Course project - Computational Geometry

Min Kyu Jung & Amirhossein Maghsdoust

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1 Abstract

Floodlight problems require the placement of floodlights in a polygon such that the entire polygon is illuminated. There are many variations of floodlight problems, one being the vertex π-floodlight problem. Such problem utilizes π angled floodlights placed only on vertices of the polygon to illuminate the entire polygon. Current progress of vertex π-floodlight problem is to determine the minimum number of vertex π-floodlight required to illuminate any polygon P. Additionally, point π-floodlight problems arose from vertex π-floodlight problems, where the placement of π-floodlights are not restricted to vertices only. A current conjecture inquires whether $\lfloor \frac{n}{3} \rfloor$ point π-floodlights are sufficient to illuminate any polygon. In this paper, we demonstrate results and attempts to find the minimum number of vertex π-floodlights to illuminate a polygon as well as proving that $\lfloor \frac{n}{3} \rfloor$ point π-floodlights are sufficient to illuminate any polygon.

2 Introduction

In this section, we will be introducing the definition, model, problem statement that this paper revolves around, the motivation of the problem, which discusses the significance the problem has on the wider academic literature, as well as the contribution of this paper.

2.1 Definitions, Model, Problem statement

2.1.1 The Problem Statement:

The problem this paper revolves around is the [1]vertex π-floodlights problem, which inquires "how many π-floodlights are always sufficient to illuminate any polygon of n vertices, with at most one floodlight placed at each vertex"?

2.1.2 Definition

To understand the vertex π-floodlights, certain definitions need to be clarified.

Definition 1 *[2]Given two points p and q of a polygon P, p is visible from q if the line segment joining p to q is totally contained in P.*

In other words, if one point can "see" another point, without any obstruction, they are visible to each other. With visibility (**Definition 1**) defined, we define illumination next:

Definition 2 *[2]A collection H of points of P illuminates P if every point u of P is visible from point p in H.*

Each point in H that illuminates a set of points in polygon P is referred to as a **floodlight**. For intuitiveness, imagine they are light sources. [2]A vertex floodlight (also referred to as a vertex guard) are floodlights that are restricted to the vertices of a polygon, as opposed to a point floodlight which can be placed anywhere in the polygon. Each floodlight have a restricted angle, and depending on the problem these angle vary. For the vertex π-floodlights, each vertex floodlight has an angle of π.

2.1.3 Model

With **Definition 2**, we understand the goal of the vertex π-floodlights i.e. find the minimum size of collection H of vertex π-floodlights that illuminates the entire polygon P.

The model is comprised of the following:

- Any polygon P with n vertices
- Vertex floodlights with π angle

2.2 Motivation

On surface value, the vertex π-floodlights problem appear to be a stimulating theoretical exercise. However, such problem has significant applications to real world problems. The following are domains where solutions to vertex π-floodlights problem provide substantial impact:

2.2.1 Target detection for robots

[3]Target detection is a problem in the robotics where robots detect one or multiple sensors, either by actively sweeping the environment with mobile sensors, or by monitoring signals emitted from fixed static sensors. The latter option is requires placing fixed static sensors across the room.

Consider a scenario where a robot has a traverse an enclosed space, and it has been given a path to follow. The path is a directional map. However, as the robots traverse, it will deviate from the map due to the drift caused by an unleveled floor. While the robot will follow accordingly to the map, the drift will cause the robot to end up in a different location or even stuck by certain obstacles like the wall. Having static sensors, or in this context, coordinates across the space will allow to robot to readjust itself accordingly to these fixed coordinates. This way when the robot does deviate from the directional map, it can re-position itself and follow the map correctly. It would be best to minimize the placement of these static coordinates so that the robot can traverse the entire space.

2.2.2 Sensor-target Surveillance

[8]Sensor-target surveillance, an application of wireless sensor networks, require sensor nodes strategically placed in a space to monitor targets, collect sensed data and transmit such data to a centralized location.

