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Exercise 1

(a) set $[A|E]$:

$$\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 - 2R_2 \to R_3 \\ R_1 - 3R_2 \to R_1 \\ R_2 + R_3 \to R_2}]{\substack{R_1 - R_2 \to R_1 \\ R_2 - R_1 \to R_2}} \begin{bmatrix} 1 & 0 & -17 & 4 & -7 & 0 \\ 0 & 1 & 0 & 11 & -2 & 1 \\ 0 & 0 & -5 & 2 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 + 17R_3 \to R_1]{R_3 \cdot (-\frac{1}{5}) \to R_3} \begin{bmatrix} 1 & 0 & 0 & -2.8 & 6.6 & -3.4 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -0.4 & 0.8 & -0.2 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -2.8 & 6.6 & -3.4 \\ 1 & -2 & 1 \\ -0.4 & 0.8 & -0.2 \end{bmatrix}$$

$$Ax = b \Rightarrow x = A^{-1} \cdot b = \begin{bmatrix} -2.8 & 6.6 & -3.4 \\ 1 & -2 & 1 \\ -0.4 & 0.8 & -0.2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

(b) set $[A|b]$:

$$\begin{bmatrix} 1 & 2 & 2 & 10 \\ 3 & 4 & 3 & 5 \end{bmatrix} \xrightarrow[R_1 + R_2 \to R_1]{R_2 - 3R_1 \to R_2} \begin{bmatrix} 1 & 0 & -1 & -15 \\ 0 & -2 & -3 & -25 \end{bmatrix} \xrightarrow{R_2 \cdot (-\frac{1}{2}) \to R_2} \begin{bmatrix} 1 & 0 & -1 & -15 \\ 0 & 1 & \frac{3}{2} & \frac{25}{2} \end{bmatrix}$$

$\therefore Ax=b$

$$\Rightarrow \begin{cases} x_1 = x_3 - 15 \\ x_2 = -\frac{3}{2}x_3 - \frac{25}{2} \end{cases} \quad \text{set } x_3=0 \text{ then } \begin{cases} x_1=-15 \\ x_2=-\frac{25}{2} \\ x_3=0 \end{cases}$$

$$\therefore x = c_1 \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -15 \\ -\frac{25}{2} \\ 0 \end{bmatrix} \quad (c_1 \text{ is any constant})$$

Exercise 2

set $[A|E]$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3-R_2 \to R_3]{\substack{R_2-2R_1 \to R_2 \\ R_3-3R_1 \to R_3}} \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \cdot (-1) \to R_3]{R_1-R_2 \to R_1} \begin{bmatrix} 1 & 0 & 5 & 3 & -1 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_2+3R_3 \to R_2]{R_1-5R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 & -2 & -6 & 5 \\ 0 & 1 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$$

$\therefore$ Matrix inverse exists.

Answer is $\begin{bmatrix} -2 & -6 & 5 \\ 1 & 4 & -3 \\ 1 & 1 & -1 \end{bmatrix}$

Exercise 3

(a) Subset of R^3 ? Yes (U is $x \geq 0, y \geq 0, z \geq 0$)

$U \neq \phi$. $0 \in U$? Yes

Closure ? No $\rightarrow$ $\lambda x < 0$ if $\lambda < 0$ and $x > 0$

$\therefore$ Answer is no

(b) Subset of R^3 ? Yes (U is $x+y+z=0$)

$U \neq \phi$. $0 \in U$? Yes

Closure ? Yes $\rightarrow$ ① $\forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in U \Rightarrow (x_1+x_2, y_1+y_2, z_1+z_2)$

$(x_1+y_1+z_1)+(x_2+y_2+z_2)=0 \in U$ ok

② $\forall (x, y, z) \in U$

$\lambda (x, y, z) = (\lambda x, \lambda y, \lambda z)$

satisfied : $\lambda (x+y+z)=0 \in U$

$\therefore$ Answer is Yes.

