Root Finding of Non-Linear Equations:

→ For powers of x which are low, we can easily find the root using approaches like:

eg $f(x) = x^2 - 2x = 0$

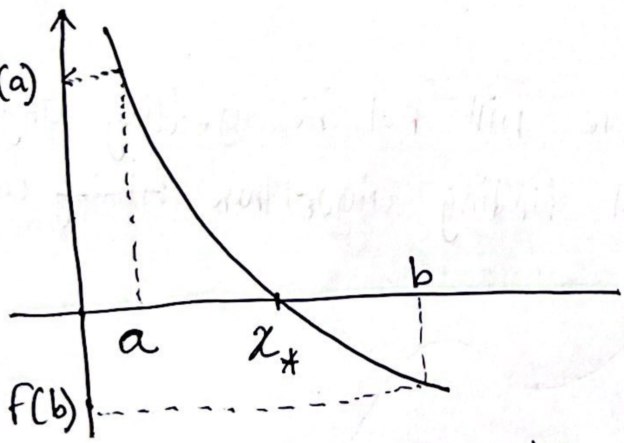
$\Rightarrow x(x-2) = 0$

$x = 2, \ x = 0$

→ But what if we are dealing with non-linear equations with high powers/ series/ polynomials/ rational functions? How can we find their root?

↳ We can apply some iterative algorithms to find the roots.

Bisection Root Finding Algorithm:



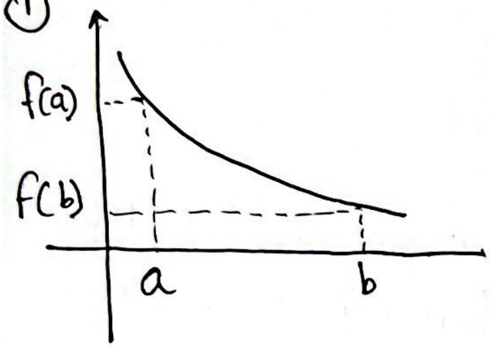
→ If root exists, the ~~graph~~ value of the function changes sign.

Here, $\left.\begin{array}{l} f(a) = +ve \\ f(b) = -ve \end{array}\right\}$ Roots lies within $[a, b]$ ~~changes~~ Therefore it changes sign

So, if $\boxed{f(a) * f(b) < 0}$ root exists between a and b.

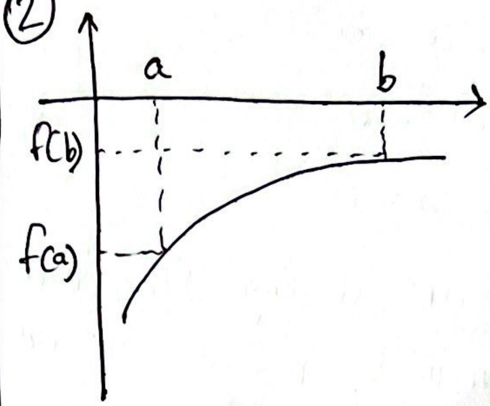
Examples:

1



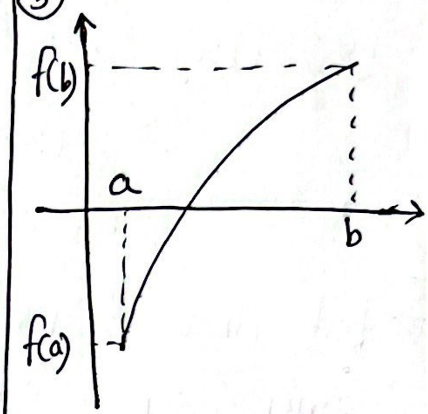
$f(a) * f(b)$
$= (+ve) * (+ve)$
$= (+ve)$
> 0
Root does not exist

