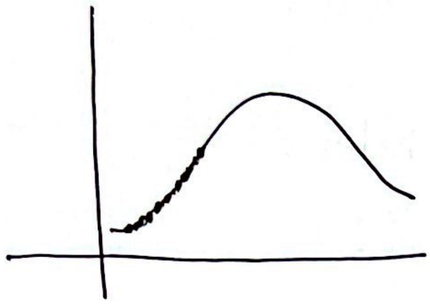
Chebyshev Nodes:

Expected:

if $n \to \infty$, error $\to 0$



But there are some functions which do not show the above properties. Those functions are called "Runge Functions"

Example:

$$f(x) = \frac{1}{1+25x^2} \quad \text{on } [-1, 1]$$

← Runge function

It gives big errors on the corners.

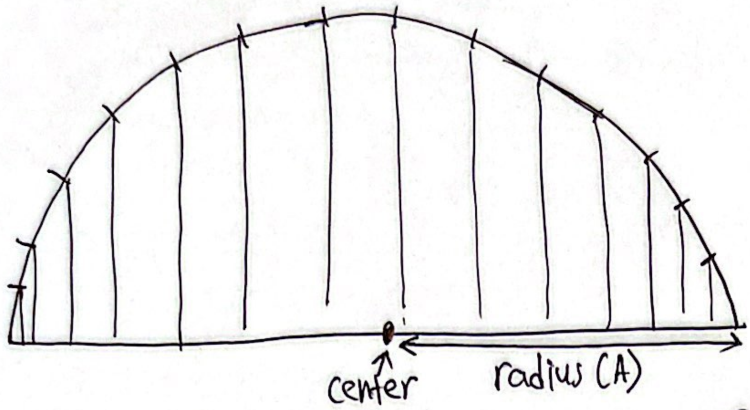
$\boxed{n \to \infty, \quad \text{error} \to \infty}$

↳ when nodes are equally spaced.

Work around:

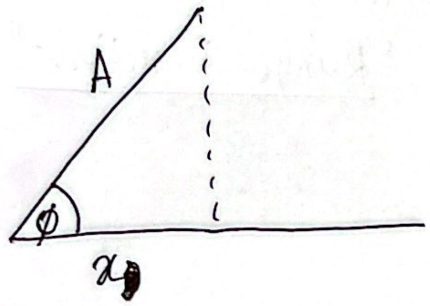
→ Do not take equally spaced nodes

→ Take more nodes at the corner.



→ Take equally spaced angles instead of nodes.

→ Then take their projection on the x-axis.



$$\cos(\phi) = \frac{x_j}{A}$$

$$x_j = A\cos(\phi)$$

$$\boxed{\phi_j = \frac{(2j+1)\pi}{2(n+1)}, \quad j = 0, 1, 2, \dots, n}$$

Formula:

$$x_j = A\cos(\phi_j) + \text{center}$$

$$= A\cos\left[\frac{(2j+1)\pi}{2(n+1)}\right] + \text{center} \qquad j = 0, 1, 2 \ldots, n$$

Example:

Runge function

$$f(x) = \frac{1}{1+25x^2} \quad \text{on the Interval } [-1, 1]$$

The above function is to be interpolated with a polynomial of degree 3.

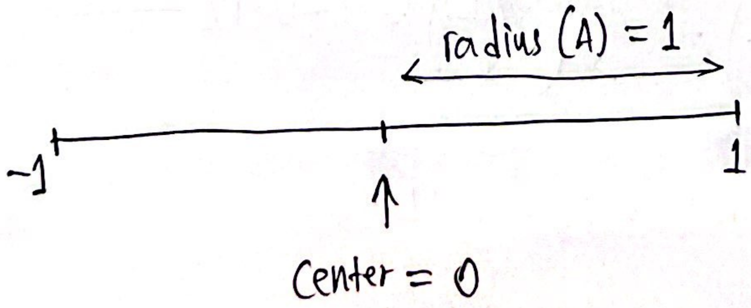
Find suitable nodes with which you would want to perform the polynomial interpolation.

