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Assignment 1
CPSC 482

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The modified algorithm is exactly the same as the original Gram-Shapley algorithm with the caveat that one several residents can accept offers from one hospital. Referring to Algorithm 1, if we have m hospitals and n residents then we will clearly have $\mathcal{O}(mn)$ complexity as each hospital only gives an offer to each of the n residents at most once and at each iteration an offer is made. We now show that algorithm produces a stable matching.

First, we note that after running the algorithm each hospital will have all its positions full, as if not that would contradict its termination which was shown previously by being $\mathcal{O}(mn)$. We now will prove the traditional stability of Algorithm 1 beginning with the first type. Namely, that there is a resident r assigned to a hospital H and r' is assigned to no hospitals but H prefers r' over r. Suppose this situation were to occur, then we know r' received an offer from H before H gave an offer to r. Therefore, in either case if r' was to accept or decline, it would imply that they have an offer from a hospital (not necessarily H). However, once a resident has an offer, they can only trade up to a higher preference offer, therefore a contradiction that r' has not been assigned to a hospital.

Finally suppose that there is an instability. That is, in the output we have the matching (H, r) and (H^*, r^*) where H prefers r^* to r and r^* prefers H to H^*. Now in the execution of the algorithm, H must have offered a position to r^* before offering one to r as offers are given in the order of preference. At that point suppose r^* rejected the offer, which means that it already had an offer from a hospital of higher preference than H, which is naturally a contradiction to r^* preferring H or over H^*. Alternatively, if r accepted the offer, then at some point it received a better offer from some hospital H'. In both of these cases, the contradiction occurs because list of preference by definition has a strict well ordering and the fact that residents only accept offers better than there current one.

Algorithm 1: Modified Gram-Shapley Algorithm

Result: Hospital Biased Stable Matching
hospitals $\leftarrow$ hospitals with individual preferences and number of positions available;
residents $\leftarrow$ residents with individual preferences;
while *There is a hospital $H \in$ hospitals that has an opening* **do**
    H offers a position to the highest preference on its list r;
    H removes r from preference list;
    if *r has not received any offers* **then**
        r tentatively accepts the offer from H;
        decrease the number of openings in H by one;
    else
        `//` r `has already has accepted an offer from` $H^* \neq H$
        if *r prefers H to H^** **then**
            r accepts offer from H;
            increase number of opening in H^* by one;
            decrease the number of opening in H by one;
        end
    end
end