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Finite-Dimensional Approximation of Constrained Tikhonov-Regularized Solutions of Ill-Posed Linear Operator Equations

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Abstract. In this paper we derive conditions under which the finite-dimensional constrained Tikhonov-regularized solutions x_{α,C_n} of an ill-posed linear operator equation $Tx = y$ (i.e., x_{α,C_n} is the minimizing element of the functional $\|Tx - y\|^2 + \alpha\|x\|^2$, $\alpha > 0$ in the closed convex set C_n, which is a finite-dimensional approximation of a closed convex set C) converge to the best approximate solution of the equation in C. Moreover, we develop an estimate for the approximation error, which is optimal for certain sets C and C_n. We present numerical results that verify the theoretical results.

1. Introduction. In many problems arising in practice one has to solve linear operator equations

$$Tx = y,$$

where x and y are elements in real Hilbert spaces X and Y, respectively, and T is a linear bounded operator from X into Y. By a solution of the equation $Tx = y$ we always mean the best-approximate solution $T^\dagger y$, where $T^\dagger$ is the Moore-Penrose inverse of T. Unfortunately, $T^\dagger$ is not bounded in general. A prominent example for the equation $Tx = y$ is a Fredholm integral equation of the first kind,

$$\int_0^1 k(t,s)x(s)\,ds = y(t), \qquad t \in [0,1],$$

$x,\ y \in L^2[0,1]$, $k \in L^2([0,1]^2)$. Here, $T^\dagger$ is bounded if and only if k is a degenerate kernel. Therefore, one has to regularize the equation $Tx = y$. A well-known and effective regularization method is Tikhonov-regularization, where the functional $\|Tx - y\|^2 + \alpha\|x\|^2$, $\alpha > 0$, is minimized in X (cf., e.g., [4]). Often, one is not interested in the solution $T^\dagger y$, but in the best-approximate solution on a certain set C, which, in the following, we assume to be closed and convex. It is thus reasonable to require that the regularized solutions should have the same properties as the unknown exact solution, e.g., it should be an element of C. Therefore, we regularize the problem

$$Tx = y \wedge x \in C$$

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by minimizing the Tikhonov functional $\|Tx - y\|^2 + \alpha\|x\|^2$, $\alpha > 0$, on C. We call the solution $x_{\alpha,C}$ of this minimum problem the "constrained Tikhonov-regularized solution." Results about convergence rates for these solutions $x_{\alpha,C}$ have been developed in [9] (cf. also [10]). For stability and convergence results see [7], [9] and [10]. Some of these results are summarized in the next section.

For numerical computation one approximates the Hilbert space X by finite-dimensional subspaces X_n. In Section 3 we are concerned with the influence of the approximation of X and C on the convergence and the convergence rates of constrained Tikhonov-regularized solutions. In contrast to the optimal estimate of the approximation error in the unconstrained case (cf., e.g., [4]), estimates for $\|x_{\alpha,C} - x_{\alpha,C_n}\|$, where x_{α,C_n} is the constrained Tikhonov-regularized solution in C_n ($\subset X_n$), in general contain terms for which only the square root of the best-possible rate of convergence of elements in C_n to $x_{\alpha,C}$ can be guaranteed (cf. [11]). We develop an estimate which implies, at least in the case that C is a ball and $C_n = C \cap X_n$, the optimal convergence rate (see Theorem 3.9 and Corollary 3.10).

In the last section we present numerical examples for integral equations of the first kind. For the sets C we have chosen the nonnegative functions on the one hand, and balls on the other hand. X_n is the space of linear splines on a uniform grid of $(n + 1)$ points in $[0, 1]$. The tables show that the convergence rates obtained confirm the theoretical results.

2. Constrained Tikhonov Regularization. Throughout this paper, let X and Y be real Hilbert spaces; T: $X \to Y$ a bounded linear operator; the set of all bounded linear operators on X into Y will be denoted by $L(X, Y)$. The inner products and norms in X and Y, though in general different, will both be denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We consider the problem of solving

$$Tx = y \quad \text{and} \quad x \in C \tag{2.1}$$

with $y \in Y$ and $\varnothing \neq C$ ($\subset X$) a convex closed set. We define now what we mean by the "solution" of (2.1).

Definition 2.1. $x_{0,C} \in C$ is called the "*C-best approximate solution*" of (2.1) if

$$\|Tx_{0,C} - y\| = \inf\{\|Tx - y\| \mid x \in C\}$$

and

$$\|x_{0,C}\| = \inf\{\|x\| \mid x \in C \text{ and } \|Tx - y\| = \|Tx_{0,C} - y\|\}.$$

Thus, a C-best approximate solution minimizes the norm of the residual on C and has minimal norm among all minimizers.

PROPOSITION 2.2. *Let R be the metric projector of Y onto $\overline{T(C)}$.*

(a) *Ry is defined as the unique element in $\overline{T(C)}$, for which*

$$(Ry - y, u - Ry) \geqslant 0 \quad \textit{for all } u \in \overline{T(C)} \tag{2.2}$$

holds.

(b) *A C-best approximate solution exists if and only if $Ry \in T(C)$; it is then unique.*

(c) *Let $Ry \in T(C)$ and let $x_{0,C}$ be defined by Definition* 2.1. *Then*

$$Tx_{0,C} = Ry \tag{2.3}$$

and

$$\|x_{0,C}\| = \inf\{\|x\| \mid x \in C \text{ and } Tx = Ry\}.$$

Proof. The proof follows from [9, Proposition 2.2, (2.3) and (2.4)] and Definition 2.1. □

We regularize the problem of solving (2.1) by solving the minimization problem

$$\min\{\|Tx - y\|^2 + \alpha\|x\|^2 \mid x \in C\}, \qquad \alpha > 0. \tag{2.4}$$

One can show that the problem (2.4) has a unique solution $x_{\alpha,C}$ for all $\alpha > 0$ and that

$$\|Tx_{\alpha,C} - Qy\|^2 + \alpha\|x_{\alpha,C}\|^2 = \inf\{\|Tx - Qy\|^2 + \alpha\|x\|^2 \mid x \in C\}, \tag{2.5}$$

where Q is the orthogonal projector onto $\overline{R(T)}$ (cf. [9, Theorem 2.3]). We call $x_{\alpha,C}$ the "constrained Tikhonov-regularized solution" of (2.1). $x_{\alpha,C}$ can also be characterized as the unique element in C such that the variational inequality

$$(T^*Tx_{\alpha,C} + \alpha x_{\alpha,C} - T^*y, h - x_{\alpha,C}) \geq 0 \quad \text{for all } h \in C \tag{2.6}$$

holds (cf. [9, (2.7)]).

In the following two theorems we show that $x_{\alpha,C}$ converges to the C-best approximate solution $x_{0,C}$ of (2.1), if $Ry \in T(C)$, and that $x_{\alpha,C}$ depends continuously on the data y for all $\alpha > 0$. Therefore, the problem of solving (2.4) is well posed.

THEOREM 2.3. *Let $T \in L(X, Y)$, $y \in Y$.*

(a) *The constrained Tikhonov-regularized solutions $x_{\alpha,C}$ converge to an element in C for $\alpha \to 0$ if and only if $Ry \in T(C)$.*

(b) *$Ry \in T(C)$ implies that $\lim_{\alpha\to 0} x_{\alpha,C} = x_{0,C}$.*

Proof. See [9, Theorem 2.4]. □

THEOREM 2.4. *Let $\alpha > 0$ and let $x_{\alpha,C}$ and $\bar{x}_{\alpha,C}$ be the constrained Tikhonov-regularized solutions for the right-hand side y and $\bar{y}$ of Eq.* (2.1), *respectively, and let Q be the orthogonal projector onto $\overline{R(T)}$. Then $\|x_{\alpha,C} - \bar{x}_{\alpha,C}\| \leq \alpha^{-1/2}\|Q(y - \bar{y})\|$ and $\|T(x_{\alpha,C} - \bar{x}_{\alpha,C})\| \leq \|Q(y - \bar{y})\|$.*

Proof. See [9, Theorem 2.5]. □

If one knows more about $x_{0,C}$ than its existence, one can also guarantee convergence rates for constrained Tikhonov-regularized solutions.

THEOREM 2.5. *Let $Ry \in T(C)$.*

(a) *If $x_{0,C} \in R(P_C T^*)$, then $\|x_{\alpha,C} - x_{0,C}\| = O(\alpha^{1/2})$ and $\|T(x_{\alpha,C} - x_{0,C})\| = O(\alpha)$. If in addition $Qy = Ry$, we even obtain $\|x_{\alpha,C} - x_{0,C}\| = o(\alpha^{1/2})$.*

(b) *Let $Qy = Ry$. Then $\|T(x_{\alpha,C} - x_{0,C})\| = O(\alpha)$ implies that $x_{0,C} \in R(P_C T^*)$.* *(P_C denotes the metric projector of X onto C.)*

Proof. See [9, Theorem 4.2]. □

THEOREM 2.6. *Let $Ry \in T(C)$ and let ∂C be twice continuously Fréchet-differentiable in a neighborhood of $x_{0,C}$; i.e., there exist $\varepsilon > 0$, $c > 0$ and a functional F: $U_\varepsilon(x_{0,C}) \to \mathbf{R}$ such that $\partial C \cap U_\varepsilon(x_{0,C}) = \{x \in U_\varepsilon(x_{0,C}) \mid F(x) = c\}$ and F is twice continuously Fréchet-differentiable. Moreover, let $F''(x_{0,C})$ be positive definite (i.e., $\gamma > 0$ exists such that $F''(x_{0,C})(z, z) \geqslant \gamma\|z\|^2$ for all $z \in X$) and let one of the following two conditions be fulfilled*:

(i) *$Ry \neq Qy$, $x_{0,C} \in N(T)^\perp$ and $Qy \in R(T)$;*

(ii) *$Ry = Qy$, $x_{0,C} \in R(P_C T^*)$ and $x_{0,C} \neq T^*\bar{u}$,*

where $\bar{u}$ is the element of minimal norm in $U := \{u \in \overline{R(T)} \mid P_C T^* u = x_{0,C}\}$. Let $\tilde{P}$ be the orthogonal projector onto $\tilde{L} := \{h \in X \mid (f_0, h) = 0\}$, where

$$f_0 = \begin{cases} \dfrac{T^*(Ry - Qy)}{\|T^*(Ry - Qy)\|} & \text{in the case (i)}, \\[2ex] \dfrac{x_{0,C} - T^*\bar{u}}{\|x_{0,C} - T^*\bar{u}\|} & \text{in the case (ii)}. \end{cases}$$

Then $\tilde{P}x_{0,C} \in R(\tilde{P}T^*T\tilde{P})$ implies that $\|x_{\alpha,C} - x_{0,C}\| = O(\alpha)$.

