Models of Infectious Diseases (II)

<Last Updated on October 14, 2020>

Taisuke Nakata

Graduate School of Public Policy
University of Tokyo

- Useful references:
  - Andrew Atkeson "On Using SIR Models to Model Disease Scenarios for COVID-19."
  - https://www.minneapolisfed.org/research/quarterly-review/on-using-sir-models-to-model-disease-scenarios-for-covid-19
  - Ben Moll "Lockdowns in SIR Models."
  - https://benjaminmoll.com/wp-content/uploads/2020/05/SIR_notes.pdf

Outline

(1) SI Model

(2) SIS Model

(3) SIR Model

SIR Model

SIR Model

- Covid-19, etc.
- After you get infected, you can recover with immunity (so that you won't get infected again).
- Initially, everyone can get infected.
- A backward-looking model (easy to solve)
- Not optimization based.

- S_t : Number of people susceptible to the disease at time t.
- I_t : Number of people with the disease at time t.
- R_t : Number of people recovered from disease at time t.
- Normalization: $S_t + I_t + R_t = 1$.

- With probability γI_t, susceptible person at time t becomes infected at time $t + 1$.
- With probability τ, infected person at time t recover without obtaining immunity at time t + 1.
- Initial condition: $I_1 = \epsilon$ and $R_1 = 0$.

- $S_{t+1} - S_t = -\gamma I_t S_t$
- $I_{t+1} - I_t = \gamma I_t S_t - \tau I_t$
- $R_{t+1} - R_t = \tau I_t$
- $S_t + I_t + R_t = 1$

Find **"steady states."**

- $S_{ss} - S_{ss} = -\gamma I_{ss} S_{ss}$

- $I_{ss} - I_{ss} = \gamma I_{ss} S_{ss} - \tau I_{ss}$

- $R_{ss} - R_{ss} = \tau I_{ss}$

- $S_{ss} + I_{ss} + R_{ss} = 1$

Thus, at a steady state, we have

- $0 = -\gamma I_{ss} S_{ss}$
- $0 = \gamma I_{ss} S_{ss} - \tau I_{ss}$
- $0 = \tau I_{ss}$
- $S_{ss} + I_{ss} + R_{ss} = 1$

- From the third equation, we have
  - $I_{ss} = 0$

- We are left with
  - $0 = \gamma * 0 * S_{ss}$
  - $S_{ss} + R_{ss} = 1$

- $0 = \gamma * 0 * S_{ss}$
- $S_{ss} + R_{ss} = 1$

- Any S_{ss} and R_{ss} satisfying the two equations above can be steady states.

- For example, $(S_{ss} = 1,\ R_{ss} = 0,\ I_{ss} = 0)$ is one steady state.

- Which steady state the economy will end up depends on the initial condition.

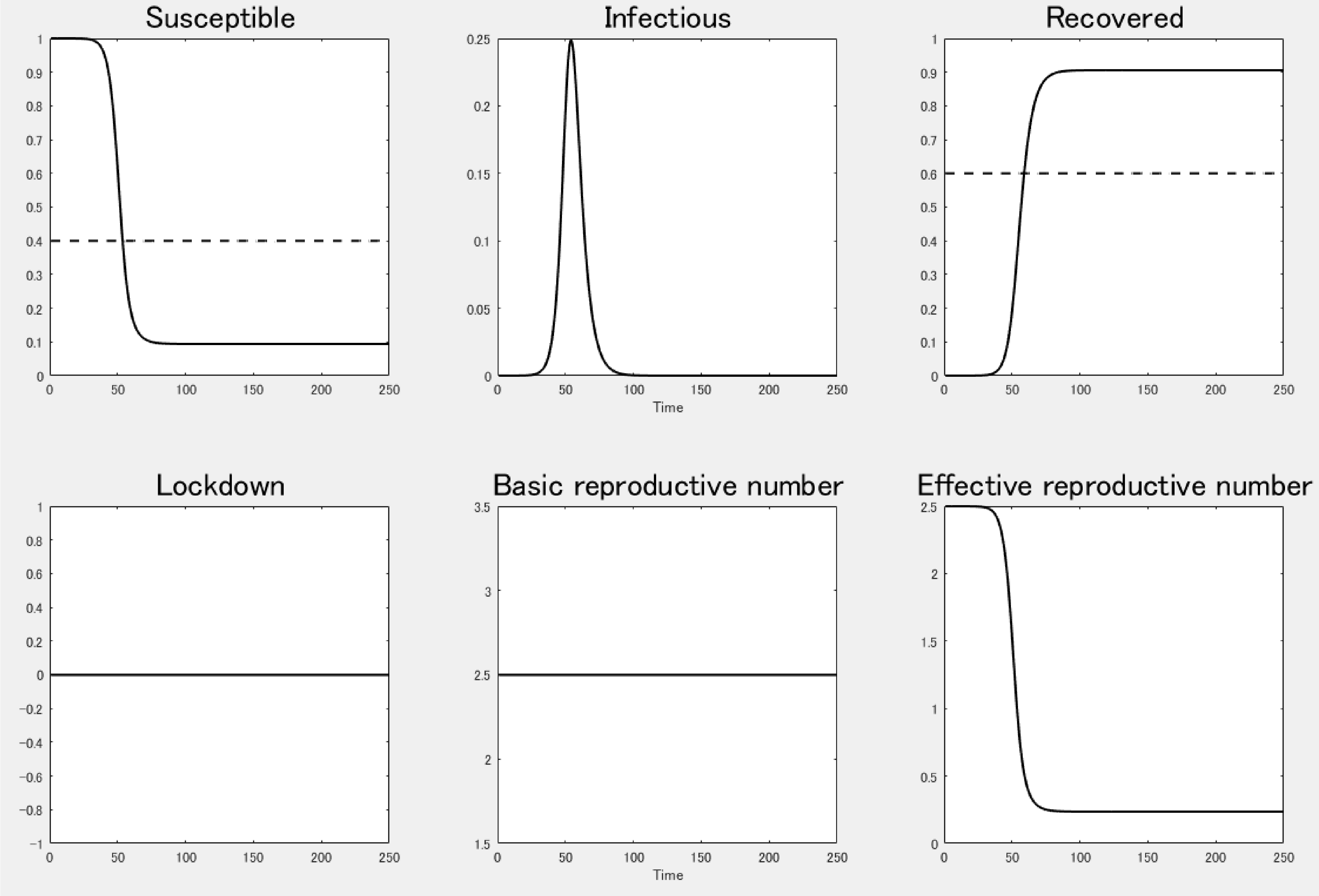
Compute **"dynamics"** (convergence towards steady state)

- Suppose that $I_1 = \epsilon$ and $R_1 = 0$ (i.e. $S_1 = 1 - \epsilon$) .

- Compute $\{I_t, S_t, R_t\}_{t=2}^{\infty}$

- How? Recursively.
  - $S_2 - S_1 = -\gamma I_1 S_1$
  - $I_2 - I_1 = \gamma I_1 S_1 - \tau I_1$
  - $R_2 - R_1 = \tau I_1$

- $S_2 = (1-\epsilon)\ - \gamma\epsilon(1-\epsilon) = (1-\epsilon)(1-\gamma\epsilon)$
- $I_2 = \ \epsilon + \gamma\epsilon(1-\epsilon) - \tau\epsilon = \epsilon\big(1+\gamma(1-\epsilon)\big) - \tau\epsilon$
- $R_2 = 0 + \ \tau\epsilon = \tau\epsilon$


Susceptible
Infectious
Recovered
Time
Lockdown
Basic reproductive number
Effective reproductive number
Time

- Basic reproductive number: $\mathrm{X}_0 = \frac{\gamma}{\tau}$

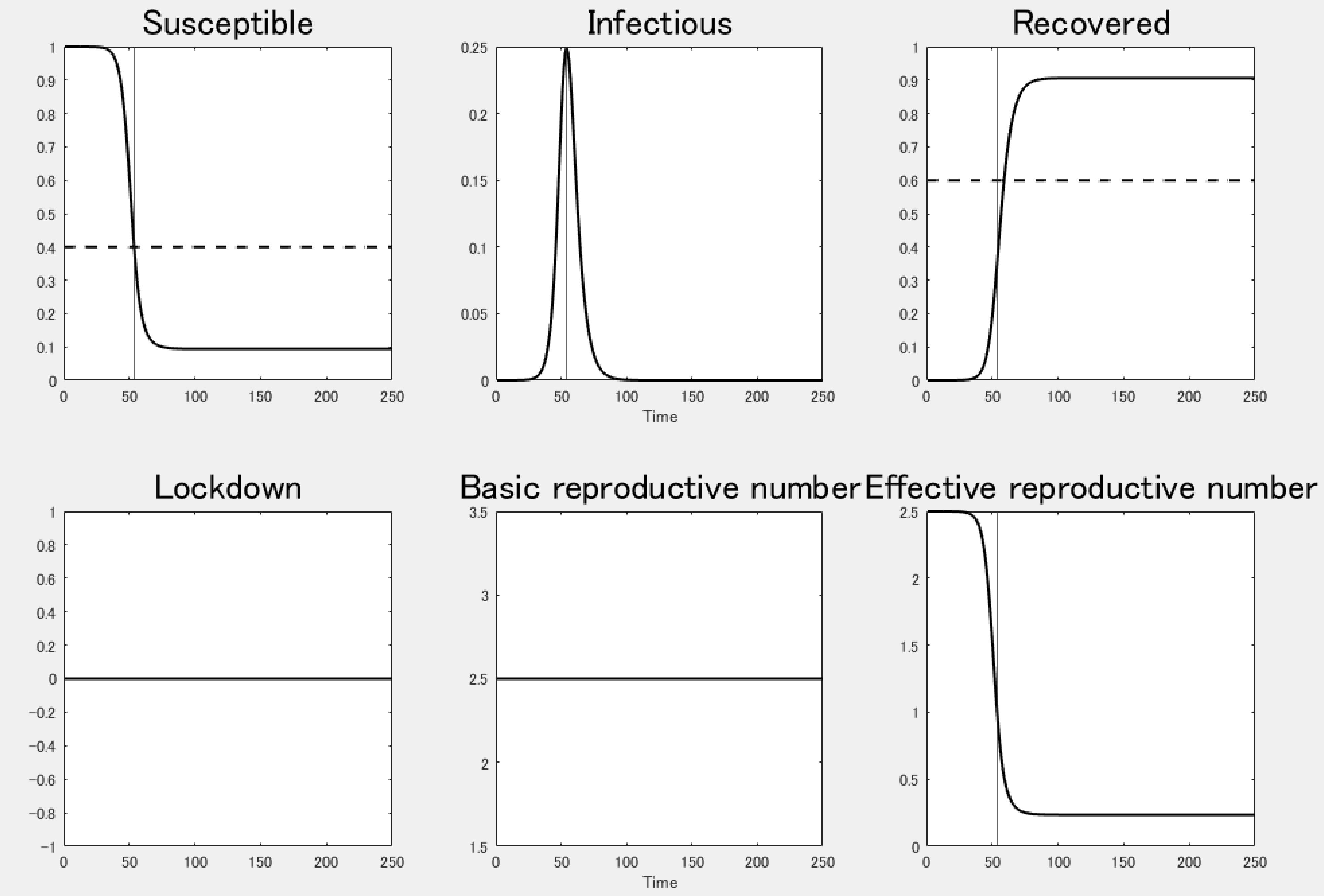
- Effective reproductive number: $\mathrm{X}_{\mathrm{t}}^{e} = \frac{\gamma}{\tau} S_t$

