第 3 週課題

来代 勝胤

2021 年 4 月 18 日

【演習 1】

Taylor-Green 渦の二次元速度場の定常解 ($t=\infty$) の $\mathbf{u}\,(u,v)$ を導出する．

(x,y) における，連続の式及びナビエ・ストークス方程式は,
以下の式 (1),(2),(3) のように表せる．

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

(x 方向)

$$-\frac{\partial u}{\partial t}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial P}{\partial x}-\nu\nabla^2 u \tag{2}$$

(y 方向)

$$-\frac{\partial v}{\partial t}=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial P}{\partial y}-\nu\nabla^2 v \tag{3}$$

また，非圧縮性粘性流体における初期条件は以下の式 (4),(5) のように与えられ,
このとき，式 (6) が成立することが知られている．

$$u=A\cos ax\sin by \tag{4}$$
$$v=B\sin ax\cos by \tag{5}$$

$$Aa+Bb=0 \tag{6}$$

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=0 \tag{7}$$

以上の式 (7) を式 (2),(3) に用いて,
(2) を x で,(3) を y でそれぞれ偏微分すると, 以下のようになる．

$$0=\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+u\frac{\partial^2 u}{\partial x^2}\right)+\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial x\partial y}\right)+\frac{1}{\rho}\frac{\partial^2 P}{\partial x^2}-\nu\nabla^2\frac{\partial u}{\partial x} \tag{8}$$
$$0=\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x\partial y}\right)+\left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}+v\frac{\partial^2 v}{\partial x^2}\right)+\frac{1}{\rho}\frac{\partial^2 P}{\partial y^2}-\nu\nabla^2\frac{\partial v}{\partial y} \tag{9}$$

ここで, 式 (8),(9) の和をとり,$-\frac{1}{\rho}\nabla^2 P$ について整理すると,

$$-\frac{1}{\rho}\nabla^2 P=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{10}$$

このとき，$\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ はそれぞれ以下のように表せる．

$$\frac{\partial u}{\partial x}=-Aa\sin ax\sin by \tag{11}$$
$$\frac{\partial u}{\partial y}=\quad Ab\cos ax\cos by \tag{12}$$
$$\frac{\partial v}{\partial x}=\quad Ba\cos ax\cos by \tag{13}$$
$$\frac{\partial v}{\partial y}=-Bb\sin ax\sin by \tag{14}$$

以上の計算結果を用いて, 式 (10) に代入して整理すると,

$$\frac{1}{\rho}\nabla^2 P = -A^2 a^2 \left(\cos 2ax + \cos 2by\right) \tag{15}$$

積分すると,

$$\frac{P}{\rho} = -\frac{1}{4}A^2 a^2 \left(\frac{1}{a^2}\cos 2ax + \frac{1}{b^2}\cos 2by\right) \tag{16}$$

また，式 (2),(3) において，$\frac{1}{\rho}\frac{\partial P}{\partial x}$，$\frac{1}{\rho}\frac{\partial P}{\partial y}$,$\nabla^2 u$,$\nabla^2 v$ はそれぞれ以下のように表せる．

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{A^2 a^2}{2a}\sin 2ax \tag{17}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial y} = \frac{A^2 a^2}{2b}\sin 2ay \tag{18}$$

$$\nabla^2 u = -A\left(a^2 + b^2\right)\cos ax \sin by \tag{19}$$

$$\nabla^2 v = -B\left(a^2 + b^2\right)\sin ax \cos by \tag{20}$$

以上の式 (17)〜(20) の計算結果及び (11)〜(14) を用いて (2),(3) に代入する．
その後，$\frac{\partial u}{\partial t}$,$\frac{\partial v}{\partial t}$ について整理すると，以下の式を得ることができる．