There are numerous constraints that can be optimized with sensor-target surveillance, one being resource or cost. In certain contexts, it is within one's interest to minimize the use of sensor nodes while gaining the maximum amount of information/sensed data.

An example of this context would be to monitor a museum. The sensor nodes used here would be surveillance camera. The museum owners have interest in monitoring every area in the museum while minimizing the cost of placing expensive cameras. By strategically placing surveillance cameras in areas with high coverage, museum owners can minimize the number of cameras to monitor the entire museum.

2.3 Paper Overview

In this paper, we aim to accomplish the following (divided by section):

1. **Background & Related Work**
   (a) ***History of vertex π-floodlight problem:*** Introduce the inception of the problem and how it has reached to the point of vertex π-floodlight problem
   (b) ***Contribution made to vertex π-floodlight problem:*** Walk through the chain of contributions made in solving the vertex π-floodlight problem
2. **Results**
   (a) A upper and lower bound of the minimum number of vertex π-floodlights to illuminate the entire polygon
   (b) An attempt to prove $\lfloor \frac{n}{3} \rfloor$ point π-floodlights are always sufficient to illuminate any polygon with n vertices.

3 Background & Related Work

3.1 History of vertex π-floodlight problem

3.1.1 Floodlight illumination problems

[2]In 1992, J. Urrutia introduced an open illumination problem using floodlights, that has become the basis of the vertex π-floodlight problem that is being discussed in this paper. This problem is referred to as the ***Stage Illumination Problem***. The model of this problem is as such:

1. A line segment l contained in the x-axis of the plane
2. A set of floodlights $F = \{f_1, \ldots, f_n\}$ with sizes $\alpha_1, \ldots, \alpha_n$ respectively
3. Each floodlights are located at some fixed points on the plane, all on the same side of l

The goal of this problem is to rotate the floodlights around their apexes such that l is completely illuminated.

In solving this open question, many variations and findings of those variations emerged into the literature. These variations and findings are but not limited to:

- ***The Floodlight Illumination Problem of the Plane:*** This problem allows floodlights to be placed anywhere in the plane, and not limited to one side of the plane like the ***Stage Illumination Problem***. Findings for this problem include:
  - [4]A theorem that states it is always possible to assign one floodlight to each point of the plane, and position the floodlights such that the whole plane is illuminated.
  - [5]The lower bound of said problem being complexity of $\Omega(nlogn)$.
- **Optimal Floodlight Illumination of Stages:** This problem places all points on the plane with positive y-coordinate and attempts to find floodlights with the minimum combined angles of all the floodlights.
  - [6]The problem can be solved in $\mathcal{O}(nlogn)$ time if two floodlights can be placed in one point of the plane.
- **Two Floodlight Illumination Problem:** The model changes from a plane to a convex polygon. The goal is to illuminate a convex polygon P using at most two floodlights in such that the sum of their sizes is minimized.
  - [7]The problem can be solved in $\mathcal{O}(n^2)$, where n is the number of vertices in the convex polygon.

3.1.2 π-floodlight problem

Among the many variations of the floodlight problem, a critical finding by Estivill-Castro and Urrutia inspired the π-floodlight problem. That finding being the following:

Theorem 1 *[7]Any orthogonal polygon with n vertices can always be illuminated using at most $\lfloor \frac{3n-4}{8} \rfloor$ orthogonal vertex floodlights. If the floodlights are allowed to be anywhere on the boundary of the polygon, $\lfloor \frac{n}{4} \rfloor$ suffice. Both bounds are tight.*

This inspired Urrutia to inquire whether there is an $\alpha < \pi$ such that any polygon P can be illuminated by placing an α-floodlight at every vertex of P. [9]Estivill-Castro, O'Rourke, Urrutia and Xu proved that there exists a polygon P where $\alpha < \pi$ cannot illuminate even by placing α floodlight at every vertex of P. Thus, the focus shifted from finding vertex α-floodlights that illuminates any polygon to finding solutions to vertex π-floodlight problems, specifically the question: [9]what is the minimum number of vertex π-floodlight required to illuminate any polygon?

