Models of Infectious Diseases (I)

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Outline

(1) SI Model

(2) SIS Model

(3) SIR Model

Key takeaways from this lecture

- Our first look at mathematical models to help better understand reality.

- These are not economic models, but share many similarities with them.

- In particular,
  - Find **“steady states.”**
  - Compute **“dynamics”**
    - Convergence towards a steady state,
    - Analyze the effect of exogenous shocks.
  - Analyze “**welfare**.”

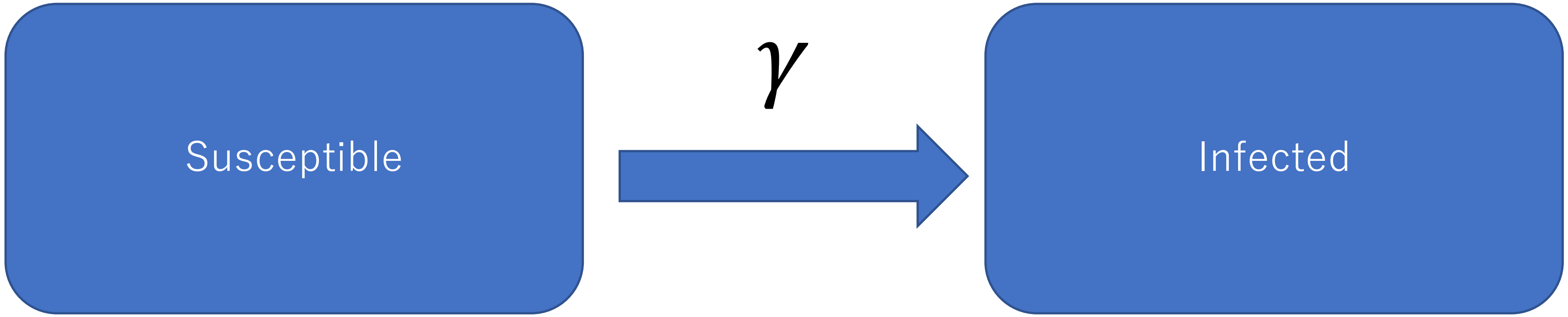
Outline

(1) SI Model

(2) SIS Model

(3) SIR Model

SI Model



SI Model

- HIV, Herpes, etc.
- Once you get infected, there is no treatment.
- A backward-looking model (easy to solve)
- Not optimization based.

- S_t : Number of people susceptible to the disease at time t.
- I_t : Number of people with the disease at time t.
- Normalization: $S_t + I_t = 1$.

- With probability γI_t, susceptible person at time t becomes infected at time $t + 1$.
- Initial condition: $I_1 = \epsilon$.

- $S_{t+1} - S_t = -\gamma I_t S_t$

- $I_{t+1} - I_t = \gamma I_t S_t$

- $S_t + I_t = 1$

Find **“steady states.”**

- A steady state is where $X_{t+1} = X_t$ for any variable X in the model.

- Let X_{ss} denote a steady state of X.

- At a steady state of SI model, we have
  - $S_{ss} - S_{ss} = -\gamma I_{ss} S_{ss}$
  - $I_{ss} - I_{ss} = \gamma I_{ss} S_{ss}$
  - $S_{ss} + I_{ss} = 1$

Thus, at a steady state, we have

- $0 = -\gamma I_{ss} S_{ss}$
- $0 = \gamma I_{ss} S_{ss}$
- $S_{ss} + I_{ss} = 1$

- The first two equations are the same in the sense that, if one is satisfied, the other is also satisfied.

- Eliminate S_{ss} to obtain

  - $0 = \gamma I_{ss}(1 - I_{ss})$

- $0 = \gamma I_{ss}(1 - I_{ss})$

- There are two steady states.

  - One steady state where $S_{ss} = 0$ and $I_{ss} = 1$.
  - The other steady state where $S_{ss} = 1$ and $I_{ss} = 0$.

Compute **“dynamics”** (convergence to steady state)

- Suppose that $I_1 = \epsilon$ (i.e. $S_1 = 1 - \epsilon$) .

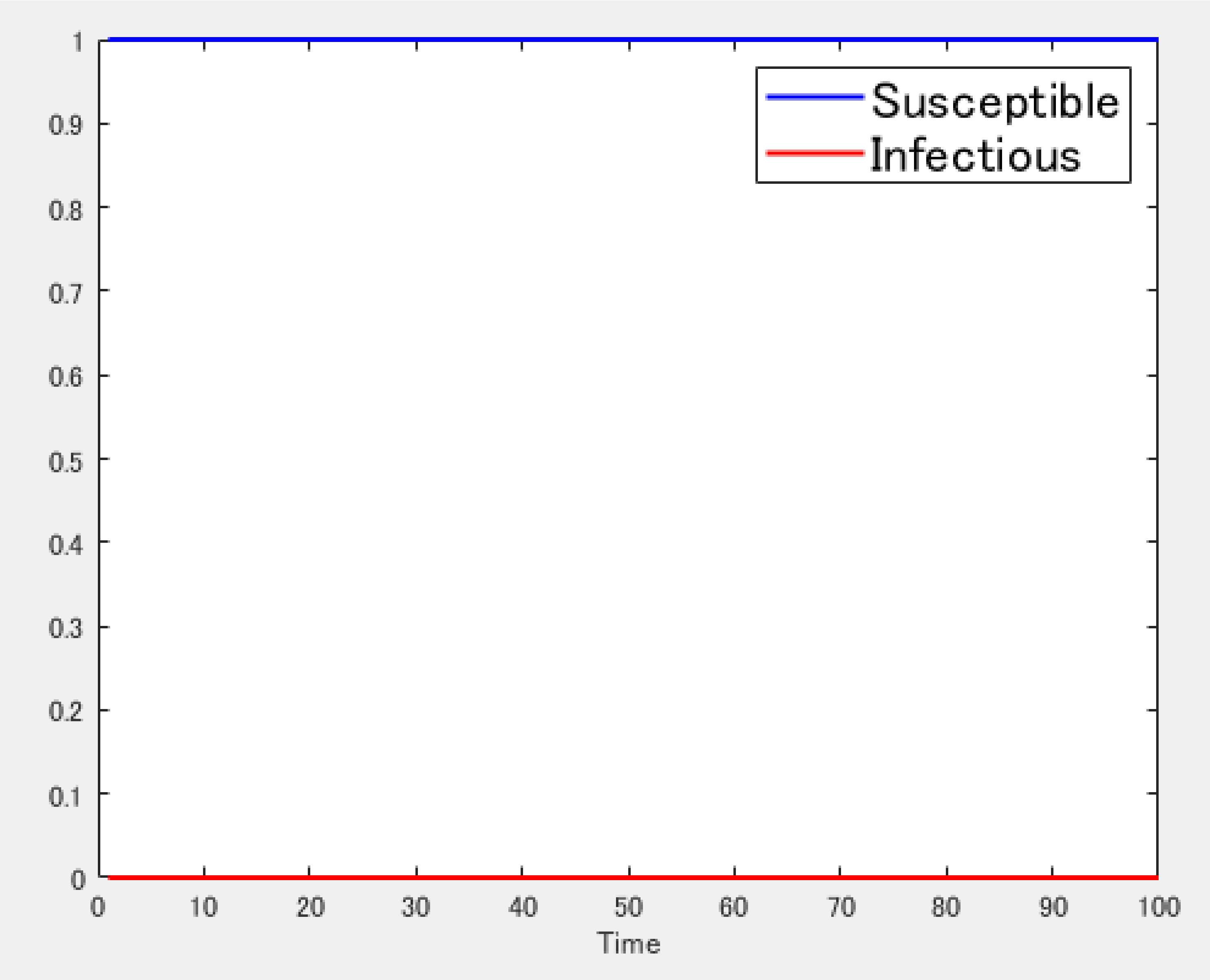
- Compute $\{I_t, S_t\}_{t=2}^{\infty}$

- How? Recursively.
  - $S_2 - S_1 = -\gamma I_1 S_1$
  - $I_2 - I_1 = \gamma I_1 S_1$
  - $S_1 + I_1 = 1$

- $S_2 = (1-\epsilon) - \gamma\epsilon(1-\epsilon) = (1-\epsilon)(1-\gamma\epsilon)$
- $I_2 = \epsilon + \gamma\epsilon(1-\epsilon) = \epsilon(1+\gamma(1-\epsilon))$

