Linear Equations:

→ System of linear equations (exponent of all variables must be 1)

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

→ Can be represented in a matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{(n \times n)\text{ matrix, } A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{(n \times 1)\text{ matrix, } x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{(n \times 1)\text{ matrix, } b}$$

$$A \cdot x = b$$

Solution:

$$x = A^{-1} \cdot b$$

Basic properties of A:

→ A should be a square matrix of shape $(n \times n)$

→ A must be non-singular [meaning $\det(A) \neq 0$]

Gaussian Elimination Method:

→ A technique which transforms matrix A into triangular form (upper or lower)

→ Solves $Ax = b$ without finding the inverse.

→ Lower triangular matrix (L), and upper triangular matrix (U) are defined as follows:

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$$

Using a (4x4) Lower triangular matrix:

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

number of operations:

$$l_{11} x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = b_1$$

$$\Rightarrow \boxed{x_1 = \frac{b_1}{l_{11}}} \longrightarrow 1 \text{ div}$$

$$l_{21} x_1 + l_{22} x_2 = b_2$$

$$\boxed{x_2 = \frac{b_2 - l_{21} x_1}{l_{22}}} \longrightarrow 1 \text{ div, } 1 \text{ mult, } 1 \text{ sub}$$

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number of operations:

$l_{31} x_1 + l_{32} x_2 + l_{33} x_3 = b_3$

$$\boxed{x_3 = \frac{b_3 - l_{31} x_1 - l_{32} x_2}{l_{33}}} \longrightarrow 1 \text{ div}, 2 \text{ mult}, 2 \text{ sub}$$

$l_{41} x_1 + l_{42} x_2 + l_{43} x_3 + l_{44} x_4 = b_4$

$$\boxed{x_4 = \frac{b_4 - l_{41} x_1 - l_{42} x_2 - l_{43} x_3}{l_{44}}} \longrightarrow 1 \text{ div}, 3 \text{ mult}, 3 \text{ sub}$$

This is a "TOP DOWN" approach because we found x_1 first, then x_2, x_3, x_4.

Total number of operations:

For ~~___~~ finding x_n, we need 1 div, $(n-1)$ mult, $(n-1)$ sub.

$$1 + (n-1) + (n-1) = 1 + 2(n-1).$$

$$\therefore \text{ total num of operations } = \sum_{j=1}^{n} [1 + 2(j-1)]$$

$$= \sum_{j=1}^{n} (2j - 1)$$

$$= 2 \sum_{j=1}^{n} j - \sum_{j=1}^{n} 1$$

$$= n^2 + n - n$$

$$= n^2$$

Gaussian Elimination Method:

→ To make a matrix into a triangular form, we apply Gaussian Elimination.

→ Need to apply row operations.

→ 1st row operation will make all elements below a_{11} into 0.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix} \begin{matrix} \\ r_2' = r_2 - \underbrace{\boxed{\frac{a_{21}}{a_{11}}}}_{m_{21}} r_1 \\ r_3' = r_3 - \underbrace{\boxed{\frac{a_{31}}{a_{11}}}}_{m_{31}} r_1 \end{matrix}$$

$$\begin{bmatrix} X & X & X & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & 0 & X & X \\ 0 & 0 & X & X \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & 0 & X & X \\ 0 & 0 & 0 & X \end{bmatrix}$$

Example:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + 2x_3 = 4$$
$$2x_1 + 12x_2 - 2x_3 = 4$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right] \begin{array}{l} \\ \rightarrow r_2' = r_2 - \frac{1}{1} r_1 \quad [x - Y] \\ \rightarrow r_3' = r_3 - \frac{2}{1} r_1 \quad [x - 2Y] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right] \begin{array}{l} \\ \\ \rightarrow r_3' = r_3 - \frac{8}{-4} r_2 \quad [x + 2Y] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right]$$

$-2x_3 = 12$	$-4x_2 + x_3 = 4$	$x_1 + 2x_2 + x_3 = 0$
$x_3 = -6$	$-4x_2 - 6 = 4$	$x_1 + 2(-2.5) + (-6) = 0$
	$x_2 = -2.5$	$x_1 = 11$