Assignment 4

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Question 1

① $\because$ P is a permutation matric, which have a single 1 in every row and column

$\therefore$ P can convert to a identity matrix just by exchanging the row or column

$\Rightarrow$ $Rank(P) = Rank(I^{n\times n}) = n$ (I is identity matrix)

$\therefore$ P is invertible

② $\because P \cdot P^{-1} = I$

$\therefore$ easily we can get P^{-1} is invertible.

Suppose $M = P \cdot P^T$, P and P^T are permutation matrix.

So when $(P^T)_{ij} = 1$, j^{th} column of M is i^{th} column of P.

$\Rightarrow$ $\because P^T_{ij} = P_{ji}$

$\therefore (P \cdot P^T)_{ii} = 1$

$\therefore$ M is an identity matrix. $M = I$, $P \cdot P^T = I$

$\therefore$ P^T can be converted to identity matrix by just changing colums or rows (because P is a permutation matrix)

$\therefore$ P^T is a permutation matrix.

Question 2

① Using Mathematical induction

(1) when $n^*=1$. since eigenvalues $\lambda_1 \neq 0$. then $x_1 \in \{\lambda_1\}$ linearly independent.

(2) suppose when $n^* = n-1$. conclusions established

To proof when $n^*=n$ conclusion established:

$\Rightarrow$ suppose eigenvectors $x_1, x_2, \cdots x_{n-1}$ linearly independent

then we need to proof:

$$k_1 x_1 + k_2 x_2 + \cdots + k_n x_n = 0 \quad (k_1, k_2, \cdots k_n \text{ is any value}) \quad ①$$

Using A left multiply:

$$k_1 A x_1 + k_2 A x_2 + \cdots + k_n A x_n = 0$$

Using $Ax = \lambda x$

$$\Rightarrow k_1 \lambda_1 x_1 + k_2 \lambda_2 x_2 + \cdots + k_n \lambda_n x_n = 0 \quad ②$$

② $- \lambda_n$ ①:

$$\Rightarrow k_1 (\lambda_1 - \lambda_k) x_1 + k_2 (\lambda_2 - \lambda_k) x_2 + \cdots + k_n (\lambda_n - \lambda_n) x_n = 0$$

$$\Rightarrow k_1 (\lambda_1 - \lambda_k) x_1 + \cdots + k_{n-1} (\lambda_{n-1} - \lambda_n) x_{n-1} = 0$$

As supposed: $n^* = n-1$ linearly independent, then:

$$k_i (\lambda_i - \lambda_k) = 0$$

$\because$ eigenvalues are distinct

$\therefore k_i = 0 \quad (i = 1, 2, \cdots n-1)$

$\therefore$ According to ①: $k_n x_n = 0$

$\because x_n$ is non-zero

$\therefore k_n = 0$

$\therefore$ Proof $x_1, x_2, \cdots x_n$ linearly independent

② $\because$ $Ax = \lambda x$

$\therefore$ $(A - \lambda E)x = 0$ (E is identity matrix)

$\because$ x are non-zero

$\therefore$ $|A - \lambda E| = 0$

$\because$ Determinant $|A - \lambda E|$ order of magnitude of the unknowns λ is $n \Rightarrow \lambda^n$ (suppose $A_{n \times n}$)

$\therefore$ For equation of $f(\lambda^n) = 0$. we have at most n distinct λ.

$\therefore$ Proof

Question 3

① Suppose set R of all lower triangular matrices.

For $\forall A, B$ which can multiply in R. Let's say: $A_{n \times n}$. $B_{n \times n}$

$$\begin{cases} A = \{a_{ij}\}, & \text{when } i<j. \quad a_{ij} = 0 \\ B = \{b_{ij}\}. & \quad b_{ij} = 0 \end{cases}$$

$C = A \cdot B = \{c_{ij}\}$

(i^{th} row of A. multiply j^{th} column of B)

$\Rightarrow$ $c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} \cdot b_{nj} = 0$

As $A \cdot B$ are lower triangular

$\Rightarrow$ when $i<j$. $b_{1j}=b_{2j}=\cdots=b_{j-1j}=0$

$$a_{i(i+1)}=a_{i(i+2)}=\cdots=a_{in}=0$$

$\therefore$ $c_{ij}=0$ when $i<j$

$\therefore$ $C=A\cdot B$ is lower triangular matrix.

