{dn}{dM}(M) b(M') \frac{dn}{dM}(M') \\ &\quad \times \int \frac{d^2\mathbf{q}}{(2\pi)^2} \int \frac{d^2\mathbf{q}'}{(2\pi)^2} \int_{k_{\perp,i}} \frac{d^2\mathbf{k}_\perp}{V(k_{\perp,i})} \int \frac{d\mathbf{k}_\parallel}{2\pi} \tilde{W}_\mathbf{q} \tilde{W}_{\mathbf{q}'} P(|\mathbf{k} - \mathbf{q}|, k_\parallel) e^{i\mathbf{k}_\parallel(\chi - \chi')} (2\pi)^2 \delta_D^2(\mathbf{q} - \mathbf{q}') \\ &\simeq \frac{1}{\Omega_S \hat{N}_h(z_L)^2} \int d\chi \chi^4 f_h(\chi; z_L)^2 \left[\int dM S(M) b(M) \frac{dn}{dM}(M) \right]^2 \chi^4 P(k_{\perp,i}) \int \frac{d^2\mathbf{q}}{(2\pi)^2} |\tilde{W}(\mathbf{l} = \chi \mathbf{q})|^2 \end{aligned} \quad (105)$$

In the following, for simplicity we consider the uniform radial selection given by

$$f_h(\chi; z_L) = \Theta(\chi_L - \chi + \Delta\chi/2) \Theta(\chi_L + \chi - \Delta\chi/2), \quad (106)$$

where $\Theta(x)$ is the Heviside-step function, defined as $\Theta(x) = 1$ if $x > 0$ otherwise $\Theta(x) = 0$, χ_L is the radial comoving distance to z_L , $\chi_L \equiv \chi(z_L)$, and $\Delta\chi$ is the radial bin width.

For a reasonably narrow radial bin of the z_L -redshift slice, satisfying $\Delta\chi/\chi_L \ll 1$, the cumulative number counts can be simplified as

$$\hat{N}_h(z_L) \simeq \bar{N}_h(z_L) [1 + \bar{b}_1 \delta_b] \quad (107)$$

$$\delta_h^W(\mathbf{x}_\perp; \chi) \equiv W(\chi \boldsymbol{\theta}) \delta_h^{2D}(\mathbf{x}_\perp, \chi) \quad (108)$$

$$\tilde{\delta}_h^W(\mathbf{k}_\perp) = \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \tilde{W}(\mathbf{q}) \tilde{\delta}_h(\mathbf{k}_\perp - \mathbf{q}) \quad (109)$$

In this paper we consider a catalog of halos, probed with galaxies and/galaxy clusters. The average, projected number density of halos is given in terms of the halo mass function as

$$\bar{N}_h(z_L) = \int d\chi \frac{d^2 V}{d\chi d\Omega} \int dM \frac{dn}{dM} S(M, \chi; z_L), \quad (110)$$

where $d^2 V/d\chi d\Omega = \chi^2$ for a flat-geometry universe, $\chi(z)$ is the comoving angular distance to redshift z , dn/dM is the three-dimensional number density of halos with masses $[M, dM]$ at $z(= z(\chi))$, and $S(M, \chi; z_L)$ is the selection function of halos at a bin of redshift around z_L . Note that the radial and angular-diameter distances are equivalent for a flat-geometry universe. For simplicity, assuming that mass and redshift for each halo is available, we consider the following, trivial selection function, defined as

$$S(\chi, M; z_L) = \Theta(\chi - \chi_L + \Delta\chi/2) \Theta(\chi_L + \Delta\chi/2 - \chi) \tilde{S}(M), \quad (111)$$

where $\Theta(x)$ is the Heviside-step function, defined as $\Theta(x) = 1$ if $x > 0$ otherwise $\Theta(x) = 0$, χ_L is the radial comoving distance to z_L , $\chi_L \equiv \chi(z_L)$, and $\Delta\chi$ is the radial bin width. $S(\chi, M; z_L)$ is nonzero in the radial range of $\chi_L - \Delta\chi/2 \leq \chi \leq \chi_L + \Delta\chi/2$. $\tilde{S}(M)$ is the selection of halo masses. We don't specify the form of $\tilde{S}(M)$ to keep generality of our discussion.

