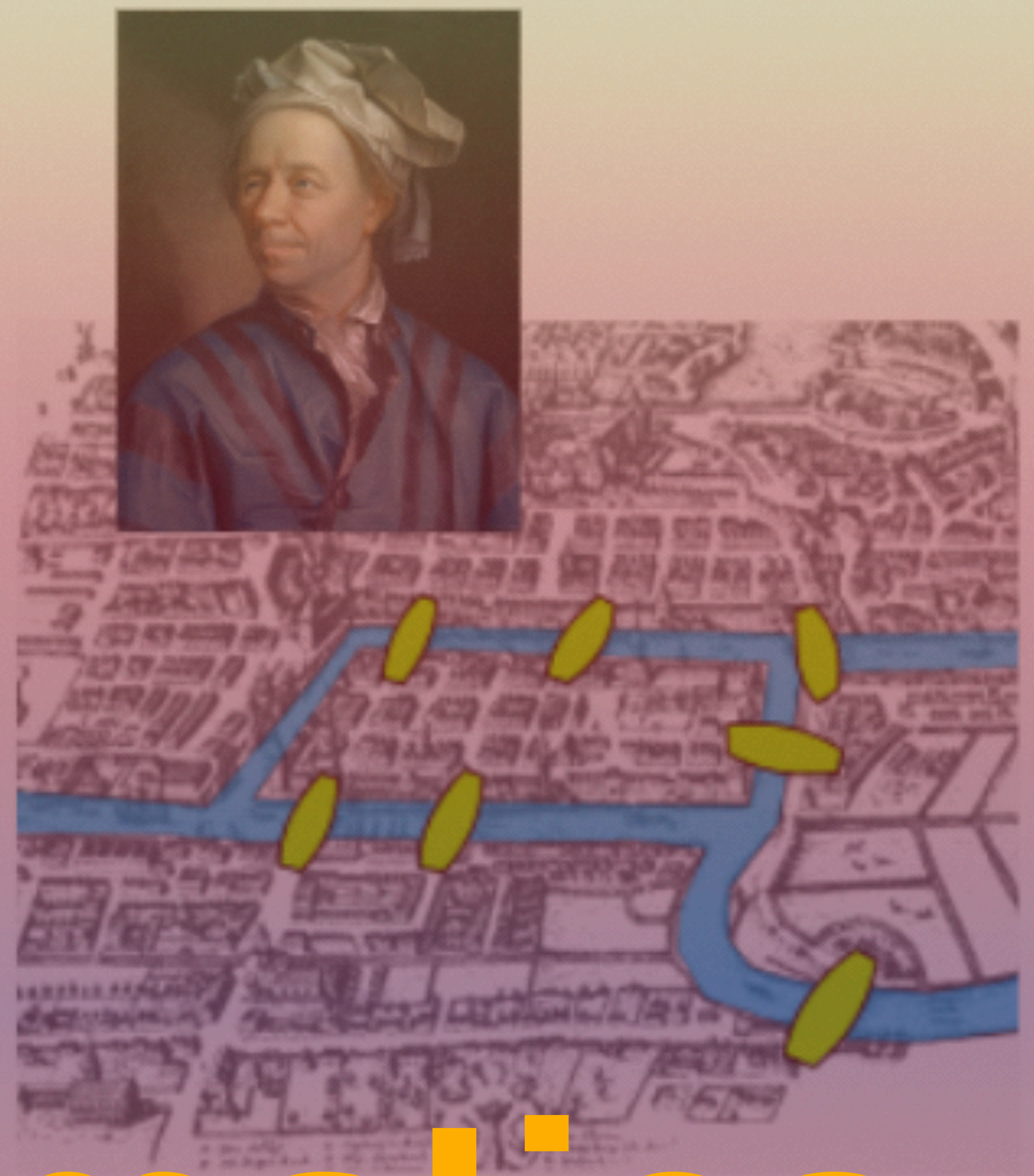
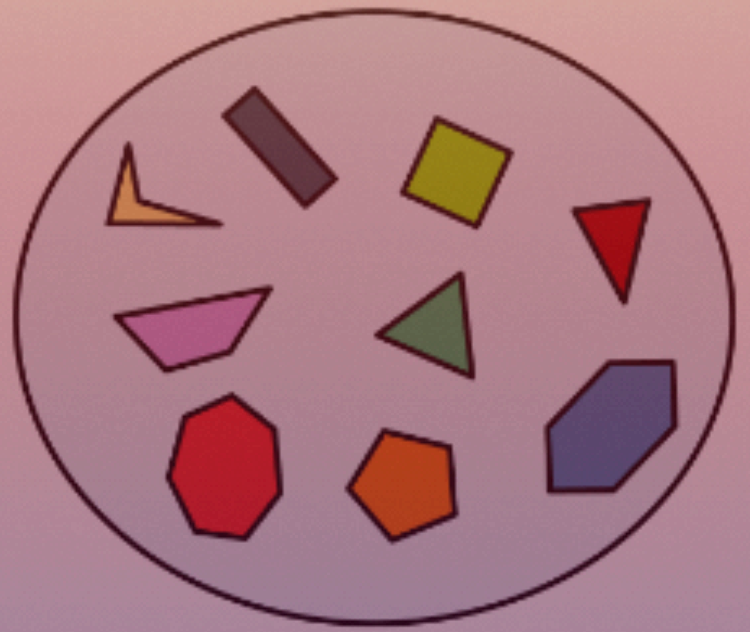
Discrete Mathematics

