Exam I Solutions

1

$$\underline{W}(n+1) = \underline{W}(n) + \mu\, \text{sign}\{e(n)\}\, \underline{X}(n)$$

$$E\{\underline{W}(n+1)\} = E\{\underline{W}(n)\} + \mu \sqrt{\frac{2}{\pi \sigma_e^2(n)}}\, E\{\underline{X}(n) e(n)\}$$

where $\sigma_e^2(n) = E\{e^2(n)\}$

$$E\{\underline{X}(n) e(n)\} = E\{X(n)\left(d(n) - X^T(n) W(n)\right)\}$$

$$= E\{\underline{X}(n) d(n)\} - E\{\underline{X}(n)\underline{X}^T(n)\, \underline{W}(n)\}$$

Using the independence assumption, this simplifies to

$$E\{\underline{X}(n) e(n)\} = \underline{P}_{Xd}(n) - \underline{R}_{XX}(n)\, E\{\underline{W}(n)\}$$

Where $\underline{P}_{Xd}(n)$ and $\underline{R}_{XX}(n)$ denote the cross correlation vector ~~betwe~~ of $\underline{X}(n)$ and $d(n)$, and the autocorrelation matrix of $\underline{X}(n)$, respectively.

$$\sigma_e^2(n) = E\left\{\left(d(n) - \underline{X}^T(n)\underline{W}(n)\right)\left(d(n) - \underline{X}^T(n)\underline{W}(n)\right)^T\right\}$$

$$= r_{dd}(n) - 2\underline{P}_{Xd}^T(n) E\{\underline{W}(n)\} + E\{\underline{W}^T(n)\, \underline{R}_{XX}(n)\, \underline{W}(n)\},$$

again using the independence assumption. We will assume that the step size is small that the variance of $\underline{W}(n)$ is negligible. With this assumption,

$$\sigma_e^2(n) = r_{dd}(n) - 2\underline{P}_{Xd}^T(n)\, E\{\underline{W}(n)\} + E\{\underline{W}^T(n)\}\underline{R}_{XX}(n) E\{\underline{W}(n)\}$$

with

$$E\{\underline{W}(n+1)\} = E\{\underline{W}(n)\} + \mu \sqrt{\frac{2}{\pi \sigma_e^2(n)}} \left(\underline{P}_{\underline{X}d}(n) - \underline{R}_{\underline{X}\underline{X}}(n) E\{\underline{W}(n)\} \right)$$

We notice that the evolution equation is similar to that of the LMS adaptive filter. For the same μ, the speed of convergence at any time is faster or slower than that of the LMS adaptive filter, depending on whether

$$\frac{2}{\pi \sigma_e^2(n)} \quad \text{is larger than or smaller}$$

than 1. In general, one should expect faster convergence behavior near steady state than away from it.

2. Since $$H(z) = \frac{1}{(1-1.2z^{-1}+0.36z^{-2})(1-1.6z^{-1}+0.64z^{-2})}$$

$$= \frac{1}{1-2.8z^{-1}+2.92z^{-2}-1.344z^{-3}+0.2304z^{-4}}$$

we have that

$$x(n) = \xi(n) + 2.8\,x(n-1) - 2.92\,x(n-2) + 1.344\,x(n-3) - 0.2304\,x(n-4)$$

implying that the (optimal) MMSE predictor is

$$\hat{x}(n) = 2.8\,x(n-1) - 2.92\,x(n-2) + 1.344\,x(n-3) - 0.2304\,x(n-4)$$

It is now straightforward to convert this direct form system to a lattice structure using the conversion equations we derived in the class. The details are omitted. (Of course, this is a privilege you don't have!!)

3. Let

$$\underline{X}(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-M) \\ d(n-1) \\ d(n-2) \\ \vdots \\ d(n-N) \end{bmatrix}$$

The augmented input vector is given by

$$\underline{X}_{aug} = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-M) \\ x(n-M-1) \\ d(n-1) \\ \vdots \\ d(n-N) \\ d(n-N-1) \end{bmatrix} = \underline{J}_1 \begin{bmatrix} x(n) \\ d(n-1) \\ \underline{X}(n-1) \end{bmatrix} = J_2 \begin{bmatrix} X(n) \\ x(n-M-1) \\ d(n-N-1) \end{bmatrix}$$

where J_1 and J_2 are appropriate permutation matrix. used to re arrange the elements so that $d(n-1)$ and $X(n-M-1)$ ~~fall~~ in the right place Note that

$JJ^T = J^TJ = I$ as we did in class.