Proof. The proof follows from [9, Lemma 5.12 and Theorem 5.13]. □

For a more general version of Theorem 2.6 see [9, Theorem 5.13].

We now assume that the exact right-hand side y of Eq. (2.1) is unknown and that only perturbed data y are available. We assume that we have the information $\|Q(y - y_\delta)\| \leqslant \delta$. Let $x_{\alpha,C}^\delta$ be the constrained Tikhonov-regularized solution of (2.1) with y replaced by y_δ. Then we obtain the following result.

THEOREM 2.7. *Let $Ry \in T(C)$ and $y_\delta \in Y$ such that $\|Q(y - y_\delta)\| \leqslant \delta$.*

(a) *If $\alpha(\delta)$ is such that $\lim_{\delta \to 0} \alpha(\delta) = 0$ and $\lim_{\delta \to 0} \delta^2/\alpha(\delta) = 0$, then $\lim_{\delta \to 0} x_{\alpha(\delta),C}^\delta = x_{0,C}$.*

(b) *$x_{0,C} \in R(P_C T^*)$ and $\alpha(\delta) \sim \delta$ imply that $\|x_{\alpha(\delta),C}^\delta - x_{0,C}\| = O(\delta^{1/2})$.*

(c) *Under the assumptions of Theorem 2.6, $\tilde{P}x_{0,C} \in R(\tilde{P}T^*T\tilde{P})$ ($\tilde{P}$ as in Theorem 2.6) and $\alpha(\delta) \sim \delta^{2/3}$ imply that $\|x_{\alpha(\delta),C}^\delta - x_{0,C}\| = O(\delta^{2/3})$.*

Proof. The proof follows from Theorems 2.3–2.6. □

For more results on constrained Tikhonov-regularized solutions, and detailed proofs, see [9] (cf. also [10]).

3. Finite-Dimensional Approximation of C-Best Approximate Solutions. For numerical computation one has to approximate the infinite-dimensional real Hilbert space X by a sequence of finite-dimensional subspaces. In the unconstrained case, algorithms for the finite-dimensional approximation of $T^\dagger y$ have been developed, e.g., in [2], [4] and [6].

We approximate X by finite-dimensional subspaces X_n ($n \in \mathbf{N}$) such that $X_1 \subset X_2 \subset \cdots$ and $\overline{\bigcup_{n \in \mathbf{N}} X_n} = X$. Moreover, we approximate the closed convex set C by closed convex sets $C_n \subset X_n$ (e.g., $C_n = C \cap X_n$) and compute the constrained Tikhonov-regularized solutions x_{α,C_n} in C_n. Now we look for conditions under which x_{α,C_n} converges to $x_{0,C}$ for $\alpha \to 0$ and $n \to \infty$.

Following [8], we define

Definition 3.1. Let C_n be a sequence of subsets in X.

(a) $$\text{s-}\underline{\lim}\; C_n := \{x \in X \mid \text{there exist a sequence } \{x_n\} \text{ and } N \in \mathbf{N} \text{ such that } x_n \in C_n \text{ for all } n \geqslant N \text{ and } x_n \to x \text{ for } n \to \infty\}.$$

$$\text{w-}\overline{\lim}\; C_n := \{x \in X \mid \text{there exist a sequence } \{x_k\}, \text{ a strictly monotonically increasing sequence } \{n_k\} \text{ and } K \in \mathbf{N} \text{ such that } x_k \in C_{n_k} \text{ for all } k \geqslant K \text{ and } x_k \rightharpoonup x \text{ for } k \to \infty\}.$$

(b) $$\lim_{n\to\infty} C_n = C \quad \text{if and only if} \quad \text{s-}\underline{\lim}\; C_n = \text{w-}\overline{\lim}\; C_n = C.$$

THEOREM 3.2. *Let $Ry \in T(C)$ and let C_n be a sequence of closed and convex subsets in X such that $\lim_{n\to\infty} C_n = C$. Moreover, let $\{x_n\}$ be a sequence in X such that $x_n \in C_n$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} x_n = x_{0,C}$, and let $\{\alpha_n\}$ be a sequence such that $\alpha_n \downarrow 0$ for $n \to \infty$. If one of the following two conditions:*

(i) $Ry = Qy$ and $\lim_{n\to\infty} \alpha_n^{-1}\|T(x_n - x_{0,C})\|^2 = 0$;

(ii) $Ry \neq Qy$, $C_n \subset C$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} \alpha_n^{-1}\|T(x_n - x_{0,C})\| = 0$

is fulfilled, then $\lim_{n\to\infty} x_{\alpha_n,C_n} = x_{0,C}$.

Proof. First we show that

$$\lim_{n\to\infty} Tx_{\alpha_n,C_n} = Ry \tag{3.1}$$

and

$$\limsup_{n\to\infty} \|x_{\alpha_n,C_n}\| \leqslant \|x_{0,C}\|. \tag{3.2}$$

Let (i) be fulfilled. Then (2.5) implies that

$$\|Tx_{\alpha_n,C_n} - Qy\|^2 + \alpha_n\|x_{\alpha_n,C_n}\|^2 \leqslant \|Tx_n - Qy\|^2 + \alpha_n\|x_n\|^2,$$

which together with (2.3) and $Ry = Qy$ implies that

$$0 \leqslant \lim_{n\to\infty} \|Tx_{\alpha_n,C_n} - Qy\|^2 \leqslant \lim_{n\to\infty}\left(\|Tx_n - Qy\|^2 + \alpha_n\|x_n\|^2\right) = 0$$

and

$$\limsup_{n\to\infty} \|x_{\alpha_n,C_n}\|^2 \leqslant \lim_{n\to\infty}\left(\alpha_n^{-1}\|T(x_n - x_{0,C})\|^2 + \|x_n\|^2\right) = \|x_{0,C}\|^2.$$

Now let (ii) be fulfilled. Since $C_n \subset C$, (2.2) (with $u = Tx_{\alpha_n,C_n}$) implies that

$$\begin{aligned}\|Tx_{\alpha_n,C_n} - Ry\|^2 &= \|Tx_{\alpha_n,C_n} - y\|^2 + 2\left(Tx_{\alpha_n,C_n} - Ry, y - Ry\right) - \|Ry - y\|^2 \\ &\leqslant \|Tx_{\alpha_n,C_n} - y\|^2 - \|Ry - y\|^2,\end{aligned}$$

and hence

$$\begin{aligned}\|Tx_{\alpha_n,C_n} - Ry\|^2 &\leqslant \|Tx_{\alpha_n,C_n} - y\|^2 - \|Ry - y\|^2 + \alpha_n\|x_{\alpha_n,C_n}\|^2 \\ &\leqslant \|Tx_n - y\|^2 - \|Ry - y\|^2 + \alpha_n\|x_n\|^2,\end{aligned}$$

which together with (2.3) implies that

$$0 \leqslant \lim_{n\to\infty} \|Tx_{\alpha_n,C_n} - Ry\|^2 \leqslant \lim_{n\to\infty}\left(\|Tx_n - y\|^2 - \|Tx_{0,C} - y\|^2 + \alpha_n\|x_n\|^2\right) = 0$$

and

$$\limsup_{n\to\infty}\|x_{\alpha_n,C_n}\|^2 \leqslant \lim_{n\to\infty}\left(\alpha_n^{-1}\left(\|Tx_n-y\|^2-\|Tx_{0,C}-y\|^2\right)+\|x_n\|^2\right)$$

$$\leqslant \lim_{n\to\infty}\left(\alpha_n^{-1}\|T(x_n-x_{0,C})\|\left(\|Tx_n-y\|+\|Tx_{0,C}-y\|\right)+\|x_n\|^2\right)=\|x_{0,C}\|^2.$$

Now let $\{\alpha_k\}$ be an arbitrary subsequence of $\{\alpha_n\}$. Then (3.2) implies that there exist a subsequence of $\{\alpha_k\}$ (again denoted by $\{\alpha_k\}$) and an element $u \in X$ such that $x_{\alpha_k,C_k} \rightharpoonup u$ for $k \to \infty$. Together with $\lim_{n\to\infty} C_n = C$, (3.1) and (3.2), we obtain that $u \in C$, $Tu = Ry$ and

$$\|u\|^2 = \lim_{k\to\infty}\left|(x_{\alpha_k,C_k}, u)\right| \leqslant \|u\| \limsup_{k\to\infty}\|x_{\alpha_k,C_k}\| \leqslant \|u\|\cdot\|x_{0,C}\|,$$

which implies that $\|u\| \leqslant \|x_{0,C}\|$. Proposition 2.2 now implies that $u = x_{0,C}$. Therefore, we have shown

$$x_{\alpha_n,C_n} \underset{n\to\infty}{\rightharpoonup} x_{0,C}. \tag{3.3}$$

Again, (3.2) implies that

$$\|x_{0,C}\|^2 = \lim_{n\to\infty}\left|(x_{0,C}, x_{\alpha_n,C_n})\right| \leqslant \|x_{0,C}\|\cdot \liminf_{n\to\infty}\|x_{\alpha_n,C_n}\|$$

$$\leqslant \|x_{0,C}\|\cdot\limsup_{n\to\infty}\|x_{\alpha_n,C_n}\| \leqslant \|x_{0,C}\|^2,$$

and hence $\lim_{n\to\infty}\|x_{\alpha_n,C_n}\| = \|x_{0,C}\|$. Together with (3.3), this implies the assertion. □

Theorem 3.2 is a qualitative convergence result. To obtain results about convergence rates, we develop an estimate for $\|x_{\alpha,C} - x_{\alpha,C_n}\|$, where we follow [11]. For the proof of the next theorem we need the following lemma.