- and so on⋯

$I_1 = 0$



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
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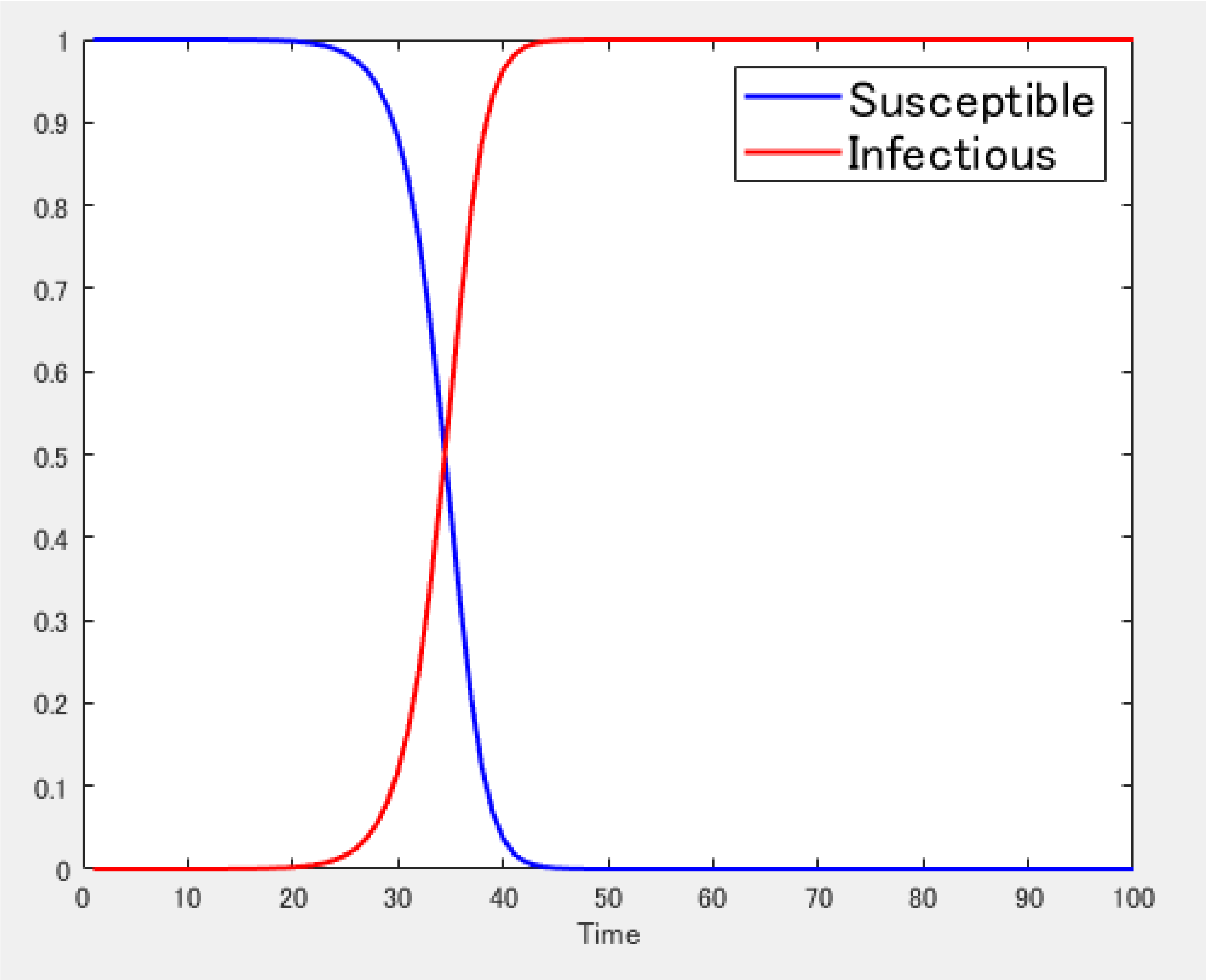
20
30
40
50
60
70
80
90
100
Time
Susceptible
Infectious

$$I_1 = \epsilon$$



1
0.9
0.8
0.7
0.6
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0.4
0.3
0.2
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20
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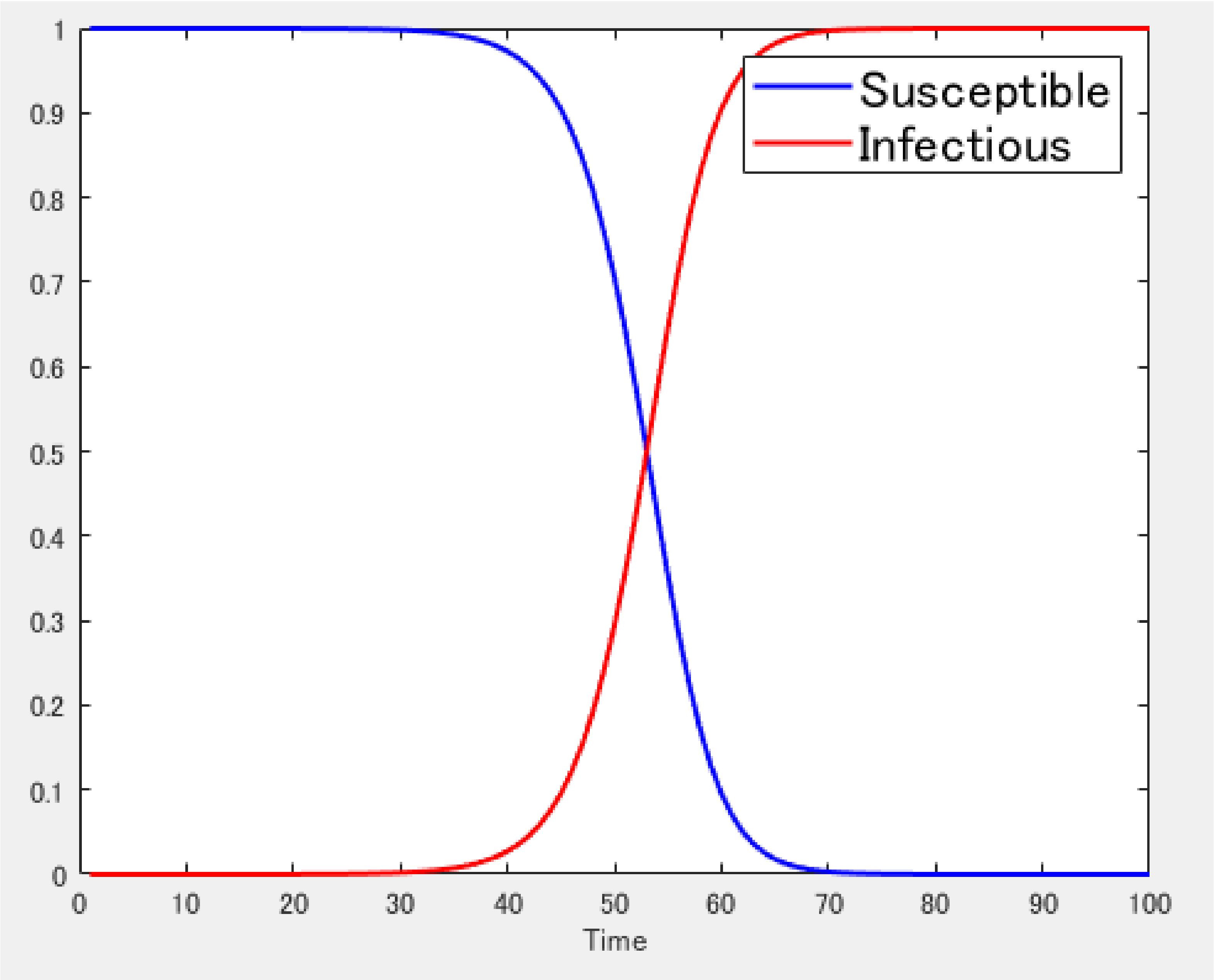
40
50
60
70
80
90
100
Time
Susceptible
Infectious

Lower γ



Susceptible
Infectious
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
80
90
100
Time

Compute **“dynamics”** (exogenous shocks)

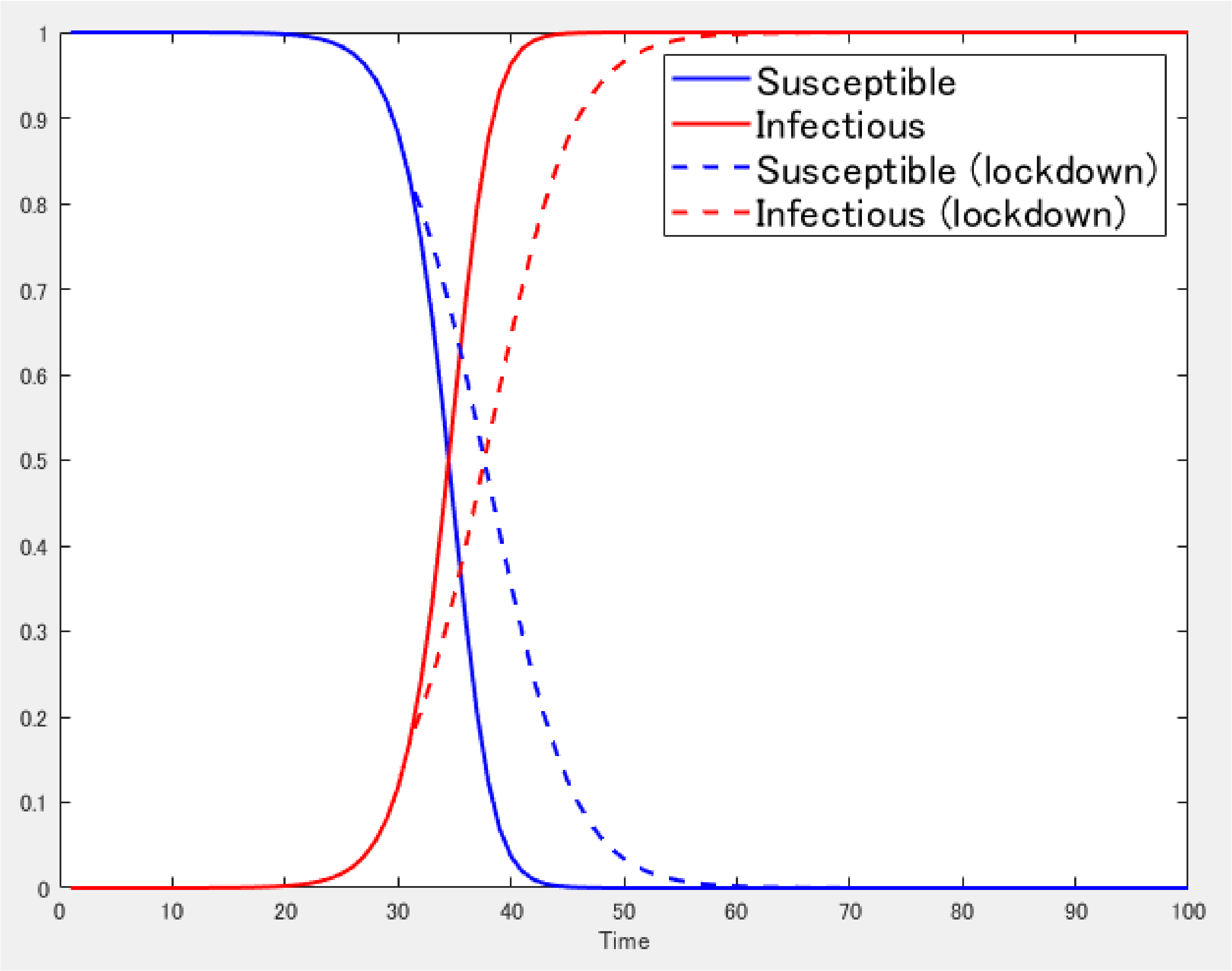
- Introduce “Lockdown” policy.
  - α_t : degree of lockdown at time t.
  - α_t is an exogenous shock.

