Linear Equations:

→ System of linear equations (exponent of all variables must be 1)

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
&\vdots \\
a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n &= b_n
\end{aligned}
$$

→ Can be represented in a matrix form:

$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{(n \times n) \text{ matrix, } A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{(n \times 1) \text{ matrix, } x}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{(n \times 1) \text{ matrix, } b}
$$

$$A \cdot x = b$$

Solution:

$$x = A^{-1} \cdot b$$

Basic properties of A:

→ A should be a square matrix of shape $(n \times n)$

→ A must be non-singular [meaning $\det(A) \neq 0$]

Gaussian Elimination Method:

→ A technique which transforms matrix A into triangular form (upper or lower)

→ Solves $Ax = b$ without finding the inverse.

→ Lower triangular matrix (L), and upper triangular matrix (U) are defined as follows:

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$$

Using a (4x4) Lower triangular matrix:

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

number of operations:

$$l_{11} x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = b_1$$

$$\Rightarrow \boxed{x_1 = \frac{b_1}{l_{11}}} \longrightarrow 1 \text{ div}$$

$$l_{21} x_1 + l_{22} x_2 = b_2$$

$$\boxed{x_2 = \frac{b_2 - l_{21} x_1}{l_{22}}} \longrightarrow 1 \text{ div}, 1 \text{ mult}, 1 \text{ sub}$$

number of operations:

$l_{31} x_1 + l_{32} x_2 + l_{33} x_3 = b_3$

$$\boxed{x_3 = \frac{b_3 - l_{31}x_1 - l_{32}x_2}{l_{33}}} \rightarrow 1 \text{ div}, 2 \text{ mult}, 2 \text{ sub}$$

$l_{41} x_1 + l_{42} x_2 + l_{43} x_3 + l_{44} x_4 = b_4$

$$\boxed{x_4 = \frac{b_4 - l_{41}x_1 - l_{42}x_2 - l_{43}x_3}{l_{44}}} \rightarrow 1 \text{ div}, 3 \text{ mult}, 3 \text{ sub}$$

This is a "TOP DOWN" approach because we found x_1 first, then x_2, x_3, x_4.

Total number of operations:

For finding x_n, we need 1 div, $(n-1)$ mult, $(n-1)$ sub.

$$1 + (n-1) + (n-1) = 1 + 2(n-1).$$

$\therefore$ total num of operations $= \sum_{j=1}^{n} [1 + 2(j-1)]$

$$= \sum_{j=1}^{n} (2j - 1)$$

$$= 2\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1$$

$$= n^2 + n - n$$

$$= n^2$$

Gaussian Elimination Method:

→ To make a matrix into a triangular form, we apply Gaussian Elimination.

→ Need to apply row operations.

→ 1^{st} row operation will make all elements below a_{11} into 0.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix} \quad \begin{array}{l} r_2' = r_2 - \underbrace{\boxed{\dfrac{a_{21}}{a_{11}}}}_{m_{21}} r_1 \\[2ex] r_3' = r_3 - \underbrace{\boxed{\dfrac{a_{31}}{a_{11}}}}_{m_{31}} r_1 \end{array}$$

$$\begin{bmatrix} X & X & X & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & 0 & X & X \\ 0 & 0 & X & X \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & 0 & X & X \\ 0 & 0 & 0 & X \end{bmatrix}$$

Example:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right] \begin{array}{ll} \\ \rightarrow r_2' = r_2 - \frac{1}{1} r_1 & [x - Y] \\ \rightarrow r_3' = r_3 - \frac{2}{1} r_1 & [x - 2Y] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right] \begin{array}{ll} \\ \\ \rightarrow r_3' = r_3 - \frac{8}{-4} r_2 & [x + 2Y] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right]$$

$-2x_3 = 12$	$-4x_2 + x_3 = 4$	$x_1 + 2x_2 + x_3 = 0$
$x_3 = -6$	$-4x_2 - 6 = 4$	$x_1 + 2(-2.5) + (-6) = 0$
	$x_2 = -2.5$	$x_1 = 11$

LU Decomposition:

→ Need to decompose matrix A into LU

$$A = \begin{bmatrix} 2 & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{bmatrix} \begin{array}{l} \\ \rightarrow R_2' = R_2 - \left(-\frac{4}{2}\right) R_1 \\ \rightarrow R_3' = R_3 - \left(\frac{6}{2}\right) R_1 \\ \rightarrow R_4' = R_4 - \left(\frac{4}{2}\right) R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & & 1 & 0 \\ 2 & & & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & -4 & -7 & -6 \\ 0 & 1 & -8 & 4 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3' = R_3 - \left(\frac{-4}{1}\right) R_2 \\ \rightarrow R_4' = R_4 - \left(\frac{1}{1}\right) R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -9 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \\ \rightarrow R_4' = R_4 - \left(\frac{-9}{-3}\right) R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Steps:

$$\boxed{A}\,x = b$$

$$\downarrow \text{decompose}$$

$$LU$$

$$L\,\boxed{Ux} = b$$

$$\downarrow$$

$$y$$

$$L\,y = b \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\text{Solve for } y \text{ [find } y_1, y_2, y_3]}$$

$$\boxed{Ux = y}$$

$$\underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\text{solve for } x \text{ [find } x_1, x_2, x_3]}$$

Example:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$$A \cdot x = b$$

$$A$$

$$\Downarrow$$

$$LU$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{array}{l} \\ \rightarrow R_2' = R_2 - \left(\frac{1}{1}\right) R_1 \\ \rightarrow R_3' = R_3 - \left(\frac{2}{1}\right) R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3' = R_3 - \left(\frac{8}{-4}\right) R_2 \end{array}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}}_{U}$$

$A \cdot x = b$

$\Downarrow$ decompose

LU

$L\boxed{U \cdot x} = b$

$\Downarrow$

y

$L\,y = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$\boxed{y_1 = 0}$

$y_1 + y_2 = 4$, $\boxed{y_2 = 4}$

$2y_1 + (-2y_2) + y_3 = 4$

$(2\times 0) + (-2\times 4) + y_3 = 4$

$\boxed{y_3 = 12}$

$U \cdot x = y$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$-2x_3 = 12$

$\boxed{x_3 = -6}$

$-4x_2 + x_3 = 4$

$-4x_2 - 6 = 4$

$\boxed{x_2 = -2.5}$

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2(-2.5) + (-6) = 0$

$\boxed{x_1 = 11}$

Advantage:

→ This method can be used to solve linear system that differ by the values of 'b' only. We need to compute L and U only once.

→ But in Gaussian Elimination Method method, if 'b' changes, we need to restart row operations from the very beginning.

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Frobenius Matrix:

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -m_{21} & 1 & 0 & 0 \\ -m_{31} & 0 & 1 & 0 \\ -m_{41} & 0 & 0 & 1 \end{bmatrix} \quad F^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_{32} & 1 & 0 \\ 0 & -m_{42} & 0 & 1 \end{bmatrix} \quad F^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m_{43} & 1 \end{bmatrix}$$

$$L = \left(F^{(1)}\right)^{-1} \left(F^{(2)}\right)^{-1} \left(F^{(3)}\right)^{-1}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{array}{l} \\ \longrightarrow R_2' = R_2 - \left(\frac{1}{1}\right) R_1 \\ \longrightarrow R_3' = R_3 - \left(\frac{2}{1}\right) R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix} \begin{array}{l} \\ \\ \longrightarrow R_3' = R_3 - \left(\frac{8}{-4}\right) R_2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$L = \left(F^{(1)}\right)^{-1}\left(F^{(2)}\right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$