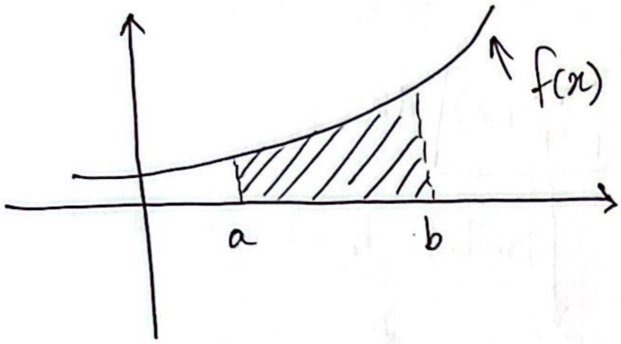
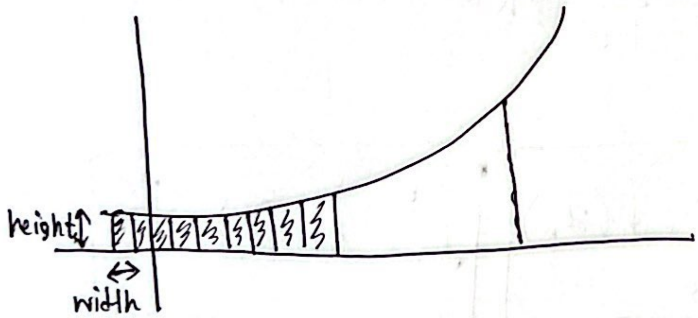
Integration:

$$I(f) = \int_a^b f(x)\,dx$$



→ Integration gives the area under $f(x)$ within the bound a & b.

→ By definition, integration is an infinite sum.



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f_i(x)}_{\text{height}}\ \underbrace{\Delta x}_{\text{width}}$$

→ Numerical integration replace function $f(x)$ with interpolating polynomial of degree n that passes through $(n+1)$ nodes

Actual Integration $= I(f) = \int_a^b f(x)\, dx$

Numerical Integration $= I_n(f) = \int_a^b \boxed{P_n(x)}\, dx$

↓ writing $P_n(x)$ using lagrange basis

This polynomial $P_n(x)$ must be interpolated with equidistant nodes $x_0, x_1, \dots x_n$ ← $P_n(x) = \sum_{i=0}^{n} l_k(x) \cdot f(x_k)$

equidistant ↓ equally spaced.

$$\therefore \quad I_n(f) = \int_a^b \sum_{i=0}^{n} l_k(x) \cdot f(x_k)\, dx$$

$$= \boxed{\sum_{k=0}^{n} f(x_k)} \underbrace{\boxed{\int_a^b l_k(x)\, dx}}_{\sigma_k}$$

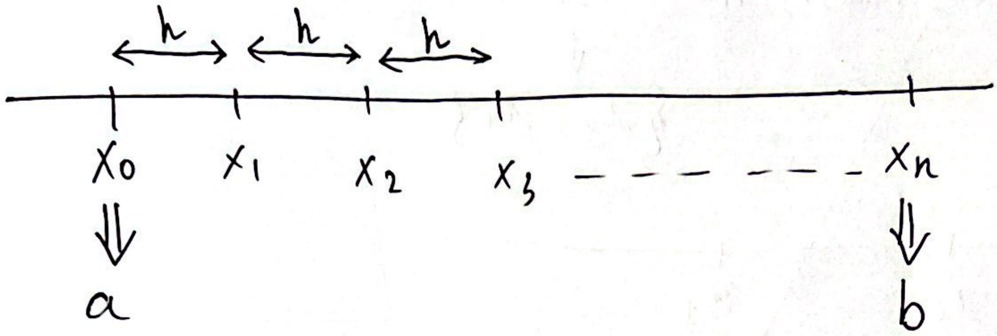
σ_k (~~weight function~~) (weighted factors)

$$\therefore \quad \boxed{I_n(f) = \sum_{k=0}^{n} \sigma_k \cdot f(x_k)}$$

↑ Newton cote's Formula.

