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Question 1

(a) ① To prog non-negative:

$p(\theta) = 30\theta^2 \cdot (1-\theta)^2$

$\because \theta^2 \geq 0 , (1-\theta)^2 \geq 0$

$\therefore p(\theta) \geq 0$

$\therefore$ non-negative

② To prog normalised:

$\because \theta \in [0,1]$

$\therefore \int_0^1 p(\theta)\, d\theta = \int_0^1 30\theta^2(1-\theta)^2\, d\theta$

$= \int_0^1 30\theta^2 - 60\theta^3 + 30\theta^4\, d\theta$

$= 10\theta^3\Big|_0^1 - 15\theta^4\Big|_0^1 + 6\theta^5\Big|_0^1$

$= 1$

$\therefore$ normalised

$\therefore$ Prog $p(\theta)$ is valid

(b) ① $p(\theta | x_{1:2} = 01) = \dfrac{p(x_{1:2}=01 | \theta) \cdot p(\theta)}{p(x_{1:2}=01)}$ (Bayes' Theorem)

$\because$ flips are indepedent

$\therefore p(x_{1:2}=01 | \theta) = p(x_1 = 0 | \theta) \cdot p(x_2 = 1 | \theta) = (1-\theta) \cdot \theta$

$\therefore p(x_{1:2}=01) = \int_0^1 p(x_{1:2}=01 | \theta) \cdot p(\theta)\, d\theta$

$$\Rightarrow \quad P(\theta \mid x_{1:2} = 01) = \frac{(1-\theta)\cdot\theta\cdot 30\theta^2(1-\theta)^2}{\int_0^1 (1-\theta)^3 \cdot 30\theta^3 \, d\theta} \qquad = 30\times\frac{14}{3}\theta^3\cdot(1-\theta)^3 = 140\,\theta^3(1-\theta)^3$$

② $$p(\theta \mid x_{1:4} = 0101) = \frac{P(x_{1:4} = 0101 \mid \theta)\cdot P(\theta)}{P(x_{1:4} = 0101)}$$ Similarly as ①

$$= \frac{(1-\theta)^2\theta^2 \cdot 30\theta^2(1-\theta)^2}{\left(\int_0^1 (1-\theta)^4 30\theta^4 \, d\theta\right)} = 21\times 30 \cdot \theta^4(1-\theta)^4$$

$$= 630\,\theta^4(1-\theta)^4$$

(c)

$$\mu = \int_0^1 p(\theta)\cdot\theta\cdot d\theta$$

① $$\mu_\theta = \int_0^1 30\theta^2(1-\theta)^2\cdot\theta\, d\theta$$

$$= \int_0^1 30\theta^5 - 60\theta^4 + 30\theta^3 \, d\theta$$

$$= \frac{1}{2} = 0.5$$

② $$\mu_{\theta_{12}} = \int_0^1 140\theta^3(1-\theta)^3\cdot\theta\, d\theta$$ (using CAS to assist with integration)

$$= \frac{1}{2} = 0.5$$

③ $$\mu_{\theta_{1234}} = \int_0^1 630\,\theta^4(1-\theta)^4\cdot\theta\, d\theta$$

$$= \frac{1}{2} = 0.5$$

(d) For $\sigma^2 = \mu(\theta^2) - [\mu(\theta)]^2$. as $\mu(\theta)$ calculated in (c)

① $\sigma_\theta^2 = \int_0^1 \theta^2 \cdot p(\theta)\, d\theta - \frac{1}{4}$

$= \int_0^1 \theta^2 \cdot 30\theta^2(1-\theta)^2\, d\theta - \frac{1}{4}$

$= \frac{2}{7} - \frac{1}{4} = \frac{1}{28}$

② $\sigma_{12}^2 = \int_0^1 \theta^2 \cdot p(\theta|x_{1:2}=01) \cdot d\theta - (\frac{3}{7})^2$

$= \int_0^1 \theta^2 \cdot 140\theta^3(1-\theta)^3\, d\theta - \frac{1}{4}$

$= \frac{5}{18} - \frac{1}{4}$

$= \frac{1}{36}$

③ $\sigma_{1234}^2 = \int_0^1 \theta^2 \cdot p(\theta|x_{1:34}=0101) \cdot d\theta - (\frac{8}{21})^2$

