Ch-1 Floating point Arithmetic

64 bit

64 integer
2 ^ number

$2^{63}-1 \rightarrow$ Largest integer

☐ Sign bit

63 bit
2^{63}

1.1 Fixed Point Numbers (real life)

$$x = \pm(d_1 d_2 \text{---} d_{k-1} \cdot d_k \ldots \text{---} d_n)_{\beta} \rightarrow \text{Base}$$

where $d_1, d_2 \text{---} d_n \in \{0, 1, \text{---} \beta-1\}$

$(10 \cdot 1)_2$

$$(d_1 d_2 \cdot d_3)_{\beta} = 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1}$$

$$= (2 \cdot 5)_{10}$$

$$(3 \cdot 14)_{10} = 3 \times 10^0 + 1 \times 10^{-1} + 4 \times 10^{-2}$$

$$3 + \frac{1}{10} + \frac{4}{100}$$

Floating point numbers ((computers use)

$F \subset \mathbb{R}$(real)

subset

base of num syste

exponent

$$F = \{ \pm (0.d_1 d_2 \ldots d_m)_\beta \times \beta^e \mid \beta, d_i, e \in \mathbb{Z}$$

$0 \le d_i \le \beta - 1, \quad e_{min} \le e \le e_{max}$

Fraction/mantissa/significand

$(123.45)_{10}$

$\Rightarrow 12.345 \times 10^1$

$\Rightarrow 0.12345 \times 10^3$

$0 \le \quad \le 1$

Fraction

$(11.101)_2$

$(1.1101)_2 \times 2^1$

$(0.11101)_2 \times 2^2$

e

fraction $\quad \beta$

Cost

we can only represent a finite set of numbers

numbers are not equally spaced

Conventions

① $\pm(0 \cdot d_1 d_2 \text{---} d_m)_\beta \, \beta^e$

sign bit

1 → to get unique representation

123 } must be unique
0123 }

$d_1 = 1$ always

② Normalized

$\pm(1 \cdot d_1 d_2 \text{---} d_m)_\beta \, \beta^e$

$(=1/\neq 1)$ যেকোনো কিছু হতে পারে d_1

③ Denormalized

$\pm(0 \cdot 1 d_1 d_2 \text{---} d_m)_\beta \, \beta^e$

hidden bit

$\beta = 2, m = 3$

(1) $(0.111)_2 \times 2^{e_{max}}$

(2) $(1.111)_2 \times 2^{e_{max}}$

(3) $(0.1111)_2 \times 2^{e_{max}}$

highest possible number

Count of $e = 4 \rightarrow -1, 0, 1, 2$

example:

$\boxed{\text{Conv 1}}$

$\beta = 2, m = 3, e_{min} = -1, e_{max} = 2$

$\pm(0.d_1d_2d_3)_2 \times 2^e \qquad e \in [-1, 2]$

1	□	□
	0	0
	0	1
	1	0
	1	1

$\times 2^e$ for each e 4 possible combo

total $\underline{4}$ e

-1, 0, 1, 2

16 possible numbers

$\boxed{\text{highest Possible Floating Point numbers:}}$

1) $\beta^{m-1} \times$ count of e

2 & 3) $\beta^m \times$ count of e

smallest possible $\boxed{\text{non-negative}}$

$(0.100)_2 \times 2^{-1}$

$2^{-1} \times 2^{-1} = \frac{1}{4}$

highest

$(0.111)_2 \times 2^2$

$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \times 4$

$\frac{7}{2}$

if negative number consider afe

(9) 2(n) $-\frac{7}{2}$ 2n

$$(0.100)_2 \times 2^{-1} = \frac{1}{4}$$

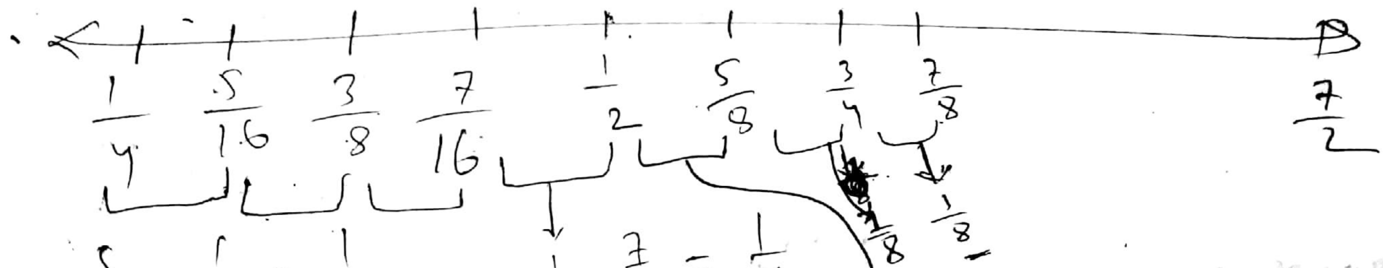
$$(0.101)_2 \times 2^{-1} = \left(\frac{1}{2} + \frac{1}{8}\right) \times \frac{1}{2} = \frac{5}{16}$$

$$(0.110)_2 \times 2^{-1} = \left(\frac{1}{2} + \frac{1}{4}\right) \times \frac{1}{2} = \frac{3}{8}$$

$$(0.111)_2 \times 2^{-1} = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \times \frac{1}{2} = \frac{7}{16}$$

$$\downarrow$$

$$(0.100)_2 \times 2^0 \rightarrow (0.101)_2 \times 2^0$$



$$\frac{5}{16} - \frac{1}{4} = \frac{1}{16} \qquad \frac{1}{2} - \frac{7}{16} = \frac{1}{16}$$

$$\frac{3}{8} - \frac{5}{16} = \frac{1}{16}$$

$$\frac{5}{8} - \frac{1}{2} = \frac{1}{8}$$

e সেইম হলে equally spaced

২ নং convention এ

0.1

$\frac{1}{2}$ ×

0 বসানো হবে না

$$(0.100) \times 2^0 = \frac{1}{2}$$

$$0.101 \times 2^0 = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

equally spaced set

IEEE standard (1985) for double precision (64-bit) arithmetic

precision

0.110 0.1101 0.111

$\beta = 2$, 52 bits for the fraction

11 bits for the exponent → range

1 bit for the sign

normalized $(1.d_1 d_2 \ldots d_{52})_2 \times 2^e$

$2^{11} = 2048$ $[0, 2047]$

$\downarrow$ e_{min} $\downarrow$ e_{max}

largest $(1.11 \ldots 1) \times 2^{2047}$

smallest $(1.00 \ldots 0) \times 2^0 = 1$

but we need 0.0000 1 (< 1)