$$\frac{\partial u}{\partial t} = -\nu\theta A \cos ax \sin by \tag{21}$$

$$\frac{\partial v}{\partial t} = -\nu\theta B \sin ax \cos by \tag{22}$$

ここで，$\theta = a^2 + b^2$ とする．
式 (21) および (22) を積分して,

$$u = -\nu\theta A t \cos ax \sin by + C_1 \tag{23}$$

$$v = -\nu\theta B t \sin ax \cos by + C_2 \tag{24}$$

ここで，式 (4)，式 (5) から，C_1 および C_2 は，以下のように表せるので,

$$C_1 = A \cos ax \sin by \tag{25}$$

$$C_2 = B \sin ax \cos by \tag{26}$$

したがって，u，v はそれぞれ以下のように表すことができる．

$$u = \left(1 - \nu\theta t\right) A \cos ax \sin by \tag{27}$$

$$v = \left(1 - \nu\theta t\right) B \sin ax \cos by \tag{28}$$

同様の手順で計算を行い，u，v を求めると，以下の式 (43),(44) のようになる．

● u，v をそれぞれ x,y で偏微分する．

$$\frac{\partial u}{\partial x} = (1 - \nu\theta At)\, a \sin ax \sin by \tag{29}$$
$$\frac{\partial u}{\partial y} = (1 - \nu\theta At)\, b \cos ax \cos by \tag{30}$$
$$\frac{\partial v}{\partial x} = (1 - \nu\theta Bt)\, a \cos ax \cos by \tag{31}$$
$$\frac{\partial v}{\partial y} = (1 - \nu\theta Bt)\, b \sin ax \sin by \tag{32}$$

● 式 (10) に代入して整理し，積分する．

$$\frac{1}{\rho}\nabla^2 P = A^2 a^2 \left(\cos 2ax + \cos 2by\right) \nu^2\theta^2 t^2 \tag{33}$$

$$\frac{P}{\rho} = \frac{1}{4} A^2 a^2 \left(\frac{1}{a^2}\cos 2ax + \frac{1}{b^2}\cos 2by\right) \nu^2\theta^2 t^2 \tag{34}$$

● $\frac{1}{\rho}\frac{\partial P}{\partial x}$，$\frac{1}{\rho}\frac{\partial P}{\partial y}$,$\nabla^2 u$,$\nabla^2 v$ をそれぞれ算出する．

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{A^2 a^2}{2a} \sin 2ax\, \nu^2\theta^2 t^2 \tag{35}$$
$$\frac{1}{\rho}\frac{\partial P}{\partial y} = \frac{A^2 a^2}{2b} \sin 2ay\, \nu^2\theta^2 t^2 \tag{36}$$

● 式 (37)，(38) をそれぞれ積分する．

$$\nabla^2 u = A\left(a^2 + b^2\right) \cos ax \sin by\, \nu^2\theta^2 t^2 \tag{37}$$
$$\nabla^2 v = B\left(a^2 + b^2\right) \sin ax \cos by\, \nu^2\theta^2 t^2 \tag{38}$$

● 式 (27)，(28) を用いて，積分定数 C_3 及び C_4 を計算する．

$$u = \frac{1}{2}\nu^2\theta^2 A t^2 \cos ax \sin by + C_3 \tag{39}$$
$$v = \frac{1}{2}\nu^2\theta^2 B t^2 \sin ax \cos by + C_4 \tag{40}$$

$$C_3 = (1 - \nu\theta t)\, A \cos ax \sin by \tag{41}$$
$$C_4 = (1 - \nu\theta t)\, B \sin ax \cos by \tag{42}$$

$$u = \left(1 - \nu\theta t + \frac{1}{2}\nu^2\theta^2 t^2\right) A \cos ax \sin by \tag{43}$$
$$v = \left(1 - \nu\theta t + \frac{1}{2}\nu^2\theta^2 t^2\right) B \sin ax \cos by \tag{44}$$

