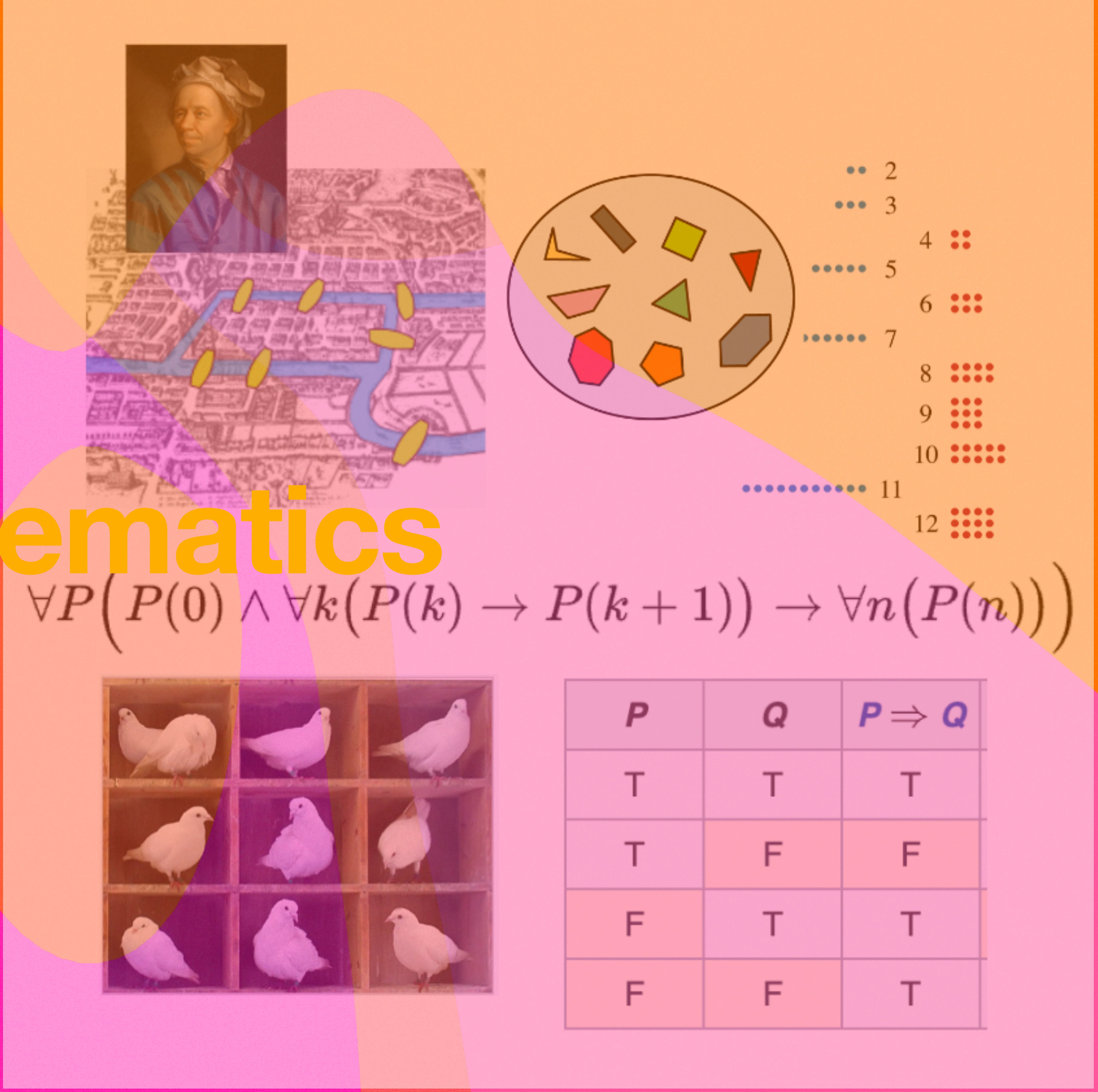
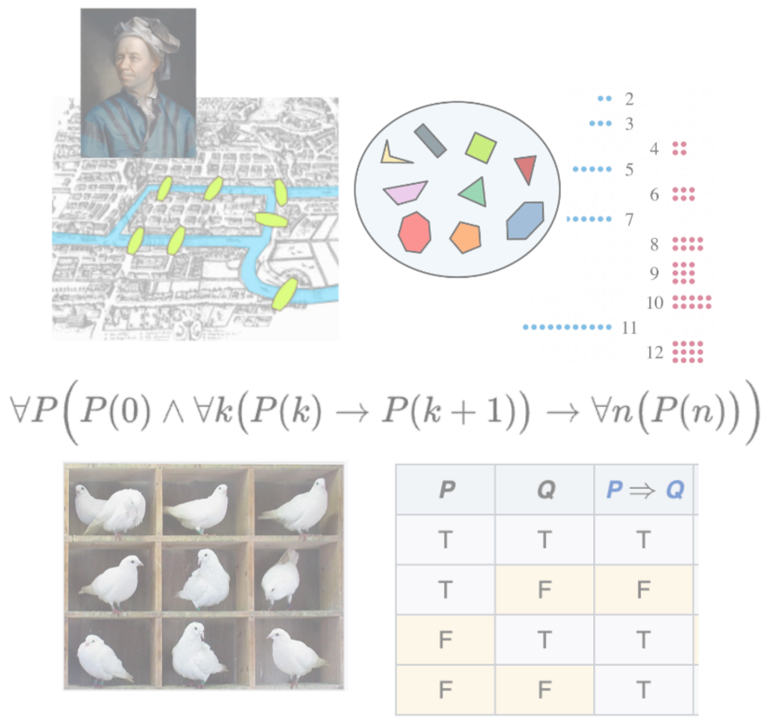
Discrete Mathematics

Partial Orders



Outline

- Antisymmetry
- Partial order relations
- Hasse diagrams
- Partially and totally ordered sets, comparability, chains
- Maximal, minimal, greatest, least elements
- Topological sorting
- Application: Scheduling



Epp: Section 8.5

Partial Order Relations

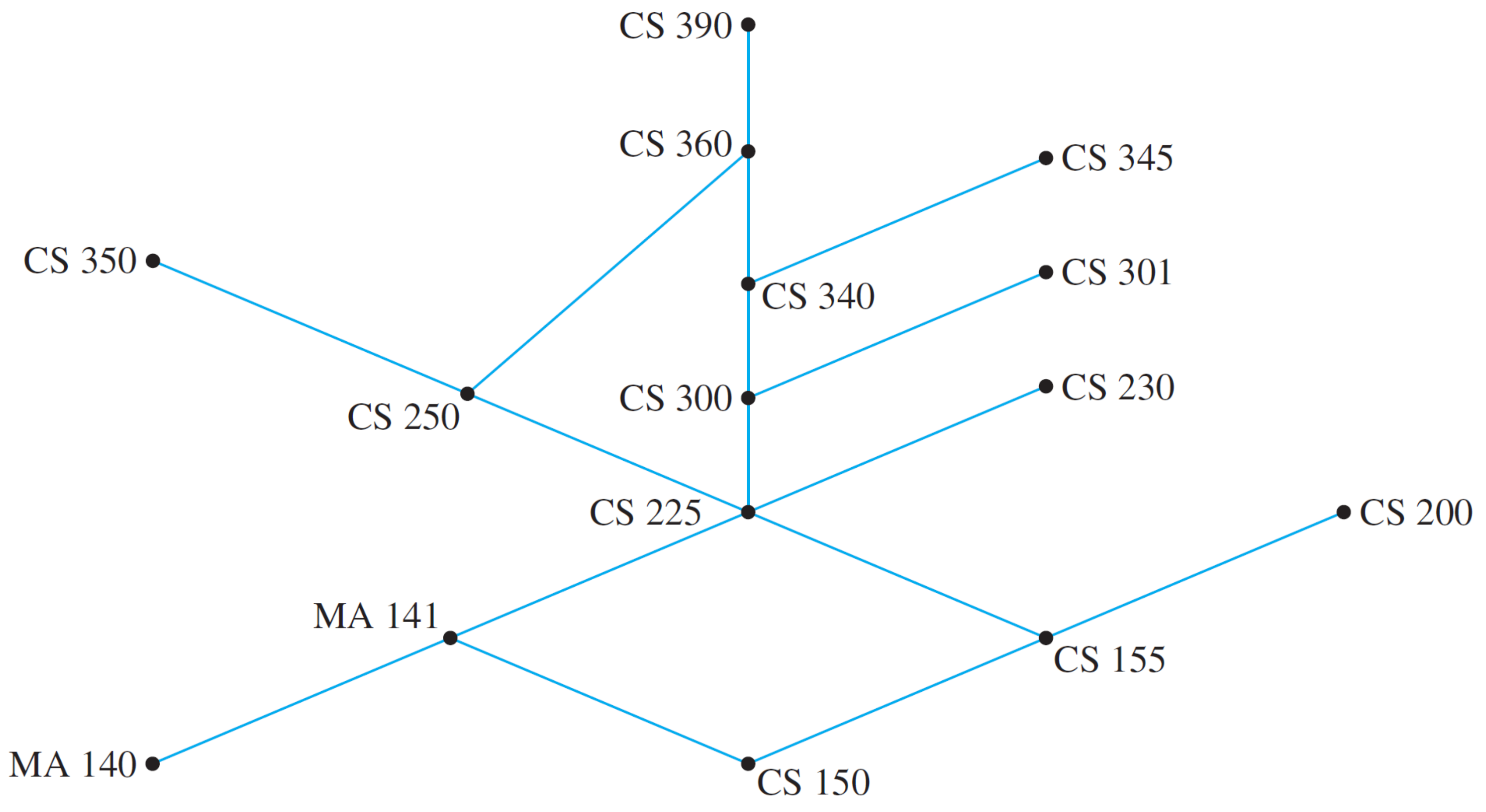
An example of ordered things

At a University, a student must take a specified set of required courses, some of which must be completed before others can be started.