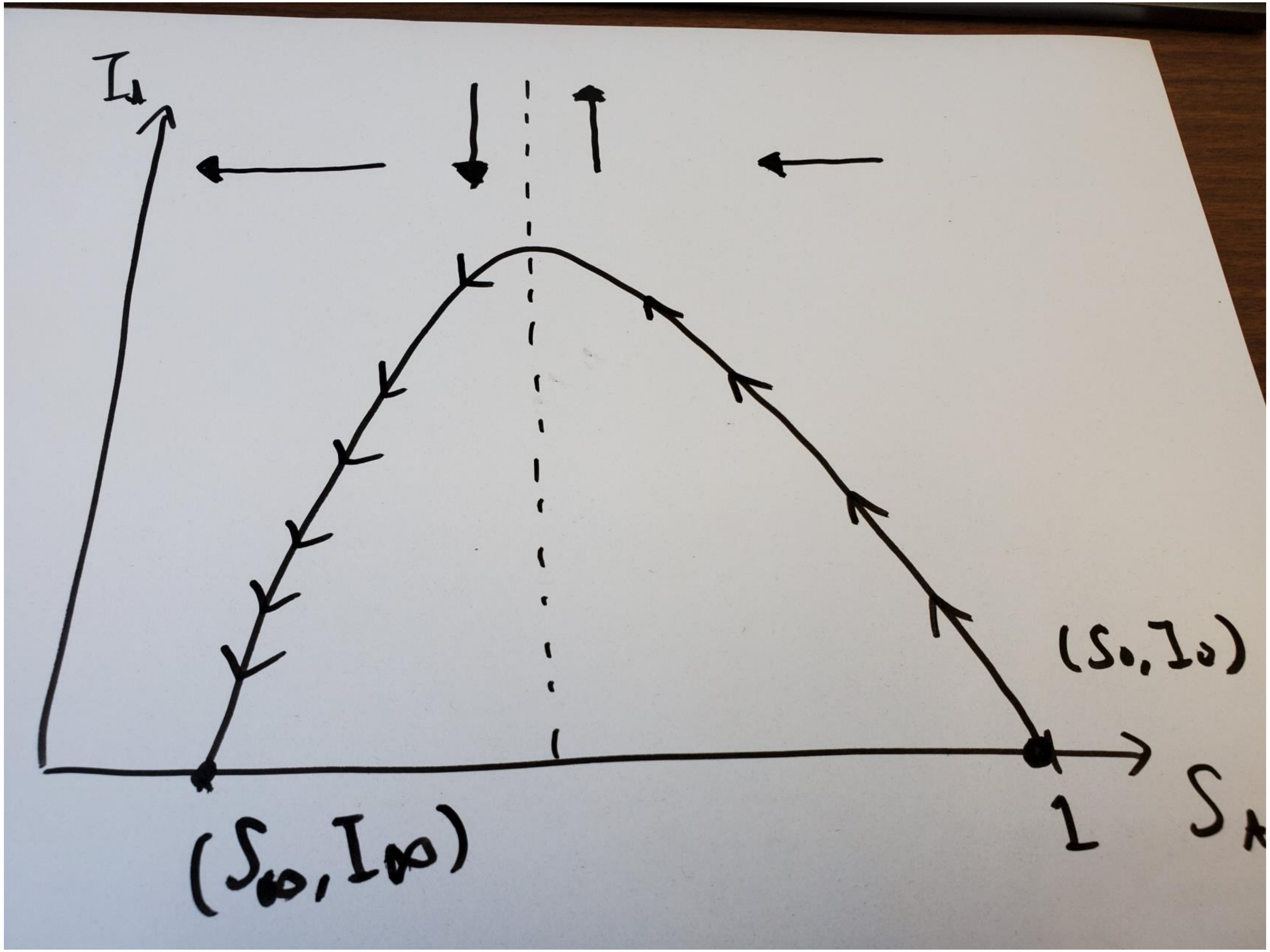
- Herd immunity threshold: $\mathrm{S}^* = \frac{1}{X_0}$ (or $\mathrm{R}^* = 1 - \mathrm{S}^*$)

- $I_{t+1} - I_t = \gamma I_t S_t - \tau I_t > 0$

$$\Rightarrow \gamma S_t - \tau > 0$$
$$\Rightarrow S_t > \frac{\tau}{\gamma} = \frac{1}{X_0} = S^*$$

- I_t peaks when $\mathrm{S_t} = S^*$.


Susceptible
Infectious
Recovered
Time
Lockdown
Basic reproductive number
Effective reproductive number
Time


I_t
(S_0, I_0)
(S_∞, I_∞)
1
S_t

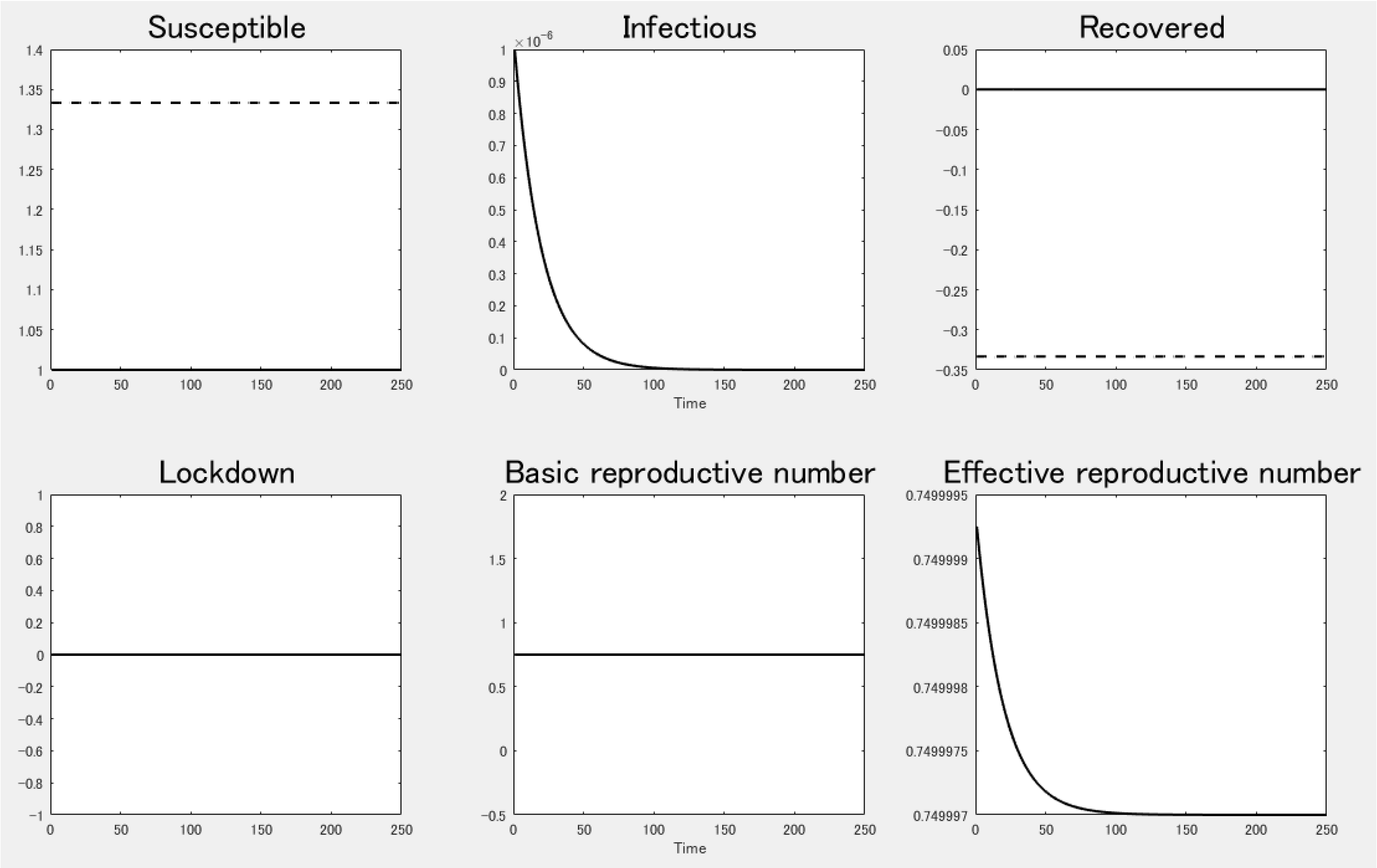
- $I_{t+1} - I_t = \gamma I_t S_t - \tau I_t < 0$