3.2 Contribution made to vertex π-floodlight problem

There has been many findings that shed light on the vertex π-floodlight problem. Here are the following findings:

- [10]O'Rourke proved that $\lceil \frac{n}{2} - 1 \rceil$ is required for monotone mountains.
- [11]Toth established $\frac{3n}{4}$ is sufficient for general orientation floodlights.

- Santos proved that $\lfloor\frac{3(n-1)}{5}\rfloor$ inward-facing floodlight are needed, and $\lfloor\frac{2(n-2)}{5}\rfloor$ for general orientation floodlights.
- [12]Speckmann and Toth showed $\lfloor\frac{n}{2}\rfloor$ and $\lfloor\frac{2n-c}{3}\rfloor$ vertex π-floodlights are sufficient for general orientation and in-ward facing, edge aligned orientation floodlights, where c is the number of convex vertices.

There are few definitions that were used in the contributions above are:

- **Monotone mountains:** Monotone mountains consists of one monotone chain, whose extreme vertices are connected by a single segment.
- **Monotone chain:** Monotone chain is a polygonal chain whose intersection with an vertical line is at most one point.
- **General orientation floodlights:** Floodlights with no restriction on its orientation.
- **Inward-facing orientation floodlights:** Floodlights facing inward the polygon, but the two sides of the polygon incident to the floodlight are disjoint.
- **Edge-aligned orientation floodlights:** Floodlights facing inwards and on the boundary of the half plane of the floodlight is collinear with one of the sides incident to v.

4 Results

For the results section, we attempt to answer the question: *what is the minimum number vertex of π-floodlight required to illuminate any polygon?* In the first section **Vertex π-floodlights**, we attempt to find the upper bound and lower bound of the minimum vertex π-floodlights required to illuminate any polygon. In the second section **Point π-floodlights**, we attempt to prove that $\lfloor\frac{n}{3}\rfloor$ point π-floodlights are sufficient to illuminate any polygon P.

4.1 Vertex π-floodlights

4.1.1 Finding the upper bound

To find the upper bound, we first try to prove the following statement: $\lfloor\frac{n}{3}\rfloor$ vertex 2π-floodlights are sufficient to illuminate any polygon. This can be proven by the induction on edges.

We know that every simple polygon can be triangulated. After the triangulation, we claim that we can color the vertices with three different color so that each adjacent vertices have different colors. To prove that, we take the three adjacent vertices in the polygon that exists in the triangulation and we color them. Then we cut that triangle, pick another triangle, color those similarly to the first one. This process is continued until the entire polygon is colored, as shown in **Figure 1 of the Appendix**.

There exists one color that has been colored at most $\lfloor\frac{n}{3}\rfloor$ times. In **Figure 2 of the Appendix**, this color is 3. We can place 2π-floodlights on such vertices to illuminate the entire polygon.

Now that we proved $\lfloor \frac{n}{3} \rfloor$ vertex 2π-floodlights are sufficient to illuminate any polygon, we can use the same reasoning to find the upper bound for vertex π floodlights. We can substitute one 2π-floodlights with two π-floodlights. This gives us $2\lfloor \frac{n}{3} \rfloor$ the upper bound for vertex π-floodlights. However, not every vertex require two π-floodlights. As shown in the **Figure 3 of the Appendix**, two of the three 3-colored vertices have convex angles, thus only one π-floodlight will be sufficient in each of those two vertices. Thus, the more accurate upper bound is $2\lfloor \frac{n}{3} \rfloor - C$, where C is the number of vertices with convex angles among the selected vertices.

However, this is not the most efficient upper bound. As shown in **Figure 4 of the Appendix**, there exists a 10-vertices polygon that requires only one vertex π-floodlight, as opposed to six vertex π-floodlights (which our upper bound would have given).

4.1.2 Finding the lower bound

To find the lower bound, we look into the following statement: $\lfloor \frac{n}{3} \rfloor$ vertex π-floodlights are sufficient to illuminate any polygon. **Figure 5 of the Appendix**, which depicts a 6-vertices polygon, is a counterexample to this claim. By the claim, such polygon can be illuminated by two vertex π-floodlights, but in actuality, at least three vertex π-floodlights are required.

