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Exercise 1

1. False

if $V \in R_{2\times 2}$, $a=(1,1)$, $b=(0,0)$, $c=(1,2)$:

then: $\langle a,c\rangle = 3 > \langle a,b\rangle + \langle b,c\rangle = 0$

$\therefore$ if $a, c \neq 0$ and $\langle a,c\rangle > 0$ and $b=0$

then $\langle a,c\rangle > \langle a,b\rangle + \langle b,c\rangle$

2. False

if $V \in R_{2\times 2}$, $a, c \neq 0$, $b=0$

then satisfied $\langle a,b\rangle = 0$ and $\langle b,c\rangle = 0$, but $\langle a,c\rangle \neq 0$ as long as $\langle a,c\rangle \neq 0$

3. True

$span(S) = k_1 v_1 + k_2 v_2 + \dots + k_n v_n$ (k_i could be any value)

$\because$ x is orthogonal to all of v_i

$\therefore$ $\langle x, v_i\rangle = 0$

$\therefore$ set $y \in span(s)$, $y = k_1v_1 + k_2v_2 + \dots + k_nv_n$

$\langle x,y\rangle = \langle x, k_1v_1 + k_2v_2 + \dots + k_nv_n\rangle$

$\because \langle x, v_i\rangle = 0$

$\therefore \langle x,y\rangle = 0$

$\therefore$ Proof

4. True

For span(S), S is a orthonormal basis of Span(S), which means any $y \in Span(S)$ can be represented by S linearly:

$$y = k_1v_1 + k_2v_2 + \dots + k_nv_n \quad (k_i \text{ could be any value})$$

if $x \neq 0$ and $x \in Span(S)$

then $x = k_1v_1 + k_2v_2 + \dots + k_nv_n$

$\because \langle x, v_i \rangle = 0$

$\therefore \langle x, x \rangle = \langle x, k_1v_1 + k_2v_2 + \dots + k_nv_n \rangle = 0 \Rightarrow x = 0$

However $x \neq 0$

$\therefore$ Disproof

5. True

if $x \in Span(S)$, then: $x = k_1v_1 + k_2v_2 + \dots + k_nv_n$ (k_i could be any value)

$\because \langle x, v_i \rangle = 0$

$\therefore \langle x, x \rangle = 0 \Rightarrow x = 0$

However $x \neq 0$

$\therefore$ Disproof

6. True

if not unique. then

$x = c_1v_1 + \cdots + c_nv_n$

$x = d_1v_1 + \cdots + d_nv_n$ where c_i and d_i are not all the same.

Let's say $d_i \neq c_i$. $(1 \le i \le n)$. other $c = d$.

then $x - x = c_1v_1 + \cdots + c_nv_n - d_1v_1 + \cdots + d_nv_n$

$= (c_1 - d_1)v_1 + \cdots + (c_i - d_i)v_i + (c_n - d_n)v_n$

$= (c_i - d_i)v_i$

$\because x - x = 0$

$\therefore (c_i - d_i)v_i = 0$

$\because c_i \neq d_i$. $v_i \neq 0$ conflict

$\therefore$ Prove that contradiction.

7. False

For example set $V \in R_{2\times2}$. $S = \{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\}$ $x = \begin{pmatrix}4\\4\end{pmatrix}$

then $x = 2v_1 + v_2$

$= 4v_1$

$= 2v_2$

$\because x = c_1v_1 + c_2v_2 + \cdots + c_nv_n$

$\therefore c_1, c_2$ can be $(2,1)$, $(4,0)$, $(0,2)$, not unique

$\therefore$ Disproof

8. False

$$\begin{array}{cccc} & v_1 & v_2 & v_3 \end{array}$$

for example set $V \in R_{3\times 3}$. $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

For v_i in S:

v_1, v_2 are linearly independent

v_2, v_3 are linearly independent

v_1 v_3 are linearly independent

For a linearly independence set S,

any vector in the set S cannot be linearly represented by other vectors

However:

$v_1 = v_2 + v_3$

$\therefore$ S is not linearly independent

$\therefore$ Disproof

Exercise 2.

if $\|\cdot\|$ is a norm, $\|\cdot\|$ should satisfy 3 properties.

1. $\|x\| \geq 0$, if $\|x\| = 0$, then $x = 0$

Obviously, $\|x\| = \sqrt{\langle x, x \rangle} \geq 0$

To proof:

$$\|x\| = \sqrt{\langle x, x \rangle} = 0$$

$\Rightarrow \langle x, x \rangle = 0$

$\Rightarrow x = 0$

$\therefore$ Proof

2. $\|\lambda x\| = |\lambda| \cdot \|x\|$

To proof:

$\|\lambda x\| = \sqrt{<\lambda x, \lambda x>}$

$\because <x, y> \leq \|x\| \|y\|$, only if $x=y$. $<x, y> = \|x\| \cdot \|y\|$

$\therefore <\lambda x, \lambda x> = \|\lambda x\| \cdot \|\lambda x\| = \|\lambda x\|^2$

$\therefore \sqrt{<\lambda x, \lambda x>} = \sqrt{\|\lambda x\|^2} = |\lambda| \cdot \|x\|$