ここで，$e^{-\nu\theta t}$ が，指数関数のテイラー展開から以下のように表せることから，

$$e^{-\nu\theta t} = 1 - \nu\theta t + \frac{1}{2}\nu^2\theta^2 t^2 - \frac{1}{3!}\nu^3\theta^3 t^3 + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}(\nu\theta t)^{n-1} \tag{45}$$

u および v が計算を繰り返し，n 回目の計算を終えたとき，
式 (4),(5),(27),(28),(43),(44) を用いて以下のように表すことができると考えられる．

$$\begin{aligned} u_n =& \left(1 - \nu\theta t + \frac{1}{2}\nu^2\theta^2 t^2 - \frac{1}{3!}\nu^3\theta^3 t^3 + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}(\nu\theta t)^{n-1}\right) \\ & A\cos ax \sin by \\ =& e^{-\nu\theta t} A\cos ax \sin by \\ =& e^{-\nu\left(a^2+b^2\right)t} A\cos ax \sin by \end{aligned} \tag{46}$$

$$\begin{aligned} v_n =& \left(1 - \nu\theta t + \frac{1}{2}\nu^2\theta^2 t^2 - \frac{1}{3!}\nu^3\theta^3 t^3 + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}(\nu\theta t)^{n-1}\right) \\ & B\sin ax \cos by \\ =& e^{-\nu\theta t} B\sin ax \cos by \\ =& e^{-\nu\left(a^2+b^2\right)t} B\sin ax \cos by \end{aligned} \tag{47}$$

したがって，u，v はそれぞれ以下のように表すことができる．

$$\begin{cases} u = e^{-\nu\left(a^2+b^2\right)t} A\cos ax \sin by \\ v = e^{-\nu\left(a^2+b^2\right)t} B\sin ax \cos by \end{cases} \tag{48}$$

ここで，式 (6) の条件から，$A = a = b = 1$,$B = -1$ とすると，
u,v はそれぞれ以下のように表される．

$$\begin{cases} u = e^{-2\nu t}\cos x \sin y \\ v = -e^{-2\nu t}\sin x \cos y \end{cases} \tag{49}$$

【演習 2】

渦度場 ω_z を導出する．

式 (49) を用いて 2 次元ベクトル場における渦度 ω_z は以下の式 (50) のように表される．
それを用いて ω_z を計算すると，式 (53) のように表すことができる．

