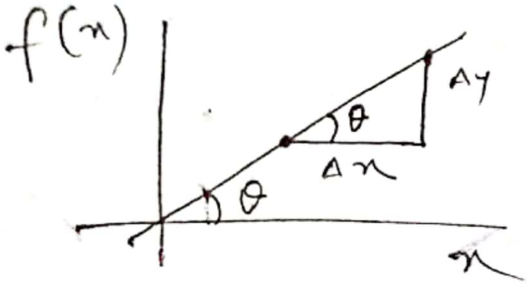
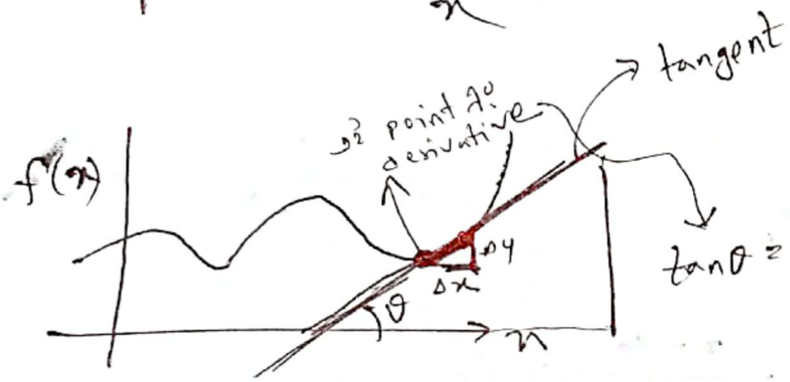
Ch-3 (Differentiation)



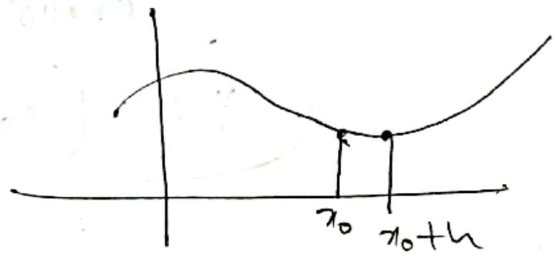
$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = f'(x) = \tan\theta$$



$$\tan\theta = \frac{\Delta y}{\Delta x}$$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$



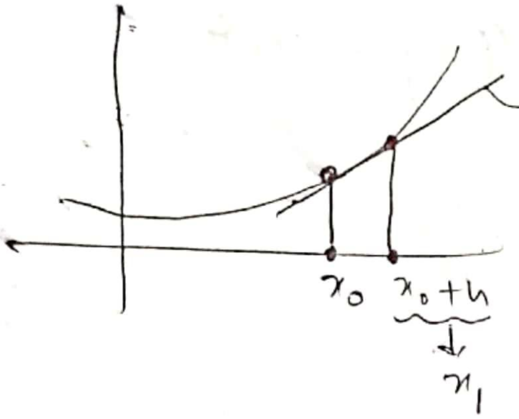
(i) if $h > 0$, forward difference

ii) if $h < 0$, backward "

Central difference:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

Forward difference:



$$P_1(x) = f(x_0).l_0(x) + f(x_1).l_1(x)$$

$$P_1(x) = f(x_0).\frac{x-x_1}{x_0-x_1} + f(x_1).\frac{x-x_0}{x_1-x_0}$$

$$f(x) = P_1(x) + \underbrace{\frac{f^{(2)}(\xi)}{2!}(x-x_0)(x-x_1)}_{\text{error (cauchy's th}^m)}$$

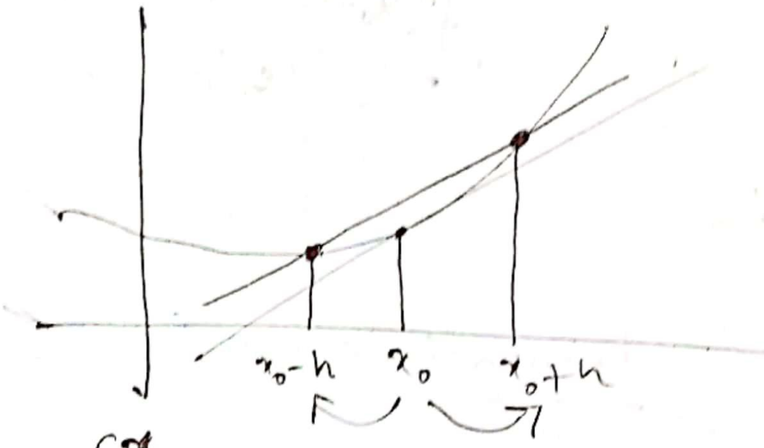
$\xi \in [x_0, x_0+h]$

$$f'(x) = \frac{1}{x_0-x_1}.f(x_0) + \frac{1}{x_1-x_0}.f(x_1) + \frac{f^{(3)}(\xi)}{2!}.\frac{d\xi}{dx}.(x-x_0)(x-x_1) + \frac{f^{(2)}(\xi)}{2!}(2x-x_0-x_1)$$

$$f'(x_0) = \frac{f(x_1)-f(x_0)}{x_1-x_0} + 0 + \frac{f^{(2)}(\xi)}{2!}(x_0-x_1)$$

$$= \frac{f(x_0+h)-f(x_0)}{h} + \boxed{\frac{f^{(2)}(\xi)}{2!}(-h)} \to \text{Truncation error}$$

