Vectors

Bruno Carpentieri
bruno.carpentieri@unibz.it

3rd floor - Room 3.10 - Phone +39 047 101 6027

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Outline

Operations on Vectors

Length and Angles

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Length and Angles

Vectors

- "*You can't add apples and oranges.*" This is the reason to introduce *vectors* containing two separate numbers v_1 and v_2.

- That pair (v_1, v_2) produces a *two-dimensional vector* $\vec{v}$ (a column vector):

$$\text{column vector} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

where $v_1 =$ *first component* of $\vec{v}$ and $v_2 =$ *second component* of $\vec{v}$.

- Note that
  - we write $\vec{v}$ as a *column*, not as a *row*,
  - we use a single letter $\vec{v}$ for this pair of numbers v_1 and v_2,
  - we write $\vec{v}$ in italic using an overrightarrow, and v_1, v_2 in *lightface italic*.

Vectors Addition

- ✔ Even if we don't add the individual components v_1 to v_2 of $\vec{v}$, we do *add vectors*.

- ✔ Vector addition:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{add to} \quad \vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

  The first components of $\vec{v}$ and $\vec{w}$ stay separate from the second components. You see the reason: we want to add apples to apples!

- ✔ Subtraction of vectors follows the same idea:

$$\vec{v} - \vec{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \end{bmatrix}.$$

Vectors Addition

- ✔ The order of addition makes no difference:

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}.$$

- ✔ Check by algebra that the first component is $v_1 + w_1$ which equals $w_1 + v_1$. Analogously for the second component.

- ✔ Check also by an example:

$$\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \quad \vec{w} + \vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}.$$

Vectors Scalar Multiplication

- ✔ The other basic operation for vectors is scalar multiplication. Vectors can be multiplied by a scalar, such as 2 or -1, or by any number c.

- ✔ The components of $c\vec{v}$ are cv_1 and cv_2: $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$. The number c is called a scalar.

- ✔ Example: there are two ways to double a vector. One way is to add $\vec{v} + \vec{v}$. The other way (the usual way) is to multiply each component by 2:

$$2\vec{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}.$$

On the other hand: $-\vec{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$.

Zero Vector

- ✔ Notice that the sum of $-\vec{v}$ and $\vec{v}$ is the zero vector. This is written as $\vec{0}$, which is not the same as the number zero!

- ✔ The vector $\vec{0}$ has components 0 and 0. We write

$$-\vec{v} + \vec{v} = \begin{bmatrix} -v_1 + v_1 \\ -v_2 + v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}.$$

- ✔ Linear algebra is built on these operations $\vec{v} + \vec{w}$ and $c\vec{v}$, that is adding vectors and multiplying by scalars.

Linear Combination

- ✔ Combining addition with scalar multiplication, we now form "*linear combinations*" of $\vec{v}$ and $\vec{w}$. The rule is: multiply $\vec{v}$ by c and multiply $\vec{w}$ by d; then add $c\vec{v} + d\vec{w}$.

Definition

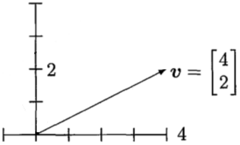
The sum of $c\vec{v}$ and $d\vec{w}$ is called a linear combination of $\vec{v}$ and $\vec{w}$.

- ✔ Four special linear combinations are:

$$
\begin{aligned}
1\vec{v} + 1\vec{w} &= \text{sum of vectors} \\
1\vec{v} - 1\vec{w} &= \text{difference of vectors} \\
0\vec{v} + 0\vec{w} &= \text{zero vector} \\
c\vec{v} + 0\vec{w} &= \text{vector } c\vec{v}
\end{aligned}
$$

Vector Representations

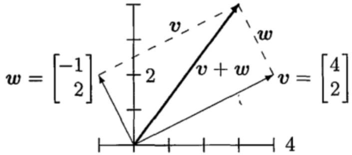
1. Two numbers. For algebra, we just need the components like 4 and 2 for $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, and the order in which they appear. Observe that $\begin{bmatrix} 4 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.
2. Arrow from $(0, 0)$. That vector $\vec{v}$ is represented by an arrow. The arrow goes $v_1 = 4$ units to the right and $v_2 = 2$ units up.



