$$f(x) = e^{x} \cos(0.5x) \qquad D_{0.1}^{(1)}$$

$$\left.\begin{matrix} h = 0.1 \\ \frac{h}{2} = 0.05 \end{matrix}\right\} \text{compute } D_h^{(1)} = \rule{6cm}{0.4pt}$$

——— X ———

Ch-4 (Non Linear Equations)

$f(x)$ is non-linear if $f(x)$ is not restricted to be degree one polynomial. $f(x)$ could be a series, polynomial, rational etc.

$\Rightarrow$ $f(x) = 0$ solution or Root of the function

It is denoted by x_*. This is also called the 'Zero' of the func.

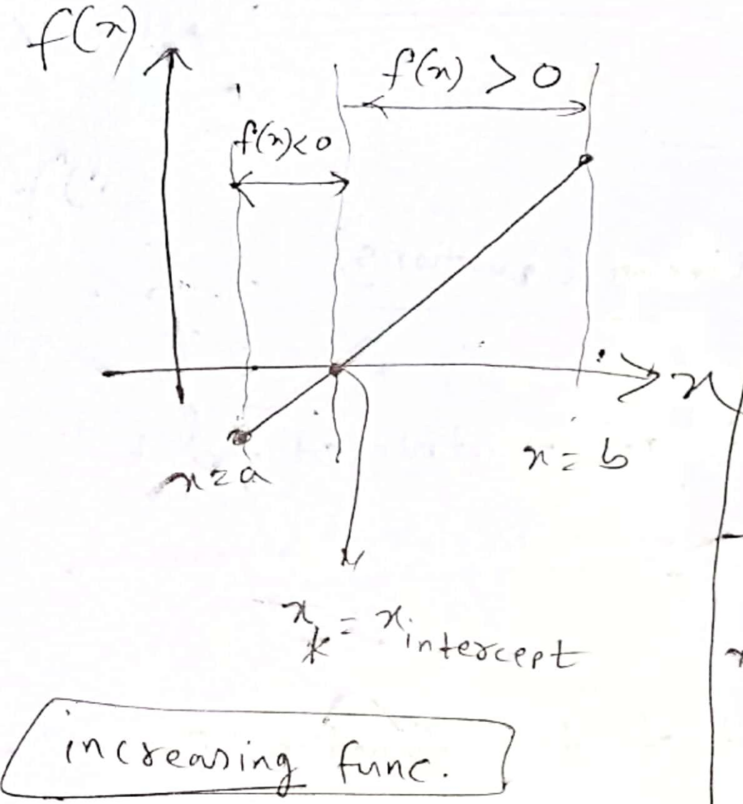
$\Rightarrow$ $f(x_*) = 0 \longrightarrow$ exact solution

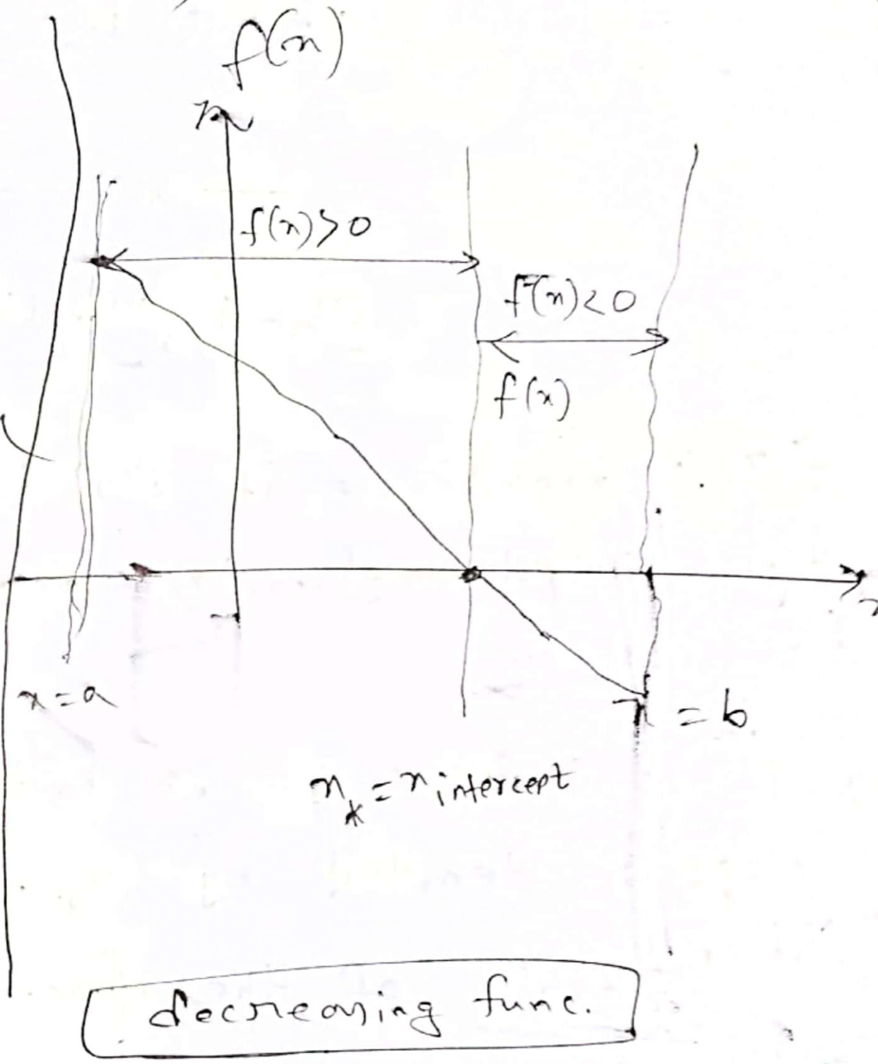
$\Rightarrow$ $(x - x_*)$ is a factor of $f(x)$

$\Rightarrow$ since $f(x_*) = 0$, x_* is also the x-intercept of $f(x)$.

$x - x_* = 0$ (exact sol^n)

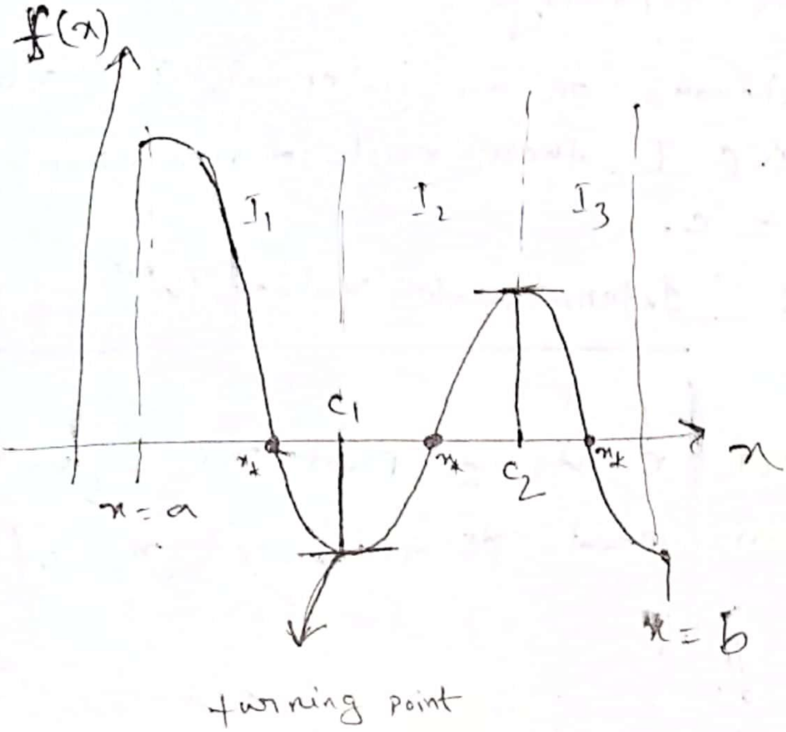
$x - x_* \approx 0$ (approximate sol^n)





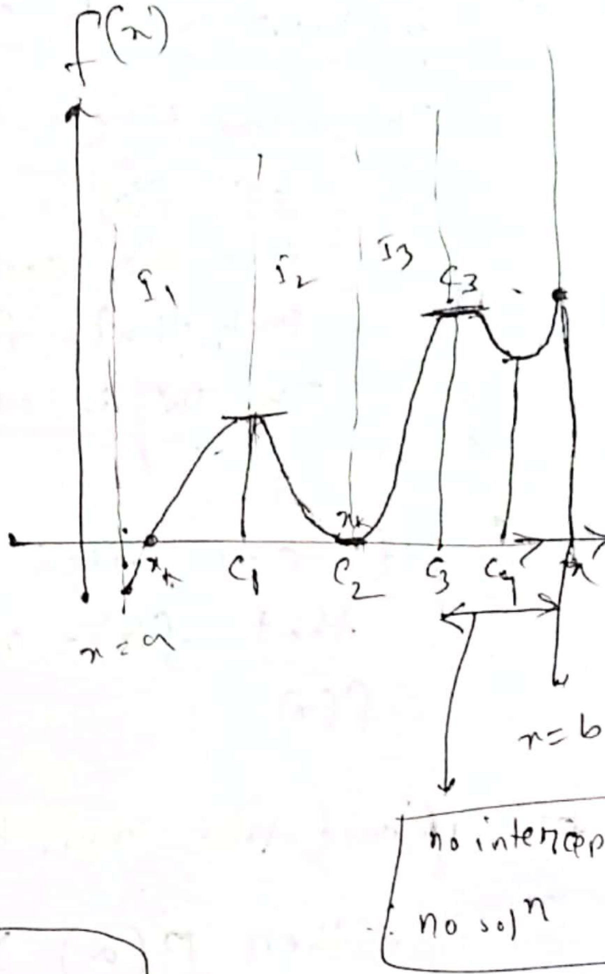
graph with single x- intercept

multiple x-intercept:



$$I = I_1 \cup I_2 \cup I_3$$

$$[a,b] = [a, c_1] \cup [c_1, c_2] \cup [c_2, b]$$



$$[a,b] = [a, c_1] \cup [c_1, c_2] \cup [c_2, c_3]$$

$$I = I_1 \cup I_2 \cup I_3$$

This is the setup of interval

· bisection method.