2



$f(a) * f(b)$
$= (-ve) * (-ve)$
$= (+ve)$
> 0
Root does not exit

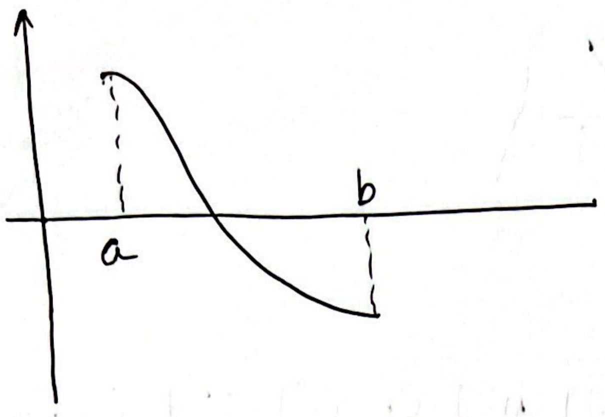
3



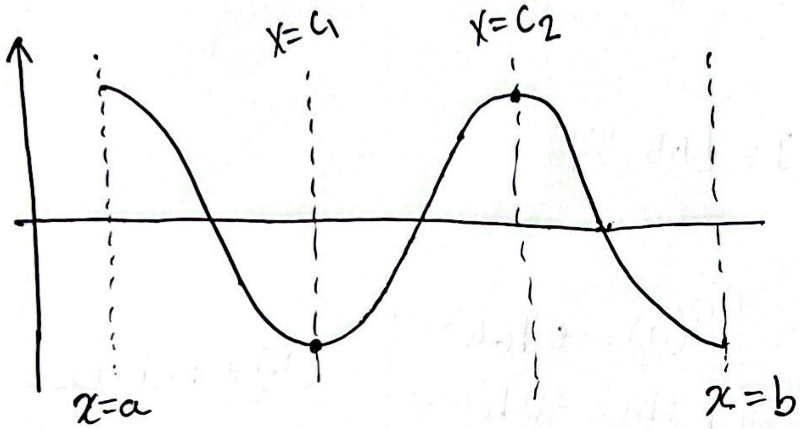
$f(a) * f(b)$
$= (-ve) * (+ve)$
$= (-ve)$
< 0
Root exists

For example 1 and 2, we will not be getting any roots if we apply ~~the~~ any root finding algorithm ~~withing~~ within the interval [a, b].

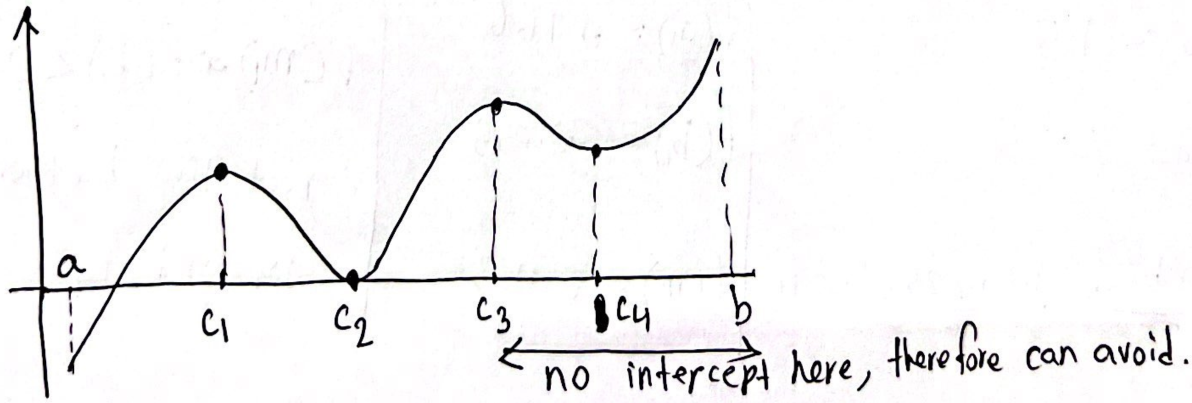
How to choose the right intervals :



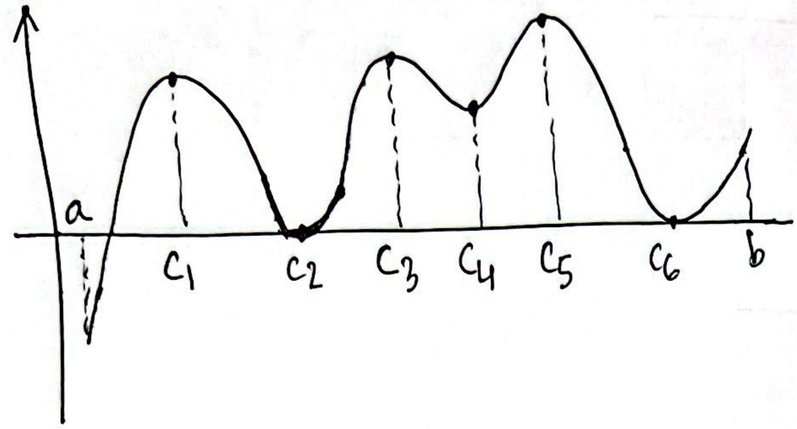
$[a, b] \rightarrow$



$$[a, b] = [a, c_1] \cup [c_1, c_2] \cup [c_2, b]$$



$$[a, b] = [a, c_1] \cup [c_1, c_2] \cup [c_2, c_3]$$



$$[a,b] = [a,c_1] \cup [c_1,c_2] \cup [c_2,c_3] \cup [c_5,c_6] \cup [c_6,b]$$

Bisection Method:

