Week 08

Date : ..../........../..........

Linear Equation

* Linear system :
  - described by a set of linear equation
    - → expressed by a set of linear variables
      - → means that the exponent of all variables must be either zero (constant) or one.

[data science, AI application, weather forecasting etc.]

* A simplest solvable linear system has the same no. of eqn. and linearly independent variables.

* Algebraically, a linear system,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

a_{n1}: row, column (in matrix)

- all $b_i = 0$: homogeneous linear system.
- Otherwise, ~~nonlinear~~ non-homogeneous

$$\Rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

$\therefore Ax = b$

$\therefore$ solution of linear system, $\boxed{x = A^{-1} b}$ only one unique solution (if $\det A \neq 0$)

* Basic properties of A :

- square matrix of order $n \times n$
- $A^T \rightarrow$ transpose of A. $(a^T)_{ij} = a_{ji}$
- A is symmetric if $A = A^T$
- A is non-singular iff $\exists$ (there exists) a solution $x \in \mathbb{R}^n$ for every $b \in \mathbb{R}^n$
- A is non-singular iff $\det(A) \neq 0$
- A " " iff there exists a unique inverse A^{-1} such that $AA^{-1} = A^{-1}A = I$

* Gaussian Elimination method is a technique that transform the matrix A into triangular form and solve $Ax = b$ for x.

- Only use ~~elim~~ elementary row or column operation.

Date : ...../...../.....

* $L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$ * $U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$

$$\det(L) = \prod_{k=1}^{k=n} l_{kk} \qquad \det(U) = \prod_{k=1}^{k=n} u_{kk}$$

$$= l_{11} l_{22} \cdots l_{nn} \qquad = u_{11} u_{22} \cdots u_{nn}$$

(7) $n = 4$, $A = L$

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$\Rightarrow$

$$l_{11} x_1 = b_1$$
$$l_{21} x_1 + l_{22} x_2 = b_2$$
$$l_{31} x_1 + l_{32} x_2 + l_{33} x_3 = b_3$$
$$l_{41} x_1 + l_{42} x_2 + l_{43} x_3 + l_{44} x_4 = b_4$$

solving these eq. will give the reesult.

• All diagonal elements $\rightarrow$ non-zero
(any triangular matrix)

(n×n) lower triangular system, $Lx = b$,

$$x_j = \frac{b_j - \sum_{k=1}^{j-1} l_{jk} x_k}{l_{jj}} \quad ; \; j = 1, 2, \dots, n$$

⇒ forward substitution method

(n×n) Upper triangular system, $Ux = b$,

$$x_j = \frac{b_j - \sum_{k=j+1}^{n} u_{jk} x_k}{u_{jj}} \quad ; \; j = n, n-1, \dots, 1$$

⇒ backward substitution method

For x_n, <u>no. of operation</u> needed,

1 divison
\+ (n-1) multiplication
\+ (n-1) substraction

$$\sum_{j=1}^{n} [1 + 2(j-1)] = \sum_{j=1}^{n} (2j-1)$$

$$= 2 \sum_{j=1}^{n} j - \sum_{j=1}^{n} 1$$

$$= 2 \cdot \frac{1}{2} n(n+1) - n$$

$$= n^2$$

↳ computational complexity

→ rough estimate of comp. cost

Date :........../........../..........

Gaussian Elimination Method

- Apply 'row operation' by column wise, on <u>augmented</u> matrix.

  → every element <u>below</u> diagonal element $= 0$

- need $(n-1)^{th}$ row operation all together.

- row multiplier, $m_{ik} = \dfrac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$; $i = k+1, k+2, \ldots, n$
  (k^{th} operation)

- For $i, j = k+1, \ldots, n$,

$$\left.\begin{aligned} a_{ij}^{k+1} &= a_{ij}^{k} - m_{ik}\, a_{kj}^{k} \\ b_i^{k+1} &= b_i^{k} - m_{ik}\, b_k^{k} \end{aligned}\right\}$$

~~ele~~ eliminate the elements in entire k^{th} column below a_{kk}

(?)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right]$$

$$m_{21} = \frac{a_{21}}{a_{11}} = 1$$

Date :........../........../..........

