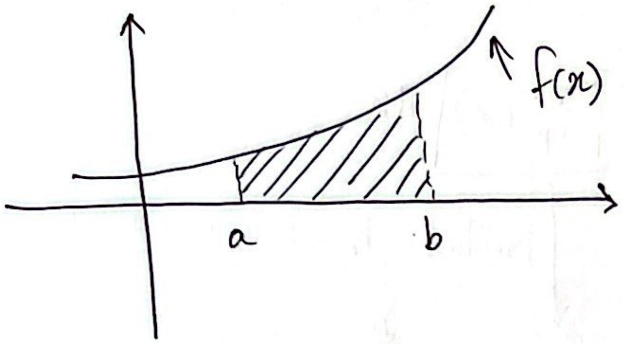
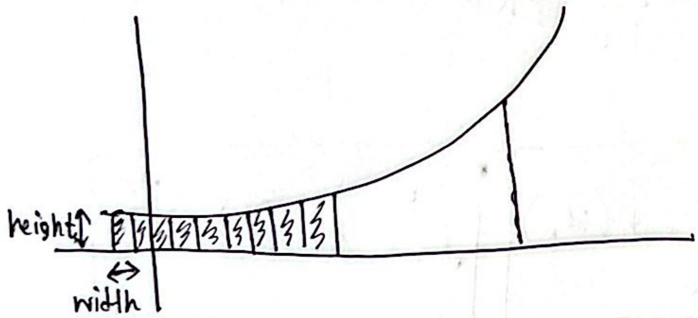
Integration:

$$I(f) = \int_a^b f(x)\, dx$$



→ Integration gives the area under f(x) within the bound a & b.

→ By definition, integration is an infinite sum.



$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{f_i(x)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$$

→ Numerical integration replace function $f(x)$ with interpolating polynomial of degree n that passes through $(n+1)$ nodes

Actual Integration $= I(f) = \int_a^b f(x)\, dx$

Numerical Integration $= I_n(f) = \int_a^b \boxed{P_n(x)}\, dx$

↓ writing $P_n(x)$ using lagrange basis

This polynomial $P_n(x)$ must be interpolated with equidistant nodes $x_0, x_1, \ldots x_n$ ← $P_n(x) = \sum_{i=0}^{n} l_K(x) \cdot f(x_K)$

equally spaced.

$$\therefore \quad I_n(f) = \int_a^b \sum_{i=0}^{n} l_K(x) \cdot f(x_K)\, dx$$

$$= \boxed{\sum_{K=0}^{n} f(x_K)} \boxed{\int_a^b l_K(x)\, dx}$$

↓

σ_K (~~weight function~~)
(weighted factors)

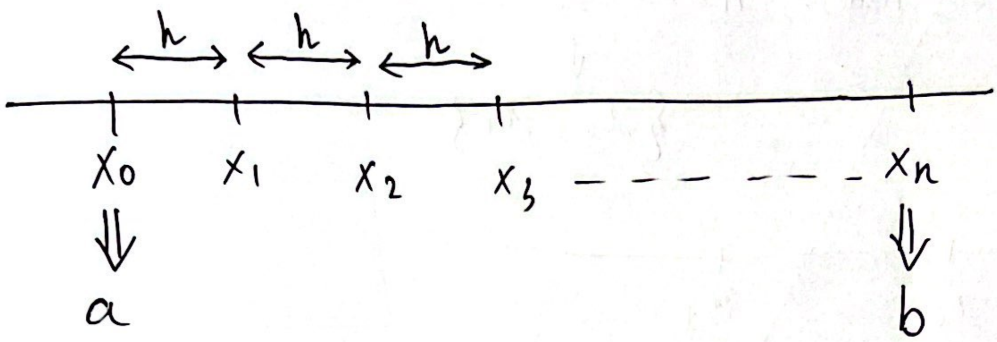
$$\therefore \quad \boxed{I_n(f) = \sum_{K=0}^{n} \sigma_K \cdot f(x_K)}$$

↑

Newton cote's Formula.

Newton Cote's Formula → Closed, open

Finding $x_0, x_1 \dots x_n$ Using closed Newton cote's Formula:



→ 'a' and 'b' are integration intervals.

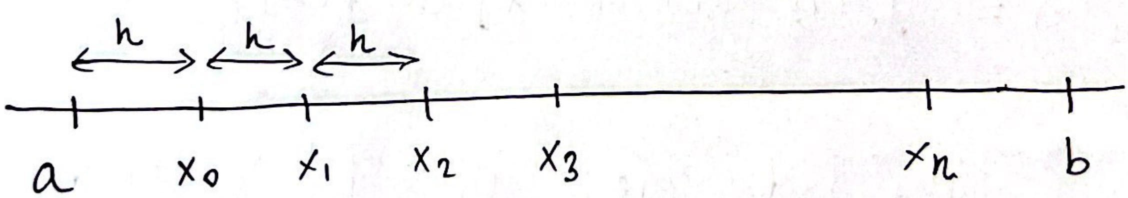
→ $\boxed{h = \dfrac{b-a}{n}}$

$x_0 = a$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

Finding $x_0, x_1 \dots x_n$ using ~~to~~ open Newton cote's Formula:



$\boxed{h = \dfrac{b-a}{n+2}}$

$x_0 = a + h$

$x_1 = a + 2h$

$x_2 = a + 3h$

Trapezium / Trapezoidal Rule:

