Super sample variance of stacked lensing

Masahiro Takada[1], Takahiro Nishimichi[1,2], Ryuichi Takahashi[3], Masamune Oguri[2] TBD
[1]*Kavli Institute for the Physics and Mathematics of the Universe (Kavli IPMU, WPI), The University of Tokyo, Chiba 277-8583, Japan*
[2]*Physics Department, The University of Tokyo, Bunkyo, Tokyo 113-0031, Japan and*
[3]*Department of Physical Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8526, Japan*

Title is still tempolary... a better title we should come up with later. To be filled ...

I. INTRODUCTION

[1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21]

II. COVARIANCE OF STACKED LENSING IN A SURVEY WINDOW

A. Halo statistics: number counts and projected power spectrum

First we consider statistical observables of halos. We assume that halos in a survey region can be identified from observables such as optical richness and X-ray observables, and then assume that halo mass and redshift of each halo are available. However, the following discussion can be extended to a more generally case where only their proxies are available.

The number density fluctuation field for halos in the mass range $[M, M+\mathrm{d}M]$ is given as

$$n_{\rm h}(\mathbf{x};M)\mathrm{d}M \simeq \frac{\mathrm{d}n}{\mathrm{d}M}\left[1+\delta_{\rm h}(\mathbf{x};M)\right] \tag{1}$$

where dn/dM is the (ensemble-average) number density of halos with masses $[M, M+\mathrm{d}M]$, and $\delta_{\rm h}(\mathbf{x})$ is the 3D number density fluctuation field of the halos.

Denoting the radial selection function by $f_{\rm h}(\chi;z_L)$ and the halo mass selection at each redshift by $S(M;\chi)$, we can define the projected number density field of halos, integrated over ranges of redshift and halo masses, in terms of the 3D density field $n_{\rm h}(\mathbf{x};\chi)$ as

$$\begin{aligned} N_{\rm h}(\mathbf{x}_\perp) &= \int\!\mathrm{d}\chi\ f_{\rm h}(\chi;z_L)\int\!\mathrm{d}M\ S(M;\chi)W(\mathbf{x}_\perp)n_{\rm h}(\mathbf{x};M,\chi)\\ &= \int\!\mathrm{d}\chi\ f_{\rm h}(\chi;z_L)\int\!\mathrm{d}M\ S(M;\chi)\frac{\mathrm{d}n}{\mathrm{d}M}\left[1+W(\mathbf{x}_\perp)\delta_{\rm h}(\chi,\mathbf{x}_\perp)\right], \end{aligned} \tag{2}$$

where we introduced the notation "z_L" in $f_{\rm h}(\chi)$ to explicitly mean that the radial selection is non-zero around a target redshift z_L, which will be below taken for the redshift slice of lensing halos. $N_{\rm h}(\mathbf{x}_\perp)$ is the projected number density field in dimension of [Mpc^{-2}], not an angular number density, and we throughout this paper employ distant observer approximation as well as flat-sky approximation. The selection functions are generally defined so as to satisfy $0\le f_{\rm h}\le 1$ and $0\le S(M;\chi)\le 1$. For an ideal, homogeneous complete selection, $f_{\rm h}=0$ or 1 and the same for $S(M)$. The vector $\mathbf{x}_\perp$ in the above equation denotes the position vector in the two-dimensional plane at distance χ, perpendicular to the line-of-sight direction, which is converted from the angular position $\boldsymbol{\theta}$ via relation $\mathbf{x}_\perp\equiv\chi\boldsymbol{\theta}$. $W(\boldsymbol{\theta})$ is a survey window, which is 1 in the measured region and 0 in the unmeasured region. Hence the survey area is given as

$$\Omega_S \equiv \int\!\mathrm{d}^2\boldsymbol{\theta}\ W(\boldsymbol{\theta}). \tag{3}$$

The "mean" surface density in the finite-area survey region is defined by the survey window average of Eq. (2):

$$\begin{aligned} \hat{N}_{\rm h}(z_L) = \frac{1}{\Omega_S}\int\!\mathrm{d}^2\boldsymbol{\theta}\ N_{\rm h}(\mathbf{x}_\perp) &\simeq \int\!\mathrm{d}\chi\ f_{\rm h}(\chi;z_L)\int\!\mathrm{d}M\ S(M;\chi)\frac{\mathrm{d}n}{\mathrm{d}M}\left[1+\frac{b(M)}{\chi^2\Omega_S(\chi)}\int\!\mathrm{d}^2\mathbf{x}_\perp\ W(\mathbf{x}_\perp)\delta_{\rm m,lin}(\chi,\mathbf{x}_\perp)\right]\\ &= \int\!\mathrm{d}\chi\ f_{\rm h}(\chi;z_L)\int\!\mathrm{d}M\ S(M;\chi)\frac{\mathrm{d}n}{\mathrm{d}M}\left[1+b(M)\delta_{\rm b}(\chi)\right]\\ &= \bar{N}_{\rm h}\left[1+\frac{1}{\bar{N}_{\rm h}}\int\!\mathrm{d}\chi\ f_{\rm h}(\chi)\bar{n}_{\rm h}^S\bar{b}^S\delta_{\rm b}\right], \end{aligned} \tag{4}$$

where quantities with hat $\hat{}$ notation, here and hereafter, denote their estimators, we have again used the conversion $\mathbf{x}_\perp = \chi\boldsymbol{\theta}$, and $\chi^2\Omega_S$ is the effective survey area at distance χ, in units of [Mpc^2]. The quantity $\delta_{\rm b}(\chi)$ is the coherent density contrast that exists across the survey region at redshfit $\chi = \chi(z)$, which we call hereafter the background mode:

$$\delta_{\rm b}(\chi) \equiv \frac{1}{\Omega_S}\int \mathrm{d}^2\boldsymbol{\theta}\ W(\boldsymbol{\theta})\delta_{\rm m,lin}(\chi,\chi\boldsymbol{\theta}) = \frac{1}{\chi^2\Omega_S(\chi)}\int \mathrm{d}^2\mathbf{x}_\perp\ W(\mathbf{x}_\perp)\delta_{\rm m,lin}(\chi,\mathbf{x}_\perp). \tag{5}$$

We here assumed that the background mode at each redshift is defined by the survey window average of Fourier modes in the two-dimensional plane perpendicular to the line-of-sight direction. Since we will later employ the Limber's approximation, we throughout this paper ignore the radial mode of the background mode which is a good approximation for the lensing statistics. In the second equality of Eq. (4), we assumed that the halo density field at large scales, after the window average, is related to the underlying mass density field as $\delta_{\rm h}(\mathbf{x};M) \simeq b(M)\delta_{\rm m,lin}(\mathbf{x})$, where $\delta_{\rm m,lin}(\mathbf{x})$ is the mass density fluctuation field in the linear regime and b is the linear halo bias, assuming that the angular average over a sufficiently wide area smooths out all the small-scale fluctuations. In the last equality we defined the ensemble-average halo number density and bias, integrated over the halo mass range, in each redshift slice at $z = z(\chi)$:

$$\begin{aligned}\bar{n}_{\rm h}^S(\chi) &\equiv \int \mathrm{d}M\ \frac{dn}{dM}S(M),\\ \bar{b}^S(\chi) &\equiv \frac{1}{\bar{n}_{\rm h}^S}\int \mathrm{d}M\ \frac{dn}{dM}S(M)\frac{\mathrm{d}n}{\mathrm{d}M}b(M),\end{aligned} \tag{6}$$

where quantities with bar notation denote their ensemble average quantities. Furthermore, $\bar{N}_{\rm h}$ in Eq. (4) is the ensemble average or global mean of the surface halo number density:

$$\bar{N}_{\rm h}(z_L) \equiv \left\langle \hat{N}_{\rm h}(z_L)\right\rangle = \int \mathrm{d}\chi\ f_{\rm h}(\chi;z_L)\bar{n}_{\rm h}^S(\chi), \tag{7}$$

where we have used $\langle\delta_{\rm b}\rangle = 0$.

B. Stacked lensing: An estimator of the projected halo-mass cross-correlation

The lensing convergence field in an angular position $\boldsymbol{\theta}$ on the sky arises from a projection of the three-dimensional mass fluctuation field $\delta_{\rm m}$, weighted by the lensing efficiency kernel:

$$\kappa^W(\boldsymbol{\theta}) \equiv W(\boldsymbol{\theta})\kappa(\boldsymbol{\theta}) = \int \mathrm{d}\chi\ f_\kappa(\chi)W(\boldsymbol{\theta})\delta_{\rm m}(\chi,\chi\boldsymbol{\theta};z) = \int \mathrm{d}\chi\ f_\kappa(\chi)\int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2}\frac{\mathrm{d}k_\parallel}{2\pi}\tilde{\delta}_{\rm m}^W(k_\parallel,\mathbf{k})e^{ik_\parallel\chi+i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} \tag{8}$$

where $\mathbf{x}_\perp = \chi\boldsymbol{\theta}$, $\delta_m(\mathbf{x})$ is the three-dimensional mass fluctuation field, and the lensing radial kernel $f_\kappa(\chi)$ is given as

$$f_\kappa(\chi) = \bar{\rho}_{\rm m0}\Sigma_{\rm cr}^{-1}(\chi). \tag{9}$$

The quantity $\Sigma_{\rm cr}$ is the critical surface mass density defined for each observer-lens-source system:

$$\Sigma_{\rm cr}^{-1}(z) \equiv 4\pi G a^{-1}\chi(z)\left[1-\chi(z)\left\langle\frac{1}{\chi_s}\right\rangle\right], \tag{10}$$

with $\langle 1/\chi_s\rangle = \int_{z_{\rm s,min}}^\infty dz_s p(z_s)[1/\chi(z_s)]\Big/\int_{z_{\rm s,min}}^\infty dz_s p(z_s)$, where $p(z_s)$ is the redshift distribution of source galaxies, and $z_{\rm s,min}$ is the minimum redshift which an observer determines to select source galaxies behind lensing halos at z_L. The factor of $\bar{\rho}_{\rm m0}$ in Eq. (8), instead of the mean density $\bar{\rho}_{\rm m}(z)$ at redshift z, is from our use of the comoving coordinates.

Before going to the detailed calculation, it would be illustrative to reexpress the convergence field as a sum of two contributions:

$$\kappa(\boldsymbol{\theta}) \simeq \bar{\rho}_{\rm m0}\Sigma_{\rm cr}^{-1}(z_L)\int_{z_L}\mathrm{d}\chi\ \delta_{\rm m}(\chi_L,\chi_L\boldsymbol{\theta}) + \bar{\rho}_{\rm m0}\int_{z\neq z_L}\mathrm{d}\chi\ \Sigma_{\rm cr}^{-1}(z)\delta_{\rm m}(\chi,\chi\boldsymbol{\theta}). \tag{11}$$

The first term on the r.h.s. denotes a lensing contribution from the mass distribution at lens redshift, assuming a thin redshift slice where the critical density $\Sigma_{\rm cr}$ is not largely varying, and the second term is the lensing contribution from the mass distribution at different redshifts, which we call the cosmic shear contribution.

The stacked lensing for sampled halos with known redshifts probes the average mass distribution surrounding the lensing halos. Observationally the stacked lensing profile can be estimated by averaging shapes of background galaxies for all the pairs of halos and source galaxies [14, 22]:

$$\widehat{\langle\Delta\Sigma\rangle}(R) \equiv \frac{1}{N_{\rm pairs}} \sum_{\text{all pairs};(l,s)} \Sigma_{\rm cr}(z_l, z_s)\epsilon_{+(s)}\Bigg|_{R=\chi(z_l)\Delta\theta_{ls}}, \tag{12}$$

where the summation runs over all the halo and source galaxy pairs each of which is separated by a particular projected distance R in the lens plane of each halo at z_l; $\Delta\theta_{ls}$ is the angular separation between lens and source in each pair, and $\chi(z_l)$ is the distance to halo at redshift z_l, and therefore $R = \chi(z_l)\Delta\theta_{ls}$. ϵ_+ is the tangential component of source galaxy ellipticity with respect to the halo center. Here we assume that redshift information of each source galaxy is also available, e.g., based on the photometric redshift estimation. Thus, *a priori* knowledge of each lens redshift allows us to probe the average projected mass profile as a function of comoving distances in each lens plane, rather than angular radii [9, 23]. We should also notice that the average is done on each source-halo basis, and therefore the stacked lensing does not suffer from a dilution of the lensing signals due to the radial projection. For simplicity we do not consider a weighting in the above average, which is often employed in an actual measurement in order to optimize the signal-to-noise ratio [e.g., 14].

The ensemble-average expectation, i.e. the cosmological signal of the stacked lensing arises from the cross-correlation between the projected number density fluctuation field of lensing halos and the lensing shear field:

$$\langle\Delta\Sigma\rangle(R) = \frac{1}{\bar{N}_{\rm h}(z_L)} \left\langle \delta N_{\rm h}(\mathbf{x}_\perp)\Sigma_{\rm cr}(\chi)\gamma_+(\mathbf{x}'_\perp)\right\rangle_{R=|\mathbf{x}_\perp-\mathbf{x}'_\perp|;\mathbf{x}'_\perp=\chi\boldsymbol{\theta}}, \tag{13}$$

where $\gamma_+(\mathbf{x}'_\perp)$ is the tangential component of the shear field with respect to halo center, in the angular direction of $\mathbf{x}'_\perp$ via the relation $\mathbf{x}'_\perp = \chi\boldsymbol{\theta}$. The use of the number density fluctuation field $\delta N_{\rm h}$ in the above defintion is due to the fact that the average of the tengential shear with respect to random points is vanishing. The average of non-lensing mode $\gamma_\times$, which is the 45-degrees-rotated shear component relative to γ_+, is vanishing,$\langle\delta N_{\rm h}\Sigma_{\rm cr}\gamma_\times\rangle = 0$, and therefore this can be used as a diagnostic of residual systematic errors such as an error due to an imperfect shape measurement.

If defining the projected cross-power spectrum of halos and the surrounding mass distribution, the stacked lensing profile is expressed as

$$\langle\Delta\Sigma\rangle(R) = \int\!\frac{kdk}{2\pi}\, C_{\rm hm}(k)J_2(kR), \tag{14}$$

where $J_2(x)$ is the 2nd-order Bessel function and the cross-power spectrum is defined as

$$C_{\rm hm}(k) = \frac{\bar{\rho}_{\rm m0}}{\bar{N}_{\rm h}(z_L)}\int\!{\rm d}\chi\, f_{\rm h}(\chi)\int\!{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M)P_{\rm hm}(k;M,\chi) = \frac{\bar{\rho}_{\rm m0}}{\bar{N}_{\rm h}(z_L)}\int\!{\rm d}\chi\, f_{\rm h}(\chi)\bar{n}_{\rm h}^S P_{\rm hm}^S(k;\chi), \tag{15}$$

where $P_{\rm hm}(k;M,\chi)$ is the 3D cross-power spectrum between the mass distribution and halos of mass M and at redshift $z = z(\chi)$, and $P^S_{\rm hm}(k;\chi)$ is the spectrum averaged by halo masses:

$$P^S_{\rm hm}(k;\chi) \equiv \frac{1}{\bar{n}^S_{\rm h}(\chi)}\int\!{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M)P_{\rm hm}(k;M). \tag{16}$$

If we employ the halo model, we can express $C_{\rm hm}(k)$ by a sum of the 1- and 2-halo terms:

$$C_{\rm hm}(k) = C^{\rm 1h}_{\rm hm}(k) + C^{\rm 2h}_{\rm hm}(k), \tag{17}$$

with

$$\begin{aligned} C^{\rm 1h}_{\rm hm}(k) &\equiv \frac{\bar{\rho}_{\rm m0}}{\bar{N}_{\rm h}(z_L)}\int_{\chi_L}\!{\rm d}\chi\, f_{\rm h}(\chi)\int\!{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M,z_L)\frac{M}{\bar{\rho}_{\rm m0}}u_M(k;\chi) = \frac{\bar{\rho}_{\rm m0}}{\bar{N}_{\rm h}(z_L)}\int_{\chi_L}\!{\rm d}\chi\, f_{\rm h}(\chi)\bar{n}^S_{\rm h}\mathcal{I}^0_1(k;\chi),\\ C^{\rm 2h}_{\rm hm}(k) &= \frac{\bar{\rho}_{\rm m0}}{\bar{N}_{\rm h}(z_L)}\int_{\chi_L}\!{\rm d}\chi\, f_{\rm h}(\chi)\int\!{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M)b(M)P^{\rm lin}_{\rm m}(k;\chi) = \frac{\bar{\rho}_{\rm m0}}{\bar{N}_{\rm h}(z_L)}\int_{\chi_L}\!{\rm d}\chi\, f_{\rm h}(\chi)\bar{n}^S_{\rm h}\bar{b}^S P^{\rm lin}_{\rm m}(k;\chi), \end{aligned} \tag{18}$$

where we introduced the notations defined as

$$\mathcal{I}^\beta_\mu(k_1,\cdots,k_\mu) \equiv \frac{1}{\bar{n}^S_{\rm h}}\int\!{\rm d}M\frac{{\rm d}n}{{\rm d}M}S(M)\left(\frac{M}{\bar{\rho}_{\rm m0}}\right)^\mu b^\beta\prod_{i=1}^{\mu}u_M(k_i), \tag{19}$$

where we have assumed that the halo power spectrum is modeled as $P_{\rm hh}(k;M,M') \simeq b(M)b(M')P^{\rm lin}_{\rm m}(k)$, and $u_M(k)$ is the Fourier transform of mass density profile for halos of mass M. Note $\mathcal{I}^{(S)1}_0 = \bar{b}^S$ (see Eq. 6).

Taking into account the survey window, we can define an estimator of the projected cross-power spectrum as

$$\hat{C}_{\rm hm}(k_i) \equiv \frac{1}{\hat{N}_{\rm h}(z_L)}\int {\rm d}\chi \int {\rm d}\chi'\ f_{\rm h}(\chi)\Sigma_{\rm cr}(\chi)f_\kappa(\chi')\int {\rm d}M\ \frac{{\rm d}n}{{\rm d}M}S(M)\frac{1}{\chi^2\Omega_S}\int_{|\mathbf{k}_\perp|\in k_i}\frac{{\rm d}^2\mathbf{k}_\perp}{A(k_i)}\int\frac{{\rm d}k_\parallel}{2\pi}\frac{{\rm d}k'_\parallel}{2\pi}\tilde{\delta}^W_{\rm h}(k_\parallel,\mathbf{k}_\perp)\tilde{\delta}^W_{\rm m}(k'_\parallel,-\mathbf{k}_\perp)e^{ik_\parallel\chi+ik'_\parallel\chi'}. \tag{20}$$

where

$$\tilde{\delta}^W_{\rm h}(k_\parallel,\mathbf{k}_\perp) \equiv \int\frac{{\rm d}^2\mathbf{q}}{(2\pi)^2}\tilde{W}(\mathbf{q})\tilde{\delta}_{\rm h}(k_\parallel,\mathbf{k}_\perp-\mathbf{q}), \tag{21}$$

$\tilde{W}(\mathbf{q};\chi) \equiv \int {\rm d}^2\mathbf{x}_\perp\ W(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}_\perp}$, the similar definition for $\tilde{\delta}^W_{\rm m}$, and quantities with tilde symbol hereafter denote their Fourier transforms. The integral $\int {\rm d}^2\mathbf{k}_\perp$ is over an annulus in $\mathbf{k}_\perp$ space of width Δk, and the area $A(k_{\perp i}) \simeq 2\pi k_{\perp i}\Delta k$. Note that the projected number density $\hat{N}_{\rm h}(z_L)$ in the denominator on the r.h.s. is an estimator of the mean projected halo number density, and therefore includes a contribution due to the background density modulation $\delta_{\rm b}$ (see Eq. 4).