- $S_{t+1} - S_t = -\gamma(1 - \alpha_t)I_t S_t$

- $I_{t+1} - I_t = \gamma(1 - \alpha_t)I_t S_t$

- $S_t + I_t = 1$

- Suppose that $\alpha_t = 0.5$ for 31<=t<=80.



Susceptible
Infectious
Susceptible (lockdown)
Infectious (lockdown)
Time
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50
60
70
80
90
100

Analyze “**welfare.**”

- Suppose that $Y_t = S_t(1-\alpha_t)$. Y_t is output.

- Define a per-period utility at time t as follows.

$$U_t = Y_t - \chi I_t$$

- Define welfare at time t as follows.

$$W_t = \sum_{k=0}^{\infty} \beta^k U_{t+k}$$
$$= U_t + \beta U_{t+1} + \beta^2 U_{t+2} + \beta^3 U_{t+3} + \cdots$$

- Note that the welfare can be written recursively as

$$W_t = \sum_{k=0}^{\infty} \beta^k U_{t+k}$$

$$= U_t + \beta U_{t+1} + \beta^2 U_{t+2} + \beta^3 U_{t+3} + \cdots$$

$$= U_t + \beta[U_{t+1} + \beta U_{t+2} + \beta^2 U_{t+3} + \cdots]$$

$$= U_t + \beta W_{t+1}$$

- Computing U_t is easy. Computing W_t is a bit tricky.

- Step 1: Compute the steady state utility, U_{ss}.

$$U_{ss} = Y_{ss} - \chi I_{ss}$$

- Step 2: Compute the steady state welfare, W_{ss}.

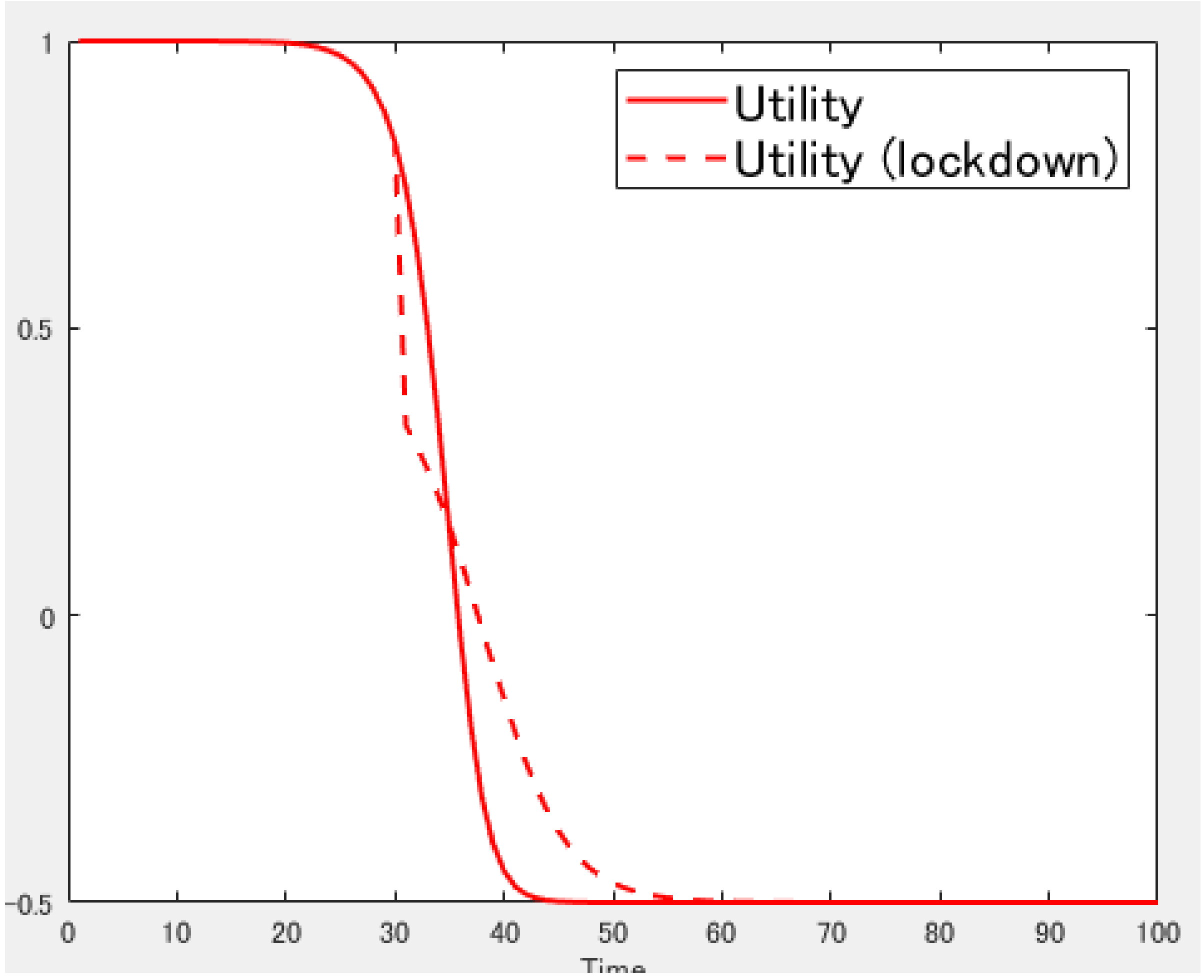
$$W_{ss} = U_{ss} + \beta W_{ss}$$
$$\Rightarrow W_{ss} = \frac{U_{ss}}{1-\beta}$$

- Step 3: Assume that, after time T, the economy is at the steady state, so that

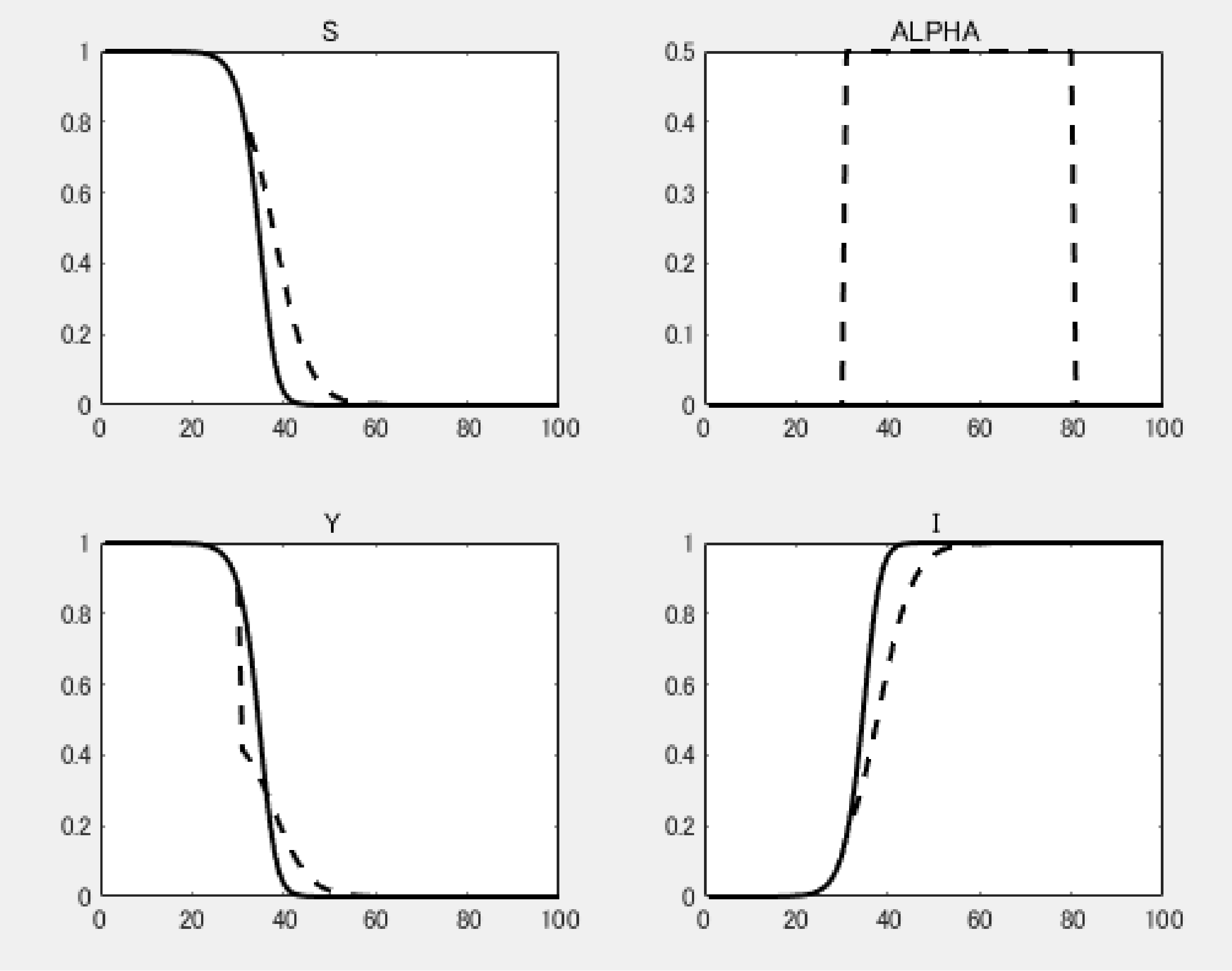
$$W_T = W_{T+1} = W_{T+2} = \cdots = W_{ss}$$