A student could not take 300 before 155.

A student could take the course in this sequence 140, 150, 141, 155, 200, 225, 230, 300, 250, 301, 340, 345, 350, 360, 390.

A student could not take all the courses in less than 7 terms.



Antisymmetry

Antisymmetry

We have defined three properties of relations: reflexivity, symmetry, and transitivity.

The three properties define equivalence relations, which give rise to partitions.

A fourth property of relations is called **antisymmetry**.

In terms of the arrow diagram of a relation, saying that **a relation is antisymmetric** is the same as saying that whenever there is an arrow going from one element to another distinct element, there is ***not*** an arrow going back from the second to the first.

Antisymmetry

• Definition

Let R be a relation on a set A. R is **antisymmetric** if, and only if,

$$\text{for all } a \text{ and } b \text{ in } A, \quad \text{if } a\ R\ b \text{ and } b\ R\ a \text{ then } a = b.$$

By taking the negation of the definition, you can see that a relation R is not antisymmetric if, and only if,

there are elements a and b in A such that aRb and bRa, but also $a \neq b$.

Example — Testing for Antisymmetry of "Divides" Relations

Let R_1 be the "divides" relation on the set of all positive integers, and let R_2 be the "divides" relation on the set of all integers.

$$\begin{array}{lccc} \text{For all } a, b \in Z^+, & a \; R_1 \; b & \Leftrightarrow & a \mid b. \\ \text{For all } a, b \in Z, & a \; R_2 \; b & \Leftrightarrow & a \mid b. \end{array}$$

a. Show that R_1 antisymmetric.

b. Show that R_2 is not antisymmetric.

Example — Solution

a. R_1 is antisymmetric.

Proof:
Suppose a and b are positive integers such that aR_1b and bR_1a.
[We must show that $a = b$.]
By definition of R_1: $a \mid b$ and $b \mid a$.

Thus, by definition of divides, there are integers k_1 and k_2 with $b = k_1a$ and $a = k_2b$.
It follows that

$$b = k_1a = k_1(k_2b) = (k_1k_2)b.$$

Dividing both sides by b gives

$$k_1k_2 = 1.$$

Example — Solution

Now since a and b are both positive integers k_1 and k_2 are both positive integers, too.

But the only product of two positive integers that equals 1 is $1 \cdot 1$.

Thus

$$k_1 = k_2 = 1$$

and so

$$a = k_2 b = 1 \cdot b = b.$$

[This is what was to be shown.]

Example — Solution

b. R_2 is not antisymmetric.

Proof by counterexample:

Let $a = 2$ and $b = -2$. Then $a \mid b$ *[since* $-2 = (-1) \cdot 2$*]* and $b \mid a$ *[since* $2 = (-1)(-2)$*]*.

Hence aR_2b and bR_2a but $a \neq b$.

Partial Order Relations

Partial Order Relations

A relation that is reflexive, antisymmetric, and transitive is called a partial order.

> **• Definition**
>
> Let R be a relation defined on a set A. R is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

Examples: Two fundamental partial order relations are the

(a) "less than or equal" relation on a set of real numbers and

(b) the "subset" relation on a set X of sets.

These can be thought of as *models* for general partial order relations.

Partial Order Relations: $\preceq$ Notation

The symbol $\preceq$ is often used to refer to a general partial order relation, and the notation $x \preceq y$ is read:

"x is less than or equal to y" or "y is greater than or equal to x"

• Notation

Because of the special paradigmatic role played by the $\leq$ relation in the study of partial order relations, the symbol $\preceq$ is often used to refer to a general partial order relation, and the notation $x \preceq y$ is read "x is less than or equal to y" or "y is greater than or equal to x."

Example — Partial Order Relations

Consider the "subset" relation on a set X of sets.

Definition: For any pair of sets $A, B \in X$ **set:**

$$A \preceq B \text{ if and only if } A \subseteq B.$$

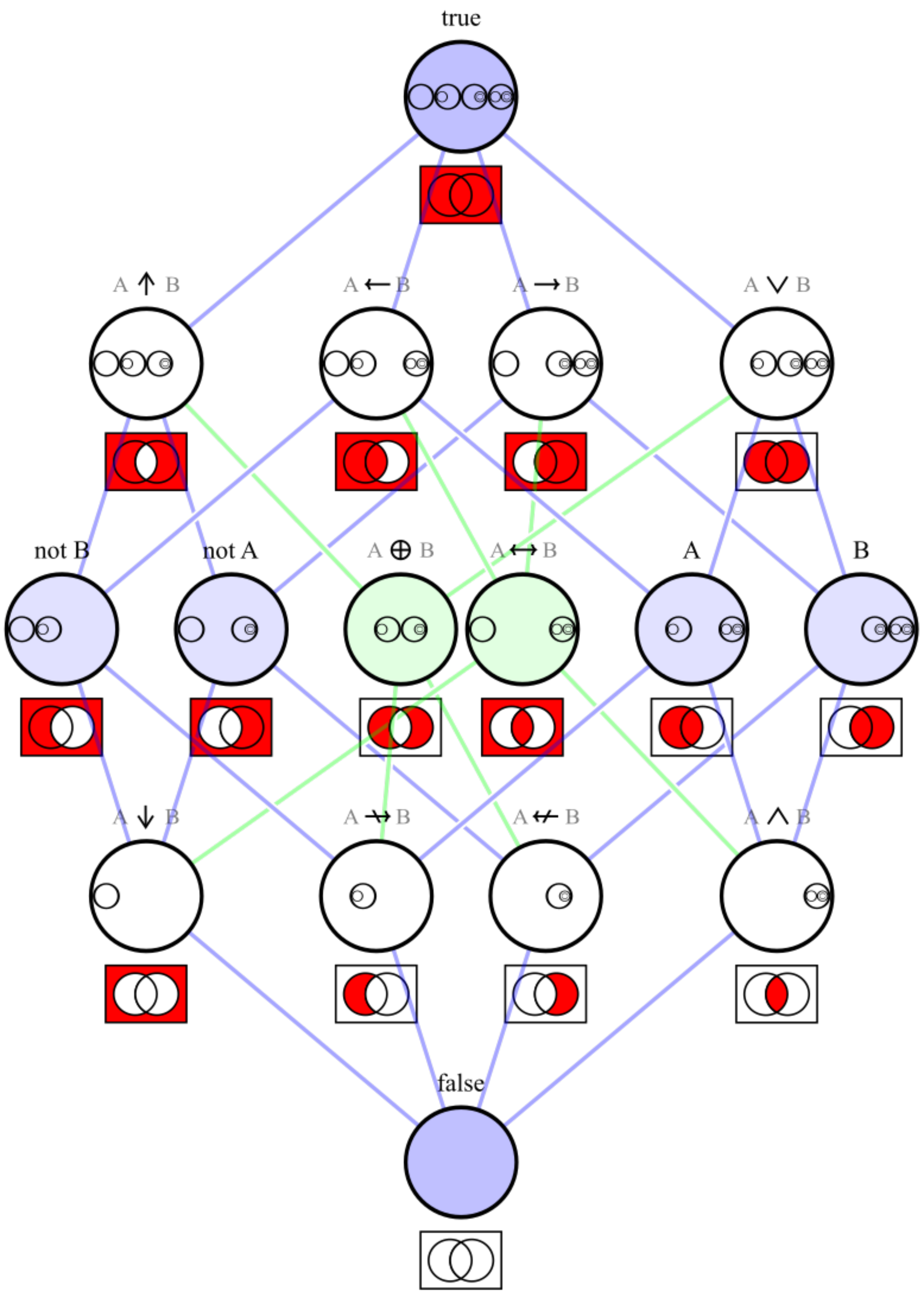
1. Why is $\preceq$ a partial order on X?
2. Can you, in general, order all the elements $x_1, x_2, x_3, x_4, \ldots$ from X in a linear kind of way?