LEMMA 3.3. *Let a, b, $c > 0$. Then $a^2 \leqslant a\cdot b + c^2$ implies $a \leqslant b + c$.*

Proof. Since $(ab/2c + c)^2 = a^2b^2/4c^2 + ab + c^2$, the inequality $a^2 \leqslant ab + c^2$ implies $a^2(1 + b^2/4c^2) \leqslant (ab/2c + c)^2$, and hence $a\cdot\sqrt{1 + b^2/4c^2} \leqslant ab/2c + c$. This implies that $a(\sqrt{b^2+4c^2} - b) \leqslant 2c^2$, and hence

$$a \leqslant \frac{2c^2}{\sqrt{b^2+4c^2}-b}\cdot\frac{\sqrt{b^2+4c^2}+b}{\sqrt{b^2+4c^2}+b} = \frac{1}{2}\left(\sqrt{b^2+4c^2}+b\right)$$

$$\leqslant \frac{1}{2}\left(\sqrt{b^2+4bc+4c^2}+b\right) = b + c. \quad \square$$

THEOREM 3.4. *Let C and C_n be closed and convex. For $\alpha > 0$ let g_α be defined by $g_\alpha := T^*Tx_{\alpha,C} + \alpha x_{\alpha,C} - T^*y$. Then*

$$\|x_{\alpha,C} - x_{\alpha,C_n}\| \leqslant \alpha^{-1/2}\|T(x_{\alpha,C}-h_n)\| + \|x_{\alpha,C}-h_n\|$$

$$+\alpha^{-1/2}\|g_\alpha\|^{1/2}\left(\|x_{\alpha,C}-h_n\| + \|x_{\alpha,C_n}-h\|\right)^{1/2}$$

for all $h \in C$ and $h_n \in C_n$.

Proof. For $\alpha > 0$ we can define the following inner product on X (cf. [4])

$$(3.4)\quad [u,v]_\alpha := (Tu,Tv) + \alpha(u,v) \quad \text{and} \quad |u|_\alpha := [u,u]_\alpha^{1/2}, \qquad u,v \in X.$$

Then (2.6) implies that

$$(3.5)\qquad [x_{\alpha,C}, h - x_{\alpha,C}]_\alpha \geqslant \left(T^*y, h - x_{\alpha,C}\right) \quad \text{for all } h \in C$$

and

$$(3.6)\qquad \left[x_{\alpha_n,C_n}, h_n - x_{\alpha_n,C_n}\right]_\alpha \geqslant \left(T^*y, h_n - x_{\alpha_n,C_n}\right) \quad \text{for all } h_n \in C_n$$

hold. Summing up (3.5) and (3.6) we obtain for all $h \in C$ and $h_n \in C_n$,

$$|x_{\alpha,C}|_\alpha^2 + |x_{\alpha,C_n}|_\alpha^2 \leqslant [x_{\alpha,C}, h]_\alpha + \left[x_{\alpha,C_n}, h_n\right]_\alpha - \left(T^*y, h - x_{\alpha,C} + h_n - x_{\alpha,C_n}\right)$$

and hence with $|x_{\alpha,C} - x_{\alpha,C_n}|_\alpha^2 = |x_{\alpha,C}|_\alpha^2 + |x_{\alpha,C_n}|_\alpha^2 - 2[x_{\alpha,C}, x_{\alpha,C_n}]_\alpha$ and (3.4) that

$$\begin{aligned}
|x_{\alpha,C} - x_{\alpha,C_n}|_\alpha^2 &\leqslant \left[x_{\alpha,C}, h - x_{\alpha,C_n}\right]_\alpha + \left[x_{\alpha,C_n}, h_n - x_{\alpha,C}\right]_\alpha \\
&\quad - \left(T^*y, h - x_{\alpha,C} + h_n - x_{\alpha,C_n}\right) \\
&= \left[x_{\alpha,C} - x_{\alpha,C_n}, x_{\alpha,C} - h_n\right]_\alpha + \left[x_{\alpha,C}, h - x_{\alpha,C_n} + h_n - x_{\alpha,C}\right]_\alpha \\
&\quad - \left(T^*y, h - x_{\alpha,C} + h_n - x_{\alpha,C_n}\right) \\
&= \left[x_{\alpha,C} - x_{\alpha,C_n}, x_{\alpha,C} - h_n\right]_\alpha + \left(g_\alpha, h - x_{\alpha,C_n} + h_n - x_{\alpha,C}\right) \\
&\leqslant |x_{\alpha,C} - x_{\alpha,C_n}|_\alpha |x_{\alpha,C} - h_n|_\alpha + \|g_\alpha\| \cdot \left\|h - x_{\alpha,C_n} + h_n - x_{\alpha,C}\right\|.
\end{aligned}$$

Together with Lemma 3.3, we obtain

$$|x_{\alpha,C} - x_{\alpha,C_n}|_\alpha \leqslant |x_{\alpha,C} - h_n|_\alpha + \|g_\alpha\|^{1/2} \cdot \left\|h - x_{\alpha,C_n} + h_n - x_{\alpha,C}\right\|^{1/2}.$$

Now (3.4) and $\|h - x_{\alpha,C_n} + h_n - x_{\alpha,C}\| \leqslant \|h - x_{\alpha,C_n}\| + \|h_n - x_{\alpha,C}\|$ imply

$$\begin{aligned}
\alpha^{1/2} \cdot \|x_{\alpha,C} - x_{\alpha,C_n}\| \leqslant |x_{\alpha,C} - x_{\alpha,C_n}|_\alpha &\leqslant \left(\|T(x_{\alpha,C} - h_n)\|^2 + \alpha\|x_{\alpha,C} - h_n\|^2\right)^{1/2} \\
&\quad + \|g_\alpha\|^{1/2}\left(\|x_{\alpha,C} - h_n\| + \|x_{\alpha,C_n} - h\|\right)^{1/2}.
\end{aligned}$$

Together with $(a^2 + b^2)^{1/2} \leqslant (a^2 + 2|ab| + b^2)^{1/2} = |a| + |b|$, we obtain

$$\begin{aligned}
\|x_{\alpha,C} - x_{\alpha,C_n}\| &\leqslant \alpha^{-1/2}\|T(x_{\alpha,C} - h_n)\| \\
&\quad + \|x_{\alpha,C} - h_n\| + \alpha^{-1/2}\|g_\alpha\|^{1/2}\left(\|x_{\alpha,C} - h_n\| + \|x_{\alpha,C_n} - h\|\right)^{1/2}. \qquad \square
\end{aligned}$$

Remark 3.5. (a) If $C_n \subset C$, then we can choose $h = x_{\alpha,C_n}$ in the estimate of Theorem 3.4.

(b) If $C = X$ and $C_n = X_n$, where X_n is a linear finite-dimensional subspace of X (unconstrained case), then $g_\alpha = 0$ and $h_n = P_n x_{\alpha,C} = P_n x_\alpha$, where P_n is the orthogonal projector onto X_n, imply the estimate

$$\|x_{\alpha,C} - x_{\alpha,C_n}\| = \|x_\alpha - x_\alpha^n\| \leqslant \alpha^{-1/2}\|T(I - P_n)x_{\alpha,C}\| + \|(I - P_n)x_{\alpha,C}\|,$$

which is the same estimate as in [4].

(c) Let

$$W_n^\alpha := \left\{h_n \in C_n \,|\, \left((T^*T + \alpha I)(x_{\alpha,C_n} - x_{\alpha,C}),\, h_n - x_{\alpha,C_n}\right) \geqslant 0\right\} \ (\subset C_n).$$

By a result of the Kuhn-Tucker theory (cf., e.g., [10, Proposition 1.2]), x_{α,C_n} is the unique element in W_n^α which minimizes $\|Tx - Tx_{\alpha,C}\|^2 + \alpha\|x - x_{\alpha,C}\|^2$ on W_n^α; hence

$$\|x_{\alpha,C_n} - x_{\alpha,C}\| \leqslant \alpha^{-1/2}\|T(h_n - x_{\alpha,C})\| + \|h_n - x_{\alpha,C}\| \quad \text{for all } h_n \in W_n^\alpha.$$

But W_n^α depends on α and, in general, $\varnothing \neq W_n^\alpha \neq C_n$.

If we only know that $h_n \in C_n$ ($\subset C$), then by Theorem 3.4 we obtain the estimate

$$\|x_{\alpha,C} - x_{\alpha,C_n}\| \leqslant \alpha^{-1/2}\|T(x_{\alpha,C} - h_n)\| + \|x_{\alpha,C} - h_n\| + \alpha^{-1/2}\|g_\alpha\|^{1/2}\|x_{\alpha,C} - h_n\|^{1/2},$$

which is not optimal with respect to convergence rates: If $Qy = Ry$ and $x_{0,C} \in R(P_C T^*)$, then Theorem 2.5 implies that $\|g_\alpha\| = O(\alpha)$; hence $\alpha^{-1/2}\|g_\alpha\|^{1/2}$ is bounded. Now let

$$h_n = P_{C_n} x_{\alpha,C},$$

where P_{C_n} is the metric projector of X onto C_n; then the third term of the estimate only converges with the rate $O(\|P_{C_n}x_{\alpha,C} - x_{\alpha,C}\|^{1/2})$, but the best possible rate of convergence of elements in C_n to $x_{\alpha,C}$ is $O(\|P_{C_n}x_{\alpha,C} - x_{\alpha,C}\|)$.

In the following we develop two estimates, one for $\|x_{\alpha,C_n} - x_{0,C}\|$, and one for $\|x_{\alpha,C} - x_{\alpha,C_n}\|$, which are both optimal with respect to convergence rates, if C is a ball (i.e., $C = \{x \in X \mid \|x - z\| \leqslant r\}$, $z \in X$, $r > 0$ and $C_n = C \cap X_n$, where X_n is a finite-dimensional subspace of X.