Using the halo power spectrum definition,

$$\left\langle \delta_{\rm h}(k_\parallel,\mathbf{k}_\perp;M)\delta_{\rm m}(k'_\parallel,\mathbf{k}'_\perp)\right\rangle = (2\pi)^3\delta_D(k_\parallel+k'_\parallel)\delta^2_D(\mathbf{k}_\perp+\mathbf{k}'_\perp)P_{\rm hm}(k;M), \tag{22}$$

and employing the Limber's approximation, we can find that the ensemble average of the power spectrum estimator gives the expectation:

$$\begin{aligned}\left\langle \hat{C}_{\rm hm}(k_i)\right\rangle &= \frac{1}{\bar{N}_{\rm h}(z_L)}\int {\rm d}\ f_{\rm h}(\chi)\Sigma_{\rm cr}(\chi)f_\kappa(\chi)\int {\rm d}M\ \frac{{\rm d}n}{{\rm d}M}S(M)\frac{1}{\chi^2\Omega_S}\int_{|\mathbf{k}_\perp|\in k_i}\frac{{\rm d}^2\mathbf{k}_\perp}{A(k_i)}\int\frac{{\rm d}^2\mathbf{q}}{(2\pi)^2}P_{\rm hm}(|\mathbf{k}_\perp-\mathbf{q}|)|\tilde{W}(\mathbf{q})|^2\\ &\simeq \frac{\bar{\rho}_{\rm m0}}{\bar{N}_{\rm h}(z_L)}\int {\rm d}\chi\ f_{\rm h}(\chi)\int {\rm d}M\frac{{\rm d}n}{{\rm d}M}S(M)P_{\rm hm}(k_i;M,\chi) = C_{\rm hm}(k_i),\end{aligned} \tag{23}$$

where we have used $P_{\rm hm}(|\mathbf{k}_\perp-\mathbf{q}|) \simeq P_{\rm hm}(k_\perp)$ over the integration range ${\rm d}^2\mathbf{q}$ where the survey window supports, and also assumed that $P_{\rm hm}(k)$ is not a rapidly varying function within the bin width Δk around the k_i-bin ($k_i \gg q$). Here and hereafter we will often omit "$\perp$" in $k_\perp$ for notational simplicity. In the second equality on the r.h.s., we used the general identity for the window function:

$$\chi^2\Omega_S = \int {\rm d}^2\mathbf{x}_\perp\ W(\mathbf{x}_\perp)^n = \int\left[\prod_{a=1}^{n}\frac{d\mathbf{q}_a}{(2\pi)^2}\tilde{W}(\mathbf{q}_a)\right](2\pi)^2\delta^2_D(\mathbf{q}_{1\dots n}), \tag{24}$$

where $\mathbf{q}_{1\dots n} \equiv \mathbf{q}_1+\mathbf{q}_2+\cdots+\mathbf{q}_n$. For $n=2$, $\chi^2\Omega_S = \Omega_S\int {\rm d}^2\mathbf{q}/(2\pi)^2\ |\tilde{W}(\mathbf{q})|^2$.

C. Covariance matrix of the stacked lensing power spectrum

Now we consider the covariance matrix of the projected power spectrum of stacked lensing, which can be defined in terms of the power spectrum estimator (Eq. 20) as

$$C_{ij} \equiv \left\langle \hat{C}_{\rm hm}(k_i)\hat{C}_{\rm hm}(k_j)\right\rangle - \left\langle \hat{C}_{\rm hm}(k_i)\right\rangle\left\langle \hat{C}_{\rm hm}(k_j)\right\rangle. \tag{25}$$

By inserting the power spectrum estimator (Eq. 20) and then computing the ensemble averages in the above equation, we will below derive an expression of the covariance matrix.

First of all, we have to bear in mind that the stacked lensing estimator depends on the local mean of the projected halo number density (Eq. 4):

$$\hat{C}_{\rm hm} \propto \frac{1}{\hat{N}_{\rm h}} \simeq \frac{1}{\bar{N}_{\rm h}}\left(1-\Delta^{\rm h}_{\rm b}\right), \tag{26}$$

where $\Delta^{\rm h}_{\rm b} \equiv (1/\bar{N}_{\rm h})\int {\rm d}\chi\ \bar{n}^S_{\rm h}\bar{b}^S\delta_{\rm b}$. The background density mode $\delta_{\rm b}$ is a statistical variable, varies with a survey window, and therefore contributes the sample variance.

If we conceptually express the power spectrum estimator as $\hat{C}_{\rm hm}(k_i) \propto (1-\Delta_{\rm b}^{\rm h})(\hat{h}\hat{\kappa})_{k_i}$, where we denote the estimators of the halo field and the lensing field as $\hat{h}$ and $\hat{\kappa}$, the covariance can be expressed by a sum of three contributions as

$$\mathbf{C} \to \left\langle (1-\Delta_{\rm b}^{\rm h})^2(\hat{h}\hat{\kappa})_{k_i}(\hat{h}\hat{\kappa})_{k_j}\right\rangle - \left\langle (\hat{h}\hat{\kappa})_{k_i}\right\rangle\left\langle (\hat{h}\hat{\kappa})_{k_j}\right\rangle = \left\langle \hat{h}_{k_i}\hat{\kappa}_{k_j}\right\rangle\left\langle \hat{\kappa}_{k_i}\hat{h}_{k_j}\right\rangle + \left\langle \hat{h}_{k_i}\hat{h}_{k_j}\right\rangle\left\langle \hat{\kappa}_{k_i}\hat{\kappa}_{k_j}\right\rangle + \left\langle (1-\Delta_{\rm b}^{\rm h})^2(\hat{h}\hat{\kappa})_{k_i}(\hat{h}\hat{\kappa})_{k_j}\right\rangle_c . \tag{27}$$

The first two terms are from products of the power spectra, $\left\langle \hat{h}\hat{\kappa}\right\rangle$, $\left\langle \hat{h}\hat{h}\right\rangle$ and $\langle\hat{\kappa}\hat{\kappa}\rangle$, which we call the Gaussian covariance term. The third term arises from the 4-point correlation function or the trispectrum in the Fourier space, which we call the non-Gasussian term. If the halo and lensing fields are both Gaussian, e.g. at linear scales, the non-Gaussian term is vanishing. We generalized the non-Gaussian term in that it includes the contribution from the modulation in the projected halo number density.

Hence, following the method in Ref. [5], the covariance matrix can be generally expressed as

$$\mathbf{C} = \mathbf{C}^{\rm Gauss} + \mathbf{C}^{T_0} + \mathbf{C}^{\rm SSC}, \tag{28}$$

where we have considered modes satisfying $k_i, k_j \gg 1/(\chi\Omega_S^{1/2})$ (χ is in the range of lens redshifts). The first term is the Gaussian term as we stated above, the second term is the non-Gaussian term arising from the trispectrum of sub-survey modes, and the third term is the super-sample covariance arising from super-survey modes expressed in terms of $\delta_{\rm b}$ in our formulation.

1. Gaussian covariance

Extending the formulation in Ref. [8] to include the redshift weight $\Sigma_{\rm cr}(\chi)$ in the estimator (Eq. 20), we can derive the Gaussian covariance term corresponding to the first and second terms in the formal expression (Eq. 27):

$$C^{\rm Gauss}(k_i,k_j) = \frac{1}{N_{\rm mode}(k_i)}\delta^K_{ij}\left[C_{\rm hm}(k_i)^2 + C^{\Sigma_{\rm cr}}_{\rm hh}(k_i)C^{\rm obs}_{\kappa}(k_i)\right], \tag{29}$$

where δ^K_{ij} is the Kronecker delta function; $\delta^K_{ij}=1$ if $k_i=k_j$ to withing the bin width, otherwise $\delta^K_{ij}=0$. The number of independent k-modes resolved in the shell, by a resolution of Fourier decomposition of a finite-area survey, is given as

$$N_{\rm mode}(k_i) \simeq \frac{2\pi k_i\Delta k\, S_{\rm eff}}{(2\pi)^2} = \frac{k_i\Delta k\,\left\langle\chi_L^2\right\rangle\Omega_S}{2\pi}, \tag{30}$$

where $S_{\rm eff}$ is the effective survey area in dimension of [Mpc^2], and $\left\langle\chi_L^2\right\rangle$ is the radial distance to lensing halos, averaged over the radial selection function: $\left\langle\chi_L^2\right\rangle \equiv \int d\chi\ f_{\rm h}(\chi;z_L)\chi^2/\int d\chi\ f_{\rm h}(\chi;z_L)$. The spectrum $C^{\Sigma_{\rm cr}}_{\rm hh}(k)$ is the projected auto-spectrum of halos including the lensing kernel weight $\Sigma_{\rm cr}(\chi)$ at each lens redshift (see Eq. 13 or 20) as well as the shot noise contamination arising from a finite number of lensing halos used in the stacked lensing measurement, while $C^{\rm obs}_{\kappa}(k)$ is the cosmic shear power spectrum including the intrinsic shape noise contamination arising from a finite number of source galaxies. These spectra are given as

$$\begin{aligned} C^{\Sigma_{\rm cr}}_{\rm hh} &= \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int d\chi\ f_{\rm h}(\chi)^2\Sigma_{\rm cr}(\chi)^2[\bar{n}^S_{\rm h}\bar{b}^S]^2P^{\rm lin}_{\rm m}(k;\chi) + \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int d\chi\ f_{\rm h}(\chi)\Sigma_{\rm cr}(\chi)^2\bar{n}^S_{\rm h}, \\ C^{\rm obs}_{\kappa}(k) &= \int d\chi\ f_{\kappa}^2P^{\rm NL}_m(k;\chi) + \frac{\sigma_\epsilon^2}{\bar{N}_{\rm sg}}, \end{aligned} \tag{31}$$

where σ_ϵ is the rms intrinsic ellipticity per component [24]. $\bar{N}_{\rm sg}$ is the mean number density of source galaxies, used for the weak lensing measurements, per unit area in dimension of [Mpc^{-2}], which is given in terms of the angular source density as $\bar{N}_{\rm sg} = \bar{n}_{\rm gs}/\left\langle\chi_L^2\right\rangle$, where $\bar{n}_{\rm sg}$ is the mean angular number density of source galaxies per unit solid steradian as often used in the literature. The piece $C^{\rm obs}_{\kappa}(k)$ in Eq. (29) describes that the mass distribution along the line-of-sight contributes statistical errors to the stacked lensing, as can be found from Eq. (11). The above formula of the Gaussian term is new in a sense that the covariance is for the stacked lensing in dimension of the surface mass density, $\langle\Delta\Sigma\rangle(R)$, while the literature usually studied the angular power spectrum.

The Kronecker delta function in Eq. (29) ensures that the Gaussian term contributes only the diagonal terms of the covariance matrix. The covariance term scales with the survey area as $C^{\rm Gauss} \propto 1/\Omega_S$.

2. Non-Gaussian covariance: the trispectrum consistency and super-sample covariance

Now we consider the non-Gaussian term of the stacked lensing covariance. As we stated above, we need to include the contributions of connected 4-point function and the modulation in the projected number density of lensing halos due to super-survey modes, $\Delta_{\rm b}^{\rm h}$ (Eqs. 26 and 27).

To derive different contributions to the non-Gaussian covariance in an efficient way, we follow the formulation used in Refs. [5, 6]. For that purpose, let us consider the contribution from the connected 4-point function, by inserting the estimator (Eq. 20) into the covariance definition (Eq. 25):

$$\left\langle (\hat{h}\hat{\kappa})_{k_i}(\hat{h}\hat{\kappa})_{k_j}\right\rangle \leftarrow \frac{\bar{\rho}_{\rm m0}^2}{\bar{N}_{\rm h}(z_L)^2}\int\!{\rm d}\chi\, f_{\rm h}(\chi)^2\int\!{\rm d}M_1\int\!{\rm d}M_2\frac{{\rm d}n}{{\rm d}M_1}S(M_1)\frac{{\rm d}n}{{\rm d}M_2}(M_2)$$
$$\times\frac{1}{(\chi^2\Omega_S)^2}\int\!\frac{{\rm d}^2\mathbf{q}}{(2\pi)}|\tilde{W}(\mathbf{q})|^2\int_{|\mathbf{k}|\in k_i}\frac{{\rm d}^2\mathbf{k}}{A(k_i)}\int_{|\mathbf{k}'|\in k_j}\frac{{\rm d}^2\mathbf{k}'}{A(k_j)}T_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q};M_1,M_2,\chi),\quad(32)$$

where we have used the Limber's approximation and $T_{\rm hmhm}$ is the 3D trispectrum at redshift $z=z(\chi)$ for the two matter fields and the two fields of halos with their masses M_1 and M_2, defined as

$$\langle\delta_{\rm h}(\mathbf{k}_1;M_1)\delta_{\rm m}(\mathbf{k}_2)\delta_{\rm h}(\mathbf{k}_3;M_2)\delta_{\rm m}(\mathbf{k}_4)\rangle\equiv(2\pi)^3\delta_D^3(\mathbf{k}_{1234})T_{\rm hmhm}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4;M_1,M_2)\quad(33)$$

The convolution with the window function in Eq. (32) means that different 4-point configurations separated by less than the Fourier width of the window function and involving contributions from super-survey modes contribute to the non-Gaussian covariances.

For squeezed configurations with $k,k'\gg q$, using the halo model approach, we can express the change in the trispectrum due to the long wavelength q-mode to leading order in q/k as

$$T_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q})\equiv T_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')+\delta T(\mathbf{k},\mathbf{k}',q)\quad(34)$$

Inserting the first term of the above equation into the covariance calculation gives the non-Gaussian term arising from the trispectrum of sub-survey modes:

$$C^{T0}(k_i,k_j)\equiv\frac{\bar{\rho}_{\rm m0}^2}{\bar{N}_{\rm h}(z_L)^2}\int\!{\rm d}\chi\, f_{\rm h}(\chi)^2\bar{n}_{\rm h}^S(\chi)^2\frac{1}{\chi^2\Omega_S}\bar{T}_{\rm hmhm}(k_i,k_j;\chi)\quad(35)$$

with

$$\bar{T}_{\rm hmhm}(k_i,k_j)=\frac{1}{(\bar{n}_{\rm h}^S)^2}\int\!{\rm d}M_1\int\!{\rm d}M_2\,\frac{{\rm d}n}{{\rm d}M_1}S(M_1)\frac{{\rm d}n}{{\rm d}M_2}S(M_2)\int_0^{2\pi}\frac{{\rm d}\phi_{\mathbf{k}\cdot\mathbf{k}'}}{2\pi}T_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}'),\quad(36)$$

where we have used $(1/\chi^2\Omega_s)\int{\rm d}^2\mathbf{q}/(2\pi)^2|\tilde{W}(\mathbf{q})|^2=1$, assumed that the trispectrum is not a rapidly varying function within the k,k'-bin width, and $\phi_{\mathbf{k}\cdot\mathbf{k}'}$ is the angle between the vectors $\mathbf{k}$ and $\mathbf{k}'$ in the 2D plane (perpendicular to the line of sight direction). C^{T0} scales with the survey area as $C^{T0}\propto 1/\Omega_S$ as the Gaussian covariance term does. In Appendix XXX we show expressions from 1 to 4-halo trispectrum terms.

Now we consider the super-sample covariance (SSC) arising from δT in Eq. (34). In doing this we use the halo model that describes the halo-matter trispectrum involving one to four halos:

$$\delta T^S_{\rm hmhm}\equiv\frac{1}{(\bar{n}_{\rm h}^S)^2}\int\!{\rm d}M_1\int\!{\rm d}M_2\frac{{\rm d}n}{{\rm d}M_1}S(M_1)\frac{{\rm d}n}{{\rm d}M_2}S(M_2)\left[T_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q})-T_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')\right],\quad(37)$$

with

$$\begin{aligned}\delta T^{S,1{\rm h}}_{\rm hmhm}&\approx 0,\\ \delta T^{S,2{\rm h}}_{\rm hmhm}&\approx P_{\rm m}^{\rm lin}(q)\mathcal{I}_1^1(k)\mathcal{I}_1^1(k'),\\ \delta T^{S,3{\rm h}(13)}_{\rm hmhm}&\approx 0,\\ \delta T^{S,3{\rm h}(22)}_{\rm hmhm}&\approx 2P_{\rm m}^{\rm lin}(q)\left[\mathcal{I}_1^1(k')\mathcal{F}(\mathbf{k},\mathbf{q})+\mathcal{I}_1^1(k)\mathcal{F}(\mathbf{k}',-\mathbf{q})\right],\\ \delta T^{S,4{\rm h}}_{\rm hmhm}&\approx 4P_{\rm m}^{\rm lin}(q)\mathcal{F}(\mathbf{k},\mathbf{q})\mathcal{F}(\mathbf{k}',-\mathbf{q}),\end{aligned}\quad(38)$$

where

$$\mathcal{F}(\mathbf{k},\mathbf{q})\equiv\left[P_{\rm m}^{\rm lin}(k)F_2(\mathbf{q},-\mathbf{k})+P_{\rm m}^{\rm lin}(|\mathbf{k}-\mathbf{q}|)F_2(\mathbf{q},\mathbf{k}-\mathbf{q})\right]\bar{b}^S\mathcal{I}_1^1(|\mathbf{k}-\mathbf{q}|).\quad(39)$$

Here the mode-coupling kernel F_2 is

$$F_2(\mathbf{k},\mathbf{q})\equiv\frac{5}{7}+\frac{1}{2}\left(\frac{1}{k^2}+\frac{1}{q^2}\right)(\mathbf{k}\cdot\mathbf{q})+\frac{2}{7}\frac{(\mathbf{k}\cdot\mathbf{q})^2}{k^2q^2}.\quad(40)$$

Taking into account the fact that the mode coupling factor F_2 has a pole when one of its arguments goes to zero, the integration of $\mathcal{F}$ over the direction of $\mathbf{k}$ is

$$\int_0^{2\pi}\frac{\mathrm{d}\phi_{\mathbf{k}}}{2\pi}\,\mathcal{F}(\mathbf{k},\mathbf{q}) \simeq \frac{1}{2}\left(\frac{24}{7}-\frac{1}{2}\frac{d\ln k^2 P^{S,2\mathrm{h}}_{\mathrm{hm}}}{d\ln k}\right)P^{S,2\mathrm{h}}_{\mathrm{hm}}(k) \tag{41}$$

where we have defined $P^{S,2\mathrm{h}}_{\mathrm{hm}}(k) = \bar{b}^S P^{\mathrm{lin}}_{\mathrm{m}}(k)$.