$$\Rightarrow \frac{\gamma}{\tau} < \frac{1}{S_t}$$

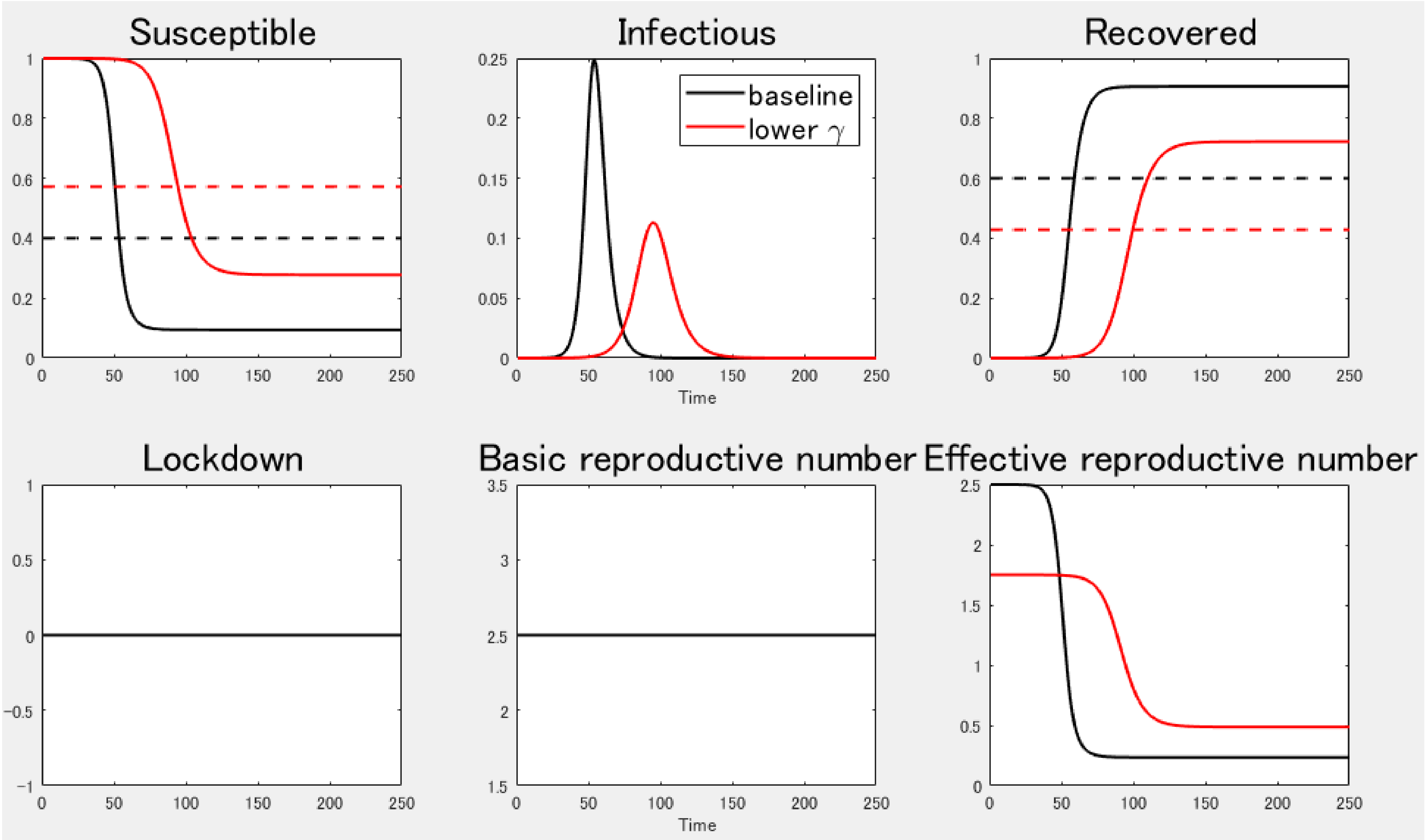
$$\Rightarrow X_0 < \frac{1}{S_t}$$

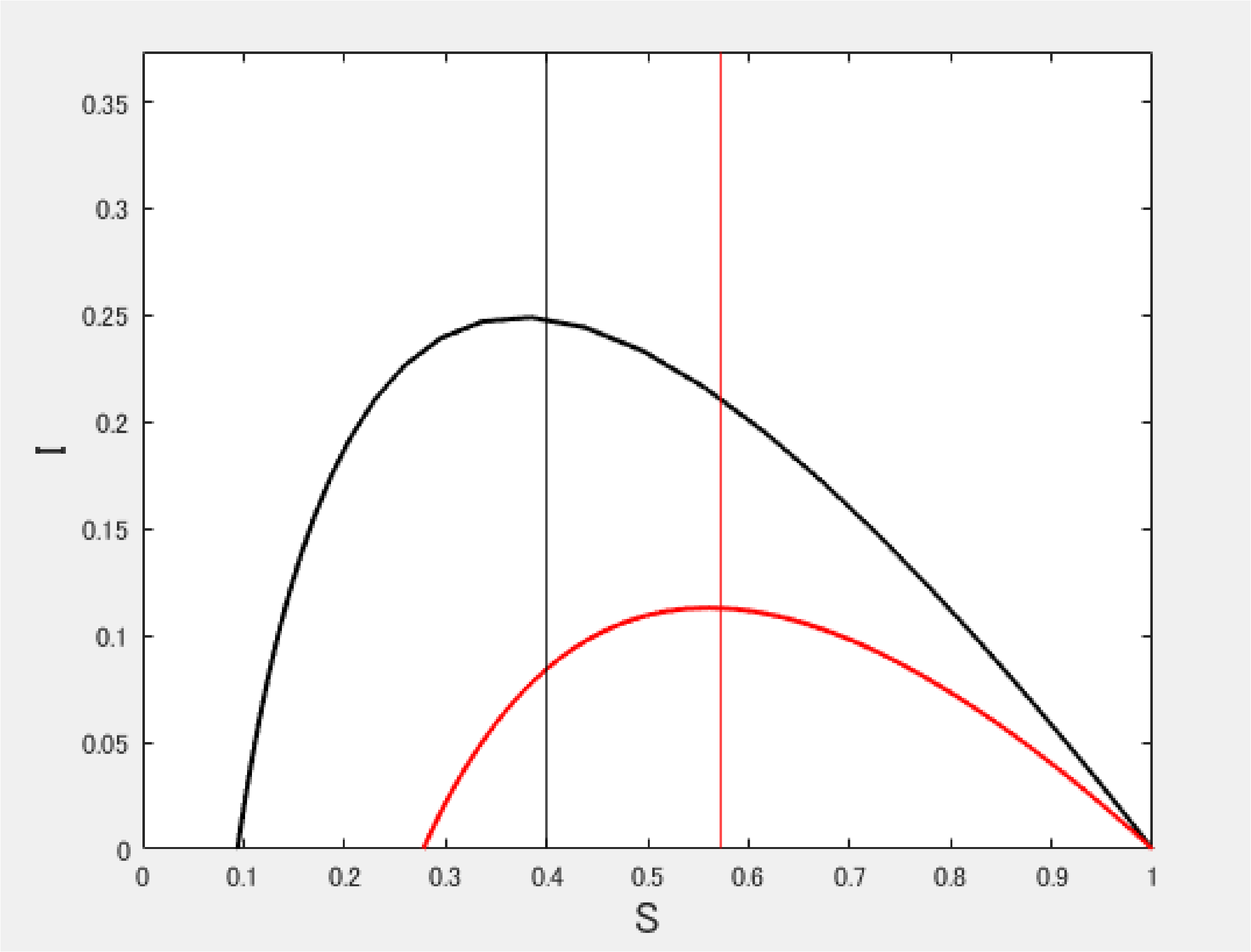
$$\Rightarrow X_0 < 1 < \frac{1}{S_t}$$

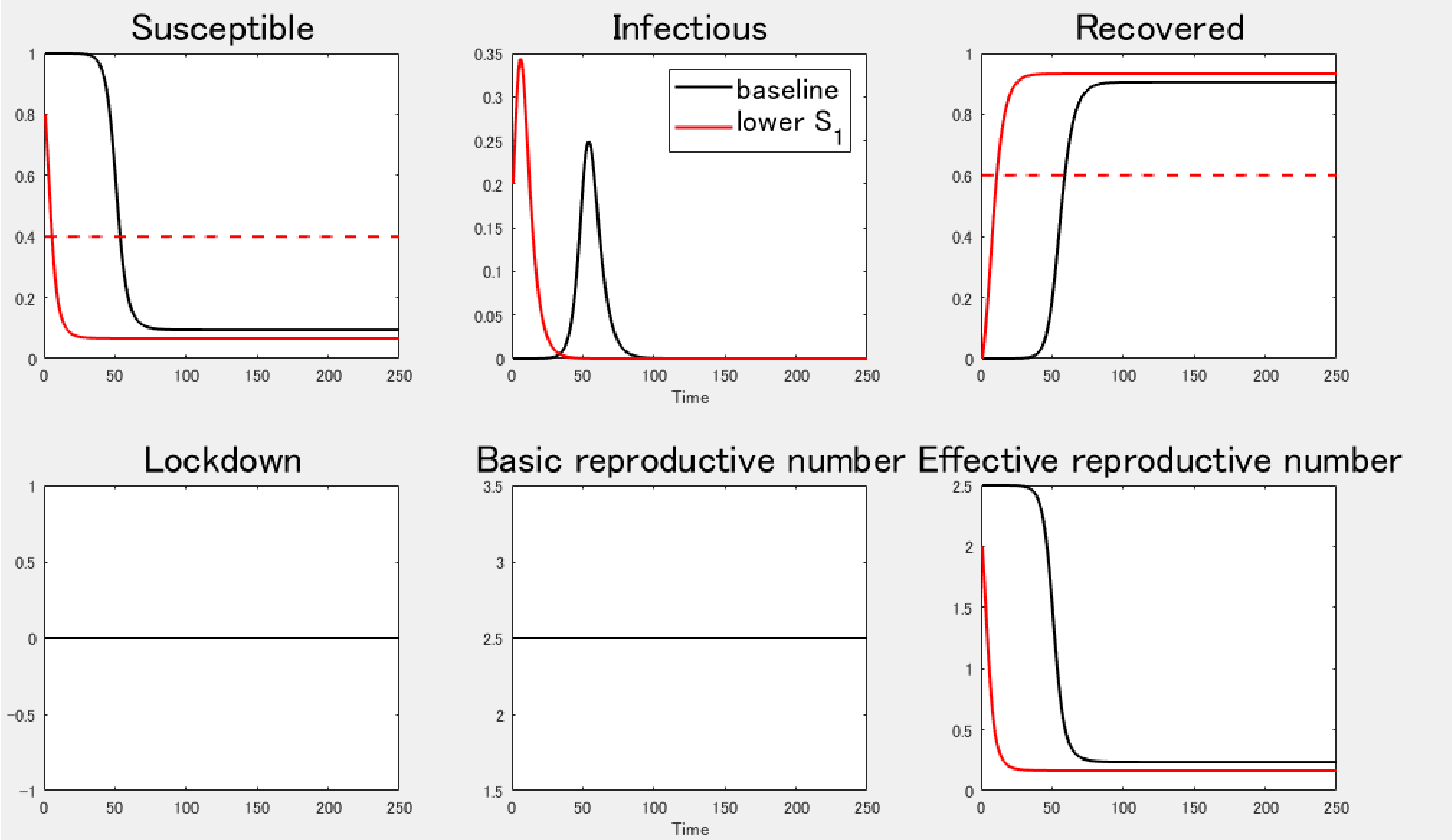
- If $X_0 < 1$, then $X_0 < \frac{1}{S_t}$ because $S_t < 1$ for any t.
- **So, if $\boldsymbol{X_0 < 1}$, $\boldsymbol{I_t}$ declines at any time.**



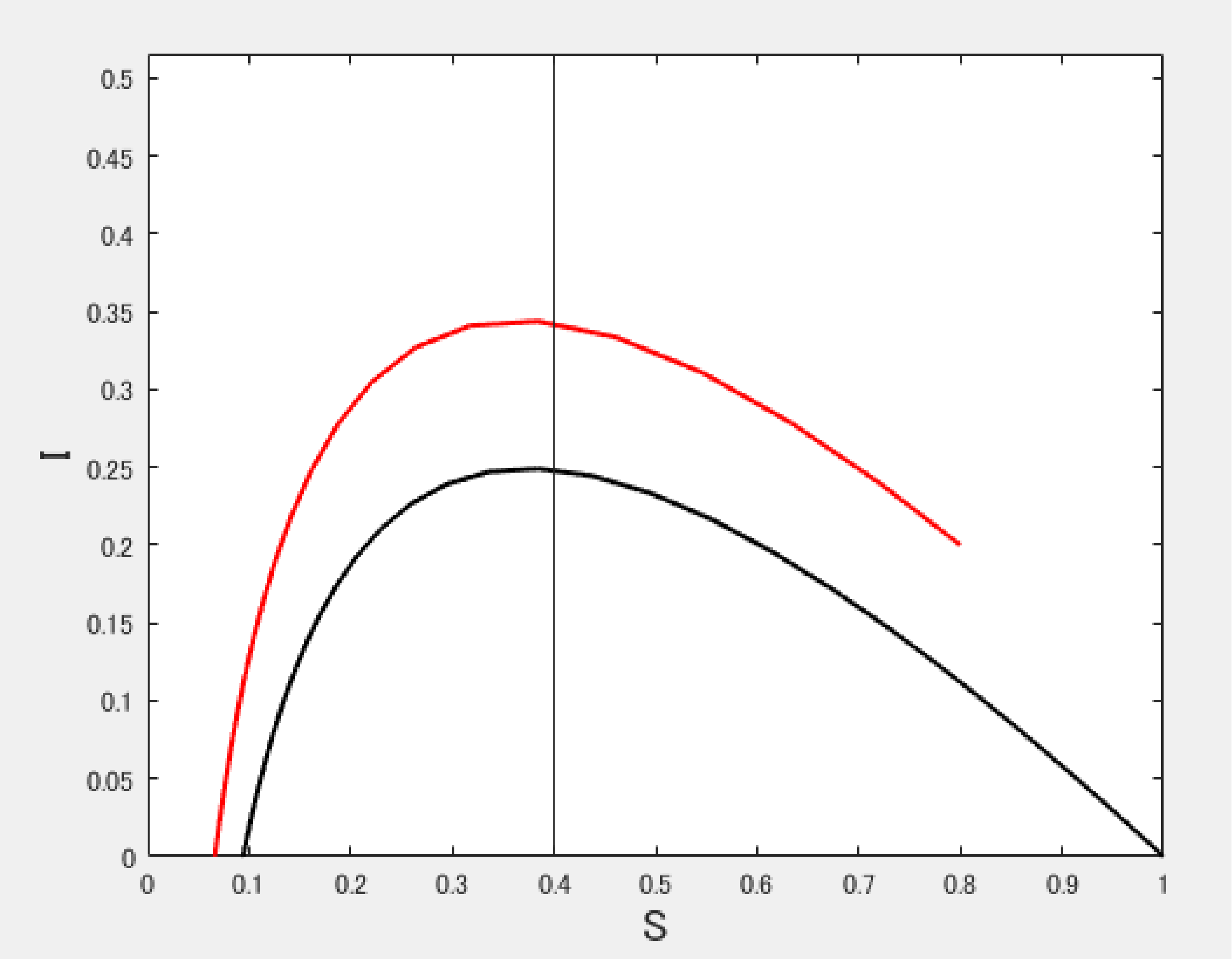
Susceptible
Infectious
Recovered
Time
Lockdown
Basic reproductive number
Effective reproductive number
Time


Susceptible
Infectious
Recovered
baseline
lower γ
Time
Lockdown
Basic reproductive number
Effective reproductive number
Time


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
I
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
S



Susceptible
Infectious
Recovered
baseline
lower S_1
Time
Lockdown
Basic reproductive number
Effective reproductive number
Time



0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
I
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
S

Compute **“dynamics”** (exogenous shocks)

- Introduce “Lockdown” policy.
  - α_t : degree of lockdown at time t.