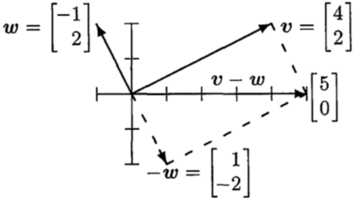
3. Point in the plane. That vector $\vec{v}$ ends at the point of coordinates $x = 4$, $y = 2$.

Graphic Vector Addition: Head to Tail Rule

- ✔ We added using the numbers, and we can also add $\vec{v} + \vec{w}$ using arrows.
- ✔ *Vector addition* (head to tail rule): *at the end of $\vec{v}$, place the start of $\vec{w}$.*



$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$$\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

- ✔ We travel along $\vec{v}$ and then along $\vec{w}$. Or we take the diagonal shortcut along $\vec{v} + \vec{w}$. Or we go along $\vec{w}$ and then $\vec{v}$. The sum is the diagonal vector $\vec{w} + \vec{v}$.

Some Comments

- ✔ The graphic representation on the left side of the previous figure confirms that $\vec{w}+\vec{v}$ gives the same answer as $\vec{v}+\vec{w}$.
- ✔ For example, for $2\vec{v}$ we double the length of the arrow.
- ✔ For $\vec{v}-\vec{w}$, we reverse $\vec{w}$ to get $-\vec{w}$. This reversing gives the subtraction on the right side of the previous figure.
- ✔ The zero vector $\vec{0}=(0,0)$ is too short to draw a decent arrow. However, we know that $\vec{v}+\vec{0}=\vec{v}$.

Vectors in Three Dimensions

✔ Summary from vectors in two dimensions:

- A vector with two components corresponds to a point in the xy plane.
- The components of $\vec{v}$ are the coordinates of the point, $x = v_1$ and $y = v_2$.
- The arrow ends at this point (v_1, v_2), when it starts from $(0, 0)$.

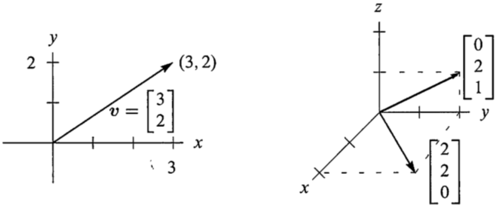
✔ Now we allow vectors to have three components (v_1, v_2, v_3). Typical vectors (still column vectors but with three components) are:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \text{and} \quad \vec{v} + \vec{w} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$

Vectors in Three Dimensions

- ✔ The vector $\vec{v}$ corresponds to an arrow in 3-space. The arrow starts at the "origin", where the xyz axes meet and the coordinates are $(0, 0, 0)$. The arrow ends at the point with coordinates v_1, v_2, v_3.

- ✔ There is a *perfect match between the column vector, the arrow from the origin and the point where the arrow ends*.



Example of 2D and 3D vectors.

Rows and Columns Format

- ✔ Notation: to save space, from now on $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is also written as $\vec{v} = (1, 1, -1)$.

- ✔ Note that $\vec{v} = (1, 1, -1)$ is not a row vector! It is actually a column vector, just temporarily lying down.

- ✔ In three dimensions, $\vec{v} + \vec{w}$ is still found a component at a time: the sum has components $v_1 + w_1$, $v_2 + w_2$ and $v_3 + w_3$. Can you see how to add vectors in 4 or 5 or n dimensions ?

- ✔ An example of linear combination of three vectors in three dimensions $\vec{u} + 4\vec{v} - 2\vec{w}$:

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}.$$

Outline

Operations on Vectors

Length and Angles

Real Dot Product or Inner Product

Definition

The real dot product or inner product of two real vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ is denoted by the symbol $\vec{v} \cdot \vec{w}$, and is given by the number

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2.$$

- ✔ Rule: the real "dot product" of $\vec{v}$ and $\vec{w}$ involves the separate products $v_1 w_1$ and $v_2 w_2$. Those two numbers are added to produce the single number $\vec{v} \cdot \vec{w}$.
- ✔ Property: obviously, the order of $\vec{v}$ and $\vec{w}$ makes no difference, $\vec{w} \cdot \vec{v} = \vec{v} \cdot \vec{w}$.
- ✔ Main point: if $\vec{v}$ and $\vec{w}$ have n real components, to compute $\vec{v} \cdot \vec{w}$, we multiply each v_i times w_i, then we add $\sum v_i w_i$.

