Newton/ Divided Difference (form) Method : – (to avoid recomputation of Lagrange form)

$x_0, x_1, \ldots x_n \rightarrow$ nodes

$\Rightarrow P_n(x)$

x_{n+1} add করলে $P_{n+1}(x)$ হবে (degree increase হবে)

$$\underbrace{P_{n+1}(x)}_{\text{new polynomial}} = \underbrace{P_n(x)}_{\text{old polynomial}} + g_{n+1}(x) \rightarrow \text{Some other polynomial}$$

$P_2(x) = x^2 + 3x + 4$ (old)

new polynomial, $P_3(x) = P_2(x) + \boxed{3x^3 + 4x^2}$

$ \downarrow g_{3}(x)$

$$P_3(x) = (x^2+3x+4) + (3x^3+4x^2)$$

$$P_3(x) = 3x^3 + 5x^2 + 3x + 4$$

$g_{n+1}(x) = P_{n+1}(x) - P_n(x)$

$$g_{n+1}(x_0) = P_{n+1}(x_0) - P_n(x_0)$$

$$= f(x_0) - f(x_0)$$

$$= 0$$

$g_{n+1}(x_i) = 0$ → all old nodes

$$g_{n+1}(x) = \underbrace{a_{n+1}}_{\boxed{\text{unknown}}} (x - x_0)(x - x_1) \ldots (x - x_n)$$

here, $\{x_0, x_1, \ldots x_n\}$ are old nodes

Now we use new node, x_{n+1}

$$P_{n+1}(x_{n+1}) = P_n(x_{n+1}) + g_{n+1}(x_{n+1})$$

$$\Rightarrow f(x_{n+1}) - P_n(x_{n+1}) = a_{n+1}(x_{n+1} - x_0)(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)$$

$$\therefore a_{n+1} = \frac{f(x_{n+1}) - P_n(x_{n+1})}{(x_{n+1} - x_0)(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)}$$

Steps:

a_{n+1} বের করে then $g_{n+1}(x)$ বের করব

Then new $P_{n+1}(x) = P_n(x) + g_{n+1}(x)$ বের করব।

Newton's Polynomials |

$$n_0(x) = 1$$

$$n_k(x) = \prod_{j=0}^{k-1} (x - x_j) \quad \text{for } k > 0$$

$$n_1(x) = (x - x_0)$$

$$n_2(x) = (x - x_0)(x - x_1)$$

$$n_3(x) = (x - x_0)(x - x_1)(x - x_2)$$

In a nutshell;

Newton's form

$$P_{n+1}(x) = P_n(x) + g_{n+1}(x)$$

where, $g_{n+1}(x) = a_{n+1} \cdot \eta_{n+1}(x)$

$$\eta_{n+1}(x) = (x-x_0)(x-x_1) \ldots (x-x_n)$$

$$a_{n+1} = \frac{f(x_{n+1}) - P_n(x_{n+1})}{(x_{n+1}-x_0)(x_{n+1}-x_1) \ldots (x_{n+1}-x_n)}$$

If we have one node x_0, we can say $P_0(x) = a_0$
= constant

Newton's Polynomial Interpolation

$\eta_K(x) \longrightarrow$ Newton's Polynomial basis

$$P_n(x) = \sum_{k=0}^{n} a_K \eta_K(x)$$

$$P_1(x) = a_0 + a_1(x-x_0)$$

$$P_1(x) = P_0(x) + g_1(x)$$

$$P_2(x) = P_1(x) + g_2(x) \longrightarrow a_2(x-x_0)(x-x_1)$$

Notation :

$$a_k = f[x_0, x_1, \ldots, x_k]$$

$$a_0 = f[x_0] = f(x_0)$$

$$a_1 = f[x_0, x_1]$$

$$a_2 = f[x_0, x_1, x_2]$$

$$\vdots$$

$$a_n = f[x_0, x_1, \ldots, x_n]$$

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \cdots$$

$$\cdots + f[x_0, x_1, \ldots, x_n](x-x_0) \cdots (x - x_{n-1})$$

Calculating f[ ]

$$f[x_0] = a_0 = f(x_0) \qquad a_0 \longrightarrow f[x_0]$$

$$f[x_0, x_1] = a_1 = \frac{f(x_1) - P_0(x_1)}{x_1 - x_0}$$

$$= \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$\vdots$$

$$f[x_0, x_1, \ldots, x_k] = a_k = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, x_2, \ldots, x_{k-1}]}{x_k - x_0}$$