Newton Cote's Formula → Closed / → open

Finding $x_0, x_1 \cdots x_n$ Using closed Newton cote's Formula:



→ 'a' and 'b' are integration intervals.

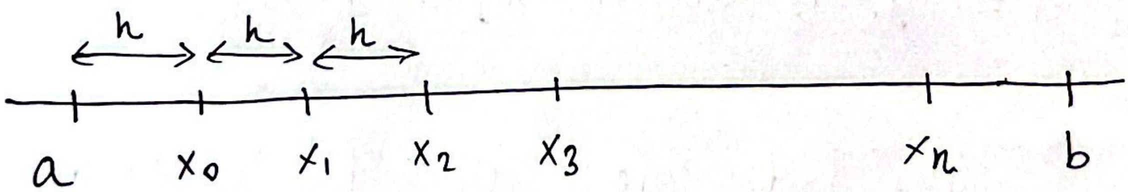
→ $\boxed{h = \dfrac{b-a}{n}}$

$x_0 = a$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

Finding $x_0, x_1 \cdots x_n$ using ~~to~~ open Newton cote's Formula:



$\boxed{h = \dfrac{b-a}{n+2}}$

$x_0 = a + h$

$x_1 = a + 2h$

$x_2 = a + 3h$

Trapezium / Trapezoidal Rule:

→ Closed Newton Cotes formula with $n=1$

$n=$ degree of polynomial $=1$

$\therefore$ number of nodes $= n+1 = 2$

$$2 \downarrow \{x_0, x_1\}$$

$$x_0 \Downarrow a \qquad x_1 \Downarrow b$$

[because closed newton cote's]

$$I_n(f) = \int_a^b P_n(x)\,dx$$

$$I_1(f) = \int_a^b P_1(x)\,dx$$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$I_1(f) = \int_a^b \left[l_0(x) f(x_0) + l_1(x) f(x_1) \right] dx$$

$$= \underbrace{\int_a^b l_0(x)\,dx}_{\sigma_0} \cdot f(x_0) + \underbrace{\int_a^b l_1(x)\,dx}_{\sigma_1} \cdot f(x_1)$$

$$\therefore I_1(f) = \sigma_0 f(x_0) + \sigma_1 f(x_1)$$

$$\sigma_0 = \int_a^b l_0(x)\, dx$$

$$= \int_a^b \frac{x - x_1}{x_0 - x_1}\, dx$$

$$= \int_a^b \frac{x - b}{a - b}\, dx$$

$$= \frac{1}{a-b} \int_a^b (x-b)\, dx$$

$$= \frac{1}{a-b} \left[\frac{x^2}{2} - bx \right]_a^b$$

$$= \frac{1}{a-b} \left(\frac{b^2}{2} - b^2 - \frac{a^2}{2} + ab \right)$$

$$= \frac{b-a}{2}$$

$$\sigma_1 = \int_a^b l_1(x)\, dx$$

$$= \frac{1}{b-a} \int_a^b (x-a)\, dx$$

$$\vdots$$

$$= \frac{b-a}{2}$$

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$$\therefore I_1(f) = \int_a^b P_1(x)\,dx$$

$$= \sigma_0 f(x_0) + \sigma_1 f(x_1)$$

$$= \sigma_0 f(a) + \sigma_1 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

$$= \frac{b-a}{2}\left(f(a) + f(b)\right)$$

$$\therefore \boxed{I_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]}$$

Example:

Find $\underbrace{I(f)}_{\text{Actual}}$ and $\underbrace{I_1(f)}_{\text{Numerical}}$ of the function e^x on interval $[0, 2]$.

