APPENDIX A

Trigonometry Review

A.1 Basic Definitions

A.1.1 Ratios on the Right Triangle

The trigonometric functions sine, cosine, and tangent are based on ratios of the sides of a right triangle, relative to one acute angle θ (Figure A.1):

$$\begin{aligned}
\sin\theta &= \text{opp/hyp} \\
\cos\theta &= \text{adj/hyp} \\
\tan\theta &= \text{opp/adj} = \sin\theta/\cos\theta
\end{aligned}$$

We also define the reciprocal functions secant, cosecant, and cotangent as follows:

$$\begin{aligned}
\sec\theta &= \text{hyp/adj} = 1/\cos\theta \\
\csc\theta &= \text{hyp/opp} = 1/\sin\theta \\
\cot\theta &= \text{adj/opp} = 1/\tan\theta \\
&= \cos\theta/\sin\theta = \sec\theta/\csc\theta
\end{aligned}$$

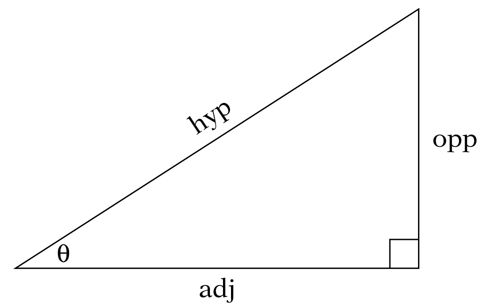

Figure A.1 Computing trigonometric functions on the right triangle.

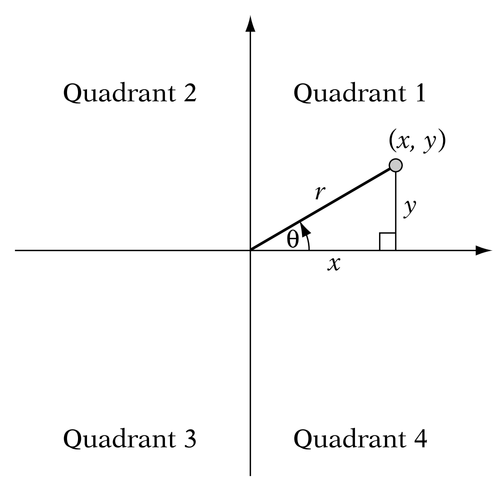

Figure A.2 Computing trigonometric functions on the standard Cartesian frame, showing the four ordered quadrants.

A.1.2 Extending to General Angles

Consider a standard Cartesian frame for $\mathbb{R}^2$. We place a line segment, or radius, with length r and one endpoint fixed at the origin. The other endpoint is located at a point (x, y). We define θ as the angle between the radius and the positive x-axis. The angle is positive if the direction of rotation from the x-axis to the radius is counterclockwise, and negative if clockwise. A full rotation is broken into 2π radians, or 360 degrees. The coordinate axes divide the plane into four quadrants: they are numbered in the order of rotation. Within this we can inscribe a right triangle, with the radius as hypotenuse and one side incident with the x-axis (Figure A.2).

We can represent the sine and cosine based on the length r of the radius and the location (x, y) of the free endpoint:

$$\sin\theta = y/r$$
$$\cos\theta = x/r$$

In this case, the tangent becomes the slope of the radius:

$$\tan\theta = y/x$$

For angles greater than $\pi/2$, the magnitude of the result is the same, but the sign may be negative depending on which quadrant the angle is in:

Functions	**Quadrant**	**Sign**
sin, csc	1,2	$+$
	3,4	$-$
cos, sec	1,4	$+$
	2,3	$-$
tan, cot	1,3	$+$
	2,4	$-$

The tangent, cotangent, secant, and cosecant all involve divisions by x or y, which may be 0. This leads to singularities at those locations, which can be seen in the function graphs in Figures A.3 through A.8. This sequence of figures shows the six trigonometric functions graphed against θ (in radians).

Also note that these functions are periodic. For example, $\sin(0) = \sin(2\pi) = \sin(-4\pi)$. In general, $\sin(x) = \sin(n \cdot 2\pi + x)$, for any integer n. The same is true

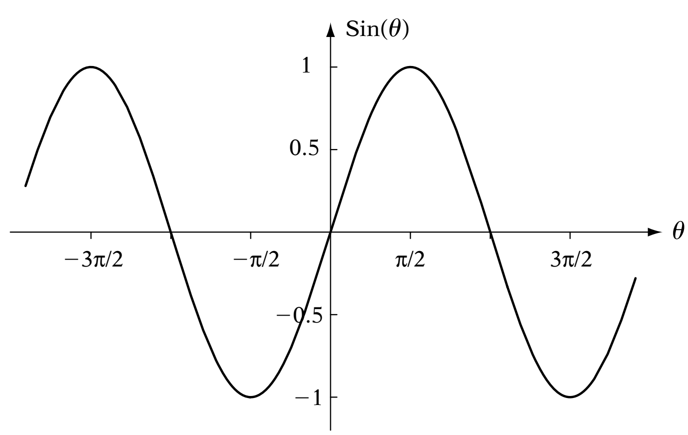

FIGURE A.3 Graph of $\sin\theta$.

