Detailed Corrected Exercises - Common Probability Distributions

Exercise 1

The weight of 80 bags of animal feed is normally distributed with a mean of 68 kg and a standard deviation of 0.6 kg.

1. How many of these bags weigh between 66.8 and 68.3 kg?
2. How many weigh less than 66.4 kg?

Detailed solution:

Let X be the random variable representing the weight of a bag. We have $X \sim \mathcal{N}(68, 0.6^2)$.

1. To find the number of bags weighing between 66.8 and 68.3 kg:

   $P(66.8 < X < 68.3) = P(\frac{66.8-68}{0.6} < Z < \frac{68.3-68}{0.6}) = P(-2 < Z < 0.5)$

   $= \Phi(0.5) - \Phi(-2)$

   From the standard normal distribution table: $\Phi(0.5) = 0.6915$ and $\Phi(-2) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$

   Therefore, $P(66.8 < X < 68.3) = 0.6915 - 0.0228 = 0.6687$

   Number of bags: $0.6687 \times 80 = 53.496$, which is 53 or 54 bags.

2. To find the number of bags weighing less than 66.4 kg:

   $P(X < 66.4) = P(Z < \frac{66.4-68}{0.6}) = P(Z < -2.67)$

   $= \Phi(-2.67) = 1 - \Phi(2.67)$

   From the table: $\Phi(2.67) = 0.9962$

   Therefore, $P(X < 66.4) = 1 - 0.9962 = 0.0038$

   Number of bags: $0.0038 \times 80 = 0.304$, which is 0 or 1 bag.

Exercise 2

Let X be a normal random variable with distribution $\mathcal{N}(\mu, \sigma)$, where $\sigma > 0$, such that $P(X < 7) = 27.43\%$ and $P(X > 14) = 21.19\%$.

1. Define μ and σ.
2. Calculate $P(6 < X < 8)$.
3. Let $Y = 2X$. What is the distribution of Y? Define $P(Y > 19)$.
4. Let $Z = X + 10$. What is the distribution of Z? Define $P(Z > 19)$.

Detailed solution:

1. To find μ and σ, we use the given information:

   $P(X < 7) = 0.2743$ implies $\Phi(\frac{7-\mu}{\sigma}) = 0.2743$ From the inverse table: $\frac{7-\mu}{\sigma} = -0.60$ (1)

   $P(X > 14) = 0.2119$ implies $1 - \Phi(\frac{14-\mu}{\sigma}) = 0.2119$ So $\Phi(\frac{14-\mu}{\sigma}) = 0.7881$ From the inverse table: $\frac{14-\mu}{\sigma} = 0.80$ (2)

   Subtracting (1) from (2), we get: $\frac{7}{\sigma} = 1.40$ so $\sigma = 5$

   Substituting in (1) or (2), we find $\mu = 10.5$

   Therefore $X \sim \mathcal{N}(10.5, 5^2)$

2. $P(6 < X < 8) = P(\frac{6-10.5}{5} < Z < \frac{8-10.5}{5}) = P(-0.9 < Z < -0.5)$
   $= \Phi(-0.5) - \Phi(-0.9)$

   From the table: $\Phi(-0.5) = 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085$ and $\Phi(-0.9) = 1 - \Phi(0.9) = 1 - 0.8159 = 0.1841$

   Therefore $P(6 < X < 8) = 0.3085 - 0.1841 = 0.1244$

3. $Y = 2X \sim \mathcal{N}(2 \times 10.5, (2 \times 5)^2) = \mathcal{N}(21, 10^2)$

   $P(Y > 19) = P(\frac{Y-21}{10} > \frac{19-21}{10}) = P(Z > -0.2) = 1 - \Phi(-0.2) = \Phi(0.2)$

   From the table: $\Phi(0.2) = 0.5793$

4. $Z = X + 10 \sim \mathcal{N}(10.5 + 10, 5^2) = \mathcal{N}(20.5, 5^2)$

   $P(Z > 19) = P(\frac{Z-20.5}{5} > \frac{19-20.5}{5}) = P(Z > -0.3) = 1 - \Phi(-0.3) = \Phi(0.3)$

   From the table: $\Phi(0.3) = 0.6179$

Exercise 3

The height in cm of 2,500 men is measured; the resulting distribution is normal with a mean of 169 cm and a standard deviation of 5.6 cm.