Following the trispectrum consistency relation introduced in Ref. [5], we can realize that the the SSC term in the squeezed trispectrum arises from the response of the power spectrum to change in the background density by a factor of $(1+\delta_{\mathrm{b}})$. Inserting Eq. (41) into Eq. (37), we can rewrite the SSC term as

$$\int_0^{2\pi}\frac{\mathrm{d}\phi_{\mathbf{k}}}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\phi_{\mathbf{k}'}}{2\pi}\delta T^S_{\mathrm{hmhm}} \approx P^{\mathrm{lin}}_{\mathrm{m}}(q)\frac{\partial P^S_{\mathrm{hm}}(k)}{\partial\delta_{\mathrm{b}}}\frac{\partial P^S_{\mathrm{hm}}(k')}{\partial\delta_{\mathrm{b}}} \tag{42}$$

where we have defined the response of the halo-matter power spectrum to the background density:

$$\frac{\partial P^S_{\mathrm{hm}}(k)}{\partial\delta_{\mathrm{b}}} \equiv \left(\frac{24}{7}-\frac{1}{2}\frac{\mathrm{d}\ln k^2 P^{S,2\mathrm{h}}_{\mathrm{hm}}(k)}{\mathrm{d}\ln k}\right)P^{S,2\mathrm{h}}_{\mathrm{hm}}(k) + \mathcal{I}^1_1(k). \tag{43}$$

Comparing the above equation with Eq. (27) in Ref. [6], which is the response of the 3D mass power spectrum, the differences, $24/7$ instead of $68/21$ and $(1/2)\mathrm{d}\ln k^2 P(k)/\mathrm{d}\ln k$ instead of $(1/3)\mathrm{d}\ln k^3 P(k)/\mathrm{d}\ln k$, are due to the fact that we consider the projected power spectrum and the super-survey mode of $\mathbf{q}$ in the 2D plan perpendicular to the line-of-sight direction; that is, we ignored the radial mode. The $24/7$ piece is called the beat coupling ("BC") effect [3, 25] that the growth of a short wavelength perturbation is enhanced in a large scale overdensity, the $\mathcal{I}^1_1$ term is the halo sample variance ("HSV") effect [2, 15] that halo number densities are also enhanced in such a region, and the derivative term is the linear dilation ("LD") effect. The dilation effect occurs because the long wavelength mode changes the local expansion factor and hence the physical size of a mode that would have comoving wavelength k in its absence.

Furthermore, including a modulation in the projected number density halos, $\hat{N}_{\mathrm{h}}$ (Eq. 26), we can realize that the background mode δ_{b} causes a modulation in the power spectrum estimator of stacked lensing, to the linear order of δ_{b}:

$$\delta\hat{C}_{\mathrm{hm}}(k)\Big|_{\delta_{\mathrm{b}}} = \frac{\bar{\rho}_{\mathrm{m0}}}{\bar{N}_{\mathrm{h}}(z_L)}\int\mathrm{d}\chi\; f_{\mathrm{h}}(\chi)\bar{n}^S_{\mathrm{h}}\frac{\partial P^S_{\mathrm{hm}}(k)}{\partial\delta_{\mathrm{b}}}\delta_{\mathrm{b}} - \frac{C_{\mathrm{hm}}(k)}{\bar{N}_{\mathrm{h}}(z_L)}\int\mathrm{d}\chi\; f_{\mathrm{h}}(\chi)\bar{n}^S_{\mathrm{h}}\bar{b}^S\delta_{\mathrm{b}}. \tag{44}$$

For a thin redshift slice of lensing halos, where $f_{\mathrm{h}}=1$ in the range $\chi = [\chi_L - \Delta\chi/2, \chi_L + \Delta\chi/2]$ and otherwise $f_{\mathrm{h}} = 0$, the power spectrum response can be simplified as

$$\begin{aligned}\delta\hat{C}_{\mathrm{hm}}(k)\Big|_{\delta_{\mathrm{b}}} &\simeq \bar{\rho}_{\mathrm{m0}}\left[\left(\frac{24}{7}-\frac{1}{2}\frac{\mathrm{d}\ln k^2 P^{S,2\mathrm{h}}_{\mathrm{hm}}}{\mathrm{d}\ln k}\right)P^{S,2\mathrm{h}}_{\mathrm{hm}} + \mathcal{I}^1_1 - \bar{b}^S P^S_{\mathrm{hm}}(k)\right]\delta_{\mathrm{b}}(\chi_L)\\ &= \bar{\rho}_{\mathrm{m0}}\frac{\partial}{\partial\delta_{\mathrm{b}}}\left[\frac{1}{\hat{n}^S_{\mathrm{h}}(\chi_L)}\int\mathrm{d}M\;\frac{\mathrm{d}n}{\mathrm{d}M}S(M)P_{\mathrm{hm}}(k;M,\chi_L)\right]\delta_{\mathrm{b}}(\chi_L),\end{aligned} \tag{45}$$

where $\hat{n}^S_{\mathrm{h}} \equiv \int\mathrm{d}M\;(\mathrm{d}n/\mathrm{d}M)S(M)\,[1+b(M)\delta_{\mathrm{b}}]$, the 3D number density of halos including the background density modulation. This corresponds to a reqsponse of the power spectrum estimator using the local mean density as discussed in Refs. [5, 6].

Hence, using Eq. (44) and the Limber's approximation, we can compute the SSC term of the non-Gaussian covariance as

$$\begin{aligned}C^{\mathrm{SSC}}(k_i,k_j) &\equiv \left\langle \delta\hat{C}_{\mathrm{hm}}(k_i)\Big|_{\delta_{\mathrm{b}}}\,\delta\hat{C}_{\mathrm{hm}}(k_j)\Big|_{\delta_{\mathrm{b}}}\right\rangle\\ &\simeq \frac{1}{\bar{N}_{\mathrm{h}}(z_L)^2}\int\mathrm{d}\chi\; f_{\mathrm{h}}(\chi)^2(\bar{n}^S_{\mathrm{h}})^2\left[\bar{\rho}^2_{\mathrm{m0}}\frac{\partial P^S_{\mathrm{hm}}(k_i)}{\partial\delta_{\mathrm{b}}}\frac{\partial P^S_{\mathrm{hm}}(k_j)}{\partial\delta_{\mathrm{b}}} - \bar{\rho}_{\mathrm{m0}}\bar{b}^S C_{\mathrm{hm}}(k_i)\frac{\partial P^S_{\mathrm{hm}}(k_j)}{\partial\delta_{\mathrm{b}}}\right.\\ &\qquad\left. -\bar{\rho}_{\mathrm{m0}}\bar{b}^S C_{\mathrm{hm}}(k_j)\frac{\partial P^S_{\mathrm{hm}}(k_i)}{\partial\delta_{\mathrm{b}}} + (\bar{b}^S)^2 C_{\mathrm{hm}}(k_i)C_{\mathrm{hm}}(k_j)\right]\frac{1}{(\chi^2\Omega_S)^2}\int\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}|\tilde{W}(\mathbf{q})|^2 P^{\mathrm{lin}}_{\mathrm{m}}(q;\chi).\end{aligned} \tag{46}$$

Compared to the results in Refs. [5, 6], the SSC arises from a line-of-sight integration of the variance of the background mode, weighted by the halo-mass power spectrum.

Henec we will use Eqs. (29), (35), and (46) to compute the full covariance matrix to compare with the simulations.

D. Super sample signal

We have so far considered the impact of super-survey modes on the stacked lensing observables as an additional statistical noise in the covariacne matrix. In a given realization of the survey volume, the super-survey effect systematically changes the stacked lensing power spectrum of sub-survey modes just like a comsmological parameter, as implied by Eq. (44):

$$\hat{C}_{\rm hm}(k_i;\delta_{\rm b}) = \hat{C}_{\rm hm}(k_i;0) + \delta\hat{C}_{\rm hm}(k_i)\Big|_{\delta_{\rm b}} A^{\rm SSC}(\delta_{\rm b}), \tag{47}$$

where $A^{\rm SSC}(\delta_{\rm b})$ is a parameter to model the shift in the power spectrum due to the super-survey mode. As pointed out in Ref. [7] [see also 26], the additive model of the super-survey effect can be realized as an additional signal rather than the noise; the parameter $A^{\rm SSC}$ can be estimated together with cosmological parameters from the measured power spectrum in the survey region. In this case, we need not include the super-survey covariance contribution in the covariance matrix when estimating parameters: $\mathbf{C} = \mathbf{C}^{\rm Gauss} + \mathbf{C}^{T_0}$. This would simplify the anlysis, and can also open up a new window of extracting the largest-scale density mode from the measured stacked lensing. For example, if we can find a super-survey mode that is too large compared to the ΛCDM model expectation, say more than 3σ, it can be a new signature beyond the standard ΛCDM model on largest scales that contain a cleaner information on the physics of the early universe such as inflation.

III. RESULTS

In this section we use the halo model to esitmate each contribution in the covariance matrix of stacked lenisng for ΛCDM model, and then compare the hlao model prediction with the ray-tracing simulations.

A. Cosmological model

We employ the flat ΛCDM model that is consistent with the Planck measurements [27] as the fiducial model. More specifically the Planck cosmology is specified by $w_b = 0.02225$, $w_{\rm cdm} = 0.1198$, $\ln(10^{10}A_s) = 3.094$, $n_s = 0.9645$, $H_0 = 67.27$ km/s/Mpc, $\Omega_{\rm m0} = 0.3156$ ($\Omega_\Lambda = 0.6844$), and $\sigma_8 = 0.831$.

IV. DISCUSSION

Acknowledgments.– We thank Wayne Hu, Yin Li and Surhud More for useful discussion. MT is supported in part by Grant-in-Aid for Scientific Research from the JSPS Promotion of Science (No. 23340061 and 26610058), MEXT Grant-in-Aid for Scientific Research on Innovative Areas (No. 15H05893 and and 15K21733) and by JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers.

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Appendix A: Projected auto-correlation function of halos

The projected number density fluctuation field of halos is

$$\widehat{\delta N}_{\rm h}(\mathbf{x}_\perp) \equiv \int d\chi\, f_{\rm h}(\chi; z_L) \int dM\, S(M) \frac{dn}{dM} W(\mathbf{x}_\perp) \delta_{\rm h}(\chi, \mathbf{x}_\perp; M). \tag{A1}$$

The Fourier transform is

$$\widetilde{\delta N}_{\rm h}(\mathbf{k}_\perp) \equiv \int d^2\mathbf{x}_\perp\, \widehat{\delta N}_{\rm h}(\mathbf{x}_\perp) e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} = \int d\chi\, f_{\rm h}(\chi; z_L) \int dM\, \frac{dn}{dM} S(M) \int \frac{dk_\parallel}{2\pi} \tilde{\delta}^W_{\rm h}(k_\parallel, \mathbf{k}_\perp) e^{ik_\parallel\chi} \tag{A2}$$

where

$$\tilde{\delta}^W_{\rm h}(k_\parallel, \mathbf{k}_\perp) \equiv \int \frac{d^2\mathbf{q}}{(2\pi)^2} \tilde{W}(\mathbf{q}) \tilde{\delta}_{\rm h}(k_\parallel, \mathbf{k}_\perp - \mathbf{q}), \tag{A3}$$

$\tilde{W}(\mathbf{q};\chi) \equiv \int d^2\mathbf{x}_\perp\, W(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}_\perp}$, and quantities with tilde symbol hereafter denote their Fourier transforms.

An estimator of the projected power spectrum of halos can be defined as

$$\begin{aligned}\hat{C}_{\rm hh}(k_{\perp,i}) \equiv\ & \frac{1}{\hat{N}_{\rm h}(z_L)^2} \int d\chi \int d\chi'\, f_{\rm h}(\chi) f_{\rm h}(\chi') \int dM \int dM'\, S(M) \frac{dn}{dM} S(M') \frac{dn}{dM'} \\ & \times \frac{1}{\chi^2 \Omega_S} \int_{|\mathbf{k}_\perp| \in k_{\perp i}} \frac{d^2\mathbf{k}_\perp}{A(k_{\perp i})} \int \frac{dk_\parallel}{2\pi} \int \frac{dk'_\parallel}{2\pi}\, \tilde{\delta}^W_{\rm h}(k_\parallel, \mathbf{k}_\perp) \tilde{\delta}^W_{\rm h}(k'_\parallel, -\mathbf{k}_\perp) e^{ik_\parallel\chi + ik'_\parallel\chi'},\end{aligned} \tag{A4}$$

where the integral $\int d^2\mathbf{k}_\perp$ is over an annulus in $\mathbf{k}_\perp$ space of width Δk, and the area $A(k_{\perp i}) \simeq 2\pi k_{\perp i}\Delta k$. Using the halo power spectrum definition,

$$\left\langle \delta_{\rm h}(k_\parallel, \mathbf{k}_\perp; M) \delta_{\rm h}(k'_\parallel, \mathbf{k}'_\perp; M') \right\rangle = (2\pi)^3 \delta_D(k_\parallel + k'_\parallel) \delta^2_D(\mathbf{k}_\perp + \mathbf{k}'_\perp) P_{\rm hh}(k; M, M'), \tag{A5}$$

and using the Limber's approximation [28], we can compute the ensemble average of the above power spectrum estimator:

$$\begin{aligned}C_{\rm hh}(k_i) = \left\langle \hat{C}_{\rm hh} \right\rangle (k_{\perp,i}) =\ & \frac{1}{\bar{N}_{\rm h}(z_L)^2} \int d\chi \int d\chi'\, f_{\rm h}(\chi) f_{\rm h}(\chi') \int dM \int dM'\, S(M) \frac{dn}{dM} S(M') \frac{dn}{dM'} \\ & \times \frac{1}{\chi^2\Omega_S} \int \frac{d^2\mathbf{k}_\perp}{A(k_{\perp i})} \int \frac{d^2\mathbf{q}}{(2\pi)^2} P_{\rm hh}\left(\sqrt{k_\parallel^2 + |\mathbf{k}_\perp - \mathbf{q}|^2}\right) |\tilde{W}(\mathbf{q})|^2 e^{ik_\parallel(\chi-\chi')} \\ \simeq\ & \frac{1}{\bar{N}_{\rm h}(z_L)^2} \int d\chi\, f_{\rm h}(\chi)^2 \int dM \int dM'\, S(M) \frac{dn}{dM} S(M') \frac{dn}{dM'} P_{\rm hh}(k_{\perp,i}, \chi; M, M') \frac{1}{\chi^2\Omega_S} \int \frac{d^2\mathbf{q}}{(2\pi)^2} |\tilde{W}(\mathbf{q})|^2 \\ =\ & \frac{1}{\bar{N}_{\rm h}(z_L)^2} \int d\chi\, f_{\rm h}(\chi)^2 \int dM \int dM'\, S(M) \frac{dn}{dM} S(M') \frac{dn}{dM'} P_{\rm hh}(k_{\perp,i}, \chi; M, M').\end{aligned} \tag{A6}$$

Here we have considered Fourier modes of $k_\perp$ staisfying $k_\perp \gg q$, and then assumed no correlation between Fourider modes of $k_\perp$ and $\delta_{\rm b}$. We have used that $P_{\rm hh}(|\mathbf{k}_\perp - \mathbf{q}|) \simeq P_{\rm hh}(k_\perp)$ over the integration range of $d^2\mathbf{q}$ which the window function supports and also assumed that $P_{\rm hh}(k)$ is not a rapidly varying function within the $k_\perp$-bin. Here and hereafter we will often omit "$\perp$" in $k_\perp$ for notational simplicity. In the third equality on the r.h.s., we used the general identity for the window function:

$$\chi^2\Omega_S = \int d^2\mathbf{x}_\perp\ W(\mathbf{x}_\perp)^n = \int\!\!\left[\prod_{a=1}^{n}\frac{d\mathbf{q}_a}{(2\pi)^2}\tilde{W}(\mathbf{q}_a)\right](2\pi)^2\delta_D^2(\mathbf{q}_{1\ldots n}), \tag{A7}$$

where $\mathbf{q}_{1\ldots n}\equiv\mathbf{q}_1+\mathbf{q}_2+\cdots+\mathbf{q}_n$. For $n=2$, $\chi^2\Omega_S=\Omega_S\int d^2\mathbf{q}/(2\pi)^2\ |\tilde{W}(\mathbf{q})|^2$.

If we employ the halo model, where the halo power spectrum is modeled as $P_{\rm hh}(k;M,M')\simeq b(M)b(M')P^{\rm lin}_{\rm m}(k)$, the projected power spectrum is rewritten as

$$C_{\rm hh}(k) = \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int d\chi f_{\rm h}(\chi)^2\left[\bar{n}_{\rm h}^S\bar{b}^S\right]^2 P^{\rm lin}_{\rm m}(k;\chi), \tag{A8}$$

where $P^{\rm lin}_{\rm m}(k)$ is the linear mass power spectrum. The projected power spectrum $C_{\rm hh}$ has a dimension of [Mpc^2].