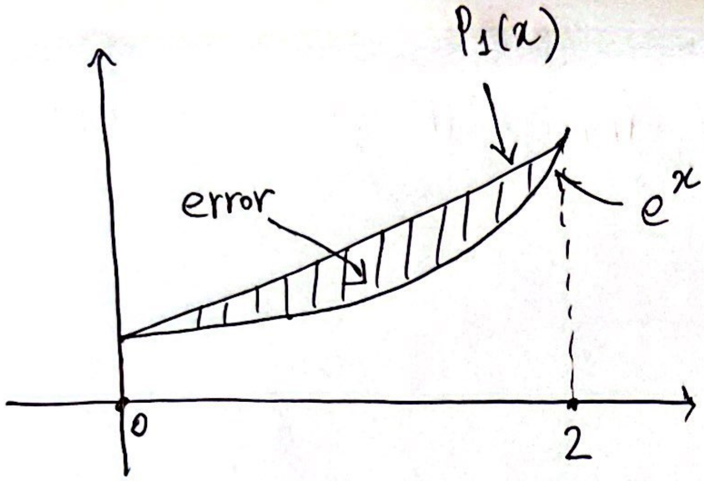
Solution:

$$I(f) = \int_0^2 e^x\,dx = \left[e^x\right]_0^2 = e^2 - e^0 = 6.389 \text{ [Actual]}$$

$$I_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{2-0}{2}\left[e^0 + e^2\right] = 8.389 \text{ [numerical approx]}$$

$$\% \text{ error} = \frac{I - I_1}{I} \times 100 = 31.3\,\%$$ [error is large bcz degree 1 polynomial is used].



→ We can find the upper bound of the error.

→ If a function $f(x)$ is interpolated by a degree n polynomial error ~~is~~ ~~for~~ can be found using cauchy's Theorem.

Upper bound error
$$|f(x) - P_n(x)| \leq \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n) \right|$$

For integration, upper bound error =

$$|I - I_n| \leq \underbrace{\left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right|}_{\text{Need to find max value within } [a;b]} \int_a^b \left| (x-x_0)(x-x_1)\cdots(x-x_n) \right| dx$$

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Example:

Computing the upper bound of error for the previous example.

→ $n = 1$

$f(x) = e^x$

$a = 0$

$b = 2$

Solution:

Finding the max of $\left|\dfrac{f^{(n+1)}(\xi)}{(n+1)!}\right|$ within $[0,2]$

$= \left|\dfrac{f^{(2)}(\xi)}{2!}\right|$

$= \left|\dfrac{e^{\xi}}{2!}\right|$

$f(x) = e^x$

$f^{(1)}(x) = e^x$

$f^{(2)}(x) = e^x$

$\therefore f^{(2)}(\xi) = e^{\xi}$

Max of $\dfrac{e^{\xi}}{2!}$ within $[0,2] = \dfrac{e^2}{2!}$

$\Rightarrow \int_a^b |(x-x_0)(x-x_1)|\,dx$

$= \int_a^b |(x-a)(x-b)|\,dx = \int_0^2 |(x^2-2x)|\,dx$

$= \left|\left[\dfrac{x^3}{3} - \dfrac{2x^2}{2}\right]_0^2\right| = \dfrac{4}{3}$

$\therefore$ upper bound of error $\leq \dfrac{e^2}{2!} \times \dfrac{4}{3} \approx 4.926$.

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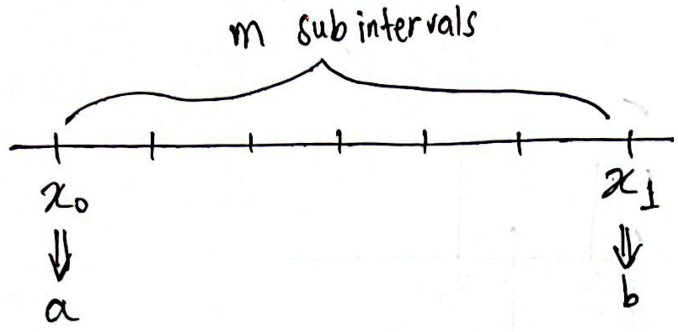
Composite Newton - Cotes Formula:

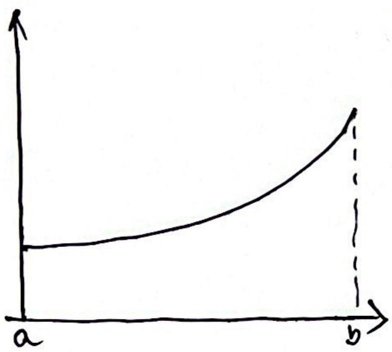
→ This method improves result without increasing num. of nodes.