Central difference:



3 nodes $\begin{cases} x_0 \\ x_1 = x_0+h \\ x_2 = x_0+2h \end{cases}$

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) + \frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$f'(x) = \frac{2x-x_1-x_2}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2x-x_0-x_2}{(x_1-x_0)(x_1-x_2)} f(x_1) +$$

$$\frac{2x-x_0-x_1}{(x_2-x_0)(x_2-x_1)} f(x_2) + \frac{f^{(3)}(\xi)}{3!}\Big[(x-x_1)(x-x_2) +$$

$$(x-x_0)(x-x_2) + (x-x_0)(x-x_1)\Big] + \frac{f^{(4)}(\xi)}{3!} \cdot \left(\frac{d\xi}{dx}\right)$$

$$(x-x_0)(x-x_1)(x-x_2)$$

$2x_0 + 2h - x_0 - x_0 - 2h = 0$

$$f'(x_1) = \frac{x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \cdot f(x_0) + \frac{2x_1 - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) +$$

$$\frac{x_1 - x_0}{(x_2 - x_0)(x_2 - x_1)} \cdot f(x_2) + \frac{f^{(3)}(\xi)}{3!} (x_1 - x_0)(x_1 - x_2)$$

$$= \frac{(x_0 + h) - x_0 - 2h}{(-h)(-2h)} \cdot f(x_0) + \frac{h}{(2h)(h)} f(x_0 + 2h) +$$

$$\frac{f^{(3)}(\xi)}{3!} (h)(-h)$$

$$= -\frac{1}{2h} f(x_0) + \frac{1}{2h} f(x_0 + 2h) - \frac{f^{(3)}(\xi)}{3!} h^2$$

$$= \frac{f(x_0 + 2h) - f(x_0)}{2h} - \underbrace{\frac{f^{(3)}(\xi)}{3!} h^2}_{\text{error}}$$

$\therefore$ error α h^2

$h < 0$

Rounding error :

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

when $h \to 0$, $f(x+h)$ ও $f(x-h)$ close to each other হবে। ওগুলো subtract করলে rounding error (loss of significance) হবে।

$$fl\left[f(x_1+h)\right] = (1+\delta_1)\, f(x_1+h)$$

$$= f(x_1+h) + \boxed{\delta_1 f(x_1+h)}$$

$$fl\left[f(x_1-h)\right] = (1+\delta_2)\, f(x_1-h)$$

$$|\delta_1|, |\delta_2| \leq \epsilon_M$$

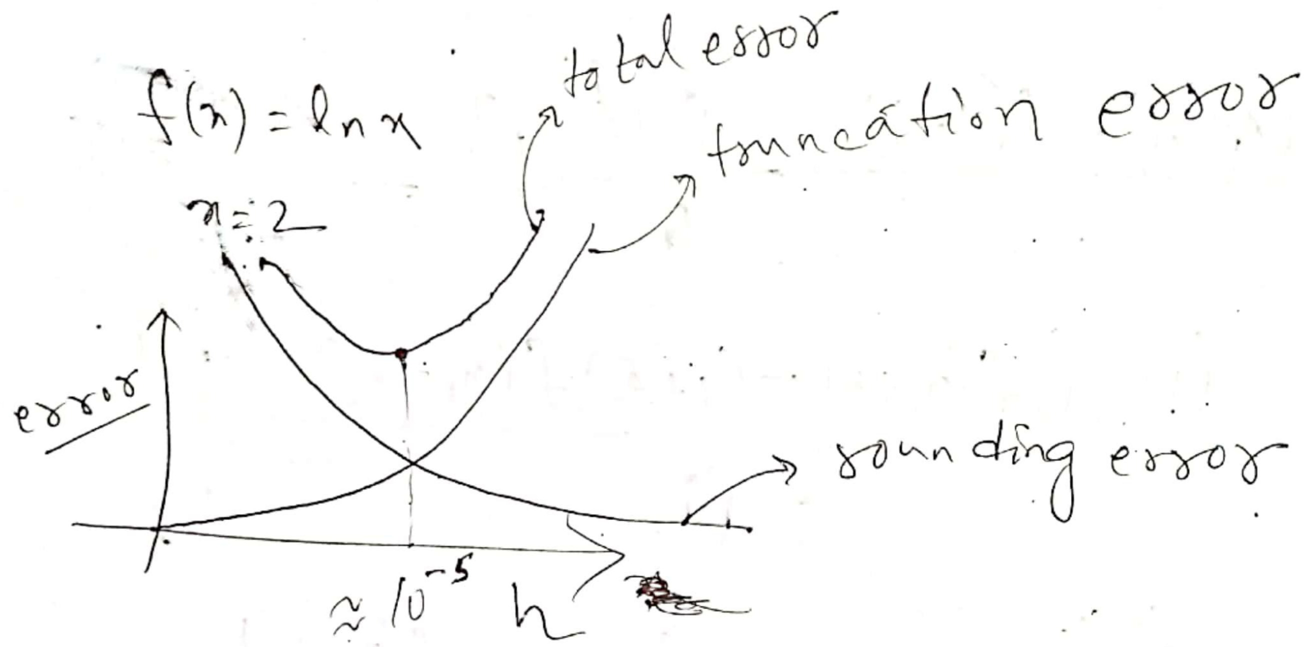
$$\left| |f'(x_1)| - \frac{fl\left[f(x_1+h)\right] - fl\left[fl(x_1-h)\right]}{2h} \right|$$