→ Closed Newton Cotes formula with $n = 1$

$n =$ degree of polynomial $= 1$

$\therefore$ number of nodes $= n + 1 = 2 \Rightarrow \{x_0, x_1\}$

$x_0 \Downarrow a \qquad x_1 \Downarrow b$ [because closed newton cote's]

$$I_n(f) = \int_a^b P_n(x)\, dx$$

$$I_1(f) = \int_a^b P_1(x)\, dx$$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$I_1(f) = \int_a^b \left[l_0(x) f(x_0) + l_1(x) f(x_1) \right] dx$$

$$= \underbrace{\int_a^b l_0(x)\, dx}_{\sigma_0} \cdot f(x_0) + \underbrace{\int_a^b l_1(x)\, dx}_{\sigma_1} \cdot f(x_1)$$

$$\therefore I_1(f) = \sigma_0 f(x_0) + \sigma_1 f(x_1)$$

$$\sigma_0 = \int_a^b l_0(x)\,dx$$

$$= \int_a^b \frac{x - x_1}{x_0 - x_1}\,dx$$

$$= \int_a^b \frac{x - b}{a - b}\,dx$$

$$= \frac{1}{a-b}\int_a^b (x-b)\,dx$$

$$= \frac{1}{a-b}\left[\frac{x^2}{2} - bx\right]_a^b$$

$$= \frac{1}{a-b}\left(\frac{b^2}{2} - b^2 - \frac{a^2}{2} + ab\right)$$

$$= \frac{b-a}{2}$$

$$\sigma_1 = \int_a^b l_1(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b (x-a)\,dx$$

$$\vdots$$

$$= \frac{b-a}{2}$$

$$\therefore I_1(f) = \int_a^b P_1(x)\,dx$$

$$= \sigma_0 f(x_0) + \sigma_1 f(x_1)$$

$$= \sigma_0 f(a) + \sigma_1 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

$$= \frac{b-a}{2}\left(f(a) + f(b)\right)$$

$$\therefore \boxed{I_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]}$$

Example:

Find $\underbrace{I(f)}_{\text{Actual}}$ and $\underbrace{I_1(f)}_{\text{Numerical}}$ of the function e^x on interval $[0,2]$.

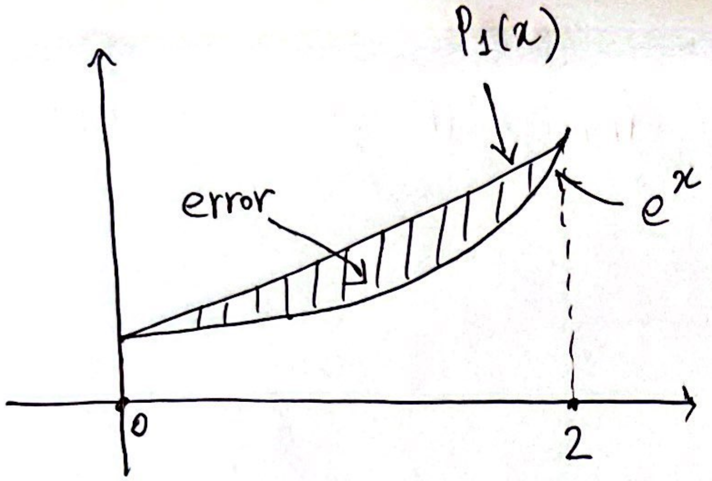
Solution:

$$I(f) = \int_0^2 e^x\,dx = \left[e^x\right]_0^2 = e^2 - e^0 = 6.389 \text{ [Actual]}$$

$$I_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{2-0}{2}\left[e^0 + e^2\right] = 8.389 \text{ [numerical approx]}$$

$$\% \text{ error} = \frac{I - I_1}{I} \times 100 = 31.3\,\%$$ [error is large bcz degree 1 polynomial is used].



→ We can find the upper bound of the error.

→ If a function $f(x)$ is interpolated by a degree n polynomial error ~~is~~ ~~for~~ can be found using cauchy's Theorem.

Upper bound error

$$|f(x) - P_n(x)| \leq \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)(x-x_1) \cdots (x-x_n) \right|$$

For integration, upper bound error =

$$|I - I_n| \leq \underbrace{\left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right|}_{\text{Need to find max value within } [a;b]} \int_a^b \left| (x-x_0)(x-x_1) \cdots (x-x_n) \right| dx$$

Example:

Computing the upper bound of error for the previous example.

→ $n = 1$

$f(x) = e^{x}$

$a = 0$

$b = 2$

Solution:

Finding the max of $\left| \dfrac{f^{(n+1)}(\xi)}{(n+1)!} \right|$ within $[0, 2]$

$$= \left| \frac{f^{(2)}(\xi)}{2!} \right|$$

$$= \left| \frac{e^{\xi}}{2!} \right|$$

$f(x) = e^{x}$

$f^{(1)}(x) = e^{x}$

$f^{(2)}(x) = e^{x}$

$\therefore f^{(2)}(\xi) = e^{\xi}$

Max of $\dfrac{e^{\xi}}{2!}$ within $[0, 2] = \dfrac{e^{2}}{2!}$

$\rightarrow \int_a^b \left| (x - x_0)(x - x_1) \right| dx$

$$= \int_a^b \left| (x-a)(x-b) \right| dx = \int_0^2 \left| (x^2 - 2x) \right| dx$$

$$= \left| \left[\frac{x^3}{3} - \frac{2x^2}{2} \right]_0^2 \right| = \frac{4}{3}$$

$\therefore$ upper bound of error $\leq \dfrac{e^2}{2!} \times \dfrac{4}{3} \approx 4.926.$