Example: Nodes $\{-1, 0, 1, 2\} \rightarrow x_0, x_1, x_2, x_3$

data $\{5, 1, 1, 11\} \rightarrow f(x_0), f(x_1), f(x_2), f(x_3)$

$$P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) +$$
$$f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$$

$x_0 = -1 \quad f[x_0] = 5$

$x_1 = 0 \quad f[x_1] = 1 \quad f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = \dfrac{1-5}{0-(-1)} = -4$

$x_2 = 1 \quad f[x_2] = 1 \quad f[x_1, x_2] = \dfrac{1-1}{1-0} = 0$

$x_3 = 2 \quad f[x_3] = 11 \quad f[x_2, x_3] = \dfrac{11-1}{2-1} = \dfrac{10}{1} = 10$

$$f[x_0, x_1, x_2] = \frac{0-(-4)}{1-(-1)} = 2$$

$$f[x_1, x_2, x_3] = \frac{10-0}{2-0} = 5$$

$$f[x_0, x_1, x_2, x_3] = \frac{5-2}{2-(-1)} = 1$$

Table

x	$f[x]$				
-1	5				
		-4			
0	1		2		
		0		1	
1	1		5		$-\frac{1}{12}$
		10		$\frac{13}{12}$	
2	11		$\frac{17}{6}$		
		$\frac{3}{2}$			
-2	5				

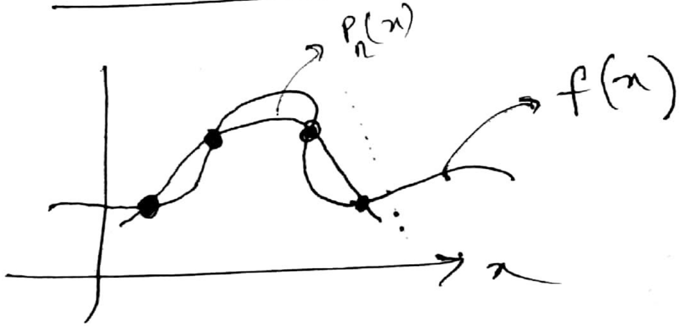
$P_3(x) = 5 + (-4)(x+1) + 2(x+1)x + 1.x(x+1)(x-$

What if I want to add a new node $x_4 = -2$ ✓

$x_4 = -2 \quad f[x_4] = 5$

$\therefore P_4(x) = P_3(x) + \underbrace{f[x_0, x_1, x_2, x_3, x_4]}_{a_4} \underbrace{x(x+1)(x-1)\cancel{(x+2)}(x-2)}_{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}$

Interpolation Error :



$f(x) \neq P_n(x)$

error occurs.

$|f(x) - P_n(x)| \sim$ error $\longrightarrow$ function of x as at different x, different value of error occurs

$\Downarrow$

problem is that $f(x)$ is unknown in real life.

error is zero in nodes (nodal points)