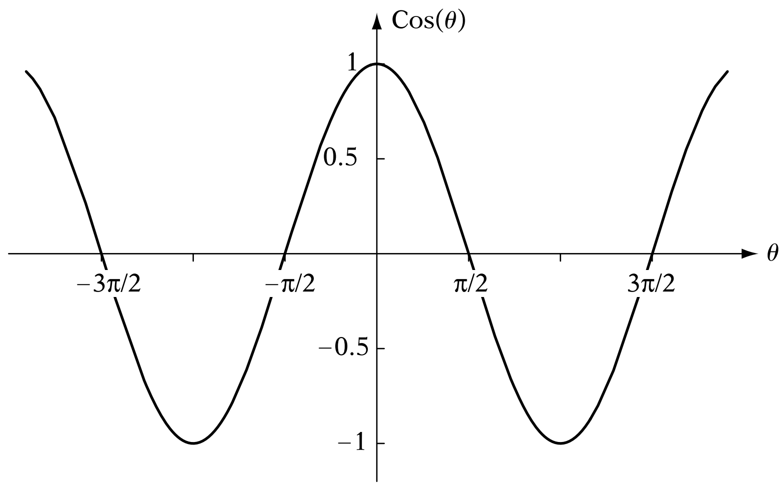

Figure A.4 Graph of $\cos\theta$.

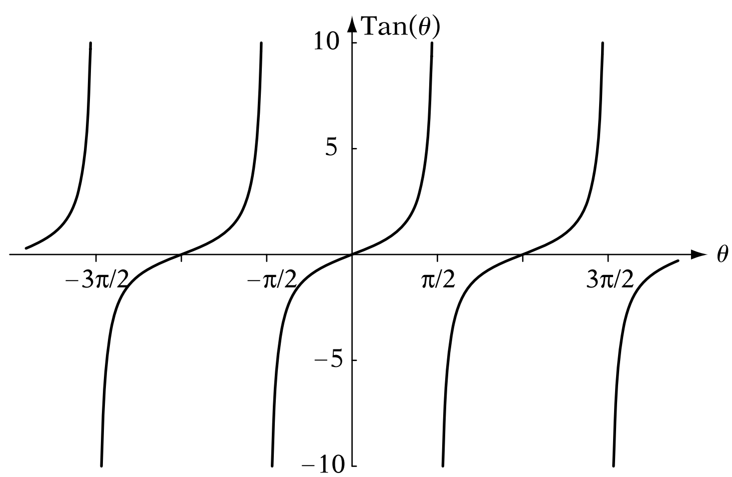

Figure A.5 Graph of $\tan\theta$.

for cosine, secant, and cosecant. Tangent and cotangent are periodic with period π: $\tan(x) = \tan(n \cdot \pi + x)$.

A.2 Properties of Triangles

There are three laws that relate angles in a triangle to sides of a triangle, using trigonometric functions. Figure A.9 shows a general triangle with sides of length a, b, and c, and corresponding opposite angles α, β, and γ.

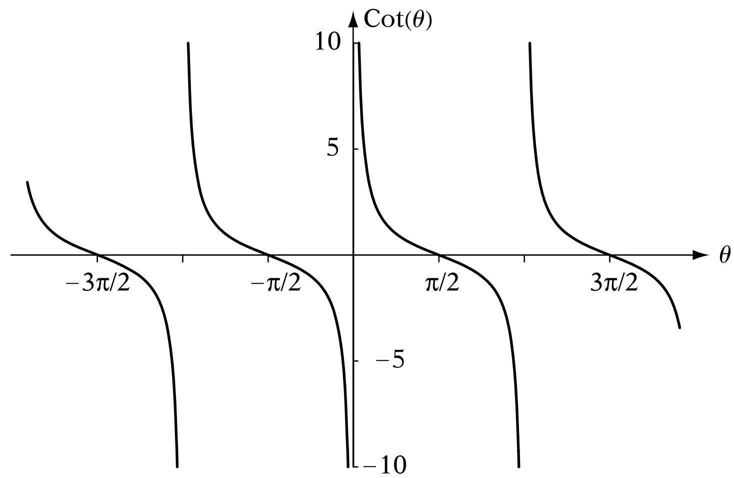

FIGURE A.6 Graph of $\cot \theta$.

