Least Square Approximation:

→ well defined linear system has equal number of variables & equations.

Example:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 9x_2 + 7x_3 = 2$$
$$2x_1 + 3x_2 + 5x_3 = 5$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -9 & 7 \\ 2 & 3 & 5 \end{bmatrix}}_{\substack{\text{square matrix} \\ \text{of } \boxed{n}\times n}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{n\times\boxed{1}} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}}_{n\times 1}$$

$$A \cdot x = b$$

→ If we have a system where number of equation > number of variables, it is called an <u>over-determined</u> system.

How do we solve over-determined system?

Example:

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 9x_2 + 7x_3 = 2$$
$$x_1 + 3x_2 + 5x_3 = 4$$
$$2x_1 + 11x_2 - 9x_3 = 5$$
$$9x_1 + x_2 - x_3 = 7$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & -9 & 7 \\ 1 & 3 & 5 \\ 2 & 11 & -9 \\ 9 & 1 & -1 \end{bmatrix}}_{\boxed{m}\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{n\times\boxed{1}} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 7 \end{bmatrix}}_{m\times 1}$$

$$A \cdot x = b$$

→ Least square approximation method is a way to find an approximate solution of an over-determined system.

$$\overset{\text{over-determined}}{\underset{(\boxed{m} \times n)}{A}} \cdot \underset{(n \times \boxed{1})}{x} = \underset{(m \times 1)}{b}$$

How to Solve such problems?

→ multiply A^T on both hand sides.

$$\underbrace{\underset{(\boxed{n} \times m)}{A^T} \; \underset{(m \times \boxed{n})}{A}}_{(n \times n)} \cdot \underbrace{\underset{(n \times 1)}{x}}_{(n \times 1)} = \underbrace{\underset{(\boxed{n} \times m)}{A^T} \cdot \underset{(m \times \boxed{1})}{b}}_{(n \times 1)}$$

Example

From polynomial chapter:

If we had $\underbrace{(n+1) \text{ nodes}}_{x_0, x_1 \cdots x_n}$, we calculated the values of $\underbrace{(n+1) \text{ coefficients}}_{a_0, a_1 \cdots a_n}$

using vandermonde matrix.

Well-defined system

$$\begin{bmatrix} 1 & x_0^1 & x_0^2 & \cdots & x_0^n \\ 1 & x_1^1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_n^1 & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

But now, lets say

we have $\underbrace{(m+1) \text{ nodes}}_{x_0, x_1 \cdots x_m}$, but we need to calculate $\underbrace{(n+1) \text{ coefficients}}_{a_0, a_1 \cdots a_n}$

[Remember $m > n$]

over-determined system

$$\begin{bmatrix} 1 & x_0^1 & x_0^2 & \cdots & x_0^n \\ 1 & x_1^1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2^1 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \\ 1 & x_m^1 & x_m^2 & \cdots & x_m^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \end{bmatrix}$$

Example:

$$\underbrace{a_0 + a_1 x}_{\text{number of coefficients} = 2}$$

We want to fit a straight line through the following nodes

$x_0 = -3$ $\quad$ $x_1 = 0$ $\quad$ $x_2 = 6$

$f(x_0) = 0$ $\quad$ $f(x_1) = 0$ $\quad$ $f(x_2) = 2$

number of nodes = 3

$$P_1(x_0) = a_0 + a_1(x_0) = f(x_0) \rightarrow a_0 + a_1(-3) = 0$$
$$P_1(x_1) = a_0 + a_1(x_1) = f(x_1) \rightarrow a_0 + a_1(0) = 0$$
$$P_1(x_2) = a_0 + a_1(x_2) = f(x_2) \rightarrow a_0 + a_1(6) = 2$$

$$\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$A \cdot x = b$$

Multiplying A^T on both sides.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

Now apply Gaussian elimination/LU/ inverse method to find the values of a_0 and a_1.

<u>Applying inverse method:</u>

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3/7 \\ 5/21 \end{bmatrix}$$

$\therefore$ $a_0 = 3/7$, $a_1 = 5/21$

$$\therefore P_1(x) = \frac{3}{7} + \frac{5}{21}x$$

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Orthogonality:

→ To understand Orthogonality, we need to understand vector dot product / inner product first.

→ Vector dot product returns a scalar value (a number)

2 types of notations → matrix notation → $x^T \cdot y$
→ vector notation → $\vec{x} \cdot \vec{y}$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

① Matrix Notation $= x^T \cdot y$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

shape → $(\boxed{1} \times 3) \; ; \; (3 \times \boxed{1})$

= ↳ returns a scalar / number

② Vector Notation $= \vec{x} \cdot \vec{y}$

$$= (1 \times 4) + (2 \times 5) + (3 \times 6)$$

↳ returns a scalar / number

Inner product with itself $= l_2$ - norm

eg. $\vec{x} \cdot \vec{x}$ or $x^T x$

Dot product:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

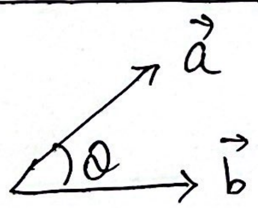
↳ returns a scalar/number.