$$f(x) = \frac{1}{x} - 0.5 \qquad I = [1.5, 3]$$

$a_0 = 1.5$

$b_0 = 3$

$m_0 = \frac{a_0 + b_0}{2} = \frac{1.5+3}{2} = 2.25$

$f(a_0) = 0.166 > 0$

$f(b_0) = -0.166 < 0$

$f(m_0) = -0.055 < 0$

$f(a_0) * f(m_0) < 0$

$\therefore$ root lies ~~beet~~ between a_0 and m_0

$a_1 = a_0 = 1.5$

$b_1 = m_0 = 2.25$

$m_1 = \frac{a_1 + b_1}{2} = \frac{1.5+2.25}{2} = 1.875$

$f(a_1) = 0.166$

$f(b_1) = -0.055$

$f(m_1) = 0.033$

$f(m_1) * f(b_1) < 0$

$\therefore$ root lies beetween m_1 and b_1

$\vdots$

Example

Use Bisection method to find solutions accurate within 10^{-3} for $f(x) = x^3 - 7x^2 + 14x - 6 = 0$ on interval $[1, 3.2]$

k	a_k	m_k	b_k	$f(a_k)$	$f(m_k)$	$f(b_k)$	$x_* \in [\ ,\ ]$
0	1	2.1	3.2	2	1.79	-0.11	[2.1, 3.2]
1	2.1	2.65	3.2	1.79	0.55	-0.11	[2.65, 3.2]
2	2.65	2.925	3.2	0.55	0.086	-0.11	[2.925, 3.2]
3	2.925	3.0625	3.2	0.86	-0.054	-0.11	[2.925, 3.0625]
⋮							
9	2.998	3.000195	3.002	1.96×10^{-3}	$\boxed{-1.95 \times 10^{-4}}$ $< 10^{-3}$	-2.3×10^{-3}	

$X_* = 3.\underbrace{000}$

Take upto 3 d.p. since error bound is 10^{-3}.

Finding number of iterations to find root :

Things that will be provided in the question:

1. interval, eg: [1.5, 3]
2. Machine epsilon / error/ accuracy, eg: 1.1×10^{-16}
   bound

Formula:

$$n \geq \frac{\log(|b-a|) - \log(\epsilon)}{\log(2)} - 1$$

put -1 if question specifies Machine epsilon or error. If question specifies accuracy, then do not put -1.

Example

interval = [1.5, 3]

$\epsilon_M = 1.1 \times 10^{-6}$

Find minimum amount of iteration required to find the root with the error bound of the machine epsilon.

Solution

$$n \geq \frac{\log|3-1.5| - \log(1.1 \times 10^{-16})}{\log(2)} - 1$$

≥ 52.59

≥ 53

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Fixed Point Iteration:

→ We want to solve $f(x) = 0$

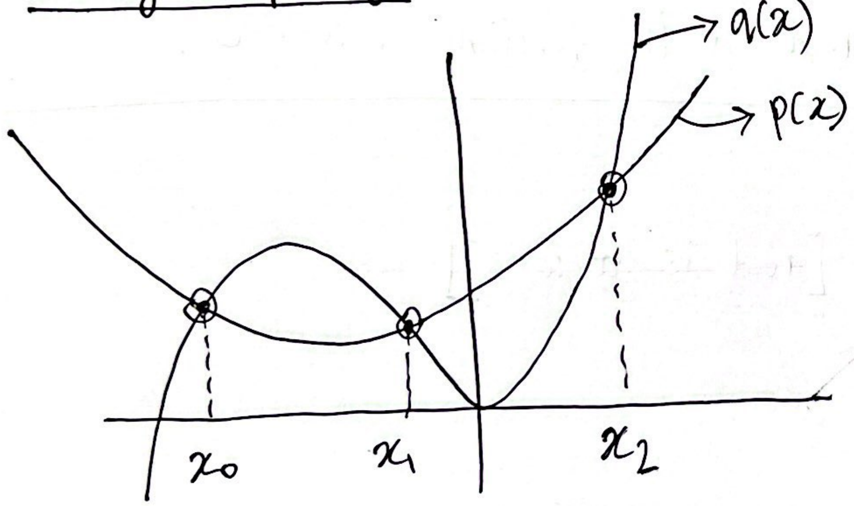
→ Need to transform $f(x) = 0$ into a new form