In the following, let X_n be a linear subspace of X and $C_n \subset X_n$ be closed and convex. By P_n we denote the orthogonal projector onto X_n. We then define

$$C_n^\alpha := \left\{h_n \in X_n \mid \left((T^*T + \alpha I)x_{\alpha,C_n} - T^*y, h_n - x_{\alpha,C_n}\right) \geqslant 0\right\}. \tag{3.7}$$

It follows from (2.6) (with C_n instead of C) that $C_n \subset C_n^\alpha$. C_n^α is closed and convex. By S_n^α we denote the metric projector of X onto C_n^α. Let

$$g_n^\alpha := P_n\left[(T^*T + \alpha I)x_{\alpha,C_n} - T^*y\right];$$

then for all $x \in X$,

$$S_n^\alpha x = \begin{cases} P_n x & \text{if } g_n^\alpha = 0, \\ P_n x + \max\left(0, \dfrac{(g_n^\alpha, x_{\alpha,C_n} - P_n x)}{\|g_n^\alpha\|^2}\right) g_n^\alpha & \text{if } g_n^\alpha \neq 0. \end{cases} \tag{3.8}$$

If $g_n^\alpha = 0$, (3.7) implies that $C_n^\alpha = X_n$ and hence $S_n^\alpha = P_n$. Now let $g_n^\alpha \neq 0$ and

$$\lambda_n^\alpha := \max\left(0, \frac{(g_n^\alpha, x_{\alpha,C_n} - P_n x)}{\|g_n^\alpha\|^2}\right).$$

By a result of the Kuhn-Tucker theory (cf., e.g., [10, Proposition 1.2]), $S_n^\alpha x$ is defined as the unique element in C_n^α such that $(S_n^\alpha x - x, h_n - S_n^\alpha x) \geqslant 0$ for all $h_n \in C_n^\alpha$. If $P_n x \in C_n^\alpha$, (3.7) implies that $\lambda_n^\alpha = 0$. Since $(P_n x - x) \in X_n^\perp$ and $(h_n - P_n x) \in X_n$ for all $h_n \in C_n^\alpha$, $(P_n x - x, h_n - P_n x) = 0$ for all $h_n \in C_n^\alpha$; hence $S_n^\alpha x = P_n x$. If $P_n x \notin C_n^\alpha$, (3.7) implies that

$$\lambda_n^\alpha = \frac{(g_n^\alpha, x_{\alpha,C_n} - P_n x)}{\|g_n^\alpha\|^2} > 0.$$

Since $(P_n x - x) \in X_n^{\perp}$, $(h_n - P_n x - \lambda_n^\alpha g_n^\alpha) \in X_n$ and $\lambda_n^\alpha > 0$,

$$\begin{aligned}&\left(P_n x + \lambda_n^\alpha g_n^\alpha - x, h_n - P_n x - \lambda_n^\alpha g_n^\alpha\right)\\ &\quad= \lambda_n^\alpha\left(g_n^\alpha, h_n - P_n x - \lambda_n^\alpha g_n^\alpha\right) + \left(P_n x - x, h_n - P_n x - \lambda_n^\alpha g_n^\alpha\right)\\ &\quad= \lambda_n^\alpha\left[\left(g_n^\alpha, h_n - P_n x\right) - \frac{\left(g_n^\alpha, x_{\alpha,C_n} - P_n x\right)}{\|g_n^\alpha\|^2}\|g_n^\alpha\|^2\right] = \lambda_n^\alpha\left(g_n^\alpha, h_n - x_{\alpha,C_n}\right) \geqslant 0\end{aligned}$$

for all $h_n \in C_n^\alpha$ (cf. (3.7)); hence $S_n^\alpha x = P_n x + \lambda_n^\alpha g_n^\alpha$.

LEMMA 3.6. *Let X_n, P_n and S_n^α be as above. Let C be defined by $C := \{x \in X \mid \|x - z\| \leqslant r\}$, where $z \in X$ and $r > 0$, and $C_n := C \cap X_n$.*

(a) *$C_n \neq \varnothing$ if and only if $r \geqslant \|(I - P_n)z\|$.*

(b) *Let $r \geqslant \|(I - P_n)z\|$. Then for all $x \in C$, $\|(P_n - S_n^\alpha)x\| \leqslant \|(I - P_n)z\|^2/r$.*

Proof. (a) Since

$$C_n = \left\{x_n \in X_n \mid \|x_n - z\| \leqslant r\right\} = \left\{x_n \in X_n \mid \|x_n - P_n z\|^2 \leqslant r^2 - \|(I - P_n)z\|^2\right\},$$

we have $C_n \neq \varnothing$ if and only if $r^2 - \|(I - P_n)z\|^2 \geqslant 0$, which is equivalent to $r \geqslant \|(I - P_n)z\|$.

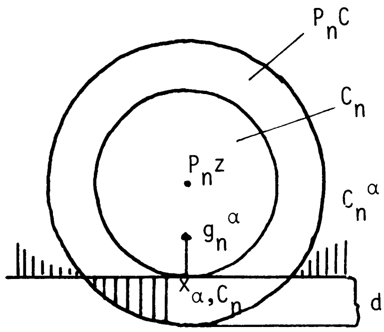

FIGURE 3.1

(b) If $g_n^\alpha = 0$, then (3.8) implies that $\|(P_n - S_n^\alpha)x\| = 0$ for all $x \in C$; hence it is clear that $\|(P_n - S_n^\alpha)x\| = 0 \leqslant \|(I - P_n)z\|^2/r$. Now let $g_n^\alpha \neq 0$; then (2.6) (with C replaced by C_n) implies that $x_{\alpha,C_n} \in \tilde{\partial} C_n$ ($\tilde{\partial}$ with respect to X_n), i.e., $\|x_{\alpha,C_n} - P_n z\|^2 = r^2 - \|(I - P_n)z\|^2$. We see from Figure 3.1 that for all $x \in C$ with $P_n x \notin C_n^\alpha$,

$$\begin{aligned}\|(P_n - S_n^\alpha)x\| &\leqslant d = r - \left(r^2 - \|(I - P_n)z\|^2\right)^{1/2}\\ &= \frac{\|(I - P_n)z\|^2}{r + \left(r^2 - \|(I - P_n)z\|^2\right)^{1/2}} \leqslant \frac{1}{r}\cdot\|(I - P_n)z\|^2.\end{aligned}$$

If $P_n x \in C_n^\alpha$, then (3.8) implies that $\|(P_n - S_n^\alpha)x\| = 0 \leqslant \|(I - P_n)z\|^2/r$. $\square$

Now we estimate $\|x_{\alpha,C_n} - x_{0,C}\|$.

THEOREM 3.7. *Let $Ry = Qy \in T(C)$, $x_{0,C} \in R(P_C T^*)$; let X_n be a linear subspace of X and $C_n \subset X_n \cap C$ be closed and convex. By P_n we denote the orthogonal projector onto X_n and by S_n^α the metric projector from X onto C_n^α, where C_n^α is defined by* (3.7).

Then

$$\frac{1}{2}\|x_{\alpha,C_n} - x_{0,C}\| \leqslant \alpha^{1/2}\|\bar{u}\| + \left(1 + \alpha^{-1/2}\|T(I - P_n)\|\right)\|(I - P_n)x_{0,C}\|$$
$$+\alpha^{-1/2}\|T(S_n^\alpha - P_n)x_{0,C}\| + \|(S_n^\alpha - P_n)x_{0,C}\|$$
$$+\max\left[0, (P_n x_{0,C}, (S_n^\alpha - P_n)x_{0,C}) - \|(I - P_n)x_{0,C}\|^2\right]^{1/2},$$

where $\bar{u}$ is the unique element of minimal norm in $U := \{u \in \overline{R(T)} \mid P_C T^* u = x_{0,C}\}$.

Proof. The existence and uniqueness of $\bar{u}$ follows from [9, Lemma 4.1]. Since $\bar{u} \in U$, we have $P_C T^*\bar{u} = x_{0,C}$ and hence $(x_{0,C} - T^*\bar{u}, h - x_{0,C}) \geqslant 0$ for all $h \in C$. Together with (3.7), (2.3), $Ry = Qy$ and $C_n \subset C$, we obtain

$$0 \leqslant \left(T^*Tx_{\alpha,C_n} + \alpha x_{\alpha,C_n} - T^*Tx_{0,C}, S_n^\alpha x_{0,C} - x_{\alpha,C_n}\right)$$
$$+\alpha\left(T^*\bar{u} - x_{0,C}, x_{0,C} - x_{\alpha,C_n} + S_n^\alpha x_{0,C} - S_n^\alpha x_{0,C}\right)$$
$$= \left((T^*T + \alpha I)(x_{\alpha,C_n} - x_{0,C}) + \alpha T^*\bar{u}, S_n^\alpha x_{0,C} - x_{\alpha,C_n} + x_{0,C} - x_{0,C}\right)$$
$$+\alpha\left(T^*\bar{u} - x_{0,C}, x_{0,C} - S_n^\alpha x_{0,C}\right),$$

which implies that

$$\left\|T(x_{\alpha,C_n} - x_{0,C})\right\|^2 + \alpha\|x_{\alpha,C_n} - x_{0,C}\|^2$$
$$\leqslant \left(\alpha T^*\bar{u}, x_{0,C} - x_{\alpha,C_n}\right)$$
$$+\left((T^*T + \alpha I)(x_{\alpha,C_n} - x_{0,C}) + \alpha T^*\bar{u}, S_n^\alpha x_{0,C} - x_{0,C}\right)$$
$$+\alpha\left(T^*\bar{u} - x_{0,C}, x_{0,C} - S_n^\alpha x_{0,C}\right)$$
$$= \left(T(x_{\alpha,C_n} - x_{0,C}), T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\right) + \alpha\left(x_{\alpha,C_n}, (S_n^\alpha - I)x_{0,C}\right)$$
$$\leqslant \left\|T(x_{\alpha,C_n} - x_{0,C})\right\| \cdot \|T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\| + \alpha\left(x_{\alpha,C_n}, (S_n^\alpha - I)x_{0,C}\right).$$

