Quantitative Uniqueness Results in Complex Analysis

Cute Results

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Holomorphic Functions

Definition

Given an open set $U \subseteq \mathbb{C}$, a function $f : U \to \mathbb{C}$ is said to be **holomorphic** if it is complex differentiable at every point in U.

Remark

Recall, a function f is said to be **complex differentiable** at z_0 if the following limit exists

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Further, if $f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic, then u and v satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Schwarz Lemma

Holomorphic functions have AMAZING properties!

Lemma (Schwarz Lemma)

If $f : \mathbb{D} \to \mathbb{D}$ *is a holomorphic function with* $f(0) = 0$*, then* $|f'(0)| \leq 1$*. If* $|f'(0)| = 1$*, then* $f(z) = f'(0)z$ *for all* $z \in \mathbb{D}$

Corollary

There is a unique holomorphic function $f : \mathbb{D} \to \mathbb{D}$ *with* $f(0) = 0$ *and* $f'(0) = 1$*, namely* $f(z) = z$

Poincaré Metric

Instead of using Euclidean distance, we use the Poincaré Metric since it has some nice properties when working inside the disk with holomorphic functions.

Definition

The Poincaré metric is defined by

$$\rho(z,w) = 2\tanh^{-1}\left(\frac{z-w}{1-\bar{z}w}\right)$$

for any $z, w \in \mathbb{D}$.

- $\mathbb{D}$ is a complete metric space under ρ
- If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic (like the functions we are studying), then

$$\rho(f(z), f(w)) \leq \rho(z,w)$$

(equality if f^{-1} is holomorphic).

Picture of the Poincaré Metric

Quantitative Schwarz Lemma

Using the Poincaré Metric, we can frame a quantitative version of the Schwarz Lemma:

Theorem

There exists a $C > 0$ *such that if* $f : \mathbb{D} \to \mathbb{D}$ *holomorphic and* $f(0) = 0$, *then*

$$\rho(f(z), z) \leq C|f'(0) - 1|e^{2\rho(z,0)}$$

for all $z \in \mathbb{D}$. *In particular, we've shown* $C = 18$ *works.*

Unlike the previous corollary where we assumed $f'(0) = 1$, we now create a bound on the difference between f and the identity (the $\rho(f(z), z)$ term) in terms of $f'(0)$ (the $|f'(0) - 1|$ term).

Next task

Now, we will try without assuming $f(0) = 0$:

Problem

There exists a $C > 0$ *such that if* $f : \mathbb{D} \to \mathbb{D}$ *holomorphic, then*

$$\rho(f(z), z) \leq C(|f(0) - 0| + |f'(0) - 1|)e^{2\rho(z,0)}$$

for all $z \in \mathbb{D}$*.*

References I

[1] Vladimir Bolotnikov. "A Uniqueness Result on Boundary Interpolation". In: *Proceedings of the American Mathematical Society* 136.5 (2008), pp. 1705–1715. ISSN: 0002-9939. URL: https://www.jstor.org/stable/20535346 (visited on 03/06/2023).

[2] Filippo Bracci, Daniela Kraus, and Oliver Roth. *A new Schwarz-Pick Lemma at the boundary and rigidity of holomorphic maps*. version: 1. Mar. 4, 2020. DOI: 10.48550/arXiv.2003.02019. arXiv: 2003.02019[math]. URL: http://arxiv.org/abs/2003.02019 (visited on 03/06/2023).