Mathematical Methods - III

Week 1

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  - We will look at some interesting applications in computer science.

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3. Numerical Integration
4. Numerical Solutions to ODEs

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Part II:

1. Modulo Arithmatic
2. Euclidian Algorithm
3. Groups
4. Lagrange's Theorem
5. RSA Algorithm
6. Polynomial Rings

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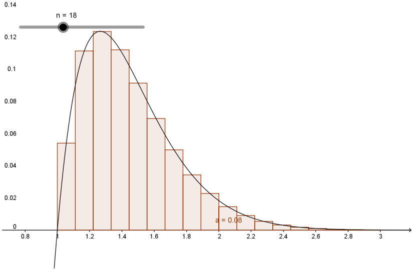
Also, observe that our results gets better and better as the divisions gets finer and finer.



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4. Eventually Monotonically increasing:
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Definition

We say $(a_n)_{n\in\mathbb{N}}$ converge to L and write $\lim\limits_{n\to\infty} a_n = L$ if

$$\forall\epsilon > 0,\ \exists N \in \mathbb{N}\ s.t.\ |a_n - L| \le \epsilon\ \ for\ \ all\ \ n \ge N$$

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We need alternative ways to get our hands on the convergence.

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Example

Consider the sequence given by

$$x_1 = 1, \qquad x_{n+1} = 3 - \frac{1}{x_n}$$

1. Show that the sequence is both bounded above and bounded below.
2. Show, using induction, that the sequence is monotonically increasing.
3. Conclude that the sequence converge to some limit and find it.
4. Explain how you can use this sequence to approximate the value of $\sqrt{5}$.

Consider the sequence given by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(\frac{3}{x_n^2} + x_n\right)$.

1. Is the sequence monotonic (i.e. monotonically decreasing or increasing)?
2. Is the sequence bounded bellow and/or bounded above?
3. Show that the sequence converge to some limit.
4. What is the limit of this sequence?

Definition

The sequence $(a_n)_{n\in\mathbb{N}}$ is said to be Cauchy if

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Fundamental importance of this result is, by showing a sequence is Cauchy one can show it converge without knowing the actual limit.

Consider a continuous function f and two points a and b such that $f(a)$ and $f(b)$ have opposite signs. A seqence (c_n) is constructed as follows.

- Begin with two points, a and b, such that $f(a)$ and $f(b)$ have opposite signs.
- Consider the midpoint between a and b; $c = \dfrac{a+b}{2}$. Set $c_1 = c$.
- Not all three of $f(a)$, $f(b)$ and $f(c)$ have the same sign. Pick the midpoint and the point with the opposite sign. Take the mid-point of the chosen points to be the next term in the sequence (c_n)
- Repeat the process.

Example

Let $f(x) = -x^5 + 2x + 2 = 0$, $a = 0$ and $b = 2$.

1. Find the first few terms of the sequence (x_n) generated by this recipe.
2. Show that $|c_n - c_{n+1}| = \alpha^n$ for some α and explicitly mention the value of α.
3. Show that this sequence is Cauchy.
4. To which value does the sequence converge?

Given a series

$$x_1 + x_2 + ... + x_i + ... = \sum_{i=1}^{\infty} x_i$$

Definition

We say the sum is L and write $\sum_{i=1}^{\infty} x_i = L$ if the sequence of n-partial sums $s_n = x_1 + x_2 + ... + x_n = \sum_{i=1}^{n} x_i$ converge to L .

1. This course will have weekly Homework.
2. HW1: Complete the first tutorial.