(e_{min} less than 0)

↳ negative

so exponent biasing

a.o

$$\left(1 \cdot d_1 d_2 \text{ ——— } d_{52}\right)_2 \times 2^{\underbrace{e-1023}_{\text{exp bias}}}$$

denormalized

$$\left(0 \cdot 1 d_1 d_2 \text{ ——— } d_{52}\right)_2 \times 2^{e-1022}$$

$$e \in [0, 2047].$$

$$(e-1022) \in [-1022, 1025]$$

$$(0 \cdot 111 \text{ ——— } 1)_2 \times 2^{1025} \quad \text{largest}$$

$$(0 \cdot 100 \text{ ——— } 0)_2 \times 2^{-1022} \quad \text{smallest}$$

$$2^{1025} \longrightarrow \pm\infty$$

$$2^{-1022} \longrightarrow \pm 0$$

$$(0 \cdot 11 \text{ ——— } 1)_2 \times 2^{1024} \simeq 1.798 \times 10^{308}$$

$$(0 \cdot 100 \text{ ——— } 0)_2 \times 2^{-1021} \simeq 2.225 \times 10^{-308}$$

$(-19.25)_{10}$ কে ৩ টা convention অনুযায়ী floating point number এ represent কর। ($\beta = 2$, $m = 4$ দেওয়া)

1st এ binary তে convert করব।

$(19)_{10} \rightarrow (10011)_2$

$$
\begin{array}{r|l}
2 & 19 \\
2 & 9 - 1 \\
2 & 4 - 1 \\
2 & 2 - 0 \\
2 & 1 - 0 \\
 & 0 - 1
\end{array}
$$

$$
\begin{array}{r}
0.25 \\
\times 2 \\
\hline
0.50 \\
\times 2 \\
\hline
1.00
\end{array}
$$

$(0.25)_{10} \rightarrow (0.01)_2$

$\therefore (19.25)_{10} \rightarrow (10011.01)_2$

1st Convention: (Lecture Note general) $(0.1001101)_2 \times 2^5$

But $m = 4$, so $\rightarrow (0.\underbrace{1001}_{m=4})_2 \times 2^5$

$d_1 = 1$

2nd: (normalized) $(1.001101)_2 \times 2^4$

but $m = 4 \longrightarrow (1.\underbrace{0011}_{m=4})_2 \times 2^4$

3rd: (denormalized) $(0.1001101)_2 \times 2^5$

But $m = 4$

$\longrightarrow (0.\underbrace{1}_{\text{এটা always 1 থাকবে (hidden bit)}}\ \underbrace{0011}_{m=4})_2 \times 2^5$

Significant figures (no decimal point → last non-zero digit)

1st non zero digit from the left, count all final zeros if they are after decimal

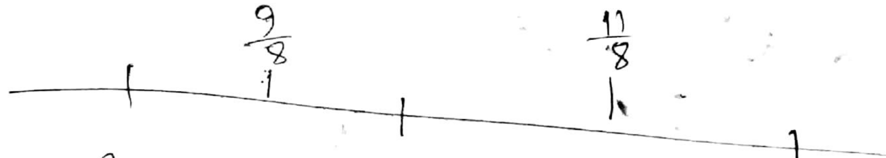
3105.0
0.031050
0.031056
31.050

} 5 sf

33.0000 → 6 sf
807000 → 3 sf
100. → 3 sf
70,000 → 1 sf
100 → 1 sf

Rounding:

mapping from R to F is called ~~rounding~~ $fl(x)$



1 $(0.100)_2 \, 2^1$

$\frac{5}{4}$ $(0.101)_2 \, 2^1$

$\frac{3}{2}$ $(0.110)_2 \, 2^1$

$\frac{9}{8}$ has neighbours 1 & $\frac{5}{4}$

last digit 0. so rounded to 1

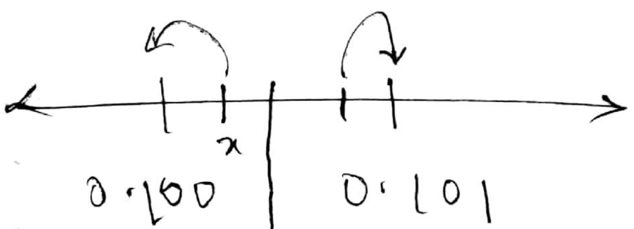
$\frac{11}{8}$ → rounded to $\frac{3}{2}$