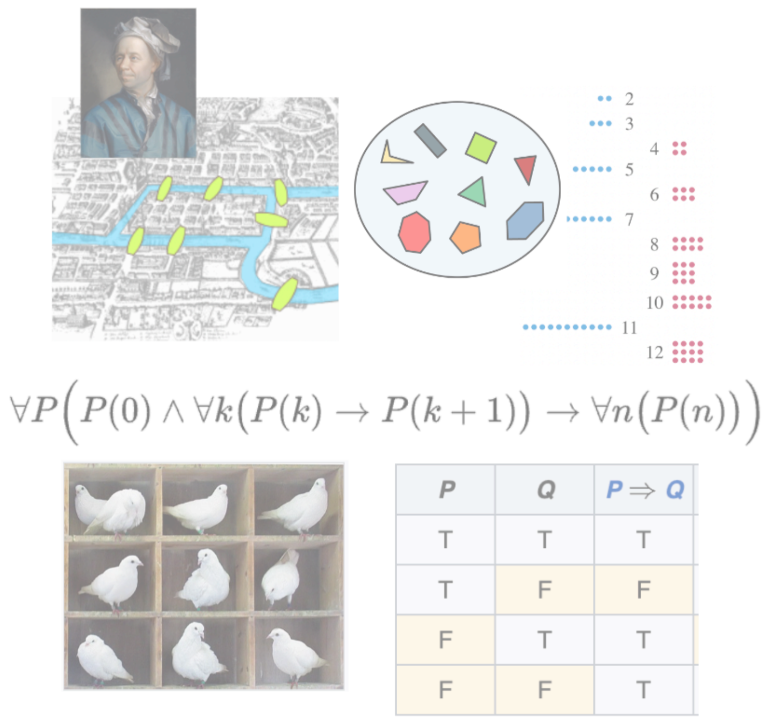
Lecture 1:
The language of mathematics





$$\forall P\Big(P(0) \land \forall k\big(P(k) \to P(k+1)\big) \to \forall n\big(P(n)\big)\Big)$$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T



Epp: Section 1.1

Variables

Variables: some historical remarks

In mathematics, a variable (from Latin variabilis, "changeable") is a symbol that represents a mathematical object.

A variable may be used to represent any kind of object, e.g. a number, a vector, a matrix, a function, the argument of a function, a set, or an element of a set.