Real Dot Product or Inner Product

✔ Example: take $\vec{v} = (4, 2)$ and $\vec{w} = (-1, 2)$. Then

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$$

✔ Example: take $\vec{v} = (1, 2, 3)$ and $\vec{w} = (3, 1, 2)$. Then

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3 + 2 + 6 = 11.$$

Complex Dot Product or Inner Product

Definition

The complex dot product or inner product of two complex vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ is denoted by the symbol $\vec{v} \cdot \vec{w}$, and is given by the number

$$\vec{v} \cdot \vec{w} = \bar{v}_1 w_1 + \bar{v}_2 w_2.$$

- ✔ Rule: the complex "dot product" of $\vec{v}$ and $\vec{w}$ involves the separate products $\bar{v}_1 w_1$ and $\bar{v}_2 w_2$. Those two numbers are added to produce the single number $\vec{v} \cdot \vec{w}$.
- ✔ Property: the order of $\vec{v}$ and $\vec{w}$ makes difference, $\vec{w} \cdot \vec{v} = \overline{\vec{v} \cdot \vec{w}}$ (see proof later).
- ✔ Main point: if $\vec{v}$ and $\vec{w}$ have n real components, to compute $\vec{v} \cdot \vec{w}$, we multiply each $\bar{v}_i$ times w_i, then we add $\sum \bar{v}_i w_i$.

Complex Dot Product or Inner Product

✔ Example: take

$$\vec{v} = \begin{bmatrix} 1+i \\ 3 \\ 1-i \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}. \text{ Then}$$

$$\begin{bmatrix} 1+i \\ 3 \\ 1-i \end{bmatrix} \cdot \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} = 2+2i.$$

$$\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1+i \\ 3 \\ 1-i \end{bmatrix} = 2-2i.$$

In general, $\vec{v} \cdot \vec{w} \neq \vec{w} \cdot \vec{v}$ for complex vectors (see next slide).

Properties of the Complex Dot Product

1. Conjugate symmetry: $\boxed{\vec{v} \cdot \vec{w} = \overline{\vec{w} \cdot \vec{v}}}$

Proof: let $\vec{v} = \begin{bmatrix} a_1 + ib_1 \\ \vdots \\ a_n + ib_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} c_1 + id_1 \\ \vdots \\ c_n + id_n \end{bmatrix}$. Then

$$\sum \bar{v}_i w_i = \sum (a_i - ib_i)(c_i + id_i) = \sum (a_i c_i + b_i d_i) + i \sum (a_i d_i - b_i c_i)$$

$$\sum \bar{w}_i v_i = \sum (c_i - id_i)(a_i + ib_i) = \sum (a_i c_i + b_i d_i) - i \sum (a_i d_i - b_i c_i)$$

2. Linearity: $\boxed{(u\vec{v}) \cdot \vec{w} = \bar{u} (\vec{v} \cdot \vec{w})}$

Proof: let $u = x + iy, \quad \vec{v} = \begin{bmatrix} a_1 + ib_1 \\ \vdots \\ a_n + ib_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} c_1 + id_1 \\ \vdots \\ c_n + id_n \end{bmatrix}$.