→ convert $f(x) = 0$ into $\boxed{g(x) = x}$

Algebra Recap:

$$\underbrace{x^2 + 5x + 7}_{p(x)} = \underbrace{x^3 + 3x^2}_{q(x)} \longrightarrow \underbrace{x^3 + 2x^2 - 5x - 7}_{f(x)}$$

Solving Graphically:

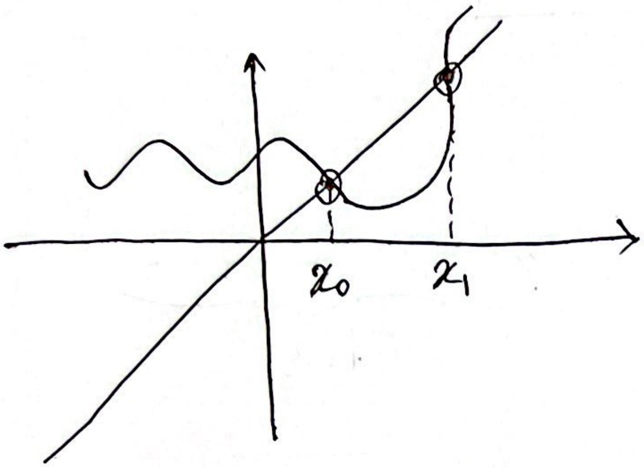


x_0, x_1, x_2 are the roots of the function $f(x)$

→ Now imagine we have a function $f(x)=0$

→ Convert the function $f(x)=0$ into $g(x)=x$

→ The intersection of $g(x)=x$ should give the root of the function $f(x)=0$.



x_0, x_1 should be the root of the function $f(x)=0$.

Example:

$f(x) = -\frac{1}{2}x + 1 = 0$ [root is at $x=2$]

$$\underbrace{\frac{x+2}{2}}_{g(x)} = \underbrace{x}_{x}$$

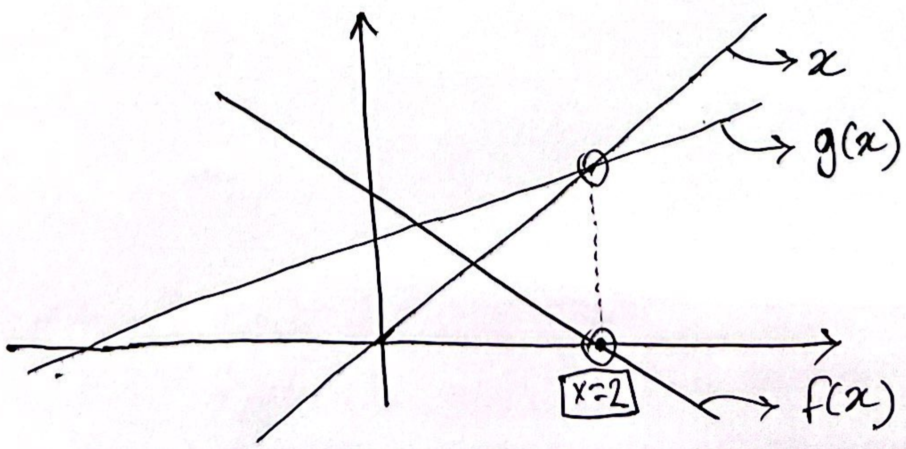
Solving numerically:

$\frac{x+2}{2} = x$

$x+2 = 2x$

$\boxed{x=2}$

Solving Graphically:



Converting $f(x)=0$ into $g(x)=x$:

$f(x) = x^2 - 2x - 3 = 0$

[Roots are at $x = -1, 3$]
[Found using middle term, or quadratic root finding formula]

