The Australian National University Semester 2, 2022
School of Computing Assignment 4 Theory Questions

COMP3670/6670: Introduction to Machine Learning

Question 1 **Permutation Matrix** (5+5=10 credits)

A *permutation matrix* is a square matrix that has exactly a single 1 in every row and column, and zeros elsewhere. For example,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let $\mathbf{P}$ be an **arbitrary** $n \times n$ permutation matrix.

1. Prove that $\mathbf{P}$ is always invertible.
2. Prove that $\mathbf{P}^T$ is a permutation matrix.

Question 2 **Distinct eigenvalues and linear independence** (20+5 credits)

Let $\mathbf{A}$ be a $n \times n$ matrix.

1. Suppose that $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and corresponding non-zero eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Prove that $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is linearly independant.

   Hint: You may use without proof the following property: If $\{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ is linearly dependent then there exists some p such that $1 \leq p < m$, $\mathbf{y}_{p+1} \in \text{span}\{\mathbf{y}_1, \ldots, \mathbf{y}_p\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_p\}$ is linearly independent.

2. Hence, or otherwise, prove that for any matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$, there can be at most n distinct eigenvalues for $\mathbf{B}$.

Question 3 **Properties of Upper Triangular** (10+15=25 credits)

1. Prove the set of all lower triangular matrices is closed under matrix multiplication.
2. Let $\mathbf{U}$ be an square $n \times n$ **lower** triangular matrix. Prove that the determinant of $\mathbf{U}$ is equal to the product of the diagonal elements of $\mathbf{U}$.

Question 4 **Eigenvalues of symmetric matrices** (15 credits)

1. Let $\mathbf{A}$ be a symmetric matrix. Let $\mathbf{v}_1$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ_1, and let $\mathbf{v}_2$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ_2. Assume that $\lambda_1 \neq \lambda_2$. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. (Hint: Try proving $\lambda_1 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2$. Recall the identity $\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a}$.)

Question 5 **Computations with Eigenvalues** (3+3+3+3+3=15 credits)

Let $\mathbf{A} = \begin{bmatrix} 6 & 4 \\ 3 & 5 \end{bmatrix}$.

1. Compute the eigenvalues of $\mathbf{A}$.

2. Find the eigenspace E_λ for each eigenvalue λ. Write your answer as the span of a collection of vectors.

3. Verify the set of all eigenvectors of $\mathbf{A}$ spans $\mathbb{R}^2$.

4. Hence, find an invertable matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

5. Hence, find a formula for efficiently [1] calculating $\mathbf{A}^n$ for any integer $n \geq 0$. Make your formula as simple as possible.

[1]That is, a closed form formula for $\mathbf{A}^n$ as opposed to multiplying $\mathbf{A}$ by itself n times over.