Example — Solution

$$X \text{ is a set of sets and } \forall A, B \in X \,.\, A \preceq B \text{ iff } A \subseteq B.$$

1. Why is $\preceq$ a partial order on X?
   reflexive: $A \subseteq A$;
   transitive: $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$;
   antisymmetric: $A \subseteq B \wedge B \subseteq A \rightarrow A = B$.

2. Can you, in general, order all the elements $x_1, x_2, x_3, x_4, \ldots$ from X in a linear kind of way?
"In a linear kind of way": **Yes! See below for the notion of a 'topological sorting'**.



Hasse diagrams

Hasse Diagrams

It is possible to associate a graph, called a Hasse diagram (after Helmut Hasse, a twentieth-century German number theorist), with a partial order relation defined on a finite set.

To obtain a Hasse diagram, proceed as follows:

- Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward.
- Then eliminate the loops at all the vertices whose existence is implied by the reflexive property, all arrows whose existence is implied by the transitive property, the direction indicators on the arrows.

Example — Constructing a Hasse Diagram

Consider the "subset" relation, $\subseteq$, on the set $\mathscr{P}(\{a, b, c\})$.

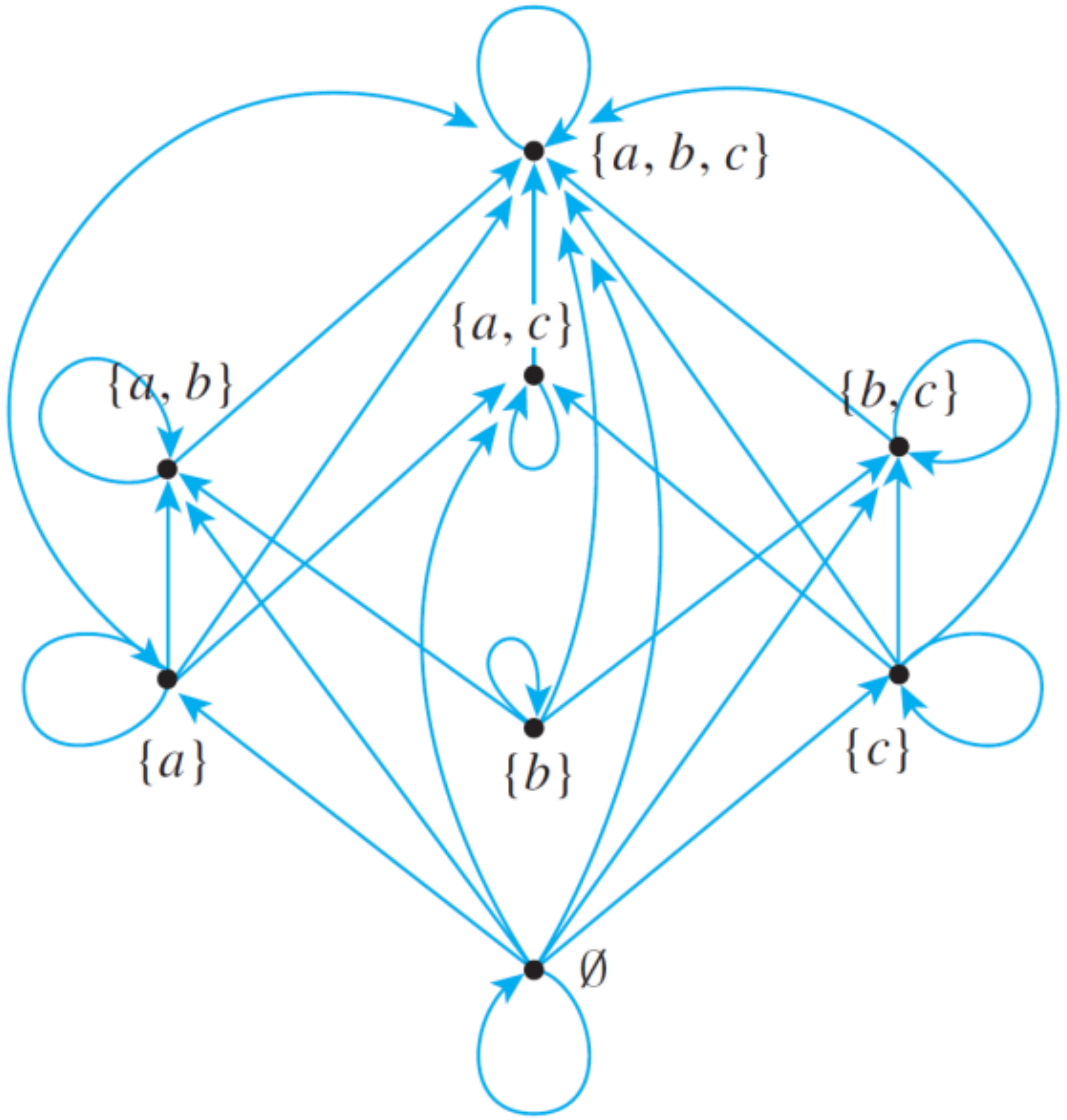
That is, for all sets U and V in $\mathscr{P}(\{a, b, c\})$:

$$U \subseteq V \iff \forall x,\ \text{if } x \in U \text{ then } x \in V.$$

Construct the Hasse diagram for this relation.

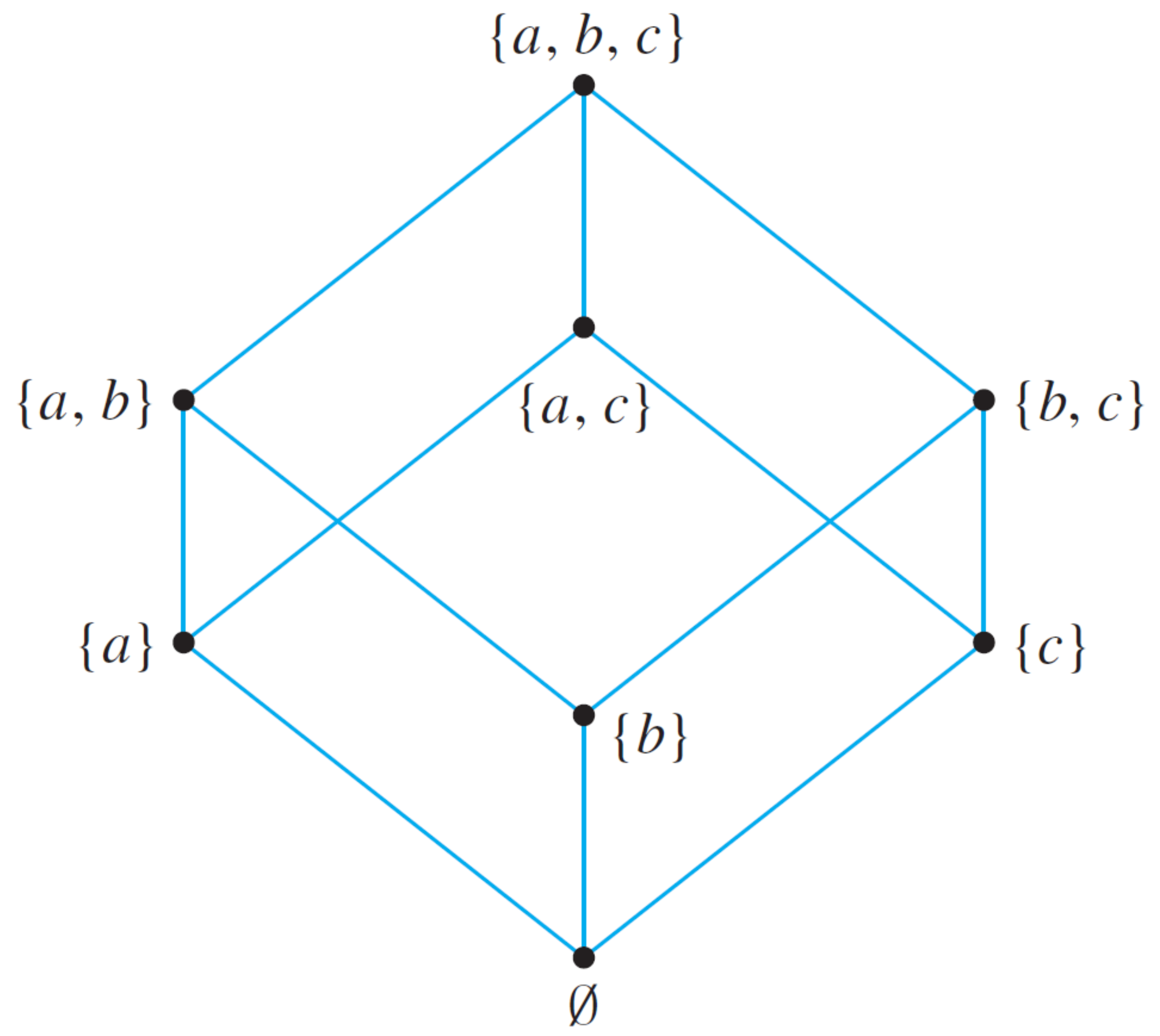
Example — Solution

The "subset" relation, ⊆, on the set $\mathscr{P}(\{a, b, c\})$.