If a finite-volume survey region is embedded into a coherent density contrast, which we hereafter call the background mode δ_b , the cumulative number of halos is modulated as

$$\hat{N}_h(z_L; \delta_b) = \int d\chi \chi^2 \int dM \frac{dn}{dM} S(M, \chi; z_L) [1 + b(M) \delta_b] = \bar{N}_h(z_L) (1 + \bar{b}_1 \delta_b) \quad (112)$$

where $b(M)$ is the linear halo bias and \bar{b}_1 is the average bias over the halo mass function:

$$\bar{b}_1 \equiv \frac{1}{\bar{N}_h(z_L)} \int d\chi \chi^2 \int dM \frac{dn}{dM} S(M, \chi; z_L) b(M) = \frac{1}{\bar{N}_h(z_L)} \int_{\chi_L} d\chi \chi^2 n_h^0(\chi), \quad (113)$$

where the integration $\int_{\chi_L} d\chi$ denotes the radial integration over the range $[\chi_L - \Delta\chi/2, \chi_L + \Delta\chi/2]$, and we introduced the collapsed notation:

$$n_h^\beta(\chi) \equiv \int dM \frac{dn}{dM} \tilde{S}(M) [b(M)]^\beta. \quad (114)$$

The background density mode is defined in terms of the survey window function $W(\chi, \boldsymbol{\theta})$ as

$$\delta_b \equiv \frac{1}{V_S} \int d\chi \chi^2 \int_{\Omega_S} d^2 \boldsymbol{\theta} W(\chi, \boldsymbol{\theta}) \delta_m(\chi, \chi \boldsymbol{\theta}). \quad (115)$$

Note that the survey window generally includes both the radial and angular windows. If the survey window has a simple geometry, without any masked region, the survey volume for the redshift slice around z_L , V_S , is simply given as $V_S \simeq \chi_L^2 \Delta\chi \Omega_S$, where Ω_S is the survey area.

The projected number density fluctuation field of halos in the radial bin around z_L is given as

$$\delta_h^{2D}(\mathbf{x}_\perp, z_L) = \frac{1}{\bar{N}_h} \int_{\chi_L} d\chi \chi^2 \int dM \frac{dn}{dM} \tilde{S}(M) \delta_h(\chi, \mathbf{x}_\perp; M), \quad (116)$$

where \mathbf{x}_\perp is the projected position vector of each halo in the perpendicular plane to the line-of-sight.

Using the Limber's approximation [16], we can express the projected power spectrum of halos as

$$\begin{aligned} P_{\text{hh}}^{2\text{D}}(k_\perp; z_L) &= \frac{1}{\hat{N}_h(z_L)^2} \int_{\chi_L} d\chi \chi^4 \int dM \frac{dn}{dM} \tilde{S}(M) \int dM' \frac{dn}{dM'} \tilde{S}(M') P_{\text{hh}}(k_\perp; M, M', z_L) \\ &= \frac{1}{\hat{N}_h(z_L)^2} \int_{\chi_L} d\chi \chi^4 [n_h^1(\chi)]^2 P_{\text{m}}^{\text{lin}}(k_\perp; \chi) \end{aligned} \quad (117)$$

where $P_{\text{hh}}(k; M, M', \chi)$ is the three-dimensional power spectrum between two halo of masses M and M' at redshift $z = z(\chi)$. In the second line on the r.h.s., we assumed that the halo power spectrum is given as $P_{\text{hh}}(k, M, M') \simeq b(M)b(M')P_{\text{m}}^{\text{lin}}(k)$ in the halo model approach, where $P_{\text{m}}^{\text{lin}}(k)$ is the linear mass power spectrum. In the following we will often denote k instead of k_\perp for notational simplification. If we further assume a thin redshift slice, $\Delta\chi/\chi_L \ll 1$, the halo power spectrum can be further simplified as

$$P_{\text{hh}}(k_\perp; z_L) \simeq \frac{\bar{b}_1^2}{\Delta\chi} P_{\text{m}}^{\text{lin}}(k). \quad (118)$$

Using the Limber's approximation, the ensemble average of the stacked lensing profile can be computed as