Root finding Method [illegible]

Interval Bisection Method:

This method depends on the following Th^m:

If $f(x)$ is continuous on an interval $I = [a, b]$, then for each $x \in I$, there exists a real number c, such that $f(x) = c$.

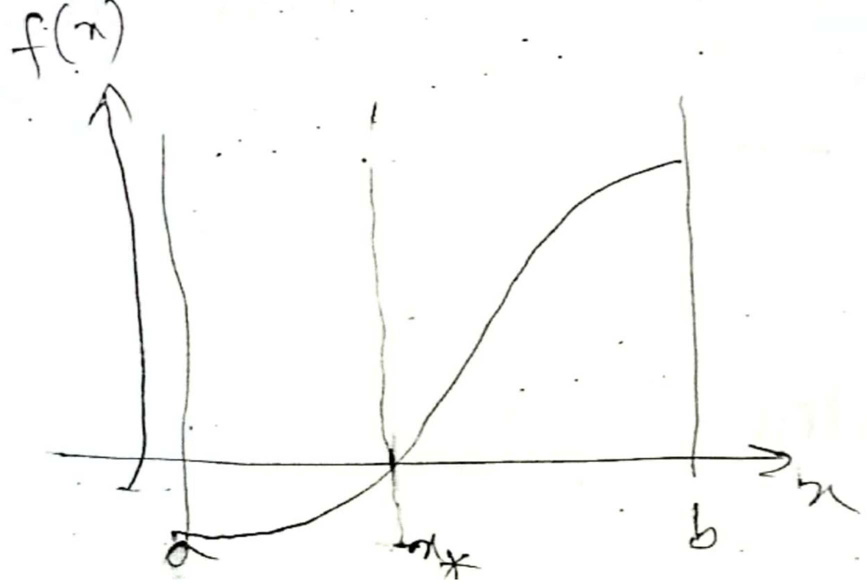
This Th^m is called 'Intermediate value Th^m'

$\Rightarrow$ If $c = 0$, then there exists a point $x \in I$, such that $f(x) = 0$. This point is the root x_* of $f(x)$.

$\Rightarrow$ Hence, we must have

$$\left.\begin{array}{ll} \text{either } f(a) > 0 & \text{and } f(b) < 0 \\ \text{or } \quad f(a) < 0 & \text{and } f(b) > 0 \end{array}\right\} \text{as } f(x) \text{ crosses the } x\text{-axis.}$$

$$\boxed{f(a) \cdot f(b) < 0}$$



$x_* \in [a, b]$

$f(a) < 0$

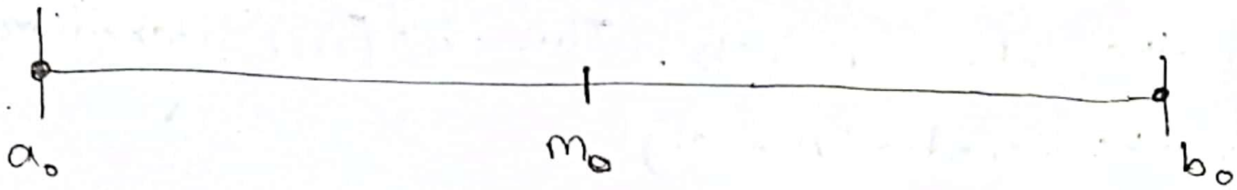
$f(b) > 0$

$f(a) \cdot f(b) < 0$

$I = [a_0, b_0]$.

$$m_0 = \frac{a_0 + b_0}{2} \implies I = [a_0, b_0] \rightarrow \underset{I_1}{[a_0, m_0]} \cup \underset{I_2}{[m_0, b_0]}$$

$$I \implies I_1 \cup I_2$$



since there is only one root, it must be in subinterval $[a_0, m_0]$ or $[m_0, b_0]$

To find which subinterval, we compute $f(m_0)$ and check if $f(m_0)$ is
$$\begin{cases} > 0 & \text{or} \\ < 0 & \text{or} \\ = 0 \end{cases}$$

if $f(m_0) = 0$, then it is the root. $\therefore \boxed{x_* = m_0}$

Note:

if $f(m_0) > 0$, sol^n or root must be in right subinterval $[m_0, b_0]$. And the error must be half the length of the interval I or less:

$$|x_* - m_0| \leq \frac{1}{2} |b_0 - a_0|$$

if $f(a) < 0$ and $f(b) > 0$ হয়, তখন $f(m_0) > 0$ হলে root left subinterval এ থাকবে।

if $f(a) > 0$ and $f(b) < 0$

This is iterative process:

→ If the error is within the bound, then m_0 is the root. So, no iteration. $(k=0)$

→ If the error is not within error bound, we repeat the above process. This is the First iteration (k is the iteration number)

→ for the first iteration, new interval $I_1 = [a_1, b_1]$

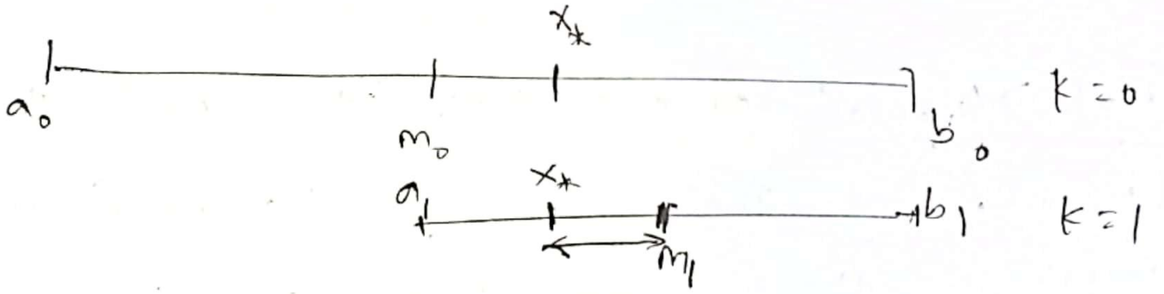
$\hookrightarrow [m_0, b_0]$

Prev process is repeated with

$$m_1 = \frac{1}{2}(b_1 + a_1)$$

Hence, $I_1 = [a_1, m_1] \cup [m_1, b_1]$

Let, $f(m_1) < 0$. So, the root must be in $[a_1, m_1]$



error now: $|x_* - m_1| \leq \frac{1}{2}|b_1 - a_1| = \frac{1}{2}|b_0 - m_0|$

$$= \frac{|b_0 - a_0|}{4}$$

if the error is within the bound, $\underline{m_1}$ is the root, if not → iteration continues.

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After n-th iteration, the error is

$$|x_* - m| \leq \frac{|b_0 - a_0|}{2^{n+1}}$$

Let δ be the desired error bound, then we must have

$$\frac{|b_0 - a_0|}{2^{n+1}} \leq \delta$$

$$\therefore n \geq \frac{\log(|b_0 - a_0|) - \log\delta}{\log(2)} - 1$$

$\boxed{\text{here, log base 2}}$

minimum number of iteration

if $a_0 = 1.5$, $b_0 = 3$, $\delta = \epsilon_m = 1.1 \times 10^{-16}$

$\therefore n \geq 53$

Example:

$$f(x) = \frac{1}{x} - 0.5 \quad \text{and} \quad I = [1.5, 3]$$

$\Rightarrow$ exact solⁿ of $f(x) = 0$ is $x_* = 2$

but we have to find approximate solⁿ by numerical method.

$\boxed{K=0}$

Solⁿ: Here, $a_0 = 1.5$, $b_0 = 3$

$f(a_0) = 0.1666 \; (>0)$

$f(b_0) = -0.1666 \; (<0)$

So, there exists a solⁿ in $[1.5, 3]$ as $f(a) \cdot f(b) < 0$

$$m_0 = \frac{1.5+3}{2} = 2.25$$

$$f(m_0) = -0.0555 \; (<0)$$

So, root must be in left subinterval $[1.5, 2.25]$

Clearly the root lies in $[a_0, m_0] \equiv [a_1, b_1]$

$= [1.5, 2.25]$

Now $\boxed{K=1}$

$$m_1 = \frac{1.5+2.25}{2} = 1.875 \text{ and}$$

$$f(m_1) = 0.0333 > 0.$$

Now solⁿ is in the interval $[m_1, b_1]$

$$\equiv [1.875, 2.25]$$

$$[a_2, b_2]$$

Now $k=2$

iteration continues

If the error is $\delta = 1.0 \times 10^{-5}$

∴ minimum number of iteration required,

$$n \geq \frac{\log(3-1.5) - \log(1\times10^{-5})}{\log(2)} - 1$$

$$n \geq 16$$

Ex: Use interval bisection method to find solutions accurate to within 10^{-3} for

$f(x) = x^3 - 7x^2 + 14x - 6 = 0$ on the interval $[1, 3.2]$ [minimum $n \geq 9$]

Soln:

$a_0 = 1$, $f(a_0) = 2 > 0$

$b_0 = 3.2$, $f(b_0) = -0.11 < 0$

as $f(a_0) \cdot f(b_0) < 0$

$\therefore$ there is a solution in $[1, 3.2]$

$[a_0, b_0]$

→ Now we compute $m_k = \dfrac{a_k + b_k}{2}$ for the iteration number k and check where is the root by comparing $f(a_k)$, $f(m_k)$, $f(b_k)$, until we obtain $|f(m_k)|$ within $10^{-3} = 0.001$.