1. What percentage of men are less than 155 cm tall?
2. What percentage of men are between 155 cm and 175 cm tall?
3. What is the interval, centered on the mean height value, that contains 80

Detailed solution:

Let X be the random variable representing a man's height. We have $X \sim \mathcal{N}(169, 5.6^2)$.

1. $P(X < 155) = P(Z < \frac{155-169}{5.6}) = P(Z < -2.5) = \Phi(-2.5)$

   From the table: $\Phi(-2.5) = 1 - \Phi(2.5) = 1 - 0.9938 = 0.0062$

   Therefore, 0.62% of men are less than 155 cm tall.

2. $P(155 < X < 175) = P(\frac{155-169}{5.6} < Z < \frac{175-169}{5.6}) = P(-2.5 < Z < 1.07)$
   $= \Phi(1.07) - \Phi(-2.5)$

   From the table: $\Phi(1.07) = 0.8577$ and $\Phi(-2.5) = 0.0062$ (calculated previously)

   Therefore $P(155 < X < 175) = 0.8577 - 0.0062 = 0.8515$

   Thus, 85.15% of men are between 155 cm and 175 cm tall.

3. To find the interval centered on the mean containing 80% of the population, we look for a such that:

   $P(169 - a < X < 169 + a) = 0.80$

   $P(-\frac{a}{5.6} < Z < \frac{a}{5.6}) = 0.80$

   $2\Phi(\frac{a}{5.6}) - 1 = 0.80$

   $\Phi(\frac{a}{5.6}) = 0.90$

   From the inverse table: $\frac{a}{5.6} = 1.28$

   $a = 1.28 \times 5.6 = 7.168$

   The interval is therefore [169 - 7.168, 169 + 7.168] = [161.832 cm, 176.168 cm].

Exercise 4

The inside diameter of a sample of 200 machine-produced pen bodies is estimated to be 0.502 cm and the average standard deviation is of the order of 0.005 cm; the distribution of these diameters is assumed to follow a normal law. These parts must pass through the fully automated assembly line, but they can only be suitable for this if and only if their diameter is between 0.496 and 0.508 cm.

1. What is the percentage of pen bodies that should be considered defective?
2. How does this percentage change if a new machine is purchased, which only admits diameters between 0.498 and 0.510 cm?

Detailed solution:
Let X be the random variable representing the diameter of a pen body. We have $X \sim \mathcal{N}(0.502, 0.005^2)$.

1. The percentage of defective pen bodies is:

   $P(X < 0.496 \text{ or } X > 0.508) = 1 - P(0.496 < X < 0.508)$
   $= 1 - [P(Z < \frac{0.508-0.502}{0.005}) - P(Z < \frac{0.496-0.502}{0.005})]$
   $= 1 - [\Phi(1.2) - \Phi(-1.2)]$

   From the table: $\Phi(1.2) = 0.8849$ and $\Phi(-1.2) = 1 - 0.8849 = 0.1151$

   Therefore, the percentage of defective pen bodies is: $1 - [0.8849 - 0.1151] = 1 - 0.7698 = 0.2302$

   Thus, 23.02% of the pen bodies are defective.

2. With the new machine, the percentage of defective pen bodies becomes:

   $P(X < 0.498 \text{ or } X > 0.510) = 1 - P(0.498 < X < 0.510)$
   $= 1 - [P(Z < \frac{0.510-0.502}{0.005}) - P(Z < \frac{0.498-0.502}{0.005})]$
   $= 1 - [\Phi(1.6) - \Phi(-0.8)]$

   From the table: $\Phi(1.6) = 0.9452$ and $\Phi(-0.8) = 1 - \Phi(0.8) = 1 - 0.7881 = 0.2119$

   Therefore, the new percentage of defective pen bodies is: $1 - [0.9452 - 0.2119] = 1 - 0.7333 = 0.2667$

   Thus, 26.67% of the pen bodies would be defective with the new machine.

Exercise 5

A study of young children shows that first words appear, on average, at 11.5 months with a standard deviation of 3.2 months. As the age distribution is normal, assess the proportion of children who have acquired their first words:

1. Before 10 months.
2. After 18 months.
3. Between 8 and 12 months.

Detailed solution:
Let X be the random variable representing the age at which a child acquires their first words. We have $X \sim \mathcal{N}(11.5, 3.2^2)$.

1. $P(X < 10) = P(Z < \frac{10-11.5}{3.2}) = P(Z < -0.46875) = \Phi(-0.46875)$

   From the table: $\Phi(-0.46875) \approx 1 - \Phi(0.47) = 1 - 0.6808 = 0.3192$

   Therefore, approximately 31.92% of children acquire their first words before 10 months.