1st row operation,

$$r_2' = r_2 - 1 \times r_1 = (1 \quad -2 \quad 2 \quad 4) - 1 \times (1 \quad 2 \quad 1 \quad 0)$$

$$= (0 \quad -4 \quad 1 \quad 4)$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right] \qquad m_{31} = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$$

$$r_3' = r_3 - 2r_1 = [0 \quad 8 \quad -4 \quad 4]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right]$$

2nd row operation :

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{8}{-4} = -2$$

$$r_3' = r_3 + 2r_2 = [0 \quad 0 \quad -2 \quad 12]$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 = 11$$

$$-4x_2 + x_3 = 4 \Rightarrow x_2 = -5/2$$

$$-2x_3 = 12 \Rightarrow x_3 = -6$$

Date :........../........../..........

* No. of operation required,

$$N = \sum_{k=1}^{n-1} \left[2(n-k)^2 + (n-k)\right]$$

$$= n(2n+1)\sum_{k=1}^{n-1} 1 - (4n+1)\sum_{k=1}^{n-1} k + 2\sum_{k=1}^{n-1} k^2$$

$$= n(2n+1)(n-1) - (4n+1)\left(\frac{1}{2}\right)(n-1)\,n + 2\left(\frac{1}{6}\right) n \; \overset{(n-1)(2n-1)}{\cancel{(n+1)(2n+1)}}$$

$$= \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n$$

- Sum of 1st n no $= \dfrac{n(n+1)}{2}$

- Sum of 1st n^2 no $= \dfrac{n(n+1)(2n+1)}{6}$

Date : ..../........../

LU Decomposition

- $A = LU$
- $F^{(1)}$ = matrix with (-row multiplier) after 1st row operation
- For $n \times n$ matrix A,

after $(n-1)^{th}$ row operation $\rightarrow U$ matrix

$$F^{(n-1)} \cdots \underbrace{F^{(2)} F^{(1)} A}_{A2} = U$$

$$\Rightarrow \quad A^{(n)} \equiv F^{(n-1)} A^{(n-1)} = U$$

$$\Rightarrow \quad FA = U \quad ; \quad F = F^{(n-1)} \cdots F^{(2)} F^{(1)}$$

- Inverse of produce of matrices = reverse order product of inverse matrices

$$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$$

$$\boxed{A = F^{-1} U = LU}$$

$$= (F^{(1)})^{-1} (F^{(2)})^{-1} \cdots (F^{(n-1)})^{-1}$$

- Inverse of C,

$$C^{-1} = \frac{1}{|\det(C)|} (\text{Adj } C)$$

Date : ..../........../..........

- $F^{(k)}$ = with -ve multipliers

$(F^{(k)})^{-1}$ = " +ve " (same)

- $$F^{-1} \equiv L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots \\ m_{21} & 1 & \cdots & \cdots & \cdots \\ m_{31} & m_{32} & 1 & \cdots & \cdots \\ m_{41} & m_{42} & m_{43} & \ddots & \vdots \\ \vdots & \vdots & \vdots & 1 & 0 \\ m_{n1} & m_{n2} & m_{n3} \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

* $Ax = b$

$\Rightarrow LUx = b$

Let,

$Ux = y$

$\therefore Ly = b$

$\therefore$ Main advantage - It can be used to solve several linear system that ~~on~~ differ by the values of b only. We need to compute L & U only once.

Date : ..../........./

(?) $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}}_{U}$

Solve for $Ly = b$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$y_1 = 0$

$y_2 = -y_1 + 4 = 4$

$2y_1 - 2y_2 + y_3 = 4$

$\Rightarrow y_3 = 4 - 2y_1 + 2y_2 = 12$

Solve for $Ux = y$,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$-2x_3 = 12$

$\Rightarrow x_3 = -6$

$-4x_2 + x_3 = 4$

$\Rightarrow x_2 = -5/2$

$x_1 + 2x_2 + x_3 = 0$

$\Rightarrow x_1 = 11$

Date : ..../......../........

Pivoting :

- used to avoid appearance of 0 along the diagonal elements in G.E. and LU decomposition.

→ multiplier for that row operation will be undefined (dividing by 0). Both method will fail.

- ~~Ued~~ Used to avoid diagonal ~~elemand~~ element difference by large order of magnitude. [Diagonal elements should be of same order of magnitudes. Else, loss of significance may occur in solution]

- Swap 2 rows or columns so that diagonal elements don't have any zeros.