**Introduction to complex function**

A complex function is a function that establishes a relation between two complex number. Suppose z and w are two complex number the function f establishes a relation between complex number x and y.

**Complex differentiability**

It is quite different than normal differentiation. Now we have two values to differentiate x and y. We need to check differentiability of a plane with x axis and y axis rather than just a single point in real functions.

**Holomorphicity and infinitesimal change**

Similar to real functions extremely small changes to the complex function causes extremely small changes to the output of a function

**Holomorphic Functions**

A complex function that is differentiable in all points of a particular domain is called a holomorphic function.

Let,

f(z) is holomorphic in a domain D if it is complex differentiable at every point in D.

This definition implies that the function z is not only continuous but infinitely differentiable within D.

**Cauchy-Riemann Relation**

Given a function , where and and are real and imaginary part of respectively. The relation is,

and

We can prove it using limit definition of derivative

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Now if we and we respectively get and

**Complex Integrals**

This is the same as integrals in real calculus, but the path of integration is now a curve in the complex plane

**Cauchy's Integral Theorem**

The integral of a holomorphic function around a closed curve in its domain is always zero. It implies that the integral of a holomorphic function depends only on the values of the function at the endpoints of the curve, rather than on the curve's path itself.

**Complex Integration: The ML Inequality**

The ML inequality is a tool used to find complex integral in a complex plane

Imagine you're going on a walk along a path in the complex plane. Along the way, you have a bag of candies f(z). Now, let's say you can always find a way to walk so that the number of candies you pick up at each step is less than or equal to some number M . This means f(z) M for all points z along your path.

The ML inequality tells you that if you know the length of your path (how far you walked), you can figure out an upper limit on the total number of candies you picked up along the way. This upper limit is ML.

**Cauchy's Integral Formula**

It says that if you have a function f(z) that's smooth inside a closed loop, you can find the value of f(z) at any point inside by averaging f(z) over, weighted by the distance from each point on to the point you're interested in, and scaled by

We prove this by first turning our closed loop into a circle around the point we're interested in. Then, using Cauchy's Integral Theorem and some clever contour manipulation, we show that the value of f(z) at the center of the circle is related to the integral of around the circle. Dividing both sides by , we get Cauchy's Integral Formula.

**Laurent series**

Suppose you have a function f(z) that has a pole at z = 1 . Around this point, f(z) behaves weirdly; it's not holomorphic there. But a Laurent series can still help us understand its behavior.

The Laurent series of f(z) around z = 1 would include terms like (the principal part) and terms like (the analytic part). The coefficients a and b capture how "strong" the singularity is and how "nice" the function behaves, respectively.

In essence, the Laurent series breaks down the function into its singular and non-singular parts, helping us understand its behavior around different points in the complex plane.

**Residue Theorem**

Imagine you have a bag of candies, and you're walking around a park along a path. The Residue Theorem says that if you count how many candies you pick up at each point where there's a pole or singularity of your function, and you add up all those counts (weighted by 2i, you'll get the total number of candies you've collected along path.

The proof uses the fact that you can break down the integral around path into integrals around small circles centered at each singularity inside the path. Then, using Cauchy's Integral Formula and the definition of residues, you can relate the integral around each small circle to the residue at the corresponding singularity. Adding up all these contributions gives you the result of the Residue Theorem.

**How to Find Residues of a Complex Function:**

**1.Identify Singularities:**

Look for points where the function behaves oddly, such as where it becomes infinite or undefined. These are called singularities.

**2. Determine Type:**

Determine what kind of singularity each point is

Poles: Points where the function blows up.

Removable Singularities: Points where the function is undefined but can be "filled in" to make it nice and smooth.

Essential Singularities: Points with really weird behavior, like oscillating or chaotic.

**3. For Poles:**

If it's a simple pole (a pole of order 1), you can find the residue by plugging the pole into a simple formula.

If it's a higher-order pole, you might need to do some extra work, like expanding the function into a fancy series.

**4.For Other Singularities:**

For removable singularities, the residue is just the value of the function at that point.

For essential singularities, things get a bit trickier and may require more advanced techniques.

**5. Evaluate the Residue:**

If it's a simple formula, just plug in the numbers.

If it's a series, identify the coefficient of a certain term in the series.

**6 .Repeat for Each Singularity:**

Go through each singularity one by one and find its residue using the method that fits.