avoids statistical bias / prolonged drift

Rounding error

$$x \longrightarrow fl(x)$$

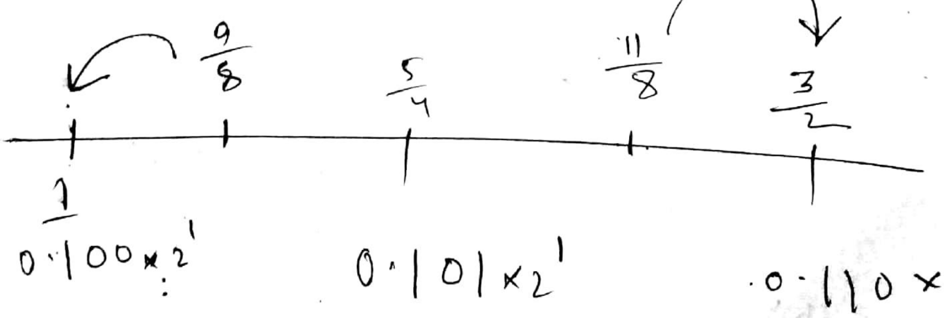
$\downarrow$ float



0.1001 → it will be rounded to the nearest even

$\beta = 2, \; m = 3, \; e_{min} = -1, \; e_{max} = 2$

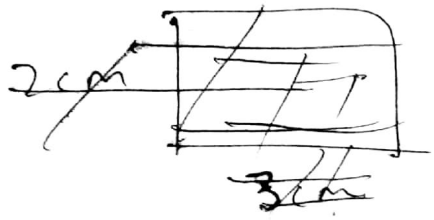
$$(0.d_1 d_2 d_3)_2 \times 2^e$$



scale invariant

Rounding error

$$\delta = \frac{|fl(x) - x|}{|x|} \times 100\%$$



$$fl(x) = x(1+\delta)$$
$$= x + x\delta$$

$x\delta$ → error

δ → scale invariant error

Machine Epsilon: ϵ_M → unit round off

scale invariant

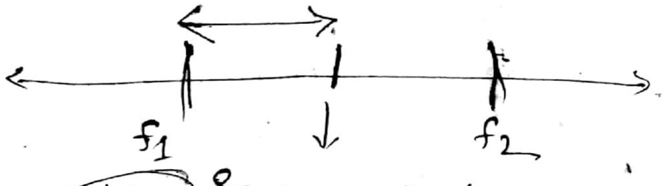
→ max possible rounding error

1) $(0.d_1d_2\_\_\_d_m)_\beta \times \beta^e$

$(0.101 - 0.100) \times \beta^e$

$\text{diff} = 0.001 \times \beta^e \to 2^{-3}$

$\sim \beta^{-m} \times \beta^e$



$0.100 \times \beta^e$ max error occurs $0.101 \times \beta^e$

where will maximum error occur

→ in the middle

$f_2 - f_1 \Rightarrow \beta^{-m} \cdot \beta^e$

$= \frac{1}{2} \beta^{-m} \cdot \beta^e$ ← max possible distance

$fl(x) - x$

max possible δ = $\dfrac{fl(x) - x \to \max}{x \to \min}$

$x \to \min$

$$(0.1000 \text{———})_{\beta} \, \beta^{e}$$

$$\min |x| = \beta^{-1} \cdot \beta^{e}$$

$$\epsilon_M = \frac{\frac{1}{2}\beta^{-m} \cdot \beta^{e}}{\beta^{-1} \cdot \beta^{e}} = \frac{1}{2}\beta^{1-m}$$

precision depends on mantissa

does not depend on exponent

2) Normalized form

$$(1 \cdot d_1 d_2 \text{———} d_m)_{\beta} \times \beta^{e}$$

$$|fl(x) - x| = \frac{1}{2}\beta^{-m} \cdot \beta^{e}$$

$$|x| = (1.000)_{\beta} \times \beta^{e} = \beta^{0}\beta^{e}$$

$$= \beta^{e}$$

1.000

1.001

β^{-m}

$$\therefore \epsilon_M = \frac{1}{2}\beta^{-m}$$

3) de normalized form

$$(0.1\,d_1 d_2 \text{———} d_m)_\beta \times \beta^e$$

$m = 3$

0.1000
0.1001

β^{-4}

β^{-m-1}

$$|fl(x) - x| = \frac{1}{2}\beta^{-m-1} \cdot \beta^e$$

$$|x| = (0.1000)_\beta \times \beta^e$$

$$= (0.1)_\beta\, \beta^e$$

$$= \beta^{-1} \cdot \beta^e.$$

$$\therefore \epsilon_M = \frac{1}{2}\beta^{-m}$$

$$|\delta| \leq \epsilon_M$$

Arithmetic with rounding error

represent x and y seperately but doesn't represent xy

$\beta = 2,\ m = 3,\ e = [-1, 2]$

$x = \frac{5}{8}$

$~~fl(x) =~~$

$= \frac{4}{8} + \frac{1}{8}$

$= \frac{1}{2} + \frac{1}{2^3}$

$= 2^{-1} + 2^{-3}$

$= (0.101)_2 \times 2^0$

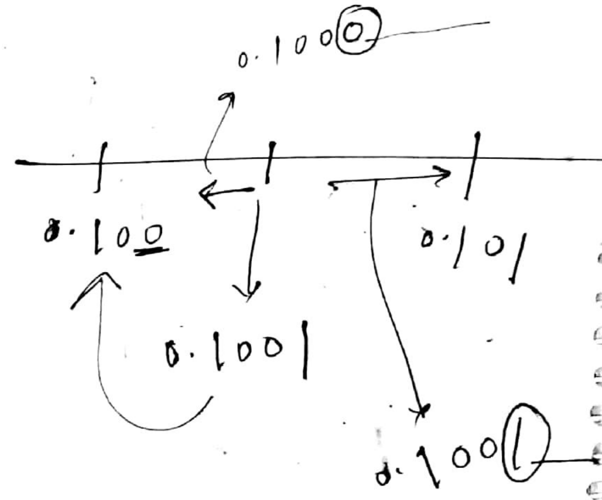
$y = \frac{7}{8} = \frac{4}{8} + \frac{2}{8} + \frac{1}{8}$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$= (0.111)_2 \times 2^0$

