Aitken Acceleration:

→ Used to accelerate convergence

Formula:

$$\boxed{\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}}$$

→ starting from x_0, every 2 iteration acceleration occurs.
eg: $x_2, x_4, x_6 \ldots$

$$x_0 \to x_1 \to x_2 \to \hat{x}_2 \to x_3 \to x_4 \to \hat{x}_4 \to x_5 \to x_6 \to \hat{x}_6$$

Example:

$f(x) = \frac{1}{x} - 0.5$ [x_* is at 2]

→ construct a $g(x)$ so that $x=2$ is a fixed point of $g(x)$.

$$g(x) = x + \frac{1}{16}\left(\frac{1}{x} - 0.5\right)$$

$$g(2) = 2$$

$$g'(x) = 1 + \frac{1}{16}\left(-\frac{1}{x^2}\right)$$

$$\lambda = |g'(2)| = 1 + \frac{1}{16}\left(-\frac{1}{2^2}\right) = 0.984375 \quad (<1)$$

→ λ is close to 1. It will converge, but it will be very slow.

→ lets start using $x_0 = 1.5$, keeping the calculations upto 7 s.f.

$$g(x) = x + \frac{1}{16}\left(\frac{1}{x} - 0.5\right), \quad x_0 = 1.5$$

$x_0 = 1.5$

$x_1 = g(x_0) = 1.510417$

$x_2 = g(x_1) = 1.520546$

$x_3 = g(x_2) = 1.530400$

$x_4 = g(x_3) =$

⋮

$x_{818} = g(x_{817}) = 1.999999$

Now, Applying Aitken Acceleration:

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

$$x_0 \rightarrow x_1 \rightarrow x_2 \xrightarrow{\text{Accelerate}} \hat{x}_2 \rightarrow x_3 \rightarrow x_4 \xrightarrow{\text{Accelerate}} \hat{x}_4 \rightarrow x_5 \rightarrow x_6 \xrightarrow{\text{Accelerate}} \hat{x}_6$$

$$\hat{x}_6 \rightarrow x_7 \rightarrow x_8 \xrightarrow{\text{Accelerate}} \hat{x}_8$$

$x_0 = 1.5$

$x_1 = g(x_0) = 1.510417$

$x_2 = g(x_1) = 1.520546$

$$\hat{x}_2 = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0} = 1.877604$$

$x_3 = g(\hat{x}_2) = 1.879641$

$x_4 = g(x_3) = 1.881642$

$$\hat{x}_4 = \hat{x}_2 - \frac{(x_3 - \hat{x}_2)^2}{x_4 - 2x_3 + \hat{x}_2} = 1.992634$$

$\vdots$

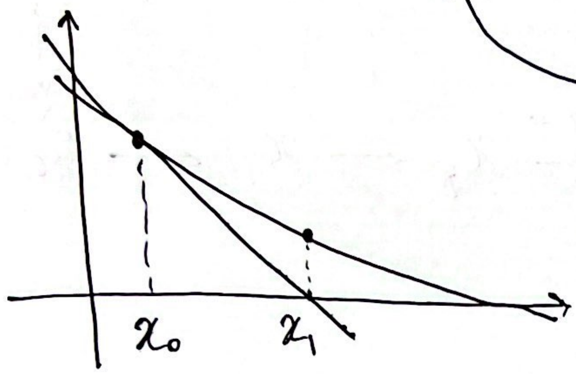
$\hat{x}_8 = 2.000000$

Secant Method / Quasi-Newton Method:

Newton's Method Recap:

$$x_{k+1} = x_k - \underbrace{\frac{f(x_k)}{f'(x_k)}}$$

→ replace $f'(x_k)$ with backward diff.



$$\text{backward differentiation} = \frac{f(x) - f(x-h)}{h}$$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \Bigg\} \text{ put this part instead of } f'(x_k)$$

$\therefore$ Iteration formula for Secant Method:

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

Note: We need 2 starting points for secant Method $(x_0 \text{ and } x_1)$.

Scanned with CamScanner

Example

$$f(x) = \frac{1}{x} - 0.5 \qquad x_0 = 0.25,\ x_1 = 0.5$$

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

$$= x_k - \frac{\left(\frac{1}{x_k} - 0.5\right)(x_k - x_{k-1})}{\left(\frac{1}{x_k} - 0.5\right) - \left(\frac{1}{x_{k-1}} - 0.5\right)}$$

k	x_k
0	0.25
1	0.5
2	0.6875
3	1.01562
4	1.3540
5	1.68205
6	1.8973
7	1.98367
8	1.99916
⋮	⋮
12	2.00000

Example:

$$f(x) = x_k^2 - 2x_k e^{-x_k} + e^{-2x_k} \quad \text{—— ①}$$

Newtons Method Iteration Formula:

$$x_{k+1} = x_k - \frac{x_k^2 - 2x_k e^{-x_k} + e^{-2x_k}}{2x_k - 2e^{-x_k} + 2x_k e^{-x_k} - 2e^{-2x_k}} \quad \text{—— ②}$$

Aitken Acceleration:

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k} \quad \text{—— ③}$$

Flow:

check if within error bound, if not, go next

$$x_0 \xrightarrow{eq\,1} f(x_0) \xrightarrow{eq\,2} x_1 \xrightarrow{eq\,1} f(x_1) \xrightarrow{eq\,2} x_2 \xrightarrow{eq\,1} f(x_2) \xrightarrow{eq\,3}$$

$$x_4 \xleftarrow{eq\,2} f(x_3) \xleftarrow{eq\,1} x_3 \xleftarrow{eq\,2} f(\hat{x}_2) \xleftarrow{eq\,1} \hat{x}_2 \leftarrow$$

$$\xrightarrow{eq\,1} f(x_4) \xrightarrow{eq\,3} \hat{x}_4 \xrightarrow{eq\,1} f(\hat{x}_4) \xrightarrow{eq\,2} x_5$$

k	x_k	$f(x_k)$	if $\lvert f(x_k)\rvert < 10^{-5}$?
0	1 $\xrightarrow{eq\,1}$	0.399576	No
	$\downarrow$ eq 2		
1	0.768941 $\xrightarrow{eq\,1}$	0.093292	No
	$\downarrow$ eq 2		
2	0.664590 $\xrightarrow{eq\,1}$	0.022532	No
	$\downarrow$ eq 3		
$\hat{2}$	0.578651 $\xrightarrow{eq\,1}$	3.2×10^{-4}	No
	$\downarrow$ eq 2		
3	0.572885 $\xrightarrow{eq\,1}$	8×10^{-5}	No
	$\downarrow$ eq 2		
4	0.570011 $\xrightarrow{eq\,1}$	2×10^{-5}	No
	$\downarrow$ eq 3		
$\hat{4}$	0.567154 $\xrightarrow{eq\,1}$	2.8×10^{-10}	Yes

Answer: $x_* = 0.56715$ (upto 5 d.p).