Set Theory

prove that any $\emptyset \subseteq$ of any other set

For any $A \subseteq B$ we have $A - B \subseteq A$ by definiton of difference

$\rightarrow A - A = \emptyset \subseteq A$

Functions

Relations

A relation is a subset of a cartesian product of two sets.

$$A \times B = ((a_1, b_1), (a_1, b_2) \ldots)$$

Definitons

> **Image**: for any ordered pairs, in any Cartesian product, the second element is called the image of the first element

> **Domain**: The set of all first elements of the ordered pairs in a relation R from a Set A to a set B.

> **Range**: The set of all second elements in a relation R from a set A to a set B.

> **Codomain**:The whole set B. Range $\subseteq$ Codomain

functions def

> A function is a relation that map each element x of a set A **with only one** element of est B

$$f : A \rightarrow Bf(x) = y$$

one to one functions (injective)

$f : A \rightarrow B$ is injective if alle elements in A have a distinct image in B

Onto function (surjective)

$f : A \rightarrow B$ is subjective if every element of B is the image of at least one element of A

One to one function (bijective)

$f : A \to B$ is bijective if it is incective and surjective.

$$f(x) = \frac{1}{x} \qquad \mathbb{D} =$$

$f : \mathbb{R} \to \mathbb{R}$

$$f : \mathbb{R} \to \mathbb{R} \ f(x) = 2x + 1$$

To show that this function is injective, we need to show that for all $x_1, x_2 \in \mathbb{R}$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

So, let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. Then we have:

$$2x_1 + 1 = 2x_2 + 1$$

Subtracting 1 from both sides, we get:

$$2x_1 = 2x_2$$

Dividing both sides by 2, we get:

$$x_1 = x_2$$

Therefore, we have shown that if $f(x_1) = f(x_2)$, then $x_1 = x_2$, which means that f is injective.

invetrible function

To be **invertable** it has to be **injective** and **surjective**.

Consider $f : x \to 2x + 3$ find $f^{-1}(x)$. Verify that your result is correct

example of invertible function:

$$\begin{aligned} y &= 2x + 3 \\ x &= \frac{y-3}{2} \\ 2y &= x - 3 \\ \to f^{-1}(x) &= \frac{x-3}{2} \end{aligned}$$

$$\begin{array}{lll} (f \circ f^{-1})(x) & \quad and \quad & (f^{-1} \circ f)(x) \\ = f(f^{-1}(x)) & & = f^{-1}(f(x)) \\ = f(\frac{x-3}{2}) & & = f^{-1}(2x+3) \\ = 2(\frac{x-3}{2}) + 3 & & = \frac{1}{2}(2x+3) - \frac{3}{2} \\ = x - 3 + 3 & & = x + \frac{3}{2} - \frac{3}{2} \\ = x & & = x \end{array}$$

check if function is sujective

to see if a functin is sujective we can check if it the domain is equal to the codomain

$$f(x) = \frac{1}{x} f:\ D = \mathbb{R} Image = \{y \in \mathbb{R} | y \neq 0\} = Range$$

$$Codomain \equiv \mathbb{R} \to 0 \in \mathbb{R}$$

example

$$f(x) = \frac{x-4}{x+2} \qquad D = \mathbb{R} - \{-2\}$$

this function is surjective because its range (image) is equal to its codomain

frequently used functions

exponential functions

Exponential functions contain a variable wirtten as an exponent, such as

$$y = ab^x$$

where a and b are two real numbers. b is called **base**, and x **exponent**.

example

Compoind interest formula

$$A = p(1+r)^t$$

is an exponential function which teh amount in the acount A depends on the principal p , the annual interest rate r and the time t.

exercise 2

$$\frac{2^{3x}2^{x+2}}{2^{2x+5}} = \frac{2^x 2^{x+2}}{2^5} = \frac{2^{x^2}}{2^3} = \frac{2^{x^2}}{2^3} = 2^{x^2-3}$$

logarithmic functions

Logarithmic functions are the inverse of exponential functions. The logarithmic function with base b is defined as

prove of logarithmic rule:

$$\begin{aligned}\ln(xy) &= \ln(x) + \ln(y) \\ n &= \ln(x) \to e^n = x \\ m &= \ln(y) \to e^m = y \\ \ln(e^n e^m) &= \ln(e^{n+m}) = n + m = \ln(x) + \ln(y) \\ &= \ln(xy)\end{aligned}$$