because at those points $f(x) = P_n(x)$

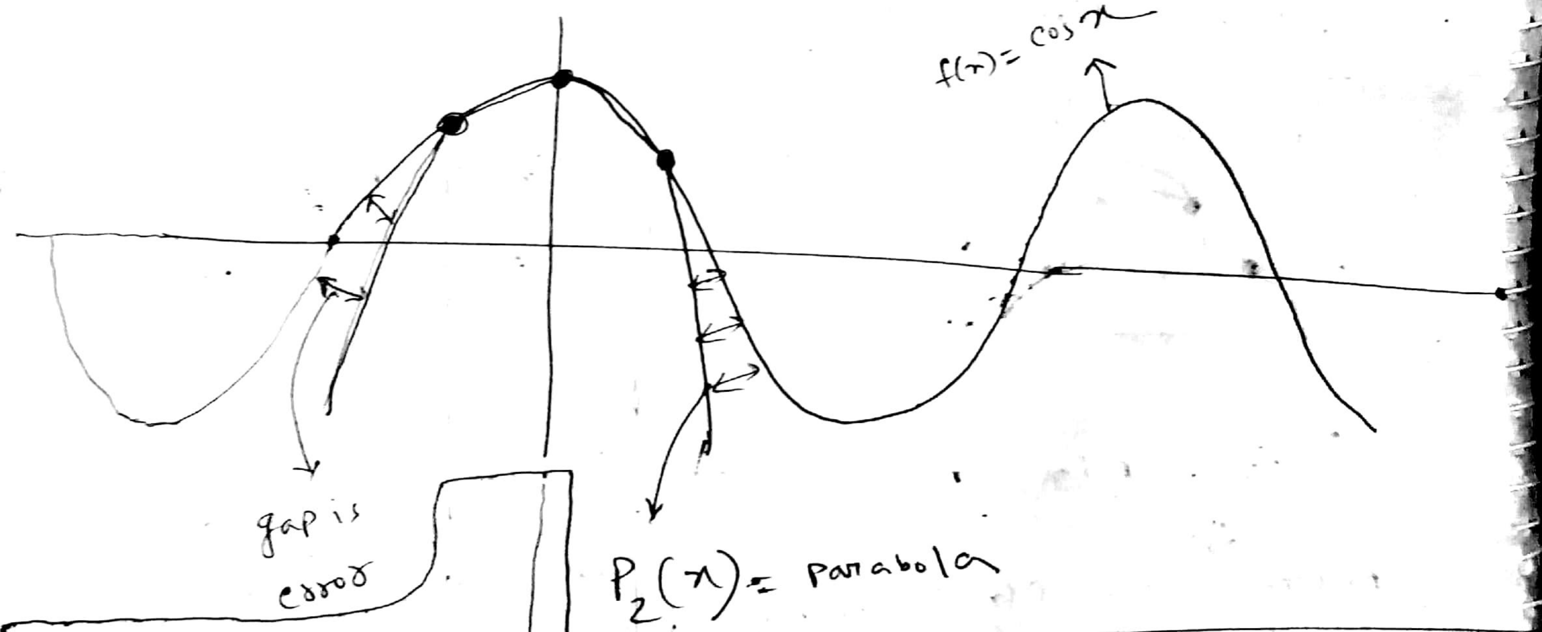
Example: $f(x) = \cos x$ nodes $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

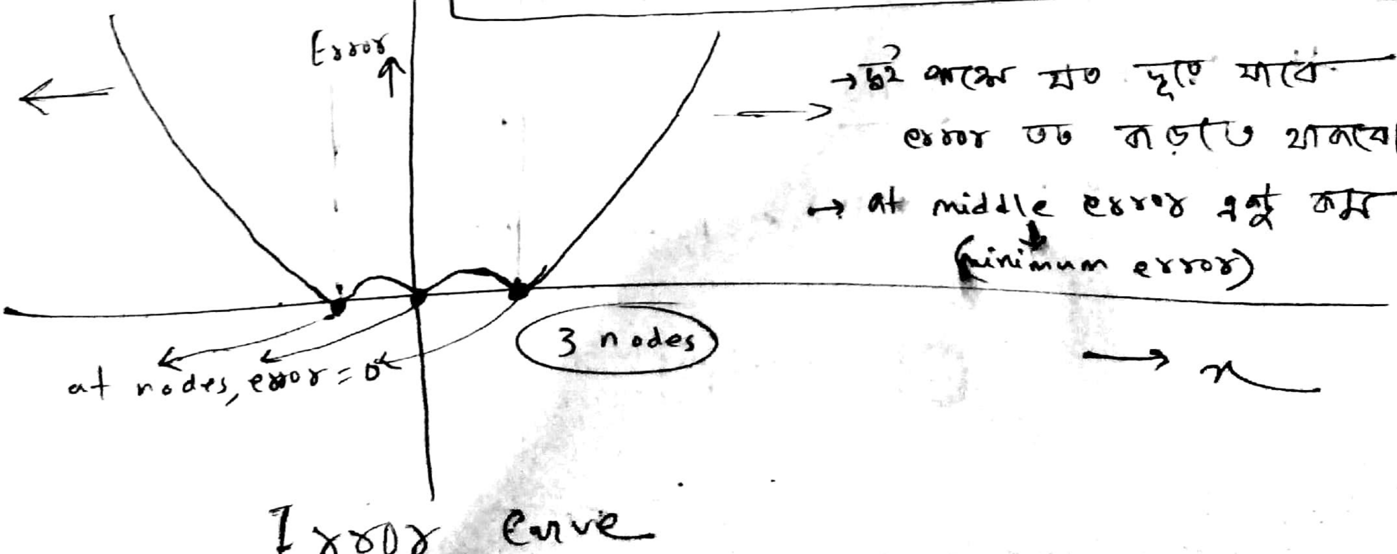
as 3 nodes, we will have 2 degree polynomial