Properties of the Complex Dot Product

$$u\vec{v} = \begin{bmatrix} u\,(a_1 + ib_1) \\ \vdots \\ u\,(a_n + ib_n) \end{bmatrix} = \begin{bmatrix} (a_1x - b_1y) + i\,(b_1x + a_1y) \\ \vdots \\ (a_nx - b_ny) + i\,(b_nx + a_ny) \end{bmatrix}.$$

$$(u\vec{v}) \cdot \vec{w} = \sum \left[c_i\,(a_ix - b_iy) + d_i\,(b_ix + a_iy)\right] +$$
$$+i \sum \left[-c_i\,(b_ix + a_iy) + d_i\,(a_ix - b_iy)\right].$$

$$\vec{v} \cdot \vec{w} = \sum (a_i - ib_i)\,(c_i + id_i) = \sum \left[(a_ic_i + b_id_i) + i\,(a_id_i - b_ic_i)\right].$$

$$\bar{u}\,(\vec{v} \cdot \vec{w}) = \sum \left[x\,(a_ic_i + b_id_i) + y\,(a_id_i - b_ic_i)\right] +$$
$$+i \sum \left[-y\,(a_ic_i + b_id_i) + x\,(a_id_i - b_ic_i)\right].$$

Properties of the Complex Dot Product

3. Linearity: $\boxed{\vec{v} \cdot (u\vec{w}) = u\,(\vec{v} \cdot \vec{w})}$

   Proof: $\vec{v} \cdot (u\vec{w}) = \overline{(u\vec{w}) \cdot \vec{v}} = \overline{\bar{u}\,(\vec{w} \cdot \vec{v})} = u\overline{\vec{w} \cdot \vec{v}} = u\,(\vec{v} \cdot \vec{w})$

4. Right-distributivity: $\boxed{(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}}$

   Proof : $(\vec{v} + \vec{w})\,\vec{u} = \sum \left(\overline{v_i + w_i}\right) u_i = \sum \left(\overline{v_i} + \overline{w_i}\right) u_i = \sum \left(\overline{v_i} u_i + \overline{w_i} u_i\right) =$
   $= \sum \overline{v_i} u_i + \sum \overline{w_i} u_i = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}.$

5. Left-distributivity: $\boxed{\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}}$.

   Proof: $\vec{u} \cdot (\vec{v} + \vec{w}) = \overline{(\vec{v} + \vec{w}) \cdot \vec{u}} = \overline{\vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}} = \overline{\vec{v} \cdot \vec{u}} + \overline{\vec{w} \cdot \vec{u}} = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$

Properties of the Complex Dot Product

7. Positive-definiteness:

$$\boxed{\vec{v} \cdot \vec{v} \geq 0}$$

$$\boxed{\vec{v} \cdot \vec{v} = 0 \iff \vec{v} = 0}$$

Proof: $\vec{v} \cdot \vec{v}$ is the sum of positive terms as
$\vec{v} \cdot \vec{v} = \bar{v}_1 v_1 + \bar{v}_2 v_2 + \cdots + \bar{v}_n v_n = |v_1|^2 + |v_2|^2 + \cdots + |v_n|^2$

8. $\vec{v} \cdot \vec{v}$ is real for all $\vec{v}$.

Proof: $\vec{v} \cdot \vec{v} = \overline{\vec{v} \cdot \vec{v}}$.

9. $-\vec{v} \cdot \vec{v} = \vec{v} \cdot (-\vec{v}) = (-\vec{v}) \cdot \vec{v}$

Proof: $-\vec{v} \cdot \vec{v} = -1\,(\vec{v} \cdot \vec{v}) = \vec{v} \cdot (-\vec{v}) = (-\vec{v}) \cdot \vec{v}$.

Properties of the Complex Dot Product

1. $\begin{bmatrix} 2+2i \\ 1+i \end{bmatrix} \cdot \begin{bmatrix} 1-2i \\ 3+2i \end{bmatrix} = \overline{\begin{bmatrix} 1-2i \\ 3+2i \end{bmatrix} \cdot \begin{bmatrix} 2+2i \\ 1+i \end{bmatrix}} = 3-7i.$

2. $\left((1+2i)\begin{bmatrix} 2+i \\ 1-i \end{bmatrix}\right) \cdot \begin{bmatrix} 1-2i \\ 3+2i \end{bmatrix} = (1-2i)\left(\begin{bmatrix} 2+i \\ 1-i \end{bmatrix} \cdot \begin{bmatrix} 1-2i \\ 3+2i \end{bmatrix}\right) = 1-2i.$

