Final Exam Questions: Fall 2021
CSE330/MAT205

(1) Each Question is numbered with double digits. (2) First digit is the set number and second digits is the question number in that set. (3) There are four sets, and one question is randomly selected from each set.

Question # 11

Question # 11: A function $f(x) = x^3 - 7x^2 + 4x + 12$ has one root in the interval $I = [4.25, 8.95]$. In the following, we would like to find the root using Newton's method along with Aitken acceleration:

1. (1 Mark) Explain an advantage of using the Quasi-Newton Method over Newton's method.
2. (7 Marks) Starting from $x_0 = 7.23$ use Newton's iteration formula, up to four iterations, *i.e.*, $k = 0, 1, 2, \cdots, 4$, to find the approximate root of $f(x)$ by applying Aitken acceleration only once appropriately. Express your result up to five decimal places where necessary.
3. (2 Marks) Suppose the actual root is 6. Calculate the percentage error of your last iteration value of root. Express your result up to three decimal places.

Question # 12

Question # 12: A function $f(x) = x^3 - 7x^2 + 4x + 12$ has real roots. Now answer the following:

1. (2 Marks) Explain how to set the interval when there are multiple x-intercept in a graph.You may sketch a graph if needed.
2. (1+4 Marks) Show that there is a root in the interval $I = [4.25, 8.95]$. Use Interval Bisection Method to find an approximate value of the root in the interval I up to four iterations.
3. (3 Marks) How many iterations will be required to find the root if the machine epsilon is 1.4×10^{-18}?

Question # 13

Question # 13: Consider the function $f(x) = x^3 - 6x^2 + 11x - 6$ which has real roots. Now answer the following:

1. (2 Marks) Find the exact roots of $f(x)$.
2. (4 Marks) Construct two different fixed point functions $g(x)$ such that $f(x) = 0$.
3. (4 Marks) Compute the convergence rate of each fixed point function $g(x)$ obtained in the previous part, and state which root it is converging to (Linear/Superlinear) or diverging.

Question # 14

Question # 14: Consider the function $f(x) = x^3 - 7x^2 + 4x + 12$ which has real roots. Now answer the following:

1. (2 Marks) Find the exact roots of $f(x)$.
2. (4 Marks) Construct two different fixed point functions $g(x)$ such that $f(x) = 0$.
3. (4 Marks) Compute the convergence rate of each fixed point function $g(x)$ obtained in the previous part, and state which root it is converging to (Linear/Superlinear) or diverging.

Question # 15

Question # 15: A function $f(x) = x^3 - 6x^2 + 11x - 6$ has real roots. Now answer the following:

1. (2 Marks) Explain how to set the interval when there are multiple x-intercepts in a graph. You may sketch a graph if needed.
2. (1+4 Marks) Verify whether a root lies within the interval $I = [2.25, 3.65]$. Use Interval Bisection Method to find an approximate value of the root in the interval I up to four iterations.
3. (3 Marks) How many iterations will be required to find the root if the machine epsilon is 1.6×10^{-18}?

Question # 16

Question # 16: A function $f(x) = x^3 - 6x^2 + 11x - 6$ has one root in the interval $I = [2.25, 4.25]$. In the following, we would like to find the root using Newton's method along with Aitken acceleration:

1. (1 Mark) Explain an advantage of using the Quasi-Newton Method over Newton's method.
2. (7 Marks) Starting from $x_0 = 3.88$ use Newton's iteration formula, up to four iterations, *i.e.,* $k = 0, 1, 2, \cdots, 4$, to find the approximate root of $f(x)$ by applying Aitken acceleration only once appropriately. Express your result up to five decimal places where necessary.
3. (2 Marks) Suppose the actual root is 3. Calculate the percentage error of your last iteration value of root. Express your result up to five decimal places.

Question # 21

Question # 21: A linear system is described by the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 + 3x_2 + 4x_3 &= 20\,. \\ 3x_1 + 4x_2 + 2x_3 &= 17 \end{aligned}$$

Solve the above linear system by answering the following:

1. (2 Marks) Does this system have any unique solution? Why or why not?
2. (3 Marks) Find the upper triangular matrix U.
3. (3 Marks) Solve the above linear system by Gaussian elimination method. Show your work.
4. (2 Marks) Suppose, you have constructed an Augmented matrix from a different linear system as given below,

$$\left(\begin{array}{ccc|c} 3 & 3 & 4 & 1 \\ 0 & 0 & -1 & 6 \\ 0 & -1 & 3 & 4 \end{array}\right)$$

   Explain why the Gaussian elimination method fails to solve this system? Also explain how we can overcome the problem to actually solve it (you do not have to solve this system)?