2. $P(X > 18) = P(Z > \frac{18-11.5}{3.2}) = P(Z > 2.03125) = 1 - \Phi(2.03125)$

   From the table: $\Phi(2.03) = 0.9788$ (rounding slightly)

   Therefore $P(X > 18) = 1 - 0.9788 = 0.0212$

   Thus, approximately 2.12% of children acquire their first words after 18 months.

3. $P(8 < X < 12) = P(\frac{8-11.5}{3.2} < Z < \frac{12-11.5}{3.2}) = P(-1.09375 < Z < 0.15625)$

   $= \Phi(0.15625) - \Phi(-1.09375)$

   From the table: $\Phi(0.15625) \approx \Phi(0.16) = 0.5636$ $\Phi(-1.09375) \approx 1 - \Phi(1.09) = 1 - 0.8621 = 0.1379$

   Therefore $P(8 < X < 12) = 0.5636 - 0.1379 = 0.4257$

   Thus, approximately 42.57% of children acquire their first words between 8 and 12 months.

Exercise 6

In 1955, Welcher (1896-1981) proposed a test to measure the IQ (Intelligence Quotient) of adults with a representative sample of the population of a given age. The performances follow a normal distribution with mean equal to 100 and standard deviation equal to 15.

1. What percentage of people have an IQ below 100?
2. What is the chance of getting an IQ including:
   (a) Between 100 and 110?
   (b) Between 95 and 100?
   (c) Between 105 and 110?
3. Is a person with a score of 69 part of the bottom 5
4. Below what IQ is a third of individuals?
5. What minimum IQ do you need to get to be in the top 5

Detailed solution:
Let X be the random variable representing the IQ. We have $X \sim \mathcal{N}(100, 15^2)$.

1. $P(X < 100) = P(Z < 0) = \Phi(0) = 0.5$ or 50%

2. (a) $P(100 < X < 110) = P(0 < Z < \frac{10}{15}) = \Phi(\frac{2}{3}) - \Phi(0)$
   From the table: $\Phi(\frac{2}{3}) = 0.7454$
   Therefore $P(100 < X < 110) = 0.7454 - 0.5 = 0.2454$ or 24.54%
   (b) $P(95 < X < 100) = P(-\frac{1}{3} < Z < 0) = \Phi(0) - \Phi(-\frac{1}{3})$
   From the table: $\Phi(-\frac{1}{3}) = 1 - \Phi(\frac{1}{3}) = 1 - 0.6293 = 0.3707$
   Therefore $P(95 < X < 100) = 0.5 - 0.3707 = 0.1293$ or 12.93%
   (c) $P(105 < X < 110) = P(\frac{1}{3} < Z < \frac{2}{3}) = \Phi(\frac{2}{3}) - \Phi(\frac{1}{3})$
   We already have $\Phi(\frac{2}{3}) = 0.7454$ and $\Phi(\frac{1}{3}) = 0.6293$
   Therefore $P(105 < X < 110) = 0.7454 - 0.6293 = 0.1161$ or 11.61%

3. To determine if a person with a score of 69 is part of the bottom 5%:

   $P(X < 69) = P(Z < \frac{69-100}{15}) = P(Z < -2.0667) = \Phi(-2.0667)$

   From the table: $\Phi(-2.0667) \approx 1 - \Phi(2.07) = 1 - 0.9808 = 0.0192$

   As 0.0192 ¡ 0.05, this person is indeed part of the bottom 5% of the distribution.

4. To find the IQ below which a third of individuals are situated, we look for x such that:

   $P(X < x) = \frac{1}{3}$
   $P(Z < \frac{x-100}{15}) = \frac{1}{3}$
   $\frac{x-100}{15} = \Phi^{-1}(\frac{1}{3}) \approx -0.4307$ (from the inverse table)
   $x = 100 - 0.4307 \times 15 = 93.5395$

   Therefore, a third of individuals have an IQ below approximately 93.54.

5. To find the minimum IQ to be in the top 5%, we look for x such that:

   $P(X > x) = 0.05$
   $P(Z > \frac{x-100}{15}) = 0.05$
   $\frac{x-100}{15} = \Phi^{-1}(0.95) = 1.6449$ (from the inverse table)
   $x = 100 + 1.6449 \times 15 = 124.6735$

   Therefore, you need a minimum IQ of approximately 124.67 to be in the top 5%.