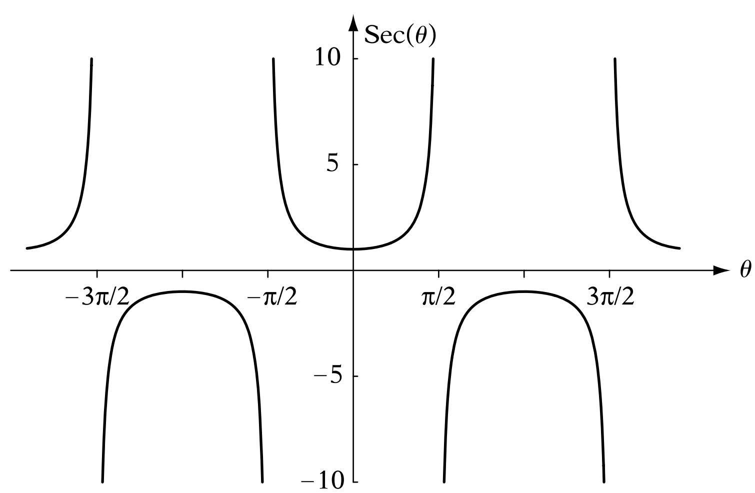

FIGURE A.7 Graph of $\sec \theta$.

The *law of sines* relates angles to their opposing sides as a constant ratio for each pair:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} \tag{A.1}$$

Recall the Pythagorean theorem,

$$c^2 = a^2 + b^2$$

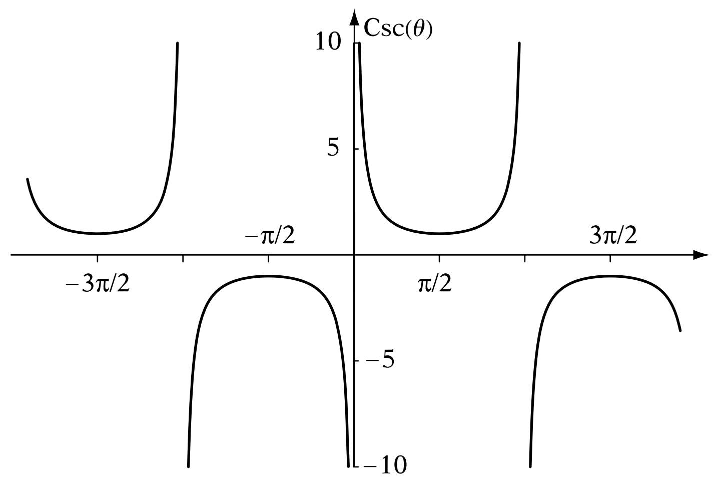

FIGURE A.8 Graph of $\csc\theta$.

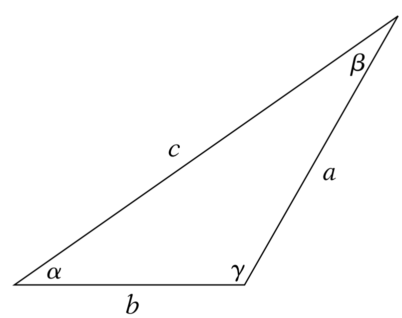

FIGURE A.9 General triangle, with sides and angles labeled.

which relates two sides of a right triangle to the hypotenuse. The *law of cosines* is an extension to this, which can be used to compute the length of a side from the length of two other sides and the angle between them:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{A.2}$$

Substituting $\pi/2$ for γ produces the specific case of the Pythagorean theorem.

The *law of tangents* relates two angles and their corresponding opposite sides:

$$\frac{a-b}{a+b} = \frac{\tan(\frac{1}{2}(\alpha-\beta))}{\tan(\frac{1}{2}(\alpha+\beta))} \tag{A.3}$$

All of these can be used to construct information about a triangle from partial data.

While not specifically one of the laws, a related set of formulas computes the area of a triangle:

$$\frac{ab \sin \gamma}{2} = \frac{bc \sin \alpha}{2} = \frac{ac \sin \beta}{2} \tag{A.4}$$

A.3 Trigonometric Identities

A.3.1 Pythagorean Identities

Again, from the Pythagorean theorem we know that

$$a^2 + b^2 = c^2$$

where c is the length of the hypotenuse and a and b are the lengths of the other two sides. In the case where the length of the hypotenuse is 1, the length of the other two sides are $\cos\theta$ and $\sin\theta$, so

$$\sin^2\theta + \cos^2\theta = 1 \tag{A.5}$$

where $\sin^2\theta = (\sin\theta)(\sin\theta)$, and similarly for $\cos^2\theta$.