Together with Lemma 3.3, we obtain

$$\left\|T(x_{\alpha,C_n} - x_{0,C})\right\| \leqslant \|T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\| + \alpha^{1/2}\max\left[0, \left(x_{\alpha,C_n}, (S_n^\alpha - I)x_{0,C}\right)\right]^{1/2},$$

and hence

$$\alpha\|x_{\alpha,C_n} - x_{0,C}\|^2 \leqslant \|T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\|^2 + \alpha\max\left[0, \left(x_{\alpha,C_n}, (S_n^\alpha - I)x_{0,C}\right)\right]$$
$$+\alpha^{1/2}\|T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\|\max\left[0, \left(x_{\alpha,C_n}, (S_n^\alpha - I)x_{0,C}\right)\right]^{1/2}$$
$$\leqslant \left(\|T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\| + \alpha^{1/2}\max\left[0, \left(x_{\alpha,C_n}, (S_n^\alpha - I)x_{0,C}\right)\right]^{1/2}\right)^2,$$

which implies that

$$\|x_{\alpha,C_n} - x_{0,C}\| \leqslant \alpha^{-1/2}\|T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\| + \max\left[0, \left(x_{0,C}, (S_n^\alpha - I)x_{0,C}\right)\right]^{1/2}$$
$$+\|x_{\alpha,C_n} - x_{0,C}\|^{1/2} \cdot \|(S_n^\alpha - I)x_{0,C}\|^{1/2}.$$

Again, Lemma 3.3 implies

$$\|x_{\alpha,C_n} - x_{0,C}\|^{1/2} \leqslant \left[\alpha^{-1/2}\|T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\|\right.$$
$$\left.+\max\left[0, \left(x_{0,C}, (S_n^\alpha - I)x_{0,C}\right)\right]^{1/2}\right]^{1/2}$$
$$+\|(S_n^\alpha - I)x_{0,C}\|^{1/2}.$$

Since $(a+b)^2 \leqslant 2(a^2+b^2)$, $(I-P_n)^2 = (I-P_n)$, $((I-P_n)x_{0,C}, P_n x_{0,C}) = 0$, and $((I-P_n)x_{0,C}, (S_n^\alpha - P_n)x_{0,C}) = 0$, this implies that

$$\begin{aligned}
\tfrac{1}{2}\|x_{\alpha,C_n} - x_{0,C}\| &\leqslant \alpha^{-1/2}\|T(S_n^\alpha - I)x_{0,C} - \alpha\bar{u}\| \\
&\qquad + \max\left[0, (x_{0,C}, (S_n^\alpha - I)x_{0,C})\right]^{1/2} + \|(S_n^\alpha - I)x_{0,C}\| \\
&\leqslant \alpha^{1/2}\|\bar{u}\| + \alpha^{-1/2}\|T(I-P_n)\|\,\|(I-P_n)x_{0,C}\| + \alpha^{-1/2}\|T(S_n^\alpha - P_n)x_{0,C}\| \\
&\quad + \|(I-P_n)x_{0,C}\| + \|(S_n^\alpha - P_n)x_{0,C}\| \\
&\quad + \max\left[0, (P_n x_{0,C}, (S_n^\alpha - P_n)x_{0,C}) - \|(I-P_n)x_{0,C}\|^2\right]^{1/2}. \quad \square
\end{aligned}$$

Remark 3.8. (a) If $P_n x_{0,C} \in C_n^\alpha$, where C_n^α is defined by (3.7), then it follows that $(S_n^\alpha - P_n)x_{0,C} = 0$. Together with Theorem 3.7, we obtain the estimate

$$\tfrac{1}{2}\|x_{\alpha,C_n} - x_{0,C}\| \leqslant \alpha^{1/2}\|\bar{u}\| + \left(1 + \alpha^{-1/2}\|T(I-P_n)\|\right)\|(I-P_n)x_{0,C}\|.$$

If we choose α_n such that $\alpha_n^{-1/2}\|T(I-P_n)\| \leqslant$ const and $\alpha_n \to 0$ for $n \to \infty$, then we obtain the convergence rate $O(\alpha_n^{1/2} + \|(I-P_n)x_{0,C}\|)$, which is optimal (see Theorem 2.5) for $x_{0,C} \in R(P_C T^*)$.

(b) If C and C_n are defined as in Lemma 3.6 and α_n is chosen such that $\alpha_n^{-1/2} \max(\|T(I-P_n)\|, \|(I-P_n)z\|) \leqslant$ const and $\alpha_n \to 0$ for $n \to \infty$, then we obtain the convergence rate $O(\alpha_n^{1/2} + \|(I-P_n)x_{0,C}\| + \|(I-P_n)z\|)$, which is again optimal.

If we know that $\|x_{\alpha,C} - x_{0,C}\| = O(\alpha)$, then the estimate of Theorem 3.7 can never be optimal with respect to α. Therefore, we develop an estimate for $\|x_{\alpha,C} - x_{\alpha,C_n}\|$, which is optimal with respect to α even in the case $\|x_{\alpha,C} - x_{0,C}\| = O(\alpha)$.

THEOREM 3.9. *Let $Ry = Qy \in T(C)$ and C_n, P_n, S_n^α as in Theorem 3.7. For $\alpha > 0$ let g_α be defined by $g_\alpha := T^*Tx_{\alpha,C} + \alpha x_{\alpha,C} - T^*y$. Then*

$$\begin{aligned}
\tfrac{1}{2}\|x_{\alpha,C} - x_{\alpha,C_n}\| &\leqslant \alpha^{-1/2}\|(I-P_n)T^*\|\left(\|(I-P_n)x_{0,C}\| + \|x_{\alpha,C} - x_{0,C}\|\right) \\
&\quad + \alpha^{-1/2}\|T(S_n^\alpha - P_n)x_{\alpha,C}\| + \|(S_n^\alpha - P_n)x_{\alpha,C}\| \\
&\quad + \alpha^{-1}\|(I-P_n)T^*T(x_{\alpha,C} - x_{0,C})\| \\
&\quad + \alpha^{-1/2}\|g_\alpha\|^{1/2}\|(S_n^\alpha - P_n)x_{\alpha,C}\|^{1/2}.
\end{aligned}$$

Proof. It follows from (3.7), (2.3), $Ry = Qy$, (2.6), and $C_n \subset C$ that

$$\begin{aligned}
0 &\leqslant \left(T^*T(x_{\alpha,C_n} - x_{0,C}) + \alpha x_{\alpha,C_n}, S_n^\alpha x_{\alpha,C} - x_{\alpha,C_n}\right) \\
&\quad + \left(T^*T(x_{\alpha,C} - x_{0,C}) + \alpha x_{\alpha,C}, x_{\alpha,C_n} - x_{\alpha,C}\right) \\
&= \left(T^*T(x_{\alpha,C} - x_{\alpha,C_n}) + \alpha(x_{\alpha,C} - x_{\alpha,C_n}), x_{\alpha,C_n} - x_{\alpha,C}\right) \\
&\quad + \left(T^*T(x_{\alpha,C_n} - x_{0,C}) + \alpha x_{\alpha,C_n}, S_n^\alpha x_{\alpha,C} - x_{\alpha,C}\right)
\end{aligned}$$

which together with $(P_n - I)x_{\alpha,C_n} = 0$ implies that

$$\begin{aligned}
&\left\|T(x_{\alpha,C} - x_{\alpha,C_n})\right\|^2 + \alpha\|x_{\alpha,C} - x_{\alpha,C_n}\|^2 \\
&\quad\leqslant \left(T^*T(x_{\alpha,C_n} - x_{0,C}) + \alpha x_{\alpha,C_n}, S_n^\alpha x_{\alpha,C} - x_{\alpha,C}\right) \\
&\quad= \left(T^*Tx_{\alpha,C} + \alpha x_{\alpha,C} - T^*y, (S_n^\alpha - I)x_{\alpha,C}\right) \\
&\qquad+\left((T^*T + \alpha I)(x_{\alpha,C_n} - x_{\alpha,C}), (S_n^\alpha - I)x_{\alpha,C}\right) \\
&\quad= \left(g_\alpha, (S_n^\alpha - P_n)x_{\alpha,C}\right) - \left(T^*T(x_{\alpha,C_n} - x_{\alpha,C}), (S_n^\alpha - I)x_{\alpha,C}\right) \\
&\qquad+\left((P_n - I)T^*T(x_{\alpha,C} - x_{0,C}), x_{\alpha,C}\right) \\
&\qquad-\left((P_n - I)T^*T(x_{\alpha,C} - x_{0,C}), x_{\alpha,C_n}\right) \\
&\qquad+\alpha\left(x_{\alpha,C},(P_n - I)x_{\alpha,C}\right) + \alpha\left(x_{\alpha,C_n} - x_{\alpha,C}, (S_n^\alpha - P_n)x_{\alpha,C}\right) \\
&\qquad+\alpha\left((P_n - I)(x_{\alpha,C_n} - x_{\alpha,C}), x_{\alpha,C}\right) \\
&\quad= \left(T(x_{\alpha,C_n} - x_{\alpha,C}), T(S_n^\alpha - I)x_{\alpha,C}\right) + \left(g_\alpha, (S_n^\alpha - P_n)x_{\alpha,C}\right) \\
&\qquad+\left(x_{\alpha,C} - x_{\alpha,C_n}, (P_n - I)T^*T(x_{\alpha,C} - x_{0,C}) - \alpha(S_n^\alpha - P_n)x_{\alpha,C}\right) \\
&\quad\leqslant \left\|T(x_{\alpha,C_n} - x_{\alpha,C})\right\| \cdot \left\|T(S_n^\alpha - I)x_{\alpha,C}\right\| + \|g_\alpha\| \left\|(S_n^\alpha - P_n)x_{\alpha,C}\right\| \\
&\qquad+\|x_{\alpha,C} - x_{\alpha,C_n}\|\left(\left\|(P_n - I)T^*T(x_{\alpha,C} - x_{0,C})\right\| + \alpha\left\|(S_n^\alpha - P_n)x_{\alpha,C}\right\|\right).
\end{aligned}$$