$fl(x) = (0.101)_2 \times 2^0$

$fl(y) = (0.111)_2 \times 2^0$



$xy \to fl(xy) = (0.100)_2 \times 2^0$

$= \frac{1}{2}$

$xy = fl(x) \times fl(y)$

$= \frac{5}{8} \times \frac{7}{8}$

$= \frac{35}{64} = \frac{32}{64} + \frac{2}{64} + \frac{1}{64}$

$= \frac{1}{2} + \frac{1}{32} + \frac{1}{64}$

$= (0.\underline{100}011)_2 \times 2^0$

$m = 3$

0.100
0.101

nearest zero

$xy = \frac{35}{64}$ rounding error

but $fl(xy) = \frac{32}{64}$

$x + y$

$fl(x) + fl(y)$

$\downarrow$

$fl(x+y)$

bux 70 float

Loss of significance

$x = fl(x)$, $y = fl(y)$

$xy \Rightarrow fl(xy)$

$x \neq fl(x)$, $y \neq fl(y)$

$fl(x) = x(1+\delta_1)$

$fl(y) = y(1+\delta_2)$

$$x \pm y = fl(x) \pm fl(y)$$

$$\simeq x(1+\delta_1) \pm y(1+\delta_2)$$

$$= (x \pm y) + x\delta_1 \pm y\delta_2$$

$$= (x \pm y)\left(1 + \frac{x\delta_1 \pm y\delta_2}{x \pm y}\right)$$

$fl(x) = x(1+\delta)$

$fl(x-y) = (x-y)(1+\delta)$

$x = 28$

$y = 27.98$

0.00001

0.000001 scale invariant error, δ

$$x - y = x - y\left(1 + \frac{x\delta_1 - y\delta_2}{x - y}\right)$$

if x & y close to each other

$x - y \to 0$

so, $\delta \to \infty$ → this phenomena is called loss of significance

adding close to 0 element

Ex: $x^2 - 56x + 1 = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$x_1 = 28 + \sqrt{783} = 55.98$

$x_2 = 28 - \sqrt{783} = \boxed{0.01786}$ calculator এ আসছে

My Computer can only calculate upto 4 significand figures

$\sqrt{783} = 27.98$ → as $\boxed{4\ sf}$ এসেছে

$\bar{x}_1 = 28 + 27.98 = 55.98$

$\bar{x}_2 = 28 - 27.98 = \boxed{0.02000}$ → 4 sf

subtracting 2 numbers which are closer to each other

so error occurs

loss of significance

$\alpha\beta = \frac{c}{a}$

$\alpha\beta = 1$

$x_1 = \alpha$

$\beta = x_2 = \frac{1}{x_1}$

to avoid subtraction closer to each other

$\beta = \frac{c}{a\alpha}$

$f(x) = e^{x} - \cos x - x$ ⟶ fluctuation হবে

$x \in [-5\times10^{-8},\ 5\times10^{-8}]$

very close to 0

$e^{0} = 1$

$\cos(0) = 1$

$\underbrace{e^{x} - \cos x}$

is not good. (∵ Loss of sig.)

how to get rid of this :

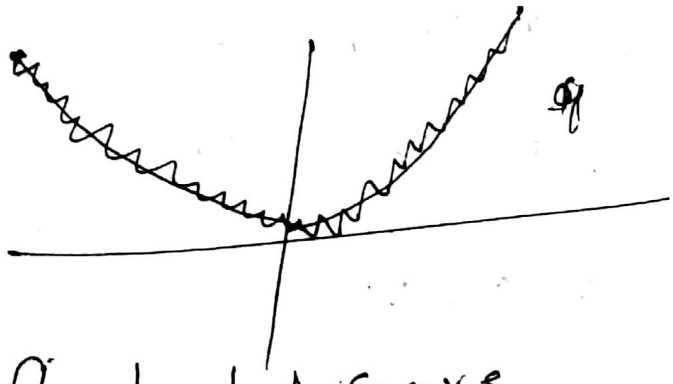
expand

Taylor series

$$\left(1 + x + \frac{x^2}{2!} + \cdots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - x$$

$$\boxed{x^2 + \frac{x^3}{6}}$$ ⟶ fluctuation curve হবে না

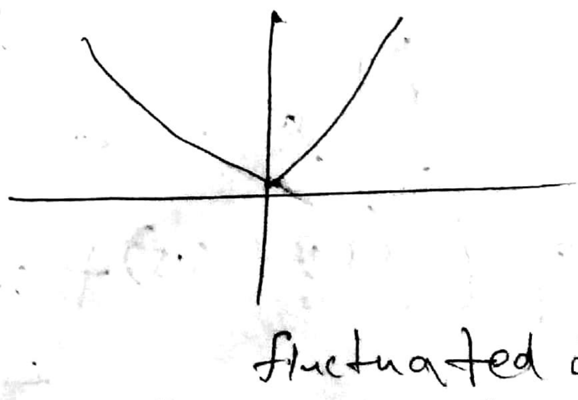
no need to subtract



fluctuated curve

↑

$e^{x} - \cos x - x$



fluctuated না

$x^2 + \frac{x^3}{6}$

upto 3 sfigure

2
1 sf

5.01 5.02

avg → 5.015

$$fl\left(\frac{5.01+5.02}{2}\right) = fl\left(\frac{10.03}{2}\right)$$

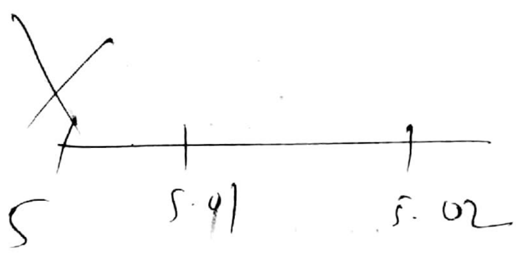
$$= \frac{fl(10.03)}{fl(2)}$$

5.00

$$= \frac{10.0}{2.00} = 5$$ (in my computer)

↓

but it occurs error because upto 3 sig fig calculation



$(a+b)+c \neq a+(b+c)$ in float point

$\boxed{2}$ digit decimal arithmetic (2 sf)

$$fl[(5.9+5.5)+0.4] = fl(fl(11.4)+0.4)$$

$$= fl(11+0.4)$$

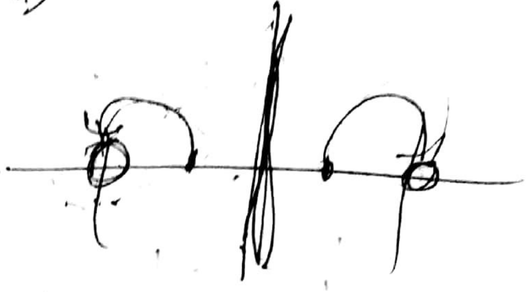
$$= 11$$

#

$$fl\left[5.9 + (5.5 + 0.4)\right] = fl\left(5.9 + 5.9\right)$$

$$= fl\,(11.8)$$

$$= 12$$



— 0 —

Ch-2

Polynomial Interpolation

polynomial of degree n

$$= P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x_n$$

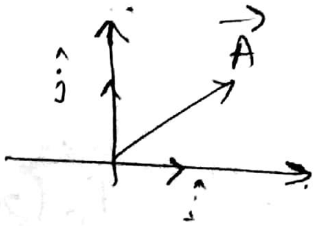
$(n+1)$ coefficients constant

: to find a polynomial $P_n(x)$ that approximates a general func f

$P_n(x_i) = f(x_i)$ at a finite set of points x_i

nodes

add vectors
multiply scalars } polynomials



dimension

basis is a set of vectors that spans the space.