In ancient works such as **Euclid**'s Elements, single letters refer to geometric points and shapes.

In the 7th century, **Brahmagupta** used different colours to represent the unknowns in algebraic equations in the Brāhma-sphuṭa-siddhānta. One section of this book is called "Equations of Several Colours”.

Source: Wikipedia entry on ‘Variable (Mathematics)’, retrieved Oct 15, 2023.

Variables: some historical remarks

At the end of the 16th century, **François Viète** (often known as “Vieta”) introduced the idea of representing known and unknown numbers by letters, nowadays called variables, and the idea of computing with them **as if** they were numbers—in order to obtain the result by a simple **replacement**.

Viète's convention was to use consonants for known values, and vowels for unknowns.

In 1637, René Descartes "invented the convention of representing unknowns in equations by x, y, and z, and knowns by a, b, and c”.

Contrarily to Viète's convention, Descartes’ naming convention is still commonly in use.

Variables: two uses — existential

“Is there a number with the following property: *doubling* **it** and *adding* 3 gives the same result as *squaring* **it**?”

In this sentence you can introduce a variable ‘x’ to replace the potentially ambiguous word “**it**”:

“Is there a number x with the property that $2x + 3 = x^2$?”

It allows you to give a *temporary name to what you are seeking* so that you can perform concrete computations with it to help discover its possible values.

Variables: two uses — universal

"No matter what number might be chosen, if **it** is greater than 2, then **its** square is greater than 4."

In this sentence you can introduce again a variable 'x' to replace the potentially ambiguous word "**it**":

"No matter what number x might be chosen, if $x > 2$, then $x^2 > 4$."

Here, it allows you to give a *temporary name to the (arbitrary) number you might choose* so that you can perform concrete computations with it to help check a certain property.

Some important kinds of mathematical statements

Three of the most important kinds of sentences in mathematics are *universal statements*, *conditional statements*, and *existential statements*:

A universal statement says that a certain property is true for all elements in a collection of objects.
(E.g., *All positive numbers are greater than* -10.)

A conditional statement says that if one thing is true then some other thing also has to be true.
(E.g., *If* 378 *is divisible by* 18, *then* 378 *is divisible by* 6.)

Given a property that may or may not be true, an existential statement says that there is at least one thing for which the property is true.
(E.g., *There is a prime number that is even*.)

Universal conditional statements

A ***universal conditional statement*** is a statement that is **both** universal and conditional.

Here is an example:

"For all animals a, if a is a dog, then a is a mammal."

Many mathematical theorems are stated as universal conditional statements.

- Universal quantification with variable a
- Fixing a **domain** (what are the things that we are talking about?)
- A conditional construction (if … then …), linking together:
  - The **precondition** for the theorem (what is the context?) about the a's
  - The **claimed property** for the objects in question, namely particular $a's$

Example: Theorem: For all natural numbers n, if n is divisible by 4, then n is divisible by 2.

Existential universal statements

An *existential universal statement* is a statement that is **primarily existential** about an object we are seeking, and states something universal about this object.

"There is a positive integer that is less than or equal to every positive integer."

In this sentence you can introduce two variables 'x' and 'y' to make it more clear:

"There is a positive integer x such that for every positive integer y we have $x \leq y$."

Universal existential statements

A *universal existential statement* is a statement that is **primarily universal** about a collection of objects, and states that some other objects exist, relative to the first one, such that some property about these pairs of objects holds.

"For any natural numbers, there is some other natural number bigger than the first."

"For all real numbers, there is a number, which when multiplied by three and subtracted from the original number, gives 0"

In these sentences you can introduce two variables 'x' and 'y' to make it more clear: E.g.

"For all real numbers x, there is a number y, such that $x - 3y = 0$"

Notice now that the choice of the value of y **depends** on the value of x .

Reformulating mathematical statements

Reformulating a mathematical statement is very useful to make it more intelligible within a certain context.