Length / magnitude of a vector:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$|a| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$$

Dot product (Second approach):



$$a \cdot b = |a|\,|b| \cos\theta$$

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|}$$

Orthogonal Vectors:

If $\angle$ between 2 vectors $= 90°$ or $\frac{\pi}{2}$

In other words, if 2 vectors are perpendicular to each other, the vectors are orthogonal.

For orthogonal vectors, their dot product is 0. Because:

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\,\underbrace{\cos\left(\frac{\pi}{2}\right)}_{\to 0}$$

$$\therefore \vec{a}\cdot\vec{b} = 0$$

or $a^T b = 0$

Lets consider a set of vector S.

$$S = \{\vec{a}, \vec{b}, \vec{c}\}$$

Set S is an orthogonal set if

$$\vec{a}\cdot\vec{b} = 0$$
$$\vec{b}\cdot\vec{c} = 0$$
$$\vec{c}\cdot\vec{a} = 0$$

i.e. each vector is perpendicular to each other.

Orthonormality:

If ① the vectors are orthogonal (dot product = 0)

② the length of the vectors = 1 (unit vectors)

Then the vectors are orthonormal.

$$\vec{a} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Checking if orthogonal:

$$\vec{a} \cdot \vec{b} = (4 \times 1) + (2 \times -3) + (1 \times 2) = 0 \quad [\therefore \text{ orthogonal}]$$

Converting into orthonormal (by making length = 1):

$$\hat{a} = \frac{\vec{a}}{|a|} = \underbrace{\frac{1}{\sqrt{21}}}_{\sqrt{4^2+2^2+1^2}} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix} \rightarrow |\hat{a}| = \sqrt{\left(\frac{4}{\sqrt{21}}\right)^2 + \left(\frac{2}{\sqrt{21}}\right)^2 + \left(\frac{1}{\sqrt{21}}\right)^2} = 1$$

$$\hat{b} = \frac{\vec{b}}{|b|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix} \rightarrow |\hat{b}| = \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{-3}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2} = 1$$

→ Process of making vectors into unit vectors (length = 1) is called normalization

→ By doing so, we only change the magnitude, not the direction.

→ Hence, they are still orthogonal

→ Since they are orthogonal and has length = 1 (unit vectors), the vectors are orthonormal.

Example:

Consider the set of vectors, S:

$$S = \left\{ \frac{1}{\sqrt{5}} (2,1)^T \ , \ \frac{1}{\sqrt{5}} (1,-2)^T \right\}$$

Show if the set S is orthonormal or not.

Solution:

$$S = \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} , \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

$$= \left\{ \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}}_{\vec{u}_1} , \underbrace{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}}_{\vec{u}_2} \right\}$$

$$\vec{u}_1 \cdot \vec{u}_2 = \left(\frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}} \times -\frac{2}{\sqrt{5}}\right) = 0 \quad [\therefore \text{orthogonal}]$$

$$|\vec{u}_1| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = 1$$

[∴ orthonormal]

$$|\vec{u}_2| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(-\frac{2}{\sqrt{5}}\right)^2} = 1$$

∴ Yes, the set of vectors are orthonormal.

Theorem:

Orthogonal / orthonormal matrices are matrices in which the column vectors form an ~~orthogonal~~ orthonormal set (each column vector has length one, and is orthogonal to all other column vectors).

For square orthonormal matrices, the inverse is simply the transpose.

Properties:

$$Q^{-1} = Q^T$$
$$Q Q^T = I$$
$$Q^T Q = I$$

Example:

$$Q = \begin{array}{cc} \vec{a} & \vec{b} \\ \left[\begin{matrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{matrix}\right. & \left.\begin{matrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{matrix}\right] \end{array}$$

$$\vec{a} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} = (1/\sqrt{2} \times 1/\sqrt{2}) + (1/\sqrt{2} \times -1/\sqrt{2}) = 0 \quad [\therefore \text{orthogonal}]$$

$$|\vec{a}| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2} = 1$$

$$|\vec{b}| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(-\tfrac{1}{\sqrt{2}}\right)^2} = 1$$

[$\therefore$ orthonormal]

→ This makes the columns of the matrix orthonormal to each other.

→ $\therefore$ The matrix Q will have the properties:
1. $Q^{-1} = Q^T$
2. $Q Q^T = I$
3. $Q^T Q = I$

Gram - Schmidt Process:

Lets have a basis $(u_1, u_2, \dots u_n)$ from a vector space. Gram - Schmidt process takes the basis $(u_1, u_2 \dots u_n)$ and forms a new orthogonal basis $(P_1, P_2 \dots P_n)$. We can later transform these orthogonal basis into orthonormal basis $(q_1, q_2 \dots q_n)$.

$$
\begin{array}{ll}
\text{Original basis :} & u_1,\ u_2,\ u_3 \\
 & \Big\downarrow \text{ Gram Schmidt Process} \\
\text{Orthogonal basis :} & P_1,\ P_2,\ P_3 \\
 & \Big\downarrow \text{ normalization} \\
\text{Orthonormal basis :} & q_1,\ q_2,\ q_3
\end{array}
$$