3. $\begin{bmatrix} 2+i \\ 1-i \end{bmatrix} \cdot \left((1+2i)\begin{bmatrix} 1-2i \\ 3+2i \end{bmatrix}\right) = (1+2i) \cdot \left(\begin{bmatrix} 2+i \\ 1-i \end{bmatrix} \cdot \begin{bmatrix} 1-2i \\ 3+2i \end{bmatrix}\right) = 1+2i.$

4. $\begin{bmatrix} 1-2i \\ 3+2i \end{bmatrix} \cdot \begin{bmatrix} 1-2i \\ 3+2i \end{bmatrix} = (1+2i)(1-2i) + (3-2i)(3+2i) = 18.$

Length of a Vector

- ✔ Important case: the dot product of a vector *with itself*. In this case $\vec{v} = \vec{w}$.

- ✔ The *length* or *norm* of a vector $\vec{v}$ is denoted by the symbol $\|\vec{v}\|$, and is given by the square root of $\vec{v} \cdot \vec{v}$:

$$\boxed{\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}}$$

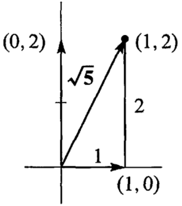
For real vectors: in two dimensions $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$, in three dimensions $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$, in four dimensions $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$, etc.

For complex vectors: in two dimensions $\|\vec{v}\| = \sqrt{|v_1|^2 + |v_2|^2}$, in three dimensions $\|\vec{v}\| = \sqrt{|v_1|^2 + |v_2|^2 + |v_3|^2}$, etc.

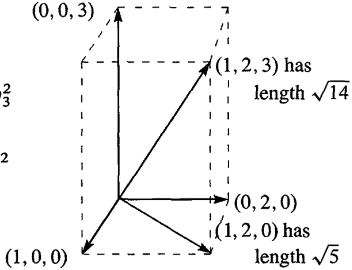
- ✔ Example: when $\vec{v} = (1, 2, 3)$, we have $\vec{v} \cdot \vec{v} = 14$ and $\|\vec{v}\| = \sqrt{14}$.

$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = 14.$$

Length of a Vector



$$\begin{aligned} v \cdot v &= v_1^2 + v_2^2 + v_3^2 \\ 5 &= 1^2 + 2^2 \\ 14 &= 1^2 + 2^2 + 3^2 \end{aligned}$$



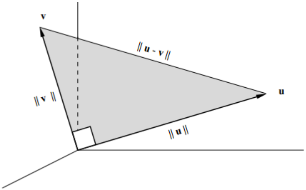
- ✔ By the calculation above, the length of $\vec{v} = (1, 2, 3)$ is $\|\vec{v}\| = \sqrt{14}$.
- ✔ Can you see the analogies with the Pythagorean theorem ? $\|\vec{v}\|$ gives the distance between the vector and the origin $\vec{0}$.

Orthogonal Vectors

Definition

Two vectors $\boxed{\text{in } \mathbb{R}^2 \text{ or } \mathbb{R}^3}$ are orthogonal (perpendicular) if the angle between them is a right angle (90°).

- ✔ But the visual concept of a right angle is not at our disposal in higher dimensions, so we must dig a little deeper. We use the Pythagorean theorem:



$\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} - \vec{v}\|^2$.

Orthogonal Vectors

- But $\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$, and $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$, so we can rewrite the Pythagorean statement as

$$0 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$
$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - (\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}) = 2\vec{u}\cdot\vec{v}.$$

- Therefore,

$$\boxed{\text{real vectors } \vec{u} \text{ and } \vec{v} \text{ are orthogonal vectors if and only if } \vec{u}\cdot\vec{v} = 0}$$

The natural extension of this provides us with a definition in more general spaces.

Definition

Two *real* vectors $\vec{x}, \vec{y}$ are said to be orthogonal (to each other) whenever $\vec{x}\cdot\vec{y} = 0$, and this is denoted by writing $\vec{x} \perp \vec{y}$.

