LU Decomposition:

→ Need to decompose matrix A into LU

$$A = \begin{bmatrix} 2 & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{bmatrix} \begin{array}{l} \\ \rightarrow R_2' = R_2 - \left(-\frac{4}{2}\right) R_1 \\ \rightarrow R_3' = R_3 - \left(\frac{6}{2}\right) R_1 \\ \rightarrow R_4' = R_4 - \left(\frac{4}{2}\right) R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & & 1 & 0 \\ 2 & & & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & -4 & -7 & -6 \\ 0 & 1 & -8 & 4 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3' = R_3 - \left(-\frac{4}{1}\right) R_2 \\ \rightarrow R_4' = R_4 - \left(\frac{1}{1}\right) R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -9 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \\ \rightarrow R_4' = R_4 - \left(\frac{-9}{-3}\right) R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Steps:

$$\boxed{A}\,x = b$$

$$\boxed{A} \xrightarrow{\text{decompose}} LU$$

$$L\,\boxed{Ux} = b$$

$$\boxed{Ux} \rightarrow y$$

$$Ly = b \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\text{Solve for } y \text{ [find } y_1, y_2, y_3]}$$

$$\boxed{Ux = y}$$

$$\underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\text{solve for } x \text{ [find } x_1, x_2, x_3]}$$

Example:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$$A \cdot x = b$$

$$A \Downarrow LU$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{array}{l} \\ \rightarrow R_2' = R_2 - \left(\frac{1}{1}\right) R_1 \\ \rightarrow R_3' = R_3 - \left(\frac{2}{1}\right) R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3' = R_3 - \left(\frac{8}{-4}\right) R_2 \end{array}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}}_{U}$$

$A \cdot x = b$

$\Downarrow$ decompose

LU

$L\boxed{U \cdot x} = b$

$\Downarrow$

y

$L\, y = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$\boxed{y_1 = 0}$

$y_1 + y_2 = 4, \boxed{y_2 = 4}$

$2y_1 + (-2y_2) + y_3 = 4$

$(2\times 0) + (-2\times 4) + y_3 = 4$

$\boxed{y_3 = 12}$

$U \cdot x = y$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$-2x_3 = 12$

$\boxed{x_3 = -6}$

$-4x_2 + x_3 = 4$

$-4x_2 - 6 = 4$

$\boxed{x_2 = -2.5}$

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2(-2.5) + (-6) = 0$

$\boxed{x_1 = 11}$

Advantage:

→ This method can be used to solve linear system that differ by the values of 'b' only. We need to compute L and U only once.

→ But in Gaussian Elimination Method method, if 'b' changes, we need to restart row operations from the very beginning.

Frobenius Matrix:

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -m_{21} & 1 & 0 & 0 \\ -m_{31} & 0 & 1 & 0 \\ -m_{41} & 0 & 0 & 1 \end{bmatrix} \quad F^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_{32} & 1 & 0 \\ 0 & -m_{42} & 0 & 1 \end{bmatrix} \quad F^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m_{43} & 1 \end{bmatrix}$$

$$L = \left(F^{(1)}\right)^{-1} \left(F^{(2)}\right)^{-1} \left(F^{(3)}\right)^{-1}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{matrix} \\ \longrightarrow R_2' = R_2 - \left(\frac{1}{1}\right) R_1 \\ \longrightarrow R_3' = R_3 - \left(\frac{2}{1}\right) R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix} \begin{matrix} \\ \\ \longrightarrow R_3' = R_3 - \left(\frac{8}{-4}\right) R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$L = \left(F^{(1)}\right)^{-1} \left(F^{(2)}\right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$