Solution:

1) a) $f(x) = 6e^{-5x}$

$x_0 = 0.2$

$h = 0.5$

Using formula of central difference,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$= \frac{f(0.7) - f(-0.3)}{2 \times 0.5}$$

$$= \frac{6e^{-5\times0.7} - 6e^{-5\times(-0.3)}}{1}$$

$$= 0.181184 - 26.8901$$

$$= -26.7089$$

(Ans)

(b) using forward difference,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \frac{f(0.7) - f(0.2)}{0.5}$$

$$= \frac{6e^{-5\times0.7} - 6e^{-5\times0.2}}{0.5}$$

$$= \frac{0.181184 - 2.20728}{0.5}$$

$$= -4.05219$$

(Ans.)

(c)

h	forward difference $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$ here, $x_0 = 2$	truncation error real derivative, $f'(2) = -30e^{-5x} = -30e^{-5\times2} = -1.36199\times10^{-3}$ $\therefore$ error $= (-1.36199\times10^{-3} - \text{forward})$
1	-2.70564×10^{-4}	-1.09143×10^{-3}
0.1	-1.07181×10^{-3}	-2.90180×10^{-4}
0.01	-1.32851×10^{-3}	-3.34800×10^{-5}
0.0001	-1.36166×10^{-3}	~~-3.3×10^{-7}~~ -3.30000×10^{-7}

h	central difference $f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$, $x_0 = 2$	truncation error: error $= (-1.36199\times10^{-3}$ - central)
1	-0.0202130	0.0188510
0.1	-1.41946×10^{-3}	5.74700×10^{-5}
0.01	-1.36257×10^{-3}	5.80000×10^{-7}
0.0001	-1.36199×10^{-3}	0.000000

d) $$D_h^{(1)} = \frac{2^2 \cdot D_{h/2} - D_h}{2^2 - 1}$$ (where $x_0 = 0.2$)

We know, $$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$D_{h/2} = \frac{f(x + \frac{h}{2}) - f(x - \frac{h}{2})}{2\times\frac{h}{2}}$$

$$\therefore D_{0.5} = \frac{f(0.2+0.5) - f(0.2-0.5)}{2\times 0.5}$$

$$= \frac{6[e^{-5\times0.7} - 6\cdot e^{-5\times -0.3}]}{1}$$

$$= -26.7089 \approx -26.7090$$

$h = 0.5$

$\frac{h}{2} = 0.25$

$$\therefore D_{0.25} = \frac{f(0.2+0.25) - f(0.2-0.25)}{2 \times 0.25}$$

$$= \frac{6\left[e^{-5 \times 0.45} - e^{-5 \times -0.05}\right]}{0.5}$$

$$= -14.1435$$

$$\therefore D_{0.5}^{(1)} = \frac{4 \cdot D_{0.25} - D_{0.5}}{4-1}$$

$$= \frac{4(-14.1435) - (-26.7090)}{3}$$

$$= -9.95500$$

(Ans)

Real derivative at $x_0 = 0.2$, $f'(0.2) = -30e^{-5 \times 0.2}$

$= -11.0364$

$\therefore$ truncation error = real derivative − approximated derivative

$= -11.0364 + 9.95500$

$= -1.08140$

(Ans)

~x~

Solution:

4) a) $f(x) = x^3 - x^2 - 9x + 9 = 0$

$\Rightarrow x^2(x-1) - 9(x-1) = 0$

$\Rightarrow (x-1)(x^2-9) = 0$

$\therefore x_k = -3, 1, 3 \longrightarrow$ actual/exact roots

(Ans.)

b) $x^3 - x^2 - 9x + 9 = 0$

1st choice:

$$9x = x^3 - x^2 + 9$$

$$x = \frac{1}{9}(x^3 - x^2 + 9) = g(x)$$

2nd choice:

$$x(x^2 - x - 9) = -9$$

$$x = \frac{-9}{x^2 - x - 9} = g(x)$$

3rd choice:

$$x = x + x^3 - x^2 - 9x + 9$$

$$x = x^3 - x^2 - 8x + 9 = g(x)$$

(c) convergence rate/ratio, $\lambda = |g'(x_*)|$

$$= \left| \frac{dg}{dx}\Big|_{x=x_*} \right|$$

For 1st case:

$$g(x) = \frac{1}{9}(x^3 - x^2 + 9)$$

$$\Rightarrow g'(x) = \frac{1}{9}(3x^2 - 2x)$$

$$\therefore \lambda = |g'(x_*)| = \begin{cases} \frac{1}{9} (<1) \text{ for } x_* = 1 \text{ (Linear convergence)} \\ \frac{33}{9} (>1) \text{ for } x_* = -3 \text{ (divergence)} \\ \frac{21}{9} (>1) \text{ for } x_* = 3 \text{ (divergence)} \end{cases}$$

$\therefore g(x)$ is converging to $x_* = 1$ for 1st case.