$\boxed{\text{make } x \text{ the subject}}$

① $x^2 - 2x - 3 = 0$

$x^2 = 2x + 3$

$\underbrace{x}_{x} = \underbrace{\sqrt{2x+3}}_{g(x)}$

② $x^2 - 2x - 3 = 0$

$\Rightarrow x(x-2) - 3 = 0$

$\Rightarrow \underbrace{x}_{x} = \underbrace{\frac{3}{x-2}}_{g(x)}$

③ $x^2 - 2x - 3 = 0$

$\Rightarrow x^2 - x - x - 3 = 0$

$\Rightarrow \underbrace{x}_{x} = \underbrace{x^2 - x - 3}_{g(x)}$

④ $x^2 - 2x - 3 = 0$

$\Rightarrow 2x^2 - 2x = x^2 + 3$

$\Rightarrow x(2x-2) = x^2 + 3$

$\Rightarrow \underbrace{x}_{x} = \underbrace{\frac{x^2+3}{2x-2}}_{g(x)}$

Root Finding Formula:

$$\boxed{g(x_k) = x_{k+1}}$$

① $g(x) = \sqrt{2x+3}$, $x_0 = 0$

$g(0) = 1.73$
$g(1.73) = 2.54$
$g(2.54) = 2.84$
$g(2.84) = 2.95$
$g(2.95) = 2.98$
$g(2.98) = 2.99$
$g(2.99) = 3$
$g(3) = 3$ → Fixed point reached.

② $g(x) = x^2 - x - 3$, $x_0 = 0$

$g(0) = -3$
$g(-3) = 9$
$g(9) = 69$
⋮

} Diverges

③ $g(x) = \dfrac{x^2+3}{2x-2}$, $x_0 = 0$

$g(0) = -1.5$
$g(-1.5) = -1.05$
$g(-1.05) = -1$
$g(-1) = -1$ → Fixed point reached.

2 Questions to answer:

① At which root will it converge to?

→ It will depend on the initial choice of x_0

→ It will also depend on the value of the converging rate, λ.

② Which form of $g(x)$ is convergent?

→ It will depend on the value of the converging rate, λ.

Elaborating the first Q/A:

① $g(x) = \sqrt{2x+3}$, $x_0 = 0$	$g(x) = \sqrt{2x+3}$, $x_0 = 42$
$g(0) = 1.73$	$g(42) = 9.33$
$g(1.73) = 2.54$	$g(9.33) = 4.65$
$g(2.54) = 2.84$	$g(4.65) = 3.51$
$g(2.84) = 2.95$	$g(3.51) = 3.17$
$g(2.95) = 2.98$	$g(3.17) = 3.06$
$g(2.98) = 2.99$	$g(3.06) = 3.02$
$g(2.99) = 3.00$	$g(3.02) = 3.01$
$g(3) = 3$	$g(3.01) = 3$
	$g(3) = 3$

→ Both $x_0 = 0$ & $x_0 = 42$ converges to the root, $x_* = 3$.

→ Even though $x_0 = 0$ is closer to $x_* = -1$, it converges to $x_* = 3$.

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② $g(x) = x^2 - x - 3, \quad x_0 = 0$

$g(0) = -3$

$g(-3) = 9$

$g(9) = 69$

$g(69) = 4.69 \times 10^3$

$\vdots \quad \}$ diverges

$g(x) = x^2 - x - 3, \quad x_0 = 42$

$g(42) = 1.72 \times 10^3$

$g(1.72 \times 10^3) = 2.95 \times 10^6$

$g(2.95 \times 10^6) = 8.72 \times 10^{12}$

$\vdots \quad \}$ Diverges

→ Both $x_0 = 0$ and $x_0 = 42$ diverges rapidly.

③ $g(x) = \dfrac{x^2+3}{2x-2}, \quad x_0 = 0$

$g(0) = -1.5$

$g(-1.5) = -1.05$

$g(-1.05) = -1$

$g(-1) = -1$

$g(x) = \dfrac{x^2+3}{2x-2}, \quad x_0 = 42$

$g(42) = 21.6$

$g(21.6) = 11.4$

$g(11.4) = 6.39$

$g(6.39) = 4.07$

$g(4.07) = 3.19$

$g(3.19) = 3.01$

$g(3.01) = 3$

$g(3) = 3$

→ $x_0 = 0$ and $x_0 = 42$ converges to 2 different roots.

→ $x_0 = 0$ converges to the nearest root, which is -1, and $x_0 = 42$ converges to the nearest root, which is 3. (42 is closer to 3 than -1).

Contraction Mapping Theory:

Find $g'(x)$

Find $g'(\text{root-1})$

Find $g'(\text{root-2})$

if $|g'(\text{root})| < 1$, $g(x)$ will be convergent.

