Mathematical Methods - III

Week 5

Jayampathy Ratnayake

Faculty of Science
Department of Mathematics
Room 209(B)
jratnaya@indiana.edu

September 11, 2017

Mathematical Methods - III

Week 5

Jayampathy Ratnayake

Faculty of Science
Department of Mathematics
Room 209(B)
jratnaya@indiana.edu

September 11, 2017

In this lesson, we will ...

In this lesson, we will ...

- Define the Taylor approximations of a function around a given point by a polynomial.

In this lesson, we will ...

- Define the Taylor approximations of a function around a given point by a polynomial.
- Use the Taylor's Theorem to approximate a function by a polynomial.

In this lesson, we will ...

- Define the Taylor approximations of a function around a given point by a polynomial.
- Use the Taylor's Theorem to approximate a function by a polynomial.
- Use the Error function to give bounds for errors associated with the approximation.

In this lesson, we will ...

- Define the Taylor approximations of a function around a given point by a polynomial.
- Use the Taylor's Theorem to approximate a function by a polynomial.
- Use the Error function to give bounds for errors associated with the approximation.
- Apply Taylor Polynomials to estimate the error of the "Mid-point rule".

- Taylor's Theorem approximates a k-times differentiable function by a polynomial, using its **derivatives**.

- Taylor's Theorem approximates a k-times differentiable function by a polynomial, using its **derivatives**.
- More precisely, given a point a and $f(a)$, $f'(a)$,...,$f^{(k)}(a)$, the Taylor's Theorem approximates the function by a polynomial of degree k.

- Taylor's Theorem approximates a k-times differentiable function by a polynomial, using its **derivatives**.
- More precisely, given a point a and $f(a)$, $f'(a)$,...,$f^{(k)}(a)$, the Taylor's Theorem approximates the function by a polynomial of degree k.
- This approximation is more accurate near a.

- Taylor's Theorem approximates a k-times differentiable function by a polynomial, using its **derivatives**.
- More precisely, given a point a and $f(a)$, $f'(a)$,...,$f^{(k)}(a)$, the Taylor's Theorem approximates the function by a polynomial of degree k.
- This approximation is more accurate near a.

Theorem (Taylor's Theorem)

If $f : \mathbb{R} \to \mathbb{R}$ *is* k*-times differentiable at the point* $a \in \mathbb{R}$,

Theorem (Taylor's Theorem)

If $f : \mathbb{R} \to \mathbb{R}$ *is k-times differentiable at the point* $a \in \mathbb{R}$*, then there is a function* $R_k : \mathbb{R} \to \mathbb{R}$ *such that,*

Theorem (Taylor's Theorem)

If $f : \mathbb{R} \to \mathbb{R}$ is k-times differentiable at the point $a \in \mathbb{R}$, then there is a function $R_k : \mathbb{R} \to \mathbb{R}$ such that,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \\ + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x)$$

Theorem (Taylor's Theorem)

If $f : \mathbb{R} \to \mathbb{R}$ is k-times differentiable at the point $a \in \mathbb{R}$, then there is a function $R_k : \mathbb{R} \to \mathbb{R}$ such that,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \\ + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x)$$

Where R_k has the property that $\lim_{x \to a} \dfrac{R_k(x)}{(x-a)^k} = 0$

Theorem (Taylor's Theorem)

If $f : \mathbb{R} \to \mathbb{R}$ is k-times differentiable at the point $a \in \mathbb{R}$, then there is a function $R_k : \mathbb{R} \to \mathbb{R}$ such that,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x)$$

Where R_k has the property that $\lim_{x \to a} \dfrac{R_k(x)}{(x-a)^k} = 0$

Remark

We will write $h_k(x) = \dfrac{R_k(x)}{(x-a)^k}$.

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the k^{th} order Taylor polynomial, of degree k,

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the k^{th} order Taylor polynomial, of degree k, at a.

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the k^{th} order Taylor polynomial, of degree k, at a.
- $f(x) = P_{k,a}(x) + R_{k,a}(x)$

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the k^{th} order Taylor polynomial, of degree k, at a.
- $f(x) = P_{k,a}(x) + R_{k,a}(x)$
- When a is clear from the context, we will simply write $f(x) = P_k(x) + R_k(x)$, omitting a from the subscript.

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the k^{th} order Taylor polynomial, of degree k, at a.
- $f(x) = P_{k,a}(x) + R_{k,a}(x)$
- When a is clear from the context, we will simply write $f(x) = P_k(x) + R_k(x)$, omitting a from the subscript.
- $R_k(x) := f(x) - P_k(x) = h_k(x)(x-a)^k$ is the error when $f(x)$ is approximated by $P_k(x)$.

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \ldots + \dfrac{f^{(k)}(a)}{k!}(x-a)^k$ is the k^{th} order Taylor polynomial, of degree k, at a.
- $f(x) = P_{k,a}(x) + R_{k,a}(x)$
- When a is clear from the context, we will simply write $f(x) = P_k(x) + R_k(x)$, omitting a from the subscript.
- $R_k(x) := f(x) - P_k(x) = h_k(x)(x-a)^k$ is the error when $f(x)$ is approximated by $P_k(x)$.
- $h = x - a$ is the step size taken from a.