Draw the directed graph of the relation in such a way that all arrows except loops point upward.

Then strip away all loops, unnecessary arrows, and direction indicators to obtain the Hasse diagram.



Partially and Totally Ordered Sets

Comparable and Non-Comparable

Definition

Suppose $\preceq$ is a partial order relation on a set A.
Elements a and b of A are said to be **comparable** if, and only if, $a \preceq b$ or $b \preceq a$.
Otherwise, a and b are called **noncomparable.**

When all the elements of a partial order relation are comparable, the relation is called a total order (or also linear order).

Definition

If R is a partial order relation on a set A, and for any two elements a and b in A we have aRb or bRa, then R is a **total order relation** on A.

Examples — Comparable and Non-Comparable

Given any two real numbers x and y, either $x \leq y$ or $y \leq x$. All pairs of real numbers are comparable.

On the other hand, given two subsets A and B of $\{a, b, c\}$, it may be the case that neither $A \subseteq B$ nor $B \subseteq A$.

For instance, let $A = \{a, b\}$ and $B = \{b, c\}$. Then $A \nsubseteq B$ and $B \nsubseteq A$.

So, A and B are non-comparable.

Examples — Total Orders

The "less than or equal to" relation $\leq$ on real numbers is a total order relation.

Many important partial order relations have elements that are not comparable and are, therefore, not total order relations.

The subset relation $\subseteq$ on $\mathscr{P}(\{a, b, c\})$ is not a total order relation because, as shown previously, the subsets $\{a, b\}$ and $\{a, c\}$ of $\{a, b, c\}$ are not comparable.

Partially and Totally Ordered Sets

Definition

Let $\preceq$ be a partial order relation on the set A. We say that A is a partially ordered set (or poset) with respect to the relation $\preceq$.

Definition

A set A is called a totally ordered set with respect to a relation $\preceq$ if, and only if: A is partially ordered with respect to $\preceq$ and $\preceq$ is a total order.

For instance, the set of real numbers is a partially ordered set with respect to the “less than or equal to” relation $\leq$, and a set of sets is a partially ordered set with respect to the “subset” relation $\subseteq$.

Also, the set of real numbers is a totally ordered set with respect to the “less than or equal to” relation $\leq$.

Chains

A set that is partially ordered but not totally ordered may have totally ordered subsets. Such subsets are called chains.

Definition

Let A be a set that is partially ordered with respect to a relation $\preceq$.

A subset $B \subseteq A$ of A is called a **chain** if, and only if, the elements in each pair of elements from B are comparable.

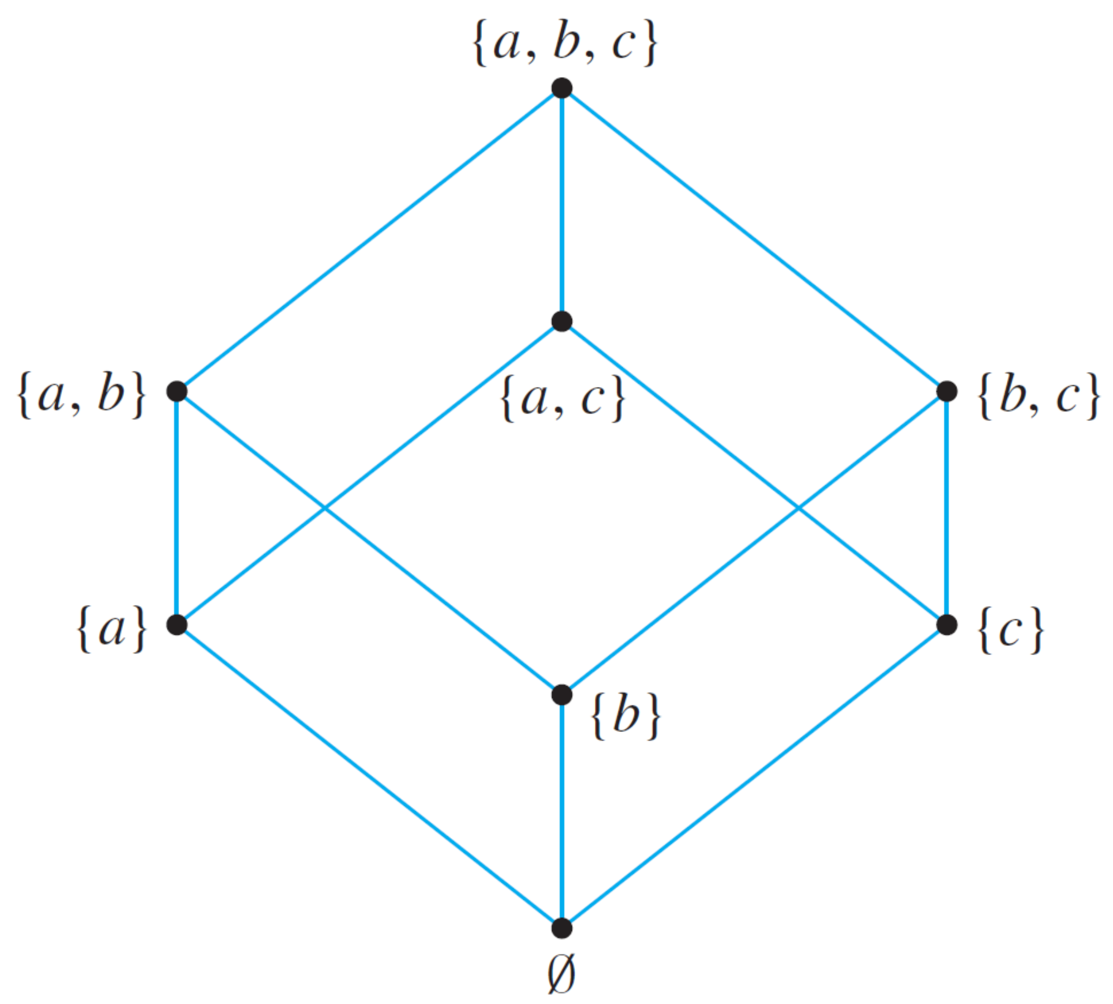
In other words: $a \preceq b$ or $b \preceq a$ for all $a, b \in B$.

The **length of a chain** is one less than the number of elements in the chain.

Observe that if B is a chain in A, then B is a totally ordered set with respect to the "restriction" of $\preceq$ to B.

Example — A Chain of Subsets

The set $\mathscr{P}(\{a, b, c\})$ is partially ordered with respect to the subset relation. Find a chain of length 3 in $\mathscr{P}(\{a, b, c\})$.