$$P_2(x) = \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}} - 1\right)x^2 + 1$$

error is defined:

$$\left|f(x) - P_2(x)\right| = \left|\cos(x) - \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}} - 1\right)x^2 - 1\right|$$





→ দুই পাশে যত দূরে যাবে error তত বাড়তে থাকবে।

→ at middle error একটু কম (minimum error)

Error curve

Theorem: (how to find upper bound of error)

Cauchy's Th^m: Let $P_n \in \mathcal{P}_n$ be the unique polynomial interpolating $f(x)$ at the $(n+1)$ distinct nodes $x_0, x_1, \dots, x_n \in [a, b]$, and let f be continuous on $[a, b]$ with $(n+1)$ continuous derivatives on (a, b).

Then for each $x \in [a, b]$, there exists a $\xi \in (a, b)$ such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}_{w(x)}$$

as we ~~don't kn~~ don't know ξ, we always take max value of $f^{(n+1)}(\xi) \longrightarrow$ upper bound.

Note:

Cauchy's th^m. এর error formal part to Taylor th^m এর error part এর মতো দেখাতে।
Just difference হল Taylor এ just ১টাই node x_0 দেওয়া থাকতো তাই $(x-x_0)$ গুণ হত Right এ শুধু।
But interpolation এ $\{x_0, x_1, \dots x_n\} \to (n+1)$ টা node

থাকবে, তাই $(x-x_0)(x-x_1)\cdots(x-x_n)$ গুণ হবে।

here, $\pi = 3.1416$

Example: $f(x) = \cos x$ $\quad \left\{ -\frac{\pi}{4}, 0, \frac{\pi}{4} \right\}$

$$\Downarrow$$

$$\{x_0, x_1, x_2\} \to 3 \text{ nodes} \Rightarrow P_2(x)$$

$P_2(x)$ এবং $f(x)$ এর gap/error বের করব।

যেহেতু $P_2(x)$ কর্তৃক interpolates হল। error শুরু হবে $f^{(3)}(\xi)$ থেকে।

$$\therefore \left| f(x) - P_2(x) \right| = \frac{f^{(3)}(\xi)}{3!} \overbrace{\left(x + \frac{\pi}{4}\right)(x - 0)\left(x - \frac{\pi}{4}\right)}^{w(x)}$$

$$\boxed{f^{(3)}(x) = \sin x}$$

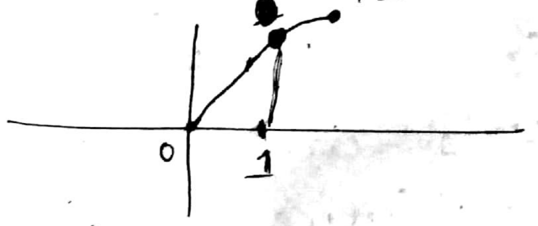
$$= \underbrace{\frac{\sin(\xi)}{6}} \underbrace{\left(x + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) x}$$

দুইটার আলাদা আলাদা max নিতে হবে।

interval of nodes will be given in question.
Let's say $[-1, 1]$. Now we have to take max value of $f^{(3)}(\xi) = \sin(\xi)$