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \ldots + \dfrac{f^{(k)}(a)}{k!}(x-a)^k$ is the k^{th} order Taylor polynomial, of degree k, at a.
- $f(x) = P_{k,a}(x) + R_{k,a}(x)$
- When a is clear from the context, we will simply write $f(x) = P_k(x) + R_k(x)$, omitting a from the subscript.
- $R_k(x) := f(x) - P_k(x) = h_k(x)(x-a)^k$ is the error when $f(x)$ is approximated by $P_k(x)$.
- $h = x - a$ is the step size taken from a.
- $\lim_{x \to a} \dfrac{R_k(x)}{(x-a)^k} = 0$ states that the order of convergence has to do something with k.

- $P_{k,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the k^{th} order Taylor polynomial, of degree k, at a.
- $f(x) = P_{k,a}(x) + R_{k,a}(x)$
- When a is clear from the context, we will simply write $f(x) = P_k(x) + R_k(x)$, omitting a from the subscript.
- $R_k(x) := f(x) - P_k(x) = h_k(x)(x-a)^k$ is the error when $f(x)$ is approximated by $P_k(x)$.
- $h = x - a$ is the step size taken from a.
- $\lim_{x\to a} \frac{R_k(x)}{(x-a)^k} = 0$ states that the order of convergence has to do something with k.

Theorem (Taylor's Theorem continued)

Moreover, given f *is* $k+1$ *times differentiable,*

Theorem (Taylor's Theorem continued)

Moreover, given f is $k+1$ times differentiable, there is a ϵ, between x and a, such that

Theorem (Taylor's Theorem continued)

Moreover, given f is $k+1$ times differentiable, there is a ϵ, between x and a, such that

$$R_k(x) = \frac{f^{(k+1)}(\epsilon)}{(k+1)!}(x-a)^{k+1}$$

Theorem (Taylor's Theorem continued)

Moreover, given f is $k+1$ times differentiable, there is a ϵ, between x and a, such that

$$R_k(x) = \frac{f^{(k+1)}(\epsilon)}{(k+1)!}(x-a)^{k+1}$$

Thus we can write Taylo's Theorem in this case as:

Theorem (Taylor's Theorem continued)

Moreover, given f is $k+1$ times differentiable, there is a ϵ, between x and a, such that

$$R_k(x) = \frac{f^{(k+1)}(\epsilon)}{(k+1)!}(x-a)^{k+1}$$

Thus we can write Taylo's Theorem in this case as:

Theorem (Taylor's Theorem (2))

If $f : \mathbb{R} \to \mathbb{R}$ is $k+1$-times differentiable at the point $a \in \mathbb{R}$, then there is a ϵ, between x and a such that,

Theorem (Taylor's Theorem continued)

Moreover, given f is $k+1$ times differentiable, there is a ϵ, between x and a, such that

$$R_k(x) = \frac{f^{(k+1)}(\epsilon)}{(k+1)!}(x-a)^{k+1}$$

Thus we can write Taylo's Theorem in this case as:

Theorem (Taylor's Theorem (2))

If $f : \mathbb{R} \to \mathbb{R}$ is $k+1$-times differentiable at the point $a \in \mathbb{R}$, then there is a ϵ, between x and a such that,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \\ + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(\epsilon)}{(k+1)!}(x-a)^{k+1}$$

Theorem (Taylor's Theorem continued)

Moreover, given f is $k+1$ times differentiable, there is a ϵ, between x and a, such that

$$R_k(x) = \frac{f^{(k+1)}(\epsilon)}{(k+1)!}(x-a)^{k+1}$$

Thus we can write Taylo's Theorem in this case as:

Theorem (Taylor's Theorem (2))

If $f : \mathbb{R} \to \mathbb{R}$ is $k+1$-times differentiable at the point $a \in \mathbb{R}$, then there is a ϵ, between x and a such that,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \\ + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(\epsilon)}{(k+1)!}(x-a)^{k+1}$$

Example

Let $f(x) = ln(x)$.

1. Find a polynomial approximation of degre 1 to f near $x = 1$.

Example

Let $f(x) = ln(x)$.

1. Find a polynomial approximation of degre 1 to f near $x = 1$.
2. What is the error estimates.

Example

Let $f(x) = ln(x)$.

1. Find a polynomial approximation of degre 1 to f near $x = 1$.
2. What is the error estimates.
3. Use this to approximate the value of $ln(1/2)$.

Example

Let $f(x) = ln(x)$.

1. Find a polynomial approximation of degre 1 to f near $x = 1$.
2. What is the error estimates.
3. Use this to approximate the value of $ln(1/2)$.
4. Give error bounds for your approximation.

Example

Let $f(x) = ln(x)$.

1. Find a polynomial approximation of degre 1 to f near $x = 1$.
2. What is the error estimates.
3. Use this to approximate the value of $ln(1/2)$.
4. Give error bounds for your approximation.
5. Find a polynomial approximation degree 3 to $f(x)$.

Example

Let $f(x) = ln(x)$.