4.1.3 Section conclusion

In this section, we attempted to find the minimum number of vertex π-floodlights sufficient to illuminate any polygon with n vertices. We could not find an exact minimum number but we managed to find a range of where the minimum should be, and that value must greater than $\lfloor \frac{n}{3} \rfloor$ and less than $2\lfloor \frac{n}{3} \rfloor$.

4.2 Point π-floodlights

4.2.1 $\lfloor \frac{n}{3} \rfloor$ point π-floodlights are sufficient

To prove that $\lfloor \frac{n}{3} \rfloor$ point π-floodlights are sufficient to illuminate any polygon, we will first demonstrate that for any polygon, we will be remove three vertices. The reason why we remove three vertices is because that will change the number of floodlights by one (as three is the denominator). When we remove three vertices in the polygon, we notice that a pentagon is removed as shown in **Figure 6 of the Appendix**.

Now we have to show that every pentagon could be illuminated with one point π-floodlight. This way, we could iteratively place one point π-floodlight in a pentagon from the polygon to illuminate the entire polygon.

Before proving that every pentagon can be illuminated with one point π-floodlight, it is important to note that we need only one point π-floodlight to illuminate the whole triangle, as shown in **Figure 7 of the Appendix**.

It is also important to note that any simple polygons can be triangulated to exactly $n - 2$ triangles. This means that the sum of the interior angles of n vertices simple polygon is $180^\circ(n - 2)$. This implies that for a polygon, the maximum number of concave angles is $n - 3$, otherwise the sum of the interior angles exceed $180^\circ(n - 2)$.

With this fact, we wil prove that every pentagon can be illuminated with one point π-floodlight. This requires three cases:

1. **Pentagon with no concave angles:** As shown in **Figure 8 of the Appendix**, one point π-floodlight is sufficient to illuminate the entire polygon, as if the polygon was encased in one larger triangle.

2. **Pentagon with one concave angle:** As shown in **Figure 9 of the Appendix**, by placing a point π-floodlight on the common edge of two triangles, we can illuminate both the triangles, and thereby the entire polygon.

3. **Pentagon with two concave angles:** As shown in **Figure 10 of the Appendix**, the variation of this pentagon is divided into two cases, referred to Case 3a and 3b. For 3a, placing a point π-floodlight on the common edge of two triangles, we can illuminate both the triangles, and thereby the entire polygon. For 3b, placing a point π-floodlight in one of the vertices as found in the subsection **Finding the upper bound**, three triangles will be illuminate that covers the entire pentagon.

Now that we have shown every pentagon can be illuminated with one point π-floodlight, we can remove a pentagon from the polygon by selecting five adjacent vertices. This way we can remove three vertices which represents one floodlight. We would iteratively do this to the remaining $n-3$ vertices polygon to show that the entire polygon can be illuminated by $\lfloor \frac{n}{3} \rfloor$ floodlights, as shown in **Figure 11 of the Appendix**.

However, there is a chance of selecting five adjacent vertices that isn't necessarily a pentagon. This can be seen in **Figure 12 of the Appendix**. Thus, instead removing five adjacent vertices, we will triangulate the polygon and remove three adjacent triangles instead. Three adjacent triangles make a polygon, and since we have already demonstrated that ever pentagon can be illuminated with one point π-floodlight, we know that the three adjacent vertices can be illuminated similarly as well. This is shown in **Figure 13 of the Appendix**, where the polygon can be illuminated with $\lfloor \frac{n-2}{3} \rfloor$ point π-floodlights, where $n-2$ is the number of triangles in the triangulation.

This leads to another problem: how do we remove three adjacent triangles such that the remaining vertices and edges for a polygon? To answer this problem, we construct a dual graph, where each triangle is a node and they are connected iff they have a common edge in the main graph. An example of such dual graph can be seen in **Figure 14 of the Appendix**.