$$\begin{aligned} \langle \Sigma \rangle(R) &\equiv \langle \Sigma_{\text{cr}}(\chi) \kappa^W(\boldsymbol{\theta}) \rangle_{R=|\mathbf{x}_\perp(z_L) - \chi(z_L)\boldsymbol{\theta}|} = \langle \delta_h(\mathbf{x}_\perp) \Sigma_{\text{cr}}(\chi) \kappa^W(\mathbf{x}'_\perp) \rangle_{R=|\mathbf{x}_\perp - \mathbf{x}'_\perp|; \chi' = \chi\boldsymbol{\theta}} \\ &= \frac{1}{\hat{N}_h(z_L)} \int d\chi d\chi' \chi^2 f_h(\chi) \Sigma_{\text{cr}}(\chi) f_\kappa(\chi') \int dM \frac{dn}{dM} S(M) b(M) W(\mathbf{x}_\perp) \langle \delta_{\text{m,lin}}(\chi, \mathbf{x}_\perp) \delta_{\text{m}}^W(\chi', \chi' \boldsymbol{\theta}) \rangle \\ &= \frac{1}{\hat{N}_h(z_L)} \int d\chi d\chi' \chi^2 \Sigma_{\text{cr}}(\chi) W_\kappa(\chi') \int dM \frac{dn}{dM} S(M, \chi; z_L) \\ &\quad \times \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{d^3 \mathbf{k}'}{(2\pi)^3} P_{\text{hm}}(k; M, \chi) (2\pi)^3 \delta_D(k_\parallel + k'_\parallel) \delta_D^2(\mathbf{k}_\perp + \mathbf{k}'_\perp) e^{ik_\parallel \chi + i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} e^{ik'_\parallel \chi' + i\mathbf{k}'_\perp \cdot \chi' \boldsymbol{\theta}} \\ &= \frac{1}{\hat{N}_h(z_L)} \int d\chi d\chi' \chi^2 \Sigma_{\text{cr}}(\chi) W_\kappa(\chi') \int dM \frac{dn}{dM} S(M, \chi; z_L) \int \frac{dk_\parallel d^2 \mathbf{k}_\perp}{(2\pi)^3} P_{\text{hm}}(k; M, \chi) e^{ik_\parallel (\chi - \chi') + i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \chi' \boldsymbol{\theta})} \\ &\simeq \frac{1}{\hat{N}_h(z_L)} \int d\chi d\chi' \chi^2 \Sigma_{\text{cr}}(\chi) W_\kappa(\chi') \int dM \frac{dn}{dM} S(M, \chi; z_L) \int \frac{dk_\parallel d^2 \mathbf{k}_\perp}{(2\pi)^3} P_{\text{hm}}(k_\perp; M, \chi) e^{ik_\parallel (\chi - \chi') + i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \chi' \boldsymbol{\theta})} \\ &= \frac{1}{\hat{N}_h(z_L)} \int_{\chi_L} d\chi \chi^2 \Sigma_{\text{cr}}(\chi) W_\kappa(\chi) \int dM \frac{dn}{dM} \tilde{S}(M) \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} P_{\text{hm}}(k_\perp; M, \chi) e^{i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \chi \boldsymbol{\theta})} \\ &= \frac{\bar{\rho}_{\text{m0}}}{\hat{N}_h(z_L)} \int_{\chi_L} d\chi \chi^2 \int dM \frac{dn}{dM} \tilde{S}(M) \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} P_{\text{hm}}(k_\perp; M, \chi) e^{i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \chi \boldsymbol{\theta})}, \end{aligned} \quad (119)$$

where $P_{\text{hm}}(k; M, z)$ is the halo-mass cross-power spectrum for halos of mass M and at redshift z . From the above equation, we can find the power spectrum of stacked lensing as

$$P_{\Delta\Sigma}(k) = \frac{\bar{\rho}_{\text{m0}}}{\hat{N}_h(z_L)} \int_{\chi_L} d\chi \chi^2 \int dM \frac{dn}{dM} \tilde{S}(M) P_{\text{hm}}(k; M). \quad (120)$$

The average excess surface mass density profile, $\langle \Delta\Sigma \rangle(R)$, is a more direct observable of the stacked lensing, and is given as

$$\langle \Delta\Sigma \rangle(R) = \int \frac{k dk}{2\pi} P_{\Delta\Sigma}(k) J_2(kR). \quad (121)$$

If we use the halo model, we can express the power spectrum by a sum of the 1- and 2-halo terms:

$$P_{\Delta\Sigma}(k) = P_{\Delta\Sigma}^{1\text{h}}(k) + P_{\Delta\Sigma}^{2\text{h}}(k), \quad (122)$$

with

$$\begin{aligned} P_{\Delta\Sigma}^{1\text{h}}(k) &\equiv \frac{\bar{\rho}_{\text{m0}}}{\hat{N}_h(z_L)} \int_{\chi_L} d\chi \chi^2 \int dM \frac{dn}{dM} S(M, z_L) M |u_M(k; z_L)| = \frac{\bar{\rho}_{\text{m0}}}{\hat{N}_h(z_L)} \int_{\chi_L} d\chi \chi^2 I_1^0(k; \chi), \\ P_{\Delta\Sigma}^{2\text{h}}(k) &= \frac{\bar{\rho}_{\text{m0}}}{\hat{N}_h(z_L)} \int_{\chi_L} d\chi \chi^2 \int dM \frac{dn}{dM} \tilde{S}(M) P_{\text{hm}}^{2\text{h}}(k; M, z_L) = \frac{\bar{\rho}_{\text{m0}}}{\hat{N}_h(z_L)} \int_{\chi_L} d\chi \chi^2 n_h^1(\chi) P_{\text{m}}^{\text{lin}}(k; \chi) \simeq \bar{b}_1 \bar{\rho}_{\text{m0}} P_{\text{m}}^{\text{lin}}(k), \end{aligned} \quad (123)$$

where we introduced the notation:

$$I_\mu^\beta(k_1, \dots, k_\mu) \equiv \int dM \frac{dn}{dM} \tilde{S}(M) \beta_1^\beta \prod_{i=1}^\mu u_M(k_i), \quad (124)$$

where $u_M(k)$ is the Fourier transform of the average mass density profile for halos of mass M .