Question # 22

Question # 22: A linear system is described by the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 + 3x_2 + 4x_3 &= 20 \\ 3x_1 + 4x_2 + 2x_3 &= 17 \end{aligned}$$

Solve the above linear system by answering the following:

1. (2 Marks) Construct the Frobenius matrices $F^{(1)}$ and $F^{(2)}$ for the above linear system.
2. (2 Marks) Find the unit lower triangular matrix L.
3. (4 Marks) Now find the solution of this linear system using LU-decomposition method. You may use the lower triangular matrix found in the previous question.
4. (2 Marks) Suppose, you have constructed an Augmented matrix from a different linear system as given below,

$$\left(\begin{array}{ccc|c} 2 & 2 & 5 & 2 \\ 0 & 0 & -1 & 12 \\ 0 & -7 & 1 & 4 \end{array}\right)$$

   Explain why the Gaussian elimination method fails to solve this system? Also explain how we can overcome the problem to actually solve it (you do not have to solve this system)?

Question # 23

Question # 23: A linear system is described by the following equations:

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_1 + x_2 - 3x_3 &= 5 \\
4x_1 - 7x_2 + x_3 &= -1
\end{aligned}
$$

Solve the above linear system by answering the following:

1. (2 Marks) Does this system have any unique solution? Why or why not?
2. (3 Marks) Find the upper triangular matrix U.
3. (3 Marks) Solve the above linear system by Gaussian elimination method. Show your work.
4. (2 Marks) Suppose, you have constructed an Augmented matrix from a different linear system as given below,

$$
\left(\begin{array}{ccc|c}
1 & -2 & 1 & 6 \\
0 & 0 & 4 & 1 \\
0 & 9 & -6 & 8
\end{array}\right)
$$

Explain why the Gaussian elimination method fails to solve this system? Also explain how we can overcome the problem to actually solve it (you do not have to solve this system)?

Question # 24

Question # 24: A linear system is described by the following equations:

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_1 + x_2 - 3x_3 &= 5 \\
4x_1 - 7x_2 + x_3 &= -1
\end{aligned}
$$

Solve the above linear system by answering the following:

1. (2 Marks) Construct the Frobenius matrices $F^{(1)}$ and $F^{(2)}$ for the above linear system.
2. (2 Marks) Find the unit lower triangular matrix L.
3. (4 Marks) Now find the solution of this linear system using LU-decomposition method. You may use the lower triangular matrix found in the previous question.
4. (2 Marks) Suppose, you have constructed an Augmented matrix from a different linear system as given below,

$$
\left(\begin{array}{ccc|c}
1 & 1 & 5 & 1 \\
0 & 0 & 3 & 2 \\
0 & 5 & -6 & 3
\end{array}\right)
$$

Explain why the Gaussian elimination method fails to solve this system? Also explain how we can overcome the problem to actually solve it (you do not have to solve this system)?

Question # 25

Question # 25: A linear system is described by the following equations:

$$
\begin{aligned}
x_1 - 3x_2 + 4x_3 &= 3 \\
2x_1 - 5x_2 + 6x_3 &= 6 \\
-3x_1 + 3x_2 + 4x_3 &= 6
\end{aligned}
$$

Solve the above linear system by answering the following:

1. (2 Marks) Does this system have any unique solution? Why or why not?
2. (3 Marks) Find the upper triangular matrix U.
3. (3 Marks) Solve the above linear system by Gaussian elimination method. Show your work.
4. (2 Marks) Suppose, you have constructed an Augmented matrix from a different linear system as given below,

$$\left(\begin{array}{ccc|c} 3 & 3 & 4 & 1 \\ 0 & 0 & -1 & 6 \\ 0 & -1 & 3 & 4 \end{array}\right)$$

Explain why the Gaussian elimination method fails to solve this system? Also explain how we can overcome the problem to actually solve it (you do not have to solve this system)?

Question # 26

Question # 26: A linear system is described by the following equations:

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 3 \\ 2x_1 - 5x_2 + 6x_3 &= 6 \\ -3x_1 + 3x_2 + 4x_3 &= 6 \end{aligned}$$

Solve the above linear system by answering the following:

1. (2 Marks) Construct the Frobenius matrices $F^{(1)}$ and $F^{(2)}$ for the above linear system.
2. (2 Marks) Find the unit lower triangular matrix L.
3. (4 Marks) Now find the solution of this linear system using LU-decomposition method. You may use the lower triangular matrix found in the previous question.
4. (2 Marks) Suppose, you have constructed an Augmented matrix from a different linear system as given below,

$$\left(\begin{array}{ccc|c} 2 & 2 & 5 & 1 \\ 0 & 0 & -1 & 6 \\ 0 & -7 & 1 & 4 \end{array}\right)$$

Explain why the Gaussian elimination method fails to solve this system? Also explain how we can overcome the problem to actually solve it (you do not have to solve this system)?