The properties of such dual graph is the following:

1. ***The dual graph is connected:*** This can be proven by contradiction, as shown in **Figure 15 of the Appendix**. If the dual graph is not connected, the corresponding polygon will not be a single polygon.

2. ***All nodes in the dual graph has either a degree of one, two or three:*** This property is trivial as each triangle can have at most three neighbors.

3. ***There are no cycles in the dual graph:*** This can be proven by contradiction. In **Figure 16 of the Appendix**, we demonstrate the only

things that produce a dual graph. Both polygons are not simple and legal polygons.

Now to remove three adjacent triangles in a polygon, we start with the node with a degree of one. We know there exists at least one node with a degree of one, otherwise, all nodes will have a degree of two which implies there is a loop. Since a loop is not legal, we know there exist at least one node with degree of one.

Once we select a node with a degree of one, one of the cases in **Figure 17 of the Appendix** can happen. If we place a point π-floodlight on one of the triangles and remove them, the remaining will still be a valid polygon.

However, our findings end here as we cannot prove that the cases in **Figure 17 of the Appendix** are the only cases in that can occur.

4.2.2 Section conclusion

In this section, we attempted to prove the following claim true: $\lfloor \frac{n}{3} \rfloor$ point π-floodlights are sufficient to illuminate any polygon. Our approach was to divide the polygon into pentagons and place one floodlight in each pentagon. However, we were not able to prove whether removing a pentagon will leave the remaining thing a polygon in which we can recursively remove more pentagons. Thus, we cannot conclude whether the claim is true.

5 Post-result literature review

While working on the results, we found a theorem that proved $\lfloor \frac{n}{3} \rfloor$ point π-floodlights being sufficient to illuminate any polygon. [13]This was proven by Tóth in 2000. Tóth's approach did not use pentagons as shown in this paper's Result section, and instead used diagonal cuts, and more complex notions beyond our comprehension. However, it is interesting to note that Tóth allows placing two floodlights in one point. While our proof for the conjecture is incomplete, if our proof was correct, then it would mean that we will not be proving the same conjecture but a more specific conjecture that is: $\lfloor \frac{n}{3} \rfloor$ *point π-floodlights are sufficient to illuminate any polygon P when there can be at most one floodlight in one point.*

6 Conclusion

In this paper, we look into two problems: 1) the minimum number of vertex *pi*-floodlights required to illuminate any polygon P and 2) whether $\lfloor \frac{n}{3} \rfloor$ point π-floodlights are sufficient to illuminate any polygon P. Our finding for the first problem is not an exact minimum number but an explanation why the number should be greater than $\lfloor \frac{n}{3} \rfloor$, and that it need not be more than $2\lfloor \frac{n}{3} \rfloor$. For the second problem, we attempt to prove the conjecture being true by demonstrating that a polygon can be deconstructed to $\lfloor \frac{n}{3} \rfloor$ pentagons, and each pentagons require only one point π-floodlight. The proof is incomplete and later literature review proved the conjecture to be true, albeit with a different, more sophisticated method.