Together with Lemma 3.3, we obtain

$$\begin{aligned}
\left\|T(x_{\alpha,C} - x_{\alpha,C_n})\right\| \leqslant{}& \left\|T(S_n^\alpha - I)x_{\alpha,C}\right\| \\
&+ \Big[\|g_\alpha\| \cdot \left\|(S_n^\alpha - P_n)x_{\alpha,C}\right\| + \|x_{\alpha,C} - x_{\alpha,C_n}\| \\
&\quad\cdot\left(\left\|(P_n - I)T^*T(x_{\alpha,C} - x_{0,C})\right\| + \alpha\left\|(S_n^\alpha - P_n)x_{\alpha,C}\right\|\right)\Big]^{1/2},
\end{aligned}$$

and hence (using the fact that for a, $b > 0$, $a^2 + ab + b^2 < (a + b)^2$),

$$\begin{aligned}
\alpha\|x_{\alpha,C} - x_{\alpha,C_n}\|^2 \leqslant{}& \Big(\left\|T(S_n^\alpha - I)x_{\alpha,C}\right\| \\
&+ \Big[\|g_\alpha\| \cdot \left\|(S_n^\alpha - P_n)x_{\alpha,C}\right\| + \|x_{\alpha,C} - x_{\alpha,C_n}\| \\
&\quad\cdot\left(\left\|(P_n - I)T^*T(x_{\alpha,C} - x_{0,C})\right\| + \alpha\left\|(S_n^\alpha - P_n)x_{\alpha,C}\right\|\right)\Big]^{1/2}\Big)^2,
\end{aligned}$$

which implies that (note that $\sqrt{a + b} \leqslant \sqrt{a} + \sqrt{b}$ for $a, b > 0$)

$$\begin{aligned}
&\|x_{\alpha,C} - x_{\alpha,C_n}\| \\
&\quad\leqslant \alpha^{-1/2}\left\|T(S_n^\alpha - I)x_{\alpha,C}\right\| + \alpha^{-1/2}\|g_\alpha\|^{1/2}\left\|(S_n^\alpha - P_n)x_{\alpha,C}\right\|^{1/2} \\
&\qquad+ \|x_{\alpha,C} - x_{\alpha,C_n}\|^{1/2}\left(\alpha^{-1}\left\|(P_n - I)T^*T(x_{\alpha,C} - x_{0,C})\right\| + \left\|(S_n^\alpha - P_n)x_{\alpha,C}\right\|\right)^{1/2}.
\end{aligned}$$

Again, Lemma 3.3 implies

$$\|x_{\alpha,C} - x_{\alpha,C_n}\|^{1/2}$$
$$\leqslant \left(\alpha^{-1/2}\|T(S_n^\alpha - I)x_{\alpha,C}\| + \alpha^{-1/2}\|g_\alpha\|^{1/2}\|(S_n^\alpha - P_n)x_{\alpha,C}\|^{1/2}\right)^{1/2}$$
$$+\left(\alpha^{-1}\|(P_n - I)T^*T(x_{\alpha,C} - x_{0,C})\| + \|(S_n^\alpha - P_n)x_{\alpha,C}\|\right)^{1/2}.$$

Since $(a+b)^2 \leqslant 2(a^2+b^2)$ and $(I-P_n)^2 = (I-P_n)$, this implies that

$$\begin{aligned}
\tfrac{1}{2}\|x_{\alpha,C} - x_{\alpha,C_n}\| &\leqslant \alpha^{-1/2}\|T(S_n^\alpha - I)x_{\alpha,C}\| + \|(S_n^\alpha - P_n)x_{\alpha,C}\| \\
&\quad + \alpha^{-1}\|(P_n - I)T^*T(x_{\alpha,C} - x_{0,C})\| \\
&\quad + \alpha^{-1/2}\|g_\alpha\|^{1/2}\|(S_n^\alpha - P_n)x_{\alpha,C}\|^{1/2} \\
&\leqslant \alpha^{-1/2}\|(I - P_n)T^*\|\left(\|(I-P_n)x_{0,C}\| + \|x_{\alpha,C} - x_{0,C}\|\right) \\
&\quad + \alpha^{-1/2}\|T(S_n^\alpha - P_n)x_{\alpha,C}\| + \|(S_n^\alpha - P_n)x_{\alpha,C}\| \\
&\quad + \alpha^{-1}\|(I - P_n)T^*T(x_{\alpha,C} - x_{0,C})\| \\
&\quad + \alpha^{-1/2}\|g_\alpha\|^{1/2}\|(S_n^\alpha - P_n)x_{\alpha,C}\|^{1/2}. \quad \square
\end{aligned}$$

If C and C_n are as in Lemma 3.3, we obtain the following corollary.

COROLLARY 3.10. *Let C be defined by $C := \{x \in X \mid \|x - z\| \leqslant r\}$, where $z \in X$ and $r > 0$. For a sequence $\{X_n\}$ of finite-dimensional linear subspaces of X, such that $X_1 \subset X_2 \subset \cdots$ and $\overline{\bigcup_{n\in\mathbf{N}} X_n} = X$, let $C_n := C \cap X_n$. Let $T \in L(X,Y)$ be compact and $y \in Y$ such that $Ry = Qy \in T(C)$. For all $n \in \mathbf{N}$, let $r \geqslant \|(I - P_n)z\|$, where P_n denotes the orthogonal projector onto X_n.*

(a) *If $x_{0,C} \in R(P_C T^*)$ and $\alpha_n := c \cdot \max\{\|(I-P_n)x_{0,C}\|^2, \|(I-P_n)z\|^2, \|(I-P_n)T^*\|^2\}$, $c > 0$, then*

$$\|x_{\alpha_n,C_n} - x_{0,C}\| = O\left(\max\left(\|(I-P_n)x_{0,C}\|, \|(I-P_n)z\|, \|(I-P_n)T^*\|\right)\right).$$

(b) *If $x_{0,C} \in R(P_C T^*)$, $x_{0,C} \neq T^*\bar{u}$ and $\tilde{P}x_{0,C} \in R(\tilde{P}T^*T\tilde{P})$, where $\tilde{P}$ is the orthogonal projector onto $\tilde{L} := \{h \in X \mid (x_{0,C} - T^*\bar{u}, h) = 0\}$, and $\alpha_n := c \cdot \max\{\|(I-P_n)x_{0,C}\|, \|(I-P_n)z\|, \|(I-P_n)T^*\|^2\}$, $c > 0$, then*

$$\|x_{\alpha_n,C_n} - x_{0,C}\|$$
$$= O\left(\max\left(\|(I-P_n)x_{0,C}\|, \|(I-P_n)z\|, \|(I-P_n)T^*\|^2, \|(I-P_n)T^*T\|\right)\right).$$

Proof. Lemma 3.6 implies that $C_n \neq \varnothing$ for all $n \in \mathbf{N}$ and $\|(S_n^\alpha - P_n)x_{\alpha,C}\| \leqslant \|(I-P_n)z\|^2/r$ for all $\alpha > 0$. Therefore, by Theorem 3.9, we obtain the estimate

$$\tfrac{1}{2}\|x_{\alpha_n,C_n} - x_{0,C}\|$$
$$\leqslant \tfrac{1}{2}\|x_{\alpha_n,C} - x_{0,C}\|$$
$$+\alpha_n^{-1/2}\|(I-P_n)T^*\|\left(\|(I-P_n)x_{0,C}\| + \|x_{\alpha_n,C} - x_{0,C}\|\right) \tag{3.9}$$
$$+\alpha^{-1/2}r^{-1}\|T\| \cdot \|(I-P_n)z\| \cdot \|(I-P_n)z\| + r^{-1}\|(I-P_n)z\|^2$$
$$+\alpha_n^{-1}\|(I-P_n)T^*T(x_{\alpha_n,C} - x_{0,C})\| + r^{-1/2}\alpha_n^{-1/2}\|g_{\alpha_n}\|^{1/2}\|(I-P_n)z\|.$$

The compactness of T guarantees (cf. [4, Lemma 4.21]) that $\|(I-P_n)T^*\| \to 0$ and $\|(I-P_n)T^*T\| \to 0$ for $n \to \infty$.

Since $x_{0,C} \in R(P_C T^*)$, Theorem 2.5 implies that $\|T(x_{\alpha,C}-x_{0,C})\| = O(\alpha)$, hence by definition of g_α (cf. Theorem 3.9) $\|g_\alpha\| = O(\alpha)$. This implies that $\alpha_n^{-1/2}\|g_{\alpha_n}\|^{1/2}$ is bounded. The choice of α_n in (a) and (b), respectively, implies that

$$\alpha_n^{-1/2}\|(I-P_n)T^*\| \quad \text{and} \quad \alpha_n^{-1/2}\|(I-P_n)z\|$$

are bounded. Therefore, (3.9) implies that

$$\begin{aligned}\|x_{\alpha_n,C_n}-x_{0,C}\| = O\Big(\max\Big(&\|x_{\alpha_n,C}-x_{0,C}\|, \|(I-P_n)x_{0,C}\|, \|(I-P_n)z\|\Big)\\ &+\alpha_n^{-1}\Big\|(I-P_n)T^*T(x_{\alpha_n,C}-x_{0,C})\Big\|\Big).\end{aligned} \tag{3.10}$$

(a) Theorem 3.2 implies that

$$\|x_{\alpha_n,C}-x_{0,C}\| = O(\alpha_n^{1/2})$$

and that $\alpha_n^{-1}\|T(x_{\alpha_n,C}-x_{0,C})\|$ is bounded. Therefore, we obtain with (3.10)

$$\|x_{\alpha_n,C_n}-x_{0,C}\| = O\Big(\max\Big(\|(I-P_n)x_{0,C}\|, \|(I-P_n)z\|, \|(I-P_n)T^*\|\Big)\Big).$$

(b) Since $\partial C = \{x \in X \mid \|x-z\|^2 = r^2\}$, $F(x) := \|x-z\|^2$ is twice continuously Fréchet-differentiable and $F''(x)$ is positive definite for all $x \in X$ (note that $F''(x)(z,z) = 2\|z\|^2$), Theorem 2.6 implies that $\|x_{\alpha_n,C}-x_{0,C}\| = O(\alpha_n)$ and $\alpha_n^{-1}\|x_{\alpha_n,C}-x_{0,C}\|$ is bounded. Therefore, we obtain from (3.10)

$$\begin{aligned}&\|x_{\alpha_n,C_n}-x_{0,C}\|\\ &\quad = O\Big(\max\Big(\|(I-P_n)x_{0,C}\|, \|(I-P_n)z\|, \|(I-P_n)T^*\|^2, \|(I-P_n)T^*T\|\Big)\Big). \quad \square\end{aligned}$$