- Step 4: Recursively compute welfare from T-1 to 1.

$$W_{T-1} = U_{T-1} + \beta W_t$$
$$W_{T-2} = U_{T-2} + \beta W_{t-1}$$
$$W_{T-3} = U_{T-3} + \beta W_{t-2}$$
$$\dots$$
$$W_1 = U_1 + \beta W_2$$



Utility
Utility (lockdown)
1
0.5
0
-0.5
0
10
20
30
40
50
60
70
80
90
100
Time


S
ALPHA
Y
I
1
0.8
0.6
0.4
0.2
0
0.5
0.4
0.3
0.2
0.1
0
20
40
60
80
100

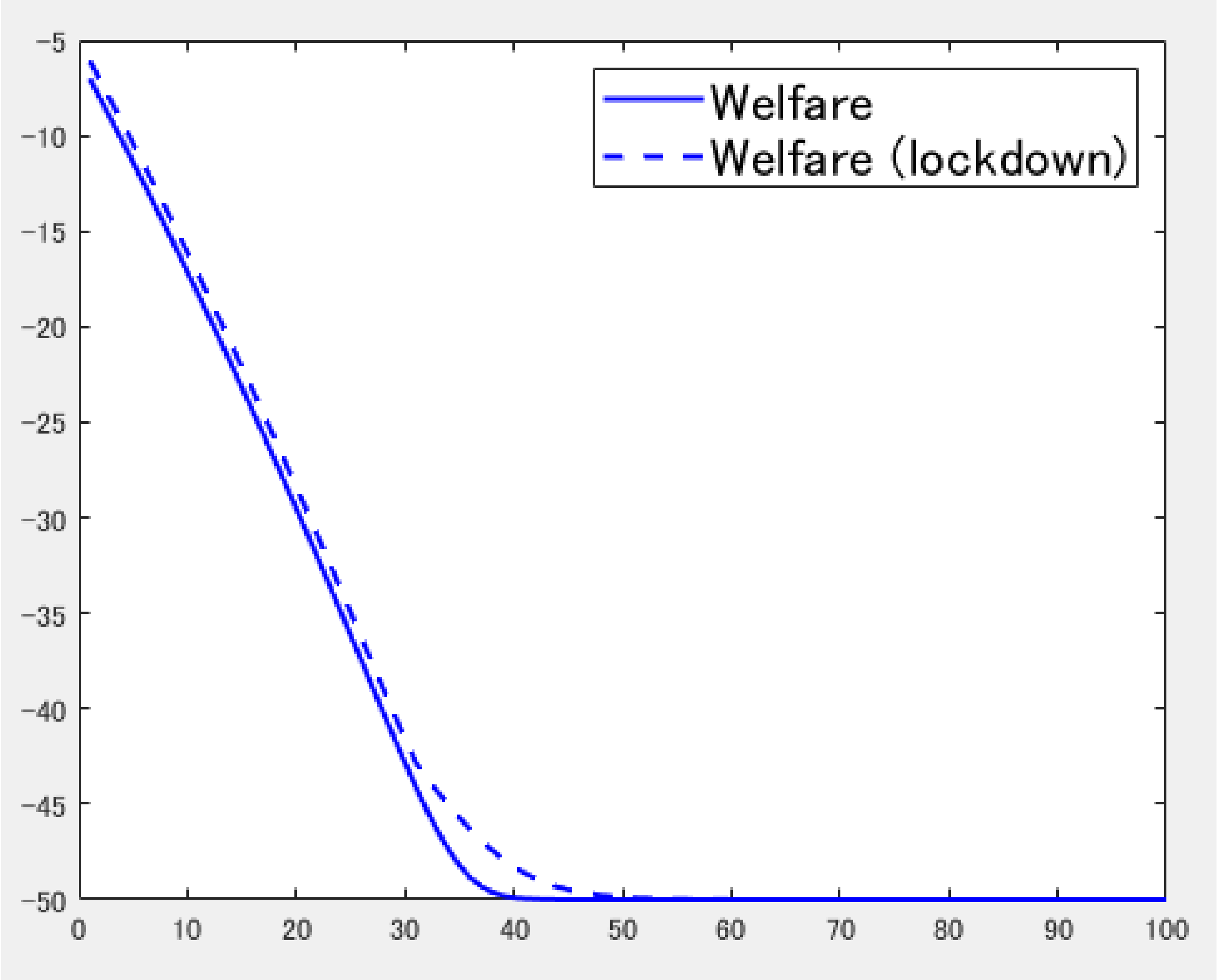
$$U_t = Y_t - \chi I_t = S_t(1 - \alpha_t) - \chi I_t$$
$$S_{t+1} - S_t = -\gamma(1 - \alpha_t) I_t S_t$$
$$I_{t+1} - I_t = \gamma(1 - \alpha_t) I_t S_t$$

- Higher α (more intense lockdown) means
  - less output (Y) today
  - less infection (I) in the future
  - More susceptible people and thus higher output in the future.

- This is an example of an **intertemoral** tradeoff.
  - An action that worsens (improves) today's utility improves (worsens) future utility flows.

- The opposite of intertemporal tradeoff is intra-temporal (or static) tradeoff.
  - In the SI model, there is no intratempotal tradeoff.

- You will see different kinds of intra- and inter-temporal tradeoffs throughout the course.
  - Today's consumption versus future consumption (inter-).
  - Today's consumption versus today's leisure (intra-).
  - Today's inflation versus future inflation (inter-).
  - Today's inflation versus today's output gap (intra-).

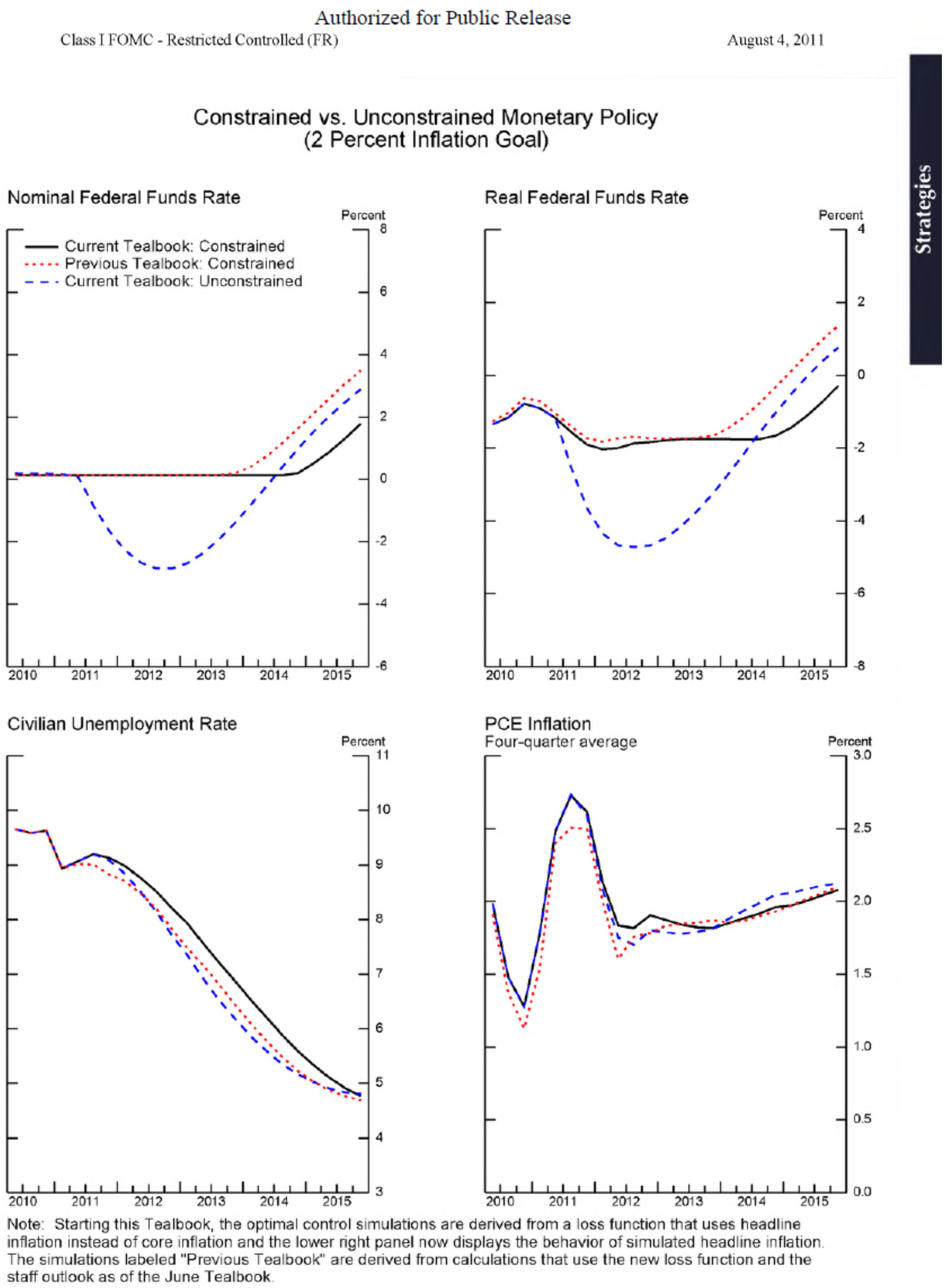

Welfare
Welfare (lockdown)
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
0
10
20
30
40
50
60
70
80
90
100

- In this example, lockdown policy improves welfare at **all time t**.