7 Bibliography

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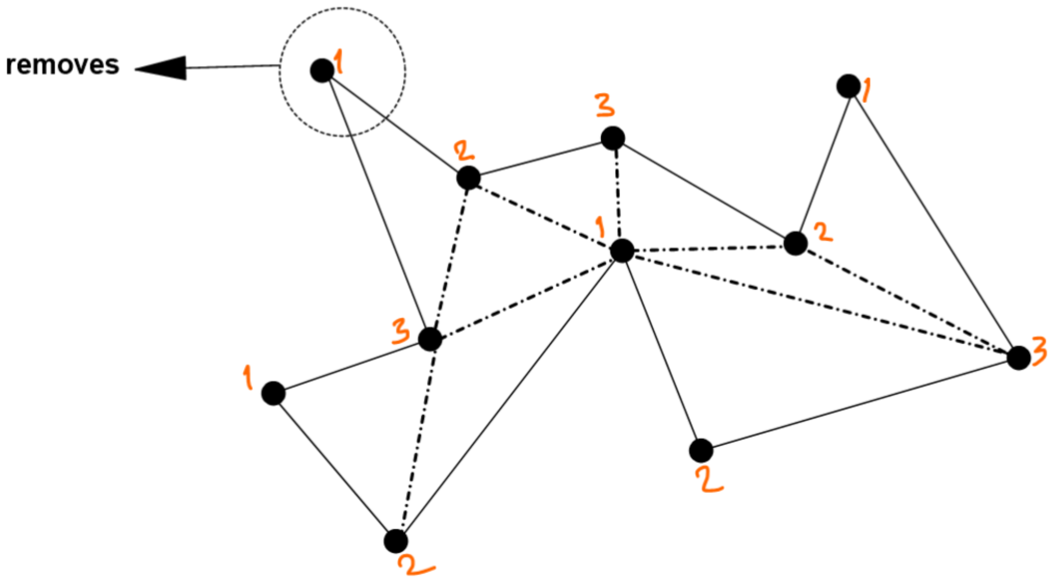
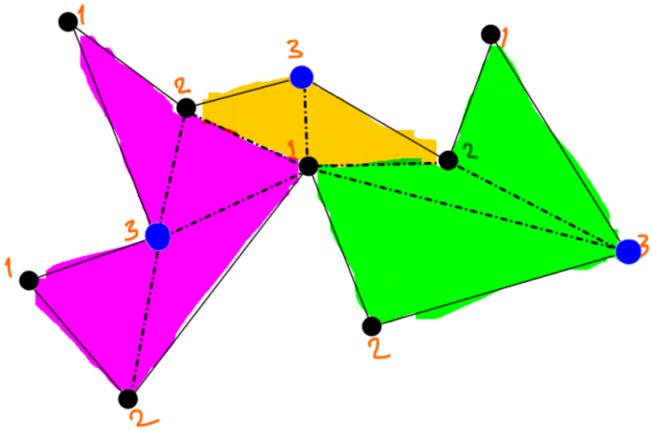
8 Appendix

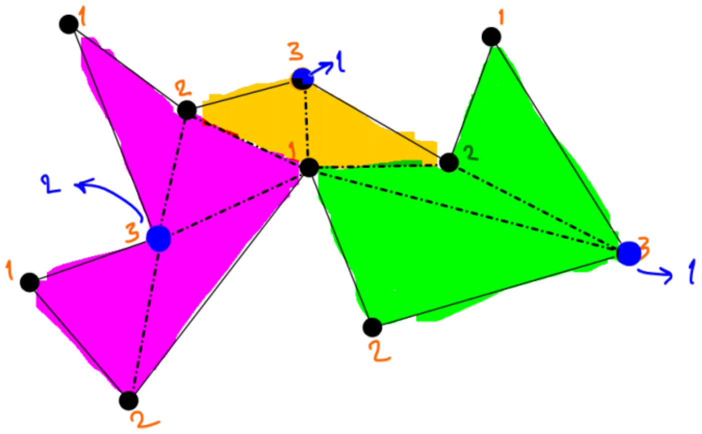
8.0.1 Figure 1



8.0.2 Figure 2



8.0.3 Figure 3



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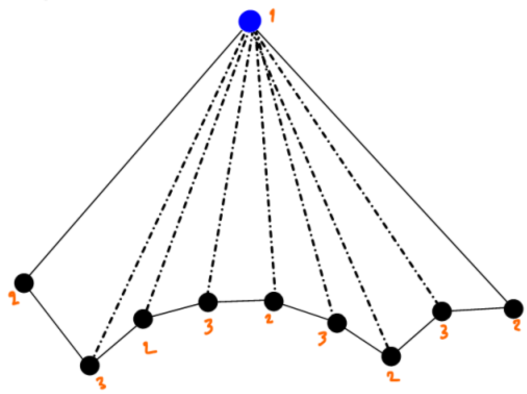