→ We present calculation in the following table: —

Note: $f(a_k)$, $f(m_k)$, $f(b_k)$ এর যে pair এ opposite sign থাকবে, সেটা next subinterval হবে।

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K	a_k	b_k	m_k	$f(a_k)$	$f(b_k)$	$f(m_k)$	$x_* \in [\,,\,]$
0	1	3.2	~~3.2~~ 2.1	$2 > 0$	$-0.11 < 0$	$1.79 > 0$	$[2.1, 3.2]$
1	2.1	3.2	2.65	$1.79 > 0$	$-0.11 < 0$	$0.55 > 0$	$[2.65, 3.2]$
2	2.65	3.2	2.925	$0.55 > 0$	$-0.11 < 0$	$0.086 > 0$	$[2.925, 3.2]$
3	2.925	3.2	3.0625	$0.86 > 0$	$-0.11 < 0$	$-0.054 < 0$	$[2.925, 3.0625]$
4		"					
⋮							
9	2.99	3.002	3.000195 $\downarrow$ 3.000	1.96×10^{-3}	-2.3×10^{-3}	-1.95×10^{-4} $(< 10^{-3})$	$\boxed{x_* = 3.000}$ Root

So iteration stops

Fixed Point Iteration :

$$f(x) = 0$$
$$\Downarrow$$
$$g(x) - x = 0$$ by using algebraic operations

Condition : Both $f(x)$ and $g(x)$ must be continuous on some interval $I = [a, b]$,

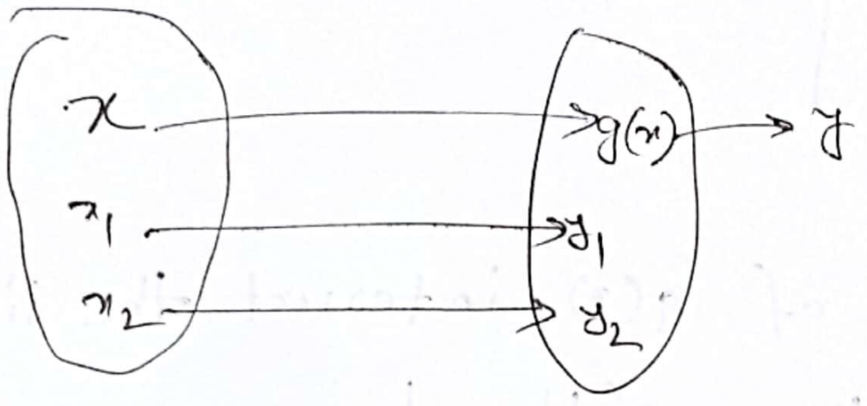
Advantage : In this method, presence of multiple roots can be taken care of which was not allowed in the interval bisection method.

* Fixed point equation/mapping

$$f(x) = 0 \Longrightarrow g(x) = x$$

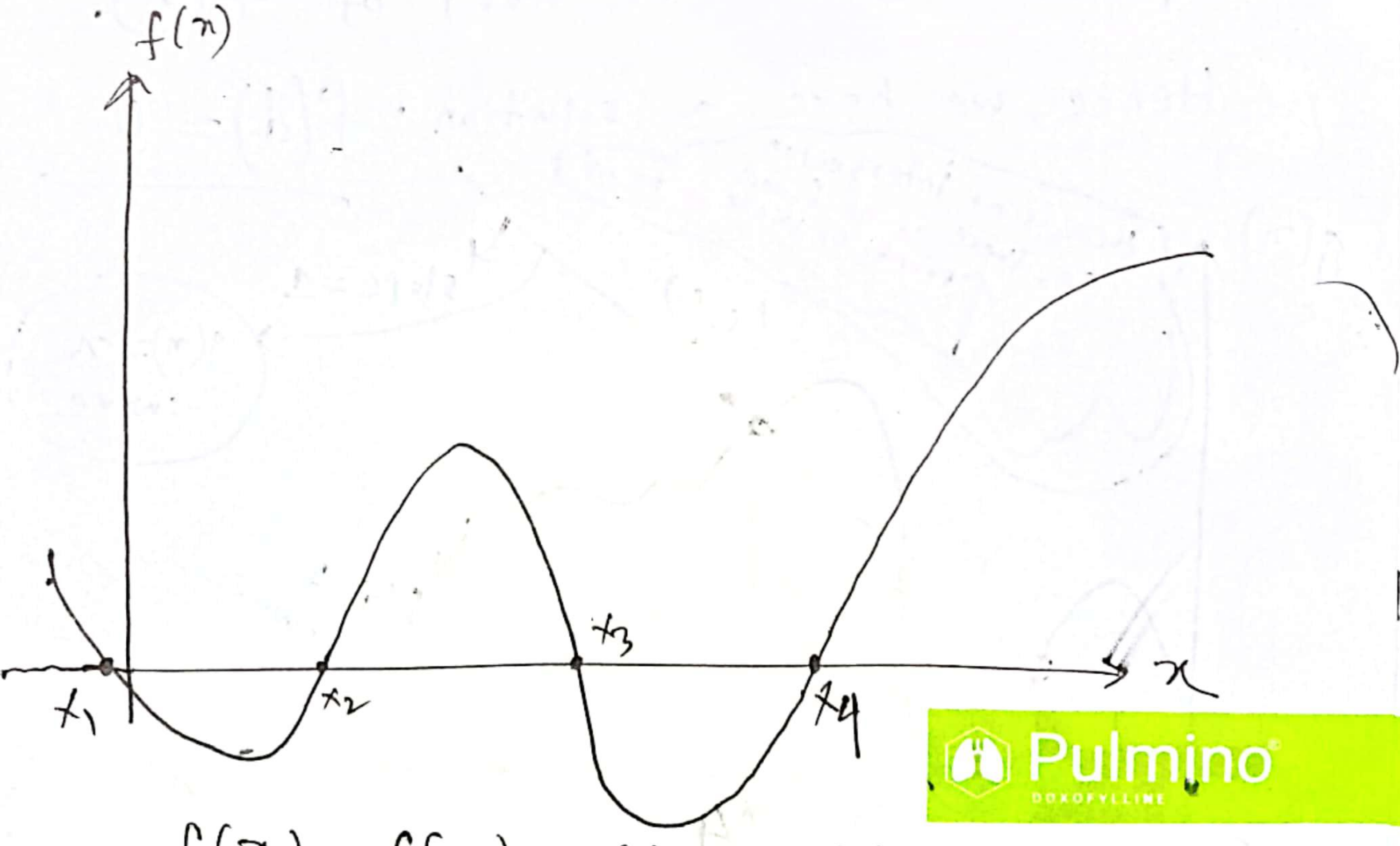
⇒ Any point x that remains same under a nontrivial mapping g is called fixed point under that mapping.

Note: g is not a trivial mapping (meaning it is not an identity transformation. In other words, it is not a multiplication by one/addition by zero).



$$y = g(x) \neq x \quad \text{(non trivial)}$$

but $g(x_0) = x_0$ 2(m x_0 is the fixed point



$$f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$$

Find such $g(x)$:

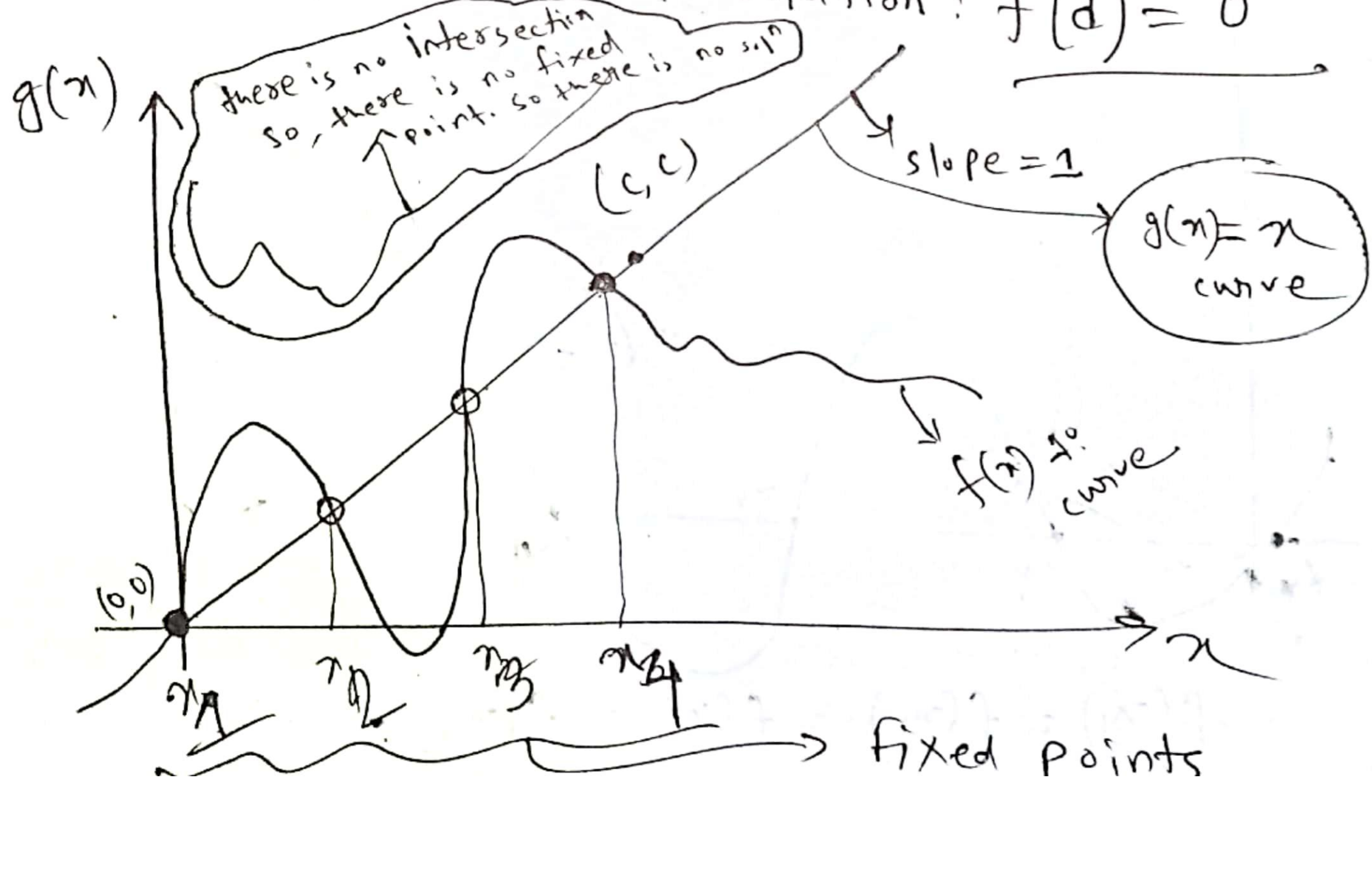
$$\left.\begin{aligned} g(x_1) &= x_1 \\ g(x_2) &= x_2 \\ g(x_3) &= x_3 \\ g(x_4) &= x_4 \end{aligned}\right\} \text{4 fixed points in this case}$$

