

第 3 週課題

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【演習 1】

Taylor-Green 渦の二次元速度場の定常解 ($t = \infty$) の $\mathbf{u}(u, v)$ を導出する .

(x, y) における , 連続の式及びナビエ・ストークス方程式は ,
以下の式 (1),(2),(3) のように表せる .

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

(x 方向)

$$-\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial x} - \nu \nabla^2 u \quad (2)$$

(y 方向)

$$-\frac{\partial v}{\partial t} = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial y} - \nu \nabla^2 v \quad (3)$$

また , 非圧縮性粘性流体における初期条件は以下の式 (4),(5) のように与えられ ,
このとき , 式 (6) が成立することが知られている .

$$u = A \cos ax \sin by \quad (4)$$

$$v = B \sin ax \cos by \quad (5)$$

$$Aa + Bb = 0 \quad (6)$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0 \quad (7)$$

以上の式 (7) を式 (2),(3) に用いて,

(2) を x で,(3) を y でそれぞれ偏微分すると, 以下のようになる.

$$0 = \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + u \frac{\partial^2 u}{\partial x^2} \right) + \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial x \partial y} \right) + \frac{1}{\rho} \frac{\partial^2 P}{\partial x^2} - \nu \nabla^2 \frac{\partial u}{\partial x} \quad (8)$$

$$0 = \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x \partial y} \right) + \left(\frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + v \frac{\partial^2 v}{\partial y^2} \right) + \frac{1}{\rho} \frac{\partial^2 P}{\partial y^2} - \nu \nabla^2 \frac{\partial v}{\partial y} \quad (9)$$

ここで, 式 (8),(9) の和をとり, $-\frac{1}{\rho} \nabla^2 P$ について整理すると,

$$-\frac{1}{\rho} \nabla^2 P = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + 2 \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) \quad (10)$$

このとき , $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ はそれぞれ以下のように表せる .

$$\frac{\partial u}{\partial x} = -Aa \sin ax \sin by \quad (11)$$

$$\frac{\partial u}{\partial y} = Ab \cos ax \cos by \quad (12)$$

$$\frac{\partial v}{\partial x} = Ba \cos ax \cos by \quad (13)$$

$$\frac{\partial v}{\partial y} = -Bb \sin ax \sin by \quad (14)$$

以上の計算結果を用いて, 式 (10) に代入して整理すると,

$$\frac{1}{\rho} \nabla^2 P = -A^2 a^2 (\cos 2ax + \cos 2by) \quad (15)$$

積分すると,

$$\frac{P}{\rho} = -\frac{1}{4} A^2 a^2 \left(\frac{1}{a^2} \cos 2ax + \frac{1}{b^2} \cos 2by \right) \quad (16)$$

また, 式 (2),(3) において, $\frac{1}{\rho} \frac{\partial P}{\partial x}, \frac{1}{\rho} \frac{\partial P}{\partial y}, \nabla^2 u, \nabla^2 v$ はそれぞれ以下のように表せる.

$$\frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{A^2 a^2}{2a} \sin 2ax \quad (17)$$

$$\frac{1}{\rho} \frac{\partial P}{\partial y} = \frac{A^2 a^2}{2b} \sin 2ay \quad (18)$$

$$\nabla^2 u = -A(a^2 + b^2) \cos ax \sin by \quad (19)$$

$$\nabla^2 v = -B(a^2 + b^2) \sin ax \cos by \quad (20)$$

以上の式 (17) ~ (20) の計算結果及び (11) ~ (14) を用いて (2),(3) に代入する.

その後, $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ について整理すると, 以下の式を得ることができる.

$$\frac{\partial u}{\partial t} = -\nu \theta A \cos ax \sin by \quad (21)$$

$$\frac{\partial v}{\partial t} = -\nu \theta B \sin ax \cos by \quad (22)$$

ここで, $\theta = a^2 + b^2$ とする.