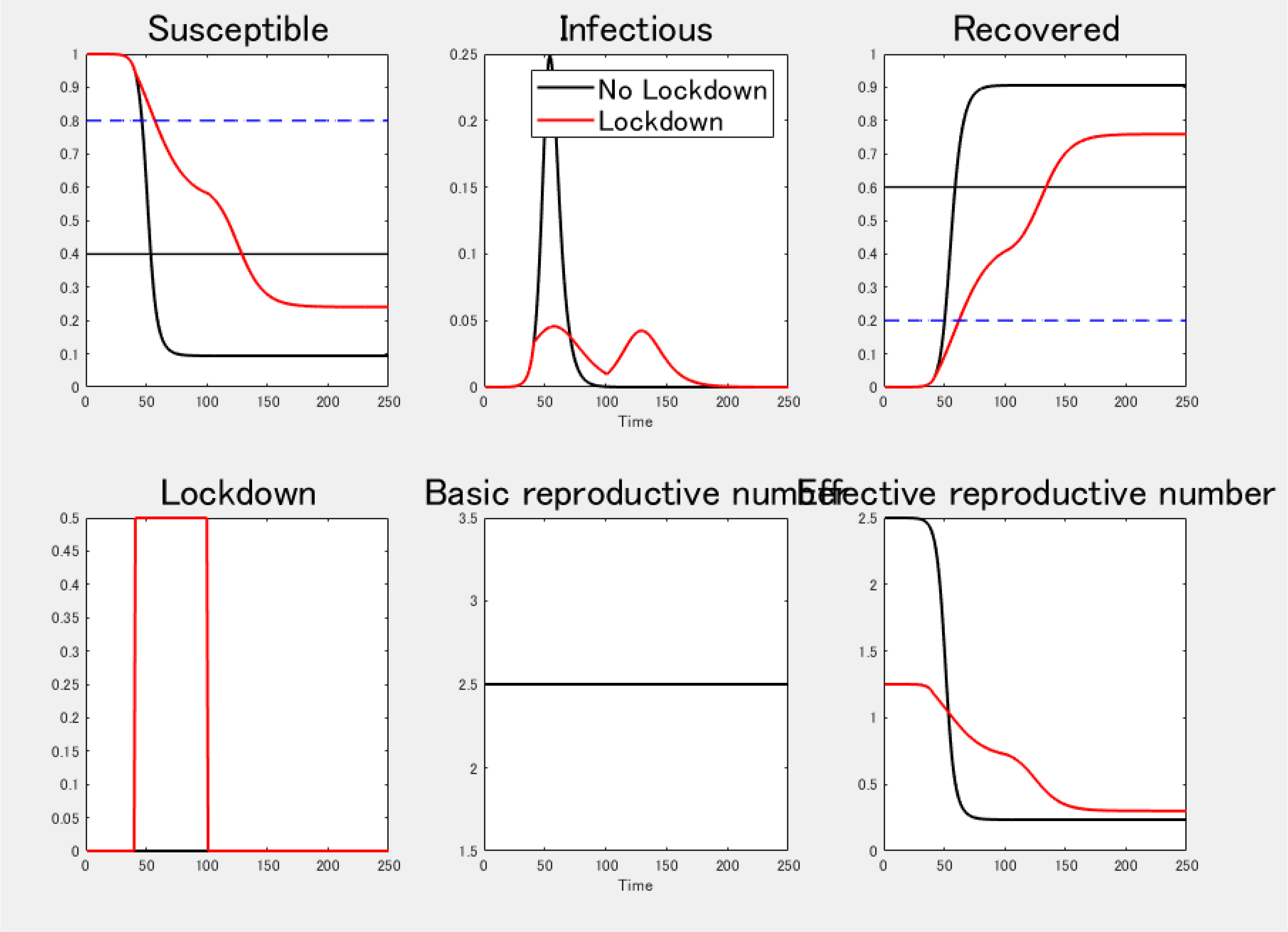
- $S_{t+1} - S_t = -\gamma(1 - \alpha_t)I_t S_t$

- $I_{t+1} - I_t = \gamma(1 - \alpha_t)I_t S_t - \tau I_t$

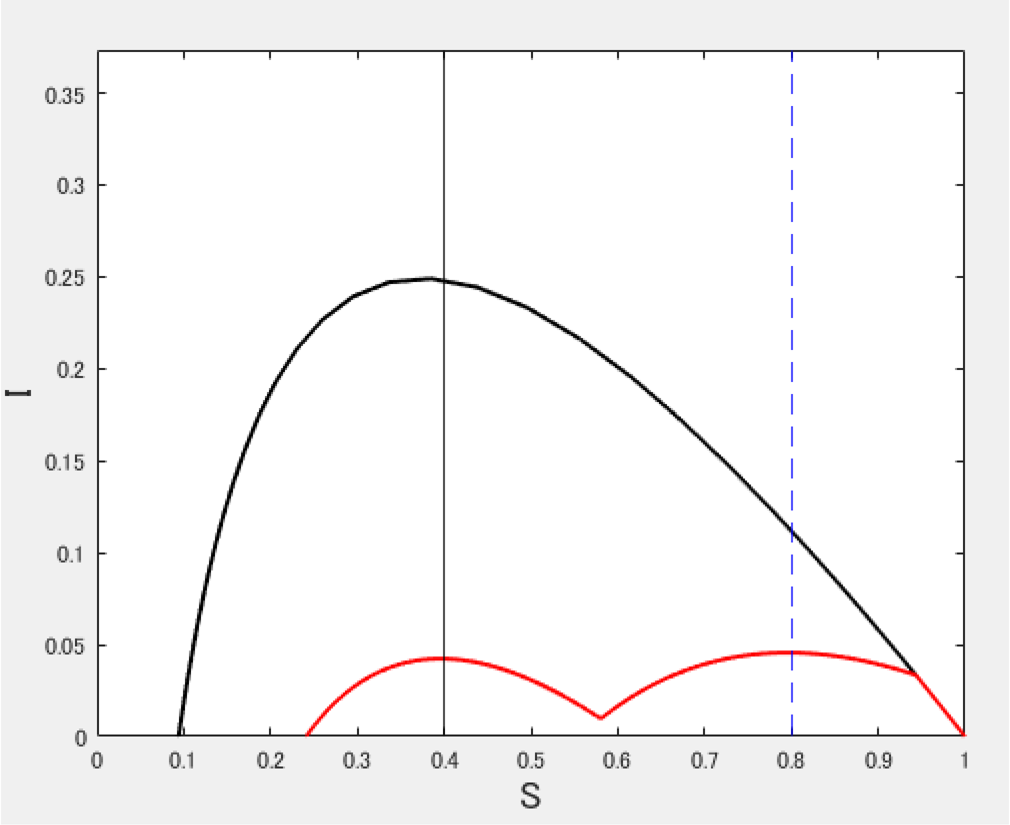
- $R_{t+1} - R_t = \tau I_t$

- $S_t + I_t + R_t = 1$

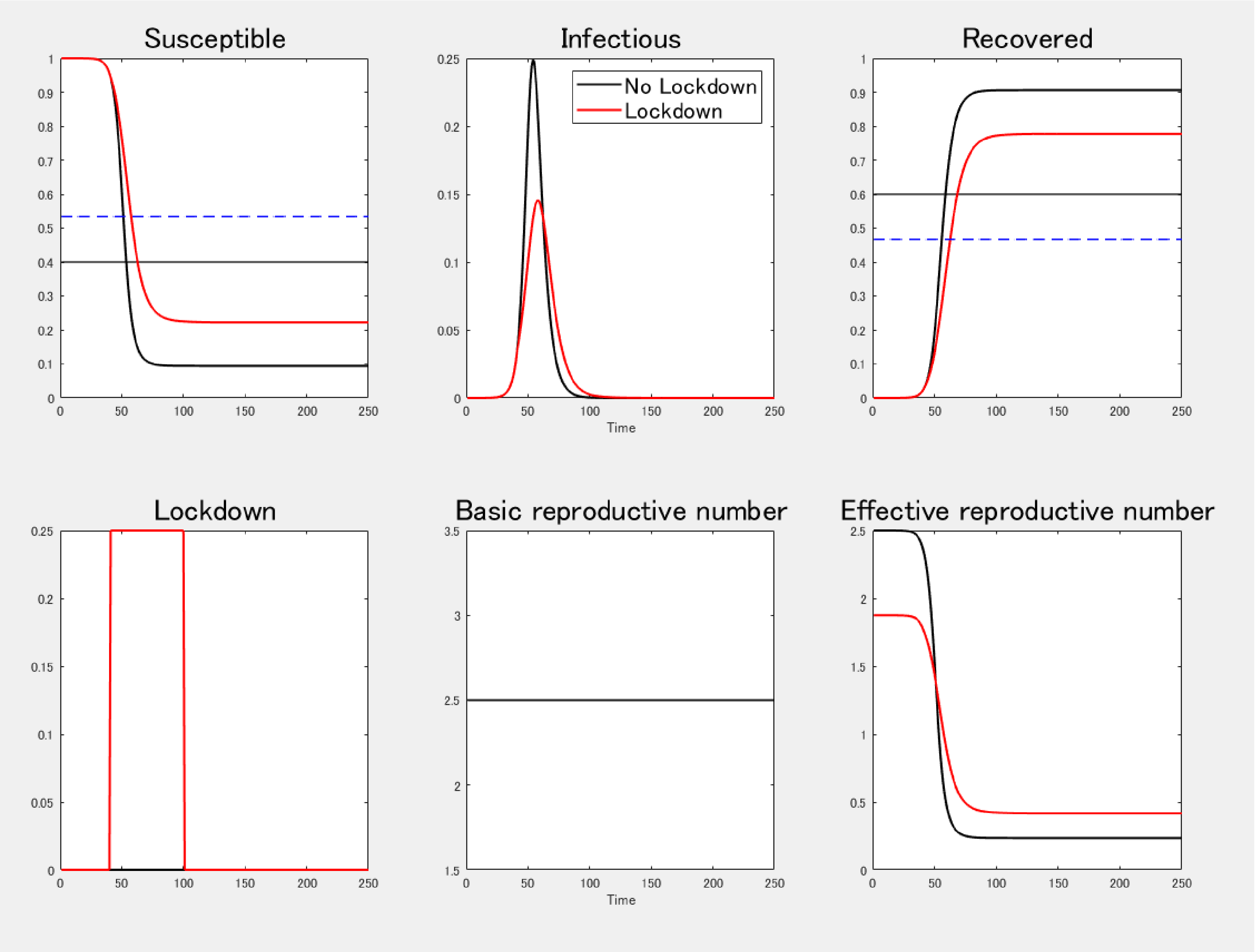
- Suppose that $\alpha_t = 0.5, 0.25, 0.75$ for all $41 \leq t \leq 100$.