→ If the graph of $g(x)$ intersect the line at $x = d$, then $g(d) = d$

→ $x = d$ is the fixed point of $g(x)$

→ $x_* = x = d$ is a root of $f(x)$.

Hence, we have a solution : $f(d) = 0$



Same fixed point numbers হয়, তবে multiple $g(x)$ থাকতে পারে। But all g are not acceptable.

Some g are converging

some are diverging.

We will discuss later ~~the~~ how to find converging g

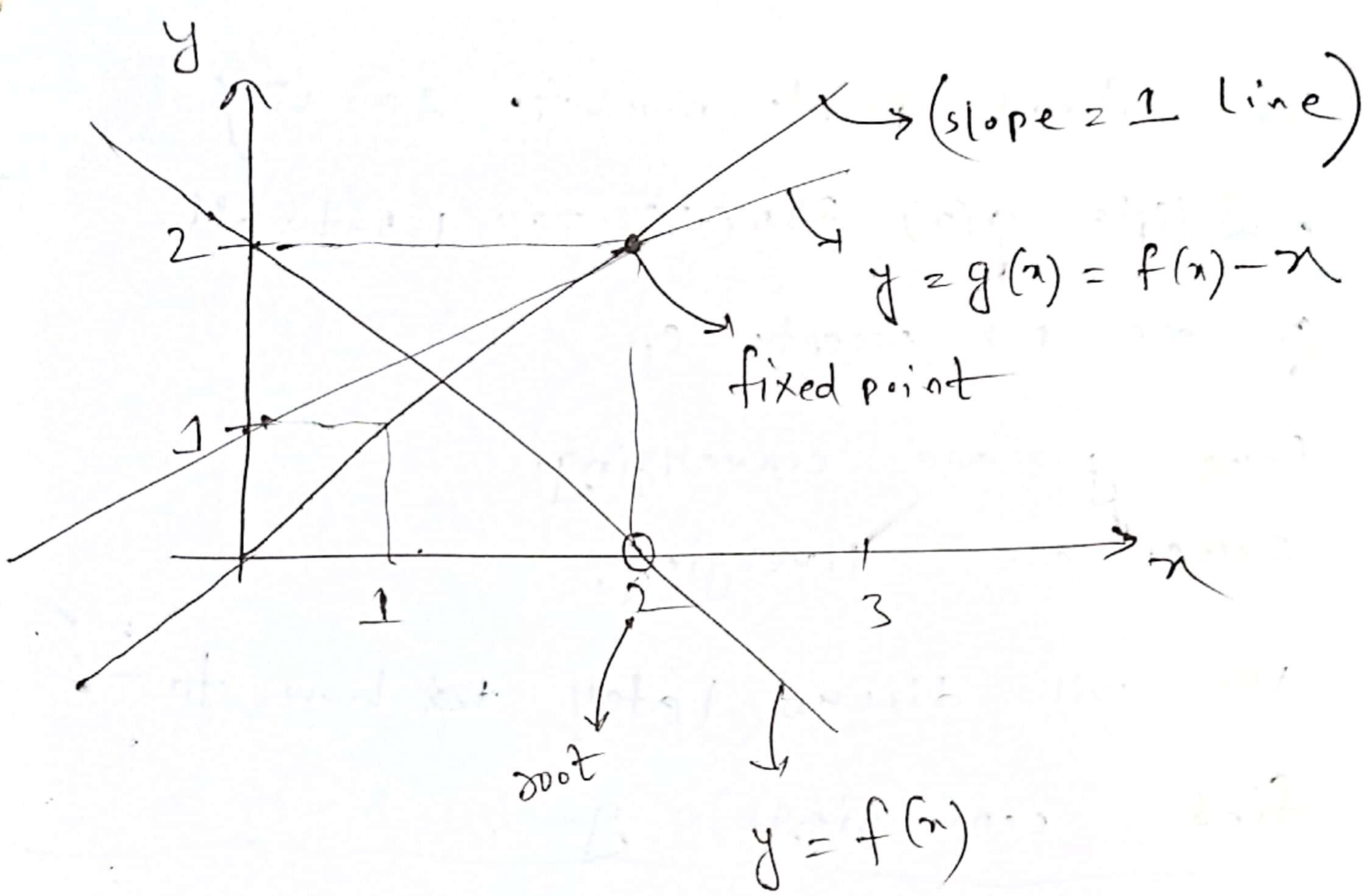
Ex: $g(x) = \dfrac{x+2}{2}$

$$\left.\begin{aligned} g(0) &= 1 \neq 0 \\ g(1) &= 1.5 \neq 1 \end{aligned}\right\} \text{not fixed point}$$

$g(2) = 2 = 2 \longrightarrow$ $x=2$ is a fixed point

$\therefore f(x) = g(x) - x = \dfrac{x+2}{2} - x = -\frac{1}{2}x + 1$

has a root at $x=2$.



Iteration process:

→ We start with an arbitrary initial point $x_0 \in I = [a, b]$

→ Compute $g(x_0)$ (1st iteration) → $K = 0$

if $g(x_0) = x_0$, it is a fixed point. done

if not, define $x_1 = g(x_0)$

$|x_1 - x_0| >$ error bound

$\downarrow$

x_0 is not my solution. iteration continues

→ Compute $g(x_1)$ (2nd iteration) → $k=1$

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if $g(x_1) = x_1$, it's a fixed point. done.

if not, define: $x_2 = g(x_1)$.

$|x_2 - x_1| >$ error bound

we repeat the process until we find difference of x less than error bound.

$\frac{\text{difference of } x}{\text{error}}$

After k-th iteration,

$$x_{k+1} = g(x_k)$$

if $|x_{k+1} - x_k| \leq \delta$, then we take

$$|x_{k+1} - x_k| \approx 0.$$

so, x_k is the fixed point of $g(x)$, and $x_k = x_k$ is the root of the function $f(x)$ within error bound δ.

Ex: $f(x) = x^2 - 2x - 3 = 0$

$$\boxed{\begin{aligned}(x-3)(x+1) &= 0\\ x_k &= -1, 3\end{aligned}}$$

exact soln/roots

we will try to find these roots numerically/approximately

Rewrite $f(x)=0$ in terms of $g(x)=x$

1) $x^2 - 2x - 3 = 0$

$$x = \sqrt{2x+3} = g(x)$$

2) $x^2 - 2x - 3 = 0$

$$x(x-2) = 3$$

$$x = \frac{3}{x-2} = g(x)$$

3) $x^2 - 2x - 3 = 0$

$$x = x^2 - x - 3 = g(x)$$

4) $x^2 - 2x - 3 = 0$

$$2x^2 - 2x = x^2 + 3$$

$$x = \frac{x^2+3}{2x-2} = g(x)$$

Ex: $f(x) = x^2 - 2x - 3 = 0$

$$\boxed{\begin{gathered}(x-3)(x+1) = 0 \\ x_k = -1, 3\end{gathered}}$$

exact sol^n/roots

we will try to find these roots numerically/approximately

Rewrite $f(x) = 0$ in terms of $g(x) = x$

1) $x^2 - 2x - 3 = 0$

$$x = \sqrt{2x+3} = g(x)$$

2) $x^2 - 2x - 3 = 0$

$$x(x-2) = 3$$

$$x = \frac{3}{x-2} = g(x)$$

3) $x^2 - 2x - 3 = 0$

$$x = x^2 - x - 3 = g(x)$$

4) $x^2 - 2x - 3 = 0$

$$2x^2 - 2x = x^2 + 3$$

$$x = \frac{x^2+3}{2x-2} = g(x)$$

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Note that, all four form of $g(x)$ above satisfy $g(-1) = -1$

$g(3) = 3$

$\therefore$ $-1, 3$ are fixed point numbers

$\rightarrow$ they are roots

But when we start the iteration with $x_0 \neq -1, 3$ the fixed point can't be obtained always.

