Orthogonality:

→ To understand orthogonality, we need to understand vector dot product / inner product first.

→ Vector dot product returns a scalar value (a number)

2 types of notations → matrix notation → $x^T \cdot y$

→ vector notation → $\vec{x} \cdot \vec{y}$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

① Matrix Notation $= x^T \cdot y$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

shape → $(\boxed{1} \times 3) \; ; \; (3 \times \boxed{1})$

= ↳ returns a scalar / number

② Vector Notation $= \vec{x} \cdot \vec{y}$

$$= (1 \times 4) + (2 \times 5) + (3 \times 6)$$

↳ returns a scalar / number

Inner product with itself $= l_2$ - norm

eg. $\vec{x} \cdot \vec{x}$ or $x^T x$

Dot product:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

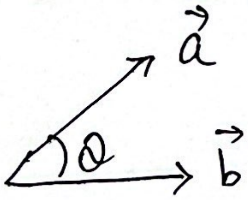
↳ returns a scalar/number.

Length / magnitude of a vector:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$|a| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$$

Dot product (Second approach):



$$a \cdot b = |a|\,|b| \cos\theta$$

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|}$$

Orthogonal Vectors:

If $\angle$ between 2 vectors $= 90^\circ$ or $\frac{\pi}{2}$

In other words, if 2 vectors are perpendicular to each other, the vectors are orthogonal.

For orthogonal vectors, their dot product is 0. Because:

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\,\underbrace{\cos\left(\frac{\pi}{2}\right)}_{\to 0}$$

$$\therefore \vec{a}\cdot\vec{b} = 0$$

or $a^T b = 0$

Lets consider a set of vector S.

$$S = \{\vec{a}, \vec{b}, \vec{c}\}$$

Set S is an orthogonal set if

$$\vec{a}\cdot\vec{b} = 0$$
$$\vec{b}\cdot\vec{c} = 0$$
$$\vec{c}\cdot\vec{a} = 0$$

i.e. each vector is perpendicular to each other.

Orthonormality:

If ① the vectors are orthogonal (dot product = 0)

② the length of the vectors = 1 (unit vectors)

Then the vectors are orthonormal.

$\vec{a} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

Checking if orthogonal:

$\vec{a} \cdot \vec{b} = (4 \times 1) + (2 \times -3) + (1 \times 2) = 0$ [∴ orthogonal]

Converting into orthonormal (by making length = 1):

$\hat{a} = \frac{\vec{a}}{|a|} = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix} \rightarrow |\hat{a}| = \sqrt{\left(\frac{4}{\sqrt{21}}\right)^2 + \left(\frac{2}{\sqrt{21}}\right)^2 + \left(\frac{1}{\sqrt{21}}\right)^2}$

$= 1$

$\sqrt{21} \leftarrow \sqrt{4^2 + 2^2 + 1^2}$

$\hat{b} = \frac{\vec{b}}{|b|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix} \rightarrow |\hat{b}| = \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{-3}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2}$

$= 1$

→ Process of making vectors into unit vectors (length = 1) is called normalization

→ By doing so, we only change the magnitude, not the direction.

→ Hence, they are still orthogonal

→ Since they are orthogonal and has length = 1 (unit vectors), the vectors are orthonormal.

Example:

Consider the set of vectors, S:

$$S = \left\{ \frac{1}{\sqrt{5}}(2,1)^T , \frac{1}{\sqrt{5}}(1,-2)^T \right\}$$

Show if the set S is orthonormal or not.

Solution:

$$S = \left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} , \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

$$= \left\{ \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}}_{\vec{u}_1} , \underbrace{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}}_{\vec{u}_2} \right\}$$

$$\vec{u}_1 \cdot \vec{u}_2 = \left(\frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}} \times -\frac{2}{\sqrt{5}}\right) = 0 \quad [\therefore \text{orthogonal}]$$

$$|\vec{u}_1| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = 1$$

$[\therefore$ orthonormal$]$

$$|\vec{u}_2| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(-\frac{2}{\sqrt{5}}\right)^2} = 1$$

$\therefore$ Yes, the set of vectors are orthonormal.

Theorem:

Orthogonal / orthonormal matrices are matrices in which the column vectors form an orthonormal ~~orthogonal~~ set (each column vector has length one, and is orthogonal to all other column vectors).

For square orthonormal matrices, the inverse is simply the transpose.

Properties:

$$Q^{-1} = Q^T$$
$$Q Q^T = I$$
$$Q^T Q = I$$

Example:

$$Q = \begin{array}{cc} \vec{a} & \vec{b} \\ \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \end{array}$$

$$\vec{a} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} = (1/\sqrt{2} \times 1/\sqrt{2}) + (1/\sqrt{2} \times -1/\sqrt{2}) = 0 \quad [\therefore \text{orthogonal}]$$

$$|\vec{a}| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2} = 1$$

$[\therefore$ orthonormal$]$

$$|\vec{b}| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(-\tfrac{1}{\sqrt{2}}\right)^2} = 1$$

→ This makes the columns of the matrix orthonormal to each other.

→ $\therefore$ The matrix Q will have the properties : ① $Q^{-1} = Q^T$

② $Q Q^T = I$

③ $Q^T Q = I$