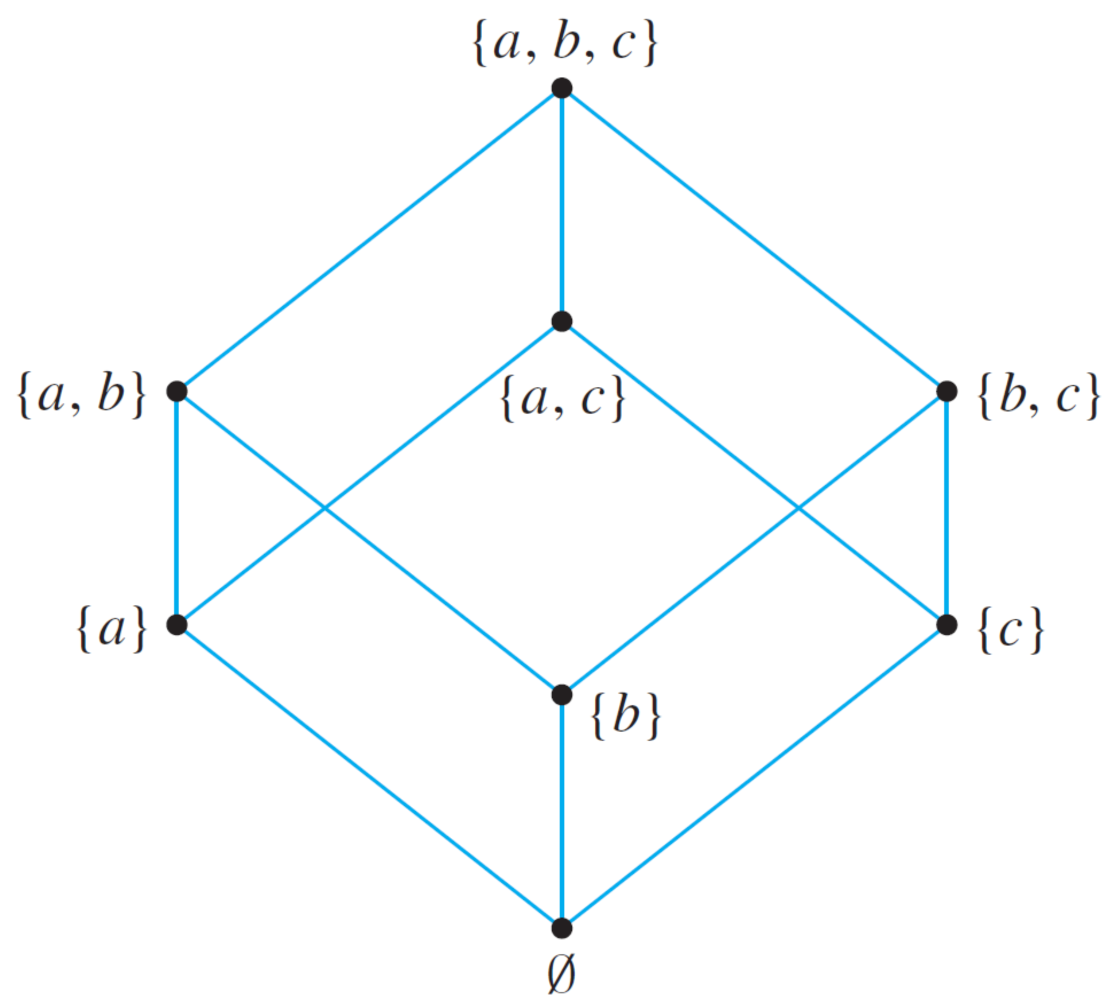


Example — Solution

The set $\mathscr{P}(\{a, b, c\})$ is partially ordered with respect to the subset relation. Find a chain of length 3 in $\mathscr{P}(\{a, b, c\})$.

Solution:

We have $\varnothing \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\}$. So the set $S = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$ is a chain of length 3 in $\mathscr{P}(\{a, b, c\})$.



Maximal, Minimal, Greatest, Least Elements

- **Definition**

Let a set A be partially ordered with respect to a relation $\preceq$.

1. An element a in A is called a **maximal element of A** if, and only if, for all b in A, either $b \preceq a$ or b and a are not comparable.
2. An element a in A is called a **greatest element of A** if, and only if, for all b in A, $b \preceq a$.
3. An element a in A is called a **minimal element of A** if, and only if, for all b in A, either $a \preceq b$ or b and a are not comparable.
4. An element a in A is called a **least element of A** if, and only if, for all b in A, $a \preceq b$.

Maximal, Minimal, Greatest, Least Elements

A greatest element is maximal, but a maximal element need not be a greatest element.

Similarly, a least element is minimal, but a minimal element need not be a least element.

Uniqueness of Greatest and Least Elements

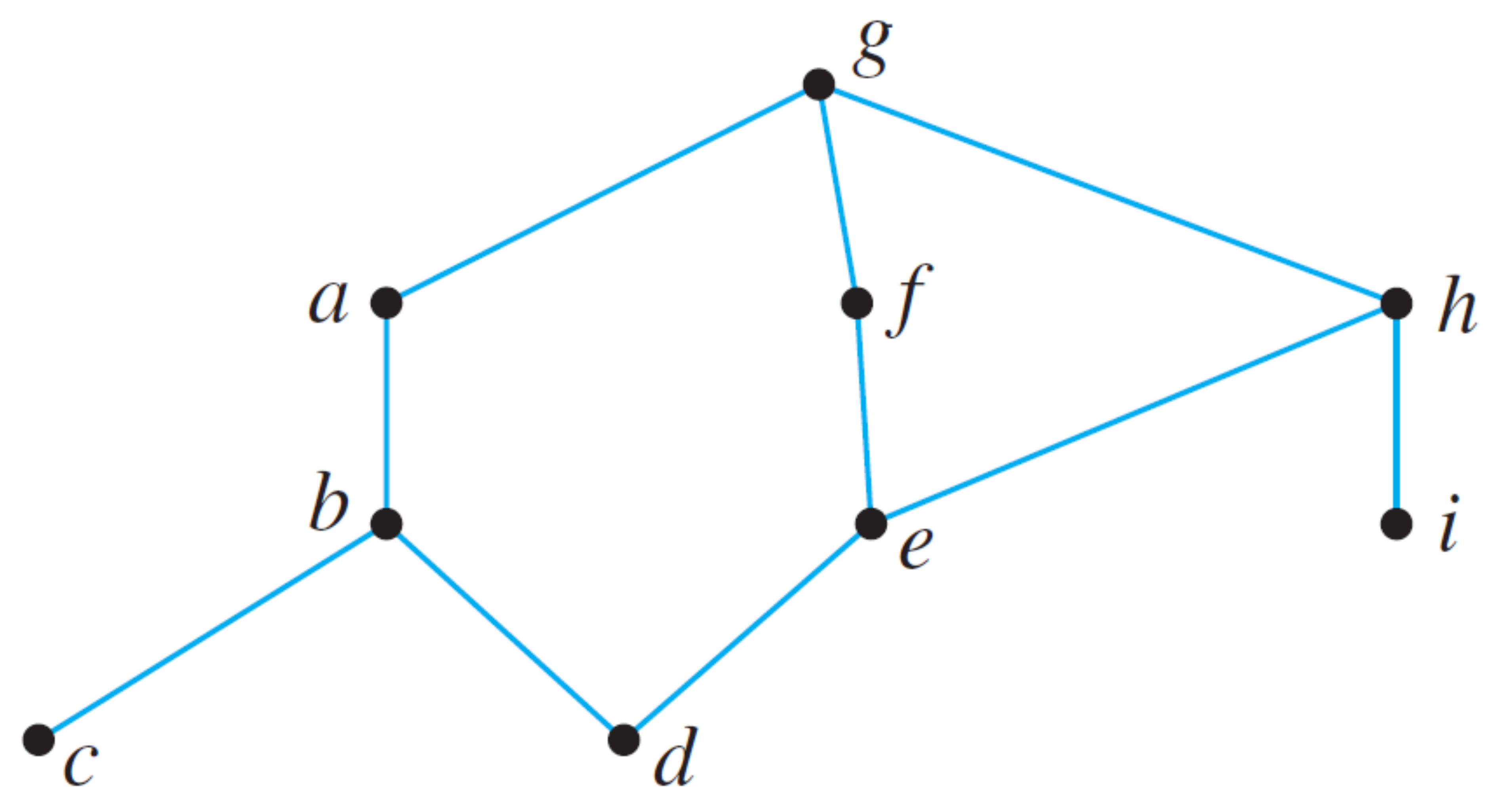
A partially ordered set can have at most one greatest element and at most one least element, but it may have more than one maximal or minimal element.

Lemma. Let $(A, \preceq)$ be a partially ordered set. If a greatest element exists, then it is unique. Likewise, if a least element exists, then it is unique.

Proof. Assume that g_1 and g_2 both satisfy the property of being greatest element. This means in particular that both $g_1 \preceq g_2$ and $g_2 \preceq g_1$ are true. By antisymmetry, this implies that $g_1 = g_2$. Therefore, there can be only one greatest element.