> Complex Numbers
>
> # Complex Numbers
>
> A complex number is a number of the form
>
> $$z = x + iy \qquad x, y \in \mathbb{R}$$
>
> Real part of z is x and the imaginary is y
>
> # Property's and logic of complex numbers
>
> if $z_1 = z_2$ then $x_1 = x_2$ and $y_1 = y_2$
>
> $$z_1 = x_1 + y_1$$
>
> $$z_2 = x_2 + y_2$$
>
> ## addition and subtraction
>
> $$\begin{aligned}z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + (iy_1 + iy_2)\end{aligned}$$
>
> Addition supports associativity and commutativity
>
> > associativity
> >
> > $$x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$$
> >
> > the order in witch you add and subtract is irrelevant
>
> > commutativity
> >
> > Example adding complex numbers
> >
> > $$z_1 + z_2 = (x_1 + y_1) + (x_2 + y_2)$$
> >
> > The statement is communitive so
> >
> > $$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)$$
>
> > ## multiplication in algebraic notation
> >
> > $$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$

> you have to multiply all terms with all terms. while multiplying i it is as multiplying $\sqrt{-1}$
>
> $$\begin{aligned}(x_1 + iy_1) \cdot (x_2 + iy_2) &= x_1x_2 + x_1y_2i + y_1x_2i + y_1y_2i^2 \\ &= x_1x_2 + i(x_1y_2 + y_1x_2) - y_1y_2\end{aligned}$$

absolute

$|z| = \sqrt{x^2 + y^2}$
the absolute number is always a $\mathbb{R}$ number. i is purposefully excluded.

Conjugate

$z = x + iy$
conjugate of z is $\overline{z} = x - iy$

$$\begin{aligned}z &= 3 + 4i \\ \overline{z} &= 3 - 4i\end{aligned}$$

$$\begin{aligned}\overline{z} \cdot z &= (x_iy)(x - iy) \\ &= x^2 - (iy^2) \\ &= x^2 - (-1y^2) \\ &= x^2 + y^2 \\ &= |z|^2\end{aligned}$$

> # inverse of a complex number in algebraic notation
>
> The inverse number if multiplied by the inverted number results in 1
>
> $$\frac{z_1}{z_2} = z_1z_2^{-1} = (x + iy)(\frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2})$$
>
> $$\begin{aligned}z^{-1} &= \frac{1}{z} \\ &= \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} \\ &= \frac{1}{z + iy} \cdot \frac{x - iy}{x - iy} \\ &= \frac{x - iy}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}\end{aligned}$$

$$\frac{z_1}{z_2} = z_1z_2^{-1} = (x + iy)(\frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2})$$

$$
\begin{aligned}
z^{-1} &= \frac{1}{z} \\
&= \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} \\
&= \frac{1}{z+iy} \cdot \frac{x-iy}{x-iy} \\
&= \frac{x-iy}{x^2+y^2} \\
&= \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}
\end{aligned}
$$

property of conjugation and absolute value

- $z_1 + z_2 = \overline{z_1} + \overline{z_2}$
- $\overline{(z_1 \cdot z_2)} = \overline{z_1} \cdot \overline{z_2}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$

Complex numbers as coordinates

> Radiant vs Degrees
>
> | Degrees | Radians |
> | --- | --- |
> | 0° | 0 |
> | 30° | π/6 |
> | 45° | π/4 |
> | 60° | π/3 |
> | 90° | π/2 |
> | 180° | π |
> | 270° | 3π/2 |
> | 360° | 2π |

Polar coordinates

To represent a complex number as a two-dimensional vector in a cartesian map. We can use an angle θ and the radius of the vector r.

$$z = r(\cos(\theta) + i\sin(\theta))$$

It can be written in a more elegant form as

$$z = re^{i\theta}$$

the reason $e^{i\theta} = \cos(\theta) + \sin(\theta)$ is because adding the taylor series of $\cos(\theta)$ and $\sin(\theta)$ it adds up to the series of $e^{i\theta}$.

conversion from polar to algebraic

to convert the polar coordinates in the algebraic representation using trigonometric functions.

$$x = r \cdot \cos(\theta)$$
$$y = r \cdot \sin(\theta)$$

conversion from algebraic to polar

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

Operators in polar form

> # multiplication in polar notation
>
> $$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

> # inverse of a complex number in polar notation
>
> $$z^{-1} = (re^{i\theta})^{-1} = r^{-1}(e^{i\theta})^{-1} = \frac{1}{r} e^{i(-\theta)}$$