Dividing equation A.5 through by $\cos^2\theta$, we get

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta \tag{A.6}$$

If we instead divide equation A.5 by $\sin^2\theta$, we get

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \csc^2\theta$$

A.3.2 Complementary Angle

If we consider one acute angle θ in a right triangle, the other acute angle is its complement $\frac{\pi}{2} - \theta$. We can compute trigonometric functions for the complementary angle by changing the sides we use when computing the ratios; for example,

$$\sin\left(\frac{\pi}{2} - \theta\right) = \text{adj/hyp} = \cos(\theta)$$

The complementary angle identities are as follows:

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \tag{A.7}$$

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \tag{A.8}$$

$$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right) \tag{A.9}$$

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \tag{A.10}$$

$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right) \tag{A.11}$$

$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) \tag{A.12}$$

A.3.3 Even–Odd

Two of the trigonometric functions, cosine and secant, are symmetric across $\theta = 0$ and are called *even* functions:

$$\cos(-\theta) = \cos\theta \tag{A.13}$$

$$\sec(-\theta) = \sec\theta \tag{A.14}$$

The remainder are antisymmetric across $\theta = 0$ and are called *odd* functions:

$$\sin(-\theta) = -\sin\theta \tag{A.15}$$

$$\csc(-\theta) = -\csc\theta \tag{A.16}$$

$$\tan(-\theta) = -\tan\theta \tag{A.17}$$

$$\cot(-\theta) = -\cot\theta \tag{A.18}$$

A.3.4 Compound Angle

For two angles α and β, the sines of the sum and difference of the angles are, respectively,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{A.19}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{A.20}$$

Similarly, the cosines of the sum and difference of the angles are

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{A.21}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{A.22}$$

These can be combined to create the compound angle formulas for the tangent:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{A.23}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \tag{A.24}$$

A.3.5 Double Angle

If we substitute the same angle θ for both α and β into the compound angle identities, we get the double angle identities:

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{A.25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{A.26}$$

The latter can be rewritten using the Pythagorean identity as

$$\cos 2\theta = 1 - 2\sin^2\theta \tag{A.27}$$

$$= 2\cos^2\theta - 1 \tag{A.28}$$

The double angle identity for tangent is

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{A.29}$$

A.3.6 Half-Angle

Equations A.27 and A.28 can be rewritten as as follows:

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \tag{A.30}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \tag{A.31}$$

Substituting $\theta/2$ for α and taking the square roots gives

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos\theta}{2}} \tag{A.32}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos\theta}{2}} \tag{A.33}$$

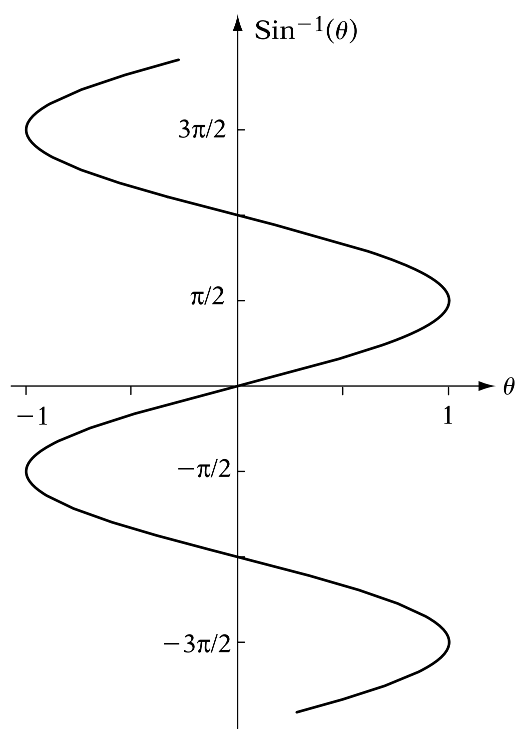

Figure A.10 Graph of arcsin θ.

Note that due to the square root, there are two choices for each identity, positive and negative — the one chosen depends on what quadrant $\theta/2$ is in.

A.4 Inverses

The trigonometric functions invert to multivalued functions because they are periodic. For example, the graph of the inverse $\sin^{-1}\theta$, or *arcsine*, can be seen in Figure A.10. Its domain is the interval $[-1, 1]$ and its range is $\mathbb{R}$.

Because of this, it is common to restrict the range of an inverse trigonometric function so that it maps only to one value, given a value in the domain. Standard choices for these restrictions are as follows:

Function	**Domain**	**Range**
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$\mathbb{R}$	$[-\pi/2, \pi/2]$