$A = A_x \hat{i} + A_y \hat{j}$

$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta}$

$\{r, \theta\}$ → basis ← $\{i, j\}$

$P_2(x) = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$

basis $\{1, x, x^2\}$

3 dimensional space

$f(x) = P_n(x) + \epsilon$

$P_n(x) = a_0 \cdot 1 + a_1 x + a_2 \cdot x^2 + \dots + a_n x^n$

basis $= \{1, x, x^2, \dots, x^n\}$

$\downarrow$

natural

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$

$P_n(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$

Function spaces

$\rightarrow$ vector space

Fourier Series

$$f(x) = \sum \left(\tilde{f} \sin(\dots) + f' \cos(\dots) \right)$$

sin, cos $\rightarrow$ basis

$\hookrightarrow \infty$ dimensional space

$f(x) = a_0 \cdot 1 + a_1 x + a_2 x^2 + \dots$ (doesn't stop)

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$P_n(x) \in V^{n+1} \rightarrow n+1$ dimensional (n+1 coefficients)

$f(x) \in V^{\infty} \rightarrow \infty$ dimensional

$\hookrightarrow \{1, x, x^2, \dots\}$

Weierstrass Approximation Th^m

For a continuous function $f(x)$ on a bounded interval, this is always possible if you take a high enough degree polynomial:

For any $f \in C([0,1])$ and any $\epsilon > 0$, there exists a polynomial such that,

$$\max_{0 \le x \le 1} |f(x) - P(x)| \le \epsilon$$

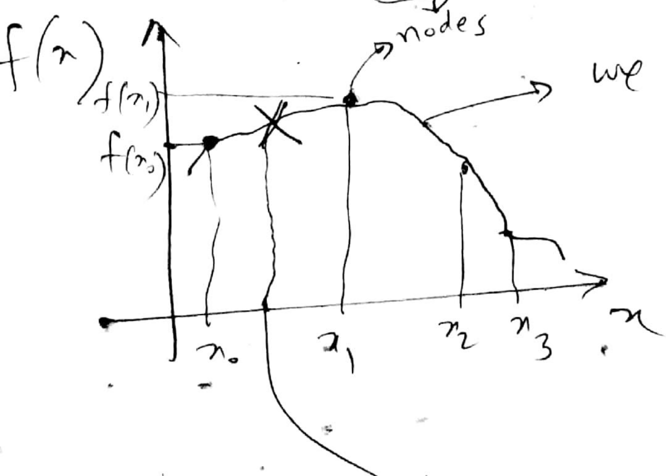
degree $\to \infty$ হলে $f(x) \approx P(x)$ হয়ে যাবে।

P. Interpolation:

$f(x) \to$ unknown

we know $x_i \to f(x_i)$

x_i → nodes



→ we have to find a polynomial that exactly goes through it

→ ML এ বেশি কাজে লাগে

→ we can predict the value upto ϵ error

x_y

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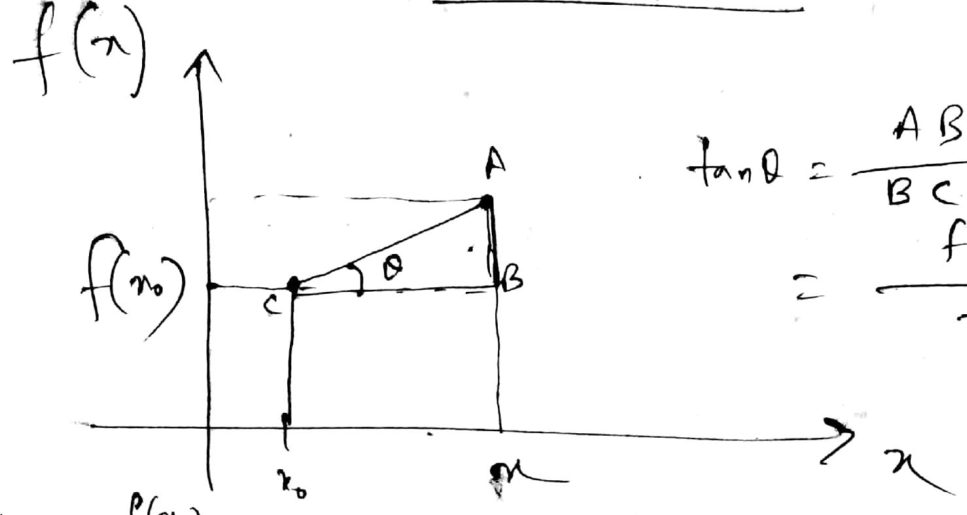
Taylor Series



$$\tan\theta = \frac{AB}{BC}$$

$$= \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

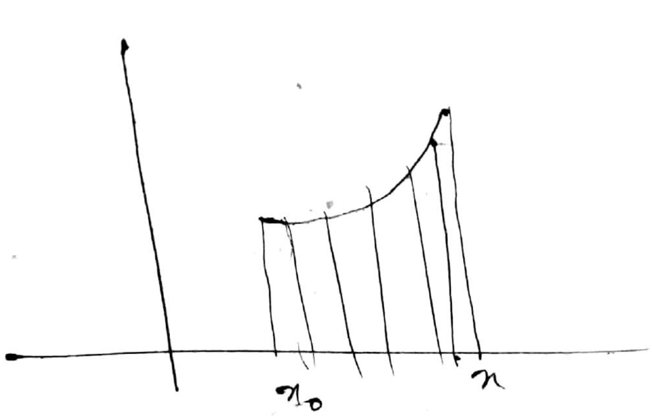
$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

if we know $f(x_0)$, $f'(x_0)$, $f''(x_0)$ ... then we can predict $f(x)$

$y = mx + c$

$y' = m$

$y'' = 0$



$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \text{---}$$

stop polynomial

ϵ error

A truncated Taylor Series is the simplest interpolating polynomial since it uses a single node x_0.