Corollary 3.10 shows that it is possible to obtain optimal convergence rates if C is a ball and if $C_n = C \cap X_n$, where X_n is a linear subspace of X. If we know only that $Ry = Qy$, $x_{0,C} \in R(P_C T^*)$, $C_n \subset C$ for all $n \in \mathbf{N}$, and $P_{C_n}x_{0,C} \to x_{0,C}$ for $n \to \infty$, where P_{C_n} denotes the metric projector from X onto C_n, then we can only guarantee the square root of the best-possible rate of convergence of elements of C_n to the C-best approximate solution of (2.1), i.e., $O(\|(P_{C_n}-I)x_{0,C}\|^{1/2})$: Since $C_n \subset C_n^\alpha$, $\|(S_n^\alpha - I)x_{0,C}\| \leqslant \|(P_{C_n}-I)x_{0,C}\|$. Now Theorem 3.7 implies that

$$\begin{aligned}\tfrac{1}{2}\|x_{\alpha_n,C_n}-x_{0,C}\| &\leqslant \|\bar{u}\|\alpha_n^{1/2} + \alpha_n^{-1/2}\|T(S_n^\alpha - I)x_{0,C}\| + \|(S_n^\alpha - I)x_{0,C}\|\\ &\qquad + \max\big[0, (x_{0,C}, (S_n^\alpha - I)x_{0,C})\big]^{1/2}\\ &\leqslant \|\bar{u}\|\alpha_n^{1/2} + \Big\|(P_{C_n}-I)x_{0,C}\Big\|^{1/2}\\ &\qquad \cdot\Big[\|T\|\cdot\alpha_n^{-1/2}\Big\|(P_{C_n}-I)x_{0,C}\Big\|^{1/2} + \Big\|(P_{C_n}-I)x_{0,C}\Big\|^{1/2} + \|x_{0,C}\|^{1/2}\Big].\end{aligned}$$

If we choose α_n such that $\alpha_n \sim \|(P_{C_n}-I)x_{0,C}\|$, then this estimate implies that

$$\|x_{\alpha_n,C_n}-x_{0,C}\| = O\Big(\Big\|(P_{C_n}-I)x_{0,C}\Big\|^{1/2}\Big).$$

If we do not know the data y exactly, but elements $y_\delta \in Y$ such that $\|Q(y-y_\delta)\| \leqslant \delta$, then we can obtain results about convergence rates in dependence on δ analogously to Theorem 2.7, using Theorem 2.4 and the fact that $\|x_{\alpha,C_n}^\delta - x_{0,C}\| \leqslant \|x_{\alpha,C_n}-x_{0,C}\| + \|x_{\alpha,C_n}^\delta - x_{\alpha,C_n}\|$.

4. Numerical Results. All results of this chapter were obtained with FORTRAN programs on an IBM 3031. We compute the constrained Tikhonov-regularized solutions x_{α_n, C_n} of linear Fredholm integral equations of the first kind,

$$\int_0^1 k(t,s)x(s)\,ds = y(t), \qquad t \in [0,1],$$

where $y \in L^2[0,1]$ and $x \in C$ ($\subset L^2[0,1]$). C is either the set of nonnegative functions, i.e., $C = \{x \in L^2[0,1] \,|\, x \geqslant 0 \text{ a.e.}\}$, or C is a ball, i.e., $C = \{x \in L^2[0,1] \,|\, \|x - z\| \leqslant r\}$, $z \in L^2[0,1]$, $r > 0$. We approximate $X := L^2[0,1]$ by the sequence of linear subspaces X_n, where X_n is the space of linear splines on a uniform grid of $(n+1)$ points in $[0,1]$. It is easy to see that $X_1 \subset X_2 \subset X_4 \subset X_8 \subset \cdots \subset X_{2^k} \subset \cdots$. C is approximated by $C_n := C \cap X_n$. For the set of nonnegative functions we used the Lemke method to obtain x_{α, C_n} and for balls we used the Wilson method. For details on these methods see [1] and [3], respectively.

We use the following notations: $e_n := \|x_{\alpha_n, C_n} - x_{0,C}\|$ and $e_n^\delta := \|x^{\delta_n}_{\alpha_n, C_n} - x_{0,C}\|$, where $x^{\delta_n}_{\alpha_n, C_n}$ is the constrained Tikhonov-regularized solution of the integral equation with y replaced by y_{δ_n} ($\|Q(y - y_{\delta_n})\| \leqslant \delta_n$),

$$\delta_n - \% := \delta_n \cdot \frac{100}{\|y\|}.$$

Example 4.1. Here the kernel is always given by

$$k(t,s) := \begin{cases} 2(s-t) + 6(s-t)^2 + 4(s-t)^3 & \text{if } s \leqslant t, \\ 2(s-t) - 6(s-t)^2 + 4(s-t)^3 & \text{if } s > t. \end{cases}$$

One can show that $T^* = -T$, $N(T) = N(T^*) = [1]$ and

$$R(T) = R(T^*) = \left\{ x \in H^3[0,1] \,|\, \int_0^1 y(s)\,ds = 0 \right.$$
$$\left. \text{and } y^{(k)}(0) = y^{(k)}(1) \text{ for } k = 0,1,2 \right\}.$$

It follows from [5] that $\|(I - P_n)T^*\| = O(n^{-2})$.

(a) $C := \{x \in X \,|\, x \geqslant 0 \text{ a.e.}\}$,

$$y(t) := \begin{cases} \frac{131}{6720} - \frac{29}{80}t - \frac{11}{20}t^2 + \frac{5}{2}t^3 + 4t^4 - \frac{128}{5}t^6 & \text{if } 0 \leqslant t < \frac{1}{4}, \\ \frac{223}{3360} - \frac{81}{80}t + \frac{49}{20}t^2 - \frac{3}{2}t^3 & \text{if } \frac{1}{4} \leqslant t \leqslant \frac{3}{4}, \\ \frac{8951}{6720} - \frac{621}{80}t + \frac{319}{20}t^2 - \frac{27}{2}t^3 + 4t^4 & \text{if } \frac{3}{4} < t \leqslant 1, \end{cases}$$

and $\|y\| \approx 0.0415$. The exact solution is

$$x_{0,C}(s) := \begin{cases} 1 + 4s - 128s^3 & \text{if } 0 \leqslant s < \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} \leqslant s \leqslant \frac{3}{4}, \\ -3 + 4s & \text{if } \frac{3}{4} < s \leqslant 1, \end{cases}$$

and $x_{0,C} \in R(P_C T^*)$.

It follows from [5] that, even though $x_{0,C} \notin H^2[0,1]$, one has $\|(I - P_{C_n})x_{0,C}\| = O(n^{-2})$ for $n = 2^k$, $k \in \mathbf{N}$. This follows from the fact that $x_{0,C|_{I_j}} \in H^2(I_j)$, $j = 1,2,3$, with $I_1 := [0, \frac{1}{4}]$, $I_2 := [\frac{1}{4}, \frac{3}{4}]$, and $I_3 := [\frac{3}{4}, 1]$.

Now Theorems 3.7 and 2.4 imply that for $\alpha_n = c_1 \cdot n^{-2}$, $\delta_n = c_2 \cdot n^{-2}$, $c_1, c_2 > 0$, we should obtain the convergence rates $e_n = O(n^{-1})$, $e_n^\delta = O(n^{-1})$.

n	α_n	e_n	$e_n \cdot n \cdot 10$
4	$6.3*10^{-6}$	$1.9*10^{-1}$	7.7
8	$1.6*10^{-6}$	$4.9*10^{-2}$	4.0
16	$3.9*10^{-7}$	$1.0*10^{-2}$	1.6
32	$9.8*10^{-8}$	$3.9*10^{-3}$	1.3
64	$2.4*10^{-8}$	$2.6*10^{-3}$	1.6

n	$\delta_n - \%$	e_n^δ	$e_n^\delta \cdot n \cdot 10$
4	$6.3*10^{-2}$	$1.9*10^{-1}$	7.7
8	$1.6*10^{-2}$	$4.9*10^{-2}$	4.0
16	$3.9*10^{-3}$	$1.0*10^{-2}$	1.7
32	$9.8*10^{-4}$	$3.9*10^{-3}$	1.3
64	$2.4*10^{-4}$	$2.6*10^{-3}$	1.7

$\alpha_n = 10^{-4} * \delta_n - \%$

The last column of each table shows that the rate obtained confirms the theoretical result.

(b) $C := \{x \in X \mid \|x - z\| \leqslant r\}$, where

$$z(t) := 7t^3 - t, \quad r^2 := \frac{317055556}{100546875} \approx 3.15,$$

$$\text{and} \quad c := \|z\|^2 - r^2 = \frac{138756944}{100546875} \approx 1.38,$$

$$y(t) := \frac{1}{6}(\lambda - 1) + \left(\frac{19}{38}\lambda - \frac{7}{10}\right)t + \frac{53}{10}(1 - \lambda)t^2 + \left(\frac{82}{15}\lambda - 5\right)t^3$$
$$+ (\lambda - 1)t^4 - \frac{7}{5}\lambda t^5 + \frac{7}{5}t^6 - \frac{2}{5}\lambda t^7,$$

$$\text{where } \lambda := \frac{8}{75}\sqrt{\frac{79263889}{1264835}} \approx 0.84, \text{ and } \|y\| \approx 0.0243.$$

The exact solution is

$$x_{0,C}(s) = \left(\frac{41}{30}\lambda - \frac{5}{4}\right) + (\lambda - 1)s - \frac{7}{2}\lambda s^2 + 7s^3 - \frac{7}{2}\lambda s^4$$

and

$$x_{0,C} \in R(P_C T^*).$$

It follows from [5] that $\|(I - P_n)x_{0,C}\| = O(n^{-2})$, $\|(I - P_n)z\| = O(n^{-2})$, and hence Corollary 3.10 and Theorem 2.4 imply that for $\alpha_n = c_1 \cdot n^{-4}$, $\delta_n = c_2 \cdot n^{-4}$, c_1, $c_2 > 0$ we should obtain the convergence rates $e_n = O(n^{-2})$, $e_n^\delta = O(n^{-2})$.

n	α_n	e_n	$e_n \cdot n^2 \cdot 10$
4	$3.9*10^{-3}$	$2.4*10^{-2}$	3.9
8	$2.4*10^{-4}$	$4.4*10^{-3}$	2.8
16	$1.5*10^{-5}$	$9.8*10^{-4}$	2.5
32	$9.5*10^{-7}$	$2.2*10^{-4}$	2.2
64	$6.0*10^{-8}$	$5.0*10^{-5}$	2.0

n	$\delta_n - \%$	e_n^δ	$e_n^\delta \cdot n^2 \cdot 10$
4	$3.9*10^{+1}$	$6.2*10^{-2}$	10.0
8	$2.4*10^{+0}$	$6.4*10^{-3}$	4.1
16	$1.5*10^{-1}$	$1.3*10^{-3}$	3.3
32	$9.5*10^{-3}$	$3.5*10^{-4}$	3.6
64	$6.0*10^{-4}$	$8.3*10^{-5}$	3.4

$\alpha_n = 10^{-4} * \delta_n - \%$

The last column of each table shows that the rate obtained confirms the theoretical result.