Question # 31

Question # 31: Consider a set of four data points: $f(0) = 3$, $f(2) = -2$, $f(-1) = 2$ and $f(1) = 1$. Find the best fit polynomial of degree two, $p_2(x)$, for the above data points using least-squares method by answering the following:

1. (2 Marks) Write down the matrices, A and b, from the above data.
2. (3 Marks) Compute the normal matrix $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}b$.
3. (5 Marks) Use the results in the previous part to compute the column matrix $x = (a_0 \;\; a_1 \;\; a_2)^{\mathrm{T}}$, where a_0, a_1 and a_2 are the coefficients of the polynomial p_2, and then write the expression of the polynomial p_2.

Question # 32

Question # 32: Consider a set of four data points: $f(0) = 3$, $f(2) = -2$, $f(-1) = 2$ and $f(1) = 1$. We now find the solution by QR-decomposition method using these four data points by answering the following:

1. (1.5 Marks) Write down the matrix A and b. Also identify the linearly independent column vectors u_1, u_2 and u_3 from the matrix A.
2. (4.5 Marks) Using the Gram-Schmidt process construct the orthonormal column matrices (or vectors) q_1, q_2 and q_3 from the linearly independent column vectors u_1, u_2 and u_3 obtained in the previous part, and then write down the Q matrix.
3. (2 Marks) Now calculate the matrix elements of R, and write down the matrix R.
4. (1 Mark) Compute Rx and $Q^{\mathrm{T}}b$, where $x = (a_0 \;\; a_1 \;\; a_2)^{\mathrm{T}}$ is a column vector with a_0, a_1 and a_2 are the coefficients of the polynomial p_2.
5. (1 Mark) Using the above result, find the values of a_0, a_1 and a_2; and write the polynomial $p_2(x)$.

Question # 33

Question # 33: Consider a set of four data points: $f(0) = 1$, $f(2) = -2$, $f(-1) = 2$ and $f(1) = 1$. Find the best fit polynomial of degree two, $p_2(x)$, for the above data points using least-squares method by answering the following:

1. (2 Marks) Write down the matrices, A and b, from the above data.
2. (3 Marks) Compute the normal matrix $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}b$.
3. (5 Marks) Use the results in the previous part to compute the column matrix $x = (a_0 \;\; a_1 \;\; a_2)^{\mathrm{T}}$, where a_0, a_1 and a_2 are the coefficients of the polynomial p_2, and then write the expression of the polynomial p_2.

Question # 34

Question # 34: Consider a set of four data points: $f(0) = 1$, $f(2) = -2$, $f(-1) = 2$ and $f(1) = 1$. We now find the solution by QR-decomposition method using these four data points by answering the following:

1. (1.5 Marks) Write down the matrix A and b. Also identify the linearly independent column vectors u_1, u_2 and u_3 from the matrix A.
2. (4.5 Marks) Using the Gram-Schmidt process construct the orthonormal column matrices (or vectors) q_1, q_2 and q_3 from the linearly independent column vectors u_1, u_2

and u_3 obtained in the previous part, and then write down the Q matrix.

3. (2 Marks) Now calculate the matrix elements of R, and write down the matrix R.
4. (1 Mark) Compute Rx and $Q^{\mathrm{T}}b$, where $x = (a_0 \;\; a_1 \;\; a_2)^{\mathrm{T}}$ is a column vector with a_0, a_1 and a_2 are the coefficients of the polynomial p_2.
5. (1 Mark) Using the above result, find the values of a_0, a_1 and a_2; and write the polynomial $p_2(x)$.

Question # 35

Question # 35: Consider a set of four data points: $f(0) = 0$, $f(2) = -1$, $f(-1) = 2$ and $f(1) = 2$. Find the best fit polynomial of degree two, $p_2(x)$, for the above data points using least-squares method by answering the following:

1. (2 Marks) Write down the matrices, A and b, from the above data.
2. (3 Marks) Compute the normal matrix $A^{\mathrm{T}}A$ and $A^{\mathrm{T}}b$.
3. (5 Marks) Use the results in the previous part to compute the column matrix $x = (a_0 \;\; a_1 \;\; a_2)^{\mathrm{T}}$, where a_0, a_1 and a_2 are the coefficients of the polynomial p_2, and then write the expression of the polynomial p_2.