$|f'(x_1)|$ → actual value

$\frac{fl[f(x_1+h)] - fl[fl(x_1-h)]}{2h}$ → computer যেটা calculate করে।

$f(x) = \ln x$

$x = 2$



we have to take optimal value of h

optimal value of $h = 10^{-5}$

Richardson Extrapolation:

$$f'(x) = \frac{f(x_1+h) - f(x_1-h)}{2h} \Rightarrow D_h \quad \text{(Central difference)}$$

D_h → Derivative of $f(x)$ at some point x

Taylor series:

$$f(x_1+h) = f(x_1) + f^{(1)}(x_1)h + \frac{f^{(2)}(x_1)}{2!}h^2 + \frac{f^{(3)}(x_1)}{3!}h^3 + \frac{f^{(4)}(x_1)}{4!}h^4 + \frac{f^{(5)}(x_1)}{5!}h^5 + \theta(h^6)$$

$$f(x_1-h) = f(x_1) - f^{(1)}(x_1)h + \frac{f^{(2)}(x_1)}{2}h^2 - \frac{f^{(3)}(x_1)}{3!}h^3 + \frac{f^{(4)}(x_1)}{4!}h^4 - \frac{f^{(5)}(x_1)}{5!}h^5 + \theta(h^6)$$

$$D_h = \frac{1}{2h}\left(2f^{(1)}(x_1)h + \frac{2f^{(3)}(x_1)}{3!}h^3 + \frac{2f^{(5)}(x_1)}{5!}h^5 + O(h^7)\right)$$

$$D_h = \underbrace{f^{(1)}(x_1)}_{\text{Exact value}} + \underbrace{\frac{f^{(3)}(x_1)}{3!}h^2 + \frac{f^{(5)}(x_1)}{5!}h^4 + O(h^6)}_{\text{error}}$$

$\boxed{h < 0 \text{ হলে dominant } h^2}$

How to get better. Let's see.

$$D_{h/2} = f^{(1)}(x_1) + \frac{f^{(3)}(x_1)}{3!}\left(\frac{h}{2}\right)^2 + \frac{f^{(5)}(x_1)}{5!}\left(\frac{h}{2}\right)^4 + O(h^6)$$

we have to remove h^2 part

$$2^2 D_{h/2} - D_h = 2^2 f^{(1)}(x_1) - f^{(1)}(x_1) + \frac{f^{(5)}(x_1)}{5!}h^4\left(\frac{1}{2^4}\times 2^2\right) - \frac{f^{(5)}(x_1)}{5!}h^4 + O(h^6)$$

$$2^2 D_{h/2} - D_h = (2^2 - 1)f'(x_1) + \left(\frac{1}{2^2} - 1\right)\frac{f^{(5)}(x_1)}{5!}h^4 + O(h^6)$$

$$\boxed{\frac{2^2 \cdot D_{h/2} - D_h}{2^2-1}} = f'(x_1) + \frac{\frac{1}{4}-1}{(2^2-1)\,5!} f^{(5)}(x_1) \cdot h^4 + O(h^6)$$

$D_h^{(1)}$

now error $\propto h^4$

error $\propto$ or $f^{(5)}(x)$

$D_h = f'(x) + c \cdot h^n + O(h^{n+1})$

$$D_{h/2} = f'(x) + c\left(\frac{h}{2}\right)^n + O(h^{n+1})$$

$\boxed{\text{cancel } h^n}$

$$D_h^{(1)} = \frac{2^n D_{h/2} - D_h}{2^n - 1}$$

$$D_h^{(1)} = f'(x_1) + c_1 h^4 + O(h^6)$$

$$D_{h/2}^{(1)} = f'(x_1) + c_1 \frac{h^4}{2^4} + O(h^6)$$

$$D_h^{(2)} = \frac{2^4 D_{h/2}^{(1)} - D_h^{(1)}}{2^4 - 1}$$

(We can eliminate h^4 term to get 6th order approximation)

$$f(x) = e^{x} \cos(0.5x)$$

$$\left.\begin{aligned} h &= 0.1 \\ \frac{h}{2} &= 0.05 \end{aligned}\right\} \text{compute } D_h^{(1)} = \underline{\hspace{6em}}$$

— x —