- In particular, government can improve welfare by implementing the lockdown at time 31.

- You could also analyze the optimal timing and intensity of lockdown in this framework, though that would be a bit technically challenging in this model.
    - Easier in the New Keynesian model.

Normative question



https://www.federalreserve.gov/monetarypolicy/files/FOMC20110809tealbookb20110804.pdf

- There are cases in which a certain policy improves welfare at the time of implementation, and but not for all dates in the future.
  - In the example above, that could happen if the solid line and dashed line crossed at some future date.

- Such policies are said to be "**time-inconsistent**" and thought to be hard to implement. Examples of time-inconsistent policy includes:
  - Bailout policy for banks by the financial regulator.
  - Inflation overshooting policy by the central bank.

Speech

March 08, 2019

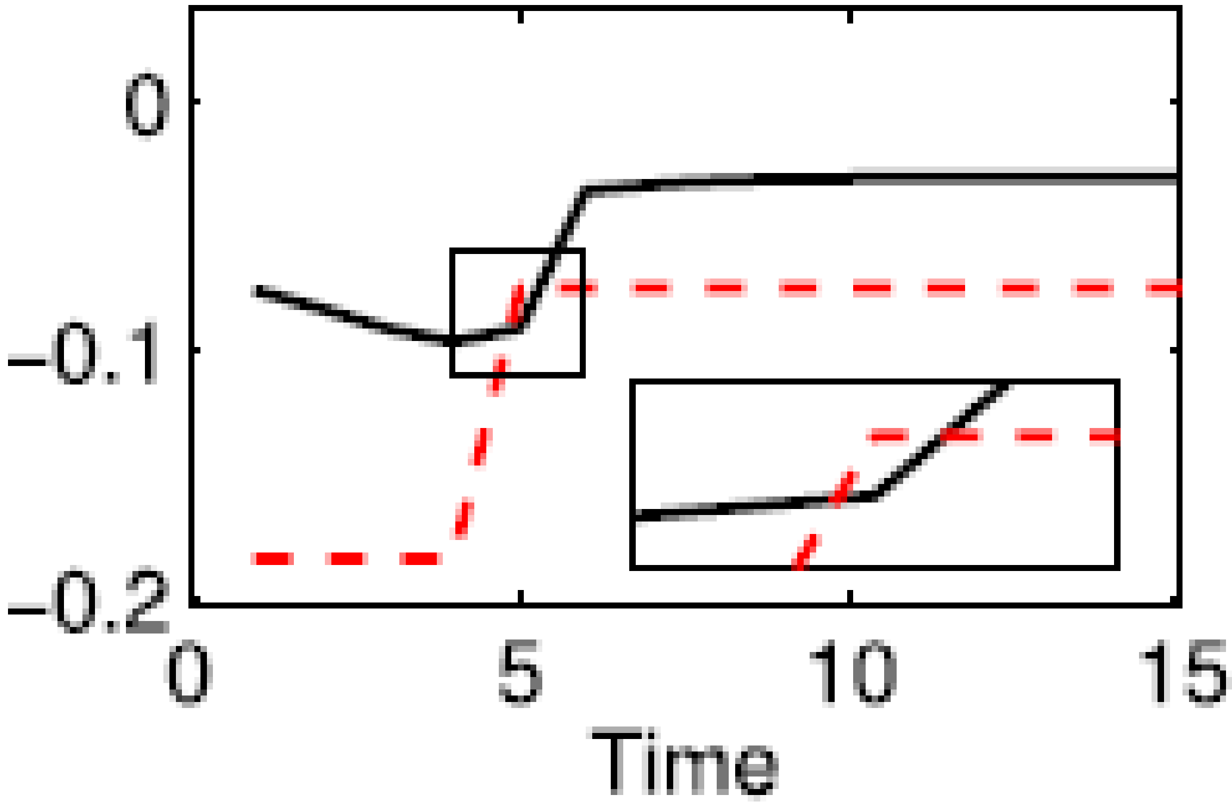
Monetary Policy: Normalization and the Road Ahead

Chair Jerome H. Powell

At the 2019 SIEPR Economic Summit, Stanford Institute of Economic Policy Research, Stanford, California

By the time of the crisis, there was a well-established body of model-based research suggesting that some kind of makeup policy could be beneficial.[12] In light of this research, one might ask why the Fed and other major central banks chose not to pursue such a policy.[13] The answer lies in the uncertain distance between models and reality. For makeup strategies to achieve their stabilizing benefits, households and businesses must be quite confident that the "makeup stimulus" is really coming. This confidence is what prompts them to raise spending and investment in the midst of a downturn. In models, confidence in the policy is merely an assumption. In practice, when policymakers considered these policies in the wake of the crisis, they had major questions about whether a central bank's promise of good times to come would have moved the hearts, minds, and pocketbooks of the public. Part of the problem is that when the time comes to deliver the inflationary stimulus, that policy is likely to be unpopular--what is known as the time consistency problem in economics.[14]

Welfare evolution associated with the inflation overshooting policy by the central bank.



Outline

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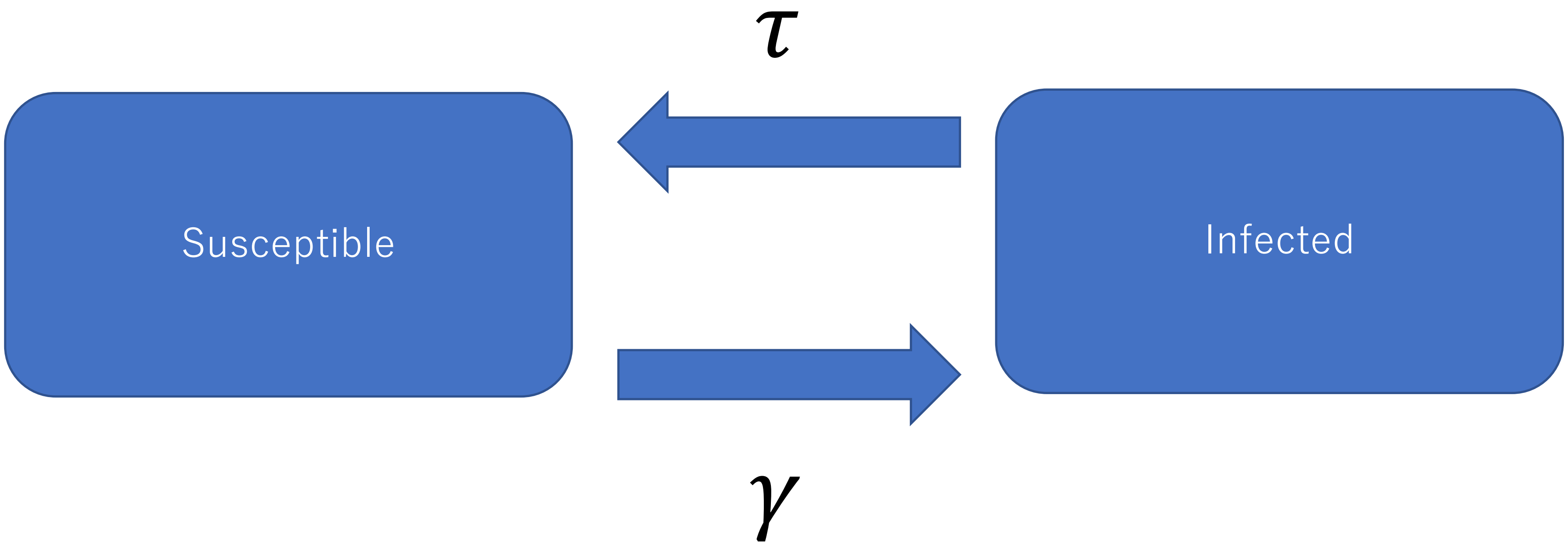
SIS Model



SIS Model

- Malaria, common cold, and chlamydia, etc.
- After you get infected, you can recover, but without obtaining immunity (so that you can get infected again).
- Everyone can get infected.
- A backward-looking model (easy to solve)
- Not optimization based.

- S_t : Number of people susceptible to the disease at time t.
- I_t : Number of people with the disease at time t.
- Normalization: $S_t + I_t = 1$.

- With probability γI_t, susceptible person at time t becomes infected at time $t + 1$.
- With probability τ, infected person at time t recover without obtaining immunity at time t + 1.
- Initial condition: $I_1 = \epsilon$.