$\sin(0.1)$ to 6 sf

$x_0 = 0$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots$$

$$f(x_0) = a_0, \quad f'(x_0) = a_1, \quad f''(x_0) = 2a_2$$

$$f'''(x_0) = 3(2)a_3$$

1 term $\rightarrow f(0.1) \approx 0.1$

2 terms $\rightarrow f(0.1) \approx 0.1 - \frac{0.1^3}{6} = 0.099833$ ____

3 terms $\rightarrow f(0.1) \approx 0.0998334$ ____

next term will be $-\frac{0.1^7}{7!} \approx \frac{-10^{-7}}{10^3} = -10^{-10}$

exact Ans: $\underbrace{0.099833}_{6\text{ sf}}4$

Proof of T. Series:

natural
$\hookrightarrow$ basis $\{1, x, x^2, \ldots\}$

$$f(x) = a_0 \cdot 1 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \ldots$$

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \ldots,$$

$$f''(x) = 2a_2 + 3a_3 2(x-x_0) + \ldots$$

$$f'''(x) = 6a_3 + \ldots$$

$$f(x_0) = a_0 \rightarrow f'(x_0) = a_1,\ f''(x_0) = \overset{2!}{2a_2},\ f'''(x_0) = \underset{\downarrow\, 3!}{3\cdot 2 a_3}$$

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Taylor's Th^m:

Let f be $(n+1)$ times differentiable on (a,b) and let $f^{(n)}$ be continuous on $[a,b]$. If $x, x_0 \in [a,b]$ then there exists $\xi \in (a,b)$ such that

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{1st } (n+1) \text{ terms in Taylor series}} + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

1st $(n+1)$ terms in Taylor series

↓

Taylor Polynomial of degree n

error

Remainder

we don't know ξ

Lagrange form of the remainder

$\xi \in (a, b)$

Example:

$x_0 = 0$

$$f(x) = \sin x = \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!}} - \frac{x^7}{7!} + \cdots$$

$P_6(x)$

Ques এ যেটা point কে interpolate করতে তাই $P_6(x)$

$$f(x) = P_6(x) + \frac{f^{(7)}(\xi)}{7!}(x-x_0)^7$$

Always $f^{(7)}(\xi)$ এর max value বসাবো

$$|f(x) - P_6(x)| = \left| \frac{f^{(7)}(\xi)}{7!}(x-x_0)^7 \right|$$

$x = 0.1$, $f^{(7)}(x) = \cancel{-\sin x} = -\cos x$

$x_0 = 0$

$$|f(0.1) - P_6(0.1)| = \frac{1}{7!}(0.1)^7 \left| -\cos(\xi) \right|$$

$7! = 5040$

$\cos(\xi) \leq 1$

$$|f(0.1) - P_6(0.1)| \leq \frac{1}{5040}(0.1)^7$$

$$\leq 1.984 \times 10^{-11}$$

approximated error

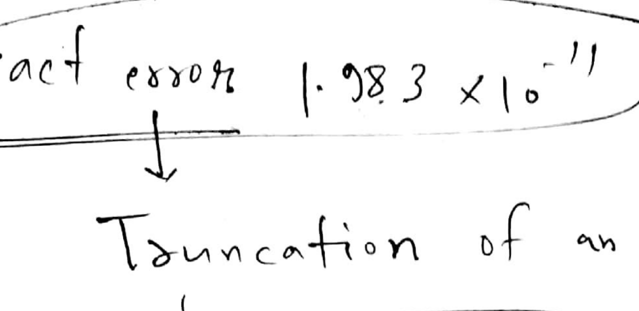
exact error 1.983×10^{-11}

Truncation of an infinite series

truncation error

Now main interpolation

Polynomial থেকে predict



$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$= \sum_{k=0}^{n} a_k x^k$$

$(a_0 \text{ —— } a_n) \rightarrow$ unknown

Conditions:

$P_n(x_0) = f(x_0)$, $P_n(x_1) = f(x_1)$ ——— $P_n(x_n) = f(x_n)$

$P_n(x_j) = f(\underbrace{x_j}_{\text{nodes}})$

$n+1$ coefficients জানার জন্য $n+1$ conditions লাগবে

if we have 2 nodes x_0, x_1
$\downarrow$
$a_0, a_1 \rightarrow$ straight line

the more nodes $\rightarrow$ the more degree polynomial
$\downarrow$
the more precision

Example: $n = 1$ (2 nodes) $\rightarrow$ polynomial of degree 1

$x_0 \longrightarrow f(x_0)$

$x_1 \longrightarrow f(x_1)$

$$P_1(x) = a_0 + a_1 x$$

$$P_1(x_0) = a_0 + a_1 x_0 = f(x_0) \quad \text{--- } (1)$$

$$P_1(x_1) = a_0 + a_1 x_1 = f(x_1) \quad \text{--- } (2)$$

subtract (But it is not recommended. We use matrix method) ~~after~~.