3. $\|x+y\| \leq \|x\| + \|y\|$

To proof:

$$\|x+y\| = \sqrt{<x+y, x+y>} = \sqrt{<x,x> + 2<x,y> + <y,y>}$$

$$\leq \sqrt{<x,x> + 2\|x\|\|y\| + <y,y>}$$

(According to CS inequality $<x,y> \leq \|x\|\|y\|$)

$$= \sqrt{<x,x> + 2\sqrt{<x,x>}\sqrt{<y,y>} + <y,y>}$$

$$= \sqrt{(\sqrt{<x,x>} + \sqrt{<y,y>})^2}$$

$$= \sqrt{<x,x>} + \sqrt{<y,y>} = \|x\| + \|y\|$$

$\therefore \|x+y\| \leq \|x\| + \|y\|$

$\therefore$ Proof

$\therefore \|\cdot\|$ is a norm

Exercise 3

1. Define $y' = X\theta + c$

then $\mathcal{L}(\theta, c) = \|y - y'\|_A^2 + \|\theta\|_B^2 + \|c\|_A^2$

$D_\theta \mathcal{L}(\theta, c) = \frac{\partial \mathcal{L}(\theta, c)}{\partial \theta} = \frac{\partial \|y-y'\|_A^2}{\partial \theta} + \frac{\partial \|\theta\|_B^2}{\partial \theta}$ (according to sum rule)

① Using chain rule:

$\frac{\partial \|y-y'\|_A^2}{\partial \theta} = \frac{\partial \|y-y'\|_A^2}{\partial y'} \cdot \frac{\partial y'}{\partial \theta} = \frac{\partial (y-y')^T \cdot A \cdot (y-y')}{\partial y} \cdot X$

$= -(y-y')^T (A + A^T) X$ $\Rightarrow$ To simplify, I use $2A$ to represent $(A+A^T)$ below as A is symmetric then $A = A^T$

$= -2 (y-y')^T \cdot A \cdot X$

$= -2 (y^T - \theta^T X^T - c^T) \cdot A \cdot X$

② $\frac{\partial \|\theta\|_B^2}{\partial \theta} = \frac{\partial \theta^T B \theta}{\partial \theta} = 2\theta^T \cdot B$

$\therefore D_\theta \mathcal{L}(\theta, c) = -2 (y^T - \theta^T X^T - c^T) AX + 2\theta^T B$

2. Let $-2 (y^T - \theta^T X^T - c^T) AX + 2\theta^T B = 0$

then

$-y^T AX + \theta^T X^T AX + c^T AX + \theta^T B = 0$

$\theta^T (B + X^T AX) = y^T AX - c^T AX$

To proof $(B+X^TAX)$ is invertible:

∵ B is positive symmetric define matrix

∴ only need to proof X^TAX is positive symmetric define matrix
then $(B+X^TAX)$ is positive symmetric define matrix

Define Y^TX^TAXY . where XY can be written by c (Y can be any vector)

∵ A is possive, symmetric define matrix

∴ Any vector $w^TAw > 0$

∴ $c^TAc > 0$, where $c = XY$.

∴ Proof $(B+X^TAX)$ is symmetric positive define matrix ⇒ invertible

∴ $\theta = [(B+X^TAX)^{-1}]^T \cdot (y^TAX - c^TAX)^T$

3. Define $y' = X\theta + c$

$$\nabla_c \mathcal{L}(\theta, c) = \frac{\partial \mathcal{L}(\theta, c)}{\partial c} = \frac{\partial \|y-y'\|_A^2}{\partial c} + \frac{\partial \|c\|_A^2}{\partial c}$$

① $$\frac{\partial \|y-y'\|_A^2}{\partial c} = \frac{\partial \|y-y'\|_A^2}{\partial y'} \cdot \frac{\partial y'}{\partial c} = -2(y-y')^TA$$
$$= -2(y^T - \theta^TX^T - c^T)A$$

② $$\frac{\partial \|c\|_A^2}{\partial c} = 2c^TA$$

∴ $$\nabla_c \mathcal{L}(\theta, c) = -2(y^T - \theta^TX^T - c^T)A + 2c^TA$$

4. let $\nabla_c \mathcal{L}(\theta, c) = 0$

then $$2c^TA = y^TA - \theta^TX^TA$$

∵ A is positive. symmetric definite matric,

∴ A is invertible

∴ $$c^T = \frac{1}{2}(y^TA - \theta^TX^TA) \cdot A^{-1}$$

$$c = \frac{1}{2}(A^{-1})^T \cdot (y^TA - \theta^TX^TA)^T$$

5. For my answer in 3.2 :

$$\theta = \left[(y^T A X - c^T A X) \cdot (B + X^T A X)^{-1} \right]^T$$

$$= \left[(B + X^T A X)^{-1} \right]^T \cdot (y^T A X - c^T A X)^T$$

$\because$ $(B + X^T A X)$ is symmetric matrix

$\therefore$ $[(B + X^T A X)^{-1}]^T = (B + X^T A X)^{-1}$

when $A = I$. $c = 0$. $B = \lambda I$.

then $\theta = (\lambda I + X^T X)^{-1} \cdot X^T y$

$\therefore$ correct. the same as 3.5