**7. Sum Up:**

Once you've found the residues at each singularity, add them all up.

**8. Use in Applications:**

Residues are often used in complex integrals, especially with the Residue Theorem, to make the math easier and solve tricky problems.

In short, finding residues is about identifying the weird points of a function and figuring out what happens there. Then, you use some formulas or series to find numbers that tell you how the function behaves at those points.

**Computing improper integrals using residue theorem**

**1. Identify Singularities**: Look for points where the function has issues, like division by zero or where it blows up. These are called singularities.

**2. Choose Contour:** Draw a closed contour that includes all the singularities you found. Keep it simple, like a circle or a rectangle.

**3. Find Residues:** For each singularity inside your contour, calculate its residue. This is like figuring out how "bad" the singularity is.

**4. Apply Residue Theorem:** Add up all the residues you found and multiply the sum by 2 i. That gives you the value of the integral.

**5. Evaluate:** Now that you have the sum of residues, you've got your answer! That's the value of the improper integral.

**Jordan’s Lemma**

Jordan's Lemma basically says that when you integrate a function times an exponential over a semicircular contour with a large radius, the integral gets smaller and smaller as the radius becomes infinite.

The proof relies on the fact that the exponential term decays very fast as z moves away from the origin.

As the radius of the semicircle grows larger, the contribution from the exponential term becomes negligible, causing the integral to approach zero.

Jordan's Lemma is particularly useful in complex analysis for evaluating contour integrals involving functions that decay rapidly as |z| increases, which often occurs in applications involving Fourier transforms and Laplace transforms.

**Integrate Fourier Integrals**

**1. Understand Fourier Integrals:** Fourier integrals represent functions as combinations of sines and cosines, allowing us to analyze periodic phenomena and decompose signals into frequency components.

**2. Identify the Integral:** Start with the Fourier integral you want to evaluate. It typically involves integrals of trigonometric functions over a certain range.

**3. Use Symmetry Properties:** If the function you're integrating has symmetry properties (even or odd), you can exploit them to simplify the integral. For example, an even function integrated over a symmetric range yields twice the integral over half the range.

**4. Leverage Fourier Transforms:** If you're dealing with complex functions, consider using Fourier transforms to convert the integral into a form that's easier to evaluate. Fourier transforms often simplify integrals involving convolution or modulation.

**5. Apply Integration Techniques:** Use standard integration techniques from calculus, such as substitution, integration by parts, or trigonometric identities, to simplify the integral. Choose the technique that best suits the form of the integrand.

**6. Use Contour Integration:** For complex integrals involving exponential functions, contour integration techniques from complex analysis can be powerful tools. Jordan's Lemma and Residue Theorem are commonly used in this context.

**7. Consider Special Functions:** Certain special functions, such as Bessel functions or the Dirac delta function, often appear in Fourier integrals. Familiarize yourself with their properties and integration techniques specific to them.

**8. Check Convergence:** Ensure that the integral converges, especially if dealing with infinite integrals. Use convergence tests to verify that the integral behaves properly.

**9. Evaluate the Integral:** Once you've simplified the integral using the above techniques, perform the necessary calculations to find its value. This may involve solving equations, evaluating limits, or manipulating expressions to arrive at a final result.

**10. Verify Results:** Double-check your solution to ensure it matches expectations and satisfies any constraints or boundary conditions imposed by the problem.

**Winding Numbers**

Winding numbers are a way to describe how many times a curve winds around a specific point. Imagine you're walking along a path that twists and turns. If you keep track of how many times you turn clockwise or counterclockwise around a point, that's the winding number. If you end up back where you started without any turns, the winding number is zero. If you turn around the point once, it's +1 (clockwise) or -1 (counterclockwise), and so on for each additional loop. Winding numbers help in understanding the topological properties of curves in space.

**Meromorphic functions**

**Rational Functions:** These are ratios of two polynomials. They're simple but can have issues at points where the denominator is zero.

Now, meromorphic functions are like a mix of holomorphic and rational functions. They are smooth and nice almost everywhere, just like holomorphic functions. However, they might have points where they misbehave, called singularities. These singularities are like glitches where the function might blow up or become undefined, similar to how a rational function might misbehave at points where its denominator becomes zero.