Let's consider the first case with $x_0 = 0$

$g(x) = \sqrt{2x+3}$ upto 3 sig. fig:

$g(0) = 1.73$

$g(1.73) = 2.54$

$g(2.54) = 2.84$

$g(2.84) = 2.95$

$g(2.95) = 2.98$

$g(2.98) = 2.99$

$g(2.99) = 3.00$

$g(\underset{x_7}{3.00}) = \overset{x_8}{3.00}$

After 7th iteration,

$x_8 - x_7 \approx 0$

$\leq \delta(10^{-3})$

Converging at fixed point

So, $x_7 = 3.00$ is the fixed point of $g(x)$ and it is also the root of $f(x)$.

Let's take 3rd expression: $g(x) = x^2 - x - 3$

(diverging)

$g(0) = -3.00$

$g(-3) = 9.00$

$g(9) = 69.0$

$g(69) = 4.669 \times 10^3$

as iteration number increases, $g(x)$ increases indefinitely

So, the iteration doesn't converge to a single value/fixed point number.

Let's take 4th expression: $g(x) = \frac{x^2+3}{2x-2}$

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start from $x_0 = 0$

$g(0) = -1.50$

$g(-1.50) = -1.05$

$g(-1.05) = -1.00$

$g(-1.00) = -1.00$

After 3rd iteration: $x_4 - x_3 \approx 0$

$< \delta\ (10^{-3})$

So, $x_3 = -1.00$ is the fixed point of $g(x)$ and it is also the root of $f(x)$.

\# We have to ~~fin~~ assume proper x_0 and $g(x) = x$ in this method for faster convergence to the solution.

Two unresolved issues:

a) how to choose x_0 to start iteration

b) which form of $g(x)$ is convergent

Both are answered by (Contraction Mapping Th^m)

* $f(x) = x^2 - 2x - 3$

$$g(x) = \begin{cases} \sqrt{2x+3} \\ x^2 - x - 3 \\ \dfrac{x^2+3}{2x-2} \end{cases}$$

* for first case, $g(x) = \sqrt{2x+3}$

if $x_0 = 0$

$g(0) = 1.73$ } 1.21

$g(1.73) = 2.54$ } 0.3

$g(2.54) = 2.84$ } 0.11

$g(2.84) = 2.95$ } 0.03

$g(2.95) = 2.98$ } 0.01

$g(2.98) = 2.99$ } 0.01

$g(2.99) = 3.00$

if $x_0 = 42$ ~~4.68~~

$g(42) = 9.33$ } ~~1.14~~ 4.68

$g(9.33) = 4.65$ } ~~0.340~~ 1.14

$g(4.65) = 3.51$ } ~~0.110~~ 0.340

$g(3.51) = 3.17$ } 0.110

$g(3.17) = 3.06$ } 0.04

$g(3.06) = 3.02$ } 0.01

$g(3.02) = 3.01$ } 0.01

$g(3.01) = 3.00$

For both values of x_0, the function $g(x)$ converges to a root 3.00

Note: even though $x_0 = 0$ is nearer to root -1, the iteration converge to 3, not -1.

the reason is that ratio λ is less than 1 requires that $x > -1$

for 2nd case of $g(x) = x^2 - x - 3$

$g(0) = -3.00$	$g(42) = 1.72 \times 10^3$
$g(-3) = 9.00$	$g(1.72 \times 10^3) = 2.95 \times 10^6$
$g(9) = 69$	$g(2.95 \times 10^6) = 8.72 \times 10^{12}$
$g(69) = 4.69 \times 10^3$	

for both choice for x_0, $g(x)$ diverges very rapidly.

It is not possible to obtain a fixed point in this case.

for 3rd case of $g(x) = \frac{x^2+3}{2x-2}$

$g(0) = -1.50$	$g(42) = 21.6$
$g(-1.5) = -1.05$	$g(21.6) = 11.4$
$g(-1.05) = -1.00$	$g(11.4) = 6.39$
	$g(6.39) = 4.07$
	$g(4.07) = 3.19$
	$g(3.19) = 3.01$
	$g(3.01) = 3.00$

Now $g(x)$ converges to other fixed point.

$x_0 = 0$ converges to nearest root $x = -1$

$x_0 = 42$ " " " " $x = 3$

If $g(x)$ is convergent, then it converges to the nearest fixed point from x_0.

for the convergent $g(x)$, the difference between successive iterated values are decreasing.

ratio, $\lambda = \left| \frac{\Delta g(x_k)}{\Delta x} \right| = \left| \frac{g(x_{k+1}) - g(x_k)}{x_{k+1} - x_k} \right| < 1$

Contraction Mapping Th^m:

if g is a contraction mapping on $L = [a, b]$ then,

1. There exists a unique fixed point $x_* \in L$ with $g(x_*) = x_*$

2. For any $x_* \in L$, the iteration $x_{k+1} = g(x_k)$ will converge to x_* as $k \to \infty$

distance before mapping $|y - x|$

" after mapping $|g(y) - g(x)|$

Since g is converging,

$$|g(y) - g(x)| \leq |y - x|$$

$$|g(y) - g(x)| = \underbrace{\lambda}_{(<1)} |y - x|$$

$\lambda \rightarrow$ real number (fraction)
$\hookrightarrow$ (convergence rate/ratio)

After k-th iteration,

$$|g(y_k) - g(x_k)| = \lambda^{k+1} |y - x| \rightarrow 0$$

as $k \rightarrow \infty$

If x and y are both fixed points,

$$\lambda = 1$$

otherwise $\lambda < 1$

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$$\frac{|g(y) - g(x)|}{|y - x|} < 1$$

Let $x \rightarrow$ fixed point
$y \rightarrow$ nearer to x

as $K \rightarrow \infty$, $|y - x| \rightarrow 0$, we can write

$$\lambda = \lim_{y \rightarrow x} \frac{|g(y) - g(x)|}{|y - x|} = |g'(x)| < 1$$

So, if derivative of g is less than one at fixed point, then iteration starting at any point near to the fixed point will converge to the fixed point.

Let's apply this :

1. for $g(x) = \sqrt{2x+3}$

$$\lambda = \left|\frac{dg}{dx}\right| = \left|\frac{1}{2} \cdot \frac{2}{\sqrt{2x+3}}\right| = \left|\frac{1}{\sqrt{2x+3}}\right|$$

$$\lambda\big|_{x=3} = \frac{1}{3} < 1$$

Also, for $x > -1$, $\lambda < 1$.

That's why, this converging to 3 for both values of x_0

2. for $g(x) = x^2 - x - 3$

$$\lambda = \left|\frac{dg}{dx}\right| = |2x - 1|$$

$$\left.\begin{array}{l} \lambda\big|_{x=3} = 5 > 1 \\ \lambda\big|_{x=-1} = 3 > 1 \end{array}\right\} \text{so, this diverges}$$

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3. for $g(x) = \dfrac{x^2+3}{2x-2}$

$$\lambda = \left|\frac{dg}{dx}\right| = \left|\frac{x^2-2x-3}{2(x-1)^2}\right|$$

$\lambda\big|_{x=-1} = 0 < 1 \rightarrow$ super Linear

$\lambda\big|_{x=3} = 0 < 1 \rightarrow$ super Linear

Convergence

since, $x_0 = 0$ is closer to -1, the iteration converged to $\underline{-1}$

Similarly $x_0 = 42$ is closer to 3, the iteration converged to 3.

Ex: $f(x) = x^3 - 2x^2 - x + 2$

a) State the roots of $f(x)$

b) Construct 3 different fixed point func $g(x)$ such that $f(x) = 0$

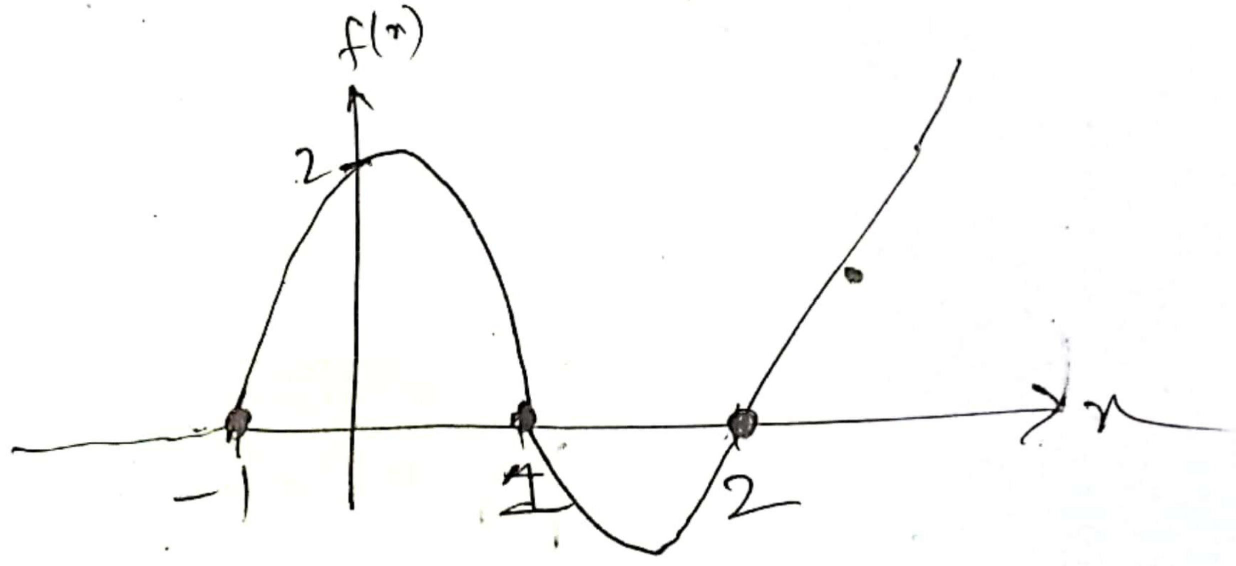
c. find the convergence rate for $g(x)$ constructed in prev part and which root it is converging to?