Example 4.2. Here the kernel is always given by

$$k(t,s) = \begin{cases} s(1-t) & \text{if } s < t, \\ t(1-s) & \text{if } s \geqslant t. \end{cases}$$

One can show that $T^* = T$, T is injective, and $R(T) = \{y \in H^2[0,1] \mid y(0) = y(1) = 0\}$. It follows from [5] that $\|(I - P_n)T^*\| = O(n^{-2})$.

(a) $C := \{x \in X \mid x \geqslant 0 \text{ a.e.}\}$,

$$y(t) := \begin{cases} \frac{5}{16}t - \frac{8}{9}t^3 + \frac{16}{27}t^4 & \text{if } 0 \leqslant t < \frac{3}{4}, \\ \frac{3}{16}(1-t) & \text{if } \frac{3}{4} \leqslant t \leqslant 1, \end{cases}$$

and $\|y\| \approx 0.0579$. The exact solution is

$$x_{0,C}(s) := \begin{cases} \frac{64}{9}s\left(\frac{3}{4} - s\right) & \text{if } 0 \leqslant s < \frac{3}{4}, \\ 0 & \text{if } \frac{3}{4} \leqslant s \leqslant 1, \end{cases}$$

and $x_{0,C} \in R(P_C T^*)$, but $x_0 = x_{0,C} \notin R(T^*)$.

It follows from [5], analogously to Example 4.1(a), that $\|(I - P_{C_n})x_{0,C}\| = O(n^{-2})$ for $n = 2^k$, $k \in \mathbf{N}$. Now Theorems 3.7 and 2.4 imply that for $\alpha_n = c_1 \cdot n^{-2}$, $\delta_n = c_2 \cdot n^{-2}$, c_1, $c_2 > 0$, we should obtain the convergence rates $e_n = O(n^{-1})$, $e_n^\delta = O(n^{-1})$.

The unconstrained Tikhonov-regularized solutions $x_{\alpha_n}^n$ $(= (T_n^* T_n + \alpha_n I)^{-1} T_n^* y$; $T_n := TP_n)$ do not converge as fast as the constrained Tikhonov-regularized x_{α_n, C_n}. (The necessary condition "$x_0 \in R(T^*)$" for the convergence rate $o(\alpha^{1/2})$ in the unconstrained case (cf. [4]) is not fulfilled). We denote $\tilde{e}_n := \|x_{\alpha_n}^n - x_{0,C}\|$, $\tilde{e}_n^\delta := \|x_{\alpha_n}^{n,\alpha_n} - x_{0,C}\|$.

n	α_n	$\tilde{e}_n$	e_n	$e_n \cdot n^2 * 10^3$
4	$6.3*10^{-5}$	$6.0*10^{-3}$	$2.3*10^{-4}$	3.6
8	$1.6*10^{-5}$	$5.3*10^{-3}$	$5.6*10^{-5}$	3.6
16	$3.9*10^{-6}$	$4.5*10^{-3}$	$1.4*10^{-5}$	3.5
32	$9.8*10^{-7}$	$3.8*10^{-3}$	$3.5*10^{-6}$	3.5
64	$2.4*10^{-7}$	$3.2*10^{-3}$	$8.6*10^{-7}$	3.5

n	α_n	$\tilde{e}_n$	e_n	$e_n \cdot n \cdot 10$
4	$6.3*10^{-5}$	$4.7*10^{-2}$	$4.6*10^{-2}$	1.8
8	$1.6*10^{-5}$	$2.5*10^{-2}$	$1.9*10^{-2}$	1.5
16	$3.9*10^{-6}$	$1.5*10^{-2}$	$9.2*10^{-3}$	1.5
32	$9.8*10^{-7}$	$8.9*10^{-3}$	$4.5*10^{-3}$	1.4
64	$2.4*10^{-7}$	$5.3*10^{-3}$	$2.2*10^{-3}$	1.4

$\alpha_n = 10^{-4} * \delta_n - \%$

The last column of each table shows that the rate obtained confirms the theoretical result.

(b) $C := \{x \in X \mid \|x - z\| \leqslant r\}$, where

$$z(t) := \frac{1}{48}(145 - 288t - 4t^3 + 2t^4), \quad r^2 = 3,$$

$$\text{and} \quad c := \|z\|^2 - r^2 = \frac{30407}{725760} \approx 0.04,$$

$$y(t) := \frac{1}{1440}(11t - 15t^2 + 6t^5 - 2t^6) \quad \text{and} \quad \|y\| \approx 0.00097.$$

The exact solution is

$$x_{0,C}(s) := \frac{1}{48}(1 - 4s^3 + 2s^4),$$

and $\tilde{P}x_{0,C} = \tilde{P}T^*T\tilde{P}1$, where $\tilde{P}$ is the orthogonal projector onto

$$\tilde{L} := \left\{h \in X \mid \left(x_{0,C} - T^*\bar{u}, h\right) = 0\right\} = \left\{h \in X \mid (z - x_{0,C}, h) = 0\right\},$$

but $x_0 = x_{0,C} \notin R(T^*)$. It follows from [5] that $\|(I - P_n)x_{0,C}\| = O(n^{-2})$, $\|(I - P_n)z\| = O(n^{-2})$, and hence Corollary 3.10 and Theorem 2.4 imply that for $\alpha_n = c_1 \cdot n^{-2}$, $\delta_n = c_2 \cdot n^{-3}$, $c_1, c_2 > 0$, we should obtain the convergence rates $e_n = O(n^{-2})$, $e_n^\delta = O(n^{-2})$. We see from the tables that the unconstrained Tikhonov-regularized solutions $x_{\alpha_n}^n$ and $x_{\alpha_n}^{n,\delta_n}$ do not converge as fast as the constrained Tikhonov-regularized solutions x_{α_n,C_n} and $x_{\alpha_n,C_n}^{\delta_n}$, respectively. (As in (a), the necessary condition "$x_0 \in R(T^*)$" for the convergence rate $o(\alpha^{1/2})$ in the unconstrained case is not fulfilled.)

n	$\delta_n - \%$	$\tilde{e}_n^\delta$	e_n^δ	$e_n^\delta \cdot n \cdot 10$
4	$6.3*10^{-1}$	$5.0*10^{-2}$	$4.8*10^{-2}$	1.9
8	$1.6*10^{-1}$	$2.5*10^{-2}$	$1.8*10^{-2}$	1.6
16	$3.9*10^{-2}$	$1.4*10^{-2}$	$8.6*10^{-3}$	1.4
32	$9.8*10^{-3}$	$9.4*10^{-3}$	$5.0*10^{-3}$	1.6
64	$2.4*10^{-3}$	$5.3*10^{-3}$	$2.2*10^{-3}$	1.4

n	$\delta_n - \%$	$\tilde{e}_n^\delta$	e_n^δ	$e_n^\delta \cdot n^2 * 10^3$
4	$1.6*10^{-1}$	$6.0*10^{-3}$	$2.3*10^{-4}$	3.7
8	$2.0*10^{-2}$	$5.3*10^{-3}$	$5.6*10^{-5}$	3.6
16	$2.4*10^{-3}$	$4.5*10^{-3}$	$1.4*10^{-5}$	3.6
32	$3.1*10^{-4}$	$3.8*10^{-3}$	$3.5*10^{-6}$	3.6
64	$3.8*10^{-5}$	$3.2*10^{-3}$	$8.7*10^{-7}$	3.6

$\alpha_n = n \cdot 10^{-4} * \delta_n - \%$

The last column of each table shows that the rate obtained confirms the theoretical result. From the third column we see that the rates for $\tilde{e}_n$ and $\tilde{e}_n^\delta$ are very slow. For more examples see [10].

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1. R. W. Cottle & G. B. Dantzig, "Complementary pivot theory of mathematical programming," *Linear Algebra Appl.*, v. 1, 1968, pp. 103–125.

2. H. W. Engl & A. Neubauer, "An improved version of Marti's method for solving ill-posed linear integral equations," *Math. Comp.*, v. 45, 1985, pp. 405–416.

3. R. Fletcher, *Practical Methods of Optimization*, Vol. 2, Wiley, New York, 1981.

4. C. W. Groetsch, *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind*, Pitman, Boston, 1984.

5. C. W. Groetsch, J. T. King & D. Murio, "Asymptotic analysis of a finite element method for Fredholm equations of the first kind," in *Treatment of Integral Equations by Numerical Methods* (C. T. H. Baker and G. F. Miller, eds.), Academic Press, London, 1982, pp. 1–11.

6. J. T. Marti, "On a regularization method for Fredholm equations of the first kind using Sobolev spaces," in *Treatment of Integral Equations by Numerical Methods* (C. T. H. Baker and G. F. Miller, eds.), Academic Press, London, 1982, pp. 59–66.

7. V. A. Morozov, *Methods for Solving Incorrectly Posed Problems*, Springer, New York, 1984.

8. U. Mosco, "Convergence of convex sets and of solutions of variational inequalities," *Adv. in Math.*, v. 3, 1969, pp. 510–585.

9. A. Neubauer, "Tikhonov-regularization of ill-posed linear operator equations on closed convex sets," *J. Approx. Theory*. (To appear.)

10. A. Neubauer, *Tikhonov-Regularization of Ill-Posed Linear Operator Equations on Closed Convex Sets*, Ph. D. Thesis, University of Linz, 1985, will appear as a book in VWGÖ (= Verlag der Wissenschaftlichen Gesellschaften Österreichs), Vienna.

11. J. T. Oden & E. B. Pires, "Error estimates for the approximation of a class of variational inequalities arising in unilateral problems with friction," *Numer. Funct. Anal. Optim.*, v. 4, 1981/82, pp. 397–412.