$\xi \in [-1, 1]$

$\pi/2 = \frac{3.1416}{2} \approx 1.5$



but interval $[-1, 1]$ so, we have to stop at 1

So, max is $\sin(1)$

if $f(x) = \cos x$

then max is $\cos(0)$

$\frac{\sin(\xi)}{6}$ এর max $\frac{\sin(1)}{6}$ at interval $[-1, 1]$

$x\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) = w(x)$ এর max নির্ণয় at $[-1,1]$

ques এ curve নাই, interval

$\Rightarrow x\left(x^2-\frac{\pi^2}{16}\right) = w(x)$

$\Rightarrow x^3 - \frac{\pi^2}{16}x = w(x)$

w(x) এর max value বের করার rule

1st derivative 0 হবে x এর value বের করি।

$w'(x) = 0$

$3x^2 - \frac{\pi^2}{16} = 0$

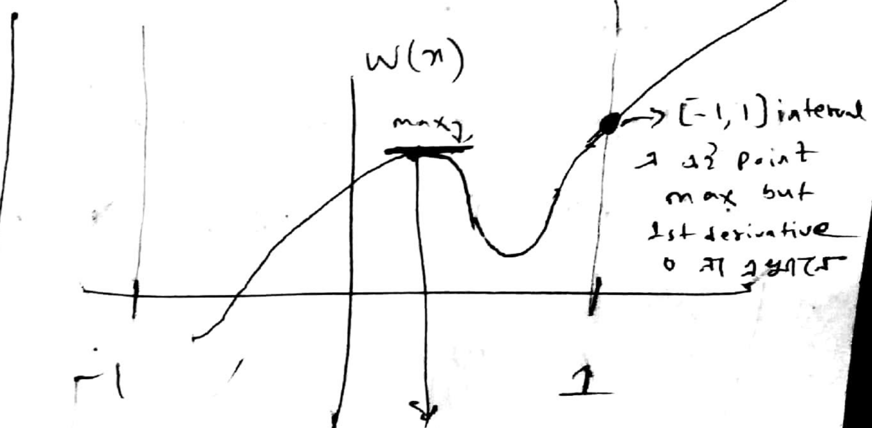
$x^2 = \frac{\pi^2}{48}$

$x = \pm\frac{\pi}{4\sqrt{3}}$

we have to find

$w(x)$ at $x = \pm\frac{\pi}{4\sqrt{3}}, \pm 1$

$[-1,1]$ interval এর corner points



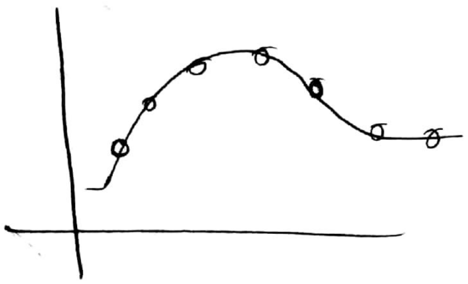
1st derivative 0 হলেও হয়তো max এখানে but interval এর মধ্যে অন্যখানে max থাকতে পারে।

So, corner point ও check করতে হবে যেটা max নিব।

x	$w(x)$
$-\frac{\pi}{4\sqrt{3}}$	0.186
$\frac{\pi}{4\sqrt{3}}$	−0.186
−1	−0.383
1	0.383 ✓ → max

So max Error $= \frac{\sin(1)}{6} \times 0.383$

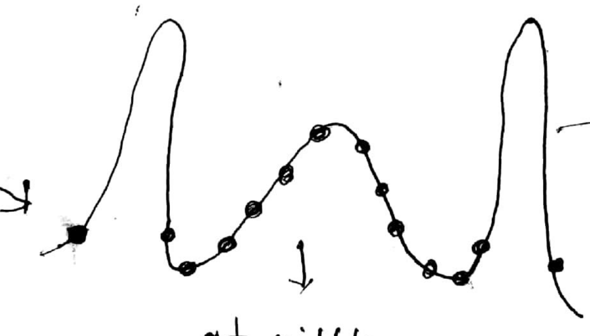
Chebyshev Nodes:



usually $n \to \infty$ হলে error $\to 0$
(number of nodes)