~~It immediately follows that~~

We already know from the discussion in the class that the LS solution has the form

$$\underline{W}(n) = \underline{W}(n-1) + \underline{k}(n)\,\epsilon(n),$$

where

$$\underline{W}(n) = [b_0(n), b_1(n), \ldots, b_M(n), a_1(n), \ldots, a_N(n)]^T$$

and

$$\underline{k}(n) = \underline{\underline{R}}_{XX}^{-1}(n)\,\underline{X}(n)$$

where

$$\underline{\underline{R}}_{XX}(n) = \sum_{k=1}^{n} \lambda^{n-\ell}\, \underline{X}(\ell)\,\underline{X}^T(\ell)$$

We also know that (from the form of $\underline{k}(n)$) ~~the~~ $\underline{k}(n)$ estimates 1 using $\underline{X}(n)$.

We follow the same procedure as before (for the single channel case) to update $\underline{k}(n)$. That is, we find the augmented gain vector and use it to ~~estimate~~ update $\underline{k}(n)$.

$$\hat{1}_{aug}(n) = \underline{k}^T(n-1)\,\underline{X}(n-1) + \text{estimate of 1 using new information in } \begin{bmatrix} x(n) \\ d(n-1) \end{bmatrix}$$

The new information is the forward prediction error in estimating $[x(n) \;\; d(n-1)]^T$ using $\underline{X}(n-1)$. Note that $\underline{A}(n)$, the forward predictor is now a matrix containing N+M+1 rows and 2 columns. Thus

(A) $\Rightarrow$ $$\hat{1}_{aug}(n) = \underline{k}^T(n-1)\,\underline{X}(n-1) + \underline{e}^T(n)\left[\begin{pmatrix} x(n) \\ d(n-1) \end{pmatrix} - \underline{A}^T(n)\,\underline{X}(n-1)\right]$$

Let $\tilde{\underline{k}}_{aug}(n)$ correspond to the coefficients associated with $\begin{bmatrix} x(n) \\ d(n-1) \\ \underline{X}(n-1) \end{bmatrix}$. That is

$$\hat{\underline{1}}_{aug}(n) = \underline{\tilde{k}}_{aug}^{T}(n) \begin{bmatrix} x(n) \\ d(n-1) \\ \underline{X}(n-1) \end{bmatrix}$$

$$= \underline{\tilde{k}}_{aug}^{T}(n) \underbrace{\underline{J}_1^T \underline{J}_1}_{\underline{I}} \begin{bmatrix} x(n) \\ d(n-1) \\ \underline{X}(n-1) \end{bmatrix}$$

$$= \underbrace{\left(\underline{\tilde{k}}_{aug}^{T}(n) \underline{J}_1^T \right)}_{\underline{k}_{aug}^{T}(n)} \underline{X}_{aug}(n)$$

Which implies that

$$\underline{k}_{aug}(n) = \underline{J}_1 \underline{\tilde{k}}_{aug}(n)$$

Let

$$\underline{\eta}(n) = \begin{bmatrix} x(n) \\ d(n-1) \end{bmatrix} - \underline{A}^T(n-1)\, \underline{X}(n-1)$$

It is not difficult to show that

$$\underline{A}(n) = \underline{A}(n-1) + \underline{k}(n-1)\, \underline{\eta}^T(n)$$

~~Substituting this result in~~ Going back to Ⓐ on page 2, we get

$$\underline{\tilde{k}}_{aug}^{T}(n)\, \underline{J}_1^T \underline{X}_{aug}(n) = \underline{k}^T(n-1)\, \underline{X}(n-1) + \underline{\rho}^T(n) \left(\begin{bmatrix} x(n) \\ d(n-1) \end{bmatrix} - \underline{A}^T(n)\, \underline{X}(n-1) \right)$$

$$= \begin{bmatrix} \underline{\rho}(n) \\ \underline{k}(n-1) - \underline{A}(n)\underline{\rho}(n) \end{bmatrix}^T \underline{J}_1^T \underline{X}_{ang}(n)$$

implying that

$$\underline{k}_{ang}(n) = \underline{J}_1 \begin{pmatrix} \underline{\rho}(n) \\ \underline{k}(n-1) - \underline{A}(n)\underline{\rho}(n) \end{pmatrix}$$

the two-element coefficient vector $\underline{\rho}(n)$ has yet to be calculated. $\underline{\rho}(n)$ will have the form

$$\underline{\rho}(n) = \underline{\alpha}^{-1}(n)\underline{f}(n)$$

where $$\underline{f}(n) = \begin{pmatrix} d(n) \\ d(n-1) \end{pmatrix} - \underline{A}^T(n)\underline{X}(n-1)$$

and

$$\underline{\alpha}(n) = \lambda\underline{\alpha}(n-1) + \underline{f}(n)\underline{\eta}^T(n)$$

(The derivation of this part is similar to the scalar case)

To do the update for $\underline{k}(n)$, we proceed in a similar way.