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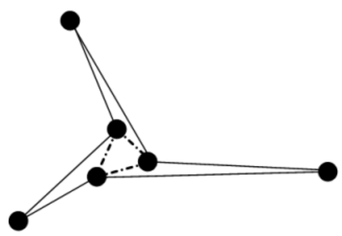

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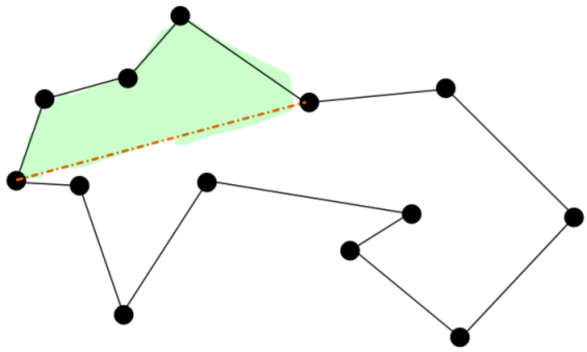

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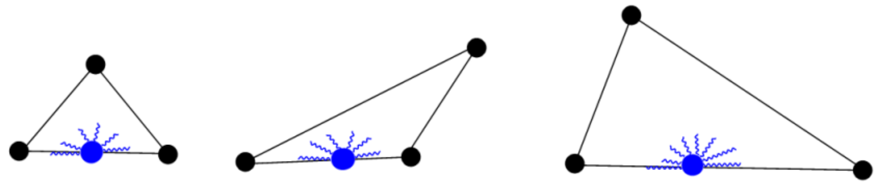

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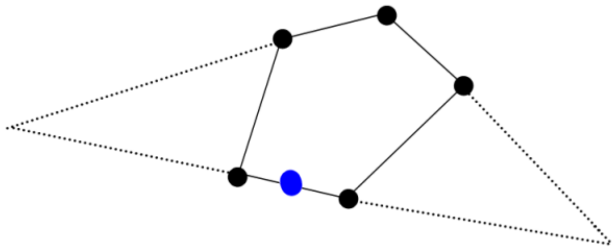
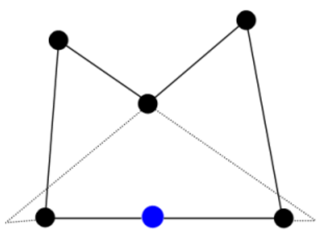


8.0.9 Figure 9



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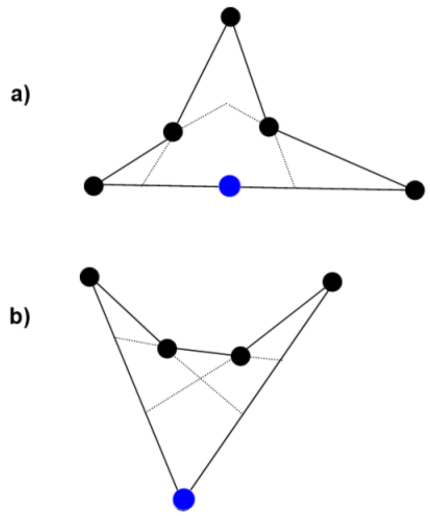
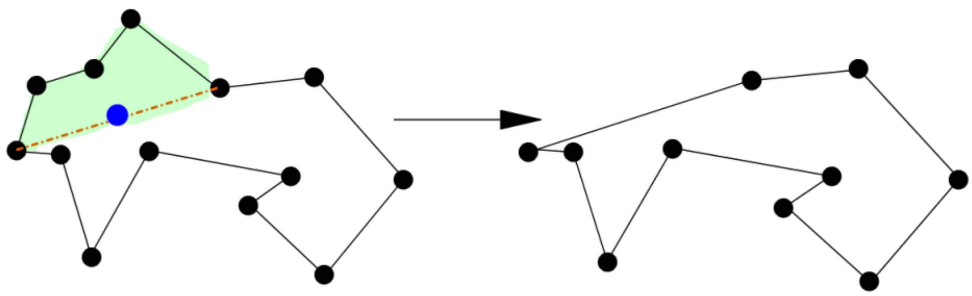
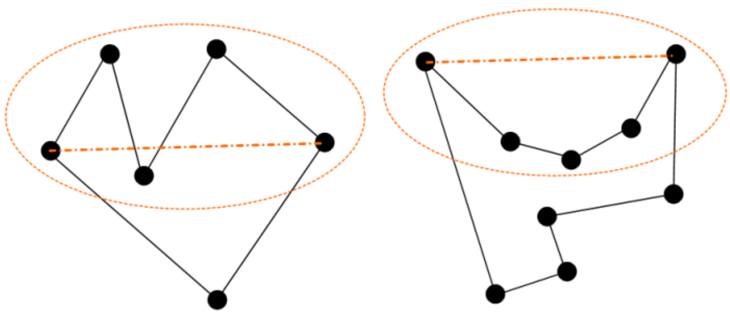