→ Basic idea is to divide the interval $[a, b]$ into m sub intervals.

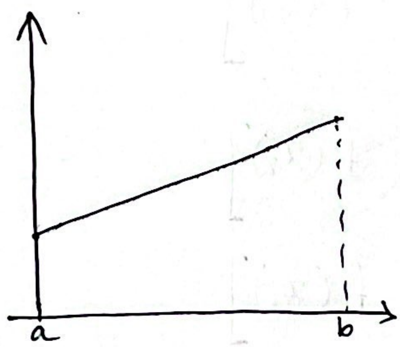
<u>when $n = 1$:</u>



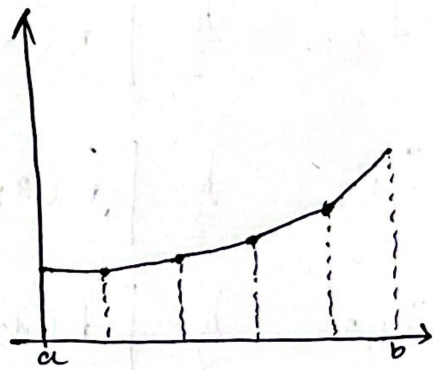
→ For each sub interval, we apply trapezium rule, then add them up.



Actual integration
$I(f)$



Newton-cotes with $n=1$
$I_1(f)$



Composite Newton Cotes with $n = 1$
$C_{1,m}(f)$

→ Total sum is denoted by $C_{1,m}(f)$ and called composite Newton Cote's.

for degree 1

for m sub interval

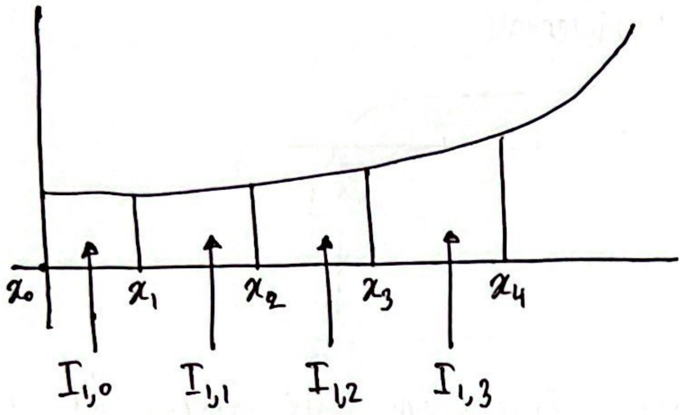
→ For m sub intervals, we define

$$\boxed{h = \frac{b-a}{m}}$$

Apply Trapezium Rule for each sub interval

$$I_1(f) = \text{Trapezium Rule} = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{h}{2}\left[f(a) + f(b)\right]$$



$$I_{1,0} = \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

$$I_{1,1} = \frac{h}{2}\left[f(x_1) + f(x_2)\right]$$

$$I_{1,2} = \frac{h}{2}\left[f(x_2) + f(x_3)\right]$$

$$\vdots$$

$$I_{1,m-1} = \frac{h}{2}\left[f(x_{m-2}) + f(x_{m-1})\right]$$

$$I_{1,m} = \frac{h}{2}\left[f(x_{m-1}) + f(x_m)\right]$$

$$C_{1,m}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \dots + 2f(x_{m-1}) + f(x_m)\right]$$

Example:

$f(x) = e^x$

$a = 0$
$b = 2$

→ Exact result $= I(f) = \int_0^2 e^x \, dx = 6.389056$

→ Composite Newton Cotes with num of subintervals $= 2$ $(m=2)$:

Step 1: Find h

$$h = \frac{b-a}{m} = \frac{2-0}{2} = 1$$

Step 2: Find $x_0, x_1, x_2 \cdots x_m$

→ Remember: If $m=2$, find x_0 to x_2
If $m=3$, find x_0 to x_3
If $m=4$, find x_0 to x_4.

