Quantitative Uniqueness Results in Complex Analysis

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The Schwarz Lemma

The Schwarz Lemma characterizes holomorphic functions from the disk to the disk which map 0 to 0 and maximize the size of $|f'(0)|$.

The Schwarz Lemma. *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$, then $|f'(0)| \leq 1$. Moreover, if $|f'(0)| = 1$, then $f(z) = f'(0)z$ for all $z \in \mathbb{D}$*

And in particular, it shows that the indentify is the only holomorphic function that maps the origin to the origin with derivative 1.

Corollary 1. *There is a unique holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$, namely $f(z) = z$.*

Poincaré Metric

For our task, the Poincaré metric is more suitable than the usual Euclidean metric. It is defined by

$$\rho(z, w) = 2\tanh^{-1}\left(\frac{z-w}{1-\bar{z}w}\right)$$

for any $z, w \in \mathbb{D}$.

This metric is more suitable because it works well in the disk and has nice properties with holomorphic functions. In particular, if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\rho(f(z), f(w)) \leq \rho(z, w)$$

(equality if f^{-1} is also holomorphic).

An Illustration of Poincaré Metric

Under the Poincaré metric, the distance from the origin to points near the boundary approaches infinity. In the figure below, all the bats are the same size under the Poincaré metric.

Fig. 1: M.C. Escher Circle Limit IV

The Quantitative Schwarz Lemma

The main question that we addressed is if f is holomorphic and $f(0) \approx 0$, $f'(0) \approx 1$, is $f(z) \approx z$? In particular, what does "$\approx$" mean?

We proved

The Quantitative Schwarz Lemma. *There exists a $C > 0$ such that if $f : \mathbb{D} \to \mathbb{D}$ holomorphic, then*

$$\rho(f(z), z) \leq C\left(\rho(f(0), 0) + |f'(0) - 1|\right) e^{2\rho(z,0)}$$

for all $z \in \mathbb{D}$.

Blaschke Products

For a finite sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{D}$ and $\theta \in \mathbb{R}$, the function

$$B(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - \alpha_k}{\overline{\alpha_k}\alpha - 1}$$

is a degree n Blaschke Product.

Corollary to Nevanlinna-Pick If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $z_0 \in \mathbb{D}$, then there exists a degree 3 Blaschke product B with $B(0) = f(0)$, $B'(0) = f'(0)$, and $B(z_0) = f(z_0)$

Sketch of Proof of the Quantitative Schwarz Lemma

We start from the right hand side.
First by Nevanlinna-Pick, we can assume f is a degree 3 blaschke prodect

$$f = B(z) = -e^{i\theta}\frac{z-\alpha_1}{z\overline{\alpha_1}-1}\frac{z-\alpha_2}{z\overline{\alpha_2}-1}\frac{z-\alpha_2}{z\overline{\alpha_2}-1}.$$

Without loss of generality, assume $|\alpha_1| \leq |\alpha_2| \leq |\alpha_3|$. Then we show that when $\max\{|f(0) - 0|, |f'(0) - 1|\} = \epsilon < \frac{1}{216}$ we have C_1, C_2, C_3 such that

$$|\alpha_1| \leq C_1\epsilon, |\alpha_2| \geq 1 - C_2\epsilon, |\alpha_3| \geq 1 - C_3\epsilon.$$

With these result we're able to give the following estimation:

$$\left(\exists C > 0\right)\left(|f(z) - z| \leq \frac{C\epsilon}{1-|z|^2} \leq \frac{C\max\{|f(0)-0|, |f'(0)-1|\}}{1-|z|^2} < \frac{C(|f(0)-0| + |f'(0)-1|)}{1-|z|^2}\right)$$

For the last term on the right, we have

$$\left|e^{\rho(z,0)}\right| = \left|\frac{1+z}{1-z}\right|^2 \geq \left|\frac{1}{1-z}\right|^2 \geq \left|\frac{1}{1-|z|^2}\right|^2$$

We need two other estimations:

$$\rho(f(z), z) \leq \frac{2|f(z)-z|}{1-|z|^2}$$

and

$$\rho(f(0), 0) \geq |f(0) - 0|$$

Put everything together, we have the desired inequality.