(B) $\Rightarrow$ $$\hat{1}_{ang}(n) = \underline{k}^T(n)\underline{X}(n) + \underline{\xi}^T(n)\left[\begin{pmatrix} d(n-M-1) \\ d(n-N-1) \end{pmatrix} - \underline{G}^T(n)\underline{X}(n)\right]$$

where we have to derive an expression for $\underline{\xi}(n)$ and $\underline{G}(n)$ is given by

$$\underline{G}(n) = \underline{G}(n-1) + \underline{k}(n)\underline{\phi}^T(n),$$

where

$$\underline{\phi}(n) = \begin{bmatrix} x(n-M-1) \\ d(n-N-1) \end{bmatrix} - \underline{G}^T(n-1)\,\underline{X}(n)$$

Rearranging terms in B,

$$\hat{1}_{aug}(n) = \underline{k}_{aug}^T(n) \begin{bmatrix} \underline{X}(n) \\ x(n-M-1) \\ d(n-N-1) \end{bmatrix}$$

$$= \begin{bmatrix} \underline{k}(n) - \underline{G}(n)\,\underline{\zeta}(n) \\ \underline{\zeta}(n) \end{bmatrix}^T \underline{J}_2^T\,\underline{X}_{aug}(n)$$

$\Rightarrow$

$$\underline{k}_{aug}(n) = \underline{J}_2 \begin{bmatrix} \underline{k}(n) - \underline{G}(n)\,\underline{\zeta}(n) \\ \underline{\zeta}(n) \end{bmatrix}$$

We find $\underline{k}(n)$ by first rearranging $\underline{k}_{aug}(n)$ calculated earlier using $\underline{J}_2^T$ as

$$\hat{\underline{k}}_{aug}(n) = J_2^T\,\underline{k}_{aug}(n)$$

~~Set~~ Then $\underline{\zeta}(n)$ is (formed by) the last two elements of $\hat{\underline{k}}_{aug}(n)$. Let $\hat{\underline{k}}'_{aug}(n)$ denote all but the last two elements of $\hat{\underline{k}}_{aug}(n)$. Then

$$\underline{k}(n) = \hat{\underline{k}}'_{aug}(n) + \underline{G}(n)\,\underline{\zeta}(n)$$

Since we need $\underline{k}(n)$ to ~~solve fo~~ update $\underline{G}(n)$ and $\underline{G}(n)$ to update $\underline{k}(n)$, we must simultaneously solve for $\underline{k}(n)$ and $\underline{G}(n)$. We write the two ~~vector~~ equations as

$$\text{C} \rightarrow \quad \underline{k}(n) - \underline{G}(n)\,\underline{\zeta}(n) = \hat{\underline{k}}'_{aug}(n)$$

$$-\underline{k}(n)\,\underline{\phi}^T(n) + \underline{G}(n) = \underline{G}(n-1)$$

Post-multiplying the last equation with $\underline{\zeta}(n)$, we get

$$\text{D} \rightarrow \quad -\underline{k}(n)\,\underline{\phi}^T(n)\,\underline{\zeta}(n) + \underline{G}(n)\,\underline{\zeta}(n) = \underline{G}(n-1)\,\underline{\zeta}(n)$$

Adding C and D we get

$$\underline{k}(n)\left(1 - \underline{\phi}^T(n)\,\underline{\zeta}(n)\right) = \hat{\underline{k}}'_{aug}(n) - \underline{G}(n-1)\,\underline{\zeta}(n)$$

This gives

$$\underline{k}(n) = \frac{\hat{\underline{k}}'_{aug}(n) - \underline{G}(n-1)\,\underline{\zeta}(n)}{\left(1 - \underline{\phi}^T(n)\,\underline{\zeta}(n)\right)}$$

We now have all the ~~equ~~ relationships needed to update the equation error, LS IIR filter. The equations are tabulated on the next page. The rest of the questions ~~can~~ can be answered as we discussed in class.