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{50}$$

$$\frac{\partial v}{\partial x} = -e^{-2\nu t}\cos x \cos y \tag{51}$$

$$\frac{\partial u}{\partial y} = e^{-2\nu t}\cos x \cos y \tag{52}$$

$$\omega_z = -2e^{-2\nu t}\cos ax \cos by \tag{53}$$

【演習 3】

プログラム

```
/****************************************************************************
PROGRAM NAME : practice_3
AUTHER : Masatsugu Kitadai
DATE : 1/4/2021
Think a Bit , Code a Bit , Test a Bit
****************************************************************************/
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <sys/stat.h>
#define x_grid 31 // number of grids in x direction
#define y_grid 31 // number of grids in y direction
const char grid_space = 1; // grid width
const char *output_data_file = "2dvec_vortex000001.dat"; // name of output file
double u[x_grid][y_grid]; // u vector array
double v[x_grid][y_grid]; // v vector array
double U[x_grid][y_grid]; // absolute vector array
double omega[x_grid][y_grid]; // vorticity array

double a[x_grid][y_grid];
double b[x_grid][y_grid];
double c[x_grid][y_grid];
double d[x_grid][y_grid];

FILE *output_file; // pointer for output file

const char *xxlabel = "{/Times-New-Roman:Italic=20 x} [pixel]";
const char *yylabel = "{/Times-New-Roman:Italic=20 y} [pixel]";
const char *cb_label = "{/Symbol:Italic=20 w}_{/Times-New-Roman:Italic=20 z} [sec]"; ///color bar range min

const double v_r = 1.0; ///magnified ratio for vector length

const int x_min = 0; ///x range min
const int x_max = 30; ///x range max
const int y_min = 0; ///y range min
const int y_max = 30; ///y range max
const int cb_min = -2; ///color bar range min
const int cb_max = 2; ///color bar range max

const char *read_file_dir = "01_plot_vec_vortex";
const char *read_file_header = "2dvec_vortex";
const char *write_file_dir = "02_splot_2dvec_vortex_map";
const char *write_file_header = "2dvec_vortex_map";

//Graph parameters for GNU
char read_file[100];
void graph_GNU(); //png & eps
FILE *gp; //gnuplot
FILE *infile;

int moveFile(const char *srcPath, const char *destPath)
{
       return !rename(srcPath, destPath);
   }

/*********************************
MAIN
*********************************/
double main()
{
int i, j;
double PI = 4.0 * atan(1.0);
```

```
// preparing for output file
output_file = fopen(output_data_file, "w");

// Array initialization
for (i = 0; i < x_grid; i++)
{
       for (j = 0; j < y_grid; j++)
       {
              u[i][j] = 0;
              v[i][j] = 0;
              U[i][j] = 0;
              omega[i][j] = 0;
          }
   }

// Calc.2D velocity vector and absolute value of velocity field, vorticity field

for (i = 0; i < x_grid; i++)
{
       for (j = 0; j < y_grid; j++)
       {
              // velocity vector field and absolute value of vector
              u[i][j] = cos(2.0 * PI / x_grid * i) * sin(2.0 * PI / y_grid * j);
              v[i][j] = -sin(2.0 * PI / x_grid * i) * cos(2.0 * PI / y_grid * j);
              U[i][j] = sqrt(u[i][j] * u[i][j] + v[i][j] * v[i][j]);

              // value of vorticity field
              omega[i][j] = -2 * cos(2.0 * PI / x_grid * i) * cos(2.0 * PI / y_grid * j);

              fprintf(output_file, "%d\t%d\t%3lf\t%.3lf\t%.3lf\t%3lf\n",
              i * grid_space, j * grid_space, omega[i][j], u[i][j], v[i][j], U[i][j]);

              //Caution : just line breaking for printing this document

              /*
              printf("%d\t,%d\t,%.3lf\t,%.3lf\t,%.3lf\t,%.3lf\n",
              i * grid_space, j * grid_space, omega[i][j], u[i][j], v[i][j], U[i][j]);
              */
          }
       fprintf(output_file, "\n");
       //printf("\n");
   }

fclose(output_file);

mode_t mode = S_IRWXU | S_IRGRP | S_IXGRP | S_IROTH | S_IXOTH;
mkdir(read_file_dir, mode);
mkdir(write_file_dir, mode);

// transfer the result file to the specified folder

if (moveFile("2dvec_vortex000001.dat", "01_plot_vec_vortex/2dvec_vortex000001.dat"))
{
       puts("transferred the file");
   }
else
   {
       puts("cannot transferred");
   }

int k = 0;
int UP = 0;
while (UP == 0)
{
k++;
sprintf(read_file, "%s//%s%06d.dat", read_file_dir, read_file_header, k);
printf("%s//%s%06d.dat\n", read_file_dir, read_file_header, k);
```

```
infile = fopen(read_file, "rb");

if (infile == NULL)
{
       printf("break!\n");
       break;
   }

fclose(infile);

if ((gp = popen("gnuplot", "w")) == NULL)
{
       printf("gnuplot is not here!\n");
       exit(0);
   }

//PNG image
fprintf(gp, "set terminal pngcairo enhanced font 'Times New Roman,15' \n");
fprintf(gp, "set output '%s//%s%06d.png'\n", write_file_dir, write_file_header, k);
fprintf(gp, "set multiplot\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "unset key\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "set size ratio -1\n"); // <steps in scan>,<steps between scans>

fprintf(gp, "set lmargin screen 0.15\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "set rmargin screen 0.85\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "set tmargin screen 0.85\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "set bmargin screen 0.15\n"); // <steps in scan>,<steps between scans>

fprintf(gp, "set xrange [%d:%d]\n", x_min, x_max); // <steps in scan>,<steps between scans>
fprintf(gp, "set xlabel '%s'offset 0.0,0.5\n", xxlabel); // <steps in scan>,<steps between scans>
fprintf(gp, "set yrange [%d:%d]\n", y_min, y_max); // <steps in scan>,<steps between scans>
fprintf(gp, "set ylabel '%s'offset 0.5,0.0\n", yylabel); // <steps in scan>,<steps between scans>

fprintf(gp, "set cblabel '%s'offset 0.0,0.0\n", cb_label);
fprintf(gp, "set cbrange['%d':'%d']\n", cb_min, cb_max);
fprintf(gp, "set colorbox vertical user origin 0.8, 0.2 size 0.025,0.6\n");
fprintf(gp, "set palette rgbformulae 22,13,-31\n");

fprintf(gp, "set pm3d map\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "splot '%s//%s%06d.dat' using 2:1:3 with pm3d, '%s//%s%06d.dat' using 2:1:($1*0.0):(%lf*$5):(%lf*
    $4):($1*0.0) with vectors head filled lt 2 lc 'black' \n", read_file_dir, read_file_header, k,
    read_file_dir, read_file_header, k, v_r, v_r);

fflush(gp); //Clean up Data

fprintf(gp, "exit\n"); // Quit gnuplot

pclose(gp);
}

return (0);
}
```