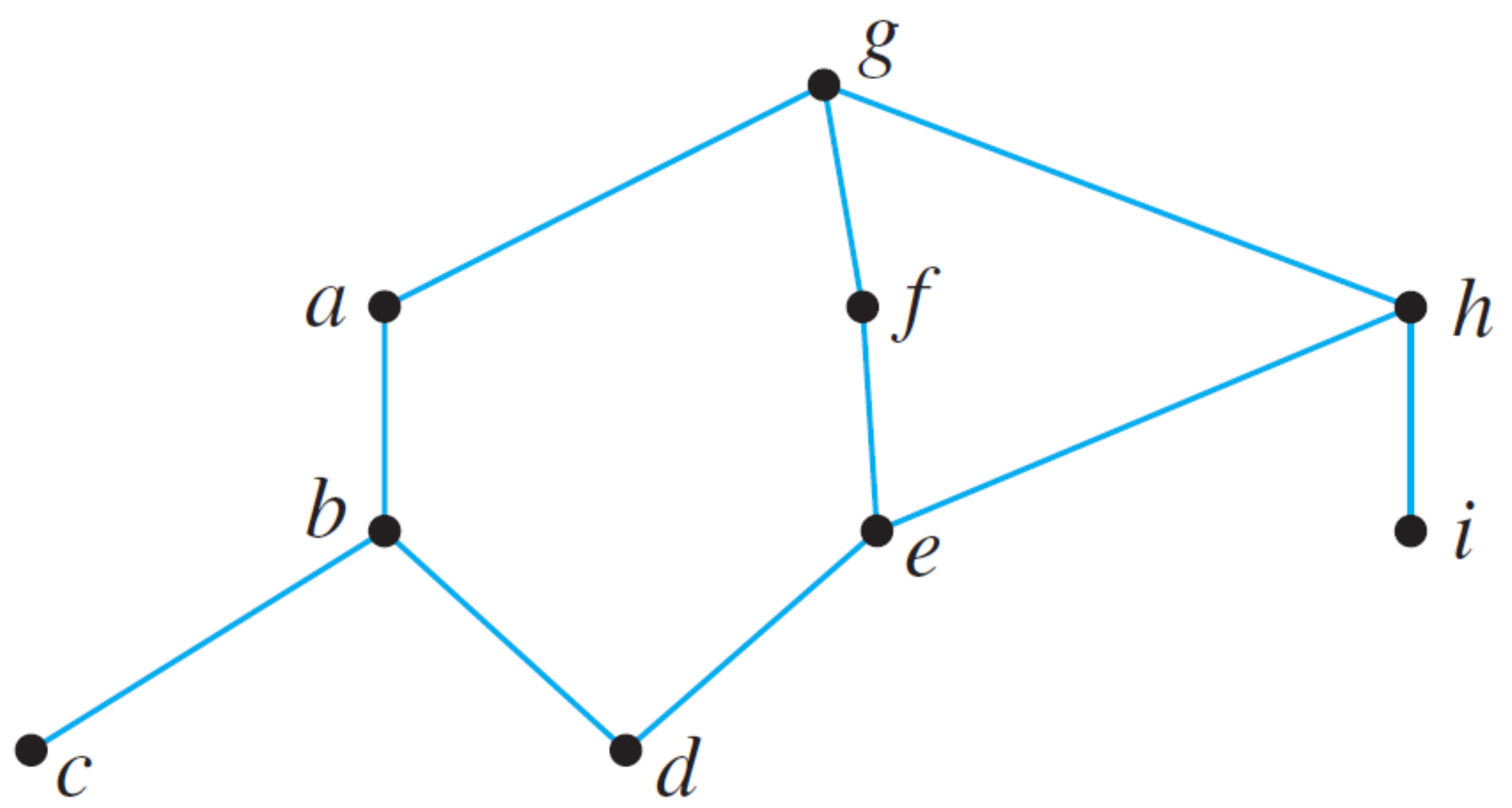
Example — Maximal, Minimal, Greatest, and Least Elements

Let $A = \{a, b, c, d, e, f, g, h, i\}$ have the partial ordering $\preceq$ defined by the following Hasse diagram.



Find all maximal, minimal, greatest, and least elements of A.

Example — Solution



There is just one maximal element, g, which is also the greatest element. The minimal elements are c, d, and i, and there is no least element.

Topological Sorting

Topological Sorting: total ordering of partial orders

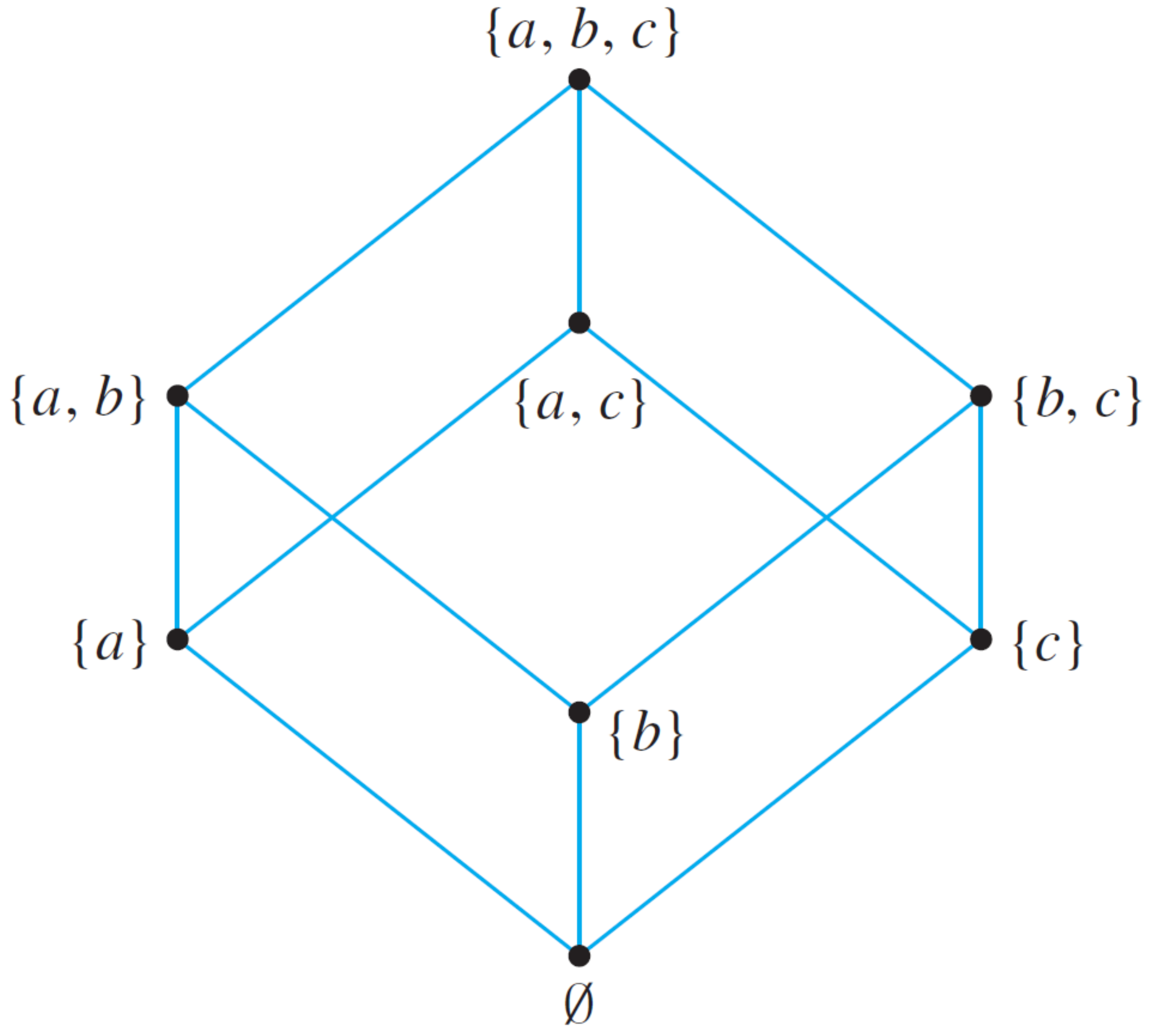
Is it possible to list linearly the sets of $\mathscr{P}(\{a, b, c\})$ in a way that is "compatible" with the subset relation $\subseteq$ in the sense that if set U is a subset of set V, then U is listed before V?

The answer, as it turns out, is yes. For instance, the following input order satisfies the given condition:

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

Another input order that satisfies the condition is

$$\emptyset, \{a\}, \{b\}, \{a, b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}.$$



Topological Sorting

- **Definition**

Given partial order relations $\preceq$ and $\preceq'$ on a set A, $\preceq'$ is **compatible** with $\preceq$ if, and only if, for all a and b in A, if $a \preceq b$ then $a \preceq' b$.

A total order that is compatible with a given partial order is called a topological sorting.

- **Definition**

Given partial order relations $\preceq$ and $\preceq'$ on a set A, $\preceq'$ is a **topological sorting** for $\preceq$ if, and only if, $\preceq'$ is a total order that is compatible with $\preceq$.