- $S_{t+1} - S_t = -\gamma I_t S_t + \tau I_t$

- $I_{t+1} - I_t = \gamma I_t S_t - \tau I_t$

- $S_t + I_t = 1$

Find **"steady states."**

- $S_{ss} - S_{ss} = -\gamma I_{ss} S_{ss} + \tau I_{ss}$

- $I_{ss} - I_{ss} = \gamma I_{ss} S_{ss} - \tau I_{ss}$

- $S_{ss} + I_{ss} = 1$

Thus, at a steady state, we have

- $0 = -\gamma I_{ss} S_{ss} + \tau I_{ss}$
- $0 = \gamma I_{ss} S_{ss} - \tau I_{ss}$
- $S_{ss} + I_{ss} = 1$

- The first two equations are the same in the sense that if one is satisfied, the other is also satisfied.

- Eliminate I_{ss} to obtain

  - $0 = \gamma I_{ss}(1 - I_{ss}) - \tau I_{ss}$

- $0 = \gamma I_{ss}(1 - I_{ss}) - \tau I_{ss}$
  $= I_{ss}(\gamma(1 - I_{ss}) - \tau)$

- There are two steady states.
  - One steady state where $S_{ss} = \frac{\tau}{\gamma}$ and $I_{ss} = 1 - \frac{\tau}{\gamma}$.
  - The other steady state where $S_{ss} = 1$ and $I_{ss} = 0$.

Compute "**dynamics**" (convergence toward steady state).

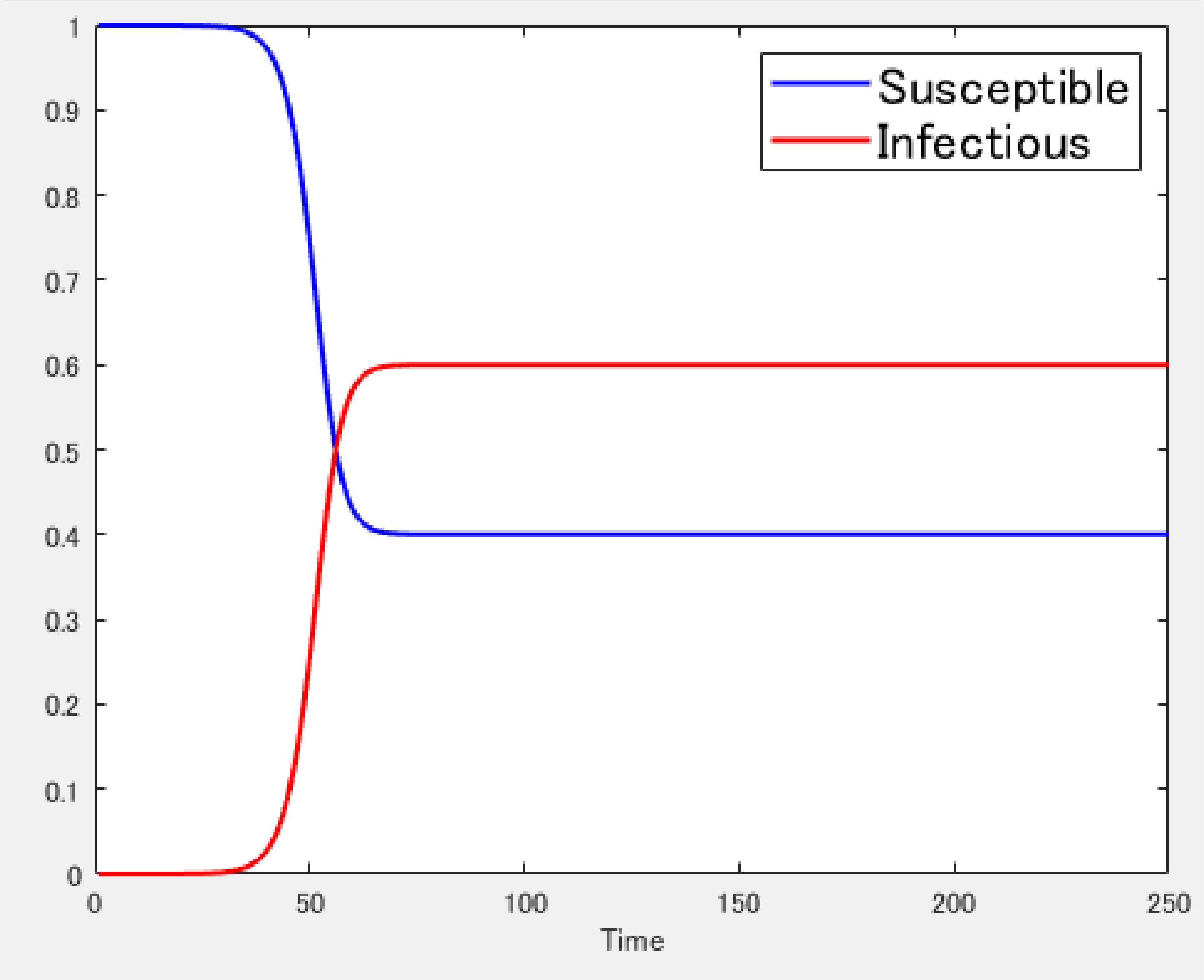
- Suppose that $I_1 = \epsilon$ (i.e. $S_1 = 1 - \epsilon$) .

- Compute $\{I_t\ , S_t\}_{t=2}^{\infty}$

- How? Recursively.
  - $S_2 - S_1 = -\gamma I_1 S_1 + \tau I_{ss}$
  - $I_2 - I_1 = \gamma I_1 S_1 - \tau I_{ss}$
  - $S_1\ +\ I_1\ =\ 1$

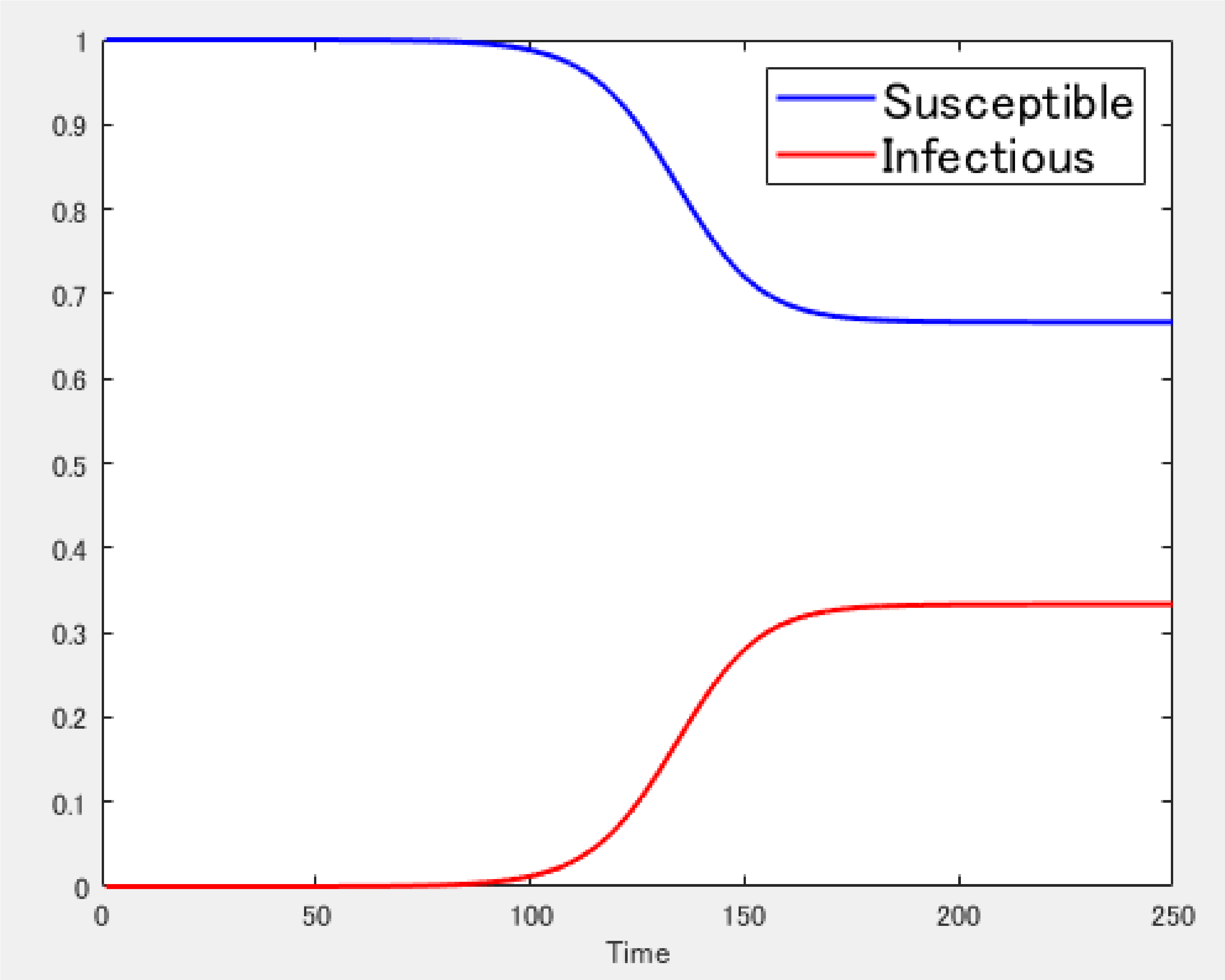
- $S_2 = (1 - \epsilon)\ - \gamma\epsilon(1 - \epsilon) + \tau\epsilon = (1 - \epsilon)(1 - \gamma\epsilon) + \tau\epsilon$
- $I_2 = \ \ \epsilon + \gamma\epsilon(1 - \epsilon) - \tau\epsilon = \epsilon\big(1 + \gamma(1 - \epsilon)\big) - \tau\epsilon$

- and so on⋯



Susceptible
Infectious
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
50
100
150
200
250
Time

Lower γ



Susceptible
Infectious
Time
0
50
100
150
200
250
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Compute “**dynamics**” (exogenous shocks).