Solution: a) $x^3 - 2x^2 - x + 2 = 0$

$$x^2(x-2) - 1(x-2) = 0$$

$$(x-2)(x^2-1) = 0$$

$$x_* = -1, 1, 2 \longrightarrow \text{actual roots}$$



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b) $x^3 - 2x^2 - x + 2 = 0$

$$\underbrace{x^3 - 2x^2 + 2}_{g(x)} = x \longrightarrow 1$$

$$x^3 - 2x^2 - x + 2 = 0$$

$$x(x^2 - 2x - 1) = -2$$

$$x = \underbrace{\boxed{\frac{-2}{x^2 - 2x - 1}}}_{g(x)} \longrightarrow 2$$

$$x^3 - 2x^2 - x + 2 = 0$$

$$x^3 - x + 2 = 2x^2$$

$$x = \sqrt{\frac{x^3 - x + 2}{2}}$$

$$x = \underbrace{\frac{1}{\sqrt{2}} \sqrt{x^3 - x + 2}}_{g(x)} \longrightarrow 3$$

(c) $\lambda = g'(x_*) = \left| \left. \frac{dg}{dx} \right|_{x = x_*} \right|$

1st case : $g(x) = x^3 - 2x^2 + 2$

$\Rightarrow g'(x) = 3x^2 - 4x$

$\Rightarrow \lambda \Big|_{x=-1} = 7 \longrightarrow$ divergence as $\lambda \geq 1$

$\lambda \Big|_{x=1} = |-1| = 1 \longrightarrow$ divergence

$\lambda \Big|_{x=2} = 4 \longrightarrow$ divergence

$$\lambda = |g'(x_*)| = \begin{cases} 7 & \text{for } x_* = -1 \\ 1 & \text{"} \quad x_* = 1 \\ 4 & \text{"} \quad x_* = 2 \end{cases}$$

as we need $\lambda < 1$ for convergence

~~all for~~ we don't have any converging point in 1st case.

<u>2nd case:</u> $g(x) = -\dfrac{2}{x^2 - 2x - 1}$ Date ....../....../...........

$$g'(x) = \frac{4(x-1)}{(x^2-2x-1)^2}$$

$$\lambda = |g'(x_k)| = \begin{cases} 2 \ (>1) \text{ for } x_k = -1 \text{ (divergence)} \\ 0 \ (<1) \text{ for } x_k = 1 \text{ (superlinear)} \\ 4 \ (>1) \text{ for } x_k = 2 \text{ (divergence)} \end{cases}$$

$g(x)$ is converging to $x_k = 1$ for 2nd choice of $g(x)$.

<u>3rd case:</u> $g(x) = \dfrac{1}{\sqrt{2}}(x^3 - x + 2)^{\frac{1}{2}}$

$$g'(x) = \frac{3x^2 - 1}{2\sqrt{2}\,(x^3 - x + 2)^{\frac{1}{2}}}$$

$$\lambda = |g'(x_k)| = \begin{cases} 0.5 \ (<1) \rightarrow \text{linear convergence for } x_k = 1 \\ 0.5 \ (<1) \rightarrow \text{linear convergence for } x_k = -1 \\ 1.375 \ (>1) \rightarrow \text{diverges for } x_k = 2 \end{cases}$$

Order of Convergence:

$$g(x) = \sqrt{2x+3} \quad (\text{fixed point } x_* = 3)$$

x_k	$3 - x_k$	$\dfrac{\lvert 3 - x_k \rvert}{\lvert 3 - x_{k-1} \rvert}$
0.00	3.00	—
1.73	1.27	0.423
2.54	0.458	0.361
2.84	0.1567	0.342
2.95	0.053	0.336
2.98	0.018	0.334
⋮	⋮	⋮
3.00	0.00	$\frac{1}{3}$

λ

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$\lambda = 0 \rightarrow$ fast/superlinear convergence

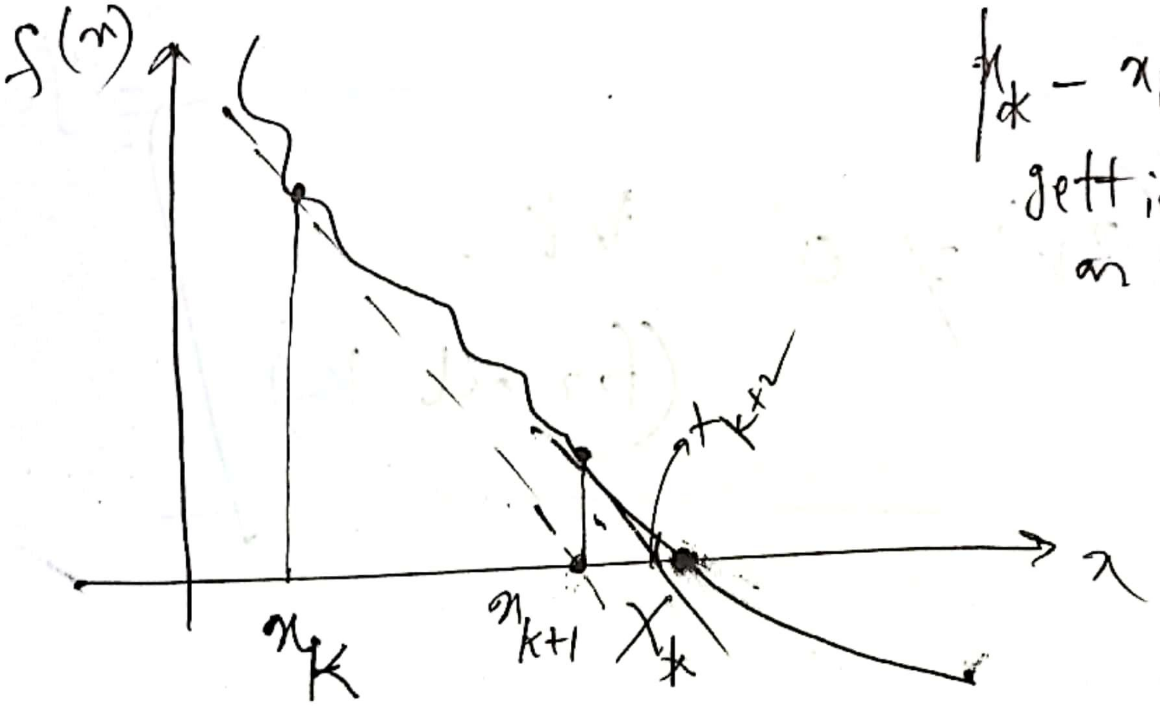
$0 < \lambda < 1 \longrightarrow$ Linear " (not so fast)

$\lambda = 1 \Rightarrow$ Points are fixed $\longrightarrow$ numerical not needed.

$\lambda > 1 \longrightarrow$ divergence.

Newton's Method :

It's a fixed point method with superlinear convergence.



$|x_* - x_K|$ is getting smaller as $k \rightarrow \infty$

x_k and x_{k+1} are on same tangent Line.

$$\text{slope} = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} = -\frac{f(x_k)}{x_{k+1} - x_k}$$

$$f(x_{k+1}) = 0$$

because x_{k+1} is the x intercept

$$f'(x_k) = -\frac{f(x_k)}{x_{k+1} - x_k}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \equiv g(x_k)$$

$$\boxed{f'(x_k) \neq 0 \quad \forall k \quad \text{(for all } k\text{)}}$$

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$x_k \to x_*$ 2(m $f(x_k) \to 0$

Rate of convergence,

$$\lambda = g'(x) = \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right)$$

$$= 1 - \frac{f'(x).f'(x) - f(x).f''(x)}{(f'(x))^2}$$

$$= \frac{f(x).f''(x)}{(f'(x))^2}$$

$$\lambda \equiv g'(x_*) = \left.\frac{f(x).f''(x)}{(f'(x))^2}\right|_{x = x_*}$$

$$= \frac{f(x_*).f''(x_*)}{(f'(x_*))^2} = 0$$

as $\lambda = 0 \to$ superlinear convergence.

using $x_* = 2$

$\alpha = 1$ (Linear convergence)

order of convergence

x_k	$\lvert 2 - x_k \rvert$	$\lambda = \dfrac{\lvert 2 - x_k \rvert}{\lvert 2 - x_{k-1} \rvert}$	$\dfrac{\lvert 2 - x_k \rvert}{\lvert 2 - x_{k-1} \rvert^2}$
1.0	1.0	—	0.5
1.5	0.5	0.5	0.5
1.875	0.125	0.25	0.5
1.9922	0.00781	0.0625	0.5
1.9999	3.05×10^{-5}	0.00391	0.5
2.0	4.6566×10^{-10}	1.52587×10^{-5}	0.5
2.0	1.0842×10^{-19}	2.32839×10^{-5}	0.5

after 6 iteration error is below ϵ_M

so, iteration stops

$\alpha = 2$ (order of convergence) → 4th column

convergence is quadratic/

Newton's Method

Example:

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$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