But there is exception $\to f(x) = \dfrac{1}{1+25x^2}$ on $[-1, 1]$

· Runge Phenomenon $\to$ getting a big spike in the corner



$\to P_n(x)$ when nodes অনেক বেশি

converges in the middle but diverges more near the ends

at middle error almost 0.
But at corners error এর amount অনেক বেশি
তাই Runge phenomenon হয়েছে।

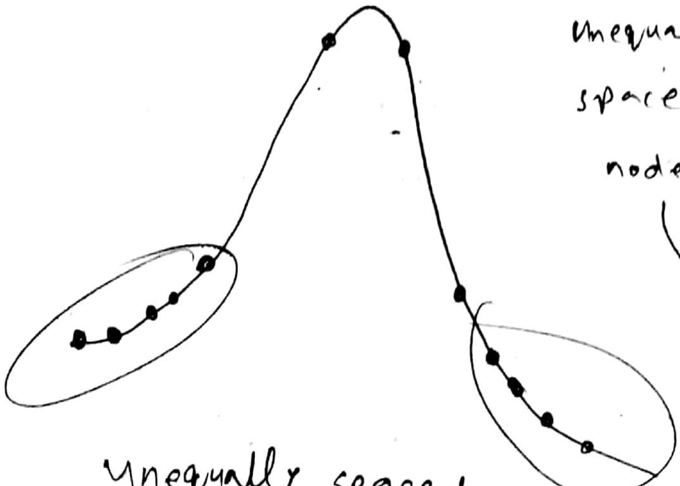
in this case, $n \to \infty \Rightarrow$ error $\to \infty$
which is exceptional

if all of the nodes are equally spaced then Runge phenomenon occurs.

$$f(x) = \frac{1}{1+25x^2} \; ; \; n = 3$$

$$x_j = \cos\left[\frac{(2j+1)\pi}{8}\right] \Rightarrow \begin{cases} x_0 = \cos\frac{\pi}{8} \\ x_1 = \cos\frac{3\pi}{8} \\ x_2 = \cos\frac{5\pi}{8} \\ x_3 = \cos\frac{7\pi}{8} \end{cases}$$

unequally spaced nodes



unequally spaced nodes (lot nodes in the corner)

chebyshev nodes

\# Runge fun^c and number of nodes is large হলে we don't take equally spaced nodes.

most efficient choice → chebyshev nodes

Derivative Conditions

A variant of the interpolation problem is to require that the interpolant matches one or more derivatives of f at each of the nodes, in addition to the functional values.

This is called osculating (Kissing) interpolation

Hermite Interpolation: we look for a polynomial that matches both $f'(x_i)$ and $f(x_i)$ at the nodes $x_i = x_0, \ldots x_n$. Since there are $2(n+1)$ $\Rightarrow 2n+2$

$(2n+2)$ coefficients

equations / conditions, so we will need a polynomial of degree $(2n+2-1) \Rightarrow (2n+1)$

Th^m: Given $(n+1)$ distinct nodes $x_0, x_1, \ldots x_n$ there exists a unique polynomial P_{2n+1} that interpolates both $f(x)$ and $f'(x)$ at these points/nodes.

Conditions:

$$\begin{cases} P(x_i) = f(x_i) \longrightarrow (n+1) \text{ conditions} \\ P'(x_i) = f'(x_i) \longrightarrow \text{extra } (n+1) \text{ conditions} \end{cases}$$

$(n+1)$ nodes are given $\longrightarrow$
- $2n+2$ conditions,
- $2n+2$ equations,
- $2n+2$ co-efficients,
- $2n+1$ degree Polynomial

$\downarrow$

$P_{2n+1}(x)$

Note:

Lagrange, Vandermond, Newton এর method এ $(n+1)$ nodes দেওয়া থাকলে $\longrightarrow$ P_n বের করতাম (degree n)

But Hermite Interpolation এ $(n+1)$ nodes দেওয়া থাকলে $\longrightarrow$ P_{2n+1} বের করব (degree 2n+1)