Question # 36

Question # 36: Consider a set of four data points: $f(0) = 0$, $f(2) = -1$, $f(-1) = 2$ and $f(1) = 2$. We now find the solution by QR-decomposition method using these four data points by answering the following:

1. (1.5 Marks) Write down the matrix A and b. Also identify the linearly independent column vectors u_1, u_2 and u_3 from the matrix A.
2. (4.5 Marks) Using the Gram-Schmidt process construct the orthonormal column matrices (or vectors) q_1, q_2 and q_3 from the linearly independent column vectors u_1, u_2 and u_3 obtained in the previous part, and then write down the Q matrix.
3. (2 Marks) Now calculate the matrix elements of R, and write down the matrix R.
4. (1 Mark) Compute Rx and $Q^{\mathrm{T}}b$, where $x = (a_0 \;\; a_1 \;\; a_2)^{\mathrm{T}}$ is a column vector with a_0, a_1 and a_2 are the coefficients of the polynomial p_2.
5. (1 Mark) Using the above result, find the values of a_0, a_1 and a_2; and write the polynomial $p_2(x)$.

Question # 41

Question # 41: Answer the following:

1. (2 Marks) Show that the upper bound error for numerical integration of a quadratic function is given by

$$\frac{f^{(3)}\left(\xi\right)}{3!}\left(x-x_0\right)\left(x-x_1\right)\ ,$$

where ξ is a value within the limits of the integral that maximizes the error.

2. (3 Marks) The vertical distance covered by a rocket from $t=8$ to $t=30$ seconds is given by

$$x=\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt\ .$$

Use single segment trapezium rule to find the distance covered.

3. (3 Marks) Find the actual distance in the previous part, and also compute the actual error.

4. (2 Marks) How can the error be decreased?

Question # 42

Question # 42: Answer the following:

1. (4 Marks) Let a function $f\left(x\right)$ be approximated by a polynomial $p_1\left(x\right)$. So, the definite integral of $f\left(x\right)$ can be approximated as in the following

$$\int_a^b f\left(x\right)dx\approx\int_{a=x_0}^{b=x_1} p_1\left(x\right)\ dx\ ,$$

where $p_1\left(x\right)$ is given in the Lagrange basis as

$$p_1\left(x\right)=\frac{x-x_1}{x_0-x_1}f\left(x_0\right)+\frac{x-x_0}{x_1-x_0}f\left(x_1\right)\ .$$

Now, derive the error term for the above polynomial function. Show all steps.

2. (2+2+2 Marks) Compute the following integration

$$\int_0^2 f\left(x\right)dx$$

numerically by using Trapezoidal and Simpson's rules if the functions $f\left(x\right)$ are given as: (i) $f\left(x\right)=\sqrt{1+x^2}$, (ii) $f\left(x\right)=\sin\left(x\right)$ and (iii) $f\left(x\right)=e^x$.

Question # 43

Question # 43: Answer the following:

1. (2 Marks) State two situations for which the Simpson's rule is limited.

2. (2 Marks) Find the exact value of the following integral

$$\int_0^{0.8} f\left(x\right)dx$$

where $f\left(x\right)=0.2+25x-200x^2+675x^3-900x^4+400x^5$.

3. (4 Marks) Use multi segment Trapezoidal rule with $m=4$ to approximate the integral

in the previous part. And also find the actual relative error.

4. (2 Marks) How can you further decrease the actual relative error?

Question # 44

Question # 44: Answer the following:

1. (5 Marks) Derive the Trapezoidal rule for multiple segments. Also show a graphical sketch.
2. (2 Marks) Using the Trapezoidal rule of Integration, integrate

$$\int_1^4 \left(1 + \frac{x^3}{400}\right) dx$$

   using single segment.

3. (3 Marks) Use the above part to show, in a single tabular form, the step size on the value of integration considering number of segment 1,2 and 4. In the table show the value of integration and relative error up to two decimal points.

Question # 45

Question # 45: Answer the following:

1. (4 Marks) Suppose you are riding a bicycle on your way home from the campus. The distance covered between $t = 4$ and $t = 16$ seconds is given by the following equation

$$x = \int_4^{16} \left(10t + 5t^2\right) dt \; .$$

   Use the trapezoidal rule to find the distance covered using segments $m = 1$ and $m = 3$.

2. (2 Marks) Find the absolute relative error for $m = 1$ and $m = 3$.
3. (1 Mark) Is the statement "The error term for Trapezoidal rule becomes identically zero for any function whose second derivative is zero" true or false?
4. (1 Mark) For any composite integration technique, is there any stability with respect to round off error?
5. (2 Marks) Write the formula of x_i and h for open Newton-Cotes interval. Also when does this phenomenon occur?