Appendix B: An estimator of the stacked lensing profile

$$\begin{aligned}
w_{\rm hh}(R;z_L) &= \left\langle\delta^{\rm 2D}_{\rm h}(\mathbf{x}_\perp;z_L)\delta^{\rm 2D}_{\rm h}(\mathbf{x}'_\perp;z_L)\right\rangle_{R=|\mathbf{x}_\perp-\mathbf{x}'_\perp|}\\
&= \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int d\chi\chi^2\int d\chi\chi'^2\int dM\frac{dn}{dM}S(M,\chi;z_L)\int dM'\frac{dn}{dM'}S(M',\chi';z_L)\\
&\qquad\times\int\frac{dk_\parallel d^2\mathbf{k}_\perp}{(2\pi)^3}\int\frac{dk'_\parallel d^2\mathbf{k}'_\perp}{(2\pi)^3}\left\langle\delta_{\rm h}(\mathbf{k};\chi,M)\delta_{\rm h}(\mathbf{k}';\chi',M')\right\rangle e^{i(\chi k_\parallel+\mathbf{x}_\perp\cdot\mathbf{k}_\perp)+i(\chi'k'_\parallel+\mathbf{x}'_\perp\cdot\mathbf{k}'_\perp)}\\
&= \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int d\chi\chi^2\int d\chi\chi'^2\int dM\frac{dn}{dM}S(M,\chi;z_L)\int dM'\frac{dn}{dM'}S(M',\chi';z_L)\\
&\qquad\times\int\frac{dk_\parallel d^2\mathbf{k}_\perp}{(2\pi)^3}P_{\rm hh}(k;M,M')e^{ik_\parallel(\chi-\chi')+i\mathbf{k}_\perp\cdot\mathbf{R}}\\
&\simeq \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int d\chi\chi^4\int dM\int dM'\ \frac{dn}{dM}S(M,\chi;z_L)\frac{dn}{dM'}S(M',\chi';z_L)\int\frac{d^2\mathbf{k}_\perp}{(2\pi)^2}P_{\rm hh}(k_\perp;M,M',z)e^{i\mathbf{k}_\perp\cdot\mathbf{R}}\\
&\simeq \frac{1}{\bar{N}_{\rm h}(z_L)^2}\chi_L^4\Delta\chi\int dM\int dM'\ \frac{dn}{dM}S(M,\chi;z_L)\frac{dn}{dM'}S(M',\chi';z_L)\int\frac{d^2\mathbf{k}_\perp}{(2\pi)^2}P_{\rm hh}(k_\perp;M,M',z)e^{i\mathbf{k}_\perp\cdot\mathbf{R}}
\end{aligned} \tag{B1}$$

Using the Limber's approximation, the ensemble average of the sacked lensing profile can be computed as

$$\begin{aligned}
\langle\Sigma\rangle(R) &\equiv \left\langle\Sigma_{\rm cr}(\chi)\kappa^W(\boldsymbol{\theta})\right\rangle_{R=|\mathbf{x}_\perp(z_l)-\chi(z_l)\boldsymbol{\theta}|} = \left\langle\delta_{\rm h}(\mathbf{x}_\perp)\Sigma_{\rm cr}(\chi)\kappa^W(\mathbf{x}'_\perp)\right\rangle_{R=|\mathbf{x}_\perp-\mathbf{x}'_\perp|;\mathbf{x}'=\chi\boldsymbol{\theta}}\\
&= \frac{1}{\hat{N}_{\rm h}(z_L)}\int d\chi d\chi'\ \chi^2 f_{\rm h}(\chi)\Sigma_{\rm cr}(\chi)f_\kappa(\chi')\int dM\ \frac{dn}{dM}S(M)b(M)W(\mathbf{x}_\perp)\left\langle\delta_{\rm m,lin}(\chi,\mathbf{x}_\perp)\delta^W_{\rm m}(\chi',\chi'\boldsymbol{\theta})\right\rangle\\
&= \frac{1}{\hat{N}_{\rm h}(z_L)}\int d\chi d\chi'\ \chi^2\Sigma_{\rm cr}(\chi)W_\kappa(\chi')\int dM\ \frac{dn}{dM}S(M,\chi;z_L)\\
&\qquad\times\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}P_{\rm hm}(k;M,\chi)(2\pi)^3\delta_D(k_\parallel+k'_\parallel)\delta^2_D(\mathbf{k}_\perp+\mathbf{k}'_\perp)e^{ik_\parallel\chi+i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}e^{ik_\parallel\chi'+i\mathbf{k}'_\perp\cdot\chi\boldsymbol{\theta}}\\
&= \frac{1}{\hat{N}_{\rm h}(z_L)}\int d\chi d\chi'\ \chi^2\Sigma_{\rm cr}(\chi)W_\kappa(\chi')\int dM\ \frac{dn}{dM}S(M,\chi;z_L)\int\frac{dk_\parallel d^2\mathbf{k}_\perp}{(2\pi)^3}P_{\rm hm}(k;M,\chi)e^{ik_\parallel(\chi-\chi')+i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\chi'\boldsymbol{\theta})}\\
&\simeq \frac{1}{\hat{N}_{\rm h}(z_L)}\int d\chi d\chi'\ \chi^2\Sigma_{\rm cr}(\chi)W_\kappa(\chi')\int dM\ \frac{dn}{dM}S(M,\chi;z_L)\int\frac{dk_\parallel d^2\mathbf{k}_\perp}{(2\pi)^3}P_{\rm hm}(k_\perp;M,\chi)e^{ik_\parallel(\chi-\chi')+i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\chi'\boldsymbol{\theta})}\\
&= \frac{1}{\hat{N}_{\rm h}(z_L)}\int_{\chi_L} d\chi\,\chi^2\Sigma_{\rm cr}(\chi)W_\kappa(\chi)\int dM\ \frac{dn}{dM}\tilde{S}(M)\int\frac{d^2\mathbf{k}_\perp}{(2\pi)^2}P_{\rm hm}(k_\perp;M,\chi)e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\chi\boldsymbol{\theta})}\\
&= \frac{\bar{\rho}_{\rm m0}}{\hat{N}_{\rm h}(z_L)}\int_{\chi_L} d\chi\,\chi^2\int dM\ \frac{dn}{dM}\tilde{S}(M)\int\frac{d^2\mathbf{k}_\perp}{(2\pi)^2}P_{\rm hm}(k_\perp;M,\chi)e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\chi\boldsymbol{\theta})},
\end{aligned} \tag{B2}$$

where $P_{\rm hm}(k;M,z)$ is the halo-mass cross-power spectrum for halos of mass M and at redshift z.

Appendix C: Halo model for the non-Gaussian covariance term C^{T0}

$$C^{T0}(k_i,k_j) = \frac{\bar{\rho}_{\mathrm{m0}}^2}{\bar{N}_{\mathrm{h}}(z_L)^2}\int \mathrm{d}\chi\, f_{\mathrm{h}}(\chi)^2(\bar{n}_{\mathrm{h}}^S)^2\bar{T}_{\mathrm{hmhm}}(k_i,k_j), \tag{C1}$$

$$T^S_{\mathrm{hmhm}}(\mathbf{k},\mathbf{k}') = \frac{1}{(\bar{n}_{\mathrm{h}}^S)^2}\int \mathrm{d}M_1\int \mathrm{d}M_2 \frac{\mathrm{d}n}{\mathrm{d}M_1}S(M_1)\frac{\mathrm{d}n}{\mathrm{d}M_2}S(M_2)T_{\mathrm{hmhm}}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') \tag{C2}$$

$$T_{\mathrm{hmhm}} = \bar{T}^{\mathrm{1h}}_{\mathrm{hmhm}} + \left(\bar{T}^{\mathrm{2h(13)}}_{\mathrm{hmhm}} + \bar{T}^{\mathrm{2h(22)}}_{\mathrm{hmhm}}\right) + \bar{T}^{\mathrm{3h}}_{\mathrm{hmhm}} + \bar{T}^{\mathrm{4h}}_{\mathrm{hmhm}} \tag{C3}$$

$$\begin{aligned}
\bar{T}^{S,\mathrm{1h}}_{\mathrm{hmhm}}(\mathbf{k},\mathbf{k}') &= \mathcal{I}^4_0(0,k,0,k')\\
\bar{T}^{S,\mathrm{2h(13)}}_{\mathrm{hmhm}}(\mathbf{k},\mathbf{k}') &= \bar{b}^S\,\mathcal{I}^1_2(k,k')P^{\mathrm{lin}}_{\mathrm{m}}(k) + I^1_1(k)\mathcal{I}^1_1(k')P^{\mathrm{lin}}_{\mathrm{m}}(k) + \bar{b}^S\,\mathcal{I}^1_2(k,k')P^{\mathrm{lin}}_{\mathrm{m}}(k') + I^1_1(k')\mathcal{I}^1_1(k)P^{\mathrm{lin}}_{\mathrm{m}}(k'),\\
\bar{T}^{S,\mathrm{2h(22)}}_{\mathrm{hmhm}}(k_i,k_j) &= \bar{b}^S\,I^1_2(k_i,k_j)P^{\mathrm{lin}}_{\mathrm{m}}(|\mathbf{k}+\mathbf{k}'|) + \mathcal{I}^1_1(k_i)\mathcal{I}^1_1(k_j)P^{\mathrm{lin}}_{\mathrm{m}}(|\mathbf{k}-\mathbf{k}'|)\\
\bar{T}^{S,\mathrm{3h}}_{\mathrm{hmhm}}(\mathbf{k},\mathbf{k}') &= \bar{b}^S\,I^1_1(k)I^1_1(k')B^{\mathrm{PT}}_{\mathrm{m}}(\mathbf{k}+\mathbf{k}',-\mathbf{k},-\mathbf{k}') + \bar{b}^S\,\mathcal{I}^1_1(k')I^1_1(k)B^{\mathrm{PT}}_{\mathrm{m}}(\mathbf{k}-\mathbf{k}',-\mathbf{k},\mathbf{k}')\\
&\quad +\bar{b}^S\,\mathcal{I}^1_1(k)I^1_1(k')B^{\mathrm{PT}}_{\mathrm{m}}(-\mathbf{k}+\mathbf{k}',\mathbf{k},-\mathbf{k}') + \bar{b}^S\,\mathcal{I}^1_1(k)I^1_1(k')B^{\mathrm{PT}}_{\mathrm{m}}(-\mathbf{k}-\mathbf{k}',\mathbf{k},\mathbf{k}')\\
\bar{T}^{S,\mathrm{4h}}_{\mathrm{hmhm}}(\mathbf{k},\mathbf{k}') &= (\bar{b}^S)^2I^1_1(k)I^1_1(k')T^{\mathrm{PT}}_{m}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')
\end{aligned} \tag{C4}$$

$$\bar{T}_{\mathrm{hmhm}} = \bar{T}^{\mathrm{1h}}_{\mathrm{hmhm}} + \left(\bar{T}^{\mathrm{2h(13)}}_{\mathrm{hmhm}} + \bar{T}^{\mathrm{2h(22)}}_{\mathrm{hmhm}}\right) + \bar{T}^{\mathrm{3h}}_{\mathrm{hmhm}} + \bar{T}^{\mathrm{4h}}_{\mathrm{hmhm}} \tag{C5}$$

$$\begin{aligned}
\bar{T}^{\mathrm{1h}}_{\mathrm{hmhm}}(\mathbf{k},\mathbf{k}') &= \mathcal{I}^4_0(0,k_i,0,k_j)\\
\bar{T}^{\mathrm{2h(13)}}_{\mathrm{hmhm}}(k_i,k_j) &= \bar{b}^S\,\mathcal{I}^1_3(0,k_i,k_j)P^{\mathrm{lin}}_{\mathrm{m}}(k_i) + I^1_1(k_i)\mathcal{I}^1_3(0,0,k_j)P^{\mathrm{lin}}_{\mathrm{m}}(k_i) + \bar{b}^S\,\mathcal{I}^1_3(0,k_i,k_j)P^{\mathrm{lin}}_{\mathrm{m}}(k_j) + I^1_1(k_j)\mathcal{I}^1_3(0,0,k_i)P^{\mathrm{lin}}_{\mathrm{m}}(k_j),\\
\bar{T}^{\mathrm{2h(22)}}_{\mathrm{hmhm}}(k_i,k_j) &= \bar{b}^S\,I^1_2(k_i,k_j)P^{\mathrm{lin}}_{\mathrm{m}}(|\mathbf{k}+\mathbf{k}'|) + \mathcal{I}^1_1(k_i)\mathcal{I}^1_1(k_j)P^{\mathrm{lin}}_{\mathrm{m}}(|\mathbf{k}-\mathbf{k}'|)\\
\bar{T}^{\mathrm{3h}}_{\mathrm{hmhm}}(\mathbf{k},\mathbf{k}') &= \bar{b}^S\,I^1_1(k)I^1_1(k')B^{\mathrm{PT}}_{\mathrm{m}}(\mathbf{k}+\mathbf{k}',-\mathbf{k},-\mathbf{k}') + \bar{b}^S\,\mathcal{I}^1_1(k')I^1_1(k)B^{\mathrm{PT}}_{\mathrm{m}}(\mathbf{k}-\mathbf{k}',-\mathbf{k},\mathbf{k}')\\
&\quad +\bar{b}^S\,\mathcal{I}^1_1(k)I^1_1(k')B^{\mathrm{PT}}_{\mathrm{m}}(-\mathbf{k}+\mathbf{k}',\mathbf{k},-\mathbf{k}') + \bar{b}^S\,\mathcal{I}^1_1(k)I^1_1(k')B^{\mathrm{PT}}_{\mathrm{m}}(-\mathbf{k}-\mathbf{k}',\mathbf{k},\mathbf{k}')\\
\bar{T}^{\mathrm{4h}}_{\mathrm{hmhm}}(\mathbf{k},\mathbf{k}') &= (\bar{b}^S)^2I^1_1(k)I^1_1(k')T^{\mathrm{PT}}_{m}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')
\end{aligned} \tag{C6}$$

$$\begin{aligned}
\langle\Sigma\rangle(R;z_L) &\equiv \langle\delta_{\mathrm{h}}(\mathbf{x}_\perp;z_L)\Sigma_{\mathrm{cr}}(z_L)\kappa(\boldsymbol{\theta})\rangle_{|\mathbf{x}_\perp-\chi_L\boldsymbol{\theta}|=R}\\
&= \frac{1}{\bar{N}_{\mathrm{h}}(z_L)}\int\mathrm{d}\chi\,\chi^2\int\mathrm{d}\chi'\,f_\kappa(\chi)\Sigma_{\mathrm{cr}}(\chi)\int\mathrm{d}M\,\frac{\mathrm{d}n}{\mathrm{d}M}S(M,\chi;z_L)\,\langle\delta_{\mathrm{h}}(\chi,\mathbf{x}_\perp;M)\delta_{\mathrm{m}}(\chi',\chi'\boldsymbol{\theta})\rangle\\
&= \frac{1}{\bar{N}_{\mathrm{h}}(z_L)}\int\mathrm{d}\chi\,\chi^2\int\mathrm{d}\chi'\,f_\kappa(z)\Sigma_{\mathrm{cr}}(\chi)\int\mathrm{d}M\,\frac{\mathrm{d}n}{\mathrm{d}M}S(M,\chi;z_L)\\
&\qquad\times\int\frac{\mathrm{d}^2\mathbf{k}_\perp\mathrm{d}k_\parallel}{(2\pi)^3}\int\frac{\mathrm{d}^2\mathbf{k}'_\perp\mathrm{d}k'_\parallel}{(2\pi)^3}\,\langle\delta_{\mathrm{h}}(\mathbf{k};M)\delta_{\mathrm{m}}(\mathbf{k}')\rangle\,e^{i(\chi k_\parallel+\mathbf{k}_\perp\cdot\mathbf{x}_\perp)+i(\chi'k'_\parallel+\mathbf{k}'_\perp\cdot\chi'\boldsymbol{\theta})}\\
&= \frac{1}{\bar{N}_{\mathrm{h}}(z_L)}\Sigma_{\mathrm{cr}}(z_L)\int\mathrm{d}\chi\,\chi^2\int\mathrm{d}\chi'\,f_\kappa(z)\int\mathrm{d}M\,\frac{\mathrm{d}n}{\mathrm{d}M}S(M,\chi;z_L)\int\frac{\mathrm{d}^2\mathbf{k}_\perp\mathrm{d}k_\parallel}{(2\pi)^3}P_{\mathrm{hm}}(k;M)e^{ik_\parallel(\chi-\chi')+i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\chi'\boldsymbol{\theta})}\\
&\simeq \frac{1}{\bar{N}_{\mathrm{h}}(z_L)}\Sigma_{\mathrm{cr}}(z_L)\int\mathrm{d}\chi\,\chi^2f_\kappa(\chi)\int\mathrm{d}M\,\frac{\mathrm{d}n}{\mathrm{d}M}S(M,\chi;z_L)\int\frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2}P_{\mathrm{hm}}(k_\perp;M)e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\chi\boldsymbol{\theta})}\\
&\simeq \frac{1}{\bar{N}_{\mathrm{h}}(z_L)}\Sigma_{\mathrm{cr}}(z_L)\Delta\chi\chi_L^2f_\kappa(\chi_L)\int\mathrm{d}M\,\frac{\mathrm{d}n}{\mathrm{d}M}S(M;z_L)\int\frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2}P_{\mathrm{hm}}(k_\perp;M)e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\chi_L\boldsymbol{\theta})}\\
&= \frac{1}{\frac{\mathrm{d}n}{\mathrm{d}M}(z_L)}\bar{\rho}_{\mathrm{m0}}\int\mathrm{d}M\,\frac{\mathrm{d}n}{\mathrm{d}M}S(M;z_L)\int\frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2}P_{\mathrm{hm}}(k_\perp;M)e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\chi_L\boldsymbol{\theta})}
\end{aligned} \tag{C7}$$

$$\mathbf{C} = \mathbf{C}^{\mathrm{Gauss}} + \mathbf{C}^{T} \tag{C8}$$

$$C_{ij}^{\rm Gauss} = \frac{2}{N_{\rm mode}(k_i)}\delta^K_{ij}\left[C_{\rm hm}(k_i)^2 + C_{\rm hh}^{\Sigma_{\rm cr}}(k_i)C_\kappa(k_i)\right] \tag{C9}$$

$$\begin{aligned}
N_{\rm mode}(k) &\simeq \int {\rm d}\chi\ f_{\rm h}(\chi)\frac{2\pi k\Delta k}{(2\pi/\chi\Theta_S)^2} = \Omega_S\int {\rm d}\chi\ f_{\rm h}(\chi)\frac{k\Delta k\chi^2}{2\pi} \simeq \frac{k\Delta k\chi_L^2\Omega_S}{2\pi}\\
C_{\rm hh}^{\Sigma_{\rm cr}}(k) &= \frac{1}{\bar N_{\rm h}(z_L)^2}\int {\rm d}\chi\, f_{\rm h}(\chi)^2\Sigma_{\rm cr}(\chi)^2\left[\int {\rm d}M\frac{{\rm d}n}{{\rm d}M}b(M)\right]^2 P_{\rm m}^{\rm lin}(k;\chi) + \frac{\int {\rm d}\chi\, f_{\rm h}(\chi)\Sigma_{\rm cr}(\chi)^2\int {\rm d}M\,\frac{{\rm d}n}{{\rm d}M}}{\bar N_{\rm h}(z_L)^2}\\
C_\kappa(k) &= \int {\rm d}\chi\, f_\kappa(\chi)^2 P_{\rm m}^{\rm NL}(k;\chi) + \frac{\sigma_\epsilon^2}{\bar n_{gs}},
\end{aligned} \tag{C10}$$

Using Limeber's approximation,

$$C_{ij}^T = \frac{1}{\bar N_{\rm h}(z_L)^2}\int {\rm d}\chi\, f_{\rm h}(\chi)^2\Sigma_{\rm cr}(\chi)^2\frac{1}{(\chi^2\Omega_S)^2}\int\frac{{\rm d}^2\mathbf{q}}{(2\pi)^2}|\tilde W(\mathbf{q})|^2\int_{k_i}\frac{{\rm d}\mathbf{k}}{V(k_i)}\int_{k_j}\frac{{\rm d}\mathbf{k}}{V(k_j)}T_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q}) \tag{C11}$$