Universal conditional statement: “For all real numbers x, if x is non-zero then x^2 is positive”:

If a real number is non-zero, then its square ___

For all non-zero real numbers x, ___

If x ___, then ___

The square of any non-zero real number is ___

All non-zero real numbers have ___

Existential universal statement: “There is a person in my class that is at least as old as every person in my class”:

Some ___ is at least as old as ___

There is a person p in my class such that ___

There is a person p in my class with the property that for every person q in my class, ___

Universal existential statement: “Every pot has a lid”:

All pots ___

For all pots p, there is ___

For all pots p, there is a lid such that ___

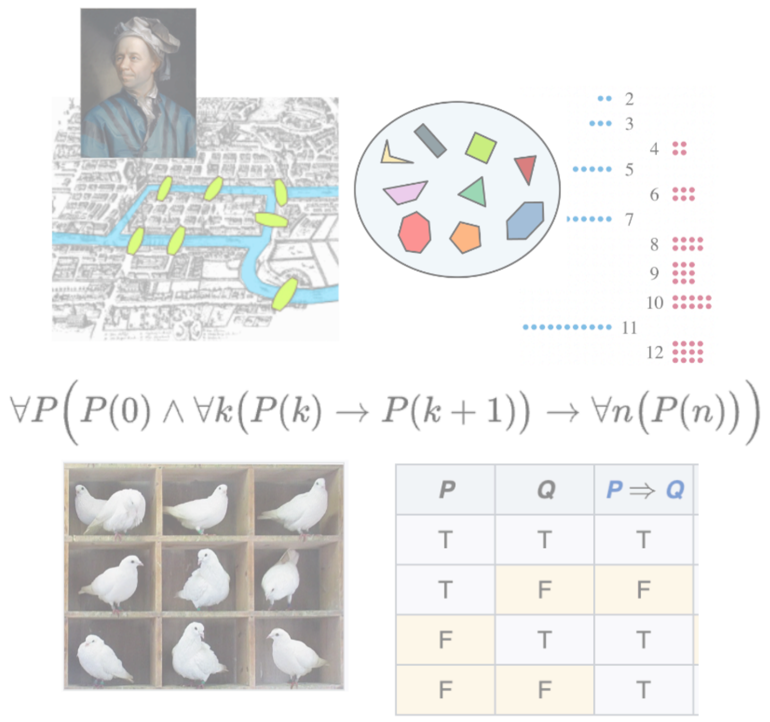
Another kind of important mathematical statement

Example of a universal equivalence statement:

For all numbers x and y, $x + y > 0$ if and only if $x > -y$.

An equivalence statement, or if and only if (iff) statement says that one thing is true exactly in those cases where another thing is true.
(E.g., $22 + 3 = 2 \times 11 + 3$ if and only if $22 = 2 \times 11$.)

It is a double conditional statement.
(E.g.,
1. if $22 + 3 = 2 \times 11 + 3$ then $22 = 2 \times 11$, **and**
2. if $22 = 2 \times 11$ then $22 + 3 = 2 \times 11 + 3$.)