式 (21) および (22) を積分して,

$$u = -\nu \theta A t \cos ax \sin by + C_1 \quad (23)$$

$$v = -\nu \theta B t \sin ax \cos by + C_2 \quad (24)$$

ここで, 式 (4), 式 (5) から, C_1 および C_2 は, 以下のように表せるので,

$$C_1 = A \cos ax \sin by \quad (25)$$

$$C_2 = B \sin ax \cos by \quad (26)$$

したがって, u, v はそれぞれ以下のように表すことができる.

$$u = (1 - \nu \theta t) A \cos ax \sin by \quad (27)$$

$$v = (1 - \nu \theta t) B \sin ax \cos by \quad (28)$$

同様の手順で計算を行い, u, v を求めると, 以下の式 (43),(44) のようになる.

u, v をそれぞれ x, y で偏微分する .

$$\frac{\partial u}{\partial x} = (1 - \nu\theta At) a \sin ax \sin by \quad (29)$$

$$\frac{\partial u}{\partial y} = (1 - \nu\theta At) b \cos ax \cos by \quad (30)$$

$$\frac{\partial v}{\partial x} = (1 - \nu\theta Bt) a \cos ax \cos by \quad (31)$$

$$\frac{\partial v}{\partial y} = (1 - \nu\theta Bt) b \sin ax \sin by \quad (32)$$

式 (10) に代入して整理し , 積分する .

$$\frac{1}{\rho} \nabla^2 P = A^2 a^2 (\cos 2ax + \cos 2by) \nu^2 \theta^2 t^2 \quad (33)$$

$$\frac{P}{\rho} = \frac{1}{4} A^2 a^2 \left(\frac{1}{a^2} \cos 2ax + \frac{1}{b^2} \cos 2by \right) \nu^2 \theta^2 t^2 \quad (34)$$

$\frac{1}{\rho} \frac{\partial P}{\partial x}, \frac{1}{\rho} \frac{\partial P}{\partial y}, \nabla^2 u, \nabla^2 v$ をそれぞれ算出する .

$$\frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{A^2 a^2}{2a} \sin 2ax \nu^2 \theta^2 t^2 \quad (35)$$

$$\frac{1}{\rho} \frac{\partial P}{\partial y} = \frac{A^2 a^2}{2b} \sin 2ay \nu^2 \theta^2 t^2 \quad (36)$$

式 (37) , (38) をそれぞれ積分する .

$$\nabla^2 u = A (a^2 + b^2) \cos ax \sin by \nu^2 \theta^2 t^2 \quad (37)$$

$$\nabla^2 v = B (a^2 + b^2) \sin ax \cos by \nu^2 \theta^2 t^2 \quad (38)$$

式 (27) , (28) を用いて , 積分定数 C_3 及び C_4 を計算する .

$$u = \frac{1}{2} \nu^2 \theta^2 A t^2 \cos ax \sin by + C_3 \quad (39)$$

$$v = \frac{1}{2} \nu^2 \theta^2 B t^2 \sin ax \cos by + C_4 \quad (40)$$

$$C_3 = (1 - \nu\theta t) A \cos ax \sin by \quad (41)$$

$$C_4 = (1 - \nu\theta t) B \sin ax \cos by \quad (42)$$

$$u = \left(1 - \nu\theta t + \frac{1}{2} \nu^2 \theta^2 t^2 \right) A \cos ax \sin by \quad (43)$$

$$v = \left(1 - \nu\theta t + \frac{1}{2} \nu^2 \theta^2 t^2 \right) B \sin ax \cos by \quad (44)$$

ここで， $e^{-\nu\theta t}$ が，指数関数のテイラー展開から以下のように表せることから，

$$e^{-\nu\theta t} = 1 - \nu\theta t + \frac{1}{2} \nu^2 \theta^2 t^2 - \frac{1}{3!} \nu^3 \theta^3 t^3 + \cdots + (-1)^{n-1} \frac{1}{(n-1)!} (\nu\theta t)^{n-1} \quad (45)$$