For squeezed configurations with $k, k' \gg q$ we can express the change in the trispectrum due to the long wavelength q mode to leading order in q/k as

$$\delta T_{\rm hmhm} \equiv T_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q}) - T_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') \tag{C12}$$

with

$$\begin{aligned}
\delta T_{\rm hmhm}^{\rm 1h} &\simeq 0\\
\delta T_{\rm hmhm}^{\rm 2h(22)} &\simeq P_{\rm m}^{\rm lin}(q)I_1^{(S)1}(k)I_1^{(S)1}(k')\\
\delta T_{\rm hmhm}^{\rm 2h(13)} &\simeq 0\\
\delta T_{\rm hmhm}^{\rm 3h} &\simeq 2P_{\rm m}^{\rm lin}(q)I_1^{(S)1}(k')\mathcal{F}(\mathbf{k},\mathbf{q}) + 2P_{\rm m}^{\rm lin}(q)I_1^{(S)1}(k)\mathcal{F}(\mathbf{k}',-\mathbf{q})\\
\delta T_{\rm hmhm}^{\rm 4h} &\simeq 4P_{\rm m}^{\rm lin}(q)\mathcal{F}(\mathbf{k},\mathbf{q})\mathcal{F}(\mathbf{k}',-\mathbf{q})
\end{aligned} \tag{C13}$$

where

$$\mathcal{F}(\mathbf{k},\mathbf{q}) \equiv \left[P_{\rm m}^{\rm lin}(k)F_2(\mathbf{q},-\mathbf{k}) + P_{\rm m}^{\rm lin}(|\mathbf{k}-\mathbf{q}|)F_2(\mathbf{q},\mathbf{k}-\mathbf{q})\right]I_0^{(S)0}I_1^{(S)1}(\mathbf{k}-\mathbf{q}) \tag{C14}$$

with the mode-coupling kernel F_2 defined as

$$F_2(\mathbf{k},\mathbf{q}) \equiv \frac{5}{7} + \frac{1}{2}\left(\frac{1}{k^2}+\frac{1}{q^2}\right)(\mathbf{k}\cdot\mathbf{q}) + \frac{2}{7}\frac{(\mathbf{k}\cdot\mathbf{q})^2}{k^2q^2} \tag{C15}$$

The latter must be handled with care since the mode coupling factor F_2 has a pole when one of its arguments goes to zero. Thus we need to consistently expand this expression in q/k (or q/k'). The result of integrating over the direction of $\mathbf{k}$ is

$$\int_0^{2\pi}\frac{{\rm d}\phi}{2\pi}\mathcal{F}(\mathbf{k},\mathbf{q}) = \frac{1}{2}\left[\frac{24}{7} - \frac{1}{2}\frac{{\rm d}\ln P_{\rm hm}^{\rm lin}(k)}{{\rm d}\ln k}\right]P_{\rm hm}^{\rm lin}(k) \tag{C16}$$

$$\frac{{\rm d}\ln P_{\rm hm}(k)}{{\rm d}\delta_{\rm b}} = \left[\frac{24}{7} - \frac{1}{2}\frac{{\rm d}\ln P_{\rm hm}^{\rm lin}(k)}{{\rm d}\ln k}\right]P_{\rm hm}^{\rm lin}(k) + I_1^{(S)1}(k) \tag{C17}$$

$$\begin{aligned}
\mathbf{C}^{\rm SSC} \leftarrow &\left\langle\left[1 - 2\int {\rm d}\chi\ f_{\rm h}(\chi)\int {\rm d}M S(M)\frac{{\rm d}n}{{\rm d}M}b\delta_{\rm b}\right]\left[\int {\rm d}\chi\ f_{\rm h}(\chi)P_{\rm hm}^S(k)\left\{1 + \frac{{\rm d}\ln P_{\rm hm}^S(k)}{{\rm d}\delta_{\rm b}}\delta_{\rm b}\right\}\right]\right.\\
&\qquad\left.\times\left[\int {\rm d}\chi\ f_{\rm h}(\chi)P_{\rm hm}^S(k)\left\{1 + \frac{{\rm d}\ln P_{\rm hm}^S(k')}{{\rm d}\delta_{\rm b}}\delta_{\rm b}\right\}\right]\right\rangle\\
\simeq &\ \frac{1}{\bar N_{\rm h}^2}\int {\rm d}\chi\ f_{\rm h}(\chi)^2\left[\frac{{\rm d}\ln P_{\rm hm}^S(k)}{{\rm d}\delta_{\rm b}}\frac{{\rm d}\ln P_{\rm hm}^S(k')}{{\rm d}\delta_{\rm b}} - 2I_0^{(S)1}\frac{{\rm d}\ln P_{\rm hm}^S(k)}{{\rm d}\delta_{\rm b}} - 2I_0^{(S)1}\frac{{\rm d}\ln P_{\rm hm}^S(k')}{{\rm d}\delta_{\rm b}}\right]\frac{1}{(\chi^2\Omega_S)^2}\int\frac{{\rm d}^2\mathbf{q}}{(2\pi)}|\tilde W(\mathbf{q})|^2P_{\rm m}^{\rm lin}(q)
\end{aligned} \tag{C18}$$

First of all, note that we define the halo number density fluctuation relative to the mean within the survey window, as shown from Eq. (4):

$$\hat{C}_{\rm hm}(k_i) = \frac{C_{\rm hm}(k_i)}{1+\delta\ln\hat{N}_{\rm h}|_{\delta_{\rm b}}} \simeq C_{\rm hm}(k_i)\left[1-\delta\ln\hat{N}_{\rm h}|_{\delta_{\rm b}}\right] \tag{C19}$$

$$\delta\ln\hat{C}_{\rm hm}(k_i)|_{\delta_{\rm b}} \simeq \frac{1}{\bar{N}_{\rm h}(z_L)}\int{\rm d}\chi\, f_{\rm h}(\chi)\int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}(M)S(M)b(M)\delta_{\rm b}(\chi) \tag{C20}$$

$$\begin{aligned}
\left\langle\delta\ln\hat{C}_{\rm hm}(k_i)|_{\delta_{\rm b}}\delta\ln\hat{C}_{\rm hm}(k_j)|_{\delta_{\rm b}}\right\rangle &= \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int{\rm d}\chi\int{\rm d}\chi'\, f_{\rm h}(\chi)f_{\rm h}(\chi')\int{\rm d}M\int{\rm d}M'\,S(M)\frac{{\rm d}n}{{\rm d}M}(M)S(M)\frac{{\rm d}n}{{\rm d}M}(M')b(M)b(M')\left\langle\delta_{\rm b}(\chi)\delta_{\rm b}(\chi')\right\rangle\\
&= \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int{\rm d}\chi\int{\rm d}\chi'\, f_{\rm h}(\chi)f_{\rm h}(\chi')\int{\rm d}M\int{\rm d}M'\,S(M)\frac{{\rm d}n}{{\rm d}M}(M)S(M')\frac{{\rm d}n}{{\rm d}M}(M')b(M)b(M')\\
&\quad\times\frac{1}{\chi^2\Omega_S\chi'^2\Omega_S}\int\frac{{\rm d}k_\parallel}{2\pi}\int\frac{{\rm d}k'_\parallel}{2\pi}\int\frac{{\rm d}^2\mathbf{q}}{(2\pi)^2}\int\frac{{\rm d}^2\mathbf{q}'}{(2\pi)^2}\tilde{W}(\mathbf{q})tW(\mathbf{q}')\left\langle\delta_{\rm m,lin}(k_\parallel,-\mathbf{q})\delta_{\rm m,lin}(k'_\parallel,-\mathbf{q}')\right\rangle e^{ik_\parallel\chi+i\chi'k'_\parallel}\\
&= \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int{\rm d}\chi\int{\rm d}\chi'\, f_{\rm h}(\chi)f_{\rm h}(\chi')\int{\rm d}M\int{\rm d}M'\,S(M)\frac{{\rm d}n}{{\rm d}M}(M)S(M')\frac{{\rm d}n}{{\rm d}M}(M')\\
&\quad\times\frac{1}{\chi^2\Omega_S\chi'^2\Omega_S}\int\frac{{\rm d}k_\parallel}{2\pi}\int\frac{{\rm d}^2\mathbf{q}}{(2\pi)^2}|\tilde{W}(\mathbf{q})|^2P^{\rm lin}_{\rm m}\left(\sqrt{k_\parallel^2+\mathbf{q}^2}\right)e^{ik_\parallel(\chi-\chi')}\\
&\simeq \frac{1}{\bar{N}_{\rm h}(z_L)^2}\int{\rm d}\chi\, f_{\rm h}(\chi)^2\left[\int{\rm d}M\,S(M)\frac{{\rm d}n}{{\rm d}M}(M)b(M)\right]^2\frac{1}{(\chi^2\Omega_S)^2}\int\frac{{\rm d}^2\mathbf{q}}{(2\pi)^2}|\tilde{W}(\mathbf{q})|^2P^{\rm lin}_{\rm m}(q)
\end{aligned} \tag{C21}$$

The covariance matrix describes expected statistical errors in measurements of the stacked lesing profile as well as how the stacked lensing power spectra of different wavenumber bins are correted with each other. The covariance matrix has three contributions: the Gaussian term, the non-Gaussian term arising from sub-volume modes, and the non-Gaussian, super-sample covariance (SSC) term arising from super-survey modes.

$$\begin{aligned}
{\rm Cov}[P_{\Delta\Sigma}(k),P_{\Delta\Sigma}(k')] &= \mathbf{C}^{\rm Gauss}+\mathbf{C}^{\rm sub-NG}+\mathbf{C}^{\rm SSC}\\
&= \frac{\delta^K_{kk'}}{N_{\rm mode}(k)}\left[P_{\Delta\Sigma}(k)^2+P^{\rm obs}_{\rm hh}(k)\Sigma_{\rm cr}(z_L)^2\chi_L^2C^{\rm obs}_\kappa(l=\chi_Lk)\right]+\frac{1}{\chi_L^2\Omega_{\rm s}}\bar{T}^{\rm 2D}_{\rm hhmm}(k,k';z_L)+\sigma_b^2\frac{\partial P_{\Delta\Sigma}(k)}{\partial\delta_b}\frac{\partial P_{\Delta\Sigma}(k')}{\partial\delta_b}
\end{aligned} \tag{C22}$$

where $N_{\rm mode}(k)$ is the number of Fourier modes taken from the survey volume, $P_{\rm hh}(k)$ is the auto-power spectrum of halos, and $C_\kappa(l)$ is the cosmic shear power spectrum.

$$\begin{aligned}
N_{\rm mode}(k) &\equiv \frac{2\pi k\Delta k}{[2\pi/(\chi_L\Theta_s)]^2} = k\Delta k\,\chi_L^2\Omega_{\rm s},\\
P^{\rm obs}_{\rm hh}(k) &= P_{\rm hh}(k)+\frac{1}{\bar{N}_{\rm h}(z_L)}\\
C^{\rm obs}_\kappa(l) &= C_\kappa+\frac{\sigma_\epsilon^2}{\bar{n}_{sg}}.
\end{aligned} \tag{C23}$$

The halo auto-power spectrum term accounts for the sample varinace arising due to a finite number of halos in the survey region. The cosmic shear term accounts for the contribution arising from the mass distribution along the line-of-sight to source redshifts, and can be computed in terms of the nonlinear mass power spectrum:

$$C_\kappa(l) = \int_{\chi\neq[\chi_L-\Delta\chi/2,\chi_L+\Delta/2]}{\rm d}\chi\;W_\kappa(\chi)^2\chi^{-2}P^{\rm NL}_{\rm m}\left(k=\frac{l}{\chi};\chi\right), \tag{C24}$$

where the line-of-sight integration excludes the contribution of lens redshifts, in order to avoid a double counting.

More precisely to estimate the sample variance contribution due to cosmic shear, we can populate exactly the same configurations of lens-source galaxies into ray-tracing simulations, and

$$\begin{aligned}
\frac{\partial\ln P_{\Delta\Sigma}(k)}{\partial\delta_b} &= -\frac{1}{\bar{N}_{\rm h}(z_L)}\frac{\partial\bar{N}_{\rm h}(z_L)}{\partial\delta_b}+\frac{1}{\bar{N}_{\rm h}(z_L)}\int{\rm d}\chi\,\chi^2\frac{\partial}{\partial\delta_b}\left[\int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M,\chi;z_L)P_{\rm hm}(k;M)\right]\\
&= -\bar{b}_1+\frac{1}{\bar{N}_{\rm h}(z_L)}\int{\rm d}\chi\,\chi^2\frac{\partial}{\partial\delta_b}\left[\int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M,\chi;z_L)P_{\rm hm}(k;M)\right]
\end{aligned} \tag{C25}$$

$$\bar{T}^{\rm 2D}_{\rm hh\Delta\Sigma\Delta\Sigma}(k,k') = \frac{\bar{\rho}^2_{\rm m0}}{\bar{N}_{\rm h}(z_L)^2}\int {\rm d}\chi\,\chi^4\int {\rm d}M{\rm d}M'\frac{{\rm d}n}{{\rm d}M}\frac{{\rm d}n}{{\rm d}M'}S(M,\chi;z_L)S(M',\chi;z_L)T_{\rm hhmm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') \tag{C26}$$

$$T = T^{\rm 1h} + \left(T^{\rm 2h(22)}+T^{\rm 2h(13)}\right) + T^{\rm 3h} + T^{\rm 4h} \tag{C27}$$

$$T^{\rm 1h}_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') = \frac{\bar{\rho}^2_{\rm m0}}{\bar{N}_{\rm h}(z_L)^2}\int {\rm d}\chi\,\chi^4\int {\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M,\chi;z_L)^2\left(\frac{M}{\bar{\rho}_{\rm m0}}\right)^2 u_M(k)u_M(k') \tag{C28}$$

$$\begin{aligned}T^{\rm 2h(13)}_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') &= \frac{\bar{\rho}^2_{\rm m0}}{\bar{N}_{\rm h}(z_L)^2}\int {\rm d}\chi\,\chi^4\int {\rm d}M_1{\rm d}M_2\,\frac{{\rm d}n}{{\rm d}M_1}\frac{{\rm d}n}{{\rm d}M_2}\\ &\times\left[P_{\rm hh}(k)\left\{S(M_1)S(M_2)\left(\frac{M_2}{\bar{\rho}_{\rm m0}}\right)^2 u_{M_2}(k)u_{M_2}(k') + \frac{M_1}{\bar{\rho}_{\rm m0}}u_{M_1}(k)S(M_2)^2\frac{M_2}{\bar{\rho}_{\rm m0}}u_{M_2}(k')\right\} + (k\leftrightarrow k')\right]\end{aligned} \tag{C29}$$

$$\begin{aligned}T^{\rm 2h(22)}_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') &= \frac{\bar{\rho}^2_{\rm m0}}{\bar{N}_{\rm h}(z_L)^2}\int {\rm d}\chi\,\chi^4\int {\rm d}M_1{\rm d}M_2\,\frac{{\rm d}n}{{\rm d}M_1}\frac{{\rm d}n}{{\rm d}M_2}\\ &\times\left[P_{\rm hh}(|\mathbf{k}+\mathbf{k}'|)S(M_1)^2\left(\frac{M_2}{\bar{\rho}_{\rm m0}}\right)^2 u_{M_2}(k)u_{M_2}(k') + P_{\rm hh}(|\mathbf{k}-\mathbf{k}'|)S(M_1)S(M_2)\frac{M_1}{\bar{\rho}_{\rm m0}}\frac{M_1}{\bar{\rho}_{\rm m0}}u_{M_1}(k')u_{M_2}(k)\right]\end{aligned} \tag{C30}$$

$$\begin{aligned}T^{\rm 3h}_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') &= \frac{\bar{\rho}^2_{\rm m0}}{\bar{N}_{\rm h}(z_L)^2}\int {\rm d}\chi\,\chi^4\int {\rm d}M_1{\rm d}M_2{\rm d}M_3\,\frac{{\rm d}n}{{\rm d}M_1}\frac{{\rm d}n}{{\rm d}M_2}\frac{{\rm d}n}{{\rm d}M_3}\\ &\times\left[B_{\rm hhh}(\mathbf{k}+\mathbf{k}',-\mathbf{k},-\mathbf{k}')S(M_1)^2\frac{M_2}{\bar{\rho}_{\rm m0}}u_{M_2}(k)\frac{M_3}{\bar{\rho}_{\rm m0}}u_{M_3}(k') + B_{\rm hhh}(\mathbf{k}-\mathbf{k}',-\mathbf{k},\mathbf{k}')S(M_1)\frac{M_1}{\bar{\rho}_{\rm m0}}u_{M_1}(k')\frac{M_2}{\bar{\rho}_{\rm m0}}u_{M_2}(k)S(M_3)\right.\\ &\left.+B_{\rm hhh}(-\mathbf{k}+\mathbf{k}',\mathbf{k},-\mathbf{k}')S(M_1)\frac{M_1}{\bar{\rho}_{\rm m0}}u_{M_1}(k)S(M_2)\frac{M_3}{\bar{\rho}_{\rm m0}}u_{M_3}(k') + B_{\rm hhh}(-\mathbf{k}-\mathbf{k}',\mathbf{k},\mathbf{k}')\left(\frac{M_1}{\bar{\rho}_{\rm m0}}\right)^2 u_{M_1}(k)u_{M_1}(k')S(M_2)S(M_3)\right]\end{aligned} \tag{C31}$$

$$\begin{aligned}T^{\rm 4h}_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') &= \frac{\bar{\rho}^2_{\rm m0}}{\bar{N}_{\rm h}(z_L)^2}\int {\rm d}\chi\,\chi^4\\ &\times\int {\rm d}M_1{\rm d}M_2{\rm d}M_3{\rm d}M_4\,\frac{{\rm d}n}{{\rm d}M_1}\frac{{\rm d}n}{{\rm d}M_2}\frac{{\rm d}n}{{\rm d}M_3}\frac{{\rm d}n}{{\rm d}M_4}S(M_1)S(M_3)T_{\rm hhhh}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')\end{aligned} \tag{C32}$$

$$T^{\rm 1h}_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q}) \simeq T^{\rm 1h}_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') \tag{C33}$$

$$T^{\rm 2h(13)}_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q}) \simeq T^{\rm 2h(22)}_{\rm hmhm}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') + P_{\rm hh}(q)S(M_1)\frac{M_1}{\bar{\rho}_{\rm m0}}u_{M_1}(k)S(M_2)\frac{M_2}{\bar{\rho}_{\rm m0}}u_{M_2}(k') \tag{C34}$$

$$\begin{aligned}T^{\rm 3h}_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q}) &\simeq T^{\rm 3h}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') + B_{\rm hhh}(\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q})S(M_1)\frac{M_1}{\bar{\rho}_{\rm m0}}u_{M_1}(k)S(M_2)\frac{M_3}{\bar{\rho}_{\rm m0}}u_{M_3}(k')\\ &+B_{\rm hhh}(-\mathbf{q},\mathbf{k},-\mathbf{k}+\mathbf{q})S(M_1)\frac{M_1}{\bar{\rho}_{\rm m0}}u_{M_1}(k')S(M_2)\frac{M_3}{\bar{\rho}_{\rm m0}}u_{M_3}(k)\end{aligned} \tag{C35}$$

$$T^{\rm 4h}_{\rm hmhm}(\mathbf{k},-\mathbf{k}+\mathbf{q},\mathbf{k}',-\mathbf{k}'-\mathbf{q}) \simeq T^{4h}_{\rm hhhh}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}') + ... \tag{C36}$$

$$\begin{aligned}B^{\rm PT}_{\rm m}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= 2F_2(\mathbf{k}_1,\mathbf{k}_2)P^{\rm lin}_{\rm m}(k_1)P^{\rm lin}_{\rm m}(k_2) + 2\text{ perm.}\\ T^{\rm PT}_{\rm m}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= 4\left[F_2(\mathbf{k}_{13},-\mathbf{k}_1)F_2(\mathbf{k}_{13},\mathbf{k}_2)P^{\rm lin}_{\rm m}(k_{13})P^{\rm lin}_{\rm m}(k_1)P^{\rm lin}_{\rm m}(k_2) + 11\text{ perm.}\right]\\ &+6\left[F_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)P^{\rm lin}_{\rm m}(k_1)P^{\rm lin}_{\rm m}(k_2)P^{\rm lin}_{\rm m}(k_3) + 3\text{ perm.}\right]\end{aligned} \tag{C37}$$

$$
\begin{aligned}
F_2(\mathbf{k}_1,\mathbf{k}_2) \equiv & \frac{5}{7}+\frac{1}{2}\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)(\mathbf{k}_1\cdot\mathbf{k}_2)+\frac{2}{7}\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2k_2^2},\\
F_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv & \frac{7}{18}\frac{\mathbf{k}_{12}\cdot\mathbf{k}_1}{k_1^2}\left[F_2(\mathbf{k}_2,\mathbf{k}_3)+G_2(\mathbf{k}_1,\mathbf{k}_2)\right]+\frac{1}{18}\frac{k_{12}^2(\mathbf{k}_1\cdot\mathbf{k}_2)}{k_1^2k_2^2}\left[G_2(\mathbf{k}_2,\mathbf{k}_3)+G_2(\mathbf{k}_1,\mathbf{k}_2)\right],\\
G_2(\mathbf{k}_1,\mathbf{k}_2) \equiv & \frac{3}{7}+\frac{1}{2}\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)(\mathbf{k}_1\cdot\mathbf{k}_2)+\frac{4}{7}\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2k_2^2}
\end{aligned}
\tag{C38}
$$

