Transforms

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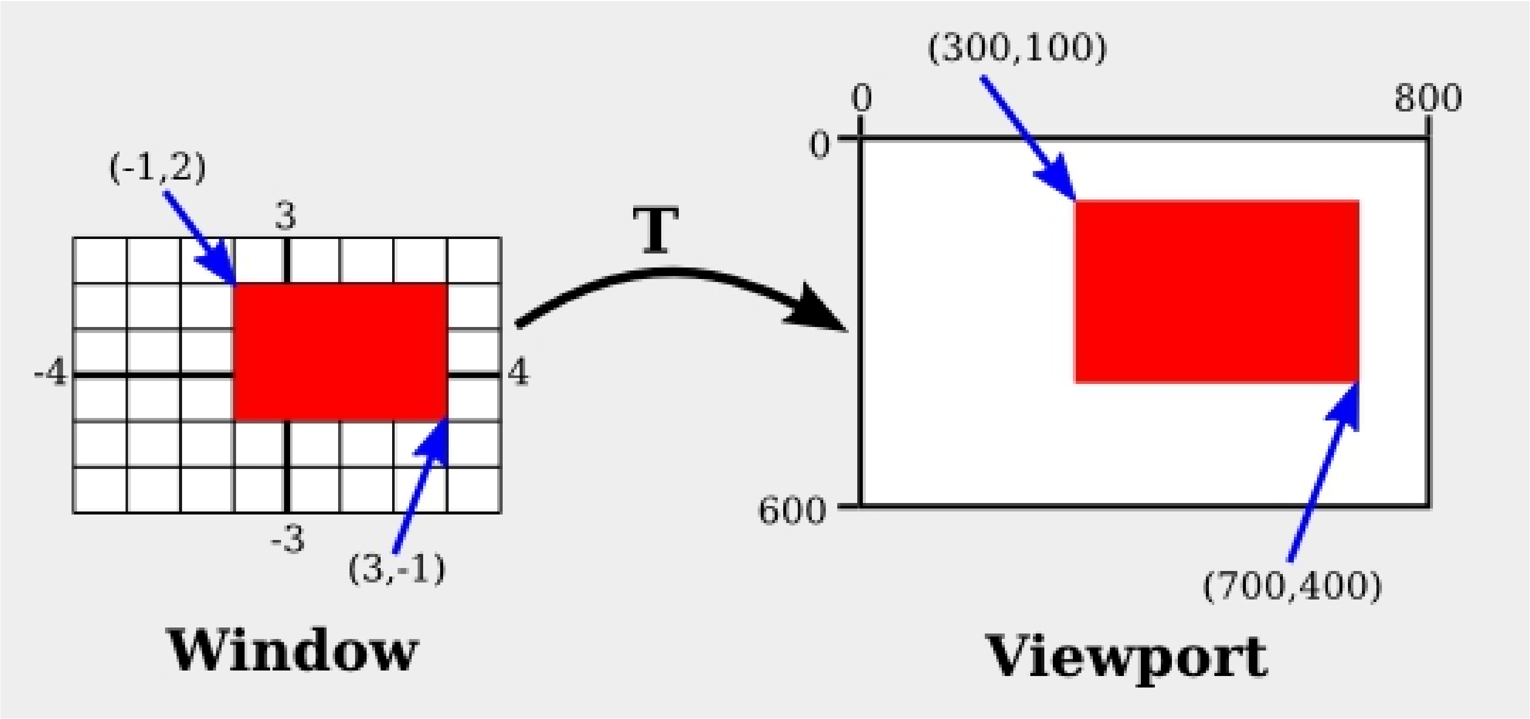
Viewing and Modeling

- How geometric transformations can be used to place graphics objects into a coordinate system
- In a typical application, we have a rectangle made of pixels, with its natural pixel coordinates, where an image will be displayed
  - This rectangle is called the **viewport**
- The coordinates that we use to define the scene are called world **coordinates**
- For 2D graphics, the world lies in a plane. We pick some rectangular area in the plane to display in the image
  - That rectangular area is called the **window**, or **view window.**
- A coordinate transform is used to map the window to the viewport.