u および v が計算を繰り返し， n 回目の計算を終えたとき，

式 (4),(5),(27),(28),(43),(44) を用いて以下のように表すことができると考えられる．

$$\begin{aligned} u_n &= \left(1 - \nu\theta t + \frac{1}{2} \nu^2 \theta^2 t^2 - \frac{1}{3!} \nu^3 \theta^3 t^3 + \cdots + (-1)^{n-1} \frac{1}{(n-1)!} (\nu\theta t)^{n-1} \right) \\ &\quad A \cos ax \sin by \\ &= e^{-\nu\theta t} A \cos ax \sin by \\ &= e^{-\nu(a^2+b^2)t} A \cos ax \sin by \end{aligned} \quad (46)$$

$$\begin{aligned} v_n &= \left(1 - \nu\theta t + \frac{1}{2} \nu^2 \theta^2 t^2 - \frac{1}{3!} \nu^3 \theta^3 t^3 + \cdots + (-1)^{n-1} \frac{1}{(n-1)!} (\nu\theta t)^{n-1} \right) \\ &\quad B \sin ax \cos by \\ &= e^{-\nu\theta t} B \sin ax \cos by \\ &= e^{-\nu(a^2+b^2)t} B \sin ax \cos by \end{aligned} \quad (47)$$

したがって， u, v はそれぞれ以下のように表すことができる．

$$\begin{cases} u = e^{-\nu(a^2+b^2)t} A \cos ax \sin by \\ v = e^{-\nu(a^2+b^2)t} B \sin ax \cos by \end{cases} \quad (48)$$

ここで，式 (6) の条件から， $A = a = b = 1, B = -1$ とすると，

u, v はそれぞれ以下のように表される．

$$\begin{cases} u = e^{-2\nu t} \cos x \sin y \\ v = -e^{-2\nu t} \sin x \cos y \end{cases} \quad (49)$$

【演習 2】

渦度場 ω_z を導出する．

式 (49) を用いて 2 次元ベクトル場における渦度 ω_z は以下の式 (50) のように表される．

それを用いて ω_z を計算すると，式 (53) のように表すことができる．

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (50)$$

$$\frac{\partial v}{\partial x} = -e^{-2\nu t} \cos x \cos y \quad (51)$$

$$\frac{\partial u}{\partial y} = e^{-2\nu t} \cos x \cos y \quad (52)$$

$$\omega_z = -2e^{-2\nu t} \cos ax \cos by \quad (53)$$

【演習 3】

プログラム

```

/*****
PROGRAM NAME : practice_3
AUTHER : Masatsugu Kitadai
DATE : 1/4/2021
Think a Bit , Code a Bit , Test a Bit
*****/
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <sys/stat.h>
#define x_grid 31 // number of grids in x direction
#define y_grid 31 // number of grids in y direction
const char grid_space = 1; // grid width
const char *output_data_file = "2dvec_vortex000001.dat"; // name of output file
double u[x_grid][y_grid]; // u vector array
double v[x_grid][y_grid]; // v vector array
double U[x_grid][y_grid]; // absolute vector array
double omega[x_grid][y_grid]; // vorticity array

double a[x_grid][y_grid];
double b[x_grid][y_grid];
double c[x_grid][y_grid];
double d[x_grid][y_grid];

FILE *output_file; // pointer for output file

const char *xxlabel = "{/Times-New-Roman:Italic=20 x} [pixel]";
const char *yylabel = "{/Times-New-Roman:Italic=20 y} [pixel]";
const char *cb_label = "{/Symbol:Italic=20 w}_{/Times-New-Roman:Italic=20 z} [sec]"; ///color bar range min

const double v_r = 1.0; ///magnified ratio for vector length

const int x_min = 0; ///x range min
const int x_max = 30; ///x range max
const int y_min = 0; ///y range min
const int y_max = 30; ///y range max
const int cb_min = -2; ///color bar range min
const int cb_max = 2; ///color bar range max

const char *read_file_dir = "01_plot_vec_vortex";
const char *read_file_header = "2dvec_vortex";
const char *write_file_dir = "02_splot_2dvec_vortex_map";
const char *write_file_header = "2dvec_vortex_map";

//Graph parameters for GNU
char read_file[100];
void graph_GNU(); //png & eps
FILE *gp; //gnuplot
FILE *infile;

int moveFile(const char *srcPath, const char *destPath)
{
    return !rename(srcPath, destPath);
}

/*****
MAIN
*****/
double main()
{
    int i, j;
    double PI = 4.0 * atan(1.0);