The lensing convergence field, more generally any two-dimensional field projected along the line-of-sight, can be expressed as

$$
\kappa(\boldsymbol{\theta})=\int \mathrm{d}\chi\ W(\chi)\delta_\mathrm{m}(\chi,\chi\boldsymbol{\theta})\simeq\sum_i\Delta\chi\ W(\chi_i)\delta_\mathrm{m}(\chi_i,\chi\boldsymbol{\theta}),
\tag{C39}
$$

where $W(\chi)$ is the radial weight function. To measaure a cross-correlation of the projected field with the distribution of halos at a particular redshfit z_L [9, 23], we can estimate the three-dimensional distribution of the underlying mass density field at the halo redshift:

$$
\hat{\delta}^\kappa_\mathrm{m}(\mathbf{k}_\perp;z_L)\equiv\int_S \mathrm{d}^2\mathbf{x}_\perp W^{-1}(z_L)\kappa(\boldsymbol{\theta})e^{-i\mathbf{k}_\perp\cdot\chi_L\boldsymbol{\theta}}\simeq\delta_\mathrm{m}(\mathbf{k}_\perp,k_\parallel;\chi_L)+\chi_L^{-2}W^{-1}(z_L)\kappa_\mathbf{l}
\tag{C40}
$$

where $\chi_L\equiv\chi(z_L)$ and $\mathbf{x}_\perp$ and $\mathbf{k}_\perp$ are the two-dimensional spatial vector and wavevector in the plane perpendicular to the line-of-sight direction.

$$
\langle\widehat{\Delta\Sigma}\rangle(k)(2\pi)^2\delta_D^2(\mathbf{k}+\mathbf{k}')\equiv\left\langle\hat{\delta}_\mathrm{h}^\mathrm{2D}(\mathbf{k},\chi)\Sigma_\mathrm{cr}(\chi)\right\rangle
\tag{C41}
$$

The two-dimensional Fourier mode in the lens plane at redshift z can be estimted as

$$
\widehat{\Delta\Sigma}(\mathbf{k}_\perp;\chi)\equiv\Sigma_\mathrm{cr}(\chi)\int_{S=\chi\Omega_S}\mathrm{d}^2\mathbf{x}_\perp\ \kappa(\boldsymbol{\theta})e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp},
\tag{C42}
$$

where $\mathbf{x}_\perp$ is the two-dimensional position vector in the plane at z, $\mathbf{x}_\perp=\chi\boldsymbol{\theta}$.

$$
\widehat{\Delta\Sigma}(\mathbf{k}_\perp)=\chi_L^2\kappa(\mathbf{l}=\chi_L\mathbf{k}_\perp)
\tag{C43}
$$

$$
\begin{aligned}
\left\langle\widehat{\Delta\Sigma}(\mathbf{k}_\perp)\widehat{\Delta\Sigma}(\mathbf{k}'_\perp)\right\rangle \ \equiv\ & P_{\Delta\Sigma}(k_\perp)(2\pi)^2\delta_D^2(\mathbf{k}_\perp+\mathbf{k}'_\perp)\\
\leftarrow\ & \chi_L^4C_\kappa(l=\chi_Lk_\perp)(2\pi)^2\delta_D^2(\mathbf{l}+\mathbf{l}')|_{\mathbf{l}=\chi_L\mathbf{k}_\perp}\\
=\ & \chi_L^2C_\kappa(l=\chi_Lk_\perp)(2\pi)^2\delta_D^2(\mathbf{k}_\perp+\mathbf{k}'_\perp)
\end{aligned}
\tag{C44}
$$

Cross-correlating the convergence field on the sky with the distribution of halos in the particular redshift slice around z_L allows us to probe the mass power spectrum at the redshift:

$$
\Delta\Sigma(R;z_L)\equiv\left\langle\delta_h^\mathrm{2D}(\mathbf{x}_\perp;z_L)\Sigma_\mathrm{cr}(z_L)\kappa(\boldsymbol{\theta})\right\rangle_{R=|\mathbf{x}_\perp-\chi_L\boldsymbol{\theta}|},
\tag{C45}
$$

where the average is doen over all the pairs of source galaxy and lensing halo that are separate by the projected distance R to within the bin width, in the plane parpendicualr to the line-of-sight. The stacked lensing profiles can be expressed in terms of teh power spectrum as

$$
\begin{aligned}
\Sigma(R;z_L)&=\int\frac{k\mathrm{d}k}{2\pi}P_{\Delta\Sigma}(k;z_L)J_0(kR),\\
\Delta\Sigma(R;z_L)&=\int\frac{k\mathrm{d}k}{2\pi}P_{\Delta\Sigma}(k;z_L)J_2(kR),
\end{aligned}
\tag{C46}
$$

where $J_0(x)$ and $J_2(x)$ are the zero-th and 2nd order Bessel functions, and the power spectrum $P_{\Delta\Sigma}(k)$ is given as

$$
P_{\Delta\Sigma}(k)=\frac{\bar{\rho}_\mathrm{m0}}{\hat{N}_\mathrm{h}(z_L)}\int\mathrm{d}\chi\,\chi^2\int\mathrm{d}M\ \frac{\mathrm{d}n}{\mathrm{d}M}S(M,z_L)P_\mathrm{hm}(k;M).
\tag{C47}
$$

Hence an estimtor of the matter power spectrum at z_L can be constructed from the cross-correlation function:

$$\hat{P}_{\rm hm}(k;z_L) = \left\langle \delta^*_{\rm h}(\mathbf{k};z_L)\hat{\delta}^{\kappa}_{\rm m}(\mathbf{k})\right\rangle \tag{C48}$$

$$P_{\Delta\Sigma}(k;z_L) = \frac{1}{\frac{{\rm d}n}{{\rm d}M}(z_L)}\bar{\rho}_{\rm m0}\int {\rm d}M\ \frac{{\rm d}n}{{\rm d}M}S(M;z_L)P_{\rm hm}(k;M). \tag{C49}$$

$$\Delta\Sigma(R;z_L) = \int \frac{k{\rm d}k}{2\pi}P_{\Delta\Sigma}(k;z_L)J_2(kR). \tag{C50}$$

$$f(\mathbf{x}_\perp) = \frac{1}{\chi_L^2\Omega_S}\sum_{\mathbf{k}} f(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} \simeq \int\frac{{\rm d}^2\mathbf{k}_\perp}{(2\pi)^2}f(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} \tag{C51}$$

$$\hat{P}_{\rm hm}(k_i) = \frac{1}{\chi_L^2\Omega_S N_{\rm mode}(k)}\sum_{\mathbf{k};k\in k_i}\hat{\delta}_{\rm h}(\mathbf{k})\widehat{\Delta\Sigma}(-\mathbf{k}) \tag{C52}$$

$$\begin{aligned}
{\rm Cov}[P_{\rm hm}(k_i),P_{\rm hm}(k_j)]^{\rm NG} &= \frac{1}{\chi_L^4\Omega_S^2 N_{\rm mode}(k)N_{\rm mode}(k')}\sum_{\mathbf{k};k\in k_i}\sum_{\mathbf{k}';k'\in k_j}\left\langle\hat{\delta}_{\rm h}(\mathbf{k})\widehat{\Delta\Sigma}(-\mathbf{k})\hat{\delta}_{\rm h}(\mathbf{k}')\widehat{\Delta\Sigma}(-\mathbf{k}')\right\rangle_{\rm c}\\
&= \frac{1}{\chi_L^4\Omega_S^2 N_{\rm mode}(k)N_{\rm mode}(k')}\sum_{\mathbf{k};k\in k_i}\sum_{\mathbf{k}';k'\in k_j}(\chi_L^2\Omega_S)\delta^K_{\mathbf{k}-\mathbf{k}+\mathbf{k}'-\mathbf{k}'}T_{\rm hh\Delta\Sigma\Delta\Sigma}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')\\
&= \frac{1}{\chi_L^2\Omega_S N_{\rm mode}(k)N_{\rm mode}(k')}\sum_{\mathbf{k};k\in k_i}\sum_{\mathbf{k}';k'\in k_j}T_{\rm hh\Delta\Sigma\Delta\Sigma}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')\\
&= \frac{\chi_L^2\Omega_S}{N_{\rm mode}(k)N_{\rm mode}(k')}\int_{k\in k_i}\frac{{\rm d}^2\mathbf{k}}{(2\pi)^2}\int_{k'\in k_j}\frac{{\rm d}^2\mathbf{k}'}{(2\pi)^2}T_{\rm hh\Delta\Sigma\Delta\Sigma}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')\\
&= \frac{1}{\chi_L^2\Omega_S}\int_{k\in k_i}\frac{{\rm d}^2\mathbf{k}}{2\pi k_i\Delta k}\int_{k'\in k_j}\frac{{\rm d}^2\mathbf{k}'}{2\pi k_j\Delta k}T_{\rm hh\Delta\Sigma\Delta\Sigma}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')\\
&= \frac{1}{4\pi f_{\rm sky}\chi_L^2}\int_{k\in k_i}\frac{{\rm d}^2\mathbf{k}}{2\pi k_i\Delta k}\int_{k'\in k_j}\frac{{\rm d}^2\mathbf{k}'}{2\pi k_j\Delta k}T_{\rm hh\Delta\Sigma\Delta\Sigma}(\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}')
\end{aligned}\tag{C53}$$

$$\begin{aligned}
\left\langle\hat{\delta}^{\rm 2D}_{\rm h}(\mathbf{k}_1)\hat{\delta}^{\rm 2D}_{\rm h}(\mathbf{k}_2)\widehat{\Delta\Sigma}(\mathbf{k}_3)\widehat{\Delta\Sigma}(\mathbf{k}_4)\right\rangle &= \frac{\Sigma_{\rm cr}(z_L)^2}{\bar{N}_{\rm h}(z_L)^2}\int {\rm d}\chi_1{\rm d}\chi_2{\rm d}\chi_3{\rm d}\chi_4\chi_1^2\chi_2^2W_\kappa(\chi_3)W_\kappa(\chi_4)\\
&\quad\times\int{\rm d}M_1{\rm d}M_2\frac{{\rm d}n}{{\rm d}M_1}\frac{{\rm d}n}{{\rm d}M_2}S(M_1,\chi_1;z_L)S(M_2,\chi_2;z_L)\\
&\quad\times\int\frac{{\rm d}k_{1\parallel}}{2\pi}\cdots\frac{{\rm d}k_{4\parallel}}{2\pi}\left\langle\delta_{\rm h}(\mathbf{k}_1)\delta_{\rm h}(\mathbf{k}_2)\delta_{\rm m}(\mathbf{k}_3)\delta_{\rm m}(\mathbf{k}_4)\right\rangle e^{ik_{1\parallel}\chi_1+\cdots+ik_{4\parallel}\chi_4}\\
= (2\pi)^2\delta_D^2(\mathbf{k}_{1\perp}+\cdot+\mathbf{k}_{4\perp})\frac{\Sigma_{\rm cr}(z_L)^2}{\bar{N}_{\rm h}(z_L)^2}&\int{\rm d}\chi_1{\rm d}\chi_2{\rm d}\chi_3{\rm d}\chi_4\chi_1^2\chi_2^2W_\kappa(\chi_3)W_\kappa(\chi_4)\\
&\quad\times\int{\rm d}M_1{\rm d}M_2\frac{{\rm d}n}{{\rm d}M_1}\frac{{\rm d}n}{{\rm d}M_2}S(M_1,\chi_1;z_L)S(M_2,\chi_2;z_L)\\
&\quad\times\int\frac{{\rm d}k_{1\parallel}}{2\pi}\cdots\frac{{\rm d}k_{3\parallel}}{2\pi}T_{\rm hhmm}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)e^{ik_{1\parallel}(\chi_1-\chi_4)+i\mathbf{k}_{2\parallel}(\chi_2-\chi_4)+i\mathbf{k}_{3\parallel}(\chi_3-\chi_4)}\\
\simeq (2\pi)^2\delta_D^2(\mathbf{k}_{1\perp}+\cdot+\mathbf{k}_{4\perp})\frac{\Sigma_{\rm cr}(z_L)^2}{\bar{N}_{\rm h}(z_L)^2}&\int{\rm d}\chi\chi^4W_\kappa(\chi)^2\int{\rm d}M_1{\rm d}M_2\frac{{\rm d}n}{{\rm d}M_1}\frac{{\rm d}n}{{\rm d}M_2}S(M_1,\chi_1;z_L)S(M_2,\chi_2;z_L)T_{\rm hhmm}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4;M_1,M_2,\chi_L)\\
\simeq (2\pi)^2\delta_D^2(\mathbf{k}_{1\perp}+\cdot+\mathbf{k}_{4\perp})\frac{\bar{\rho}^2_{\rm m0}}{\bar{N}_{\rm h}(z_L)^2}&\chi_L^4\Delta\chi\int{\rm d}M_1{\rm d}M_2\frac{{\rm d}n}{{\rm d}M_1}\frac{{\rm d}n}{{\rm d}M_2}S(M_1,\chi_1;z_L)S(M_2,\chi_2;z_L)T_{\rm hhmm}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4;\chi_L)\\
= (2\pi)^2\delta_D^2(\mathbf{k}_{1\perp}+\cdot+\mathbf{k}_{4\perp})\frac{\bar{\rho}^2_{\rm m0}}{\frac{{\rm d}n}{{\rm d}M}(z_L)^2\Delta\chi}&\int{\rm d}M_1{\rm d}M_2\frac{{\rm d}n}{{\rm d}M}\frac{{\rm d}n}{{\rm d}M}S(M,z_L)S(M',z_L)T_{\rm hhmm}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4;M,M',\chi_L)
\end{aligned}\tag{C54}$$

$$
\begin{aligned}
\delta_{\rm h}(\mathbf{x}_1) &= \frac{1}{\bar{n}_{\rm h}} n_{\rm h}(\mathbf{x}_1) - 1 = \frac{1}{\frac{{\rm d}n}{{\rm d}M}} \sum_i \delta_D^3(\mathbf{x}_i - \mathbf{x}_1) - 1 \\
\delta_{\rm m}(\mathbf{x}_1) &= \frac{1}{\bar{\rho}_{\rm m0}} \sum_i m_i u_{m_i}(\mathbf{x}_1 - \mathbf{x}_i) - 1
\end{aligned} \tag{C55}
$$

$$
\xi_{4,{\rm hhmm}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \langle \delta_{\rm h}(\mathbf{x}_1)\delta_{\rm h}(\mathbf{x}_2)\delta_{\rm m}(\mathbf{x}_3)\delta_{\rm m}(\mathbf{x}_4)\rangle \tag{C56}
$$

$$
\begin{aligned}
\frac{{\rm d}n}{{\rm d}M}^2 \rho_{\rm m0}^2 \xi^{\rm 1h}_{4,{\rm hhmm}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \left\langle \sum_i \delta_D^3(\mathbf{x}_1 - \mathbf{x}_i)\delta_D^3(\mathbf{x}_2 - \mathbf{x}_i) M_i u_{M_i}(\mathbf{x}_3 - \mathbf{x}_i) M_i u_{M_i}(\mathbf{x}_4 - \mathbf{x}_i) \right\rangle \\
&= \int {\rm d}M \frac{{\rm d}n}{{\rm d}M} M^2 \int d\mathbf{x} \delta_D^3(\mathbf{x}_1 - \mathbf{x})\delta_D^3(\mathbf{x}_2 - \mathbf{x}) u_M(\mathbf{x}_3 - \mathbf{x}) u_M(\mathbf{x}_4 - \mathbf{x}) \\
&= \int {\rm d}M \frac{{\rm d}n}{{\rm d}M} M^2 \int d\mathbf{x} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d^3\mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1 + \cdots + \mathbf{k}_4\cdot\mathbf{x}_4)} e^{-i\mathbf{x}\cdot(\mathbf{k}_1 + \cdot + \mathbf{k}_4)} u_M(k_3) u_M(k_4) \\
&= \int {\rm d}M \frac{{\rm d}n}{{\rm d}M} M^2 \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d^3\mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1 + \cdots + \mathbf{k}_4\cdot\mathbf{x}_4)} u_M(k_3) u_M(k_4) (2\pi)^3 \delta_D^3(\mathbf{k}_1 + \cdots + \mathbf{k}_4) \\
&\to T^{\rm 1h}_{\rm hhmm}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \frac{1}{\frac{{\rm d}n}{{\rm d}M}^2} \int {\rm d}M \frac{{\rm d}n}{{\rm d}M} \left(\frac{M}{\bar{\rho}_{\rm m0}}\right)^2 u_M(k_3) u_M(k_4)
\end{aligned} \tag{C57}
$$