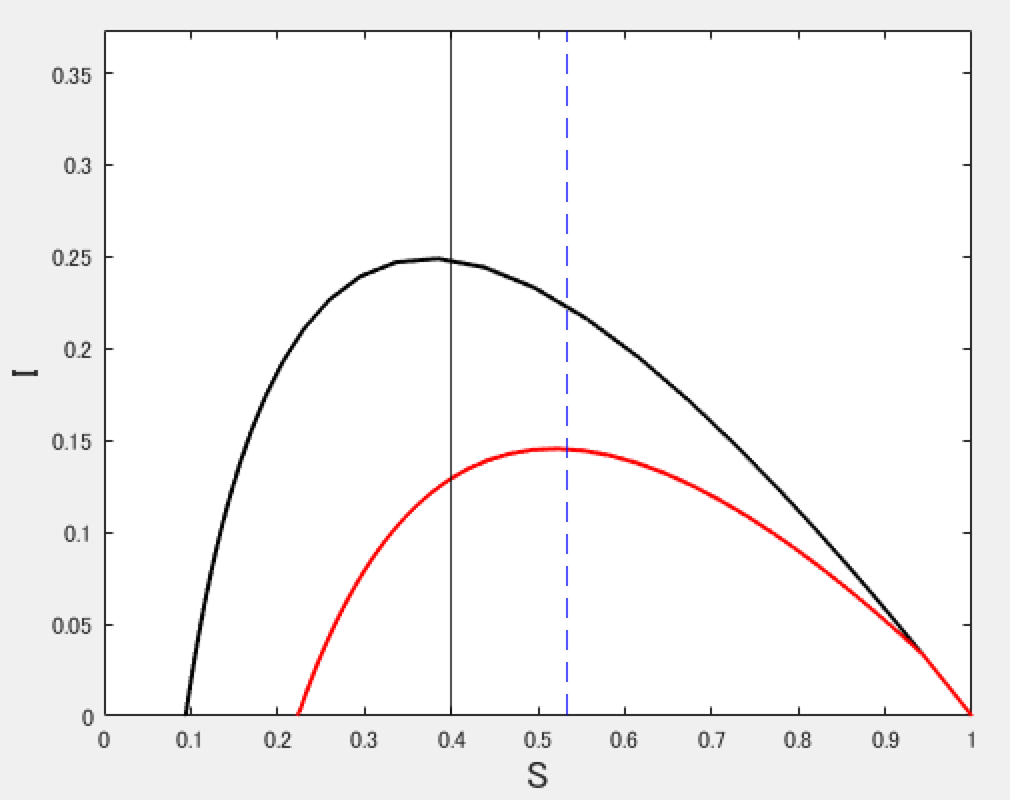


Susceptible
Infectious
Recovered
No Lockdown
Lockdown
Time
Lockdown
Basic reproductive number
Effective reproductive number
Time

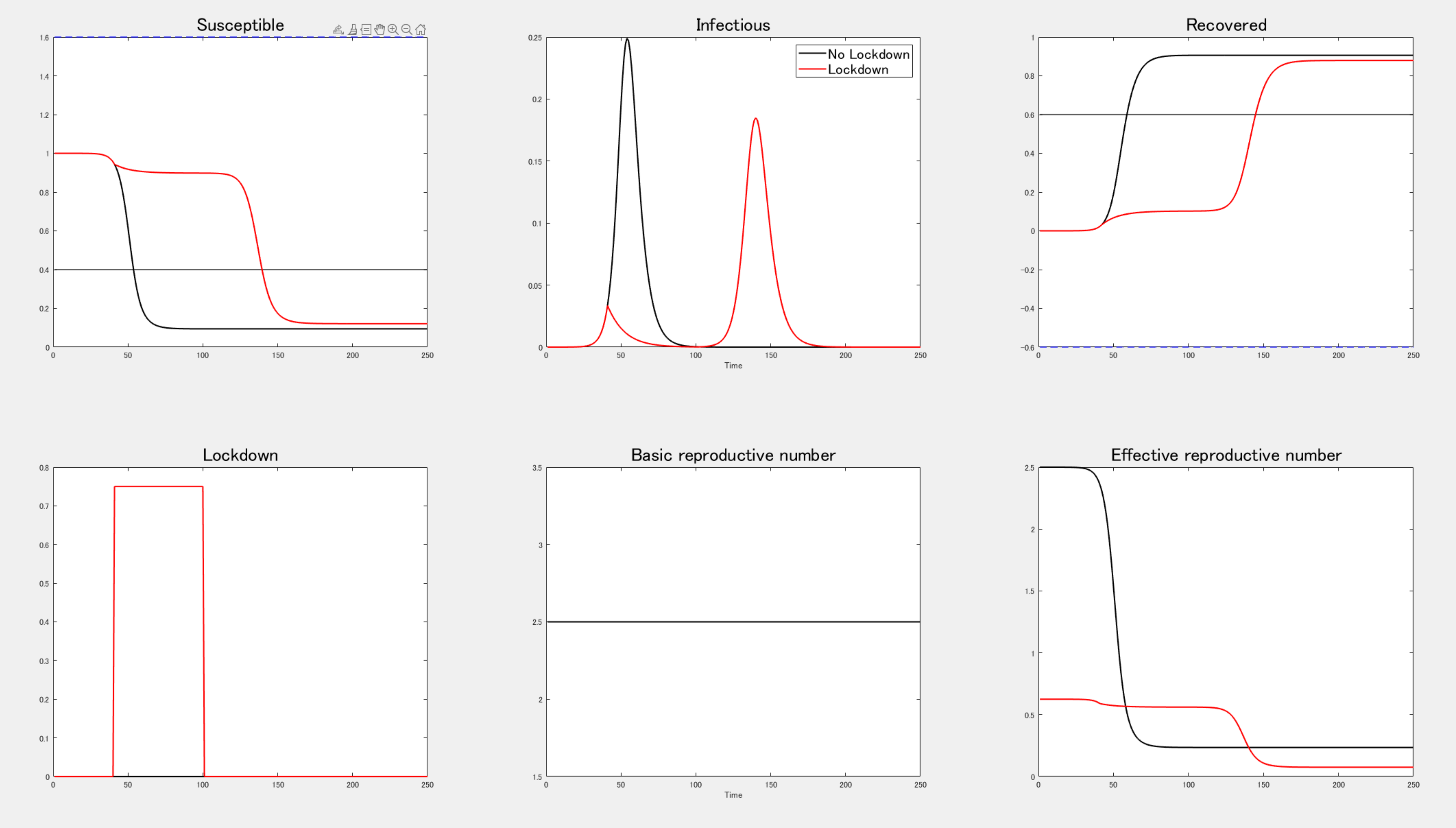


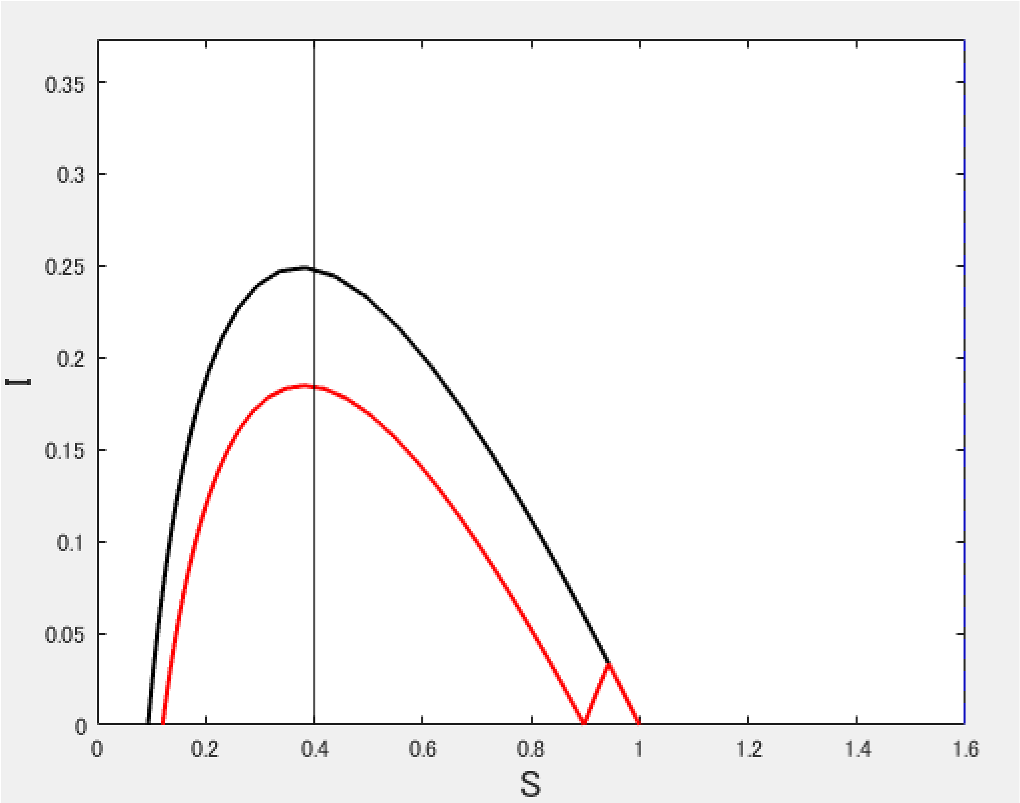
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
I
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
S


Susceptible
Infectious
Recovered
No Lockdown
Lockdown
Time
Lockdown
Basic reproductive number
Effective reproductive number
Time



0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
I
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
S


Susceptible
Infectious
No Lockdown
Lockdown
Recovered
Time
Lockdown
Basic reproductive number
Effective reproductive number
Time


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
I
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
S

Analyze **"welfare."**

- Suppose that $y_t = (S_t + R_t)(1 - \alpha_t)$. Y_t is output.