(c) Subset of R^3 ? Yes (U is $x=0$ or $y=0$ or $z=0$)

$U \neq \phi$, $0 \in U$? Yes

Closure ? No.

① $\forall (x_1, y_1, z_1) \in U$.

$\forall (x_2, y_2, z_2) \in U$

Let's say $x_1 = 0$. $y_1, z_1 \neq 0$

$y_1 = 0$ $x_2, y_2 \neq 0$.

then $(x_1+x_2, y_1+y_2, z_1+z_2) \notin U$

$\begin{cases} x_1+x_2 \neq 0 \\ y_1+y_2 \neq 0 \\ z_1+z_2 \neq 0 \end{cases}$

$\therefore$ Answer is No

(d) (U is $Ax=b$)
subset of R^3 ? Not really:
if $R(A) < R(A,b)$, Solutions $x = \phi \Rightarrow$ NO
if $R(A) = R(A,b)$, Solutions x exists $\Rightarrow$ Yes

$U \neq \phi$. $0 \in U$? Not really:
if $b=0$. Yes.
if $b \neq 0$. $0 \notin U$. no.

Closure ?

① $\forall (x_1, x_2) \in U$
$A(x_1+x_2) = Ax_1 + Ax_2 = 2b$. $\Rightarrow$ only if $b=0$. satisfied

② $\forall x \in U$.
$A(\lambda x) = \lambda Ax = \lambda b$ $\Rightarrow$ only if $b=0$. satisfied

$\therefore$ Answer is: only if $b=0$, All solutions x set is subspace of R^3

Exercise 4:

(a) $\because T(V) = W$ is a linear transformation
$\therefore W = \lambda V$, exists a λ. satisfied $W = \lambda V$.
$\therefore T(0) = 0$

* where λ is a matrix with constant, representing scaling on each dimension

(b) $\because$ linear transformation $T(V) = W = \lambda V$

$\therefore$ Let's say $C = \{c_1, c_2, \cdots c_n\}$

$T(cV) = \lambda cV$

$\because$ λ. c is constant for each elements in V

$\therefore$ $\lambda cV = c\lambda V = cT(V)$

$\therefore$ $T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n)$

(c) $\because$ $\{v_1 \cdots v_n\}$ is a set of dependent vectors in V.

$\therefore$ $k_1v_1 + k_2v_2 + \cdots + k_nv_n = 0$ (1)

exists at least one $k_i \neq 0$. satisfied equation (1).

To prove $\{w_1, w_2, \cdots w_n\}$ is a set of dependent vectors in W.

$q_1w_1 + q_2w_2 + \cdots + q_nw_n = 0$. (2)

$\because T(V) = W = \lambda V$

$\therefore$ Equation (2) can be written as:

$q_1\lambda v_1 + q_2\lambda v_2 + \cdots + q_n\lambda v_n = 0$

$\because$ q and λ is constant.

$\therefore$ Define $c = q\lambda$

$\therefore$ Equation (2) can be written as:

$c_1 v_1 + \cdots c_n v_n = 0$ (3)

$\therefore$ exists at least one $c_i \neq 0$ satisfied Equation (3)

$\because \lambda \neq 0$

$\therefore$ exists at least one $q_i \neq 0$ satisfied Equation (2)

$\therefore \{w_1, w_2, \cdots w_n\}$ is a set of dependent vectors in W.

(d) $\because S$ is a linear transformation

$\therefore S(w) = X = \lambda_1 w$

$L(v) = S(T(v))$ can be written as:

$L(v) = \lambda_1 (T(v))$

$= \lambda_1 \cdot \lambda V$

$\because \lambda_1 \cdot \lambda$ is constant for each elements in V

$\therefore$ Defined $k = \lambda_1 \cdot \lambda$

$\therefore L(v) = kV$

$\therefore L$ is a linear transformation.