```

```

// preparing for output file
output_file = fopen(output_data_file, "w");

// Array initialization
for (i = 0; i < x_grid; i++)
{
    for (j = 0; j < y_grid; j++)
    {
        u[i][j] = 0;
        v[i][j] = 0;
        U[i][j] = 0;
        omega[i][j] = 0;
    }
}

// Calc.2D velocity vector and absolute value of velocity field, vorticity field

for (i = 0; i < x_grid; i++)
{
    for (j = 0; j < y_grid; j++)
    {
        // velocity vector field and absolute value of vector
        u[i][j] = cos(2.0 * PI / x_grid * i) * sin(2.0 * PI / y_grid * j);
        v[i][j] = -sin(2.0 * PI / x_grid * i) * cos(2.0 * PI / y_grid * j);
        U[i][j] = sqrt(u[i][j] * u[i][j] + v[i][j] * v[i][j]);

        // value of vorticity field
        omega[i][j] = -2 * cos(2.0 * PI / x_grid * i) * cos(2.0 * PI / y_grid * j);

        fprintf(output_file, "%d\t%d\t%.3lf\t%.3lf\t%.3lf\t%.3lf\n",
            i * grid_space, j * grid_space, omega[i][j], u[i][j], v[i][j], U[i][j]);

        //Caution : just line breaking for printing this document

        /*
        printf("%d\t%d\t%.3lf\t%.3lf\t%.3lf\t%.3lf\n",
            i * grid_space, j * grid_space, omega[i][j], u[i][j], v[i][j], U[i][j]);
        */
    }
    fprintf(output_file, "\n");
    //printf("\n");
}

fclose(output_file);

mode_t mode = S_IRWXU | S_IRGRP | S_IXGRP | S_IROTH | S_IXOTH;
mkdir(read_file_dir, mode);
mkdir(write_file_dir, mode);

// transfer the result file to the specified folder

if (moveFile("2dvec_vortex000001.dat", "01_plot_vec_vortex/2dvec_vortex000001.dat"))
{
    puts("transferred the file");
}
else
{
    puts("cannot transferred");
}

int k = 0;
int UP = 0;
while (UP == 0)
{
    k++;
    sprintf(read_file, "%s//s%06d.dat", read_file_dir, read_file_header, k);
    printf("%s//s%06d.dat\n", read_file_dir, read_file_header, k);
}

```

```

infile = fopen(read_file, "rb");

if (infile == NULL)
{
    printf("break!\n");
    break;
}

fclose(infile);

if ((gp = popen("gnuplot", "w")) == NULL)
{
    printf("gnuplot is not here!\n");
    exit(0);
}

//PNG image
fprintf(gp, "set terminal pngcairo enhanced font 'Times New Roman,15' \n");
fprintf(gp, "set output '%s//%s%06d.png'\n", write_file_dir, write_file_header, k);
fprintf(gp, "set multiplot\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "unset key\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "set size ratio -1\n"); // <steps in scan>,<steps between scans>

fprintf(gp, "set lmargin screen 0.15\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "set rmargin screen 0.85\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "set tmargin screen 0.85\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "set bmargin screen 0.15\n"); // <steps in scan>,<steps between scans>

fprintf(gp, "set xrange [%d:%d]\n", x_min, x_max); // <steps in scan>,<steps between scans>
fprintf(gp, "set xlabel '%s'offset 0.0,0.5\n", xlabel); // <steps in scan>,<steps between scans>
fprintf(gp, "set yrange [%d:%d]\n", y_min, y_max); // <steps in scan>,<steps between scans>
fprintf(gp, "set ylabel '%s'offset 0.5,0.0\n", ylabel); // <steps in scan>,<steps between scans>

fprintf(gp, "set cblabel '%s'offset 0.0,0.0\n", cb_label);
fprintf(gp, "set cbrange['%d': '%d']\n", cb_min, cb_max);
fprintf(gp, "set colorbox vertical user origin 0.8, 0.2 size 0.025,0.6\n");
fprintf(gp, "set palette rgbformulae 22,13,-31\n");

fprintf(gp, "set pm3d map\n"); // <steps in scan>,<steps between scans>
fprintf(gp, "splot '%s//%s%06d.dat' using 2:1:3 with pm3d, '%s//%s%06d.dat' using 2:1:($1*0.0):(%lf*$5):(%lf*$4):($1*0.0) with vectors head filled lt 2 lc 'black' \n", read_file_dir, read_file_header, k,
    read_file_dir, read_file_header, k, v_r, v_r);

fflush(gp); //Clean up Data

fprintf(gp, "exit\n"); // Quit gnuplot

pclose(gp);
}

return (0);
}