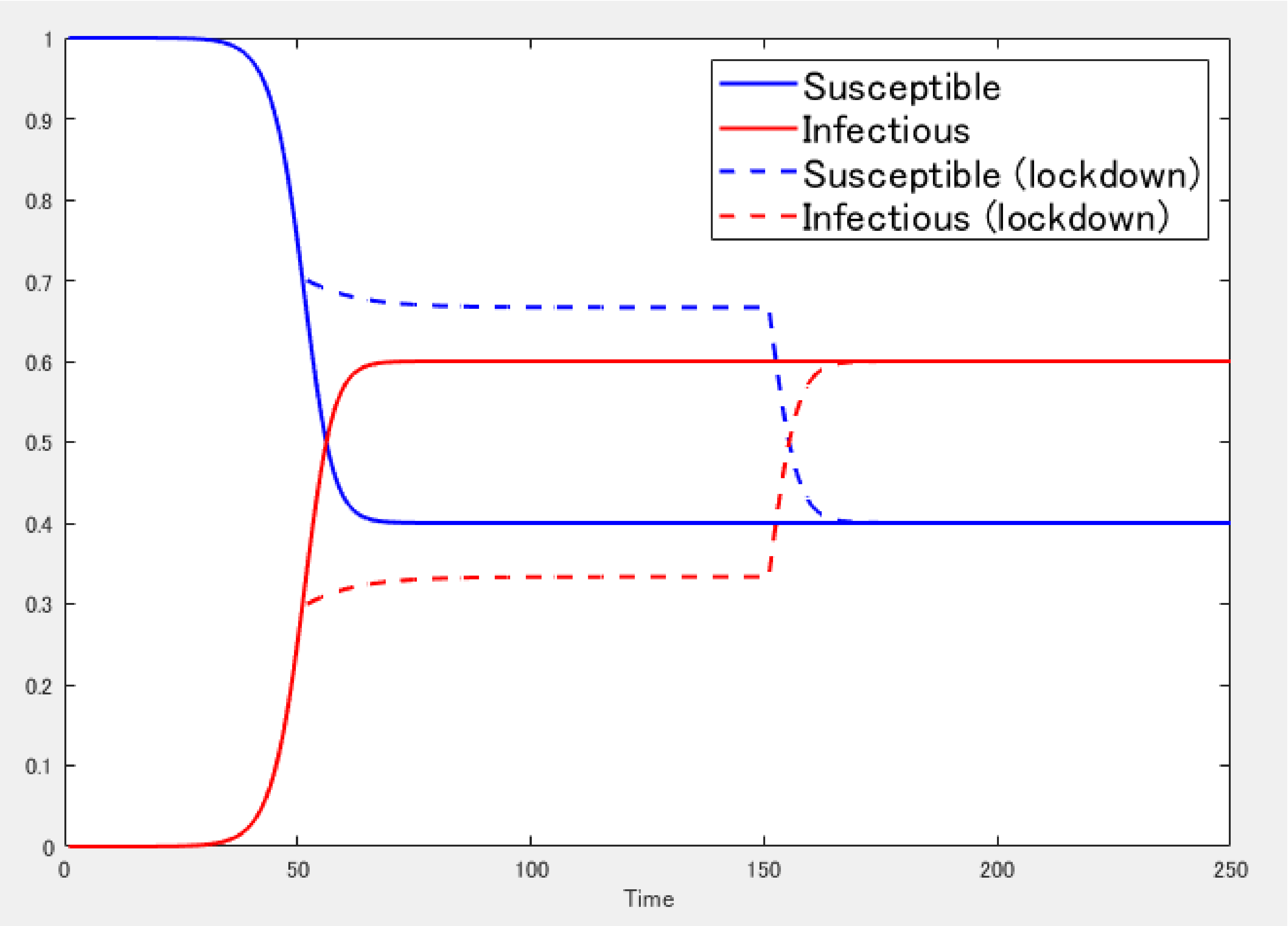
- Introduce “Lockdown” policy.
  - α_t : degree of lockdown at time t.

- $S_{t+1} - S_t = -\gamma(1-\alpha_t)I_t S_t + \tau I_{ss}$

- $I_{t+1} - I_t = \gamma(1-\alpha_t)I_t S_t - \tau I_{ss}$

- $S_t + I_t = 1$

- Suppose that $\alpha_t = 0.4$ for 51<=t<=150.



Susceptible
Infectious
Susceptible (lockdown)
Infectious (lockdown)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
50
100
150
200
250
Time

Analyze "**welfare.**"

- Suppose that $Y_t = S_t(1-\alpha_t)$. Y_t is output.

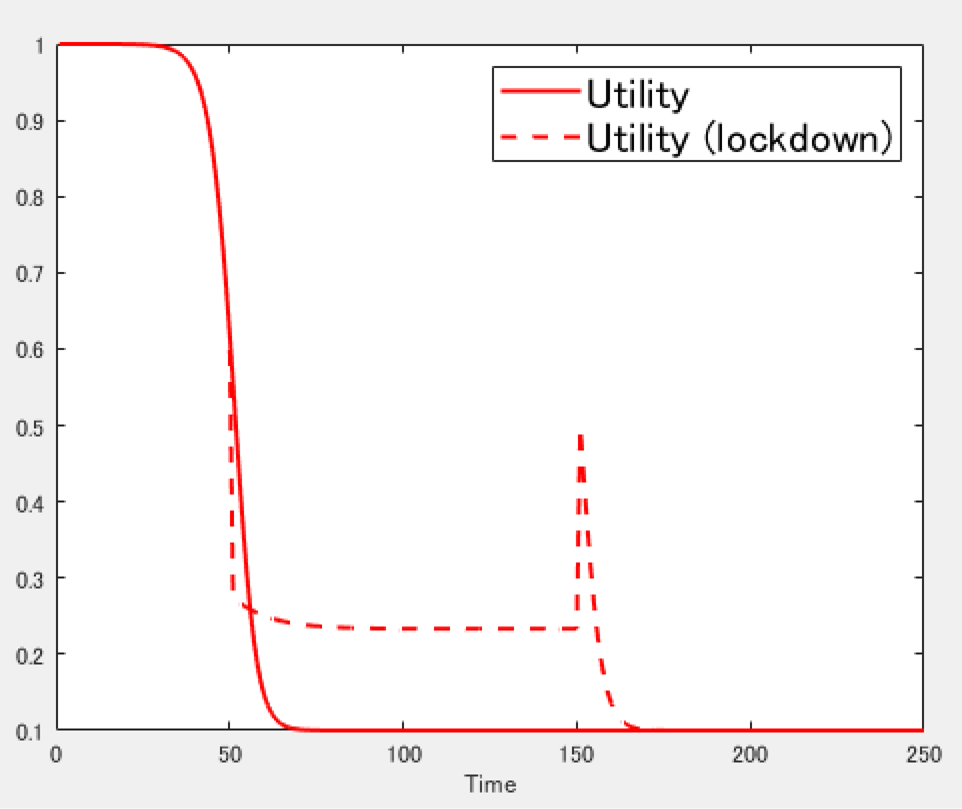
- Define a per-period utility at time t as follows.