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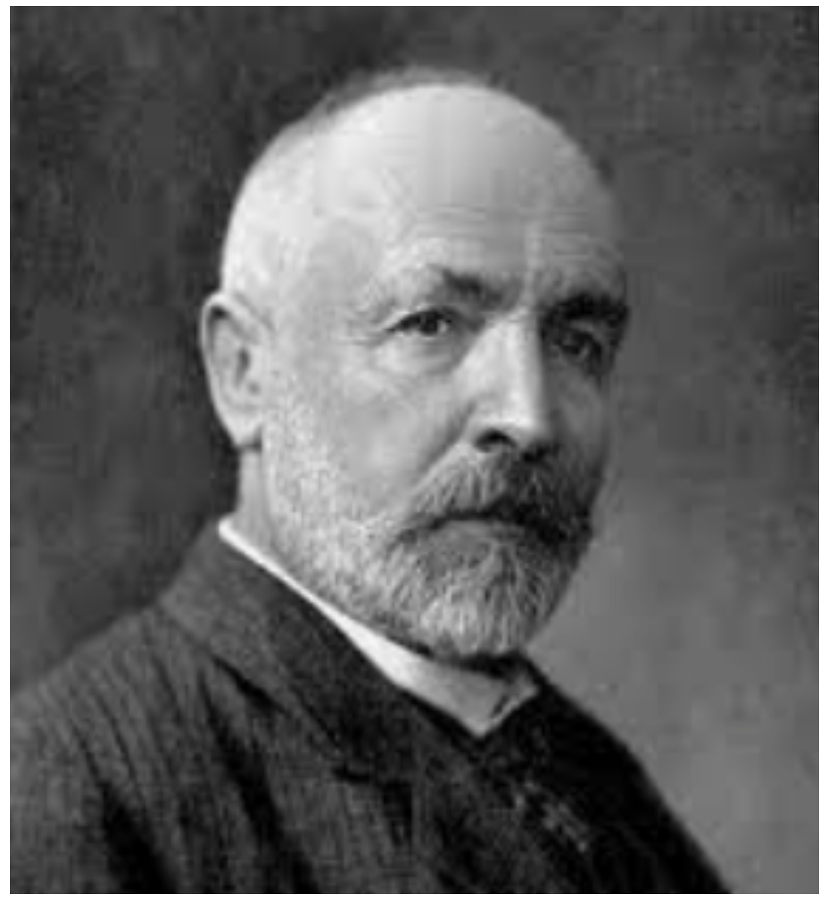
The language of sets

The language of sets

Use of the word set as a formal mathematical term was introduced in 1879 by **Georg Cantor** (1845–1918).



For most mathematical purposes we can think of a set intuitively, simply as a collection of objects, which are elements of it.

Examples:

- If C is the set of all countries that are currently in the United Nations, then Italy is an element of C; or
- If I is the set of all integers from 1 to 100, then the number 57 is an element of I.

The language of sets

Notation: If S is a set, the notation $x \in S$ means that the object x is an element of the set S; the notation $x \notin S$ means that the object x is not an element of the set S.

A set is completely determined by what its elements are (**axiom of extensionality**).
That is, two sets A and B **are equal** if and only if, for any x, we have that
$x \in A$ if and only if $x \in B$.
Notice that extensionality gives us an **identity criterion for sets:** two sets are the same exactly when they have the same elements as members.

Notation: A set may be specified using the **set-roster notation** by writing all its elements between braces.
(E.g., $\{1,2,3\}$ denotes the set whose elements are 1,2, and 3, and nothing else.)

Sometimes used for large sets, we can use the *ellipsis* …
(E.g., $\{1,2,3,\ldots,100\}$ to refer to all the integers from 1 to 100; or $\{2,4,6,8,10,\ldots\}$ to refer to all positive even numbers.)

Example — using the set-roster notation

a. Let $A = \{1,2,3\}$, $B = \{3,2,1\}$, and $C = \{1,1,2,3,3,3\}$.
What are the elements of A, B and C?
How are A, B and C related?

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1,\{1\}\}$?

d. For each non-negative integer n, let $U_n = \{-n, n\}$.
Find U_1, U_2, and U_0.

Example — solution

a. A, B and C have exactly the same three elements 1, 2 and 3.
Therefore, A, B and C are simply different ways to represent **the same set**.

b. $\{0\} \neq 0$ because 0 is the symbol that represents the number zero,
and $\{0\}$ is the set with one element, namely 0.