Topological Sorting

Constructing a Topological Sorting

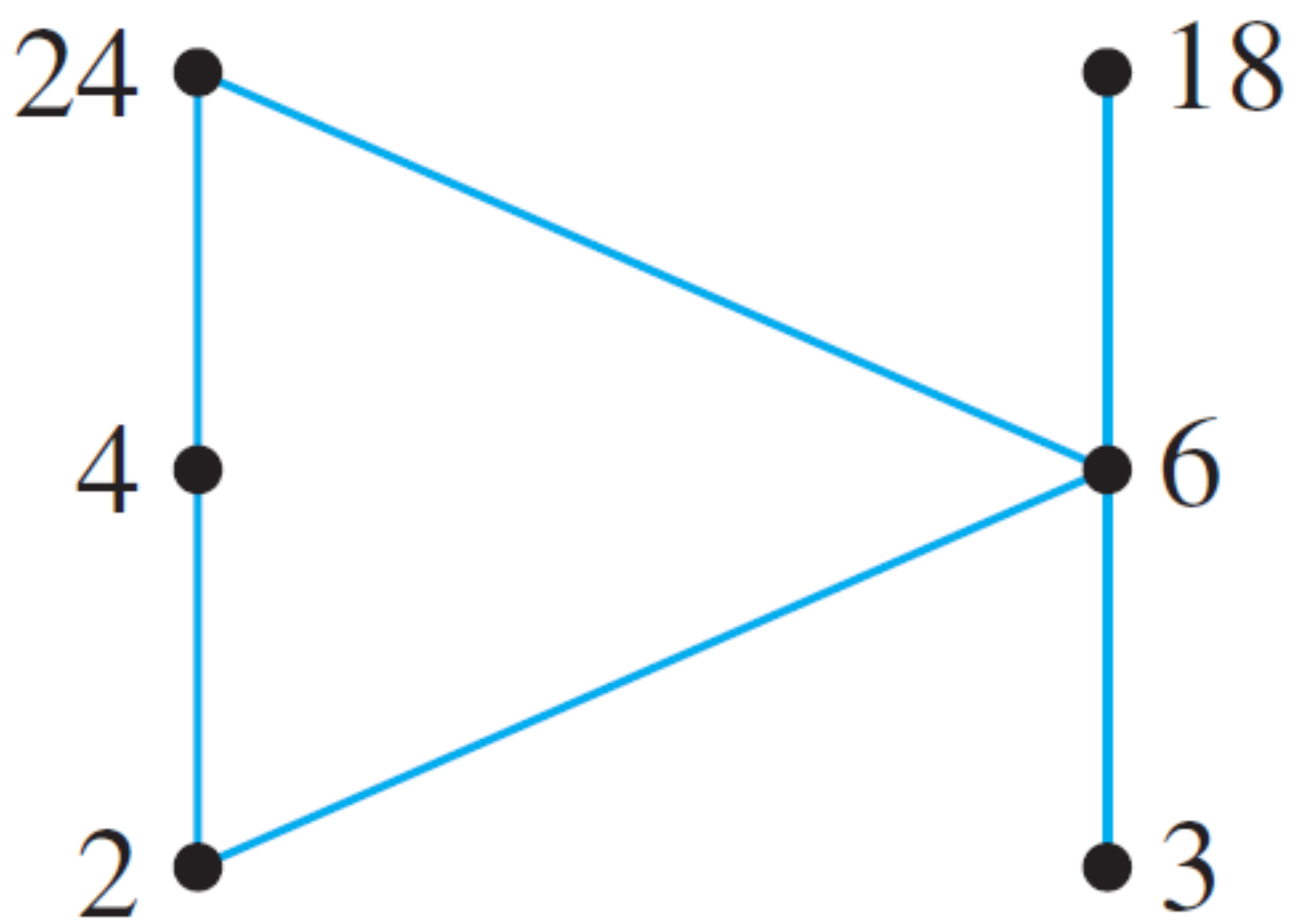
Let $\preceq$ be a partial order relation on a non-empty finite set A.
To construct a topological sorting, do the following:

1. Pick any minimal element x in A. *(It exists since $A \neq \emptyset$)*
2. Set $A' := A - \{x\}$
3. Repeat (a)—(c) while $A' \neq \emptyset$
   (a) Pick any minimal element $y \in A'$
   (b) Define $x \preceq' y$
   (c) Set $A' := A' - \{y\}$ and $x := y$

Example — A Topological Sorting

Consider the set $A = \{2,3,4,6,18,24\}$ ordered by the "divides" relation $|$. (Here, $\preceq\, = \,|$.)

The Hasse diagram of this relation is the following:



The ordinary "less than or equal to" relation $\leq$ on this set is a topological sorting for it since for positive integers a and b, if $a \mid b$ then $a \leq b$.

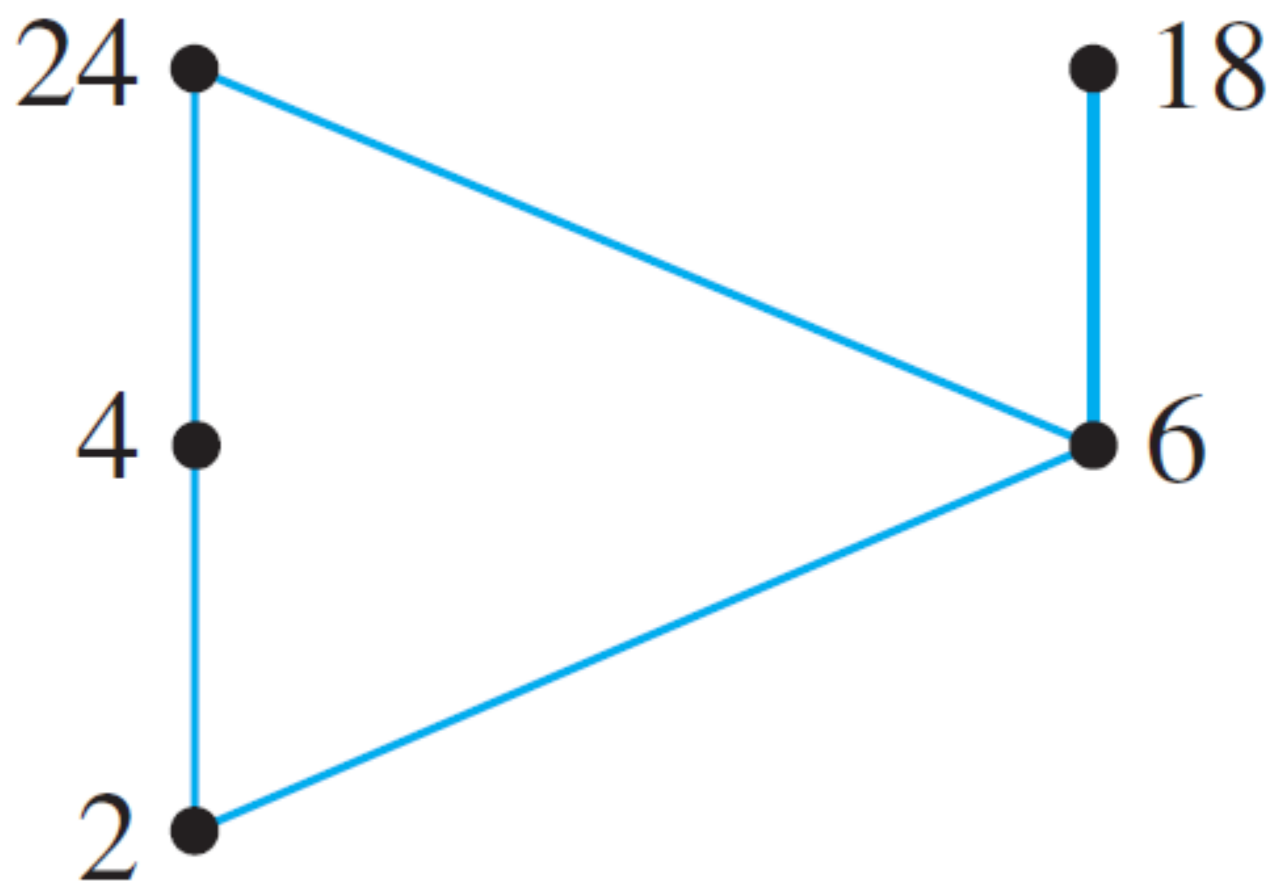
Find another topological sorting $\preceq'$ for this set.

Example — Solution

The set has two minimal elements: 2 and 3. Either one may be chosen; say you pick 3.

The beginning of the total order is: **total order** 3.

Set $A' = A - \{3\}$ You can indicate this by removing 3 from the Hasse diagram.



Next choose a minimal element from $A - \{3\}$. Only 2 is minimal, so you must pick it.

Example — Solution

The total order thus far is: **total order** $3 \preceq' 2$.

Set $A' = (A - \{3\}) - \{2\} = A - \{3{,}2\}$.

You can indicate this by removing 2 from the Hasse diagram.

Choose a minimal element from $A - \{3{,}2\}$.

Example — Solution

Again you have two choices: 4 and 6.

Say you pick 6.

The total order for the elements chosen thus far is: **total order** $3 \preceq' 2 \preceq' 6$.

You continue in this way until every element of A has been picked. One possible sequence of choices gives the total order: $3 \preceq' 2 \preceq' 6 \preceq' 18 \preceq' 4 \preceq' 24$.

Example — Solution

You can verify that this order is compatible with the "divides" partial order by checking that for each pair of elements

$$a \text{ and } b \text{ in } A \text{ such that } a \mid b, \text{ then } a \preceq' b.$$

Note that it is not the case that if $a \preceq' b$ then $a \mid b$.

An Application: Scheduling

An Application

At an automobile assembly plant, the job of assembling an automobile can be broken down into these tasks:

1. Build frame.
2. Install engine, power train components, gas tank.
3. Install brakes, wheels, tires.
4. Install dashboard, floor, seats.
5. Install electrical lines.
6. Install gas lines.
7. Install brake lines.
8. Attach body panels to frame.
9. Paint body.

An Application

Certain of these tasks can be carried out at the same time, whereas some cannot be started until other tasks are finished.