$\lambda = |g'(\text{root})|$

converging rate

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① $g(x) = \sqrt{2x+3} = (2x+3)^{\frac{1}{2}}$

$g'(x) = \frac{1}{\cancel{2}}(2x+3)^{-\frac{1}{2}}\cancel{(2)}$

$= (2x+3)^{-\frac{1}{2}}$

$= \dfrac{1}{\sqrt{2x+3}}$

$\lambda = |g'(-1)| = 1$ (not <1)

$\lambda = |g'(3)| = \frac{1}{3}$ (<1)

$\therefore$ both $x_0=0$, $x_0 = 42$ converges to the root $x_* = 3$.

② $g(x) = x^2 - x - 3$

$g'(x) = 2x - 1$

$\lambda = |g'(-1)| = |-3| = 3$ (not <1)

$\lambda = |g'(3)| = 5$ (not <1)

$\therefore$ no choices of x_0 will converge to any root. It will all diverge.

③ $g(x) = \dfrac{x^2+3}{2x-2}$

$\lambda = |g'(-1)| = 0$ (<1)

$\lambda = |g'(3)| = 0$ (<1)

$\therefore$ will converge to both $x_* = -1$ & 3.

Since $x_0 = 0$ is closer to -1, it will converge to -1.

$x_0 = 42$ is closer to 3, it will converge to 3.

Example:

$$f(x) = x^3 - 2x^2 - x + 2$$

(a) State the roots of the function $f(x)$

(b) Construct 3 different fixed point function $g(x)$ such that $f(x) = 0$

(c) Find the converging rate of $g(x)$ and which root it will converge to.

Solution:

(a) $f(x) = x^3 - 2x^2 - x + 2 = 0$

$$\Rightarrow x^2(x-2) - 1(x-2) = 0$$

$$\Rightarrow (x^2-1)(x-2) = 0$$

$$x_* = \begin{matrix} +1 \\ -1 \\ +2 \end{matrix}$$

(b) (i) $x^3 - 2x^2 - x + 2 = 0$

$$x = \underbrace{x^3 - 2x^2 + 2}_{g(x)}$$

(ii) $x^3 - 2x^2 - x + 2 = 0$

$$\Rightarrow x(x^2 - 2x - 1) = -2$$

$$\Rightarrow x = \frac{-2}{x^2 - 2x - 1}$$

(iii) $x^3 - 2x^2 - x + 2 = 0$

$$\Rightarrow 2x^2 = x^3 - x + 2$$

$$\Rightarrow x^2 = \frac{1}{2}(x^3 - x + 2)$$

$$\Rightarrow \underbrace{x}_{x} = \underbrace{\frac{1}{\sqrt{2}}\sqrt{x^3 - x + 2}}_{g(x)}$$

(c) (i) $g(x) = x^3 - 2x^2 + 2$

$g'(x) = 3x^2 - 4x$

$$\lambda = |g'(x_*)| = \left\{ \begin{array}{l} 7 \text{ at } x_* = -1 \\ 1 \text{ at } x_* = 1 \\ 4 \text{ at } x_* = 2 \end{array} \right\} \text{ Divergent}$$

(ii) $g(x) = \dfrac{-2}{x^2 - 2x - 1}$

$g'(x) = \dfrac{4(x-1)}{(x^2 - 2x - 1)^2}$

$$\lambda = |g'(x_*)| = \begin{cases} 2 \text{ at } x_* = -1 \\ 0 \text{ at } x_* = 1 \\ 4 \text{ at } x_* = 2 \end{cases}$$

$\therefore$ $g(x)$ will converge to $x_* = 1$

(iii) $g(x) = \frac{1}{\sqrt{2}} (x^3 - x + 2)^{\frac{1}{2}}$

$g'(x) = \dfrac{3x^2 - 1}{2\sqrt{2}\,(x^3 - x + 2)^{1/2}}$

$$\lambda = |g'(x_*)| = \begin{cases} 0.5 \text{ when } x_* = -1 \\ 0.5 \text{ when } x_* = 1 \\ 1.375 \text{ when } x_* = 2 \end{cases}$$

$\therefore$ $g(x)$ will converge to $x_* = -1$ & 1.

Order of convergence:

1. $\lambda = 0 \rightarrow$ Super linear convergence
   - $\rightarrow$ fastest convergence
   - $\rightarrow$ less iterations required to converge.

2. $0 < \lambda < 1 \rightarrow$ linear convergence
   - $\rightarrow$ will converge
   - $\rightarrow$ But not as fast as $\lambda = 0$.

3. $\lambda = 1 \rightarrow$ ignore

4. $\lambda > 1 \rightarrow$ Diverge.