At Lagrange:

$$P_n(x) = \sum_{k=0}^{n} f(x_k) \cdot \underbrace{l_k(x)}_{\text{basis fn } (n+1)}$$

But,

At hermite, number of basis elements will be $(2n+2)$

$\{1, x, x^2, \ldots\ldots x^{2n+1}\}$ $\quad$ $\{0, 1, \ldots\ldots 2n+1\}$

$(2n+2)$ basis

$(n+1)$ basis ← → $(n+1)$ basis

$\downarrow$

$h_k(x)$

$\downarrow$

It's like lagrange basis

$\downarrow$

$\hat{h}_k(x)$

$\downarrow$

এটা আলাদা ভাবে বের করব →

$$\therefore P_{2n+1}(x) = \sum_{k=0}^{n} \Big(\underbrace{f(x_k)\cdot h_k(x)}_{\text{Lagrange Polynomial}} + \underbrace{f'(x_k)\cdot \hat{h}_k(x)}_{\text{extra terms}} \Big)$$

1) $h_k(x_j) = \delta_{kj}$	3) $\hat{h}_k(x_j) = 0$
2) $h'_k(x_j) = 0$	4) $\hat{h}'_k(x_j) = \delta_{kj}$

$\delta_{kj} = 1$ if $k = j$

$= 0$ if $k \neq j$

$$\begin{cases} h_k(x) = l_k^2(x)\left(\underline{a_k}(x - x_k) + \underline{b_k}\right) \\ \hat{h}_k(x) = l_k^2(x)\left(\underline{\hat{a}_k}(x - x_k) + \underline{\hat{b}_k}\right) \end{cases}$$

we have 4 unknown $a_k, \hat{a}_k, b_k, \hat{b}_k$

4 equations likbo এখন →

1) $h_k(x_k) = 1 \Rightarrow b_k = 1$ $\quad [\,l_k(x_k) = 1\,]$

2) $h_k'(x_k) = 0$

$x = x_k$ করলে 0 $\quad$ 1

$$h_k'(x) = 2\, l_k(x)\, l_k'(x) \left(a_k (x - x_k) + b_k\right) + l_k^2(x)\, a_k$$

$$h_k'(x_k) = 2\, l_k(x_k)\, l_k'(x_k) + l_k^2(x_k) \cdot a_k = 0$$

$$\therefore a_k = -2\, l_k'(x_k)$$

3) $\hat{h}_k(x_k) = 0 \Longrightarrow \hat{b}_k = 0$

4) $\hat{h}_k'(x_k) = 1 \Longrightarrow \hat{a}_k = 1$

$$\therefore h_k(x) = l_k^2(x) \left[-2\, l_k'(x_k)(x - x_k) + 1\right]$$

$$= l_k^2(x) \left[1 - 2(x - x_k)\, l_k'(x_k)\right]$$

$$\hat{h}_k(x) = l_k^2(x)(x - x_k)$$

Step:

1. l_k, l_k' বের করব lagrange দিয়ে
2. h_k, $\hat{h}_k$ বের করব formula দিয়ে
3. $P_{2n+1}(x)$ বের করব finally

Example: $f(x) = \sin x$; nodes $\left\{ 0, \frac{\pi}{2} \right\}$

2 nodes $(n+1)$

$\therefore n = 1$

$P_{2n+1}(x)$ বের করতে হবে। $\therefore$ Ques $\boxed{P_3(x) = ?}$

degree 3

$x_0 = 0$ | $x_1 = \frac{\pi}{2}$

$f(x_0) = 0$ ✓ | $f(x_1) = 1$ ✓

$(0, 0)$ $\left(\frac{\pi}{2}, 1\right)$

$(x_0, f(x_0))$ $(x_1, f(x_1))$

Ques এ এভাবে co-ordinate আকারেও দিতে পারে। সেটা দেখে বুঝতে হবে x_i কোনটা, $f(x_i)$ কোনটা