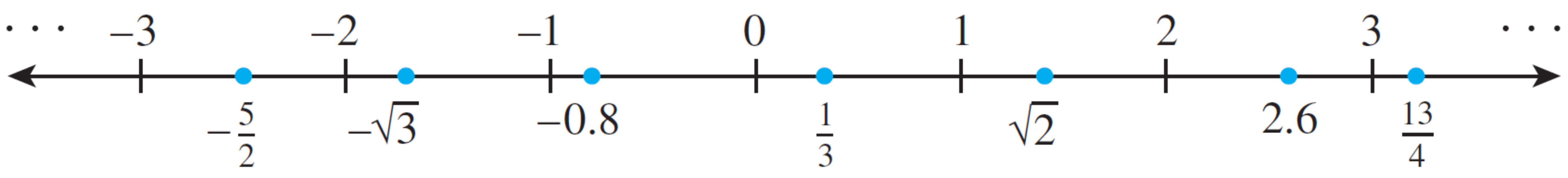
c. The set $\{1,\{1\}\}$ has two elements:
the integer 1 and the set whose only element is 1.

d. $U_1 = \{1, -1\}, U_2 = \{2, -2\}, U_0 = \{-0,0\} = \{0\}$.

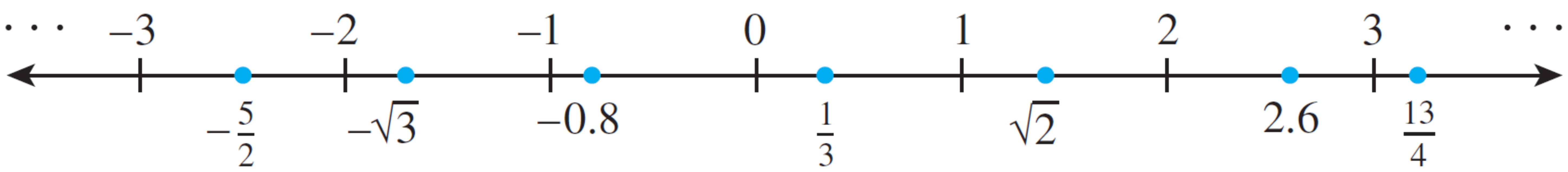
The language of sets

Certain sets of numbers are so frequently referred to that they are given special symbolic names.

Symbol	Set
$\mathbb{R}$ or $\mathbf{R}$	Set of all reals numbers
$\mathbb{Z}$ or $\mathbf{Z}$	Set of all integers
$\mathbb{N}$ or $\mathbf{N}$	Set of all integers greater or equal than 0 (set of natural numbers)
$\mathbb{Q}$ or $\mathbf{Q}$	Set of all rational numbers, or quotients of integers



Intermission: continuous vs discrete



The real number line is called continuous, because it is imagined to have no "holes".

The set of integers corresponds to a collection of points located at fixed intervals.

Thus every integer is a real number, and because the integers are all separated from each other, the set of integers is called ***discrete***.

The name ***discrete mathematics*** comes from the distinction between continuous and discrete mathematical objects.

Typical discrete objects are integers, finite orders, trees, and graphs.

The language of sets: set-builder notation

We showed that sets can be defined by listing all their elements.

Another, and more powerful, way to specify a set uses what is called the *set-builder notation*.

Notation: Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all** elements x in S **such that** $P(x)$ is true.

We denote this set formally as follows:

$$\{x \in S \mid P(x)\}$$

$P(x)$ is a predicate. More on that later.

Example — using the set-builder notation

Given that $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{Z}$ the set of all integers, and $\mathbb{Z}^+$ the set of all positive integers, describe each of the following sets.

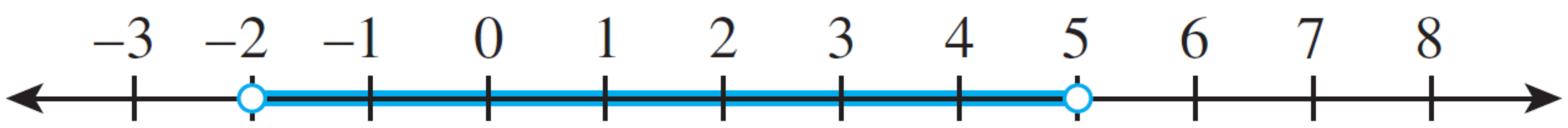
a. $\{x \in \mathbb{R} \mid -2 < x < 5\}$