1. Find a polynomial approximation of degre 1 to f near $x = 1$.
2. What is the error estimates.
3. Use this to approximate the value of $ln(1/2)$.
4. Give error bounds for your approximation.
5. Find a polynomial approximation degree 3 to $f(x)$.
6. Use this to approximate $ln(1/2)$. Give error bounds.

Example

Let $f(x) = ln(x)$.

1. Find a polynomial approximation of degre 1 to f near $x = 1$.
2. What is the error estimates.
3. Use this to approximate the value of $ln(1/2)$.
4. Give error bounds for your approximation.
5. Find a polynomial approximation degree 3 to $f(x)$.
6. Use this to approximate $ln(1/2)$. Give error bounds.

Example

Let $f(x) = \sin(x)$

1. Find a polynomial approximation of degree 1 to f near $x = 0$.
2. Use this to approximate the value of $\sin(1)$.
3. Give error bounds for your approximation.

Example

Let $f(x) = \sin(x)$

1. Find a polynomial approximation of degre 1 to f near $x = 0$.
2. Use this to approximate the value of $\sin(1)$.
3. Give error bounds for your approximation.
4. Find a polynomial approximation degree 4 to $f(x)$ near $x = 0$.
5. Use this to approximate $\sin(1)$. Give error bounds.

Example

Let $f(x) = \sqrt{x}$

1. Find a polynomial approximation of degree 1 to f near $x = 4$.
2. Use this to approximate the value of $\sqrt{3.9}$. Give error bounds for your approximation.

Example

Let $f(x) = \sqrt{x}$

1. Find a polynomial approximation of degree 1 to f near $x = 4$.
2. Use this to approximate the value of $\sqrt{3.9}$. Give error bounds for your approximation.
3. Find a polynomial approximation degree 2 to $f(x)$ near $x = 4$.
4. Use this to approximate $\sqrt{3.9}$. Give error bounds.

Example

Let $f(x) = x^2$. Use Taylor's theorem to find polynomial approximations of,

- degree 1, near $x = 0$
- degree 2, near $x = 0$
- degree 3, near $x = 0$

Example

Let $f(x) = x^2$. Use Taylor's theorem to find polynomial approximations of,

- degree 1, near $x = 0$
- degree 2, near $x = 0$
- degree 3, near $x = 0$
- degree 1, near $x = 1$
- degree 2, near $x = 1$
- degree 3, near $x = 1$

Example

Let $f(x) = x^2$. Use Taylor's theorem to find polynomial approximations of,

- degree 1, near $x = 0$
- degree 2, near $x = 0$
- degree 3, near $x = 0$
- degree 1, near $x = 1$
- degree 2, near $x = 1$
- degree 3, near $x = 1$
- degree k, near $x = a$, where $k > 2$.

Example

Find the values of

- $P_k(a)$ and
- $R_k(a)$.

Example

Suppose $f(1) = 0$, $f'(1) = 0.2$, $f''(1) = -2$ and $f'''(1) = 10$.

- Use this information to approximate $f(1.01)$.

Example

Suppose $f(1) = 0$, $f'(1) = 0.2$, $f''(1) = -2$ and $f'''(1) = 10$.

- Use this information to approximate $f(1.01)$.
- If $|f^{(4)}(x)| \leq 10$ for all x between 1 and 1.01, then estimate the error in the above approximation.

Problem

Let $f(x) = e^x$*.*

- *Find the Taylor polynomial,* $P_k(x)$*, of degree* k *to* f *near* a*.*
- *What is the error?*
- *Find a bound for error (independent of* x*), when using* $P_k(x)$ *to estimate* $f(x)$ *in the interval* $[a, a+h]$*.*

Problem

Consider $k+1$ *equally spaced points* x_0,..., x_i,...,x_k. *Let* $y_i = f(x_i)$.

1. *Then one can find the degree* k *polynomial,* $P_{k,h}$*, interpolation of these points* (x_i, y_i)*, approximating* f *in the interval* $[x_0, x_k]$.

Problem

Consider $k+1$ *equally spaced points* $x_0,\ldots,\ x_i,\ldots,x_k$. *Let* $y_i = f(x_i)$.

1. *Then one can find the degree* k *polynomial,* $P_{k,h}$, *interpolation of these points* (x_i, y_i), *approximating* f *in the interval* $[x_0, x_k]$.
2. *If the distance between two consecutive points is* h, *then*

$$x_1 = x_0 + h, \cdots, x_i = x_0 + ih, \cdots, x_k = x_0 + kh$$

Problem

Consider $k+1$ equally spaced points x_0,..., x_i,...,x_k. Let $y_i = f(x_i)$.

1. *Then one can find the degree k polynomial, $P_{k,h}$, interpolation of these points (x_i, y_i), approximating f in the interval $[x_0, x_k]$.*
2. *If the distance between two consecutive points is h, then*

$$x_1 = x_0 + h, \cdots, x_i = x_0 + ih, \cdots, x_k = x_0 + kh$$

Show that

$$\lim_{h \to 0} P_{k,h} = P_{k,x_0}$$

The degree k Taylor polynomial at x_0.