So, meromorphic functions are essentially complex functions that behave well most of the time but can have occasional "hiccups" at certain points.

**Argument Principle**

Imagine you have a complex function (a function that takes complex numbers as inputs and outputs complex numbers) defined in a region of the complex plane. The argument principle relates to the behavior of this function around its zeros (points where the function equals zero) and poles (points where the function becomes infinite).

**1. Zeros and Poles:** Zeros are where the function vanishes (output equals zero), and poles are where the function blows up (output becomes infinite).

**2. Counting Loops**: Imagine drawing a loop around the zeros and poles of the function within the region of interest.

**3. Count the Turns:** As you trace along the loop, count how many times the angle of the function's output changes as you move around each zero or pole in a counterclockwise direction.

**4. Net Turn:** The net count of these angle changes (including positive and negative changes) is equal to the difference between the number of zeros and the number of poles enclosed by the loop, counting multiplicities (how many times each zero or pole is repeated).

**5. Argument Principle Equation:** This can be mathematically expressed as the contour integral of the function divided by the function itself along the loop, which is equal to 2 i times the difference between the number of zeros and the number of poles.

**Rouche’s Theorem**

Imagine you have two complex functions, let's call them f(z) and g(z) , defined on a region in the complex plane. Rouche's Theorem helps us determine how many zeros f(z) and f(z) + g(z) share in that region.

**1. Function Comparison:** Rouche's Theorem compares the sizes of f(z) and g(z) in a specific region. More precisely, it looks at how much bigger f(z) is than g(z) in that region.

**2. Counting Zeros:** If |f(z)| > |g(z)| on the boundary of a closed curve within that region, then f(z) and f(z) + g(z) have the same number of zeros inside that closed curve.

**3. Intuitive Understanding:** You can think of f(z) as the dominant function and g(z) as a smaller "perturbation" or "correction" to f(z). If f(z) is significantly larger than g(z) around the boundary of the curve, then the zeros of f(z) won't change much when you add g(z) .

**4. Rigorous Statement**: Mathematically, Rouche's Theorem says that if f(z) and g(z) are analytic functions inside and on a simple closed curve C , and |f(z) + g(z)| > |f(z)| for all z on C , then f(z) and f(z) + g(z) have the same number of zeros inside C , counting multiplicities.

It seems there might be a typo in your question. Did you mean "Branch points"? If so, I can explain that for you. Branch points are essential in complex analysis, especially when dealing with multivalued functions. Let me explain:

**Branch Cuts**

To make sense of multivalued functions, we often introduce what we call "branch cuts." These are like lines or curves in the complex plane where we decide to "cut" the function so that it becomes single-valued.

**Branch Points**

A branch point is a point in the complex plane where a multivalued function behaves in a special way. Specifically, at a branch point, you can't choose a single continuous value for the function. Instead, as you go around a branch point, the function may change its value.

**Complex integration using Branch Cuts**

Complex integration using branch cuts involves choosing appropriate branch cuts to make a function single-valued, enabling us to integrate it along a contour in the complex plane. Here's how it generally works:

**1. Identify Multivalued Functions:** First, identify the function you want to integrate. Ensure it's a multivalued function, such as the complex logarithm or square root.

**2. Choose Branch Cuts:** Select a branch cut for the multivalued function. A common choice is a straight line in the complex plane, often along the negative real axis or some other line where the function is not defined. The choice of branch cut depends on the function and the contour you're integrating along.

**3. Define Branches:** Once you've chosen the branch cut, define branches of the function. For example, for the complex logarithm, you might define the principal branch where the imaginary part lies in the range (-π, π].

**4. Integrate Along Contour:** With the function properly defined with branch cuts and branches, integrate it along the desired contour in the complex plane. Ensure that the contour does not cross any branch cuts, as this could lead to inconsistencies in the integration.

**5. Account for Branch Cuts:** When integrating across branch cuts, you may need to make adjustments to account for the discontinuity in the function. This could involve adding or subtracting terms to compensate for the change in value across the branch cut.

**6. Evaluate Integral:** Once the contour integral is set up, evaluate it using standard techniques of complex analysis, such as parameterization and residue calculus if necessary.

**7. Check Consistency:** After integrating, ensure that your result is consistent with the behavior of the function and the chosen branches. If the function is well-defined and single-valued along the contour, the integral should yield a valid result.