Solution:

degree $(n) = 3$

$\therefore$ number of nodes required $= n+1 = 4$.

Interval: $[-1, 1]$



$$x_j = A\cos(\phi_j) + \text{center}$$

$$= 1 \cdot \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right] + 0$$

$$= \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right]$$

$j = 0, 1 \ldots, n$

$= 0, 1, 2, 3$ ← since $n=3$ in our question.

<u>When $j=0$</u>

$$x_0 = \cos\left(\frac{[2(0)+1]\pi}{2(3+1)}\right) = \cos\left(\frac{\pi}{8}\right) = 0.92$$

<u>When $j=1$</u>

$$x_1 = \cos\left(\frac{[2(1)+1]\pi}{2(3+1)}\right) = \cos\left(\frac{3\pi}{8}\right) = 0.38$$

<u>When $j=2$</u>

$$x_2 = \cos\left(\frac{[2(2)+1]\pi}{2(3+1)}\right) = \cos\left(\frac{5\pi}{8}\right) = -0.38$$

<u>when $j=3$</u>

$$x_3 = \cos\left(\frac{[2(3)+1]\pi}{2(3+1)}\right) = \cos\left(\frac{7\pi}{8}\right) = -0.92$$

$\therefore$ The chebyshev nodes are: $x_0 = 0.92$

$x_1 = 0.38$

$x_2 = -0.38$

$x_3 = -0.92$

Hermitte Interpolation:

Previously only one condition used to be fulfilled:

$$p(x_i) = f(x_i)$$

Now, along with the previous condition, one more condition is to be fulfilled:

$$p'(x_i) = f'(x_i)$$

Previously,

If I am given $(n+1)$ nodes, degree of polynomial was $P_n(x)$

Now, using Hermitte Interpolation:

If I am given $(n+1)$ nodes, degree of polynomial will be $P_{2n+1}(x)$

Using Natural Basis:

$$P_n(x) = \sum_{k=0}^{n} a_k x^k = a_0 x^0 + a_1 x^1 + \cdots a_n x^n$$

Using Lagrange Basis

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + \cdots$$

Using Hermitte Basis:

$$P_{2n+1}(x) = f(x_k)\,\underline{\underline{h_k(x)}} + f'(x_k)\,\underline{\underline{\hat{h}_k(x)}}$$

$$\boxed{\begin{aligned} h_k(x) &= \left[1 - 2(x - x_k)\,l_k'(x_k)\right] l_k^2(x) \\ \hat{h}_k(x) &= (x - x_k)\, l_k^2(x) \end{aligned}}$$

Example:

$f(x) = \sin(x)$

$x_0 = 0 \qquad x_1 = \frac{\pi}{2}$

$f(x_0) = 0 \qquad f(x_1) = 1$

$f'(x_0) = 1 \qquad f'(x_1) = 0$

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f'(x_0) = f'(0) = \cos(0) = 1$

$f'(x_1) = f'(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$

$\rightarrow$ $\left.\begin{aligned} \text{Number of nodes} &= 2 \\ n+1 &= 2 \\ n &= 1 \end{aligned}\right\}$ $P_{2n+1}(x) = P_{2(1)+1}(x) = P_3(x)$

$$P_3(x) = \cancelto{0}{f(x_0)}\, h_0(x) + \cancelto{1}{f'(x_0)}\, \hat{h}_0(x) + \cancelto{1}{f(x_1)}\, h_1(x) + \cancelto{0}{f'(x_1)}\, \hat{h}_1(x)$$

$$P_3(x) = \hat{h}_0(x) + h_1(x)$$

$$h_k(x) = \left[1 - 2(x - x_k)\, l_k'(x_k)\right] l_k^2(x)$$

$$h_1(x) = \left[1 - 2(x - x_1)\, l_1'(x_1)\right] l_1^2(x)$$

$x_0 = 0$, $x_1 = \frac{\pi}{2}$

$\uparrow$ (from question)