Viewing and Modeling



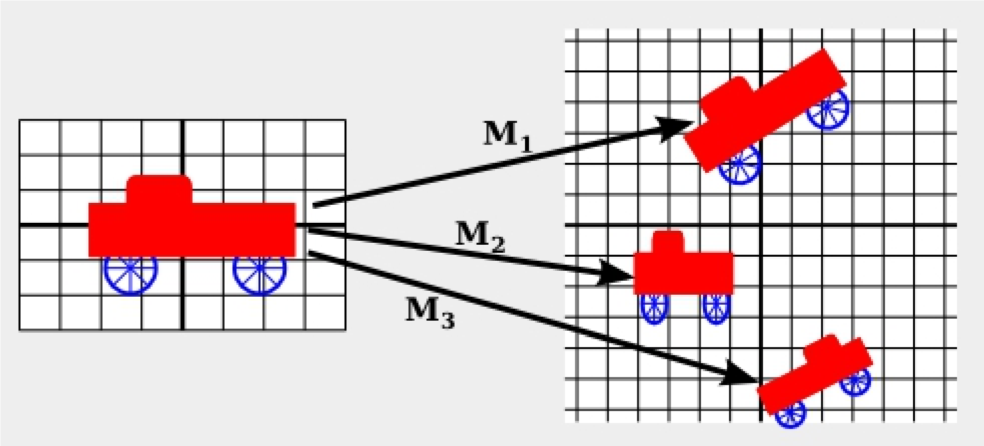
In this illustration, **T** represents the coordinate transformation

Viewing and Modeling

- **T** is a function that takes world coordinates (*x*,*y*) in some window and maps them to pixel coordinates **T**(*x*,*y*) in the viewport
- From the above example; T(x,y) = (800*(x+4)/8, 600*(3-y)/6)
  - Look at the rectangle with corners at (-1,2) and (3,-1) in the window. When this rectangle is displayed in the viewport, it is displayed as the rectangle with corners **T**(-1,2) and **T**(3,-1). In this example, **T**(-1,2) = (300,100) and **T**(3,-1) = (700,400)
- Coordinate transformations are easier to do that than to work directly with viewport coordinates
- Suppose that we want to define some complex object, and suppose that there will be several copies of that object in our scene.
- The coordinates that we use to define an object are called object coordinates for the object.

Viewing and Modeling

- We need to transform the object coordinates that we used to define the object into the world coordinate system that we are using for the scene.
- The transformation that we need is called a **modeling transformation**.
- This picture illustrates an object defined in its own object coordinate system and then mapped by three different modeling transformations into the world coordinate system:

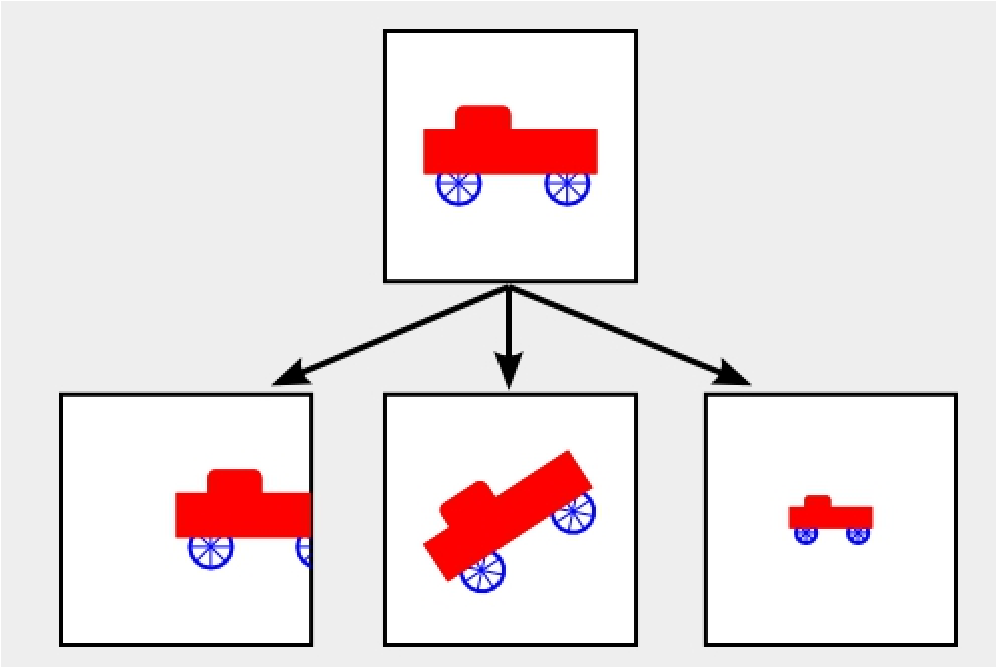


Viewing and Modeling

- Remember that in order to view the scene, there will be another transformation that maps the object from a view window in world coordinates into the viewport.
- The choice of a view window tells which part of the scene is shown in the image
  - Moving, resizing, or even rotating the window will give a different view of the scene

Viewing and Modeling

- Suppose we make several images of the same car:



Viewing and Modeling

- What happened between making the top image in this illustration and making the image on the bottom left?
  - Either the car was moved to the *right*, or the view window that defines the scene was moved to the *left*
- Similarly, what happens between the top picture and the middle picture on the bottom?
  - Either the car rotated counterclockwise, or the window was rotated clockwise
- Finally, the change from the top picture to the one on the bottom right could happen because the car got smaller or because the window got larger.

Viewing and Modeling

- Important general idea: When we modify the view window, we change the coordinate system that is applied to the viewport.
- But in fact, this is the same as leaving that coordinate system in place and moving the objects in the scene instead
- So, there is no essential distinction between transforming the window and transforming the object.
- Mathematically, you specify a geometric primitive by giving coordinates in some natural coordinate system, and the computer applies a sequence of transformations to those coordinates to produce, in the end, the coordinates that are used to actually draw the primitive in the image
- You will think of some of those transformations as modeling transforms and some as coordinate transforms, but to the computer, it's all the same

Translation

- A translation transform simply moves every point by a certain amount horizontally and a certain amount vertically
- If (x,y) is the original point and $(x1,y1)$ is the transformed point, then the formula for a translation is

  $x1 = x + e$

  $y1 = y + f$

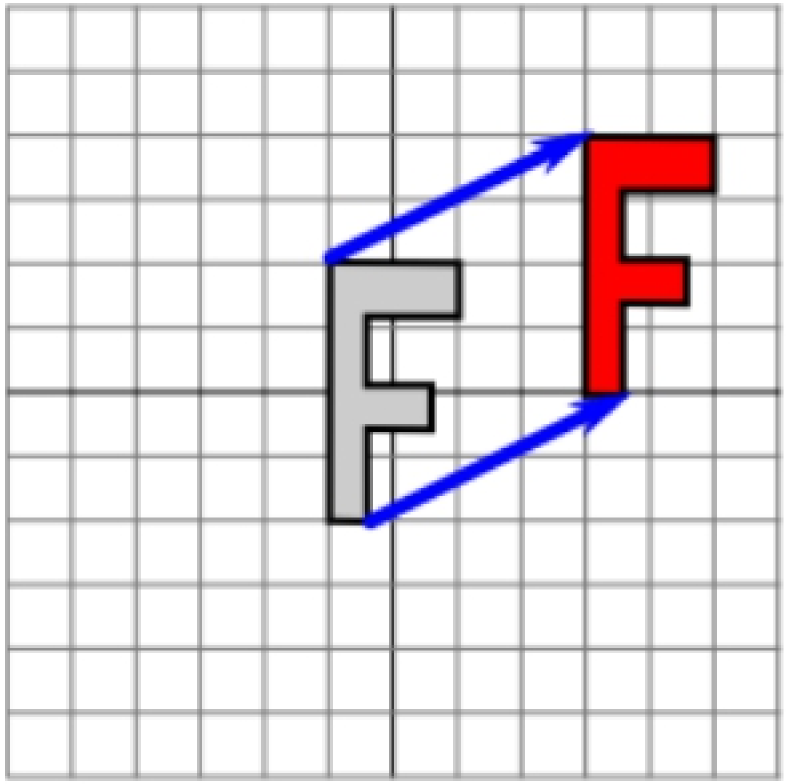
  - where *e* is the number of units by which the point is moved horizontally and *f* is the amount by which it is moved vertically
- A 2D graphics system will typically have a function to apply a translate transformation such as