Introducing Jets

For a holomorphic function $f : \mathbb{D} \to \mathbb{D}$, and a point $w_0 \in \mathbb{D}$, the k-jet of f at w_0 is defined to be a vector $(f(w_0), f'(w_0), \ldots, f^{(k)}(w_0))$ in $\mathbb{C}^{k+1}$. Denote it as $\mathcal{J}^k_{w_0}(f)$.

Lemma. *For two Blaschke products B_1, B_2 of degree at most k, if there exist k distinct points $w_1, \ldots, w_k$ in $\mathbb{D}$, and $n_1, \ldots, n_k$ that are positive integers such that for all $i = 1, \ldots, k$ $\mathcal{J}^{n_i}_{w_i}(B_1) = \mathcal{J}^{n_1}_{w_i}(B_2)$, then we have $B_1 = B_2$.*

Future Directions

We're going to study jets and avionics instead of Complex Analysis.
There are many ways to generalize Schwarz lemma. One that we're still working on is approximating functions with the "norm" of jets.
For holomorphic function f from $\mathbb{D} \to \mathbb{D}$, the "norm" of its k-jet at point w_0, denoted as $||\mathcal{J}^k_w(f)||_{\text{hyp}}$ is defined to be

$$\left(\sum_{i=0}^{k} |(\varphi_{f(w)} \circ f \circ \varphi_w)^{(i)}(0)|^2\right)^{\frac{1}{2}}$$

where $\varphi_\alpha(z) = \frac{z-\alpha}{z\overline{\alpha}-1}$, which is the Möbius transformation and $|\cdot|$ denotes the absolute value of a complex number.
We have shown that

- $||\mathcal{J}^k_w(f)||_{\text{hyp}} \geq 0$, and $||\mathcal{J}^k_w(f)||_{\text{hyp}} = 0$ if and only if $\mathcal{J}^k_w(f) = 0$.
- $||\mathcal{J}^k_w(f)||_{\text{hyp}} = ||\mathcal{J}^k_w(\varphi_\alpha \circ f)||_{\text{hyp}}$.
- $||\mathcal{J}^k_w(f)||_{\text{hyp}} = ||\mathcal{J}^k_{\varphi_\beta^{-1}(w)}(f \circ \varphi_\beta)||_{\text{hyp}}$.

We will work on defining the distance between two jets with such "norm", and try to find cute estimations.

References

[1] Vladimir Bolotnikov. "A Uniqueness Result on Boundary Interpolation". In: *Proceedings of the American Mathematical Society* 136.5 (2008), pp. 1705–1715. ISSN: 0002-9939. URL: https://www.jstor.org/stable/20535346 (visited on 03/06/2023).

[2] Filippo Bracci, Daniela Kraus, and Oliver Roth. *A new Schwarz-Pick Lemma at the boundary and rigidity of holomorphic maps*. version: 1. Mar. 4, 2020. DOI: 10.48550/arXiv.2003.02019. arXiv: 2003.02019[math]. URL: http://arxiv.org/abs/2003.02019 (visited on 03/06/2023).

[3] Escher. *Cirkellimiet IV (Hemel en hel)*. Museum Escher in Het Paleis. July 14, 2018. URL: https://www.escherinhetpaleis.nl/escher-vandaag/cirkellimiet-iv-hemel-hel/ (visited on 04/26/2023).

[4] Stephan Ramon Garcia, Javad Mashreghi, and William T. Ross. *Finite Blaschke Products and Their Connections*. 1st ed. 2018. Cham: Springer International Publishing : Imprint: Springer, 2018. 1 p. ISBN: 9783319782478.