1) $P_1 = u_1$

2) $P_2 = u_2 - \dfrac{u_2 \cdot P_1}{P_1 \cdot P_1} P_1$

3) $P_3 = u_3 - \dfrac{u_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \dfrac{u_3 \cdot P_2}{P_2 \cdot P_2} P_2$

Example:

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

1) $P_1 = u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2) $$P_2 = u_2 - \frac{u_2 \cdot P_1}{P_1 \cdot P_1} P_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{(1\times1) + (0\times-1) + (1\times1)}{(1\times1) + (-1\times-1) + (1\times1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

③ $P_3 = U_3 - \frac{U_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{U_3 \cdot P_2}{P_2 \cdot P_2} P_2$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{(1\times1)+(1\times-1)+(2\times1)}{(1\times1)+(-1\times-1)+(1\times1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{(1\times\frac{1}{3})+(1\times\frac{2}{3})+(2\times\frac{1}{3})}{(\frac{1}{3}\times\frac{1}{3})+(\frac{2}{3}\times\frac{2}{3})+(\frac{1}{3}\times\frac{1}{3})} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} 5/6 \\ 5/3 \\ 5/6 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

P_1, P_2, P_3 form an orthogonal basis

If we want orthonormal basis, we can devide each vectors by its length. (normalization).

① $q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

② $q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{6}/3} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}$

③ $q_3 = \frac{P_3}{|P_3|} = \frac{1}{\sqrt{2}/2} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$

QR Decomposition:

Any real (mxn) matrix "A" with m>n can be written in the form :

$$A = QR$$

Where Q is a (mxn) matrix with orthonormal columns

R is an upper triangular matrix of shape (nxn)

orthonormal matrix. Therefore has the properties:

$$Q Q^T = I_{nn}$$

$$Q^T Q = I_{nn}$$

$$\underset{m \times n}{A} = \underset{m \times n}{Q} \; \underset{n \times n}{R}$$

→ Multiplying Q^T on both sides.

$$Q^T A = \boxed{Q^T Q} R$$

$$Q^T A = \cancel{I}^{1} R$$

$$Q^T A = R$$

$$\boxed{R = Q^T A}$$

Example of QR Decomposition:

$$A = QR$$

Starting with A

$$A = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & & & \\ u_{m1} & u_{m2} & & u_{mn} \end{bmatrix}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$u_1 \quad u_2 \quad u_n$$

convert u into orthogonal vectors p (Gram-smidt process)

convert p into orthonormal vectors q (normalization)

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\downarrow \quad \downarrow$$

$$u_1 \quad u_2$$

$$p_1 = u_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$P_2 = U_2 - \frac{U_2 \cdot P_1}{P_1 \cdot P_1} P_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{(1\times3) + (2\times6) + (2\times0)}{(3\times3) + (6\times6) + (0\times0)} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\therefore P_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

P_1 and P_2 are orthogonal.

→ Now convert P_1 and P_2 into unit vectors. We will call the unit vectors q_1 and q_2.

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \\ 0 \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore q_1 = \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

q_1 and q_2 are orthonormal

$$\therefore Q = \begin{bmatrix} \sqrt{5}/5 & 0 \\ 2\sqrt{5}/5 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} \sqrt{5}/5 & 2\sqrt{5}/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\sqrt{5} & \sqrt{5} \\ 0 & 2 \end{bmatrix}$$

$$A = Q \quad R$$

$$\begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5}/5 & 0 \\ 2\sqrt{5}/5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & \sqrt{5} \\ 0 & 2 \end{bmatrix}$$

Check if $Q \cdot R = A$ in your calculator.

Example 2

$x_0 = -3 \qquad x_1 = 0 \qquad x_2 = 6$

$f(x_0) = 0 \qquad f(x_1) = 0 \qquad f(x_2) = 2$

$$\overset{A}{\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}} \cdot \overset{x}{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$$u_1 \qquad u_2$$

$$p_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$p_2 = u_2 - \frac{u_2 \cdot p_1}{p_1 \cdot p_1} p_1$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - \frac{(-3 \times 1) + (0 \times 1) + (6 \times 1)}{(1 \times 1) + (1 \times 1) + (1 \times 1)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix}$$

P_1 and P_2 are orthogonal vectors.

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{42}} \begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{bmatrix} \longrightarrow \text{matrix } Q \text{ has orthonormal columns.}$$

$$R = Q^T A$$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}$$

$$Ax = b$$

Applying Least square approximation method by applying A^T on both sides.

$$A^T A x = A^T b$$

$$\downarrow \quad \downarrow \quad \downarrow \qquad \downarrow \quad \downarrow$$

$$(QR)^T (QR) x = (QR)^T b$$

$$R^T \boxed{Q^T Q} R x = R^T Q^T b$$

$$\downarrow$$

I_{nn} [because Q = orthonormal matrix]

$$\cancel{R^T} R x = \cancel{R^T} Q^T b$$

$$\boxed{R x = Q^T b}$$

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$$Rx = Q^T b$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ 5\sqrt{2}/\sqrt{21} \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 5/21 \end{bmatrix}$$