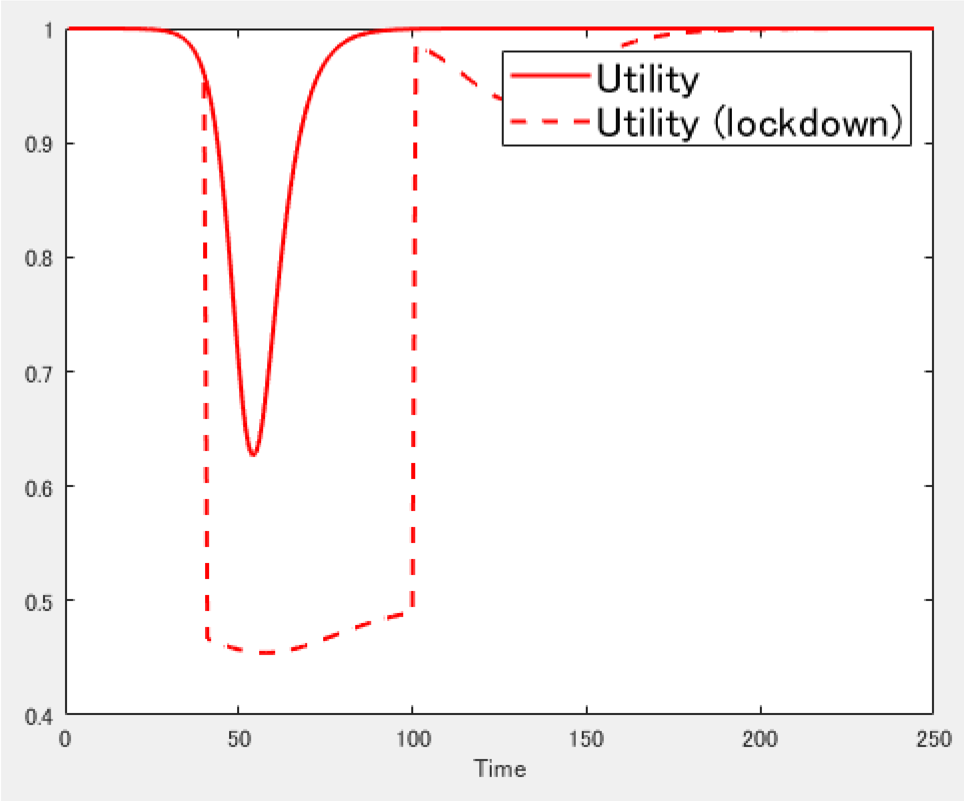
- Define a per-period utility at time t as follows.

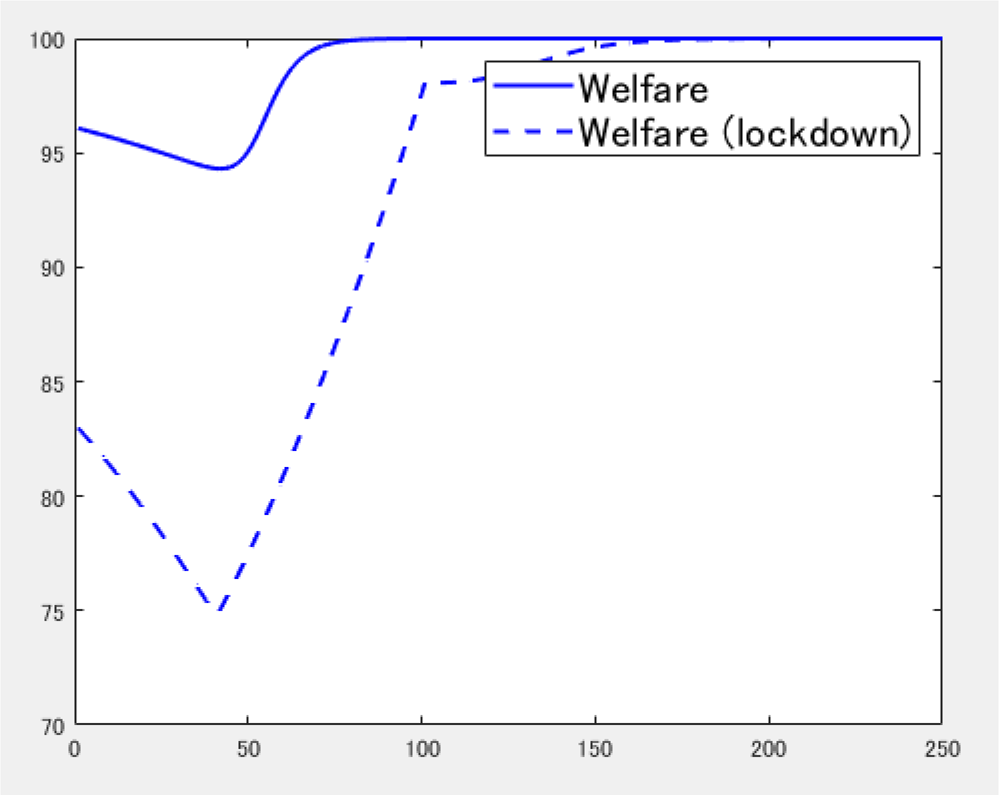
$$U_t = Y_t - \chi I_t$$

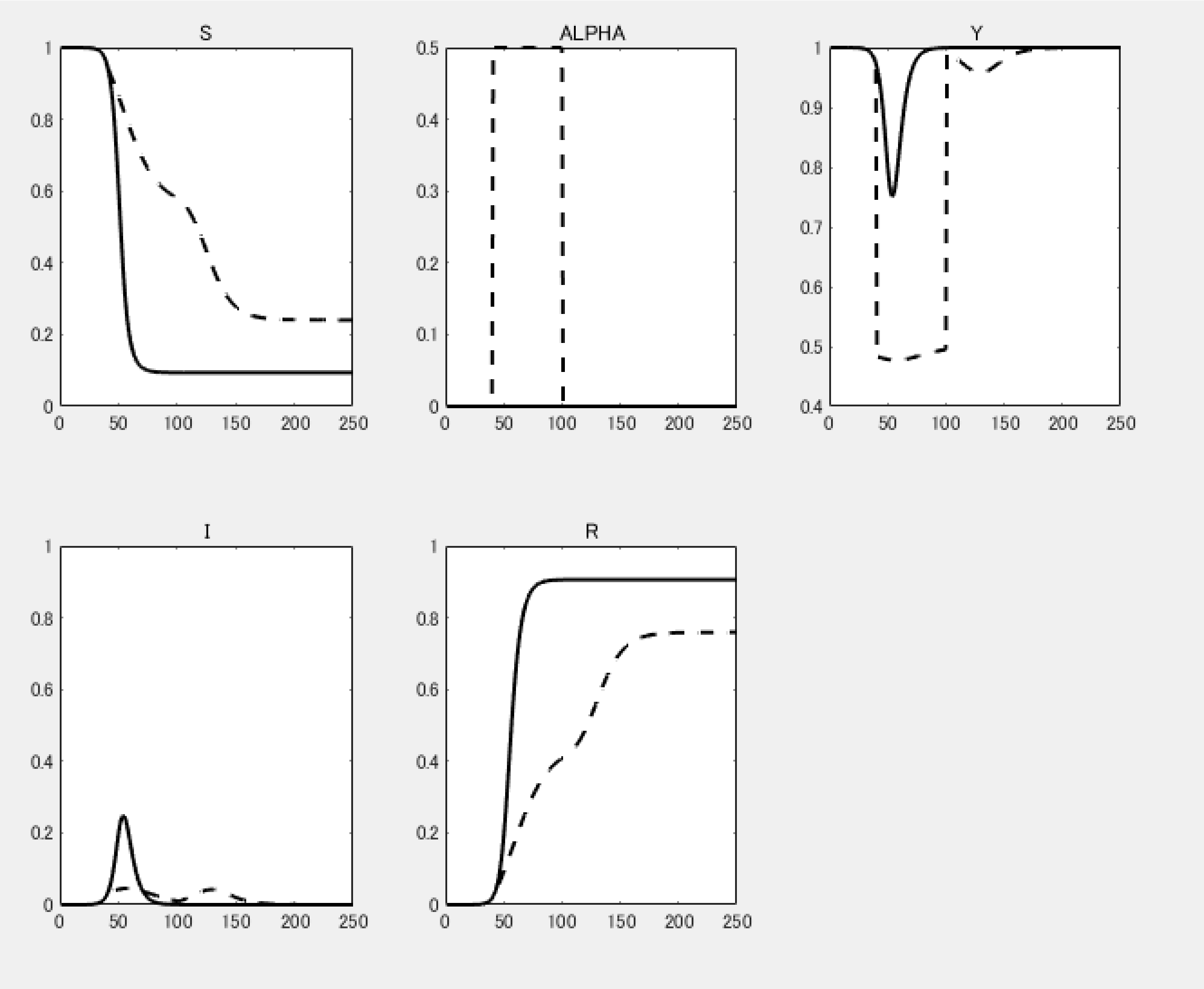
- Define welfare at time t as follows.

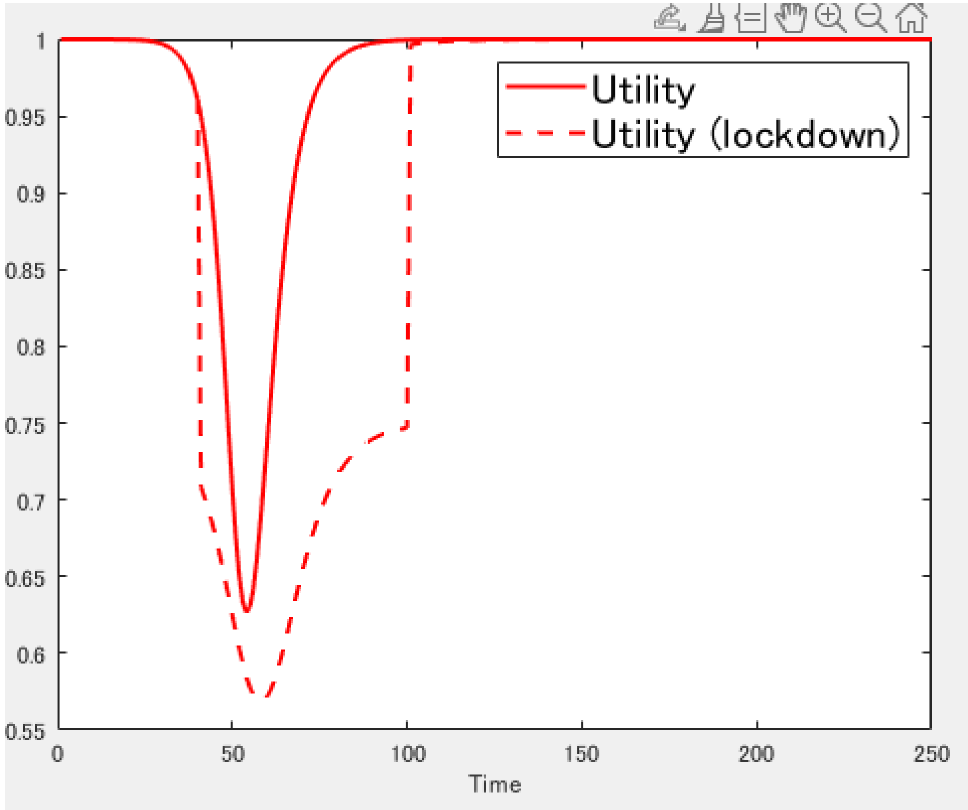
$$W_t = \sum_{k=0}^{\infty} \beta^k U_{t+k}$$
$$= U_t + \beta U_{t+1} + \beta^2 U_{t+2} + \beta^3 U_{t+3} + \cdots$$