$$
\begin{aligned}
\frac{{\rm d}n}{{\rm d}M}^2 \rho_{\rm m0}^2 \xi^{\rm 2h-(13)}_{4,{\rm hhmm}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \left\langle \sum_{i,j;i\neq j} \delta_D^3(\mathbf{x}_1 - \mathbf{x}_i)\delta_D^3(\mathbf{x}_2 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_3 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_4 - \mathbf{x}_j) \right\rangle \\
&\quad + \left\langle \sum_{i,j;i\neq j} M_i u_{M_i}(\mathbf{x}_3 - \mathbf{x}_i) \delta_D^3(\mathbf{x}_1 - \mathbf{x}_j)\delta_D^3(\mathbf{x}_2 - \mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_4 - \mathbf{x}_j) \right\rangle + (\text{perm.}) \\
= \int d^3\mathbf{y}_1 d^3\mathbf{y}_2 \int {\rm d}M_1 {\rm d}M_2 \frac{{\rm d}n}{{\rm d}M_1} \frac{{\rm d}n}{{\rm d}M_2} \xi_{\rm hh}(\mathbf{y}_1 - \mathbf{y}_2; M_1, M_2) &\left[\delta_D^3(\mathbf{x}_1 - \mathbf{y}_1)\delta_D^3(\mathbf{x}_2 - \mathbf{y}_2) M_2^2 u_{M_2}(\mathbf{x}_3 - \mathbf{y}_2) u_{M_2}(\mathbf{x}_4 - \mathbf{y}_2)\right. \\
&\quad \left. + M_1 u_{M_1}(\mathbf{x}_3 - \mathbf{y}_1)\delta_D^3(\mathbf{x}_1 - \mathbf{y}_2)\delta_D^3(\mathbf{x}_2 - \mathbf{y}_2) M_2 u_{M_2}(\mathbf{x}_4 - \mathbf{y}_2) + (\text{perm.})\right] \\
= \int d^3\mathbf{y}_1 d^3\mathbf{y}_2 \int {\rm d}M_1 {\rm d}M_2 \frac{{\rm d}n}{{\rm d}M_1} \frac{{\rm d}n}{{\rm d}M_2} &\int \frac{d^3\mathbf{q}}{(2\pi)^3} P_{hh}(q; M_1, M_2) e^{i\mathbf{q}\cdot(\mathbf{y}_1 - \mathbf{y}_2)} \\
\times \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d^3\mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1 + \cdots + \mathbf{k}_4\cdot\mathbf{x}_4)} &\left[e^{-i\mathbf{k}_1\cdot\mathbf{y}_1} e^{-i\mathbf{y}_2\cdot(\mathbf{k}_2 + \mathbf{x}_3 + \mathbf{k}_4)} M_2^2 u_{M_2}(k_3) u_{M_2}(k_4)\right. \\
&\quad \left. + e^{-i\mathbf{y}_1\cdot\mathbf{k}_3} e^{-i\mathbf{y}_2\cdot(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_4)} M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + (\text{perm.})\right] \\
= \int {\rm d}M_1 {\rm d}M_2 \frac{{\rm d}n}{{\rm d}M_1} \frac{{\rm d}n}{{\rm d}M_2} &\int \frac{d^3\mathbf{q}}{(2\pi)^3} P_{hh}(q; M_1, M_2) \\
\times \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d^3\mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1 + \cdots + \mathbf{k}_4\cdot\mathbf{x}_4)} &\left[(2\pi)^3\delta_D^3(\mathbf{q} - \mathbf{k}_1)(2\pi)^3\delta_D^3(\mathbf{q} + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4)\right. \\
&\quad \left. + (2\pi)^3\delta_D^3(\mathbf{q} - \mathbf{k}_3)(2\pi)^3\delta_D^3(\mathbf{q} + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_4) M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + (\text{perm.})\right] \\
= \int {\rm d}M_1 {\rm d}M_2 \frac{{\rm d}n}{{\rm d}M_1} \frac{{\rm d}n}{{\rm d}M_2} &\int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d^3\mathbf{k}_4}{(2\pi)^3} (2\pi)^3 \delta_D^3(\mathbf{k}_{1234}) e^{i(\mathbf{k}_1\cdot\mathbf{x}_1 + \cdots + \mathbf{k}_4\cdot\mathbf{x}_4)} \\
&\times \left[P_{\rm hh}(k_1) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) + P_{\rm hh}(k_3) M_1 M_2 u_{M_1}(k_3) u_{M_2}(k_4) + (\text{perm.})\right] \\
\to T^{\rm 2h(13)}_{4,{\rm hhmm}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \int {\rm d}M_1 {\rm d}M_2 \frac{{\rm d}n}{{\rm d}M_1} \frac{{\rm d}n}{{\rm d}M_2} &\left[\{P_{\rm hh}(k_1) + P_{\rm hh}(k_2)\} M_2^2 u_{M_2}(k_3) u_{M_2}(k_4)\right. \\
&\quad \left. + P_{\rm hh}(k_3) M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + P_{\rm hh}(k_4) M_1 u_{M_1}(k_4) M_2 u_{M_2}(k_4)\right]
\end{aligned} \tag{C58}
$$

$$
\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}M}^2 \rho_{\mathrm{m}0}^2 \xi_{4,\mathrm{hhmm}}^{2\mathrm{h}-(22)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) &= \left\langle \sum_{i,j;i\neq j} \delta_D^3(\mathbf{x}_1-\mathbf{x}_i)\delta_D^3(\mathbf{x}_2-\mathbf{x}_i) M_j u_{M_j}(\mathbf{x}_3-\mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_4-\mathbf{x}_j)\right\rangle \\
&\quad + \left\langle \sum_{i,j;i\neq j} \delta_D^3(\mathbf{x}_1-\mathbf{x}_i)\delta_D^3(\mathbf{x}_2-\mathbf{x}_i) M_i u_{M_j}(\mathbf{x}_3-\mathbf{x}_j) M_j u_{M_j}(\mathbf{x}_4-\mathbf{x}_j)\right\rangle + (\text{perm.}) \\
&= \int d^3\mathbf{y}_1 d^3\mathbf{y}_2 \int \mathrm{d}M_1 \mathrm{d}M_2 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2} \xi_{\mathrm{hh}}(\mathbf{y}_1-\mathbf{y}_2; M_1, M_2)\Big[\delta_D^3(\mathbf{x}_1-\mathbf{y}_1)\delta_D^3(\mathbf{x}_2-\mathbf{y}_1) M_2^2 u_{M_2}(\mathbf{x}_3-\mathbf{y}_2) u_{M_2}(\mathbf{x}_4-\mathbf{y}_2) \\
&\qquad + \delta_D^3(\mathbf{x}_1-\mathbf{y}_1) M_1 u_{M_1}(\mathbf{x}_3-\mathbf{y}_1)\delta_D^3(\mathbf{x}_2-\mathbf{y}_2) M_2 u_{M_2}(\mathbf{x}_4-\mathbf{y}_2) + (\text{perm.})\Big] \\
&= \int d^3\mathbf{y}_1 d^3\mathbf{y}_2 \int \mathrm{d}M_1 \mathrm{d}M_2 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2} \int \frac{d^3\mathbf{q}}{(2\pi)^3} P_{hh}(q; M_1, M_2) e^{i\mathbf{q}\cdot(\mathbf{y}_1-\mathbf{y}_2)} \\
&\qquad \times \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\cdots\frac{d^3\mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+\mathbf{k}_4\cdot\mathbf{x}_4)}\Big[e^{-i\mathbf{y}_1\cdot(\mathbf{k}_1+\mathbf{k}_2)} e^{-i\mathbf{y}_2\cdot(\mathbf{k}_3+\mathbf{k}_4)} M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \\
&\qquad + e^{-i\mathbf{y}_1\cdot(\mathbf{k}_1+\mathbf{k}_3)} e^{-i\mathbf{y}_2\cdot(\mathbf{k}_2+\mathbf{k}_4)} M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + (\text{perm.})\Big] \\
&= \int \mathrm{d}M_1 \mathrm{d}M_2 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2} \int \frac{d^3\mathbf{q}}{(2\pi)^3} P_{hh}(q; M_1, M_2) \\
&\qquad \times \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\cdots\frac{d^3\mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+\mathbf{k}_4\cdot\mathbf{x}_4)}\Big[(2\pi)^3\delta_D^3(\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2)(2\pi)^3\delta_D^3(\mathbf{q}+\mathbf{k}_3+\mathbf{k}_4) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \\
&\qquad + (2\pi)^3\delta_D^3(\mathbf{q}-\mathbf{k}_1-\mathbf{k}_3)(2\pi)^3\delta_D^3(\mathbf{q}+\mathbf{k}_2+\mathbf{k}_4) M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4) + (\text{perm.})\Big] \\
&= \int \mathrm{d}M_1 \mathrm{d}M_2 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\cdots\frac{d^3\mathbf{k}_4}{(2\pi)^3} (2\pi)^3\delta_D^3(\mathbf{k}_{1234}) e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+\mathbf{k}_4\cdot\mathbf{x}_4)} \\
&\qquad \times \Big[P_{\mathrm{hh}}(k_{12}) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) + P_{\mathrm{hh}}(k_{13}) M_1 M_2 u_{M_1}(k_3) u_{M_2}(k_4) + (\text{perm.})\Big] \\
\to T_{4,\mathrm{hhmm}}^{2\mathrm{h}(22)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= \int \mathrm{d}M_1 \mathrm{d}M_2 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2}\Big[P_{\mathrm{hh}}(k_{12}) M_2^2 u_{M_2}(k_3) u_{M_2}(k_4) \\
&\quad + \{P_{\mathrm{hh}}(k_{13}) + P_{\mathrm{hh}}(k_{14})\} M_1 u_{M_1}(k_3) M_2 u_{M_2}(k_4)\Big] \qquad \text{(C59)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}M}^2 \rho_{\mathrm{m0}}^2 \xi^{\mathrm{3h}}_{4,\mathrm{hhmm}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) &= \left\langle \sum_{i,j,k} \delta_D^3(\mathbf{x}_1-\mathbf{x}_i)\delta_D^3(\mathbf{x}_2-\mathbf{x}_i) M_j u_{M_j}(\mathbf{x}_3-\mathbf{x}_j) M_k u_{M_k}(\mathbf{x}_4-\mathbf{x}_k)\right\rangle \\
&\quad + \left\langle \sum_{i,j,k} \delta_D^3(\mathbf{x}_1-\mathbf{x}_i) M_i u_{M_i}(\mathbf{x}_3-\mathbf{x}_i)\delta_D^3(\mathbf{x}_2-\mathbf{x}_j) M_k u_{M_k}(\mathbf{x}_4-\mathbf{x}_k)\right\rangle + (\mathrm{perm.}) \\
&= \int d^3\mathbf{y}_1 d^3\mathbf{y}_2 d^3\mathbf{y}_3 \int \mathrm{d}M_1\mathrm{d}M_2\mathrm{d}M_3 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2}\frac{\mathrm{d}n}{\mathrm{d}M_3}\int \frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3} B_{\mathrm{hhh}}(\mathbf{q}_1,\mathbf{q}_2,-\mathbf{q}_{12}) e^{i\mathbf{q}_1\cdot(\mathbf{y}_1-\mathbf{y}_3)+i\mathbf{q}_2\cdot(\mathbf{y}_2-\mathbf{y}_3)} \\
&\quad \times \Big[\delta_D^3(\mathbf{x}_1-\mathbf{y}_1)\delta_D^3(\mathbf{x}_2-\mathbf{y}_1) M_2 u_{M_2}(\mathbf{x}_3-\mathbf{y}_2) M_3 u_{M_3}(\mathbf{x}_4-\mathbf{y}_3) \\
&\qquad + \delta_D^3(\mathbf{x}_1-\mathbf{y}_1) M_1 u_{M_1}(\mathbf{x}_3-\mathbf{y}_1)\delta_D^3(\mathbf{x}_2-\mathbf{y}_2) M_2 u_{M_2}(\mathbf{x}_4-\mathbf{y}_3) \\
&\qquad + M_1^2 u_{M_1}(\mathbf{x}_3-\mathbf{y}_1) u_{M_1}(\mathbf{x}_4-\mathbf{y}_1)\delta_D^3(\mathbf{x}_1-\mathbf{y}_2)\delta_D^3(\mathbf{x}_1-\mathbf{y}_3)\Big] \\
&= \int d^3\mathbf{y}_1 d^3\mathbf{y}_2 d^3\mathbf{y}_3 \int \mathrm{d}M_1\mathrm{d}M_2\mathrm{d}M_3 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2}\frac{\mathrm{d}n}{\mathrm{d}M_3}\int \frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3} B_{\mathrm{hhh}}(\mathbf{q}_1,\mathbf{q}_2,-\mathbf{q}_{12}) e^{i\mathbf{q}_1\cdot(\mathbf{y}_1-\mathbf{y}_3)+i\mathbf{q}_2\cdot(\mathbf{y}_2-\mathbf{y}_3)} \\
&\quad \times \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\cdots\frac{d^3\mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+\mathbf{k}_4\cdot\mathbf{x}_4)}\Big[e^{-i\mathbf{y}_1\cdot(\mathbf{k}_1+\mathbf{k}_2)} e^{-i\mathbf{y}_2\cdot\mathbf{k}_3} e^{-i\mathbf{y}_3\cdot\mathbf{k}_4} M_2 u_{M_2}(k_3) M_3 u_{M_3}(k_4) \\
&\qquad + e^{-i\mathbf{y}_1\cdot(\mathbf{k}_1+\mathbf{k}_3)} e^{-i\mathbf{y}_2\cdot\mathbf{k}_2} e^{-i\mathbf{y}_3\cdot\mathbf{k}_4} M_1 u_{M_1}(k_3) M_3 u_{M_3}(k_4) \\
&\qquad + e^{-i\mathbf{y}_1\cdot(\mathbf{k}_3+\mathbf{k}_4)} e^{-i\mathbf{y}_2\cdot\mathbf{k}_1} e^{-i\mathbf{y}_3\cdot\mathbf{k}_2} M_1^2 u_{M_1}(k_3) u_{M_1}(k_4)\Big] \\
&= \int \mathrm{d}M_1\mathrm{d}M_2\mathrm{d}M_3 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2}\frac{\mathrm{d}n}{\mathrm{d}M_3}\int \frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3} B_{\mathrm{hhh}}(\mathbf{q}_1,\mathbf{q}_2,-\mathbf{q}_{12}) \\
&\quad \times \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\cdots\frac{d^3\mathbf{k}_4}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+\mathbf{k}_4\cdot\mathbf{x}_4)}\Big[(2\pi)^9\delta_D^3(\mathbf{q}_1-\mathbf{k}_1-\mathbf{k}_2)\delta_D^3(\mathbf{q}_2-\mathbf{k}_3)\delta_D^3(\mathbf{q}_1+\mathbf{q}_2+\mathbf{k}_4) M_2 u_{M_2}(k_3) M_3 u_{M_3}(k_4) \\
&\qquad + (2\pi)^9\delta_D^3(\mathbf{q}_1-\mathbf{k}_1-\mathbf{k}_3)\delta_D^3(\mathbf{q}_2-\mathbf{k}_2)\delta_D^3(\mathbf{q}_1+\mathbf{q}_2+\mathbf{k}_4) M_1 u_{M_1}(k_3) M_3 u_{M_3}(k_4) \\
&\qquad + (2\pi)^9\delta_D^3(\mathbf{q}_1-\mathbf{k}_3-\mathbf{k}_4)\delta_D^3(\mathbf{q}_2-\mathbf{k}_1)\delta_D^3(\mathbf{q}_1+\mathbf{q}_2+\mathbf{k}_2) M_1^2 u_{M_1}(k_3) u_{M_1}(k_4)\Big] \\
&= \int \mathrm{d}M_1\mathrm{d}M_2\mathrm{d}M_3 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2}\frac{\mathrm{d}n}{\mathrm{d}M_3}\int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\cdots\frac{d^3\mathbf{k}_4}{(2\pi)^3}(2\pi)^3\delta_D^3(\mathbf{k}_{1234}) e^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\cdots+\mathbf{k}_4\cdot\mathbf{x}_4)} \\
&\quad \times \big[B_{\mathrm{hhh}}(\mathbf{k}_{12},\mathbf{k}_3,\mathbf{k}_4) M_2 u_{M_2}(k_3) M_3 u_{M_3}(k_4) + B_{\mathrm{hhh}}(\mathbf{k}_{13},\mathbf{k}_2,\mathbf{k}_4) M_1 u_{M_1}(k_3) M_3 u_{M_3}(k_4) \\
&\qquad + B_{\mathrm{hhh}}(\mathbf{k}_{34},\mathbf{k}_1,\mathbf{k}_2) M_1^2 u_{M_1}(k_3) u_{M_1}(k_4)\Big] \\
\rightarrow T^{3h}_{\mathrm{hhh}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= \int \mathrm{d}M_1\mathrm{d}M_2\mathrm{d}M_3 \frac{\mathrm{d}n}{\mathrm{d}M_1}\frac{\mathrm{d}n}{\mathrm{d}M_2}\frac{\mathrm{d}n}{\mathrm{d}M_3}\big[B_{\mathrm{hhh}}(\mathbf{k}_{12},\mathbf{k}_3,\mathbf{k}_4) M_2 u_{M_2}(k_3) M_3 u_{M_3}(k_4) \\
&\qquad + B_{\mathrm{hhh}}(\mathbf{k}_{13},\mathbf{k}_2,\mathbf{k}_4) M_1 u_{M_1}(k_3) M_3 u_{M_3}(k_4) + B_{\mathrm{hhh}}(\mathbf{k}_{34},\mathbf{k}_1,\mathbf{k}_2) M_1^2 u_{M_1}(k_3) u_{M_1}(k_4) + (\mathrm{perm.})\Big]
\end{aligned}
\tag{C60}
$$

$$
\Omega_S = \int\left[\prod_i^n \frac{d\mathbf{q}_i}{(2\pi)^2}\right]\tilde{W}(\mathbf{q}_1) t W(\mathbf{q}_2)\cdots\tilde{W}(\mathbf{q}_n) = \int\left[\prod_i^{(n-1)} \frac{d\mathbf{q}_i}{(2\pi)^2}\right]\tilde{W}(\mathbf{q}_1)\tilde{W}(\mathbf{q}_2)\cdots\tilde{W}(-\mathbf{q}_1-\mathbf{q}_2-\cdots-\mathbf{q}_{n-1})
\tag{C61}
$$

$$
\begin{aligned}
\hat{N}_{\mathrm{h}}(z_L) &\equiv \int \mathrm{d}^2\boldsymbol{\theta}\; W(\boldsymbol{\theta})\int \mathrm{d}\chi\;\frac{\mathrm{d}^2V}{\mathrm{d}\chi d\Omega}\int \mathrm{d}M\; S(M) f_{\mathrm{h}}(\chi;z_L) n_{\mathrm{h}}(\mathbf{x};M) \\
&= \int \mathrm{d}^2\boldsymbol{\theta}\; W(\boldsymbol{\theta})\int \mathrm{d}\chi\,\chi^2 f_{\mathrm{h}}(\chi;z_L)\int \mathrm{d}M\; S(M)\frac{\mathrm{d}n}{\mathrm{d}M}(M;\chi)\left[1+b(M)\delta_{\mathrm{m}}(\chi,\chi\boldsymbol{\theta})\right] \\
&= \Omega_S\int \mathrm{d}\chi\,\chi^2 f_{\mathrm{h}}(\chi;z_L)\int \mathrm{d}M\; S(M)\frac{\mathrm{d}n}{\mathrm{d}M}(M;\chi)\left[1+b(M)\delta_{\mathrm{b}}(\chi)\right],
\end{aligned}
\tag{C62}
$$

where quantities with hat notation ˆ , here and hereafter, denote their estimator, the quantity with bar notation denotes the ensemble average expectation, and the background mode is defined as

$$
\delta_{\mathrm{b}}(\chi) \equiv \frac{1}{\Omega_S}\int \mathrm{d}^2\boldsymbol{\theta}\; W(\boldsymbol{\theta})\delta_{\mathrm{m}}(\chi,\chi\boldsymbol{\theta}) = \frac{1}{A_S(\chi)}\int \mathrm{d}^2\mathbf{x}_\perp\; W(\mathbf{x}_\perp)\delta_{\mathrm{m}}(\chi,\mathbf{x}_\perp),
\tag{C63}
$$

