Differentiation:

$$\frac{d}{dx}(c \cdot x^n) = n \cdot c \cdot x^{n-1}$$

$$f(x) = x^3 - 4x + 1$$

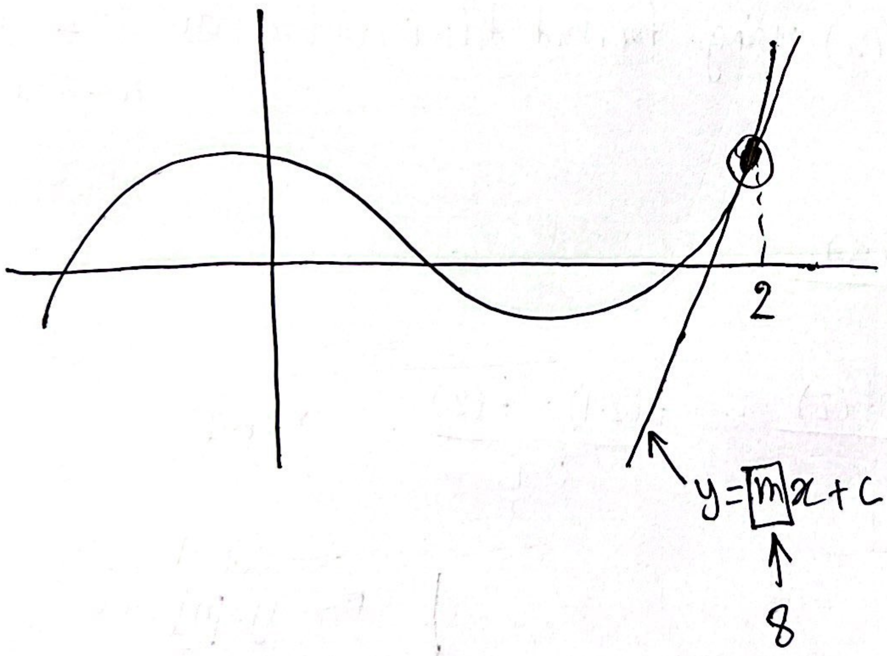
$$f'(x) = 3x^2 - 4$$

$$f'(2) = 3(2)^2 - 4$$

$$= 8$$

↳ What is the meaning of this number, 8?

Understanding its significance through a graph.



Numerical Approach:

Forward Differentiation:

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$h \rightarrow$ step size

$\hookrightarrow$ Assign some small value to h.

The smaller the value, the more closer to actual value (more accurate)

Example:

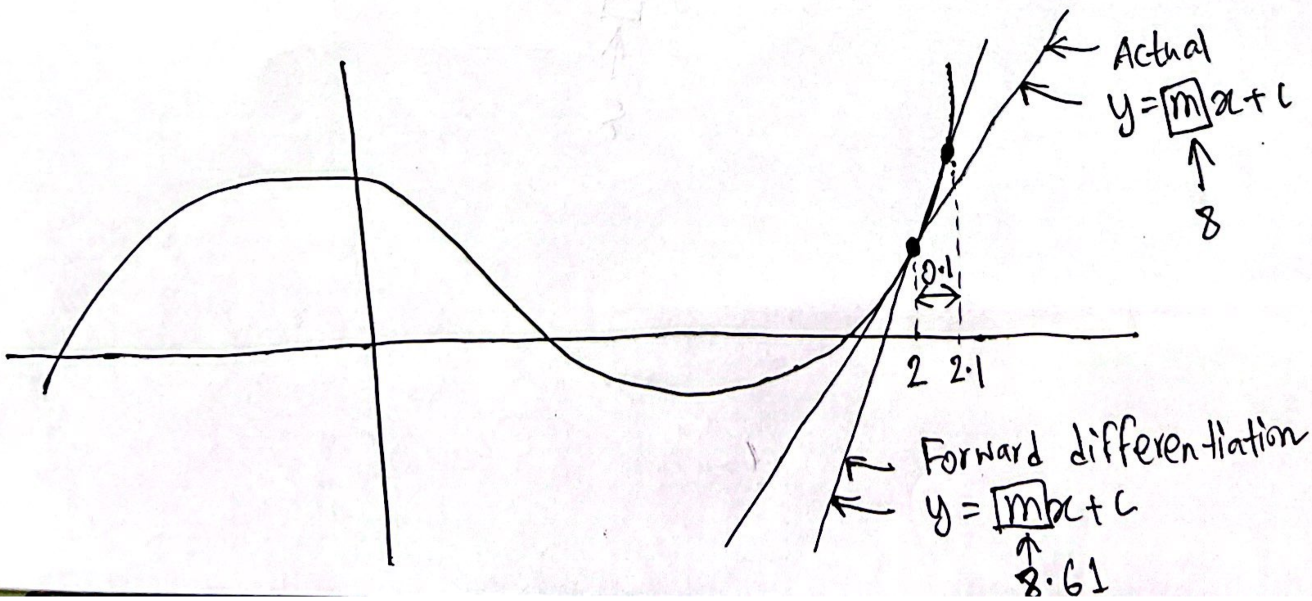
$f(x) = x^3 - 4x + 1$

Find the value of $f'(x)$ using forward differentiation at $x=2$, and $h=0.1$.

Solution:

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$f'(2) = \frac{f(2+0.1) - f(2)}{0.1} = \frac{f(2.1) - f(2)}{0.1} = 8.61$$



Backward Differentiation:

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

Example:

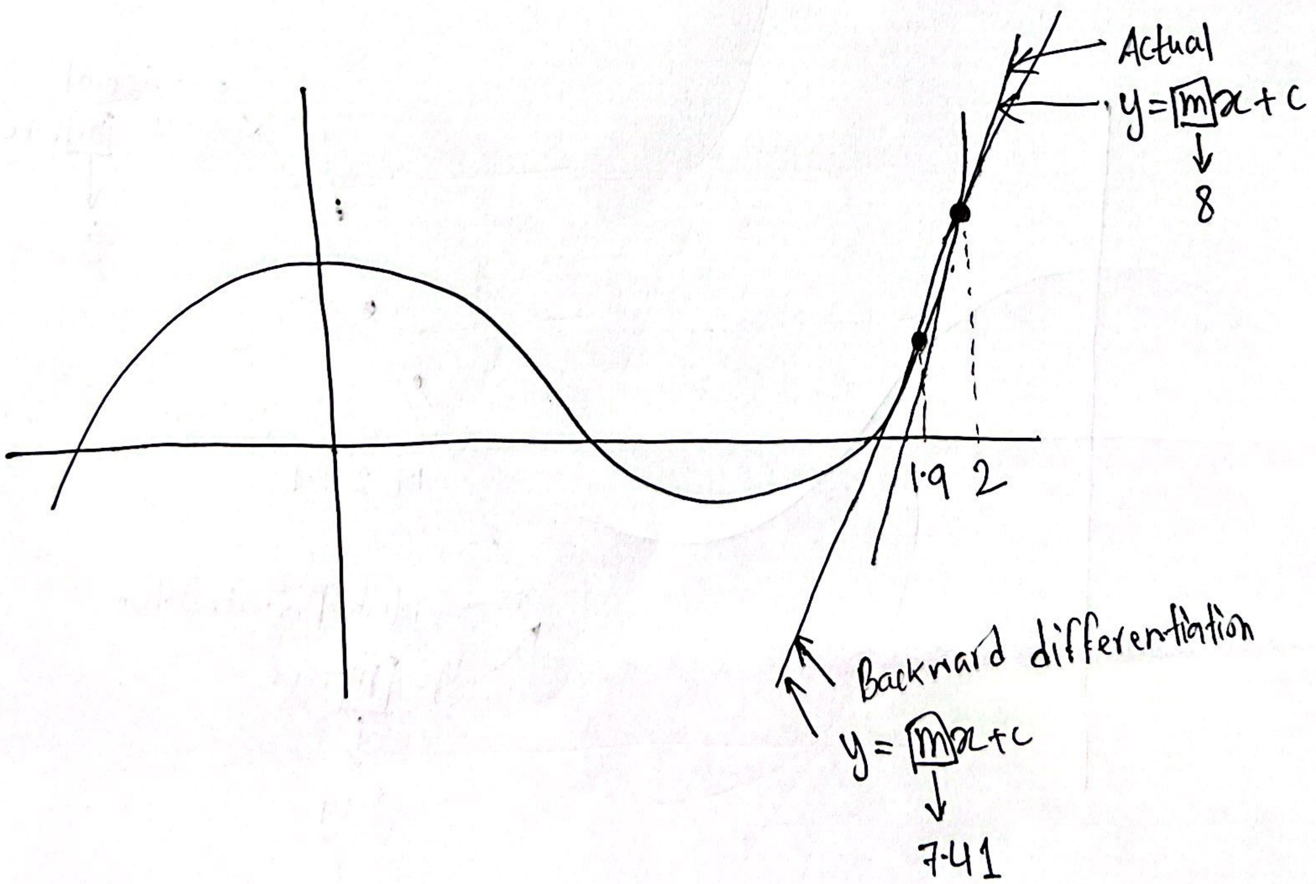
$f(x) = x^3 - 4x + 1$

Find the value of $f'(x)$ using Backward Differentiation at $x = 2$, $h = 0.1$

Solution:

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

$$f'(2) = \frac{f(2) - f(2-0.1)}{0.1} = \frac{f(2) - f(1.9)}{0.1} = 7.41$$



Central Differentiation:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Central difference gives less error than forward and backward differentiation.

Example:

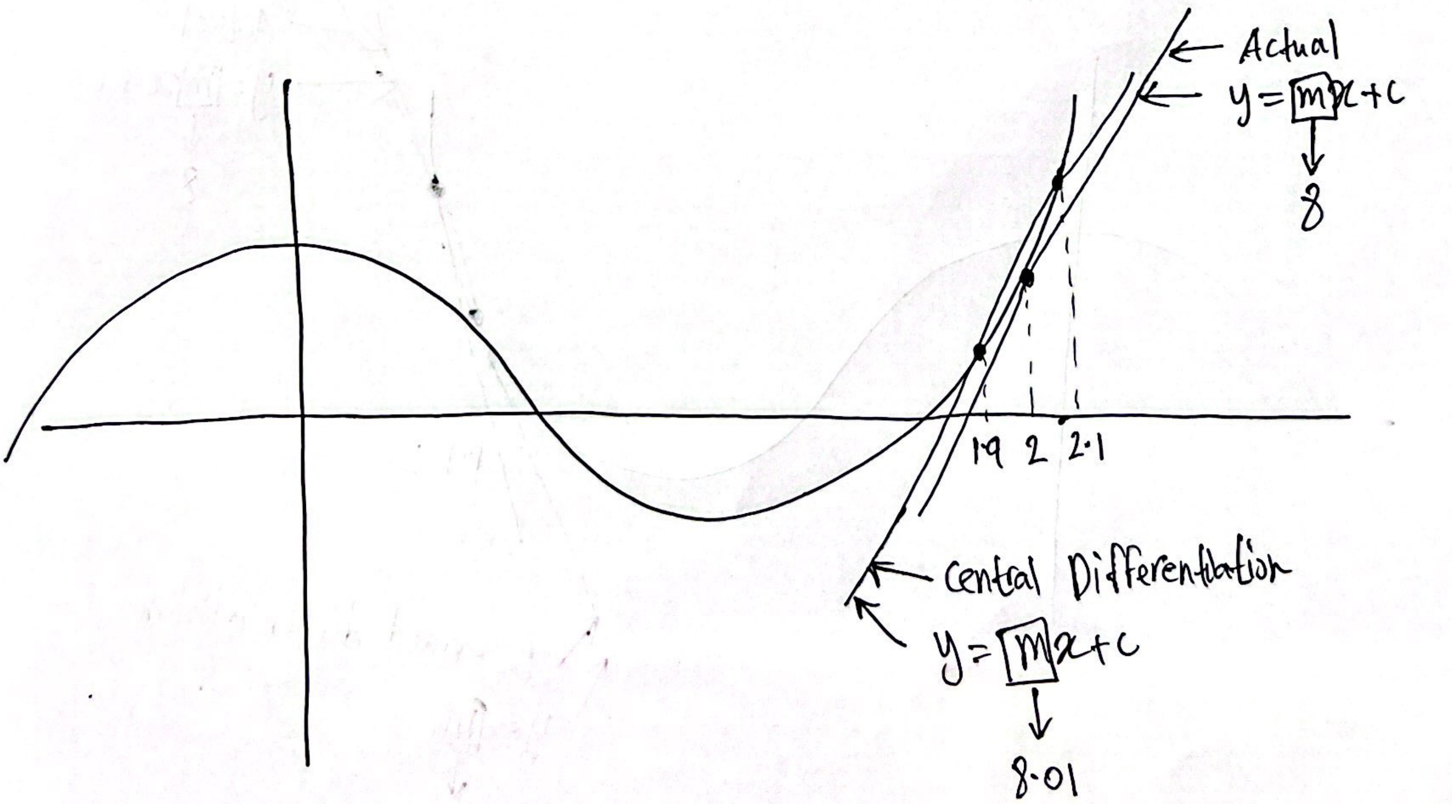
$f(x) = x^3 - 4x + 1$

Find the value of $f'(x)$ using centrall differentiation at $x = 2$
$h = 0.1$

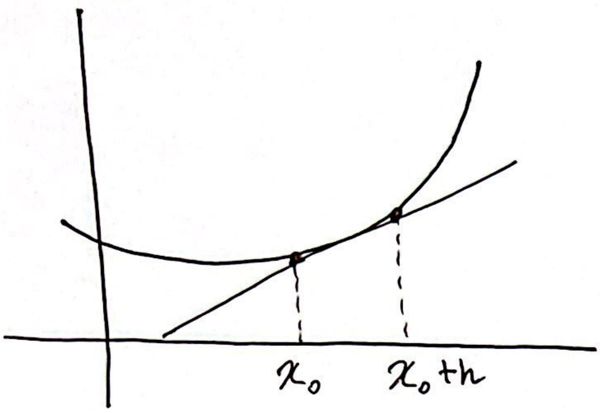
Solution:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(2) = \frac{f(2+0.1) - f(2-0.1)}{2(0.1)} = \frac{f(2.1) - f(1.9)}{0.2} = 8.01$$



Forward Difference



$$\begin{array}{cc} x_0 & x_1 \\ \downarrow & \downarrow \end{array}$$

Interpolating a polynomial using x_0, x_0+h

$$P_1(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x)$$

$$= \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$f(x) = \underbrace{\frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)}_{P_1(x)} + \underbrace{\overbrace{\frac{f''(\xi)}{2}}^{u} \overbrace{(x-x_0)(x-x_1)}^{v}}_{\text{error}}$$

$$f'(x) = \frac{1}{x_0-x_1} f(x_0) + \frac{1}{x_1-x_0} f(x_1) + \frac{f'''(\xi)}{2} \frac{d\xi}{dx} \cancelto{0}{(x-x_0)(x-x_1)}$$

$$+ \frac{f''(\xi)}{2} (2x - x_0 - x_1)$$

Plugging $x = x_0$

$$f'(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{f''(\xi)}{2} (x_0 - x_1)$$

$x_1 = x_0+h$

$$= \frac{f(x_0+h) - f(x_0)}{h} + \boxed{\frac{f''(\xi)}{2} (-h)}$$

truncation error.

$$\boxed{\text{Error} \propto h}$$

Example:

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$f'(2) = \frac{1}{2} = 0.5$

Using Forward differentiation:

h	$f'(x_0)$	Truncation Error
	$\dfrac{f(x_0+h) - f(x_0)}{h}$	$\lvert$Actual value − Forward diff value$\rvert$
	$\rightarrow \dfrac{\ln(2+h) - \ln(2)}{h}$	$\rightarrow \lvert 0.5$ − Forward diff value$\rvert$
1	0.405465	0.0945349
0.1	0.487902	0.0120984
0.01	0.498754	0.00124585
0.001	0.499875	0.00012

Decreasing on a ~~scalee~~ scale of 10, just like h

[if h is divided by 10, error also gets divided by 10]

[$\therefore$ error $\propto h$]

Scanned with CamScanner

Backward Differentiation:

Error $\propto h$ [Derivation same as Forward Difference]

Central Differentiation:

Error $\propto h^2$ $\left[\frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(\xi)}{3!} h^2 \right]$

For central differentiation, error becomes small when $h < 1$.

Example

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$f'(2) = \frac{1}{2} = 0.5$

h	$f'(x_0)$	Truncation Error
	$\boxed{\frac{f(x_0+h) - f(x_0-h)}{2h}}$	$\lvert$ Actual value − central diff value $\rvert$
1	0.549306	0.0493061
0.1	0.500417	0.000417293
0.01	0.5000004	4.16673×10^{-6}
0.001	0.5000000	4.16666×10^{-8}

[If h is divided by 10, error gets divided by 100
$\therefore$ error $\propto h^2$]