  translate(e, f)

Translation

- The translation would apply to everything that is drawn after the command is given.
- That is, for all subsequent drawing operations, e would be added to the x-coordinate and f would be added to the y-coordinate.
- E.g. Suppose that you draw an "F" using coordinates in which the "F" is centered at (0,0). If you say translate(4,2) before drawing the "F", then every point of the "F" will be moved horizontally by 4 units and vertically by 2 units before the coordinates are actually used, so that after the translation, the "F" will be centered at (4,2):

Translation



- The light gray "F" in this picture shows what would be drawn without the translation; the dark red "F" shows the same "F" drawn after applying a translation by (4,2).
- The top arrow shows that the upper left corner of the "F" has been moved over 4 units and up 2 units.
- Every point in the "F" is subjected to the same displacement

Translation

- Note: when you give the command translate(e,f), the translation applies to all the drawing that you do after that, not just to the next shape that you draw
- If you apply another transformation after the translation, the second transform will not replace the translation. It will be combined with the translation, so that subsequent drawing will be affected by the combined transformation.
- For example, if you combine translate(4,2) with translate(-1,5), the result is the same as a single translation, translate(3,7).
- Also remember that you don't compute coordinate transformations yourself. You just specify the original coordinates for the object (that is, the object coordinates), and you specify the transform or transforms that are to be applied

Rotation

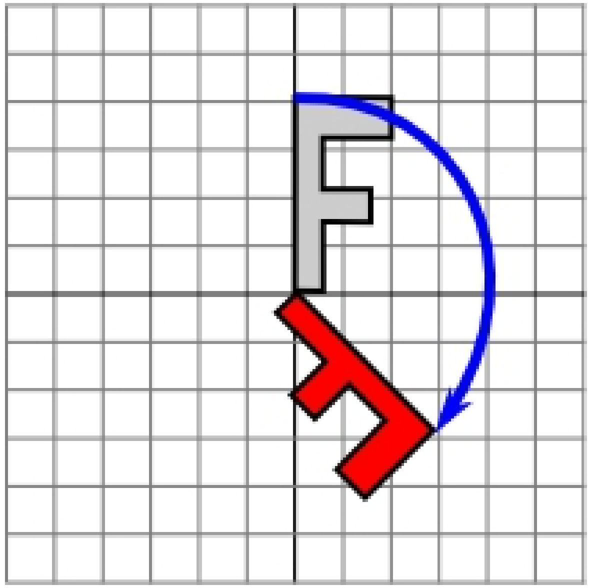
- A rotation transform, for our purposes here, rotates each point about the origin, (0,0).
- Every point is rotated through the same angle, called the angle of rotation measured either in degrees or in radians.
- A rotation with a positive angle rotates objects in the direction from the positive x-axis towards the positive y-axis.
- When rotation through an angle of r radians about the origin is applied to the point (x,y), then the resulting point (x1,y1) is given by:

  x1 = cos(r) * x - sin(r) * y

  y1 = sin(r) * x + cos(r) * y

Rotation

- Here is a picture that illustrates a rotation about the origin by the angle negative 135 degrees:



- light gray "F" is the original shape, and the dark red "F" is the shape that results if you apply the rotation. The arrow shows how the upper left corner of the original "F" has been moved.
- A 2D graphics API would typically have a command *rotate*(*r*) to apply a rotation. The command is used **before** drawing the objects to which the rotation applies.