$\therefore$ For R:

$\forall A,B\in R$, $A+B\in R$

$\forall A\in R$, $\lambda\in R^{\infty}$, $\lambda A\in R$

$\forall A,B\in R$, $A\cdot B\in R$

$\therefore$ Proof

② $$|U|=\begin{vmatrix} U_{11} & 0 & \cdots & 0 \\ U_{12} & U_{22} & \ddots & \vdots \\ \vdots & & & \vdots \\ U_{1n} & U_{2n} & \cdots & U_{nn} \end{vmatrix} = U_{11}\cdot(-1)^{1+1}\cdot\begin{vmatrix} U_{22} & 0 & \cdots & 0 \\ U_{23} & U_{33} & \ddots & \vdots \\ \vdots & & & \vdots \\ U_{2n} & & \cdots & U_{nn} \end{vmatrix}$$

$$= U_{11}\cdot U_{22}\cdot(-1)^{2+2}\begin{vmatrix} U_{33} & 0 & \cdots & 0 \\ U_{34} & & \ddots & \vdots \\ \vdots & & & \\ U_{3n} & & \cdots & U_{nn} \end{vmatrix} \quad \overset{\cdots}{\Rightarrow}$$

$$= U_{11}\cdot U_{22}\cdot U_{33}\cdots U_{nn}$$

$\therefore$ Proof

Question 4

To proof $v_1^T \cdot v_2 = 0$

We have :

$$Av_1 = \lambda_1 v_1 \quad , \quad Av_2 = \lambda_2 v_2 \quad \text{. do transpose}$$

$$\Rightarrow \quad v_1^T A^T = \lambda_1 v_1^T$$

$$\Rightarrow \quad v_1^T A^T \cdot v_2 = \lambda_1 v_1^T \cdot v_2 \quad \text{(right multiply } v_2\text{)} \quad ①$$

Left multiply v_1^T on $Av_2 = \lambda_2 v_2$

$$\Rightarrow \quad v_1^T A v_2 = \lambda_2 v_1^T v_2 \quad ②$$

② - ① : $\quad 0 = (\lambda_2 - \lambda_1) v_1^T v_2$

$\because \lambda_1 \neq \lambda_2$

$\therefore v_1^T v_2 = 0$

$\therefore$ Proof

Question 5

① $(A - \lambda E) = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & 4 \\ 3 & 5-\lambda \end{vmatrix} = 0 \quad \Rightarrow (6-\lambda)\cdot(5-\lambda) - 12 = 0$$

$$\Rightarrow \quad \lambda^2 - 11\lambda + 18 = 0 \quad \Rightarrow \quad \lambda_1 = 2 \ , \ \lambda_2 = 9$$

② $\lambda_1 = 2$: $Av_1 = \lambda_1 v_1$

$\Rightarrow \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \cdot v_1 = 0$, Let $v_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 4v_{11} + 4v_{12} \\ 3v_{11} + 3v_{12} \end{pmatrix} = 0 \Rightarrow V_{11} = -V_{12}$

$\Rightarrow V_1 = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, k could be any value

$E\lambda_1 = \left\{ k \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{matrix} k \in R \\ k \neq 0 \end{matrix} \right\}$

$\lambda_2 = 9$

$\Rightarrow \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \cdot V_2 = 0$, Let $V_2 = \begin{pmatrix} V_{21} \\ V_{22} \end{pmatrix}$

$\Rightarrow \begin{bmatrix} -3V_{21} + 4V_{22} \\ 3V_{21} - 4V_{22} \end{bmatrix} = 0 \Rightarrow 4V_{22} = 3V_{21} \Rightarrow V_{21} = \frac{4}{3} V_{22}$

$\therefore V_2 = \frac{4}{3} \cdot k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E\lambda_2 = \left\{ k \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}, \begin{matrix} k \in R \\ k \neq 0 \end{matrix} \right\}$

③ For v_1 and v_2,

obviously, $m\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $n\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$ are linearly independent.

$\therefore$ 2 linearly indepent vectors

can spans R^2

$\therefore$ Proof

④ For $Av = \lambda v$

$\Rightarrow \quad Av = \lambda v \cdot E$

$\Rightarrow \quad A = \lambda v \cdot E \cdot v^{-1}$

$\Rightarrow \quad A = v \cdot \lambda E \cdot v^{-1}$

$\therefore$ $P = v$, let's say $k=1$, then $P = \begin{bmatrix} 1 & \frac{4}{3} \\ -1 & 1 \end{bmatrix}$

P is invertible obviously, $\text{rank}(P) = 2$

$D = \lambda E = \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix}$

⑤

$$A^2 = A \cdot A = P \cdot D \cdot P^{-1} \cdot P \cdot D \cdot P^{-1} = PD^2P^{-1} \quad (\because P^{-1} \cdot P = I)$$

$$A^3 = A^2 \cdot A = PD^2P^{-1} \cdot PDP^{-1} = PD^3P^{-1}$$

$\therefore$ $A^n = PD^nP^{-1}$, which can be

$$= \begin{bmatrix} 1 & \frac{4}{3} \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^n & 0 \\ 0 & 9^n \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{4}{3} \\ -1 & 1 \end{bmatrix}^{-1}$$