```

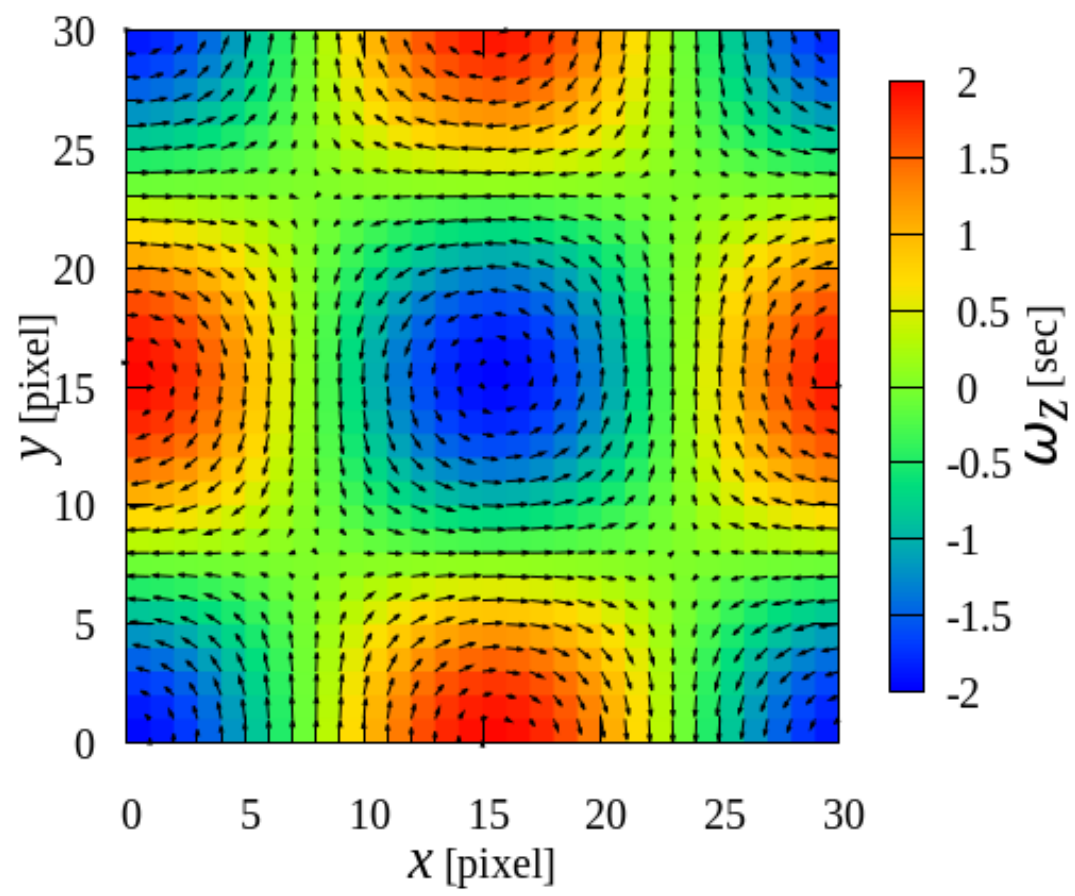


図 1 gnuplot で作図した速度場ベクトル及び渦度場