For 2nd case:

$$g(x) = \frac{-9}{x^2 - x - 9}$$

$$\therefore g'(x) = \frac{9(2x-1)}{(x^2-x-9)^2}$$

$$\lambda = |g'(x_*)| = \begin{cases} \frac{9}{81} \ (<1) \text{ for } x_* = 1 \ \text{(Linear convergen} \\ \frac{63}{9} \ (>1) \text{ for } x_* = -3 \ \text{(divergence)} \\ \frac{45}{9} = 5 \ (>1) \text{ for } x_* = 3 \ \text{(divergence)} \end{cases}$$

$\therefore g(x)$ is converging to $x_* = 1$ for 2nd case.

For 3rd case:

$$g(x) = x^3 - x^2 - 8x + 9$$

$$g'(x) = 3x^2 - 2x - 8$$

$$\lambda = |g(x_*)| = \left.\begin{cases} 7 \ (>1) \text{ for } x_* = 1 \\ 25 \ (>1) \text{ for } x_* = -3 \\ 13 \ (>1) \text{ for } x_* = 3 \end{cases}\right\} \rightarrow \text{divergence}$$

as we need $\lambda < 1$ for convergence,

we don't have any converging point in 3rd case.

(Ans.)

(d) $\epsilon_M = 10^{-3}$ (max error bound)

$x_0 = 0$

$g(x) = \frac{1}{9}(x^3 - x^2 + 9)$

k=0 $\quad g(0) = 1.000$

k=1 $\quad g(1.000) = 1.000 \Rightarrow x_1 - x_0 \approx 0.000 \; (< 10^{-3})$

$\therefore$ 1.000 is the fixed point of $g(x)$ and it is also the root of $f(x)$.

(Ans.)

OR, $\quad g(x) = \dfrac{-9}{x^2 - x - 9}$

$g(0) = 1.000$

$g(1.000) = 1.000 \longrightarrow$ fixed point of $g(x)$ and root of $f(x)$.

$\boxed{\therefore x_k = 1.000}$

— x —

Solution:

2.(b) $D_h = \dfrac{f(x+h) - f(x-h)}{2h}$

$$f(x+h) = f(x) + f'(x).h + \frac{f^{(2)}(x)}{2!}h^2 + \frac{f^{(3)}(x)}{3!}.h^3 + \frac{f^{(4)}(x)}{4!}.h^4 + \frac{f^{(5)}(x)}{5!}.h^5 + \cdots$$

$$f(x-h) = f(x) - f^{(1)}(x).h + \frac{f^{(2)}(x)}{2!}.h^2 - \frac{f^{(3)}(x)}{3!}.h^3 + \frac{f^{(4)}(x)}{4!}.h^4 - \frac{f^{(5)}(x)}{5!}.h^5 + \cdots$$

$$\therefore D_h = \frac{1}{2h}\left[2f^{(1)}(x).h + \frac{2f^{(3)}(x)}{3!}.h^3 + \frac{2f^{(5)}(x)}{5!}.h^5 + O(h^7)\right]$$

$$D_h = f^{(1)}(x) + \frac{f^{(3)}(x)}{3!}.h^2 + \frac{f^{(5)}(x)}{5!}.h^4 + O(h^6)$$

$$\therefore D_{h/3} = f^{(1)}(x) + \frac{f^{(3)}(x)}{3!}.\left(\frac{h}{3}\right)^2 + \frac{f^{(5)}(x)}{5!}.\left(\frac{h}{3}\right)^4 + O(h^6)$$

$$\Rightarrow 3^2.D_{h/3} = 3^2 f^{(1)}(x) + \frac{f^{(3)}(x)}{3!}.\frac{h^2}{1} + \frac{1}{9}.\frac{f^{(5)}(x)}{5!}.h^4 + O(h^6)$$

$$\Rightarrow 3^2 \cdot D_{h/3} - D_h = f'(x)\left(3^2 - 1\right) - \frac{8}{9} \cdot \frac{f^{(5)}(x)}{5!} \cdot h^4 + o(h$$

$$\Rightarrow \frac{3^2 \cdot D_{h/3} - D_h}{3^2 - 1} = f'(x) - \frac{1}{9} \cdot \frac{f^{(5)}(x)}{5!} \cdot h^4 + o(h^6)$$

$$\Rightarrow D_h^{(1)} = f'(x) - \frac{1}{9} \cdot \frac{f^{(5)}(x)}{5!} \cdot h^4 + o(h^6).$$

(Ans)

—×—