8.0.11 Figure 11



8.0.12 Figure 12



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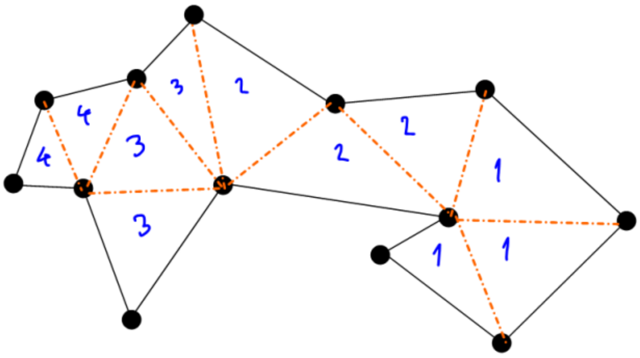

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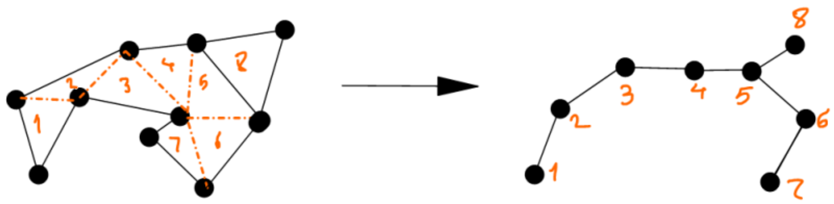
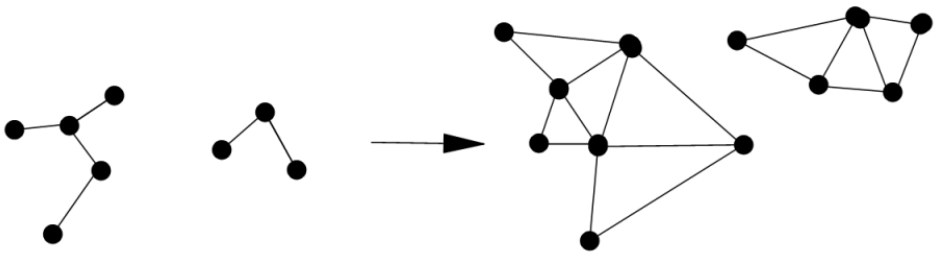
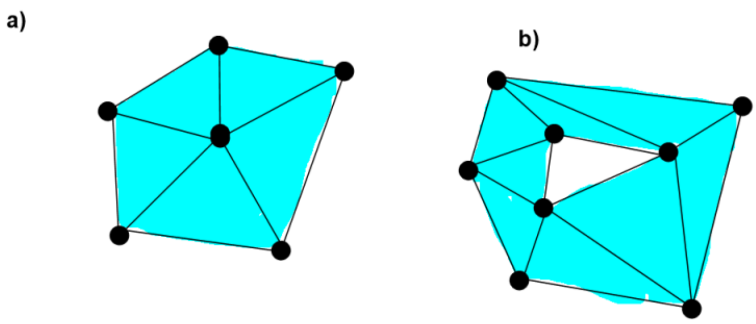


8.0.15 Figure 15



8.0.16 Figure 16



8.0.17 Figure 17