$X_0 = a = 0$ [since trapezium rule follows closed newton cotes]

$X_1 = X_0 + h = 0 + 1 = 1$

$X_2 = X_1 + h = 1 + 1 = 2$

Step 3: Find $C_{1,m}(f)$

$$C_{1,2}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$
$$= \frac{1}{2}\left[e^0 + 2e^1 + e^2\right]$$
$$= 6.91281$$

→ <u>Composite Newton Cotes with num of sub intervals = 3 (m=3):</u>

$$h = \frac{b-a}{m} = \frac{2-0}{3} = \frac{2}{3}$$

<u>Find x_0 to x_3:</u>

$x_0 = a = 0$

$x_1 = x_0 + h = 0 + \frac{2}{3} = \frac{2}{3}$

$x_2 = x_1 + h = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

$x_3 = x_2 + h = \frac{4}{3} + \frac{2}{3} = 2$

$$c_{1,3}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{2/3}{2}\left[e^{0} + 2e^{2/3} + 2e^{4/3} + e^{2}\right]$$

$$= 6.62395$$

→ <u>Composite Newton Cotes with $m=4$:</u>

$$c_{1,4} = \frac{0.5}{2}\left[e^{0} + 2e^{0.5} + 2e^{1} + 2e^{1.5} + e^{2}\right] = 6.52161$$

$$\boxed{\text{Error decreases as } m \text{ increases}}$$

Simpson's Rule:

Trapezium Rule $= \int_a^b P_1(x)\,dx$

Simpson's Rule $= \int_a^b P_2(x)\,dx$

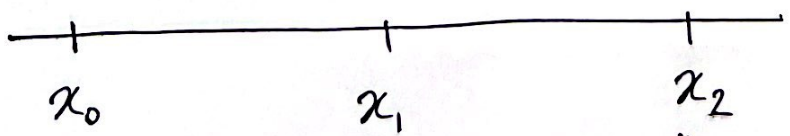
$$I_2(f) = \int_a^b \underbrace{P_2(x)}_{P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)}\,dx$$

$$I_2(f) = \int_a^b \left[l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) \right] dx$$

$$= \underbrace{\int_a^b l_0(x)\,dx}_{\sigma_0} \cdot f(x_0) + \underbrace{\int_a^b l_1(x)\,dx}_{\sigma_1} \cdot f(x_1) + \underbrace{\int_a^b l_2(x)\,dx}_{\sigma_2} \cdot f(x_2)$$

$$\therefore\ I_2(f) = \sigma_0 f(x_0) + \sigma_1 f(x_1) + \sigma_2 f(x_2)$$

Here, since $n = 2$, number of nodes $= n+1 = 3 \rightarrow \{x_0, x_1, x_2\}$



$x_0 \Downarrow a$

$x_1 \Downarrow x_1 = x_0 + h$

$m = x_1 = \dfrac{a+b}{2}$

$x_2 \Downarrow b$

$$\sigma_0 = \int_a^b l_0(x)\,dx$$

$$= \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx$$

$$= \int_a^b \frac{(x-m)(x-b)}{(a-m)(a-b)}\,dx$$

$$= \frac{1}{(a-m)(a-b)} \int_a^b (x-m)(x-b)\,dx$$

$$= \frac{1}{6}(b-a)$$

$$\sigma_1 = \int_a^b l_1(x)\,dx$$

$$= \vdots$$

$$= \frac{2}{3}(b-a)$$

$$\sigma_2 = \int_a^b l_2(x)\,dx$$

$$= \vdots$$

$$= \frac{1}{6}(b-a)$$

$$I_2(f) = \sigma_0 f(x_0) + \sigma_1 f(x_1) + \sigma_2 f(x_2)$$

$$= \sigma_0 f(a) + \sigma_1 f(m) + \sigma_2 f(b)$$

$$= \frac{1}{6}(b-a) f(a) + \frac{2}{3}(b-a) f(m) + \frac{1}{6}(b-a) f(b)$$

$$= \frac{b-a}{6}\left[f(a) + 4f(m) + f(b) \right]$$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Exactness:

For numerical integration, upper bound of error =

$$|I - I_n| = \left|\frac{1}{(n+1)!} f^{(n+1)}(\xi)\right| \int_a^b \left|(x-x_0)(x-x_1)\dots(x-x_n)\right| dx$$

→ If $f^{(n+1)}(\xi) = 0$, error $= 0$

→ In that case, Newton Cotes will give exact answers

→ The above formula was derived using Cauchy's Theorem.

Cauchy's Theorem:

$$|f(x) - P_n(x)| = \underbrace{\left|\frac{1}{(n+1)!} f^{(n+1)}(\xi) \quad (x-x_0)(x-x_1)\dots(x-x_n)\right|}_{\text{error}}$$

$$f(x) - P_n(x) = \text{error}$$

$$f(x) = P_n(x) + \text{error}$$

→ If error $= 0$,

$$\boxed{f(x) = P_n(x)}$$

→ $f(x)$ is a n-degree polynomial

→ $f(x)$ is a polynomial itself.

→ If $f(x)$ itself is a polynomial, $I_n(f)$ will give exact result since error $=0$.

→ ~~The~~ This implies that trapezium rule $I_1(f)$ is exact for all functions $f(x) = P_1(x)$

→ In other words, if we have a degree 1 polynomial, $P_1(x)$ and we apply both the actual integration, $I(f)$, and numerical integration, $I_1(f)$, we will get the exact result.

Definition:

The degree of exactness is the largest integer, n, for which the formula is exact for all polynomials, $P_n(x)$.

Example:

Find (a) Actual integration, $I(f)$

(b) Newton Cote's integral using $n=2$, $I_2(f)$

for the following functions:

① $f(x) = 1$

② $f(x) = x$

③ $f(x) = x^2$

④ $f(x) = x^3$

⑤ $f(x) = x^4$

① $f(x) = 1$

(a) Exact $= I(f) = \int_a^b 1\, dx = b-a$

match / zero error

(b) Newton cotes $= I_2(f) = \frac{b-a}{6}\left[1+4+1\right] = b-a$

---

② $f(x) = x$

(a) Exact $= I(f) = \int_a^b x\, dx = \frac{1}{2}(b^2-a^2)$

match/ zero error

(b) Newton Cotes $= I_2(f) = \frac{b-a}{6}\left[a + 4\left(\frac{a+b}{2}\right) + b\right] = \frac{1}{2}(b^2-a^2)$

---

③ $f(x) = x^2$

(a) Exact $= I(f) = \int_a^b x^2\, dx = \frac{1}{3}(b^3-a^3)$

match

(b) Newton Cotes $= I_2(f) = \frac{b-a}{2}\left[a^2 + 4\left(\frac{a+b}{2}\right)^2 + b^2\right] = \frac{1}{3}(b^3-a^3)$

---

④ $f(x) = x^3$

(a) Exact $= I(f) = \int_a^b x^3\, dx = \frac{1}{4}(b^4-a^4)$

Match

(b) Newton cotes $= I_2(f) = \frac{b-a}{6}\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right] = \frac{1}{4}(b^4-a^4)$

⑤ $f(x) = x^4$

(a) Exact $= I(f) = \int_a^b x^4\, dx = \frac{1}{5}(b^5 - a^5)$

(b) Newton Cotes $= I_2(f) = \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] \neq \frac{1}{5}(b^5 - a^5)$

$\therefore$ Above result shows that simpson's formula, $I_2(f)$, gives exact result upto degree 3 polynomial, and error becomes non-zero from degree 4 polynomial and higher.

→ Degree of exactness is exactly 3 for Simpson's Rule, $I_2(f)$.