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{\frac{\pi}{2} - 0} = \frac{2}{\pi} x$$

$$l_1'(x) = \frac{2}{\pi}$$

$$\therefore h_1(x) = \left[1 - 2(x - x_1)\, l_1'(x_1)\right] l_1^2(x)$$

$$= \left[1 - 2\left(x - \frac{\pi}{2}\right)\left(\frac{2}{\pi}\right)\right]\left[\frac{2}{\pi} x\right]^2$$

$$= \frac{4}{\pi^2} x^2 \left(3 - \frac{4}{\pi} x\right)$$

$$\hat{h}_k(x) = (x - x_k)\, l_k^2(x)$$

$$\hat{h}_0(x) = (x - x_0)\, l_0^2(x)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - \frac{\pi}{2}}{0 - \frac{\pi}{2}} = 1 - \frac{2}{\pi} x$$

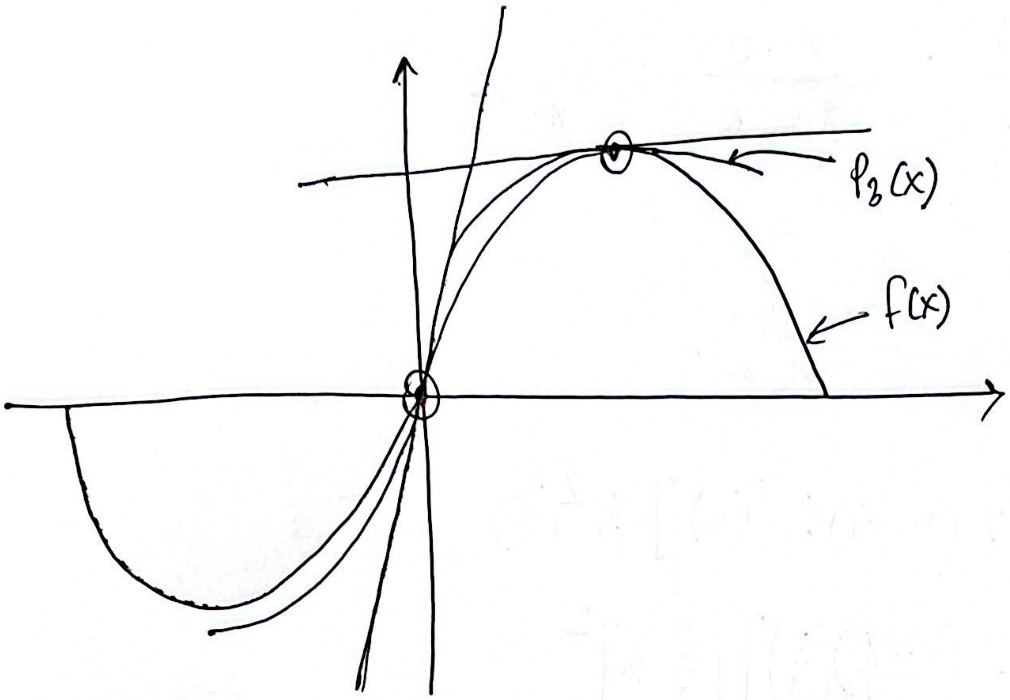
$$\hat{h}_0(x) = (x - x_0)\, l_0^2(x)$$

$$= (x - 0)\left(1 - \frac{2}{\pi} x\right)^2$$

$$= x\left(1 - \frac{2}{\pi} x\right)^2$$

$$P_3(x) = \hat{h}_0(x) + h_1(x)$$

$$= x\left(1-\frac{2}{\pi}x\right)^2 + \frac{4}{\pi^2}x^2\left(3-\frac{4}{\pi}x\right)$$



$$P_n(x_i) = f(x_i)$$

$$P_n'(x_i) = f'(x_i)$$

Very good interpolation with 2 nodes only.

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