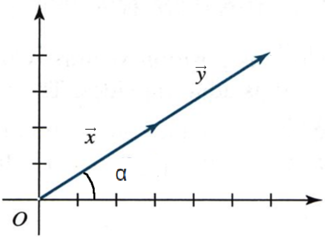
Example

1. $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$ is orthogonal to $\vec{v} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ -4 \end{bmatrix}$ because $\vec{u} \cdot \vec{v} = 0$.

2. $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are *not* orthogonal because $\vec{u} \cdot \vec{v} \neq 0$.

Parallel vectors

- Now that orthogonal vectors $\vec{x}$ and $\vec{y}$ are defined, how can parallel vectors be defined ? Let $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ be parallel vectors.



- From trigonometry we have

$$\begin{cases} x_1 = \|\vec{x}\| \cos\alpha \\ x_2 = \|\vec{x}\| \sin\alpha \end{cases}, \quad \begin{cases} y_1 = \|\vec{y}\| \cos\alpha \\ y_2 = \|\vec{y}\| \sin\alpha \end{cases}$$

- Therefore, we can write

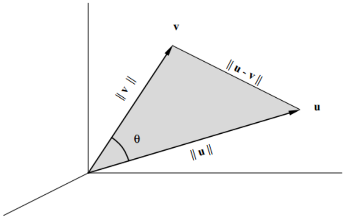
$$\begin{cases} y_1 = \|\vec{y}\| \dfrac{\|\vec{x}\|}{\|\vec{x}\|} \cos\alpha = \dfrac{\|\vec{y}\|}{\|\vec{x}\|} x_1 \\ y_2 = \|\vec{y}\| \dfrac{\|\vec{x}\|}{\|\vec{x}\|} \sin\alpha = \dfrac{\|\vec{y}\|}{\|\vec{x}\|} x_2 \end{cases}$$

- All in all, two vectors $\vec{x}$ and $\vec{y}$ are parallel if $\vec{y} = t\vec{x}$, with $t \in \mathbb{R}$. The definition is generalized to higher-dimensional vectors.

Angles

✔ Now that "right angles" in higher dimensions make sense, how can more general angles be defined ? Use the *law of cosines* rather than the Pythagorean theorem.

✔ Recall that *the law of cosines in* $\boxed{\mathbb{R}^2 \text{ or } \mathbb{R}^3}$ says



$$\boxed{\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\|\vec{u}\|\,\|\vec{v}\|\cos\theta}$$

✔ If $\vec{u} \perp \vec{v}$ are orthogonal, the law of cosines reduces to the Pythagorean theorem.

Angles

✔ In general,

$$\cos\theta = \frac{\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2}{2\,\|\vec{u}\|\,\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})}{2\,\|\vec{u}\|\,\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}.$$

The Cauchy-Bunyakovsky-Schwarz inequality (see next slides) states that $|\vec{u}\cdot\vec{v}| \leqslant \|\vec{u}\|\,\|\vec{v}\|$. It guarantees that $|\cos\theta| \leq 1$, or $-1 \leq \cos\theta \leq 1$. Hence there is a unique value θ in $[0,\pi]$ such that $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$.

Definition

The radian measure of the angle between real nonzero vectors $\vec{x}, \vec{y}$ is defined to be the number θ in $[0,\pi]$ such that

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}}$$

Example

✔ Exercise. From

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}.$$

determine the angle between

$$u = \begin{bmatrix} -4 \\ 2 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}.$$

✔ Solution. Compute $\cos\theta = 2/(5\cdot 3) = 2/15$, and use the inverse cosine function to conclude that $\theta = 1.437$ radians (rounded).