$$a_1 (x_0 - x_1) = f(x_0) - f(x_1)$$

$$a_1 = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$a_0 + \frac{f(x_0) - f(x_1)}{x_0 - x_1} \cdot x_0 = f(x_0)$$

$$a_0 = f(x_0) - \frac{x_0 f(x_0) - x_0 f(x_1)}{x_0 - x_1}$$

$$a_0 = \frac{\cancel{x_0 f(x_0)} - x_1 f(x_0) - \cancel{x_0 f(x_0)} + x_0 f(x_1)}{x_0 - x_1}$$

$$a_0 = \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1}$$

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} f(x_0) \\ f(x_1) \end{pmatrix}$$

$$= \frac{1}{\underset{x_1 - x_0}{\det}} \begin{pmatrix} x_1 & -x_0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} f(x_0) \\ f(x_1) \end{pmatrix}$$

general for Vandermonde Matrix Method

$x_0, x_1, \_\_\_\_ , x_n \rightarrow (n+1)$ nodes

$\downarrow \quad \downarrow \quad \downarrow$ $\quad n+1$ conditions

$f(x_0) \quad f(x_1) \quad f(x_n)$ $\quad n+1$ co-efficients

$\Rightarrow P_n \rightarrow \{a_0, a_1, \_\_\_\_ a_n\}$

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$P_n(x_i) = f(x_i)$$

$$a_0 + a_1 x_0 + a_2 x_0^2 + \_\_\_\_ + a_n x_0^n = f(x_0)$$
$$a_0 + a_1 x_1 + a_2 x_1^2 + \_\_\_\_ + a_n x_1^n = f(x_1)$$
$$a_0 + a_1 x_2 + a_2 x_2^2 + \dots + a_n x_2^n = f(x_2)$$
$$\vdots$$
$$a_0 + a_1 x_n + a_2 x_n^2 + \_\_\_\_ + a_n x_n^n = f(x_n)$$

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}}_{F}$$

Vandermonde matrix

$VA = F$

$A = V^{-1} \cdot F$

$\downarrow$

invertible হতে হবে — (all nodes distinct হতে হবে)

$\boxed{\det(V) \neq 0}$

(same x_i multiple বার থাকলে det(v) = 0 হবে)

$$\det(V) = \prod_{0 \le i < j \le n} (x_i - x_j)$$

$$= (x_0 - x_1)(x_1 - x_2) \cdots (x_{n-1} - x_n)$$

$\boxed{\prod_{n=1}^{3} n = 1 \cdot 2 \cdot 3}$

$\neq 0$ হবে if all nodes ~~are~~ have to be distinct

~~Example~~ Th^m (Existence/Uniqueness):

~~Given~~ there is a unique P_n that interpolates $f(x)$ at the nodes.

Proof: P_n q_n

we have to show $r_n = 0$

$r_n = P_n - q_n$

$\downarrow$

has a degree $\le n$

$$r_n(x_i) = P_n(x_i) - q_n(x_i)$$
$$= f(x_i) - f(x_i)$$
$$= 0$$

r_n has $(n+1)$ roots

n degree $\Rightarrow$ n roots

r_n can't have $n+1$ roots

$r_n(x) = 0$

$\therefore P_n(x) = r_n(x)$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2}{\pi} \\ 0 \end{pmatrix}$$

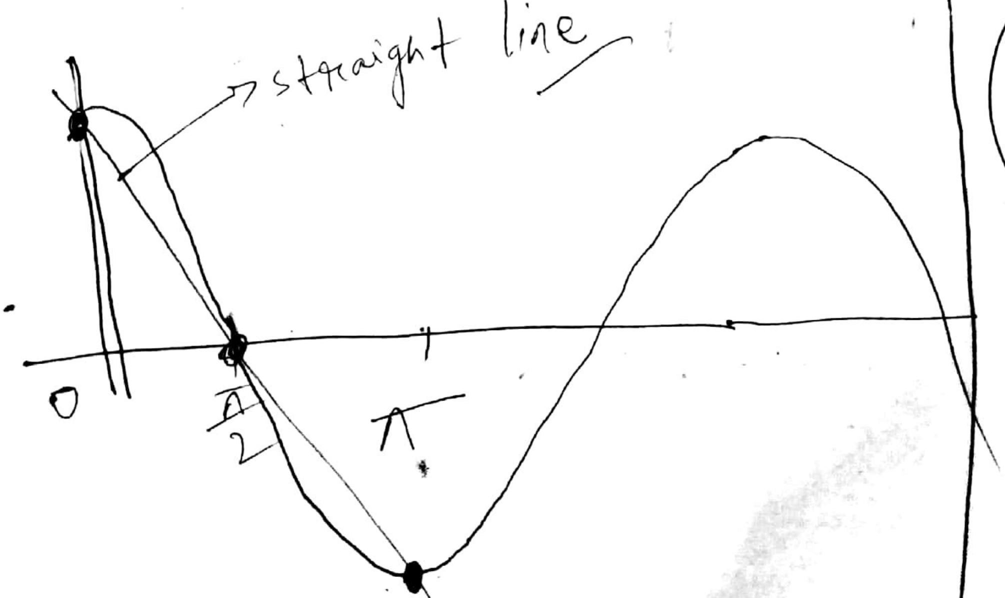
$f(x) = \cos x$

$x_0 = 0, \quad x_1 = \frac{\pi}{2}, \quad x_2 = \pi$

$f(x_0) = 1 \quad f(x_1) = 0 \quad f(x_2) = -1$

3 nodes $\longleftrightarrow$ $P_2 = ?\ (a_0 + a_1 x + a_2 x^2)$

$P_2(x) = 1 - \frac{2}{\pi} x \qquad (a_2 = 0)$



$$\begin{array}{ccc} x_0 & x_1 & x_2 \\ 0 & 2\pi & 4\pi \\ \downarrow & \downarrow & \downarrow \\ \frac{1}{f(x_0)} & \frac{1}{f(x_1)} & \frac{1}{f(x_2)} \end{array}$$

$P_2(x) = 1$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Lagrange Form :

basis: we will choose a basis, $l_K(x)$ such that

$$P_n(x) = \sum_{K=0}^{n} f(x_K) \cdot l_K(x)$$

$f(x_K)$ → co-efficients

$l_K(x)$ → basis

$$= f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + \_\_\_ + f(x_n)\, l_n(x)$$

