MATHEMATICAL METHODS - III (2017)
TUTORIAL 2

Problem 1. Consider Problems 1-5 of Tutorial 1. All these problems are fixed point iterations and have the form

$$x_0 = a$$
$$x_n = F(x_{n-1})$$

Answer the following questions for each of these problems. When the initial value is not given take $a = 1$.

(1) Identify the function F that is being iterated.
(2) Each of these schemes is derived to solve a particular fixed point problem. What is it? Identify the problem (equation) that is being solved.
(3) Plot $y = F(x)$ and $y = x$ on the same plot and illustrate the first few iterates graphically.
(4) Does the scheme converge?
(5) Find $F'(x)$ and discuss the applicability of the Theorem (Week 3) to prove the convergence.

Problem 2. Consider the sequence given by $x_0 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$. We showed that this sequence converge and computed its limit. Using a similar idea, write down a iterative scheme to find the value of $\sqrt{2}$. You need to verify the convergence, its limit and in the case it does not directly compute $\sqrt{2}$, how you can use the proposed scheme to estimate $\sqrt{2}$.

Problem 3. Suppose the Bisection method is used to find a solution to $x - \sin(x) - \frac{1}{2} = 0$.

(1) Which of the following intervals can be used to initiate the Bisection Algorithm.

(a) $[-\pi, 0]$
(b) $[0, 1]$
(c) $[0, \pi]$
(d) $[0, \frac{\pi}{4}]$
(e) *None of the above.*

(2) When you apply the Bisection Algorithm once, what is the resulting interval, approximation of the root and the error, respectively.
(a) *interval* $=[-\frac{\pi}{2}, 0]$, *approximated root* $= -\frac{\pi}{4}$ *and Error* $\leq \frac{\pi}{4}$
(b) *interval* $=[0, \frac{\pi}{2}]$, *approximated root* $= \frac{\pi}{4}$ *and Error* $\leq \frac{\pi}{4}$
(c) *interval* $=[0, \frac{1}{2}]$, *approximated root* $= -\frac{1}{4}$ *and Error* $\leq \frac{1}{4}$
(d) *interval* $=[0, \frac{\pi}{8}]$, *approximated root* $= \frac{\pi}{16}$ *and Error* $\leq \frac{\pi}{16}$
(e) *None of the above.*

(3) Taking the initial guess in the first part to be the 0-th iterate, how many iterates do you need to guarantee that the error is less than 10^{-2}.

(a) *6*
(b) *7*
(c) *8*
(d) *9*
(e) *None of the above.*

Problem 4. Suppose you want to use the Bisection method, with the initial interval $[0, 3]$, to find a solution to an equation $f(x) = 0$. Taking the initial guess as the 0-th iterate, find the smallest number of iterates that need to guarantee that the error is less than 10^{-2}.

(1) *6*
(2) *7*
(3) *8*
(4) *11*
(5) *None of the above.*

Problem 5.

(1) Show that the function $f(x) = \cos(x) - x$ has a zero in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
(2) Use the Bisection algorithm with three iterates to estimate a root.
(3) What can you say about the error after n iterates.
(4) Find the number of iterates required to achieve an accuracy of 10^{-3}.

Problem 6.

(1) Draw the graph of $y = \cos(x)$ for $x \in [-\pi, \pi]$.
(2) Also draw the graph of $y = x$ on the same plot.
(3) Draw a cob-web to explain the first few iterations of the fixed point problem $x = \cos(x)$, with your choice of an initial guess.
(4) Compare your answer with the answer to the previous problem. Which method converge faster?

Problem 7. Use Newton Rapson to find a solution to the following.

(1) $x^3 + x^2 - 1 = 0$
(2) $\cos(x) - x = 0$
(3) $10x = e^{-x}$
(4) $x = 1 + 0.5\sin(x)$

Problem 8. Set up Newton's scheme of iteration to find the square root of a positive number N. Using it, approximate $\sqrt{12}$. What is the error involved in your estimate?

Problem 9. Find an approximate solution, with error bounds, to $x^3 - 9x + 1 = 0$ in $[2, 4]$, using

(1) *bisection method,*
(2) *fixed point iteration and*
(3) *Newton Rapson.*

Problem 10. Use an iterative method to determine a solution to the following equations.

(1) $10x = e^{-x}$
(2) $x = 1 + 0.5\sin(x)$

Problem 11. Find the limit of the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(\frac{1}{ax_n} + x_n\right)$.

(1) $\frac{1}{\sqrt{a}}$
(2) a
(3) $\sqrt{a}$
(4) *limit does not exist*
(5) *none of the above*

Problem 12. Find the limit of the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(\frac{3}{x_n^2} + x_n\right)$.

(1) $\sqrt[3]{3}$
(2) $\frac{1}{\sqrt[3]{3}}$
(3) $\frac{1}{\sqrt{3}}$
(4) *limit does not exist*
(5) *none of the above*

Problem 13. Show that the n^{th} partial sums of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ forms a Cauchy sequence. Hence, conclude that the series converge.