$= \int_0^1 630\, \theta^6 (1-\theta)^4\, d\theta - \frac{1}{4}$

$= \frac{3}{11} - \frac{1}{4}$

$= \frac{1}{44}$

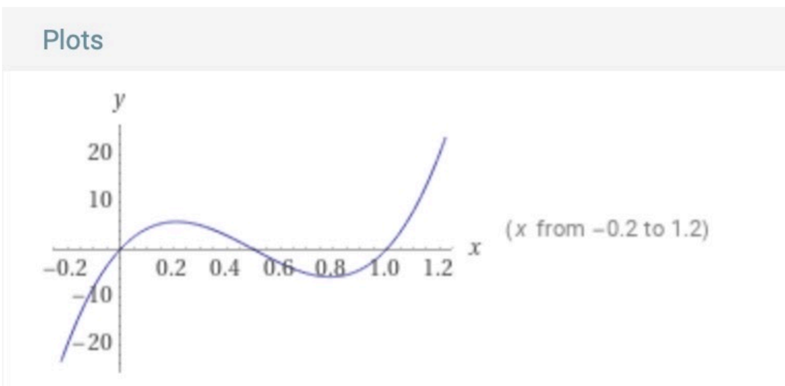
(e) $\theta_{MAP} = \arg\max\limits_{\theta} P(\theta|x) = \arg\max\limits_{\theta} \frac{P(x|\theta)\cdot P(\theta)}{P(x)} \propto \arg\max\limits_{\theta} P(x|\theta)\cdot P(\theta)$

① $\theta_{MAP_0} = \arg\max\limits_{\theta} 30\theta^2(1-\theta)^2$ (where $\theta \in [0,1]$)

$\Rightarrow 60\theta\cdot(2\theta^2-3\theta+1)=0$

(using CAS to assist with differentiation)

the plots is shown below:



∵ when $\theta < 0.5$, $(30\theta^2(1-\theta)^2)'' > 0$.

$\theta > 0.5$ $(30\theta^2(1-\theta)^2)'' < 0$

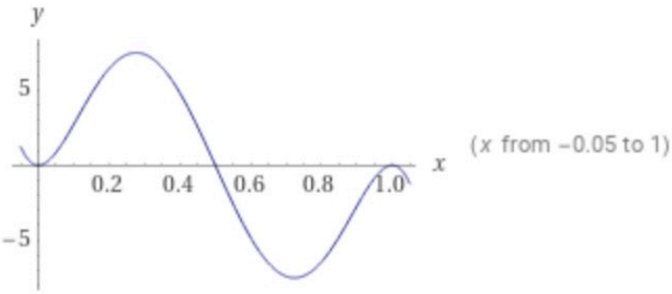
∴ we can get $\theta_{MAP_0} = 0.5$ by $2\theta^2-3\theta+1=0$

② $\theta_{MAP_{12}} = \arg\max\limits_{\theta} 140\theta^3(1-\theta)^3$ ($\theta \in [0,1]$)

$\Rightarrow -420(\theta-1)^2\theta^2(2\theta-1)$

the plots is shown below

Plots



Similarly as ①

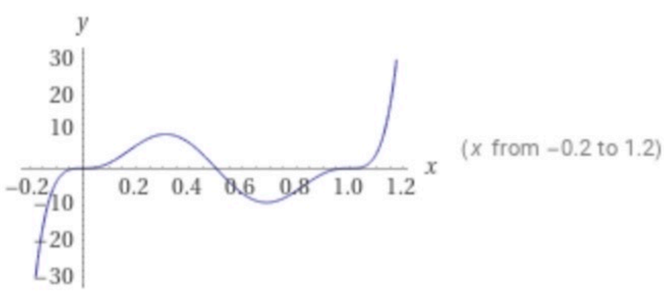
we can get $\theta_{map} = 0.5$

③ $\theta_{map_{1234}} = \underset{\theta}{\arg\max}\ 630\,\theta^4(1-\theta)^4$

$\Rightarrow 2520\,\theta^3(\theta-1)^3(2\theta-1) = 0$

the plots is shown below:

