Two Channel Levinson-Durbin Algorithm.

This problem is essentially the same as the single-channel case. The main difference is that some of the coefficients are matrices instead of scalars.

Backward prediction errors

$$\underline{b}_i(n) = \underline{x}(n-i) - \sum_{k=0}^{i-1} \underline{g}_{k,i}\, \underline{x}(n-k)$$

where $\underline{g}_{k,i}$ is a 2×2-element matrix.

Forward prediction errors

$$\underline{f}_i(n) = \underline{x}(n) - \sum_{k=1}^{i} \underline{a}_{k,i}\, \underline{x}(n-k)$$

Where $\underline{a}_{k,i}$ is also a 2x2-element matrix.

We can update the forward prediction errors in the following manner:

$$\underline{f}_i(n) = \underline{f}_{i-1}(n) - \underline{\rho}^{(f)}_i\, \underline{b}_{i-1}(n-1)$$

($\underline{\rho}^{(f)}$ is a 2x2-element matrix) | (Same arguments as in the single-channel case.)

Expanding both sides of the above equation, we get

$$\underline{x}(n) - \sum_{k=1}^{i} \underline{a}_{k,i}\, \underline{x}(n-k) = \underline{x}(n) - \sum_{k=1}^{i-1} a_{k,i-1}\, x(n-k)$$
$$- \underline{\rho}_i^{(f)} \left(\underline{x}(n-i) - \sum_{k=0}^{i-1} \underline{g}_{k,i-1}\, x(n-1-k) \right)$$

Equating the coefficients of $x(n-k)$, we get

$$\underline{a}_{i,i} = \underline{\rho}_i^{(f)}$$

$$\underline{a}_{k,i} = \underline{a}_{k,i-1} - \underline{\rho}_i^{(f)} \underline{g}_{k-1,i-1}$$
$$; k = 1, 2, \cdots i-1$$

Similarly, working with the backward prediction errors, we have

$$\underline{b}_i(n) = b_{i-1}(n-1) - \rho_i^{(b)} \underline{f}_{i-1}(n)$$

Expanding this gives

$$x(n-i) - \sum_{k=0}^{i-1} \underline{g}_{k,i}\, x(n-k) = \underline{x}(n-i) - \sum_{k=0}^{i-2} \underline{g}_{k,i-1}\, x(n-1-k)$$
$$-\underline{\rho}_i^{(b)} \left(\underline{x}(n) - \sum_{k=1}^{i-1} \underline{a}_{k,i-1}\, \underline{x}(n-k) \right)$$

Equating Coefficients, we get

$$\underline{g}_{0,i} = \underline{\rho}_i^{(b)}$$

$$\underline{g}_{k,i} = \underline{g}_{k-1,i-1} - \underline{\rho}_i^{(b)} \underline{a}_{k,i-1} \quad ; k = 1, 2, \ldots i-1$$

$$\underline{\rho}_i^{(b)} = \underbrace{\left(E\{\underline{f}_{i-1}(n)\,\underline{f}^T_{i-1}(n)\}\right)^{-1}}_{\begin{array}{c}\underline{\sigma}^2_{i-1,f}\\ \text{We should have}\\ \text{evaluated it}\\ \text{in Stage } i-1\\ (2\times 2 \text{ - matrix})\end{array}} \underbrace{E\{\underline{f}_{i-1}(n)\,\underline{x}^T(n-i)\}}_{\begin{array}{c}\text{Will find now}\\ \text{by expanding}\\ \underline{f}_{i-1}(n)\end{array}}$$

$$E\{\underline{f}_{i-1}(n)\,\underline{x}^T(n-i)\} = E\left\{\left(\underline{x}(n) - \sum_{k=1}^{i-1} \underline{a}_{k,i-1}\,\underline{x}(n-k)\right)\underline{x}^T(n-i)\right\}$$

$$= \underline{r}_{xx}(i) - \sum_{k=1}^{i-1} \underline{a}_{k,i-1}\,\underline{r}_{xx}(i-k).$$

Each autocorrelation here is a 2x2-element matrix

This implies that

$$\underline{\rho}_i^{(b)} = \left(\underline{\sigma}^2_{i-1,f}\right)^{-1}\left\{\underline{r}_{xx}(i) - \sum_{k=1}^{i-1} \underline{a}_{k,i-1}\,\underline{r}_{xx}(i-k)\right\}$$