Exercise 5:

(a) $\because$ Ω is symmetric

$\therefore$ $\Omega(x,y) = \Omega(y,x)$

$\because$ Ω is linear in the first argument

Define (For $\Omega(x,y)$)

$$\Omega(\lambda x_1 + \varphi x_2, y) = \lambda \Omega(x_1, y) + \varphi \Omega(x_2, y)$$

$\because$ $\Omega(x,y) = \Omega(y,x)$

For $\Omega(y,x)$

$$\Omega(\lambda y_1 + \varphi y_2, x) = \lambda \Omega(y_1, x) + \varphi \Omega(y_2, x)$$

$\therefore$ Ω is linear in the second argument

$\therefore$ Ω is bilinear

(b)

① symmetric $\langle x,y\rangle = \langle y,x\rangle$

$\langle y,x\rangle = y_1x_1 + y_2x_2 - (y_1+y_2+x_1+x_2) = \langle x,y\rangle$

Satisfied

② positive definite

$$<x,x> = x_1^2 + x_2^2 - (x_1+x_1+x_2+x_2)$$
$$= x_1^2 + x_2^2 - 2(x_1+x_2) = (x_1-1)^2 + (x_2-1)^2 - 2$$

if $x_1 = 1$. $x_2 = 1$. $<x,x> < 0$. dissatisfied

③ Bilinearity . set $x' \in R^2$

$$<cx+dx', y> = (cx_1+dx_1')y_1 + (cx_2+dx_2')y_2$$
$$-(y_1+y_2+cx_1+dx_1'+cx_2+dx_2')$$

$$= cx_1y_1 + cx_2y_2 + dx_1'y_1 + dx_2'y_2$$
$$-y_1 - y_2 - cx_1 - dx_1' - cx_2 - dx_2' \quad ③$$

$$c<x,y> = cx_1y_1 + cx_2y_2 + c(x_1+x_2+y_1+y_2) \quad ①$$

$$d<x',y> = cx_1'y_1 + cx_2'y_2 + c(x_1'+x_2'+y_1+y_2) \quad ②$$

$\because$ ① + ② $\neq$ ③

$\therefore$ dissatisfied

$\therefore$ Ansner is only satisfied Symmetric axiom.

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Exercise 6

(a) $\because$ x, y are orthogonal

$\therefore \langle x, y \rangle = 0$

Defined λ_1, λ_2 satisfied:

$\lambda_1 x + \lambda_2 y = 0$

Use x to do inner products operation:

$\because \langle x, y \rangle = 0$

$\therefore \lambda_1 \langle x, x \rangle = 0$

$\because \langle x, x \rangle = \|x\|^2 \neq 0$

$\therefore \lambda_1 = 0$

Use y to do inner products operation:

Proof $\lambda_2 = 0$

$\because x_1 = \lambda_2 = 0$

$\therefore$ x, y are linearly independent.

(b) if x, y are linearly independent:

$$\lambda_1 x + \lambda_2 y = 0 \quad , \quad \lambda_1 = 0 , \lambda_2 = 0$$

use x and y to do the inner products operation respectively:

then $\lambda_1 \langle x, x \rangle + \lambda_2 \langle y, x \rangle = 0$ and $\lambda_1 \langle x, y \rangle + \lambda_2 \langle y, y \rangle = 0$

$\because x, y \neq 0$

$\therefore \langle x, x \rangle > 0 , \langle y, y \rangle > 0$

$\because \lambda_1 = \lambda_2 = 0$

$\therefore \langle x, y \rangle , \langle y, x \rangle$ can be any value

$\therefore$ Disprove.

If x, y are linearly independent, they may not be orthogonal.

(c) For statements (a). (Changed)

if x or/and y is zero. then they are linearly dependant.
else: they are linearly independent

Answer: if remove the restriction, statements (a) can't be proved.

For statements (b). (Does not hold)

if x or y is zero. then they are linearly dependent. The condition does not hold. As zero is linearly dependent with any vector.