Plots

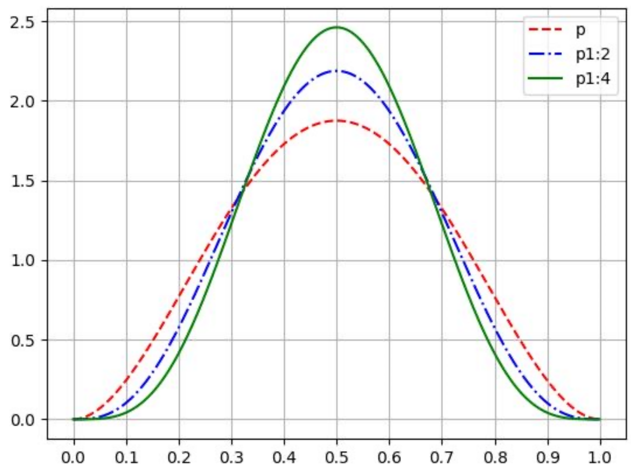


Similarly as ① ②:

we can get $\theta_{map_{1234}} = 0.5$

Posterior	PDF	μ	σ^2	θ_{MAP}
$P(\theta)$	$30\theta^2(1-\theta)^2$	$\frac{1}{2}$	$\frac{1}{28}$	$\frac{1}{2}$
$P(\theta \mid x_{1:2}=01)$	$140\theta^3(1-\theta)^3$	$\frac{1}{2}$	$\frac{1}{36}$	$\frac{1}{2}$
$P(\theta \mid x_{1:4}=0101)$	$630\theta^4(1-\theta)^4$	$\frac{1}{2}$	$\frac{1}{44}$	$\frac{1}{2}$

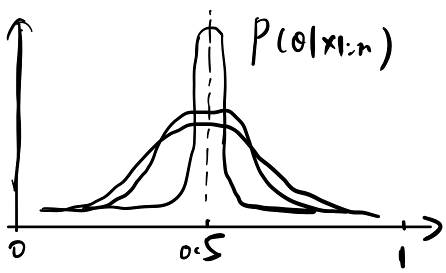
(f)



(g) ① when $x_{1:n} = 010101\cdots$, $P(\theta \mid x_{1:n})$ will have more chance on 0.5. which means more and more close to $P(x=1) = P(x=0) = \frac{1}{2}$

② For large n. μ still $\frac{1}{2}$. σ^2 getting smaller and smaller. θ_{MAP} still $\frac{1}{2}$.

③



Question 2

(a) if $\phi = 1$, then $P(\hat{X}=1 \mid X=1) = 1$

$P(\hat{X}=0 \mid X=0) = 1$ always return correct answer without errors

if $\phi = 0.5$. then each return's confidence is unreliable as it has same chance report right or wrong answers.

if $\phi = 0$ then it always report the opposite answer.

(b) Known $P(X=0 \mid \theta) = 1-\theta$, $P(X=1 \mid \theta) = \theta$

$\because$ $X=0$ and $X=1$ i.i.d

$\therefore$ $P(\hat{X}=0 \mid \theta) = P(\hat{X}=0 \mid \theta, X=0) + P(\hat{X}=0 \mid \theta, X=1)$

$= (1-\theta)\cdot\phi + \theta\cdot(1-\phi)$

$P(\hat{X}=1 \mid \theta) = P(\hat{X}=1 \mid \theta, X=0) + P(\hat{X}=1 \mid \theta, X=1)$

$= (1-\theta)\cdot(1-\phi) + \theta\cdot\phi$

(c) Bayes' Theorem

$$P(\theta \mid \hat{X}=0) = \frac{P(\hat{X}=0 \mid \theta)\cdot P(\theta)}{P(\hat{X}=0)}$$

$$= \frac{\left[(1-\theta)\phi + \theta \cdot (1-\phi)\right] \cdot P(\theta)}{P(\hat{x}=0)}$$

where $P(\hat{x}=0) = \int_0^1 P(\hat{x}=0 \mid \theta) \cdot P(\theta) \cdot d\theta \qquad (\theta \in [0,1])$

$$= \int_0^1 \left[(1-\theta)\phi + \theta(1-\phi)\right] \cdot P(\theta)\, d\theta$$

$$\therefore P(\theta \mid \hat{x}=0) = \frac{\left[(1-\theta)\phi + \theta \cdot (1-\phi)\right] \cdot P(\theta)}{\int_0^1 \left[(1-\theta)\phi + \theta(1-\phi)\right] \cdot P(\theta)\, d\theta}$$

① when $\phi = 1$

$$P(\theta \mid \hat{x}=0) = \frac{(1-\theta) \cdot P(\theta)}{\int_0^1 (1-\theta) \cdot P(\theta)\, d\theta}$$