Similarly

$$\underline{\rho}_i^{(f)} = \left(\underline{\sigma}^2_{i-1,b}\right)^{-1} E\{\underline{b}_{i-1}(n-1)\,\underline{x}^T(n)\}$$

$$= \left(\underline{\sigma}^2_{i-1,b}\right)^{-1}\left\{\left(\underline{x}(n-i) - \sum_{k=0}^{i-2} \underline{g}_{k,i-1}\,\underline{x}(n-1-k)\right)\underline{x}^T(n)\right\}$$

$$= \left(\underline{\sigma}^2_{i-1,b}\right)^{-1}\left\{\underline{r}_{xx}(-i) - \sum_{k=0}^{i-2} \underline{g}_{k,i-1}\,\underline{r}_{xx}(-k-1)\right\}$$

Recognizing that

$$\underline{r}_{xx}(-i) = E\{\underline{x}(n-i)\,\underline{x}^T(n)\} = \left(E\{\underline{x}(n)\,\underline{x}^T(n-i)\}\right)^T$$

$$= \underline{r}_{xx}^T(i),$$

we can write the previous expression as

$$\underline{\rho}_i^{(f)} = \left(\underline{\sigma}_{i-1,b}^2\right)^{-1}\left\{\underline{r}_{xx}^T(i) - \sum_{k=0}^{i-2} \underline{g}_{k,i-1}\,\underline{r}_{xx}^T(k+1)\right\}$$

~~Substituting for~~

$$= \left(\underline{\sigma}_{i-1,b}^2\right)^{-1}\left\{\underline{r}_{xx}^T(i) - \sum_{k=1}^{i-1} \underline{g}_{k-1,i-1}\,\underline{r}_{xx}^T(k)\right\}$$

What is left is the derivation of the expressions for $\underline{\sigma}_{i,b}^2$ and $\underline{\sigma}_{i,f}^2$.

$$\underline{\sigma}_{i,f}^2 = E\Bigg\{\Bigg(\overbrace{\underline{x}(n) - \sum_{k=1}^{i-1}\underline{a}_{k,i-1}\,\underline{x}(n-k)}^{(i-1)\text{th order prediction error}} - \underline{\rho}_i^{(f)}\underline{b}_{i-1}(n-1)\Bigg)$$

$$\cdot\left(\underline{x}(n) - \sum_{k=1}^{i-1}\underline{a}_{k,i-1}\,x(n-k) - \underline{\rho}_i^{(f)}\,\underline{b}_{i-1}(n-1)\right)^T\Bigg\}$$

$$= \underline{\sigma}_{i-1,f}^2 + \underline{\rho}_i^{(f)}\,\sigma_{i-1,b}^2\left(\underline{\rho}_i^{(f)}\right)^T$$

$$- E\{\underline{x}(n)\,\underline{b}_{i-1}^T(n-1)\}\left(\underline{\rho}_i^{(f)}\right)^T$$

$$- \underline{\rho}_i^{(f)}\,E\{\underline{b}_{i-1}(n-1)\,\underline{x}^T(n)\}$$

with the rest of the terms going to zero because of the orthogonality principle.

From earlier derivations, we have that

$$\underline{\rho}_i^{(f)} = \left(\sigma_{i-1,b}^2\right)^{-1} E\left\{\underline{b}_{i-1}(n-1)\,\underline{x}^T(n)\right\}$$

implying that

$$E\left\{\underline{b}_{i-1}(n-1)\,\underline{x}^T(n)\right\} = \sigma_{i-1,b}^2\,\underline{\rho}_i^{(f)}$$

Substituting this in the last equation on page 4, we get

Check eqns There may be errors.

$$\begin{aligned}\sigma_{i,f}^2 = \sigma_{i-1,f}^2 &+ \underline{\rho}_i^{(f)}\,\sigma_{i-1,b}^2\left(\underline{\rho}_i^{(f)}\right)^T \\ &- \left(\underline{\rho}_i^{(f)}\right)^T \sigma_{i-1,b}^2\left(\underline{\rho}_i^{(f)}\right)^T \\ &- \left(\underline{\rho}_i^{(f)}\right)\sigma_{i-1,b}^2\left(\underline{\rho}_i^{(f)}\right)\end{aligned}$$

