Mathematical Methods - III

Week 3

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$$F(x) = x.$$

Example

Consider the following examples:

1. $f(x) = \ln x - x + 3 = 0$
2. $f(x) = \cos x - x = 0$
3. $f(x) = \sqrt{x^3 - 5} = 0$
4. $f(x) = x^3 - x + 1 = 0$
5. $0.5\sin(x) - x + 1 = 0$
6. $2\sin(x) - x + 3 = 0$

Express these as a problem of finding a fixed point (a fixed point problem). There are more than one way of doing each of these!

Fixed point iteration

If x_0 is a fixed point of F, then it is a solution to $F(x) = x$.

Algorithm (Fixed point iteration)

x_0 - initial guess
Iteration: $x_{n+1} = F(x_n)$

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- Prove the above theorem.
- Effectiveness of solving an equation using Fixed point iterations depends on how you pose the problem as a "fixed point problem".

1. There is no uniqe way to present a proble as a fixed point problem.

Example

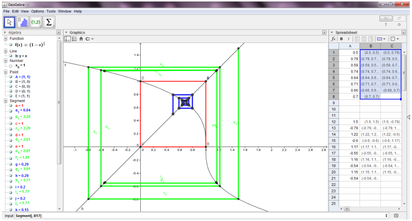
Express $f(x) = x^3 - x + 1 = 0$ as a fixed point problem in two different ways.

2. Convergence of the sequence depends on F.

Example

1. Consider $f(x) = x^3 - x + 1 = 0$. This can be written as a fixed point problem $x = x^3 + 1 = F(x)$, where $F(x) = x^3 + 1$.
2. Starting from $x_0 = 0.5$, find the first few terms generated by iterating x_0 with respect to F.
3. Consider the same problem, but now written as the fixed point problem $x = \sqrt[3]{1-x} = G(x)$, where $G(x) = \sqrt[3]{1-x}$.
4. Starting from $x_0 = 0.5$, find the first few terms generated by iterating x_0 with respect to G.
5. What can you say about the two iterations?

Geometric interpritation of fixed point iterations and "Cobweb Plots".



1. The figure shows three sequence with respect to the fixed point iteration of $x = \sqrt[3]{1-x} = G(x)$
2. Red with the initial point $x_0 = 1$. It generates a periodic orbit $1, 0, 1, 0, \ldots$. This sequence does not converge.
3. Blue with the initial point $x_0 = 0.5$. It generates a converging sequence.
4. Green with the initial point $x_0 = 1.5$. This sequence does not converge.

One can show that the fixed point iteration $F(x) = x$ converge when F is continuously differentiable function with bounded derivatives (lessthan 1).

Theorem

Suppose that

1. $x_n \in [a, b]$ *for all* n, *and*
2. $|F'(x)| \leq k < 1$ *for some* $0 < k < 1$, *for all* $x \in [a, b]$.

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Example

Can you use the Mean-Value theorem to give an errorbound in this case?

Example

Show that the fixed point iteration $x = \sqrt[3]{1-x} = G(x)$ with $x_0 = 0.5$ converge.

Example

Consider the sequence given by

$$x_1 = 1, \qquad x_{n+1} = 3 - \frac{1}{x_n}$$

1. The above scheme is derived to solve a particular fixed point problem. What is it?
2. Show that the sequence is both bounded above and bounded below.
3. Conclude that the sequence converge to some limit and find it.

Example

Consider the sequence given by

$$x_1 = 1$$
$$x_{n+1} = 1 + \frac{1}{x_n}$$

1. The above scheme is derived to solve a particular fixed point problem. What is it?
2. Is the sequence monotonic (i.e. monotonically decreasing or increasing)?
3. Is the sequence bounded bellow and/or bounded above?
4. Show that the sequence converge to some limit and find it.

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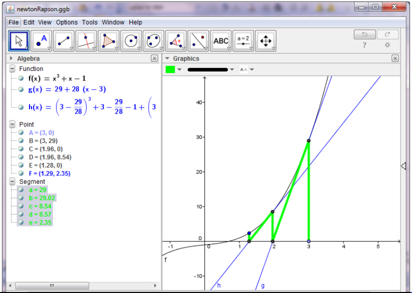
4. Need to know the derivative of f.



Algorithm (Newton-Rapson)

To solve $f(x) = 0$:
x_0 - initial guess (choosen manualy)
Iteration: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Example

When using Newton-Rapson to solve $f(x) = 0$:

- What is the function that is being itterated?

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  - Gives a canonical function to iterate.
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  - ...
- What about the convergence?

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4. Hence disscuss the convergence of Newton-Rapson.

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Example

Find a solution to $f(x) = x^3 + x - 1 = 0$ using Newton-Rapson.