② when $\phi = 0.5$

$$P(\theta \mid \hat{x}=0) = \frac{\frac{1}{2} \cdot P(\theta)}{\int_0^1 \frac{1}{2} P(\theta)\, d\theta} = \frac{P(\theta)}{\int_0^1 P(\theta)\, d\theta} = P(\theta)$$

$(\because \int_0^1 P(\theta)\, d\theta = 1)$

③ when $\phi = 0$

$$P(\theta \mid \hat{x}=0) = \frac{\theta P(\theta)}{\int_0^1 \theta P(\theta)\, d\theta}$$

Observation:

From $\phi=1$ to $\phi=0.5$, $\phi=0$. Correct → wrong report.

$\phi=1$ rely on appropriate corresponding $\theta \to X=0$.

$\phi=0.5$ totally rely on prior $P(\theta)$

$\phi=0$ rely on wrong corresponding $\theta \to X=1$

(d) As calculated in (c)

$$P(\theta|\hat{x}=0) = \frac{[(1-\theta)\phi + \theta(1-\phi)]\,30\theta^2(1-\theta)^2}{\int_0^1 [(1-\theta)\phi + \theta\cdot(1-\phi)]\cdot 30\theta^2(1-\theta)^2\,d\theta}$$

$$= \frac{[(1-\theta)\phi + \theta(1-\phi)]\cdot 30\theta^2(1-\theta)^2}{\int_0^1 [30\theta^2(1-\theta)^3\phi + 30\theta^3(1-\theta)^2(1-\phi)]\,d\theta}$$

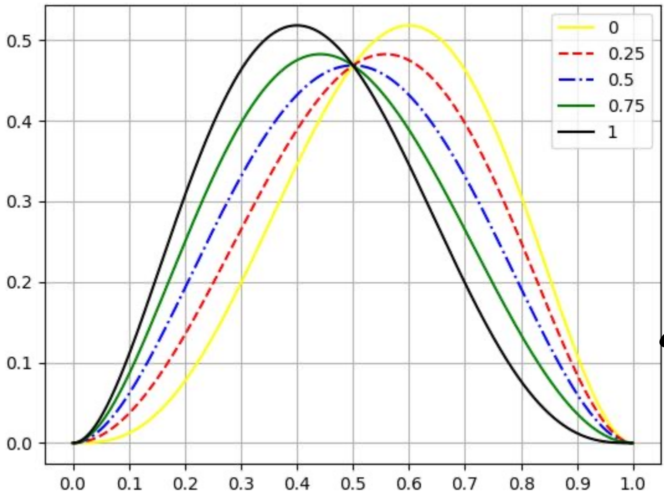
Using CAS to assist with integration, denominator can be calculated as $-\frac{1}{2}$

$$\therefore P(\theta|\hat{x}=0) = 2\left[[(1-\theta)\phi + \theta(1-\phi)]\cdot 30\theta^2(1-\theta)^2\right]$$

$$= 60\left[\theta^2(1-\theta)^3\phi + \theta^3(1-\theta)^2\cdot(1-\phi)\right]$$

$$= 60\theta^2(1-\theta)^2\cdot[(1-\theta)\cdot\phi + \theta\cdot(1-\phi)]$$

(e)



① The shape of distribution will swing to the left, which means $p(\theta|\hat{x})$ is changing on $p(\hat{x}|x)$ distribution.

② ϕ more close to 1 means the observation $\hat{x}$ has more probability to be real x. so $p(\theta|\hat{x})$ is more close to real $p(\theta|x)$.

Question 3

$$F(x) = P(X \le x) = \int x^2 + \frac{2}{3}x + \frac{1}{3}\, dx$$
$$= \frac{1}{3}x^3 + \frac{1}{3}x^2 + \frac{1}{3}x$$

$\because Y = X^2 + 2 \quad , x \in [0,1]$

$\therefore X = \sqrt{Y-2}$

For $F(y) = P(X \le \sqrt{x-2}) = F(\sqrt{x-2}) = \frac{1}{3}(x-2)^{\frac{3}{2}} + \frac{1}{3}(x-2) + \frac{1}{3}(x-2)^{\frac{1}{2}}$

$\therefore P(Y) = \frac{1}{2}\sqrt{y-2} + \frac{1}{3} + \frac{1}{6}\frac{1}{\sqrt{y-2}}$