$\mathbf{x}_\perp \equiv \chi\boldsymbol{\theta}$, and $A_S(\chi) \equiv \chi^2\Omega_S$ is the effective survey area of the plane at distance χ, in units of [Mpc^2]. Eq. (5) shows that the background mode at each redshift is from the average of Fourier modes perpendicular to the line-of-sight direction. If ignoring the effect of radial Fourier mode on clustering observables, whcih is a good approximation for lensing observables, the background mode can be realized to cause a shift in the mean mass density in each redshift plane. Note that, throughout this paper, we employ a flat-geometry universe, where the radial and angular-diameter distances are the same, and we emloy the flat sky approximation. The esemble average expectation can be compuated as

$$\begin{aligned}\bar{N}_{\rm h}(z_L) \equiv \left\langle \hat{N}_{\rm h}(z_L)\right\rangle &= \int {\rm d}^2\boldsymbol{\theta}\ W(\boldsymbol{\theta}) \int {\rm d}\chi\, \chi^2 f_{\rm h}(\chi; z_L) \int {\rm d}M\, S(M) \frac{{\rm d}n}{{\rm d}M}(M;\chi)\left[1 + b(M)\left\langle\delta_{\rm m}(\chi, \chi\boldsymbol{\theta})\right\rangle\right] \\ &= \Omega_S \int {\rm d}\chi\, \chi^2 f_{\rm h}(\chi; z_L) \int {\rm d}M\, S(M) \frac{{\rm d}n}{{\rm d}M}(M;\chi), \end{aligned} \tag{C64}$$

where we have uesd the fact $\langle\delta_{\rm b}\rangle = 0$.

Similarly, taking into account the survey window, we define an estimator of the number desnity fluctuation field of halos of mass M and at redschift χ as

$$\delta n_{\rm h}^W(\mathbf{x}_\perp; M, \chi) = \frac{{\rm d}n}{{\rm d}M}(M;\chi) b(M) W(\boldsymbol{\theta}) \delta_{\rm m}(\chi, \chi\boldsymbol{\theta}). \tag{C65}$$

Since we later consider the projected field, we focus on the field in the two-diemnsinal plane perpendicular to the line-of-sight direction; hence, a mapping between the angular position and the comoving, perpendicular position vector for each halo is given as $\mathbf{x}_\perp = \chi\boldsymbol{\theta}$. This is feasible as redshift of each is assumed to be known. Integrating the above field in the light cone, weighted by the selection function yields an estimator of the projected number density fluctuation field:

$$\Sigma_{\rm h}^W(\boldsymbol{\theta}) \equiv \int {\rm d}\chi\, \chi^2 f_{\rm h}(\chi; z_L) \int {\rm d}M\, S(M) \delta n_{\rm h}^W(M;\chi). \tag{C66}$$

The Fourier transform of $\delta n_{\rm h}$ is

$$\widetilde{\delta n}_{\rm h}^W(\mathbf{k}_\perp, \chi) \equiv \int {\rm d}^2\mathbf{x}_\perp\ \delta n_{\rm h}^W(\mathbf{x}_\perp) e^{-i\mathbf{x}_\perp\cdot\mathbf{k}_\perp} = \frac{{\rm d}n}{{\rm d}M}(M;\chi) b(M) \int \frac{{\rm d}k_\parallel}{2\pi} \tilde{\delta}_{\rm m}^W(\mathbf{k}_\perp, k_\parallel; \chi) e^{i\chi k_\parallel}, \tag{C67}$$

where

$$\tilde{\delta}_{\rm m}^W(\mathbf{k}, k_\parallel; \chi) \equiv \int \frac{{\rm d}^2\mathbf{q}}{(2\pi)^2} \tilde{W}(\mathbf{l} = \chi\mathbf{q}) \tilde{\delta}_{\rm m}(k_\parallel, \mathbf{k}_\perp - \mathbf{q}), \tag{C68}$$

and $\tilde{W}(\mathbf{q};\chi) \equiv \int {\rm d}^2\mathbf{x}_\perp\ W(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}_\perp}$ with $\mathbf{x}_\perp = \chi\boldsymbol{\theta}$.

$$\hat{P}_{\rm hh}(k_{\perp,i}) \equiv \frac{1}{\hat{N}_{\rm h}(z_L)^2} \int {\rm d}\chi \int {\rm d}\chi' f_{\rm h}(\chi) f_{\rm h}(\chi') \frac{1}{\chi^2\Omega_S} \int \frac{{\rm d}^2\mathbf{k}_\perp}{A(k_{\perp i})} \widetilde{\delta n}_{\rm h}^W(\mathbf{k}_\perp, \chi) \widetilde{\delta n}_{\rm h}^W(-\mathbf{k}_\perp, \chi') \tag{C69}$$

Let us define an estimator of the projected power spectrum of halos in the light cone as

$$\hat{P}_{\rm hh}(k_{\perp,i}) = \frac{1}{\Omega_S \hat{N}_{\rm h}(z_L)^2} \int {\rm d}\chi \int {\rm d}\chi' \chi^2 \chi'^2 f_{\rm h}(\chi; z_L) f_{\rm h}(\chi'; z_L) \int {\rm d}M \int {\rm d}M'\ S(M) S(M') \int_{k_{\perp i}} \frac{{\rm d}^2\mathbf{k}_\perp}{V(k_{\perp,i})} \widetilde{\delta n}_{\rm h}^W(\mathbf{k}_\perp, \chi) \widetilde{\delta n}_{\rm h}^W(-\mathbf{k}_\perp, \chi) \tag{C70}$$

$$\begin{aligned}\left\langle \hat{P}_{\rm hh}(k_{\perp,i})\right\rangle &= \frac{1}{\Omega_S \hat{N}_{\rm h}(z_L)^2} \int {\rm d}\chi \int {\rm d}\chi' \chi^2 \chi'^2 f_{\rm h}(\chi; z_L) f_{\rm h}(\chi'; z_L) \int {\rm d}M \int {\rm d}M'\ S(M) S(M') \chi^2 \chi'^2 b(M) \frac{{\rm d}n}{{\rm d}M}(M) b(M') \frac{{\rm d}n}{{\rm d}M}(M') \\ &\quad \times \int \frac{{\rm d}^2\mathbf{q}}{(2\pi)^2} \int \frac{{\rm d}^2\mathbf{q}'}{(2\pi)^2} \int_{k_{\perp i}} \frac{{\rm d}^2\mathbf{k}_\perp}{V(k_{\perp,i})} \int \frac{{\rm d}k_\parallel}{2\pi} \tilde{W}_{\mathbf{q}} \tilde{W}_{\mathbf{q}'} P(|\mathbf{k} - \mathbf{q}|, k_\parallel) e^{ik_\parallel(\chi - \chi')} (2\pi)^2 \delta_D^2(\mathbf{q} - \mathbf{q}') \\ &\simeq \frac{1}{\Omega_S \hat{N}_{\rm h}(z_L)^2} \int {\rm d}\chi \chi^4 f_{\rm h}(\chi; z_L)^2 \left[\int {\rm d}M\, S(M) b(M) \frac{{\rm d}n}{{\rm d}M}(M)\right]^2 \chi^4 P(k_{\perp,i}) \int \frac{{\rm d}^2\mathbf{q}}{(2\pi)^2} |\tilde{W}(\mathbf{l} = \chi\mathbf{q})|^2 \end{aligned} \tag{C71}$$

In the following, for simplicity we consider the uniform radial selection given by

$$f_{\rm h}(\chi; z_L) = \Theta(\chi_L - \chi + \Delta\chi/2)\Theta(\chi_L + \chi - \Delta\chi/2), \tag{C72}$$

where $\Theta(x)$ is the Heviside-step function, defined as $\Theta(x) = 1$ if $x > 0$ otherwise $\Theta(x) = 0$, χ_L is the radial comoving distance to z_L, $\chi_L \equiv \chi(z_L)$, and $\Delta\chi$ is the radial bin width.

For a reasonably narrow radial bin of the z_L-redshift slice, satisfying $\Delta\chi/\chi_L \ll 1$, the cumulative number counts can be simplified as

$$\hat{N}_{\rm h}(z_L) \simeq \bar{N}_{\rm h}(z_L)\left[1 + \bar{b}_1\delta_{\rm b}\right] \tag{C73}$$

$$\delta_{\rm h}^W(\mathbf{x}_\perp;\chi) \equiv W(\chi\boldsymbol{\theta})\delta_{\rm h}^{\rm 2D}(\mathbf{x}_\perp,\chi) \tag{C74}$$

$$\tilde{\delta}_{\rm h}^W(\mathbf{k}_\perp) = \int\frac{{\rm d}^2\mathbf{q}}{(2\pi)^2}\tilde{W}(\mathbf{q})\tilde{\delta}_{\rm h}(\mathbf{k}_\perp - \mathbf{q}) \tag{C75}$$

In this paper we consider a catalog of halos, probed with galaxies and/galaxy clusters. The average, projected number density of halos is given in terms of the halo mass function as

$$\bar{N}_{\rm h}(z_L) = \int{\rm d}\chi\,\frac{{\rm d}^2V}{{\rm d}\chi d\Omega}\int{\rm d}M\frac{{\rm d}n}{{\rm d}M}S(M,\chi;z_L), \tag{C76}$$

where ${\rm d}^2V/{\rm d}\chi d\Omega = \chi^2$ for a flat-geometry universe, $\chi(z)$ is the comoving angular distance to redshift z, $dn/{\rm d}M$ is the three-dimensional number density of halos with masses $[M, {\rm d}M]$ at $z(= z(\chi))$, and $S(M,\chi;z_L)$ is the selection function of halos at a bin of redshift around z_L. Note that the radial and angular-diameter distances are equivalent for a flat-geometry universe. For simplicilty, assuming that mass and redshift for each halo is available, we consider the following, trivial selection function, defined as

$$S(\chi, M; z_L) = \Theta(\chi - \chi_L + \Delta\chi/2)\Theta(\chi_L + \Delta\chi/2 - \chi)\tilde{S}(M), \tag{C77}$$

where $\Theta(x)$ is the Heviside-step function, defined as $\Theta(x) = 1$ if $x > 0$ otherwise $\Theta(x) = 0$, χ_L is the radial comoving distance to z_L, $\chi_L \equiv \chi(z_L)$, and $\Delta\chi$ is the radial bin width. $S(\chi, M; z_L)$ is nonzero in the radial range of $\chi_L - \Delta\chi/2 \le \chi \le \chi_L + \Delta\chi/2$. $\tilde{S}(M)$ is the selection of halo masses. We don't specify the form of $\tilde{S}(M)$ to keep generality of our discussion.

If a finite-volume survey region is embedded into a coherent density contrast, which we hereafter call the background mode $\delta_{\rm b}$, the cumulative number of halos is modulated as

$$\hat{N}_{\rm h}(z_L;\delta_{\rm b}) = \int{\rm d}\chi\,\chi^2\int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M,\chi;z_L)\,[1 + b(M)\delta_{\rm b}] = \bar{N}_{\rm h}(z_L)(1 + \bar{b}_1\delta_{\rm b}) \tag{C78}$$

where $b(M)$ is the linear halo bias and $\bar{b}_1$ is the average bias over the halo mass function:

$$\bar{b}_1 \equiv \frac{1}{\bar{N}_{\rm h}(z_L)}\int{\rm d}\chi\,\chi^2\int{\rm d}M\frac{{\rm d}n}{{\rm d}M}S(M,\chi;z_L)b(M) = \frac{1}{\bar{N}_{\rm h}(z_L)}\int_{\chi_L}{\rm d}\chi\,\chi^2 n_{\rm h}^0(\chi), \tag{C79}$$

where the integration $\int_{\chi_L}{\rm d}\chi$ denotes the radial integration over the range $[\chi_L - \Delta\chi/2, \chi_L + \Delta\chi/2]$, and we introduced the collapsed notation:

$$n_{\rm h}^\beta(\chi) \equiv \int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}\tilde{S}(M)\,[b(M)]^\beta\,. \tag{C80}$$

The background density mode is defined in terms of the survey window funciton $W(\chi,\boldsymbol{\theta})$ as

$$\delta_{\rm b} \equiv \frac{1}{V_S}\int{\rm d}\chi\,\chi^2\int_{\Omega_s}{\rm d}^2\boldsymbol{\theta}\;W(\chi,\boldsymbol{\theta})\delta_{\rm m}(\chi,\chi\boldsymbol{\theta}). \tag{C81}$$

Note that the survey window generally includes both the radial and angular windows. If the survey window has a simple geometry, without any masked region, the survey volume for the redshift slice around z_L, V_S, is simply given as $V_S \simeq \chi_L^2\Delta\chi\Omega_S$, where Ω_S is the survey area.

The projected number density fluctuation field of halos in the radial bin around z_L is given as

$$\delta_{\rm h}^{\rm 2D}(\mathbf{x}_\perp, z_L) = \frac{1}{\hat{N}_{\rm h}}\int_{\chi_L}{\rm d}\chi\,\chi^2\int{\rm d}M\frac{{\rm d}n}{{\rm d}M}\tilde{S}(M)\delta_{\rm h}(\chi,\mathbf{x}_\perp;M), \tag{C82}$$

where $\mathbf{x}_\perp$ is the projected position vector of each halo in the perpendicular plane to the line-of-sight.

Using the Limber's appxoimation [28], we can express the projected power spectrum of halos as

$$P^{\rm 2D}_{\rm hh}(k_\perp;z_L)=\frac{1}{\hat{N}_{\rm h}(z_L)^2}\int_{\chi_L}{\rm d}\chi\,\chi^4\int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}\tilde{S}(M)\int{\rm d}M'\,\frac{{\rm d}n}{{\rm d}M'}\tilde{S}(M')P_{\rm hh}(k_\perp;M,M',z_L)$$
$$=\frac{1}{\hat{N}_{\rm h}(z_L)^2}\int_{\chi_L}{\rm d}\chi\,\chi^4\left[n^1_{\rm h}(\chi)\right]^2P^{\rm lin}_{\rm m}(k_\perp;\chi) \tag{C83}$$

where $P_{\rm hh}(k;M,M',\chi)$ is the three-dimensinal power spectrum between two halo of masses M and M' at redshift $z=z(\chi)$. In the second line on the r.h.s., we assumed that the halo power spectrum is given as $P_{\rm hh}(k,M,M')\simeq b(M)b(M')P^{\rm lin}_{\rm m}(k)$ in the halo model approach, where $P^{\rm lin}_{\rm m}(k)$ is the linear mass power spectrum. In the following we will often denote k instead of $k_\perp$ for notational simplification. If we further assume a thin redshift slice, $\Delta\chi/\chi_L\ll 1$, the halo power spectrum can be further simplified as

$$P_{\rm hh}(k_\perp;z_L)\simeq\frac{\bar{b}_1^2}{\Delta\chi}P^{\rm lin}_{\rm m}(k). \tag{C84}$$

From the above equation, we can find the power spectrum of stacked lensing as

$$P_{\Delta\Sigma}(k)=\frac{\bar{\rho}_{\rm m0}}{\hat{N}_{\rm h}(z_L)}\int_{\chi_L}{\rm d}\chi\,\chi^2\int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}\tilde{S}(M)P_{\rm hm}(k;M). \tag{C85}$$

The average excess surface mass density profile, $\langle\Delta\Sigma\rangle(R)$, is a more direct observable of the stacked lensing, and is given as

$$\langle\Delta\Sigma\rangle(R)=\int\frac{k{\rm d}k}{2\pi}P_{\Delta\Sigma}(k)J_2(kR). \tag{C86}$$

If we use the halo model, we can express the power spectrum by a sum of the 1- and 2-halo terms:

$$P_{\Delta\Sigma}(k)=P^{\rm 1h}_{\Delta\Sigma}(k)+P^{\rm 2h}_{\Delta\Sigma}(k), \tag{C87}$$

with

$$P^{\rm 1h}_{\Delta\Sigma}(k)\equiv\frac{\bar{\rho}_{\rm m0}}{\hat{N}_{\rm h}(z_L)}\int_{\chi_L}{\rm d}\chi\,\chi^2\int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}S(M,z_L)M|u_M(k;z_L)|=\frac{\bar{\rho}_{\rm m0}}{\hat{N}_{\rm h}(z_L)}\int_{\chi_L}{\rm d}\chi\,\chi^2I^0_1(k;\chi),$$
$$P^{\rm 2h}_{\Delta\Sigma}(k)=\frac{\bar{\rho}_{\rm m0}}{\hat{N}_{\rm h}(z_L)}\int_{\chi_L}{\rm d}\chi\,\chi^2\int{\rm d}M\,\frac{{\rm d}n}{{\rm d}M}\tilde{S}(M)P^{2h}_{\rm hm}(k;M,z_L)=\frac{\bar{\rho}_{\rm m0}}{\hat{N}_{\rm h}(z_L)}\int_{\chi_L}d\chi\,\chi^2n^1_{\rm h}(\chi)P^{\rm lin}_{\rm m}(k;\chi)\simeq\bar{b}_1\bar{\rho}_{\rm m0}P^{\rm lin}_{\rm m}(k), \tag{C88}$$

where we introduced the notation:

$$I^\beta_\mu(k_1,\cdots,k_\mu)\equiv\int{\rm d}M\frac{{\rm d}n}{{\rm d}M}\tilde{S}(M)\beta^\beta_1\prod_{i=1}^{\mu}u_M(k_i), \tag{C89}$$

where $u_M(k)$ is the Fourier transform of the average mass density profile for halos of mass M.