b. $\{x \in \mathbb{Z} \mid -2 < x < 5\}$

c. $\{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$

Example — solution

a. $\{x \in \mathbb{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between –2 and 5. It is pictured as:



b. $\{x \in \mathbb{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between –2 and 5. It is equivalent to: $\{-1,0,1,2,3,4\}$

c. Since all the integers in $\mathbb{Z}^+$ are positive, $\{x \in \mathbb{Z}^+ \mid -2 < x < 5\} = \{1,2,3,4\}$

Subsets

Subsets

A basic relation between sets is that of subset.

Definition: If A and B are sets, then A is a subset of B, written $A \subseteq B$, if and only if every element of A is also an element of B.

Symbolically:

$$A \subseteq B \text{ iff for all objects } x, \text{ if } x \in A \text{ then } x \in B.$$

We also say that "A is contained in B" and "B contains A".

Non-subsets and strict subsets

It follows from the definition of subset that for a set A *not* to be a subset of a set B, noted $A \nsubseteq B$, means that there is at least one element of A that is not an element of B. This is in the form of an `existential conjunctive statement'. Symbolically:

> **Definition:** If A and B are sets then A is not a subset of B, written $A \nsubseteq B$ **iff** there is at least one object x such that $x \in A$ **and** $x \notin B$**.**

A closely related notion is the one of strict subset:

> **Definition:** If A and B are sets, then A is a strict (or proper) subset of B, written $A \subset B$, if and only if every element of A is also an element of B, and there is at least one element of B that is not in A.

Example — distinction between $\in$ and $\subseteq$

Which of the following are true statements?

a. $2 \in \{1,2,3\}$

b. $\{2\} \in \{1,2,3\}$

c. $2 \subseteq \{1,2,3\}$

d. $\{2\} \subseteq \{1,2,3\}$

e. $\{2\} \subseteq \{\{1\}, \{2\}\}$

f. $\{2\} \in \{\{1\}, \{2\}\}$

Example — solution

Only (**a**), (**d**), and (**f**) are true. E.g., **(a)** is true because the number 2 is an element of $\{1,2,3\}$.

For (**b**) to be true, the set $\{1,2,3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1,2,3\}$ are 1, 2, and 3, and none are equal to $\{2\}$, not even 2. Hence (**b**) is false.

For (**c**) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of $\{1,2,3\}$. This is not the case, so (**c**) is false.

For (**e**) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are $\{1\}$and $\{2\}$. But 2 is not equal to either $\{1\}$or $\{2\}$, and so (**e**) is false.

Cartesian products

Ordered pairs vs “sets of two things”

Notation: Given the elements a and b, the **notation** (a, b) denotes the ordered pair consisting of a as the **first** elements and b as the **second** element.

Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Examples:

- Is $(1,2) = (2,1)$?
- Is $(3,5/10) = (\sqrt{9},1/2)$?
- What is the first element of $(1,1)$?
- Are there x, y, z such that $(x, y) = (y, z)$?

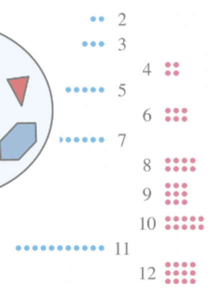
Cartesian product

Definition: Given sets A and B, the Cartesian product of A and B, denoted $A \times B$ is the set of all ordered pairs (a, b) where a is in A and b is in B.

Symbolically:
$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Example: Let $A = \{1,2,3\}$ and $B = \{u, v\}$:

- Find $A \times B$
- Find $B \times A$
- Find $B \times B$
- How many elements are there in $A \times B, B \times A, B \times B$
- Describe $\mathbb{R} \times \mathbb{R}$



$$\forall P\Big(P(0) \land \forall k(P(k) \to P(k+1)) \to \forall n(P(n))\Big)$$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Next lecture:

The language of logic