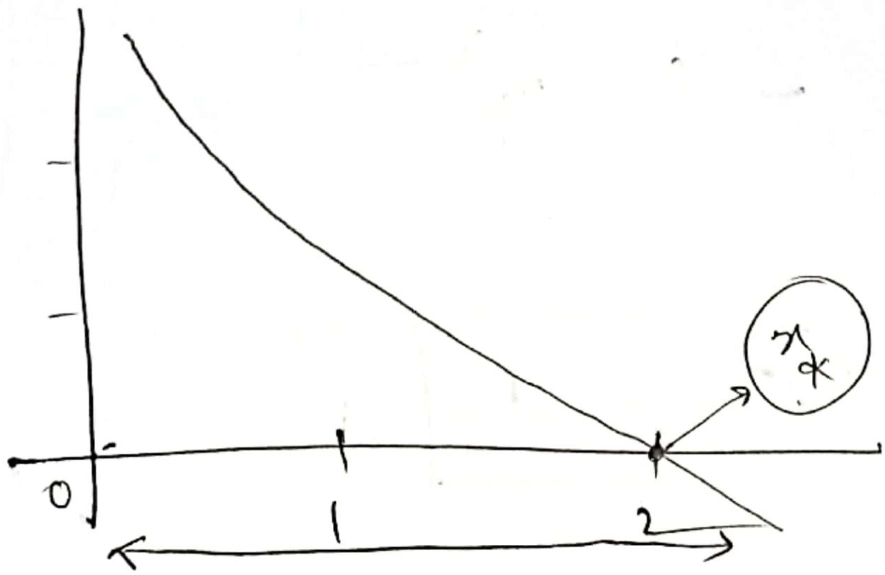
$f(x) = \frac{1}{x} - a$; $\boxed{a > 0}$

Clearly, $f(x) = 0$ [illegible] $\frac{1}{x} - a = 0$

$$x = \frac{1}{a}$$

actual root $\leftarrow$ $\therefore x^* = \frac{1}{a}$ is the root of $f(x)$.

$a = 0.5$ $\quad \therefore x^* = \frac{1}{0.5} = 2$



$\boxed{x_0 \in [0, 2]}$

starting x

if $x_0 \in \left(0, \frac{1}{a}\right) = (0, 2)$

iteration will converge.

if x_0 is too large $\longrightarrow$ diverge

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{\frac{1}{x_k} - 0.5}{\frac{d}{dx}\left(\frac{1}{x_k} - 0.5\right)}$$

$$x_{k+1} = 2x_k - 0.5\, x_k^2$$

let's assume $\boxed{x_0 = 1}$

using $x_* = 2$

x_k	$\|2-x_k\|$	$\alpha = 1$ (linear convergence) λ: $\frac{\|2-x_k\|}{\|2-x_{k-1}\|}$	order of convergence $\frac{\|2-x_k\|}{\|2-x_{k-1}\|^2}$
1.0	1.0	—	0.5
1.5	0.5	0.5	0.5
1.875	0.125	0.25	0.5
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after 6 iteration error is below E_M so, iteration stops

$\alpha = 2$ (order of convergence) → 4th column

convergence is quadratic/

for $h(x)$ if $h'(x) = 0$ $\longrightarrow \frac{\Delta h}{\Delta x}$

means $\rightarrow$ $\Delta h \longrightarrow 0$ faster than

$\Delta x \longrightarrow 0$

This is superlinear convergence.

convergence rate, $\lambda = |g'(x_*)|$

$$= \left| \lim_{x_k \to x_*} \frac{g(x_k) - g(x_*)}{x_k - x_*} \right|$$

This is convergence of order one

(linear/super linear)

we diefine order of convergence, α

লামডা

$$\lambda = |g'(x_*)| = \left| \lim_{x_k \to x_*} \frac{g(x_k) - g(x_*)}{(x_k - x_*)^{\alpha}} \right|$$

α can be fraction also $<$ infinity

$\dfrac{\Delta h}{\Delta x}$

for $h(x)$ if $h'(x) = 0$

means $\rightarrow$ $\Delta h \rightarrow 0$ faster than $\Delta x \rightarrow 0$

This is <u>superlinear convergence</u>.

convergence rate, $\lambda = |g'(x_*)|$

$$= \left| \lim_{x_k \to x_*} \frac{g(x_k) - g(x_*)}{x_k - x_*} \right|$$

This is convergence of <u>order one</u>

(linear/super linear)

we define <u>order of convergence</u>, α $\downarrow$ আলফা

$$\lambda = |g'(x_*)| = \left| \lim_{x_k \to x_*} \frac{g(x_k) - g(x_*)}{(x_k - x_*)^{\alpha}} \right|$$

<u>α can be fraction also</u> $<$ infinity

* There is one more fact we need to be careful about.

as $f'(x_k) \neq 0$, we need to take extra precaution to choose initial point x_0. otherwise, we might run into an <u>infinite loop</u>.

Caution: turning point of $f(x)$ should not fall betw^n x_k & x_{k+1}

* $x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_k \longrightarrow x_{k+1} \longrightarrow \cdots$

if turning point of $f(x)$ falls between x_k and x_{k+1}, then

$f'(x_k)$ and $f'(x_{k+1})$ will have opposite signs.

So, Instead of converging to a fixed point/root, the iteration will <u>fluctuate</u> around the turning point indefinitely.

* There is one more fact we need to be careful about.

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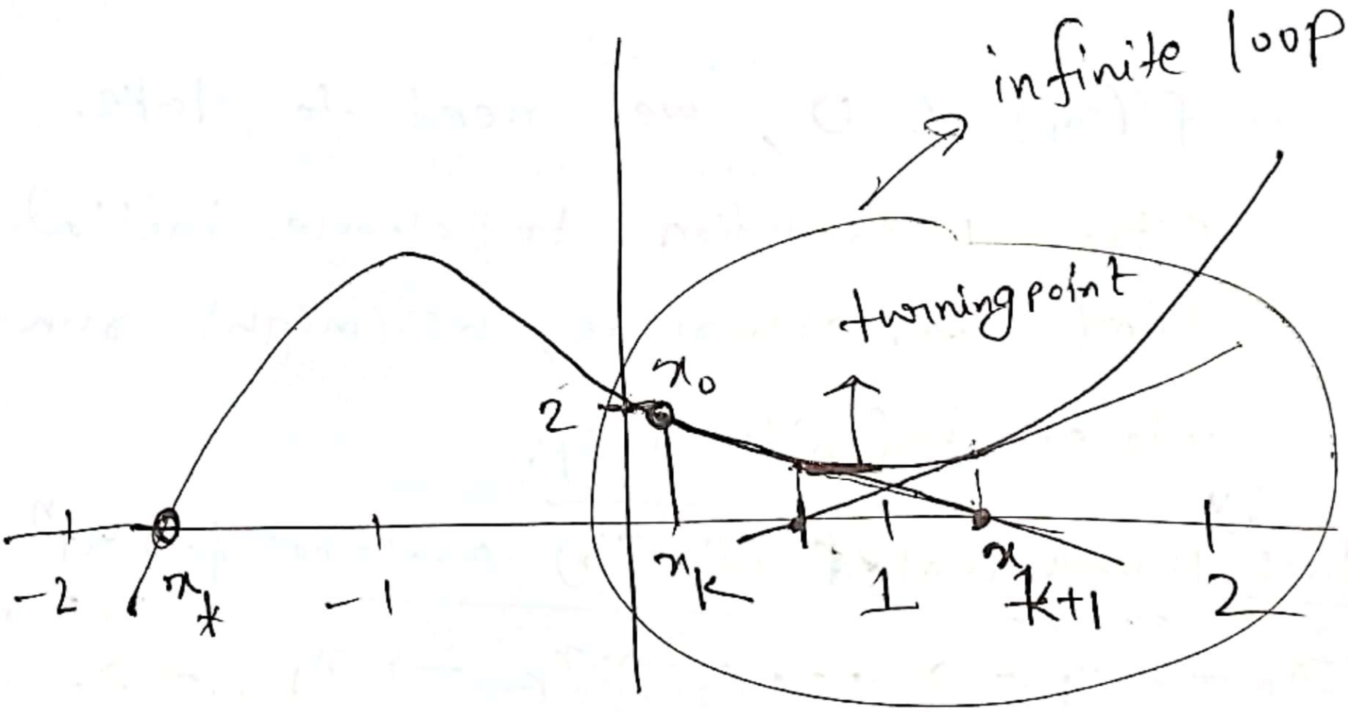
if turning point of $f(x)$ falls between x_K and x_{K+1}, then $f'(x_K)$ and $f'(x_{K+1})$ will have opposite signs.

So, Instead of converging to a fixed point/root, the iteration will fluctuate around the turning point indefinitely.

$f(x) = x^3 - 2x + 2$



turning point is betn x_k and x_{k+1}

So, here is an infinite loop.

$x_0 = 0$ or -0.5 lead to an infinite loop.