Combining Transformations

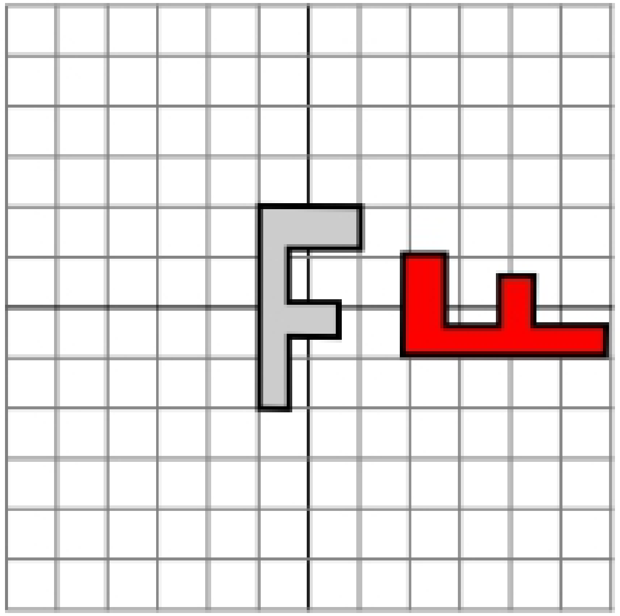
- Suppose that before drawing some object, you say

  translate(4,0)

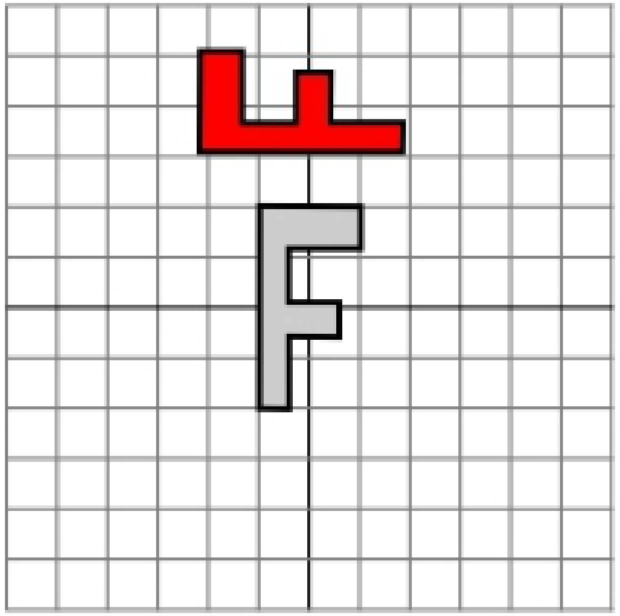
  rotate(90 degrees)

- The translation will then apply to all subsequent drawing. But, because of the rotation command, the things that you draw after the translation are rotated objects.
- That is, the translation applies to objects that have already been rotated.
- An example is shown on the left in the illustration below, where the light gray "F" is the original shape, and red "F" shows the result of applying the two transforms to the original. The original "F" was first rotated through a 90 degree angle, and then moved 4 units to the right

Combining Transformations



Rotate then translate



Translate then rotate

Note that the order in which the transforms are applied is important

Combining Transformations

- For another example of applying several transformations, suppose that we want to rotate a shape through an angle r about a point (p,q) instead of about the point (0,0).
- We can do this by first moving the point (p,q) to the origin, using translate(-p,-q).
- Then we can do a standard rotation about the origin by calling rotate(r).
- Finally, we can move the origin back to the point (p,q) by applying translate(p,q). Keeping in mind that we have to write the code for the transformations in the reverse order, we need to say

```
translate(p,q)
rotate(r)
translate(-p,-q)
```

- Note: Some graphics APIs let us accomplish this transform with a single command such as rotate(r,p,q). This would apply a rotation through the angle r about the point (p,q).)