$l_i(x)$ [illegible]

$$P_n(x_0) = f(x_0)$$

$$l_0(x_0) = 1, \quad l_1(x_0) = 0 \quad \_\_\_$$

$$P_n(x_1) = f(x_1)$$

$l_1(x_1) = 1$, all the rest 0

$f(x) = 0$ and 5, $f(5) = 0$

$f(x) \propto (x-5)$

$f(x) = K(x-5)$

$f(5) = 0$ & $f(8) = 0$

$f(x) = K(x-5)(x-8)$

$l_0(x) = 0$ at $x_1, \ldots x_n$

$l_0(x) \cancel{\propto} K(x-x_1)(x-x_2) \ldots (x-x_n)$

$l_0(x_0) = 1$

$$l_0(x) = \frac{(x-x_1)(x-x_2) \ldots (x-x_n)}{(x_0-x_1)(x_0-x_2) \ldots (x_0-x_n)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2) \ldots (x-x_n)}{(x_1-x_0)(x_1-x_2) \ldots (x_1-x_n)}$$

$$l_K(x) = \prod_{\substack{j=0 \\ j \neq K}}^{n} \frac{x-x_j}{x_K-x_j}$$

$$l_k(x_j) = \delta_{kj} \qquad \delta_{kj} = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases}$$

$\downarrow$

Kronecker delta

$l_0(x_0)$, $l_1(x_1)$ $\rangle$ 1

$$P_1(x) = a_0 + a_1 x$$

$$= \underbrace{\frac{x - x_1}{x_0 - x_1}}_{l_0(x)} f(x_0) + \underbrace{\frac{x - x_0}{x_1 - x_0}}_{l_1(x)} f(x_1)$$

Ex :

$$P_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + \_\_\_ l_n(x) f(x_n)$$

$f(x) = \cos x$ $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$ nodes (3 nodes তাই $P_2(x)$)

$\quad x_0 \quad x_1 \quad x_2$

P_2

$$f(x_0) = \frac{1}{\sqrt{2}}, \; f(x_1) = 1, \; f(x_2) = \frac{1}{\sqrt{2}}$$

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0)\left(x - \frac{\pi}{4}\right)}{\left(-\frac{\pi}{4}\right)\left(-\frac{\pi}{2}\right)}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x\left(x + \frac{\pi}{4}\right)}{\frac{\pi}{2} \cdot \frac{\pi}{4}}$$

$$l_0(x) = \frac{8}{\pi^2} x\left(x - \frac{\pi}{4}\right)$$

$$l_1(x) = \frac{-16}{\pi^2}\left(x + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$l_3(x) = \frac{8}{\pi^2} x\left(x + \frac{\pi}{4}\right)$$

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

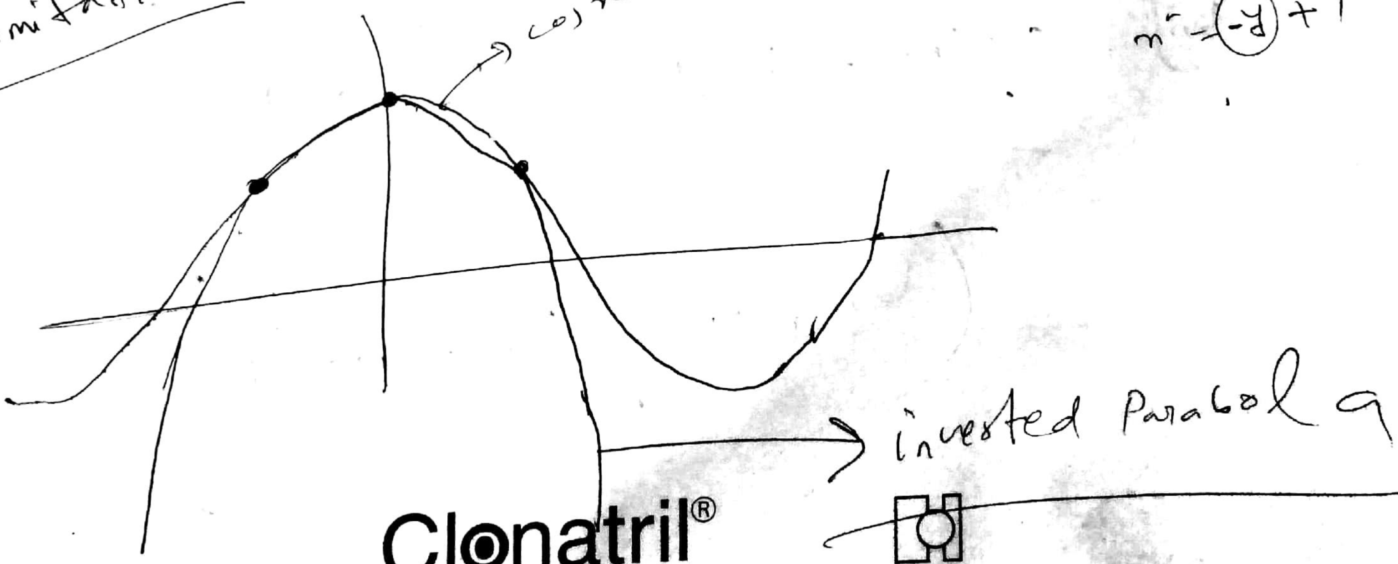
$$= \frac{1}{\sqrt{2}} \frac{8}{\pi^2} x\left(x - \frac{\pi}{4}\right) + \left(-\frac{16}{\pi^2}\right)\left(x + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} \frac{8}{\pi^2} x\left(x + \frac{\pi}{4}\right)$$

$$P_2(x) = \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}} - 1\right) x^2 + 1$$

· limitation: recompute করা লাগতে new node add করলে

$$y = -x^2 + 1$$

$$x^2 = -y + 1$$



Vandermonde Method এ n টি nodes দেয়া থাকলে (n × n) dimension এর Vandermonde matrix বানাতে হবে। n টি unknown co-efficients থাকে।

(n-1) degree Polynomial বানাতে হবে।

যেমন: 3 টা node $\{x_0, x_1, x_2\}$ দেয়া থাকলে (3 × 3) order/dimension এর V matrix বানাই। এবং 2 degree Polynomial $P_2(x)$ বানাই। যেখানে 3 টা co-efficients থাকে $\{a_0, a_1, a_2\}$

Vandermonde or Lagrange যে Method এ solve করি না কেন always same nodes এর জন্য Unique polynomial-ই পাব।
(same)

↓

(Uniqueness/Existence Th^m)