$f'(x) = \cos x$

$f'(x_0) = 1$ ✓

$f'(x_1) = 0$ ✓

$$P_3(x) = \overset{\nearrow 0}{f(x_0)}\, h_0(x) + \overset{\nearrow 1}{f(x_1)}\, h_1(x) +$$

$$\underset{\downarrow 1}{f'(x_0)}\, \hat{h}_0(x) + \underset{\searrow 0}{f'(x_1)}\, \hat{h}_1(x)$$

$$= \underline{h_1(x)} + \underline{\hat{h}_0(x)}$$

$$h_1(x) = \left(1 - 2(x - x_1)\, l_1'(x)\right) l_1^2(x)$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{\frac{\pi}{2} - 0} = \frac{2x}{\pi}$$

$$l_1'(x) = \frac{2}{\pi}$$

$$h_1(x) = \frac{4}{\pi^2} x^2 \left(3 - \frac{4}{\pi} x\right)$$

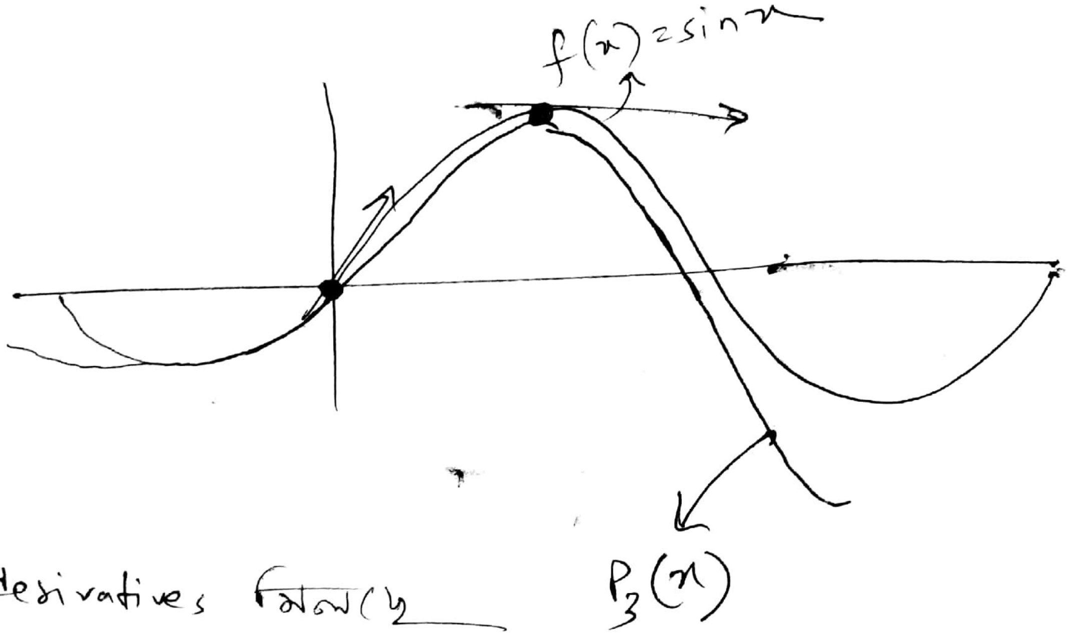
$$\hat{h}_0(x) = (x - x_0)\, l_0^2(x)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - \frac{\pi}{2}}{0 - \frac{\pi}{2}} = 1 - \frac{2x}{\pi}$$

$$\hat{h}_0(x) = x\left(1 - \frac{2x}{\pi}\right)^2$$

$$\therefore P_3(x) = x\left(1 - \frac{2}{\pi}x\right)^2 + \frac{4}{\pi^2} x^2 \left(3 - \frac{4}{\pi} x\right)$$

(Ans)



as derivatives মিলেছে

direction মিলেছে

তাই মাত্র ২টা node নেওয়ার পরও almost perfect polynomial পেয়ে গেছি যেটা আসল f(x) এর মতো দেখতে।

—⊙—

* Polynomial is also called interpolant/interpolating Polynomial

$f(x) = \tan x$

$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x$

Ques → hermite interpolant বের করতে দিলে বুঝতে হবে hermite interpolation use করে polynomial বের করা।

$l_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x+1}{1+1} = \frac{1}{2}(x+1)$

—X—

End of Ch-2