Task	Immediately Preceding Tasks	Time Needed to Perform Task
1		7 hours
2	1	6 hours
3	1	3 hours
4	2	6 hours
5	2, 3	3 hours
6	4	1 hour
7	2, 3	1 hour
8	4, 5	2 hours
9	6, 7, 8	5 hours

Let T be the set of all tasks, and consider the partial order relation $\preceq$ defined on T as follows:

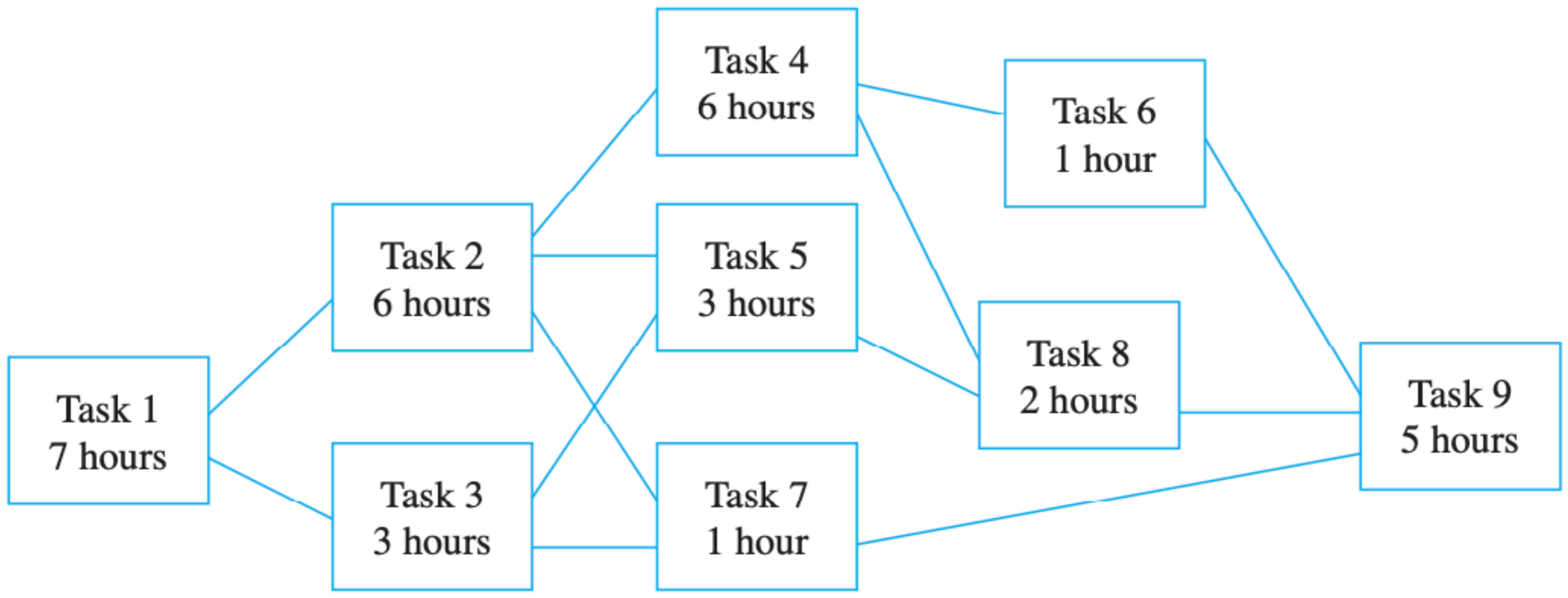
For all tasks y and y in T,

$$x \preceq y \iff (x = y) \vee (x \text{ preceeds } y)$$

The Hasse diagram for the relation can be used to answer some interesting questions.

An Application

In scheduling, the Hasse diagram is usually turned sideways (left to right).

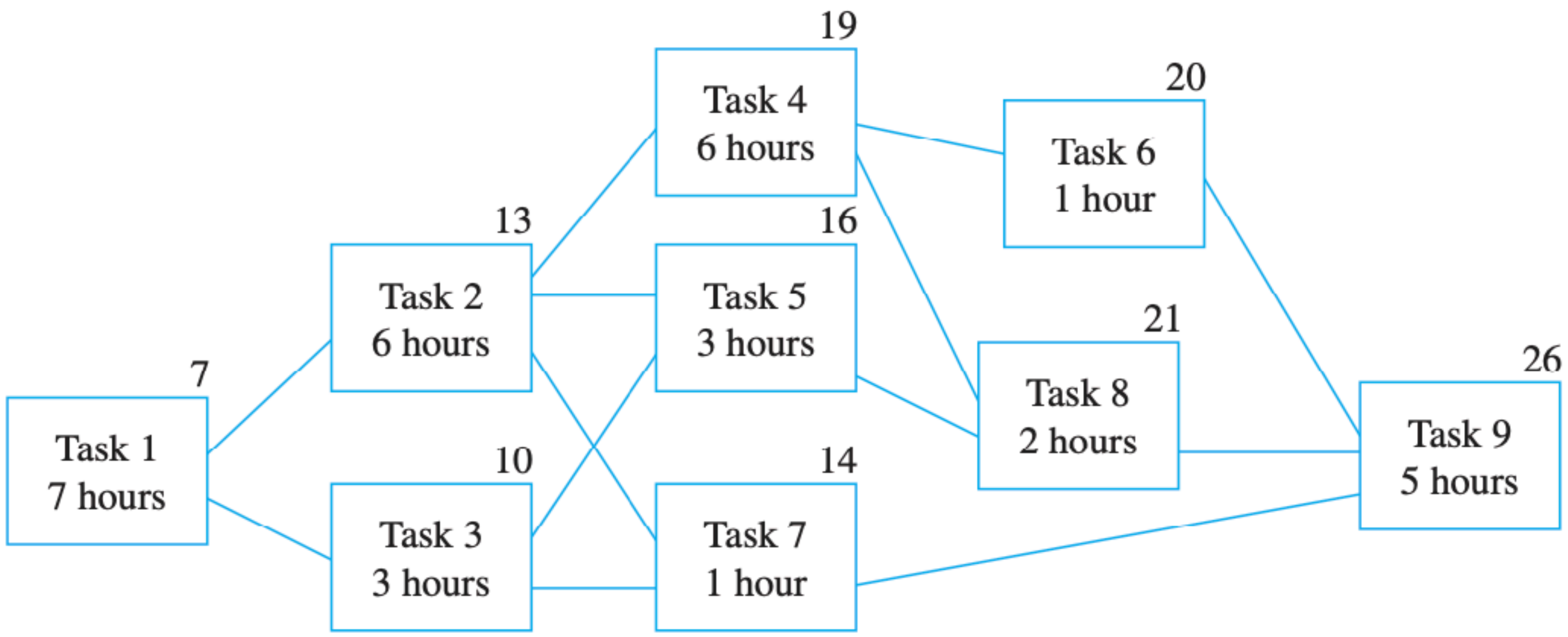


What is the **minimum time** required to assemble a car (to complete Task 9)?

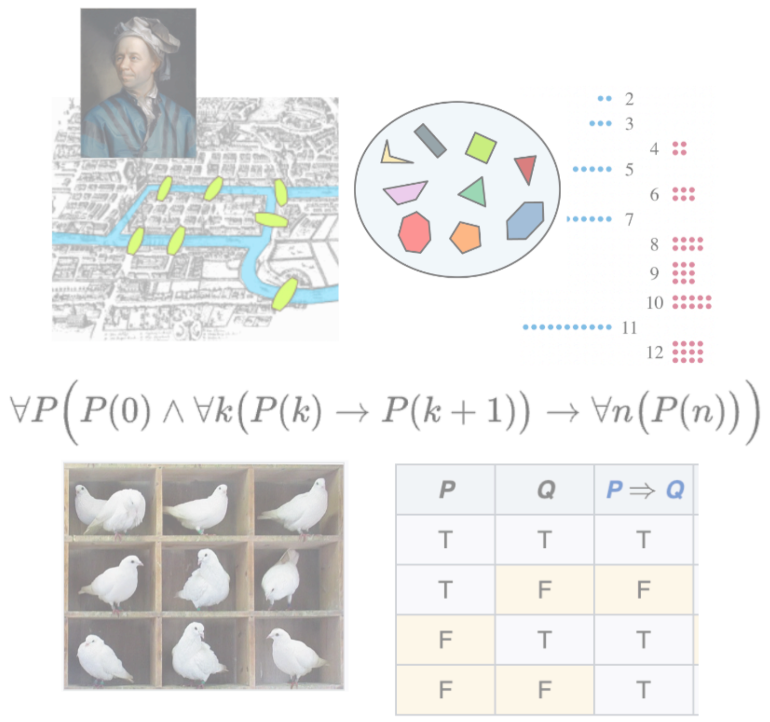
Are there **critical tasks**, that is, tasks that if delayed, cause a delay in the total assembly time?

An Application

The minimum time is 26 hours.



Note that the minimum time required to complete tasks 1, 2, 4, 8, and 9 in sequence is exactly 26 hours. This means that a delay in performing any one of these tasks causes a delay in the total time required for assembly of the car. For this reason, the path through tasks 1, 2, 4, 8, and 9 is called a critical path.



Next lecture:

Elementary Graph Theory