$$U_t = Y_t - \chi I_t$$

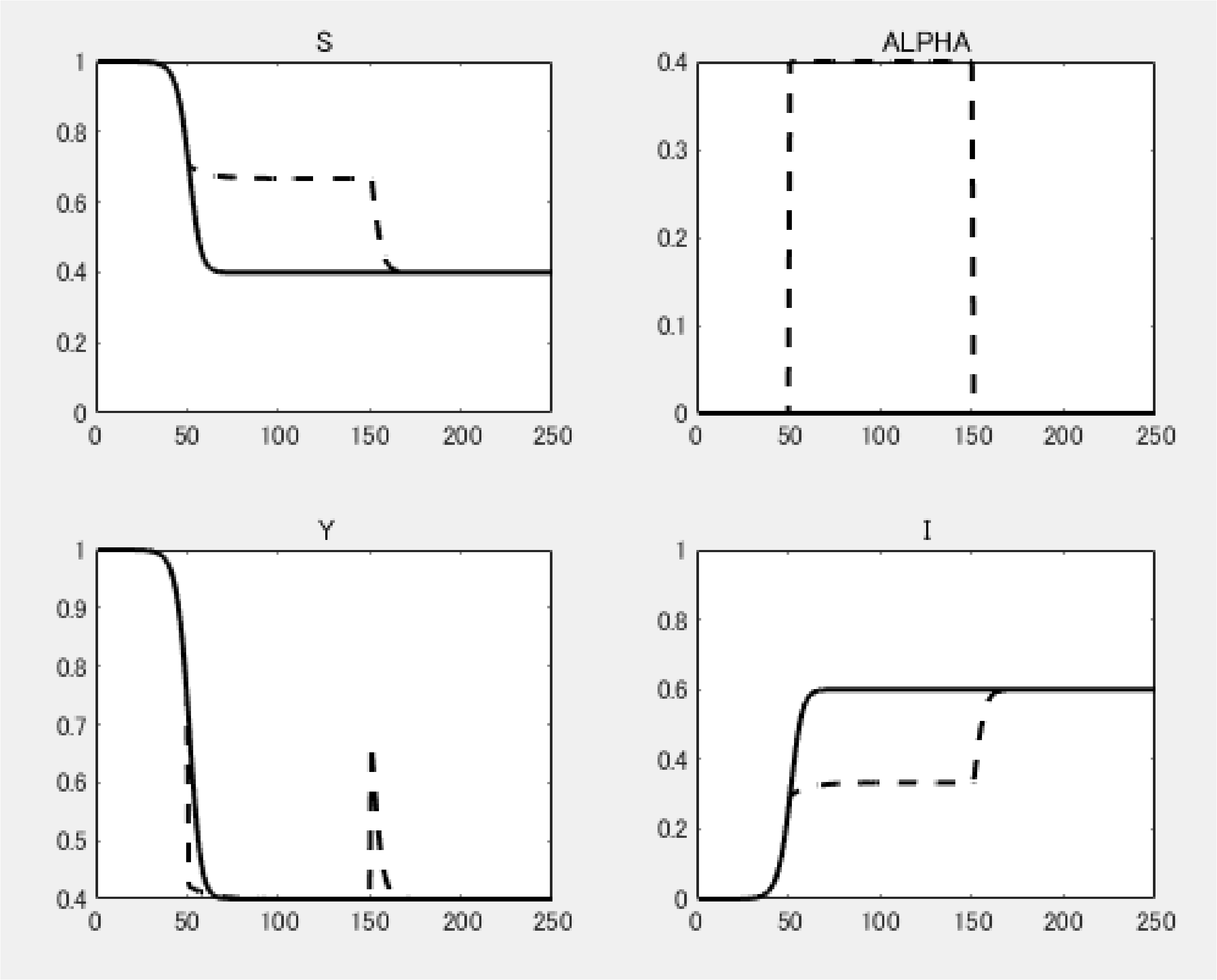
- Define welfare at time t as follows.

$$W_t = \sum_{k=0}^{\infty} \beta^k U_{t+k}$$

$$= U_t + \beta U_{t+1} + \beta^2 U_{t+2} + \beta^3 U_{t+3} + \cdots$$



Utility
Utility (lockdown)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
50
100
150
200
250
Time


S
ALPHA
Y
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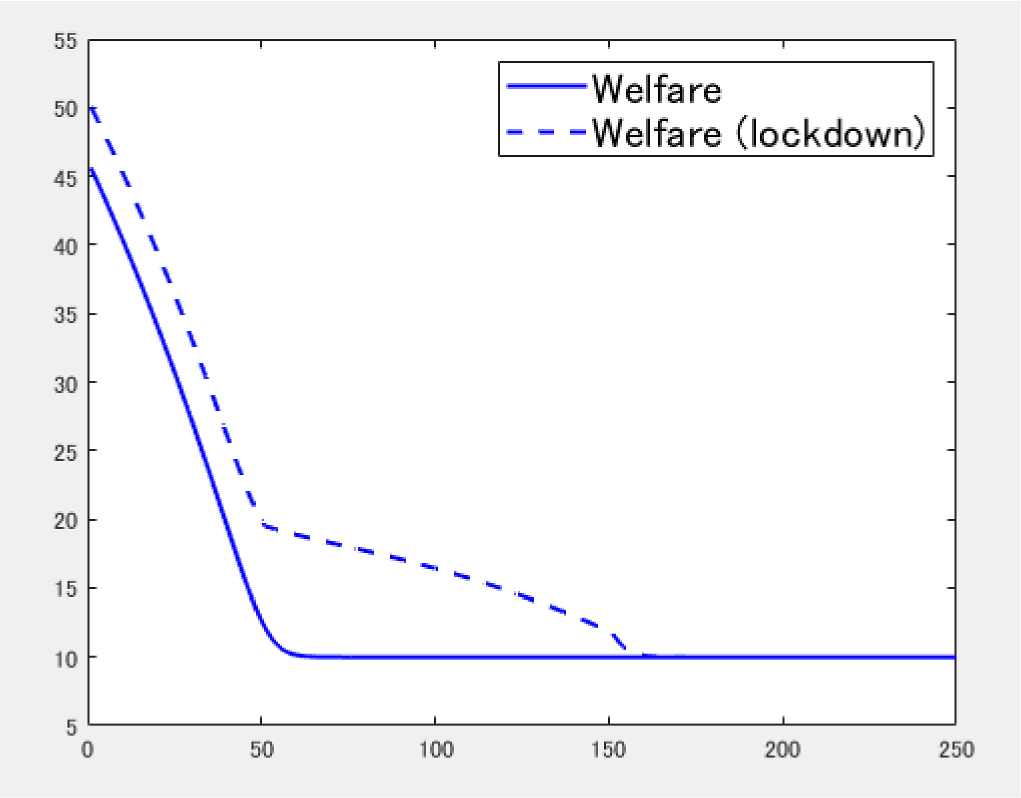
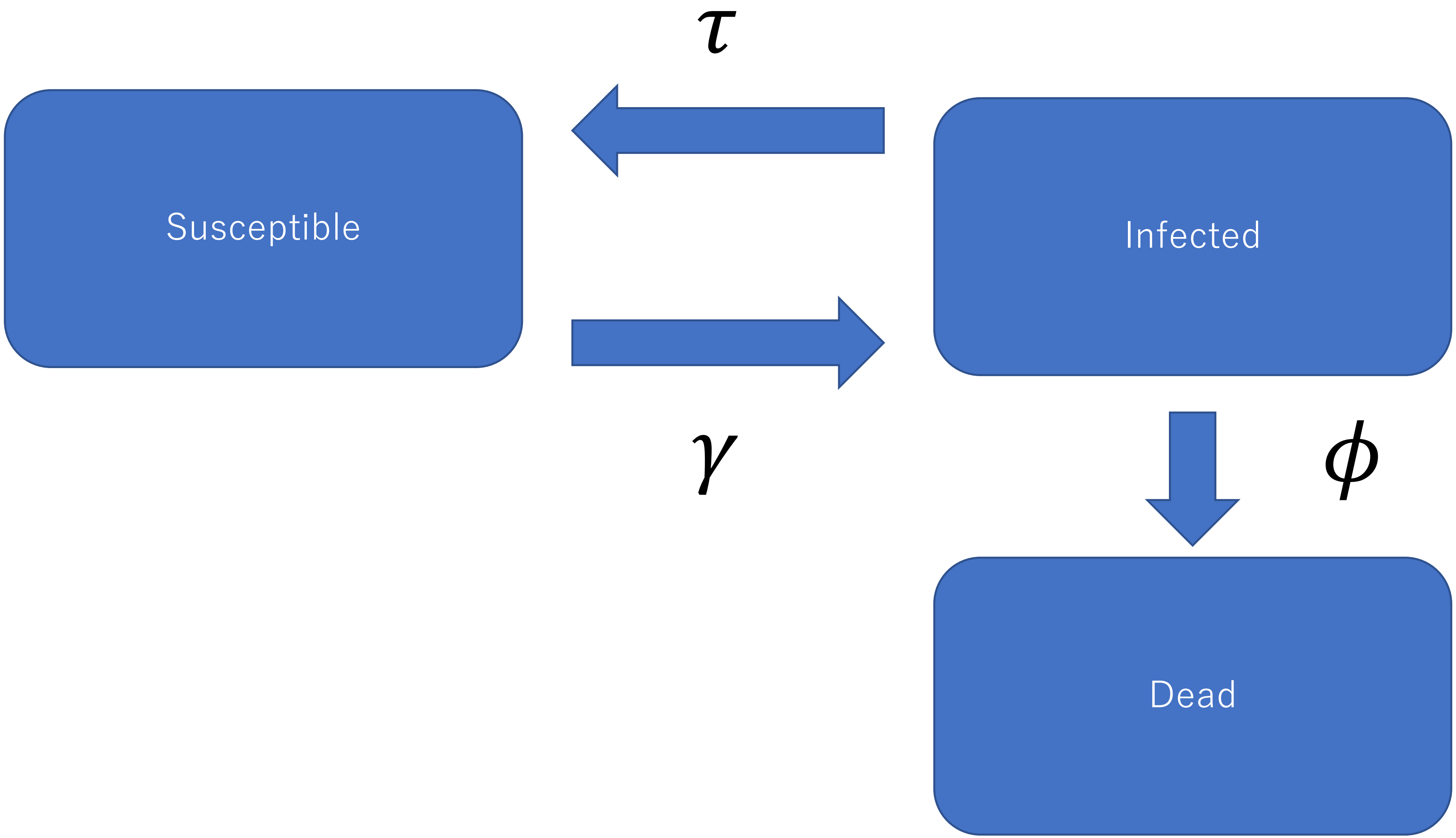

Welfare
Welfare (lockdown)
55
50
45
40
35
30
25
20
15
10
5
0
50
100
150
200
250

Exercises

1. SIS-D Model

- $S_{t+1} - S_t = -\gamma I_t S_t + \tau I_t$
- $I_{t+1} - I_t = \gamma I_t S_t - \tau I_t - \phi I_t$
- $D_{t+1} - D_t = \phi I_t$
- $S_t + I_t + D_t = 1$

SIS-D Model



- Examine the steady state of the model. If there are more than one steady state, make sure you compute all of them.

- Compute convergence to a steady state (with non-zero S).

- Analyze the effect of a lockdown policy.
    - $\alpha_t = 0.5$ for 51<=t<=150.

- Analyze the welfare effect of a lockdown policy.