Cauchy-Bunyakovsky-Schwarz (CBS) Inequality

Theorem

The inequality is defined for general complex vectors. For all vectors $\vec{x}, \vec{y} \in \mathbb{C}^n$,

$$\boxed{|\vec{x} \cdot \vec{y}| \leqslant \|\vec{x}\| \, \|\vec{y}\|}$$

Equality holds if and only if $\vec{y} = c\vec{x}$ *for* $c = \vec{x} \cdot \vec{y} / \vec{x} \cdot \vec{x}$. □

Proof. The case $\vec{x} = 0$ is trivial. For $\vec{x} \neq 0$, we show that $\|\vec{x}\| \, \|\vec{y}\| \geqslant |\vec{x} \cdot \vec{y}| \, 0$, or, equivalently, $\|\vec{x}\|^2 \, \|\vec{y}\|^2 \geqslant |\vec{x} \cdot \vec{y}|^2$. To simplify, we divide both sides by $\|\vec{x}\|^2 \neq 0$. We set $c = \vec{x} \cdot \vec{y} / \vec{x} \cdot \vec{x} = \vec{x} \cdot \vec{y} / \|\vec{x}\|^2$ so that $\vec{x} \cdot (\vec{y} - c\vec{x}) = \vec{x} \cdot \vec{y} - c\vec{x} \cdot \vec{x} = 0$:

$$\frac{\|\vec{y}\|^2 \, \|\vec{x}\|^2 - (\vec{x} \cdot \vec{y})\,(\vec{y} \cdot \vec{x})}{\|\vec{x}\|^2} = \vec{y} \cdot \vec{y} - c(\vec{y} \cdot \vec{x}) = \vec{y} \cdot \vec{y} - \vec{y} \cdot (c\vec{x}) = \vec{y} \cdot (\vec{y} - c\vec{x}) =$$

$$\vec{y} \cdot (\vec{y} - c\vec{x}) - c\vec{x} \cdot (\vec{y} - c\vec{x}) = (\vec{y} - c\vec{x}) \cdot (\vec{y} - c\vec{x}) = \|\vec{y} - c\vec{x}\|^2 \geqslant 0.$$

Cauchy-Bunyakovsky-Schwarz (CBS) Inequality

Since $\vec{y} \cdot \vec{x} = \overline{\vec{x} \cdot \vec{y}}$, it follows that $(\vec{x} \cdot \vec{y})\,(\vec{y} \cdot \vec{x}) = |\vec{x} \cdot \vec{y}|^2$, so

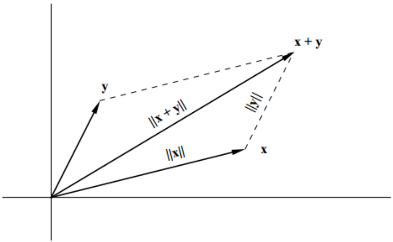
$$0 \leqslant \|c\vec{x} - \vec{y}\|^2 = \frac{\|\vec{y}\|^2\,\|\vec{x}\|^2 - |\vec{x} \cdot \vec{y}|^2}{\|\vec{x}\|^2}. \tag{1}$$

Now, $\|\vec{x}\|^2 > 0$ implies $0 \leq \|\vec{y}\|^2\,\|\vec{x}\|^2 - |\vec{x} \cdot \vec{y}|^2$, and thus the CBS inequality is obtained.

Equality: note that if $\vec{y} = c\vec{x}$, then $|\vec{x} \cdot \vec{y}| = |\vec{x} \cdot (c\vec{x})| = |c(\vec{x} \cdot \vec{x})| = |c|\ \|\vec{x}\|^2 = \|\vec{x}\|\ |c|\ \|\vec{x}\| = \|\vec{x}\|\|\vec{y}\|$. Conversely, if $|\vec{x} \cdot \vec{y}| = \|\vec{x}\|\|\vec{y}\|$, then from (1), $\|c\vec{x} - \vec{y}\| = 0$, and hence $c\vec{x} - \vec{y} = 0$, or $c = \vec{x} \cdot \vec{y}/\vec{x} \cdot \vec{x}$.

Triangle Inequality

- ✔ The CBS inequality is important because it helps to establish that the geometry in higher-dimensional spaces is consistent with the visual geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- ✔ Consider the situation depicted in the figure below in $\mathbb{R}^2$



Imagine traveling from the origin to the point $\vec{x}$ and then moving from $\vec{x}$ to the point $\vec{x} + \vec{y}$.