Similarly one can show that

$$\begin{aligned}\sigma_{i,b}^2 = \sigma_{i-1,b}^2 &+ \underline{\rho}_i^{(b)}\,\sigma_{i-1,f}^2\left(\underline{\rho}_i^{(b)}\right)^T \\ &- \left(\underline{\rho}_i^{(b)}\right)^T \sigma_{i-1,f}^2\left(\underline{\rho}_i^{(B)}\right)^T \\ &- \left(\underline{\rho}_i^{(b)}\right)\sigma_{i-1,f}^2\left(\underline{\rho}_i^{(b)}\right)\end{aligned}$$

The initialization of the ~~recursive~~ recursions are done as follows.

$$\underline{\rho}_1^{(f)} = \underline{r}_{xx}^{-1}(0)\; E\{\underline{x}(n-1)\,\underline{x}^T(n)\}$$

$$= \underline{r}_{xx}^{-1}(0)\;\underline{r}_{xx}^T(1) \qquad = \underline{a}_{1,1}$$

$$\underline{\rho}_1^{(b)} = \underline{r}_{xx}^{-1}(0)\; E\{x(n)\,x^T(n-1)\}$$

$$= r_{xx}^{-1}(0)\;\underline{r}_{xx}(1) \qquad = \underline{g}_{0,1}$$

$$\underline{f}_0(n) = \underline{b}_0(n) = \underline{x}(n)$$

$$\sigma_{0,f}^2 = \sigma_{0,b}^2 = \underline{r}_{xx}(0)$$

The recursion for the ith stage is as follows:

$$\underline{\rho}_i^{(b)} = \left(\sigma_{i-1,f}^2\right)^{-1}\left\{\underline{r}_{xx}(i) - \sum_{k=1}^{i-1}\underline{a}_{k,i-1}\,\underline{r}_{xx}(i-k)\right\}$$

$$\underline{\rho}_i^{(f)} = \left(\sigma_{i-1,b}^2\right)^{-1}\left\{\underline{r}_{xx}^T(i) - \sum_{k=1}^{i-1}\underline{g}_{k-1,i-1}\,\underline{r}_{xx}^T(k)\right\}$$

$$\underline{a}_{i,i} = \underline{\rho}_i^{(f)}$$

$$\underline{a}_{k,i} = \underline{a}_{k,i-1} - \underline{\rho}_i^{(f)}\,\underline{g}_{k-1,i-1} \quad ; k=1,2,\cdots,i-1$$

$$g_{0,i} = \underline{\rho}_i^{(b)}$$

$$g_{k,i} = g_{k-1,i-1} - \rho_i^{(b)}\,a_{k,i-1} \quad ; k=1,2,\cdots,i-1$$

$$\underline{\sigma}_{i,f}^{2} = \underline{\sigma}_{i-1,f}^{2} + \underline{\rho}_i^{(f)} \underline{\sigma}_{i-1,b}^{2} \left(\underline{\rho}_i^{(f)}\right)^T - \left(\underline{\rho}_i^{(f)}\right)^T \underline{\sigma}_{i-1,b}^{2} \left(\underline{\rho}_i^{(f)}\right)^T - \left(\underline{\rho}_i^{(f)}\right) \underline{\sigma}_{i-1,b}^{2} \left(\underline{\rho}_i^{(f)}\right)$$

$$\underline{\sigma}_{i,b}^{2} = \underline{\sigma}_{i-1,b}^{2} + \underline{\rho}_i^{(b)} \underline{\sigma}_{i-1,f}^{2} \left(\underline{\rho}_i^{(b)}\right)^T - \left(\underline{\rho}_i^{(b)}\right)^T \underline{\sigma}_{i-1,f}^{2} \left(\underline{\rho}_i^{(b)}\right)^T - \left(\underline{\rho}_i^{(b)}\right) \underline{\sigma}_{i-1,f}^{2} \left(\underline{\rho}_i^{(b)}\right)$$

Please check the algebra! I did this quickly and may have made mistakes!