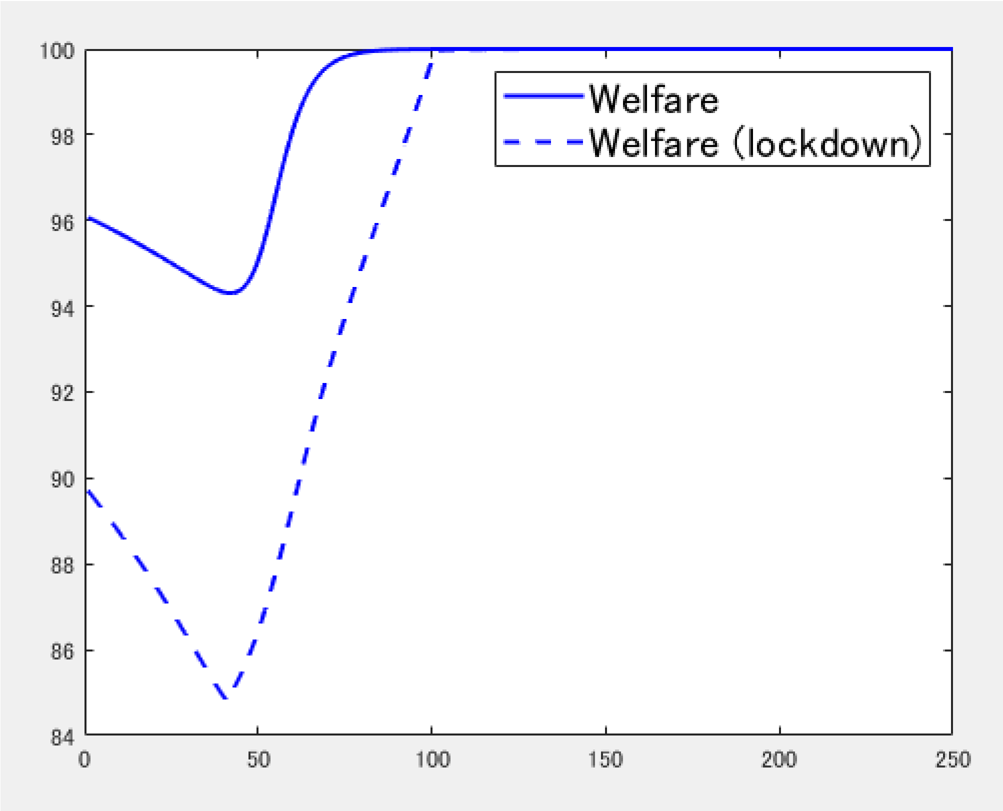
- Suppose that $\alpha_t = 0.5, 0.25, 0.75$ for all $41 \leq t \leq 100$.

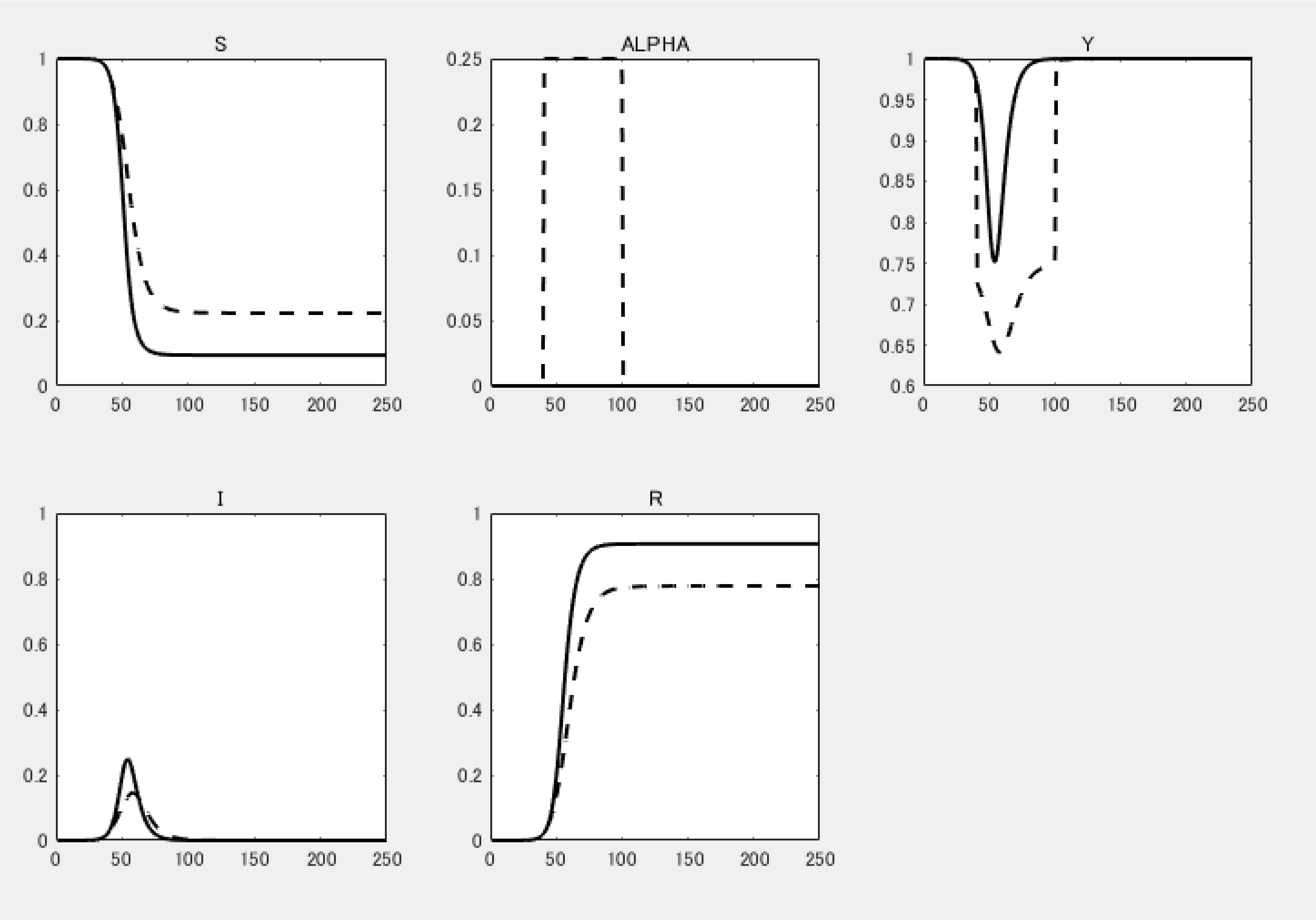

1
0.9
0.8
0.7
0.6
0.5
0.4
0
50
100
150
200
250
Time
Utility
Utility (lockdown)

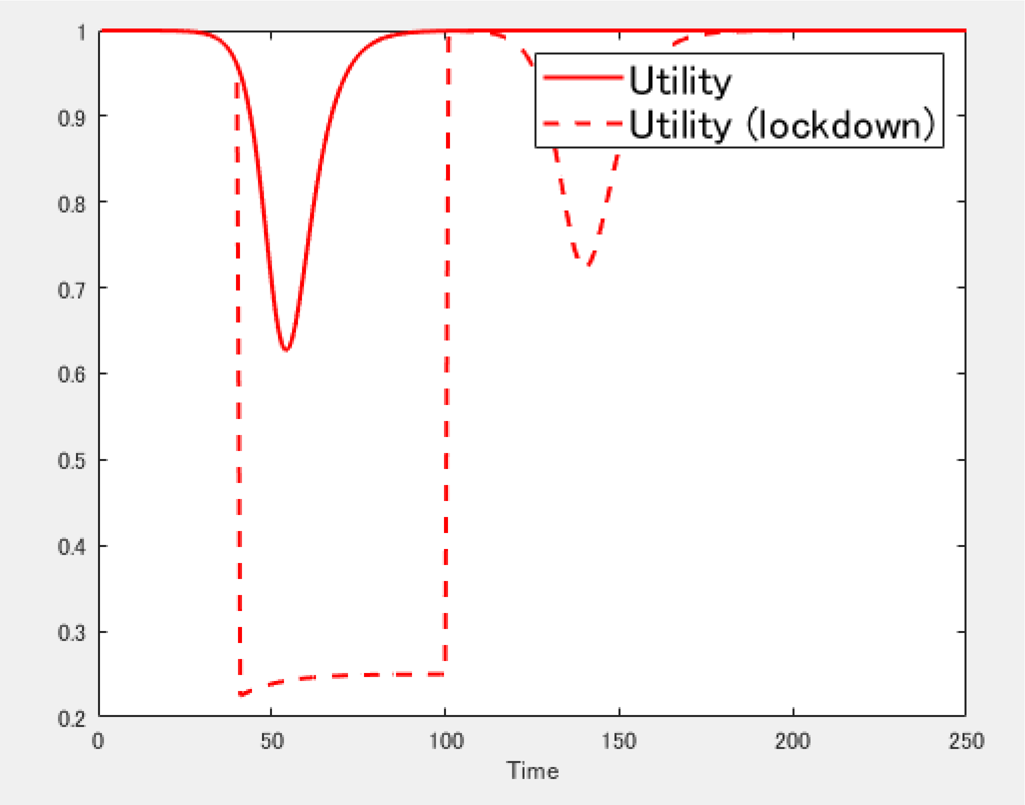

100
95
90
85
80
75
70
0
50
100
150
200
250
Welfare
Welfare (lockdown)


S
ALPHA
Y
I
R

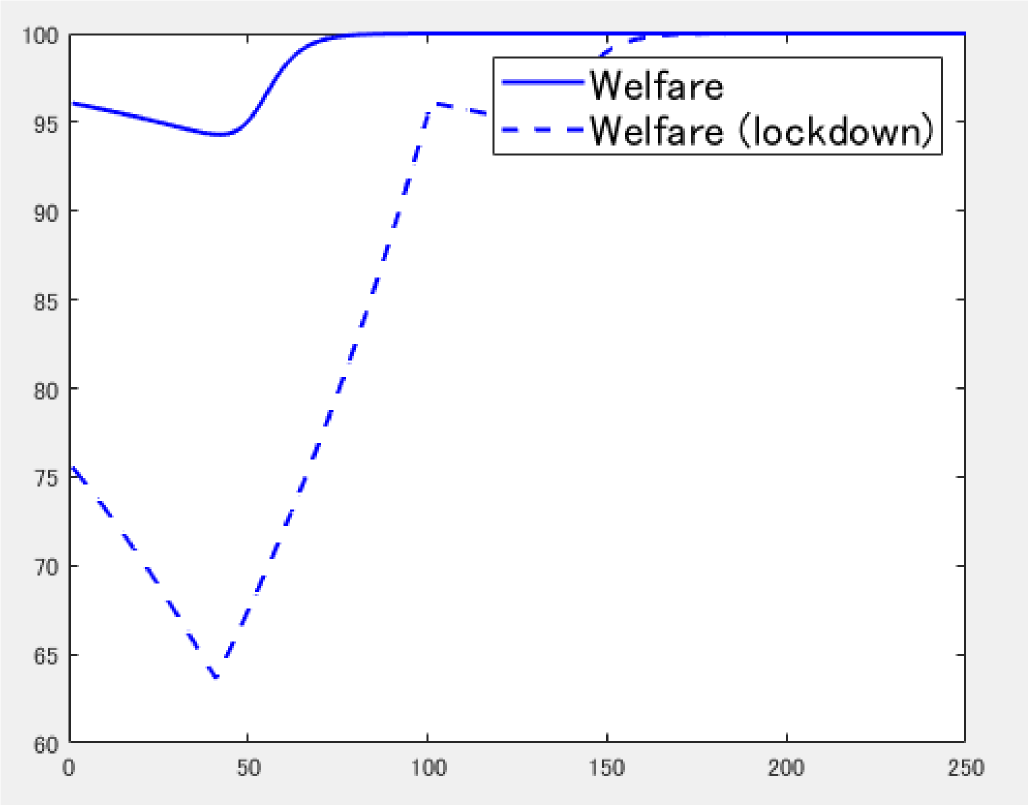

Utility
Utility (lockdown)
Time
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0
50
100
150
200
250