Clearly, you have traveled a distance that is at least as great as the direct distance from the origin to $\vec{x} + \vec{y}$ along the diagonal of the parallelogram.

In other words,it's visually evident that

$$\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|.$$

Triangle Inequality

- ✔ The previous observation is known as the triangle inequality.
- ✔ In higher-dimensional spaces we cannot visualize the geometry with our eyes, and it is unclear whether or not the triangle inequality remains valid.
- ✔ The CBS inequality is precisely what is required to prove that, in this respect, the geometry of higher dimensions is no different than that of the visual spaces.
- ✔ We prove the triangle inequality in the next slide. Note that this inequality is valid for general complex vectors.

Triangle Inequality

Theorem

For all $\vec{x}, \vec{y} \in \mathbb{C}^n$

$$\boxed{\|\vec{x}+\vec{y}\| \leqslant \|\vec{x}\| + \|\vec{y}\|} \quad \square$$

Proof. Consider $\vec{x}$ and $\vec{y}$ to be column vectors, and write

$$\begin{aligned} \|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y}) = \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y} \\ &= \|\vec{x}\|^2 + \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + \|\vec{y}\|^2 . \end{aligned}$$

If $z = a + ib$, then $z + \bar{z} = 2a = 2Re(z)$ and $|z|^2 = a^2 + b^2 \geq a^2 \Rightarrow |z| \geq Re(z)$.

Using the fact that $\vec{y}\cdot\vec{x} = \overline{\vec{x}\cdot\vec{y}}$ together with the CBS inequality yields

$$\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} = 2\,\mathrm{Re}\,(\vec{x}\cdot\vec{y}) \leqslant 2\,|\vec{x}\cdot\vec{y}| \leqslant 2\,\|\vec{x}\|\,\|\vec{y}\| .$$

Therefore,

$$\|\vec{x}+\vec{y}\|^2 \leqslant \|\vec{x}\|^2 + 2\,\|\vec{x}\|\,\|\vec{y}\| + \|\vec{y}\|^2 = (\|\vec{x}\| + \|\vec{y}\|)^2 . \quad \square$$

Triangle Inequality

1. It's not difficult to see that the triangle inequality can be extended to any number of vectors in the sense that

$$\boxed{\left\|\sum_i \vec{x}_i\right\| \leqslant \sum_i \|\vec{x}_i\|}$$

2. Furthermore,it follows as a corollary that for real or complex numbers, the triangle inequality for scalars (vectors with only one component)

$$\boxed{\left|\sum_i c_i\right| \leqslant \sum_i |c_i|}$$

Backward Triangle Inequality

- The triangle inequality produces an upper bound for a sum, but it also yields the following lower bound for a difference called backward triangle inequality

$$\boxed{|\|\vec{x}\| - \|\vec{y}\|| \leqslant \|\vec{x} - \vec{y}\|}$$

- This is a consequence of the triangle inequality because

$$\|\vec{x}\| = \|\vec{x} - \vec{y} + \vec{y}\| \leqslant \|\vec{x} - \vec{y}\| + \|\vec{y}\| \Rightarrow \|\vec{x}\| - \|\vec{y}\| \leqslant \|\vec{x} - \vec{y}\|$$

and

$$\|\vec{y}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{x}\| \Rightarrow -(\|\vec{x}\| - \|\vec{y}\|) \leqslant \|\vec{x} - \vec{y}\| \, .$$

Unit Vector

Definition

A unit vector $\vec{u}$ is a vector whose lengths equal one. Then $\|\vec{u}\| = 1$ and also $\vec{u} \cdot \vec{u} = 1$.

- ✔ Note: the word "unit" is always indicating that some measurement equals "one".
- ✔ Example: in four dimensions, take $\vec{u} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Then $\vec{u} \cdot \vec{u} = 1$.
- ✔ Property: for a vector $\vec{v}$, $\vec{u} = \vec{v}/\|\vec{v}\|$ is a unit vector in the same direction as $\vec{v}$. Hence a unit vector can be obtained by dividing a general vector $\vec{v}$ by $\|\vec{v}\|$.