Mathematical Methods - III

Week 2

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Consider a continuous function f and two points a and b such that $f(a)$ and $f(b)$ have opposite signs. A seqence (c_n) is constructed as follows.

- Begin with two points, a and b, such that $f(a)$ and $f(b)$ have opposite signs.
- Consider the midpoint between a and b; $c = \dfrac{a+b}{2}$. Set $c_1 = c$.
- Not all three of $f(a)$, $f(b)$ and $f(c)$ have the same sign. Pick the midpoint and the point with the opposite sign. Take the mid-point of the chosen points to be the next term in the sequence (c_n)
- Repeat the process.

Example

Let $f(x) = -x^5 + 2x + 2 = 0$, $a = 0$ and $b = 2$.

1. Find the first few terms of the sequence (x_n) generated by this recipe.
2. What does this sequence approximate?
3. How can we guarantee that this is finding what we expect to find?
4. Show that $|c_n - c_{n+1}| = \alpha^n$ for some α and explicitly mention the value of α.
5. Show that this sequence is Cauchy.
6. To which value does the sequence converge? Justify your answer.

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An x^* such that $f(x^*) = 0$ is called a **zero** or a **root** of f.

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Example

Use the bisection method to find a solution to the equation $f(x) = sin(x) - x^3 + 2x + 2 = 0$.

Problem

Try applying bisection method to find a solution to $x - \frac{1}{x^2} + 2 = 0$. Take $[-2, 1]$ to be the initial interval.

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What are your observations?

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- What are the errors associated?

Theory behind the Bisection Method

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Remark

Bisection method is guaranteed to work only when the function is continuous in the initial interval.

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- Find the number of iterates required to achieve an accuracy of 10^{-k}.

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Remark

A disadvantage of the above definitions is that one needs to know the true value to compute the error.

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Let S_n be a sequence of approximations converging to L (obtained by a discretization with decreasing step sizes h_n).

Definition

We say that $e_n = |S_n - L|$ is $O(f(n))$, if there is a constant C such that

$$|S_n - L| \leq Cf(n) \text{ for all } n.$$

We write

$$Error = e_n = |S_n - L| = O(f(n)).$$

- We say S_n converge to L with order p, if $f(n) = n^{-p}$.
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- What are the drawbacks of this method?

Error Analysis for the Bisection Method

Suppose we apply the Bisection Method to the function f starting with interval $[a, b]$.

1. Find the maximum possible error after n iterates.
2. Show that when f is continuous Bisection method always converge.
3. Find the order of convergence of the Bisection method.

Problem

Find a root of the function $f(x) = x^3 - x + 1$ *with error lessthan* 10^{-3}.

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- Solve the equation $x = \cos x$. This amounts to finding a root of the function $g(x) = \cos x - x$.
- Solve $x^3 + 1 = x$.
- Solve $x = (x-1)^{\frac{1}{3}}$.
- How can you compare (2) and (3)?

More sophisticated methods than the Bisection Method can be developed around the idea of "Fixed point iterations".