$$\underline{\eta}(n) = \begin{bmatrix} x(n) \\ d(n-1) \end{bmatrix} - \underline{A}^T(n-1)\,\underline{X}(n-1)$$

$$\underline{A}(n) = \underline{A}(n-1) + \underline{k}(n-1)\,\underline{\eta}^T(n)$$

$$\underline{f}(n) = \begin{bmatrix} x(n) \\ d(n-1) \end{bmatrix} - \underline{A}^T(n)\,\underline{X}(n-1)$$

$$\underline{\alpha}(n) = \lambda\underline{\alpha}(n-1) + \underline{\eta}(n)\,\underline{f}^T(n)$$

$$\underline{\rho}(n) = \underline{\alpha}^{-1}(n)\,\underline{f}(n)$$

$$\underline{k}_{aug}(n) = \underline{J}_1 \begin{bmatrix} \underline{\rho}(n) \\ \underline{k}(n-1) - \underline{A}(n)\underline{\rho}(n) \end{bmatrix}$$

$$\hat{\underline{k}}_{aug}(n) = \underline{J}_2^T\,\underline{k}_{aug}(n)$$

$\underline{\zeta}(n)$ = Vector formed by last two elements of $\hat{\underline{k}}_{aug}(n)$

$\hat{\underline{k}}'_{aug}(n)$ = Vector formed by all but the last two elements of $\hat{\underline{k}}_{aug}(n)$

$$\underline{\phi}(n) = \begin{bmatrix} x(n-N-1) \\ d(n-N-1) \end{bmatrix} - \underline{G}^T(n-1)\,\underline{X}(n)$$

$$\underline{k}(n) = \frac{1}{1-\underline{\phi}^T(n)\underline{\zeta}(n)} \left[\hat{\underline{k}}'_{aug}(n) - \underline{G}(n-1)\underline{\zeta}(n) \right]$$

$$\underline{G}(n) = \underline{G}(n-1) + \underline{k}(n)\,\underline{\phi}^T(n)$$

$$\epsilon(n) = d(n) - \underline{W}^T(n-1)\,\underline{X}(n)$$

$$\underline{W}(n) = \underline{W}(n-1) + \underline{k}(n)\,\epsilon(n)$$

$$e(n) = d(n) - \underline{W}^T(n)\,\underline{X}(n)$$

A.

Let

$$\underline{R} = \begin{bmatrix} r_{00} & \underline{p}^T \\ \underline{p} & \underline{R}' \end{bmatrix}$$

where r_{00} is the first entry of $\underline{R}$, $\underline{p}^T$ represents the first row without the first element and $\underline{R}'$ denotes the matrix obtained by removing the first row and first column of $\underline{R}$.

Then

$$\begin{bmatrix} 1 \\ -W(n) \end{bmatrix}^T \underline{R} \begin{bmatrix} 1 \\ -\underline{W}(n) \end{bmatrix} = r_{00} - 2\underline{W}^T(n)\underline{p} + \underline{W}^T(n)\underline{R}'\underline{W}(n)$$

$$\frac{\partial J(n)}{\partial W(n)} = \frac{\left\{ \left(r_{00} - 2\underline{W}^T(n)\underline{p} + \underline{W}^T(n)\underline{R}'\underline{W}(n) \right)\left(-2e(n)X(n) \right) - e^2(n)\left(-2\underline{p} + 2\underline{R}'\underline{W}(n) \right) \right\}}{\left(\begin{pmatrix} 1 \\ -\underline{W}(n) \end{pmatrix}^T \underline{R} \begin{bmatrix} 1 \\ -\underline{W}(n) \end{bmatrix} \right)^2}$$

$$= -\frac{2e(n)\underline{X}(n)}{\begin{pmatrix} 1 \\ -\underline{W}(n) \end{pmatrix}^T \underline{R} \begin{pmatrix} 1 \\ -\underline{W}(n) \end{pmatrix}} - \frac{2e^2(n)\left(\underline{R}'\underline{W}(n) - \underline{p} \right)}{\left(\begin{pmatrix} 1 \\ -\underline{W}(n) \end{pmatrix}^T \underline{R} \begin{pmatrix} 1 \\ -\underline{W}(n) \end{pmatrix} \right)^2}$$

Let

$$\bar{e}(n) = \frac{e(n)}{\begin{pmatrix} 1 \\ -\underline{W}(n) \end{pmatrix}^T \underline{R} \begin{pmatrix} 1 \\ -\underline{W}(n) \end{pmatrix}}$$

Then

$$J(n) = -2\bar{e}(n)X(n) - \bar{e}^2(n)\left(\underline{R}'\underline{W}(n) - \underline{P}\right)$$

The coefficient update equation is

$$\underline{W}(n+1) = \underline{W}(n) + \mu\, \bar{e}(n)\left\{\underline{X}(n) + \bar{e}(n)\left(\underline{R}'\underline{W}(n) - \underline{P}\right)\right\}$$