MATHEMATICAL METHODS - III (2017)
TUTORIAL 3

Problem 1. A function passes through the points $(1, 0)$, $(2, 2)$, $(3, 5)$ and $(4, -1)$.

(1) Find a degree 3 polynomial, passing through these points.
(2) If the function also passes through the point $(5, -2)$, find a degree 4 polynomial passing through all 5 points.
(3) Using the answer to the above part as an polynomial approximation to f, find an estimate for the integral $\int_1^5 f(x)dx$.

Problem 2.

(1) Use Lagrange method to find a degree 3 polynomial passing through the points $(-1, 2)$, $(0, 3)$, $(1, -1)$ and $(2, 0)$. Simplify your answer.
(2) The function f is passing through the above 4 points. Use the polynomial to estimate the value $f(1.5)$.
(3) It is observed, at a later time, that the function also passes through the point $(-2, 4)$. Find a degree 4 polynomial passing through these points and use it to estimate the value $f(1.5)$. How does it compare to the previous answer?

Problem 3.

(1) Find a quadratic polynomial passing through the points (a_1, b_1), (a_2, b_2) and (m, b_3), where $m = \dfrac{a_1 + a_2}{2}$ is the midpoint in between a_1 and a_2.
(2) hence find an estimate for the integral of the funtion f in the interval $[a_1, a_2]$, $\displaystyle\int_{a_1}^{a_2} f(x)dx$, in terms of $f(a_1)$, $f(a_2)$ and $f(m)$ with m as above.

Problem 4. Given a polynomial $p(x)$ passing through (x_i, y_i) for $i = 1, ...k$, which of the following polynomial(s) pass through (x_i, y_i)'s and the additional point (a, b), where $a \neq x_1, ...x_k$.

(1) $p(x) + (b - p(a))\dfrac{(x - x_1)(x - x_2)...(x - x_k)}{(a - x_1)(a - x_2)...(a - x_k)}$
(2) $p(x) + b(x - a)$
(3) $p(x) + b(x - x_1)(x - x_2)...(x - x_k)$
(4) $p(x)(x - a) + b\dfrac{(x - x_1)(x - x_2)...(x - x_k)}{(a - x_1)(a - x_2)...(a - x_k)}$
(5) none of the above.

Problem 5. Find a degree 2 polynomial passing through the three points $(1, 2)$, $(2, 3)$ and $(3, -2)$.

(1) $(x - 1)(x - 2)(-3)$
(2) $2(x - 2)(x - 3) + 3(x - 1)(x - 3) - 2(x - 1)(x - 2)$
(3) $(x - 2)(x - 3) - 3(x - 1)(x - 3) - (x - 1)(x - 2)$
(4) $\dfrac{1}{2}(x - 2)(x - 3) - (x - 1)(x - 3) + \dfrac{1}{2}(x - 1)(x - 2)$
(5) None of the above.

Problem 6. Estimate $\sqrt{3.9}$ using the Taylor polynomial of degree 2 around the point $x = 4$, of the function $\sqrt{x}$.

Problem 7. Suppose you need to approximate $f(x) = \sin(x)$ in the interval $[-\pi/2, \pi/2]$ using a Taylor polynomial, $p(x)$, around 0.

(1) What is the smallest degree of p required to guarantee that error $|p(x) - f(x)|$ is less than 0.1 for all $x \in [-\pi/2, \pi/2]$.
(2) What is the polynomial which corresponds to the previous part.

Problem 8. Suppose $f(1) = 0$, $f'(1) = 0.2$, $f''(1) = -2$ and $f'''(1) = 10$.

(1) Use this information to find a polynomial approximating f near 1.

(a) $0.2x - 2x^2 + 10x^3$
(b) $0.2(x-1) - 2(x-1)^2 + 10(x-1)^3$
(c) $0.2x - x^2 + \frac{10}{6}x^3$
(d) $0.2(x-1) - (x-1)^2 + \frac{5}{3}(x-1)^3$
(e) None of the above.

(2) If $|f^{(4)}(x)| \leq 10$ for all x between 1 and 1.1, then estimate the error when $f(1.1)$ is approximated using the above polynomial.

(a) $\frac{1}{24000}$ *(b)* $\frac{1}{1000}$ *(c)* $\frac{1}{10000}$ *(d)* $\frac{1}{240000}$ *(e) None of the above.*

Problem 9 (2014)**.** Suppose you need to approximate $f(x) = \ln(x+1)$ in the interval $[-\frac{1}{2}, \frac{1}{2}]$ using a Taylor polynomial, $p(x)$, around 0.

(1) The degree three approximation of $f(x)$ around 0 is,

(a) $\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1$ *(b)* $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 1$ *(d)* $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x$
(c) $x^3 - x^2 + x$ *(e)* None of the above.

(2) Use the standard error bounds of the Taylor polynomial approximation to determine the smallest degree required to gurantee that the polynomial approximation of that degree computes values of $f(x) = \ln(x+1)$ in the given interval accurate to *two* decimal places.

(a) 4
(b) 3
(c) 2
(d) 1
(e) None of the above.

Problem 10 (2016)**.** Find the degree two Taylor polynomial of $f(x) = \sqrt{x}$ around 1.

(a) $1 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$
(b) $1 + \frac{1}{2}x - \frac{1}{4}x^2$
(c) $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$
(d) $1 + \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2$
(e) None of the above.

Problem 11 (2016)**.** Suppose $|f^{(3)}(x)| \leq 10$ for all x between 1 and 1.1. Based on this information, estimate the standard error bound when $f(1.1)$ is approximated using the degree two Taylor polynomial around 1.

(a) $\frac{1}{600}$ *(b)* $\frac{1}{6000}$ *(c)* $\frac{1}{3000}$ *(d)* $\frac{1}{300}$ *(e)* None of the above.