Scaling

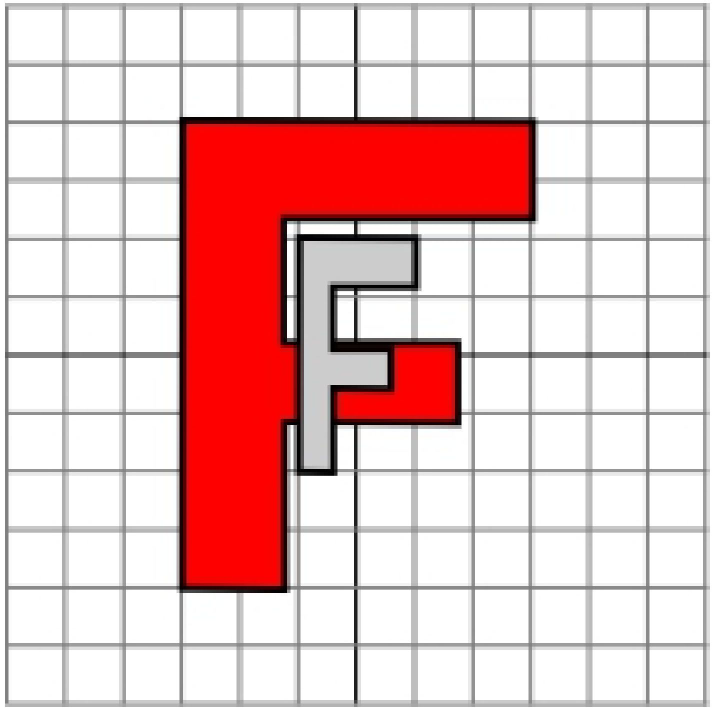
- A scaling transform can be used to make objects bigger or smaller.
- Mathematically, a scaling transform simply multiplies each x-coordinate by a given amount and each y-coordinate by a given amount. That is, if a point (x,y) is scaled by a factor of a in the x direction and by a factor of d in the y direction, then the resulting point (x1,y1) is given by

  $x1 = a * x$

  $y1 = d * y$

- If you apply this transform to a shape that is centered at the origin, it will stretch the shape by a factor of a horizontally and d vertically.
- Here is an example, in which the original light gray "F" is scaled by a factor of 3 horizontally and 2 vertically to give the final dark red "F":

Scaling



- The common case where the horizontal and vertical scaling factors are the same is called uniform scaling. Uniform scaling stretches or shrinks a shape without distorting it
- When scaling is applied to a shape that is not centered at (0,0), then in addition to being stretched or shrunk, the shape will be moved away from 0 or towards 0
- If you want to scale about a point other than (0,0), you can use a sequence of three transforms, similar to what was done in the case of rotation