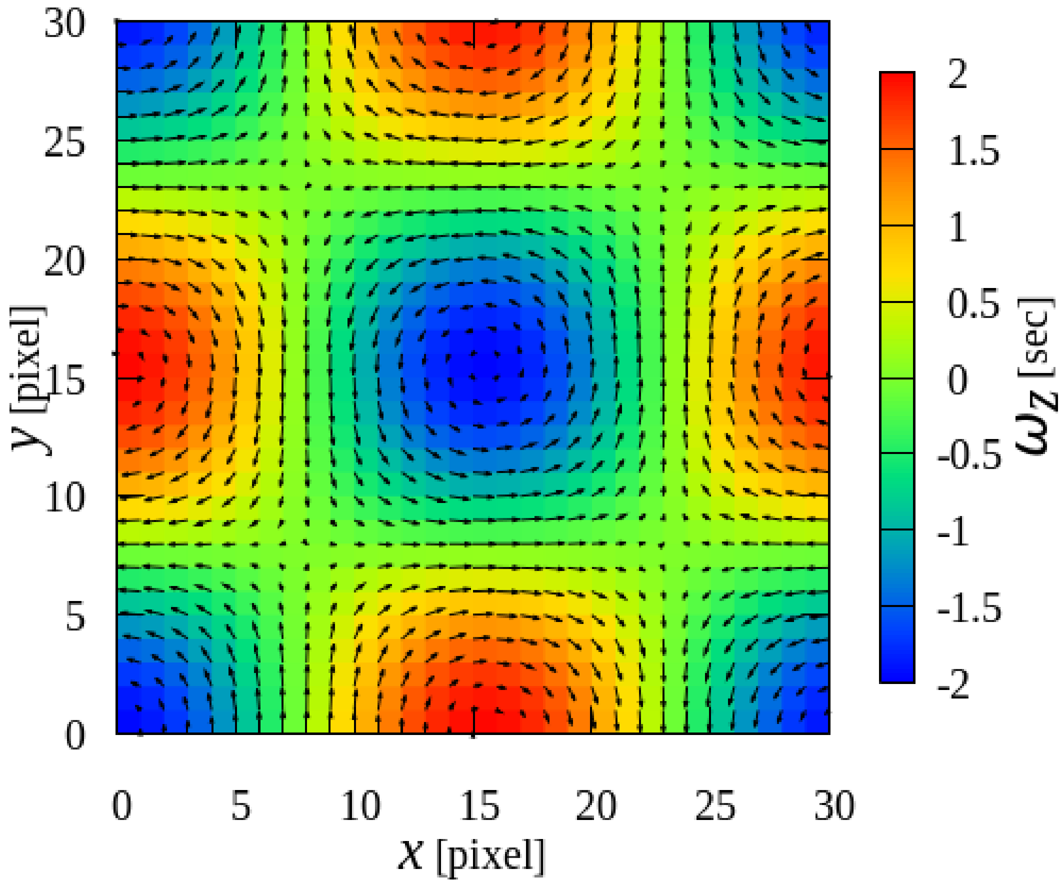

図１ gnuplot で作図した速度場ベクトル及び渦度場