Welfare
Welfare (lockdown)
100
98
96
94
92
90
88
86
84
0
50
100
150
200
250

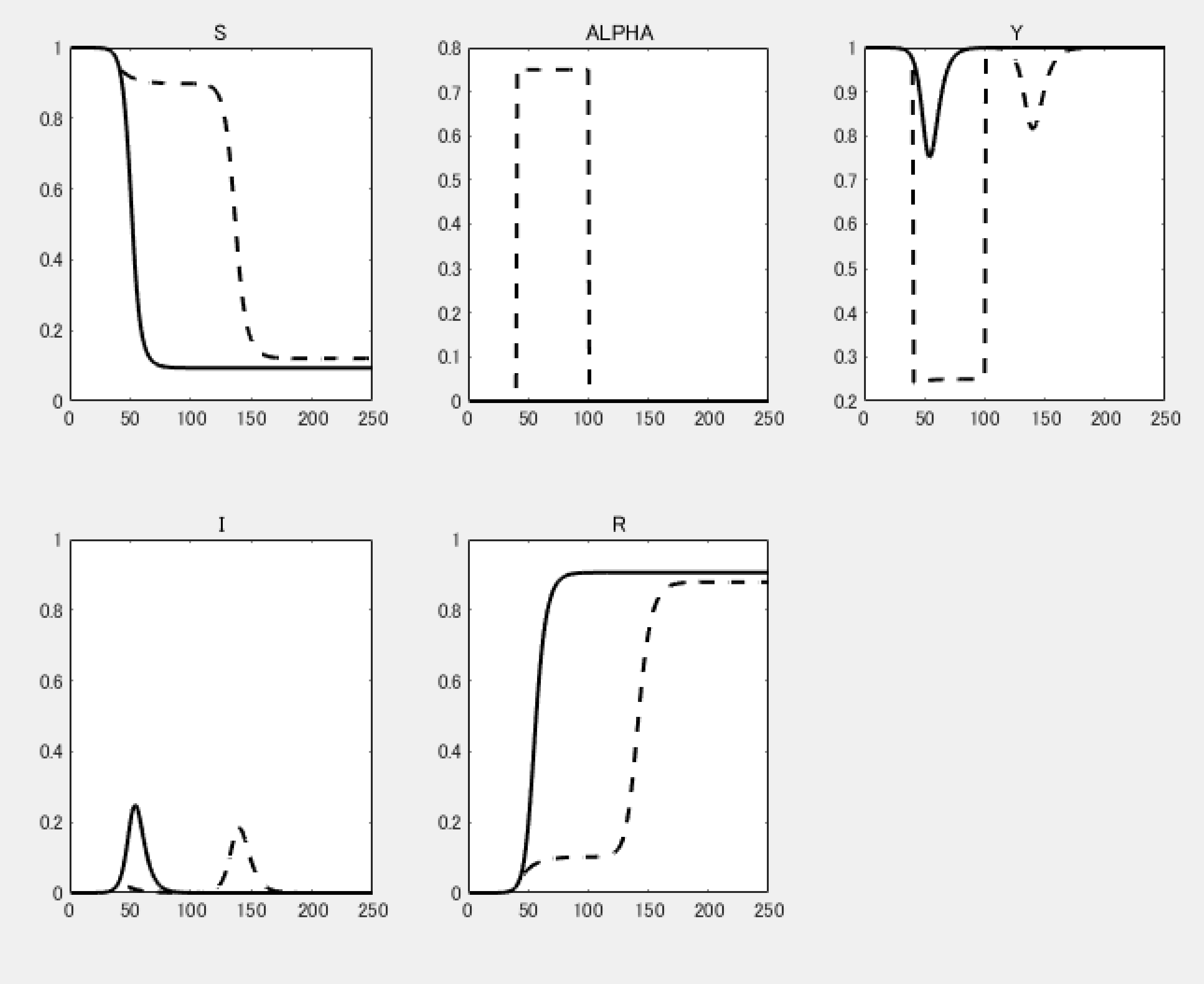

S
ALPHA
Y
I
R



Utility
Utility (lockdown)
Time
0
50
100
150
200
250
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Welfare
Welfare (lockdown)
0
50
100
150
200
250
60
65
70
75
80
85
90
95
100

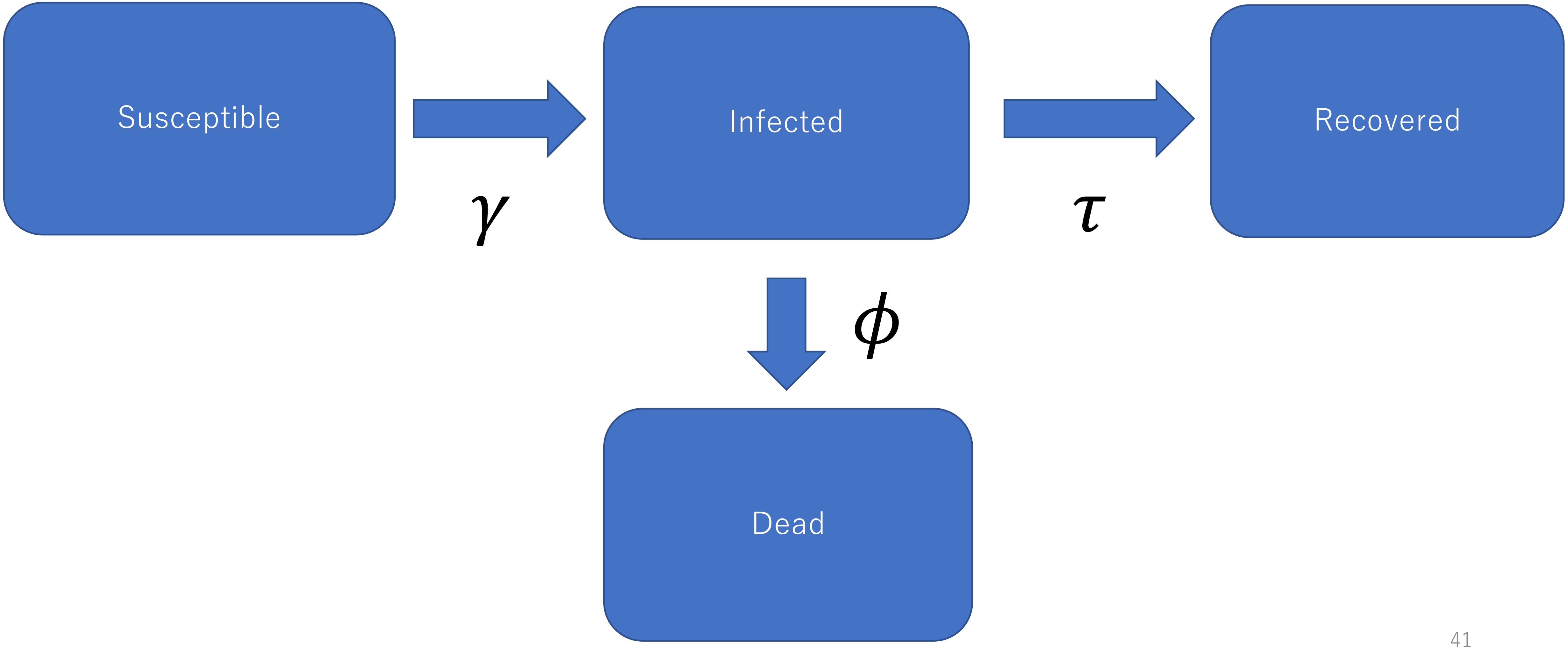

S
ALPHA
Y
I
R

Exercises

1. SIR-D Model

- $S_{t+1} - S_t = -\gamma I_t S_t$
- $I_{t+1} - I_t = \gamma I_t S_t - \tau I_t - \phi I_t$
- $R_{t+1} - R_t = \tau I_t$
- $D_{t+1} - D_t = \phi I_t$
- $S_t + I_t + R_t + D_t = 1$

SIR-D Model



- Examine the steady state of the model.
- Compute convergence to a steady state.
- Analyze the effect of a lockdown policy.
- Analyze the welfare effect of a lockdown policy.

2. SEIR Model

- $S_{t+1} - S_t = -\gamma I_t S_t$

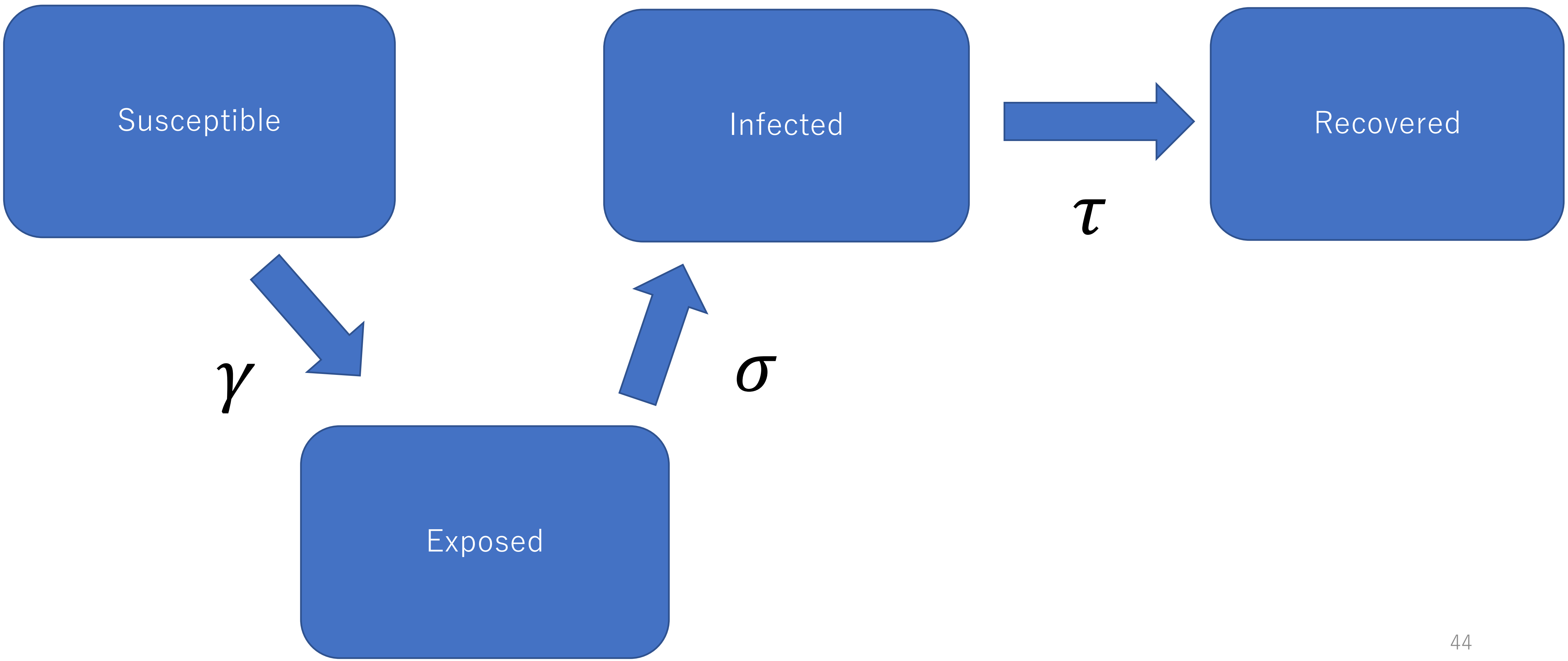
- $E_{t+1} - E_t = \gamma I_t S_t - \sigma E_t$

- $I_{t+1} - I_t = \sigma E_t - \tau I_t$

- $R_{t+1} - R_t = \tau I_t$

- $S_t + E_t + I_t + R_t = 1$

SEIR Model



- Examine the steady state of the model.
- Compute convergence to a steady state.
- Analyze the effect of a lockdown policy.
- Analyze the welfare effect of a lockdown policy.