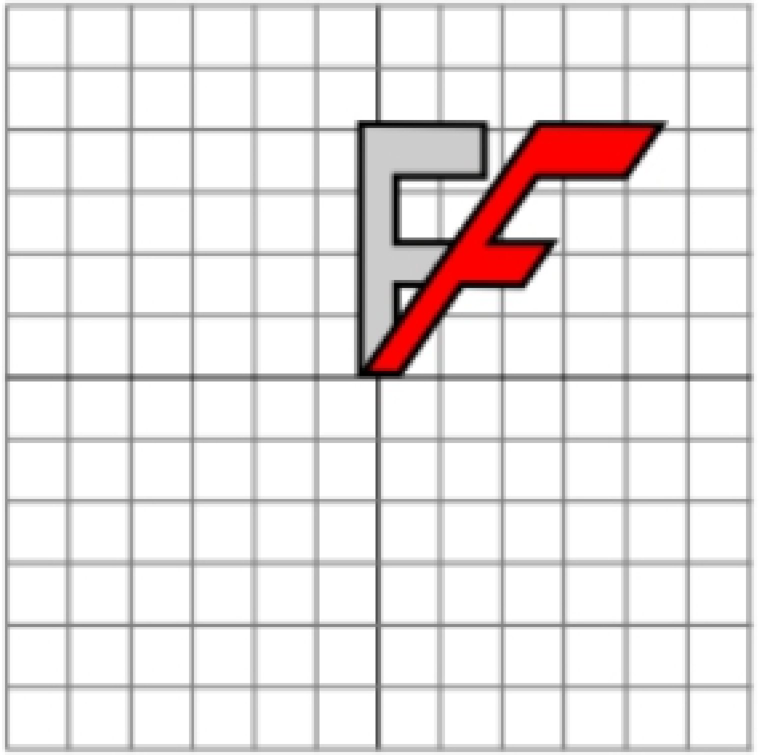
Scaling

- A 2D graphics API can provide a function *scale*(*a*,*d*) for applying scaling transformations.
- As usual, the transform applies to all *x* and *y* coordinates in subsequent drawing operations.
- Note that negative scaling factors are allowed and will result in reflecting the shape as well as possibly stretching or shrinking it.
- For example, *scale*(1,-1) will reflect objects vertically, through the *x*-axis
- Note that a transform that is made from translations and rotations, with no scaling, will preserve length and angles in the objects to which it is applied. It will also preserve aspect ratios of rectangles. Transforms with this property are called "**Euclidean.**"

Shear

- We will look at one more type of basic transform, a shearing transform
- Although shears can in fact be built up out of rotations and scalings if necessary, it is not really obvious how to do so
- A shear will "tilt" objects. A horizontal shear will tilt things towards the left (for negative shear) or right (for positive shear)
- A vertical shear tilts them up or down. Here is an example of horizontal shear:

Shear



- A horizontal shear does not move the x-axis. Every other horizontal line is moved to the left or to the right by an amount that is proportional to the y-value along that line.
- When a horizontal shear is applied to a point (x,y), the resulting point (x1,y1) is given by

  x1 = x + b * y

  y1 = y

for some constant shearing factor b.

Similarly, a vertical shear with shearing factor c is given by the equations

  x1 = x

  y1 = c * x + y

- Shear is occasionally called "skew," but skew is usually specified as an angle rather than as a shear factor.

Window-to-Viewport

- The last transformation that is applied to an object before it is displayed in an image is the window-to-viewport transformation
- This maps the rectangular view window in the xy-plane that contains the scene to the rectangular grid of pixels where the image will be displayed
- assume here that the view window is not rotated; that it, its sides are parallel to the x- and y-axes. In that case, the window-to-viewport transformation can be expressed in terms of translation and scaling transforms.
- Let's look at the typical case where the viewport has pixel coordinates ranging from 0 on the left to width on the right and from 0 at the top to height at the bottom. And assume that the limits on the view window are left, right, bottom, and top.
- In that case, the window-to-viewport transformation can be programmed as:

```
scale( width / (right-left), height / (bottom-top) );
translate( -left, -top )
```

Window-to-Viewport

- These should be the last transforms that are applied to a point. Since transforms are applied to points in the reverse of the order in which they are specified in the program, they should be the first transforms that are specified in the program.
- To see how this works, consider a point (x,y) in the view window. (This point comes from some object in the scene.
- Several modeling transforms might have already been applied to the object to produce the point (x,y), and that point is now ready for its final transformation into viewport coordinates.)
- The coordinates (x,y) are first translated by (-left,-top) to give (x-left,y-top). These coordinates are then multiplied by the scaling factors shown above, giving the final coordinates

  x1 = width / (right-left) * (x-left)

  y1 = height / (bottom-top) * (y-top)

