Least Square Approximation:

→ Well defined linear system has equal number of variables & equations.

Example:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 9x_2 + 7x_3 &= 2 \\ 2x_1 + 3x_2 + 5x_3 &= 5 \end{aligned} \rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -9 & 7 \\ 2 & 3 & 5 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}}_{b}$$

square matrix of $\boxed{n} \times n$ | $n \times \boxed{1} \rightarrow n \times 1$

→ If we have a system where number of equation > number of variables, it is called an <u>over-determined</u> system.

How do we solve over-determined system?

Example:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 9x_2 + 7x_3 &= 2 \\ x_1 + 3x_2 + 5x_3 &= 4 \\ 2x_1 + 11x_2 - 9x_3 &= 5 \\ 9x_1 + x_2 - x_3 &= 7 \end{aligned} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & -9 & 7 \\ 1 & 3 & 5 \\ 2 & 11 & -9 \\ 9 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

$\boxed{m} \times n$ | $n \times \boxed{1}$ | $m \times 1$

$$A \cdot x = b$$

→ Least square approximation method is a way to find an approximate solution of an over-determined system.

$$\underbrace{\overset{\text{over-determined}}{A}}_{(\boxed{m} \times n)} \cdot \underbrace{x}_{(n \times \boxed{1})} = \underbrace{b}_{(m \times 1)}$$

How to solve such problems?

→ multiply A^T on both hand sides.

$$\underbrace{\underbrace{A^T}_{(\boxed{n} \times m)} \; \underbrace{A}_{(m \times \boxed{n})}}_{(n \times n)} \cdot \underbrace{\underbrace{x}_{(n \times 1)}}_{(n \times 1)} = \underbrace{\underbrace{A^T}_{(\boxed{n} \times m)} \cdot \underbrace{b}_{(m \times \boxed{1})}}_{(n \times 1)}$$

Example

From polynomial chapter:

If we had $\underbrace{(n+1)}_{x_0, x_1 \cdots x_n}$ nodes, we calculated the values of $\underbrace{(n+1)\text{ coefficients}}_{a_0, a_1 \cdots a_n}$

using vandermonde matrix.

Well-defined system

$$\begin{bmatrix} 1 & x_0^1 & x_0^2 & \cdots & x_0^n \\ 1 & x_1^1 & x_1^2 & \cdots & x_1^n \\ \vdots \\ 1 & x_n^1 & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

But now, lets say

we have $\underbrace{(m+1)\text{ nodes}}_{x_0, x_1 \cdots x_m}$, but we need to calculate $\underbrace{(n+1)\text{ coefficients}}_{a_0, a_1 \cdots a_n}$

[Remember $m > n$]

over-determined system

$$\begin{bmatrix} 1 & x_0^1 & x_0^2 & \cdots & x_0^n \\ 1 & x_1^1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2^1 & x_2^2 & \cdots & x_2^n \\ \vdots \\ 1 & x_m^1 & x_m^2 & \cdots & x_m^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \end{bmatrix}$$

$$\underbrace{a_0 + a_1 x}_{\text{number of coefficients } = 2}$$

Example:

We want to fit a straight line through the following nodes

$x_0 = -3$ $\quad x_1 = 0$ $\quad x_2 = 6$

$$\underbrace{f(x_0) = 0 \qquad f(x_1) = 0 \qquad f(x_2) = 2}_{\text{number of nodes } = 3}$$

$$\begin{aligned} P_1(x_0) &= a_0 + a_1(x_0) = f(x_0) \rightarrow a_0 + a_1(-3) = 0 \\ P_1(x_1) &= a_0 + a_1(x_1) = f(x_1) \rightarrow a_0 + a_1(0) = 0 \\ P_1(x_2) &= a_0 + a_1(x_2) = f(x_2) \rightarrow a_0 + a_1(6) = 2 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$A \cdot x = b$$

Multiplying A^T on both sides.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

Now apply Gaussian elimination / LU / inverse method to find the values of a_0 and a_1.

<u>Applying inverse method:</u>

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3/7 \\ 5/21 \end{bmatrix}$$

$\therefore a_0 = 3/7, \quad a_1 = 5/21$

$\therefore P_1(x) = \frac{3}{7} + \frac{5}{21} x$