To find correct x_0, we need to find turning points. (critical points)

These are:

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$$f'(x) = 0$$

$$3x^2 - 2 = 0$$

$$x = \pm\sqrt{\frac{2}{3}}$$

and $x_0 > \sqrt{\frac{2}{3}}$

Clearly, $x_0 < -\sqrt{\frac{2}{3}} \approx -0.81649$, otherwise
↳ should be   infinite loop

if we choose $x_0 = 0$

$$x_1 = 0 - \frac{f(0)}{f'(0)} = 1$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 0$$

Loop

$\boxed{\text{infinite loop}}$

because turning point $\sqrt{\frac{2}{3}} = 0.81649$

lies between 0 and 1.

if $x_0 = -1$ (< -0.816)

converges to fixed point

$$x_* = -1.769$$

Summary: Not all value of x_0 lead to converging iteration even though x_0 is close to x_*.

we have to ensure that turning point doesn't lie betn $\underbrace{x_k \text{ and } x_{k+1}}$

(successive iterated points)

Newton's Method for Systems

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Jacobian matrix, $J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$

$$x_{k+1} = x_k - J^{-1}(x_k) \cdot f(x_k)$$

Ex: $xy - y^3 - 1 = 0$

$x^2 y + y - 5 = 0$

$x_0 = 2, \; y_0 = 3$

$$J = \begin{pmatrix} y & x - 3y^2 \\ 2xy & x^2 + 1 \end{pmatrix}$$

$$J^{-1}(x, y) = \frac{1}{y(x^2+1) - 2xy(x - 3y^2)} \begin{pmatrix} x^2+1 & -x+3y^2 \\ -2xy & y \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J^{-1}(x_0, y_0) \cdot f(x_0, y_0)$$

$$\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} - J^{-1}(x_4, y_4) \cdot f(x_4, y_4)$$

$$= \begin{pmatrix} 2.00 \\ 1.00 \end{pmatrix}$$

Aitken
~~Atiken~~ Acceleration:

We take 3 nearest iteration points

$$x_k,\ x_{k+1},\ x_{k+2}$$

$$\lambda = \frac{x_* - x_{k+2}}{x_* - x_{k+1}} = \frac{x_* - x_{k+1}}{x_* - x_k}$$

$$(x_* - x_{k+2})(x_* - x_k) = (x_* - x_{k+1})^2$$

$$x_* = \frac{x_{k+2} \cdot x_k - x_{k+1}^2}{x_{k+2} - 2x_{k+1} + x_k}$$

$$x_* = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

$$\hat{x}_{k+2} = \left(x_k - \frac{(\Delta x_k)^2}{\Delta x_{k+1} - \Delta x_k}\right)$$

accelerated value $= \left(x_k - \frac{(\Delta x_k)^2}{\Delta(\Delta x_k)}\right)$

$$\hat{x}_{k+2} = x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k}$$

$x_0 \to x_1 \to x_2 \xrightarrow{\text{accelerated}} \hat{x}_2 \to$ same

$x_2 \to x_3 \to x_4 \xrightarrow{\text{accelerated}} \hat{x}_4$

$\Downarrow$

$x_4 \to x_5 \to \hat{x}_6$

Next Example এ,

Fixed Point Iteration Method কে fast converging করতে Aitken acceleration use করবো।

Example: $f(x) = \frac{1}{x} - 0.5$

$x_* = 2$

$$x = g(x) = x + \frac{1}{16} f(x)$$

$$= x + \frac{1}{16}\left(\frac{1}{x} - 0.5\right)$$

$g(x) = x$ কেন?

এমনে হবে যেন

$f(x) = 0$ হয়

$x = \boxed{x + \frac{1}{16} f(x)} \to g(x)$

$\Rightarrow f(x) = 0$

$g(2) = 2 \longrightarrow$ fixed point of $g(x)$

$$\lambda = |g'(x_*)| = \left.\frac{dg}{dx}\right|_{x = x_*}$$

$$= 1 + \frac{1}{16}\left(-\frac{1}{x^2}\right)\Big|_{x=2}$$

$$= 0.9843 < 1$$

linear convergence

but slowly. So acceleration is needed.

Without acceleration করলে $\longrightarrow$

$x_0 = 1.5$

$x_1 = g(x_0) = 1.510417 \qquad |x_1 - x_*| = 0.4895$

$x_2 = g(x_1) = 1.5205 \qquad |x_2 - x_*| = 0.479$

$\vdots$

$x_{818} = g(x_{817}) = 1.999 \qquad |x_{818} - x_*| = 0.00$

(818 টা iteration is impossible to do)

So, we apply ~~aitken~~ acceleration
aitken

aitken :-

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$x_0 = 1.5$

$x_1 = g(x_0) = 1.510417 \Rightarrow |x_1 - x_*| = 0.48958$

$x_2 = g(x_1) = 1.520554 \Rightarrow |x_2 - x_*| = 0.479445$

$$\hat{x}_2 = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0}$$

$= 1.877604 \Rightarrow |\hat{x}_2 - x_*| = 0.122396$

$x_3 = g(\hat{x}_2)$

$x_4 = g(x_3)$

$$\hat{x}_4 = \hat{x}_2 - \frac{(x_3 - \hat{x}_2)^2}{x_4 - 2x_3 + \hat{x}_2} \Rightarrow |\hat{x}_4 - x_*| = 0.007366$$

$= 1.992634$

$\hat{x}_8 = g(x_7) = 2.00 \Rightarrow |x_8 - x_*| = 0.0000$

(root)

After 8th iteration, we can get answer.

Secant Method : (one kind of Quasi-Newton Method)

Recall Newton's method of finding root →

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

we have to iteratively check ag ain and again if $f'(x_k) \neq 0$

if $f'(x_k)$ is too small, loss of significance can occur.

So, it is a drawback.

⇒ To avoid drawback, we replace $f'(x_k)$ by an easily computable function g_k which is approximately equal to $f'(x_k)$.

This technique is known as Quasi-Newton Method.

⇒ If we choose g_k to be backward difference betn nearest points, it is called Secant method.

The backward difference for the nearest iteration points is:

$$g_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

So, iteration formula for secant method

$$x_{k+1} = x_k - \frac{f(x_k) \cdot (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

Clearly, iteration starts at x_0 and x_1

$\therefore$ 1st iterated point x_2.

advantage: only one function is evaluated per iteration, Just $f(x)$

$\boxed{\text{no need for derivative of } f(x)}$

Note: Newton এ $f(x)$ ও $f'(x)$ both লাগে।

Secant/Quasi-newton এ Just $f(x)$ লাগে।

__Example:__ $f(x) = \frac{1}{x} - 0.5$

$x_0 = 0.25, \quad x_1 = 0.5$

K	x_K	$\frac{\lvert x_* - x_K \rvert}{\lvert x_* - x_{K-1} \rvert} = \lambda$
2	0.6875	0.75
3	1.01562	0.75
4	1.3540	0.65625
5	1.68205	0.4921
6	1.8973	0.32229
7	1.98367	0.15897
8	1.99916	0.05134
⋮	⋮	⋮
12	2.0	0.00000

Here, convergence is very fast.
and in 12 Iteration, ϵ_m is achieved
as $\lambda \to 0 \to$ superlinear convergence.
$\hookrightarrow$ decreasing

$f(x) = x^2 - 2xe^{-x} + e^{-2x}$. Find solution of $f(x) = 0$ within 10^{-5} by using

a) Newton's method

b) By using ~~Atiken~~ Aitken acceleration (discuss later)

Note: i) find root x_*

↓

if $|x_* - x_k| < 10^{-5}$, then x_k is the root

ii) find solution of $f(x) = 0$

↓

if $f(x_k) < 10^{-5}$, then x_k is the solution of $f(x) = 0$

When root is unknow/irrational

Sol^n: a) $f'(x) = 2x - 2e^{-x} - 2x(-e^{-x}) - 2e^{-2x}$

Newton's method, (iteration formula)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow x_{k+1} = x_k - \frac{x_k^2 - 2x_k e^{-x_k} + e^{-2x_k}}{2x_k - 2e^{-x_k} + 2x_k e^{-x_k} - 2e^{-2x_k}}$$

(k = iteration number)

Starting point : $x_0 = 1$ ($k=0$)

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k	x_k	$f(x_k)$	if $f(x_k) < 10^{-5}$
0	1	0.39958	No
1	0.76894	0.093292	NO
2	0.66459	0.022532	No
3	0.615033	0.005537	NO
4	0.590884	0.001372	NO
5	0.37896	0.000342	NO
6	0.57304	0.000085	NO
7	0.57008	2×10^{-5}	NO
8	0.568615 (Ans)	0.5×10^{-5} $(< 10^{-5})$	yes ✓

$f(x_8) \approx 0$ (Ans: ~~[illegible]~~ 0.568615)

b) Aitken Acceleration:

$$f(x_k) = x_k^2 - 2x_k e^{-x_k} + e^{-2x_k} \quad \text{—— (1)}$$

$$x_{k+1} = x_k - \frac{x_k^2 - 2x_k e^{-x_k} + e^{-2x_k}}{2x_k - 2e^{-x_k} + 2x_k e^{-x_k} - 2e^{-2x_k}} \quad \text{—— (2)}$$

for acceleration,

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k} \quad \text{—— (3)}$$

$x_0 = 1$

$$x_0 \xrightarrow{(1)} f(x_0) \xrightarrow{(2)} x_1 \xrightarrow{(1)} f(x_1) \xrightarrow{(2)} x_2 \xrightarrow{(1)} f(x_2) \xrightarrow{(3)} \hat{x}_2 \xrightarrow{(1)} f(\hat{x}_2) \xrightarrow{(2)} x_3 \rightarrow x_4 \rightarrow \hat{x}_4 \ldots$$

$\boxed{\text{per 3rd point, we accelerate}}$

k	x_k	$f(x_k)$	if $f(x_k) < 10^{-5}$
0	1	0.3995	NO
1	0.7189	0.09329	NO
2	0.6645	0.022532	NO
$\hat{2}$	0.5786	3.2×10^{-4}	NO
3	0.57028	8×10^{-5}	NO
4	0.57001	2×10^{-5}	NO
$\hat{4}$	0.56715 (Ans)	2.8×10^{-10} ($<< 10^{-5}$)	Yes

Ans if $f(x) = 0$

$\boxed{\text{is } 0.567154}$

$f(\hat{x}_4) \approx 0$