- Note that the point (left,top) is mapped to (0,0), while the point (right,bottom) is mapped to (width,height), which is just what we want

Matrices and Vectors

- The transforms that are used in computer graphics can be represented as matrices, and the points on which they operate are represented as vectors
- Recall that a matrix, from the point of view of a computer scientist, is a two-dimensional array of numbers, while a vector is one-dimensional
- Matrices and vectors are studied in the field of mathematics called linear algebra. Linear algebra is fundamental to computer graphics
- In fact, matrix and vector math is built into GPUs

Matrices and Vectors

- The vectors that we need are lists of two, three, or four numbers. They are often written as (x,y), (x,y,z), and (x,y,z,w).
- A matrix with N rows and M columns is called an "N-by-M matrix." For the most part, the matrices that we need are N-by-N matrices, where N is 2, 3, or 4. That is, they have 2, 3, or 4 rows and columns, and the number of rows is equal to the number of columns.
- If A and B are two N-by-N matrices, then they can be multiplied to give a product matrix C = AB. If A is an N-by-N matrix, and v is a vector of length N, then v can be multiplied by A to give another vector w = Av.
- The function that takes v to Av is a transformation; it transforms any given vector of length N into another vector of length N. A transformation of this form is called a linear transformation.

Matrices and Vectors

- Now, suppose that A and B are N-by-N matrices and v is a vector of length N. Then, we can form two different products: $A(Bv)$ and $(AB)v$.
  - It is a central fact that these two operations have the same effect.
  - That is, we can multiply v by B and then multiply the result by A, or we can multiply the matrices A and B to get the matrix product AB and then multiply v by AB.
- Rotation and scaling, as it turns out, are linear transformations. That is, the operation of rotating (x,y) through an angle d about the origin can be done by multiplying (x,y) by a 2-by-2 matrix.
- Let's call that matrix R_d. Similarly, scaling by a factor a in the horizontal direction and b in the vertical direction can be given as a matrix $S_{a,b}$. If we want to apply a scaling followed by a rotation to the point $v = (x,y)$, we can compute **either** $R_d(S_{a,b}v)$ **or** $(R_dS_{a,b})v$.

Matrices and Vectors

- So what? Well, suppose that we want to apply the same two operations, scale then rotate, to thousands of points, as we typically do when transforming objects for computer graphics
- The point is that we could compute the product matrix $R_d S_{a,b}$ once and for all, and then apply the combined transform to each point with a single multiplication
- This means that if a program says

```
rotate(d)
scale(a,b)
  .
  . // draw a complex object
  .
```

the computer doens't have to keep track of two separate operations.

Matrices and Vectors

- However, **Translation is not a linear transformation.** To bring translation into this framework, we do something that looks a little strange at first:
- Instead of representing a point in 2D as a pair of numbers (*x*,*y*), we represent it as the triple of numbers (*x*,*y*,1). That is, we add a one as the third coordinate. It then turns out that we can then represent rotation, scaling, and translation—and hence any affine transformation—on 2D space as multiplication by a 3-by-3 matrix.
- The matrices that we need have a bottom row containing (0,0,1). Multiplying (*x*,*y*,1) by such a matrix gives a new vector (*x1*,*y1*,1). We ignore the extra coordinate and consider this to be a transformation of (*x*,*y*) into (*x1*,*y1*). For the record, the 3-by-3 matrices for translation ($T_{a,b}$), scaling ($S_{a,b}$), and rotation (R_d) in 2D are

Matrices and Vectors

- For the record, the 3-by-3 matrices for translation ($T_{a,b}$), scaling ($S_{a,b}$), and rotation (R_d) in 2D are

$$\mathbf{T_{a,b}} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{S_{a,b}} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R_d} = \begin{pmatrix} \cos(d) & -\sin(d) & 0 \\ \sin(d) & \cos(d) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Questions??

- End