Analysis II

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25 febbraio 2024

Indice

1 The vector space $\mathbb{R}^n$

1.1 Operations

Addition and scalar multiplications are defined as follows:

$$\mathbf{x}+\mathbf{y}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}+\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}=\begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

$$\lambda\mathbf{x}=\lambda\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

$$\lambda\mathbf{x}+\mathbf{y}=\lambda\mathbf{x}+\lambda\mathbf{y}$$

Scalar product, $\langle\cdot,\cdot\rangle:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, in the vector space $\mathbb{R}^n$ is defined as:

$$\langle\mathbf{x},\mathbf{y}\rangle=\sum_{k=1}^{n}x_k y_k=\mathbf{x}^{\mathbf{T}}\mathbf{y}=\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The scalar product satisfies the following properties:
1. Positive definiteness: $\langle\mathbf{x},\mathbf{x}\rangle\geq 0\forall x$ with $\langle\mathbf{x},\mathbf{x}\rangle=0\overset{\text{iff}}{\longleftrightarrow}x=0$
2. Simmetry: $\langle x,y\rangle=\langle y,x\rangle$
3. Bilinearity: $\langle\alpha x+\beta y,z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle\ \forall x,y,z\in\mathbb{R}^n$ and $\forall\alpha,\beta\in\mathbb{R}$

1.2 Euclidean Norm on $\mathbb{R}^n$

The function $\|\cdot\|_2:\mathbb{R}^n\to\mathbb{R}$ is defined by

$$\|x\|_2=\sqrt{\langle x,x\rangle}=\left(\sum_{k=1}^{n}x_k^2\right)^{\frac{1}{2}}$$

The Euclidean norm on $\mathbb{R}^n$ has the following properties:
1. Non-negativity: $\|\mathrm{x}\|_2\geqslant 0$ for all $\mathrm{x}\in\mathbb{R}^n$, with eequality $\overset{\text{iff}}{\longleftrightarrow}$ $\mathbf{x}=\mathbf{0}$.
2. Homogeneity: $\|\lambda\cdot\mathbf{x}\|_2=|\lambda|\cdot\|\mathbf{x}\|_2$ for all $\lambda\in\mathbb{R}$ and $\mathbf{x}\in\mathbb{R}^n$
3. Triangle inequality: $\|\mathbf{x}+\mathbf{y}\|_2\leqslant\|\mathbf{x}\|_2+\|\mathbf{y}\|_2\forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ (also called Cauchy Schwartz inequality)

$$\left|\langle x,y\rangle\right|\leq\|x\|_2\left\|y\right\|_2\text{ (Angle Formula)}$$

if **x** and **y** are orthogonal then:

$$\left\|x+y\right\|_2^2=\|x\|_2^2+\left\|y\right\|_2^2\text{ (Pythagoras)}$$

<u>Definition:</u>
The Euclidean distance on $\mathbb{R}^n$ is the function $d(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ given by:

$$d(\mathbf{x},\mathbf{y}) := \|x-y\|_2 = \sqrt{(x_1-y_1)^2 + \ldots + (x_n-y_n)^2}$$

This function outputs the distance between two points in $\mathbb{R}^n$ and it satisfies the following 3 properties:
1. Non-negativity: $d(\mathbf{x},\mathbf{y}) \geqslant 0$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, with equality $\overset{\text{iff}}{\longleftrightarrow}$ x = y.
2. Symmetry: $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$.
3. Triangle inequality: $d(\mathbf{x},\mathbf{y}) \leqslant d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y})$.

1.3 Topology on $\mathbb{R}^n$

<u>Definition:</u> Open Ball
Let $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$. The set

$$B(a,r) = \{x \in \mathbb{R}^n : d(x,a) < r\}$$

is called the open ball of radius r centered at $\mathbf{a}$.
If and are two distinct points then: if $\mathbf{x},\mathbf{y} \in \mathbb{R}^n and \mathbf{x} \neq \mathbf{y}$ then we can find a sufficiently small open ball centered in $\mathbf{x}$ and another centered in $\mathbf{y}$ such that the two balls don't touch. Open balls are open sets[1]
<u>Definition:</u> Open set
A subset $U \subseteq \mathbb{R}^n$ is open if $\forall \mathbf{x} \in U\ \exists \varepsilon > 0 :$ the open ball $B(x,\varepsilon)$ is contained in U.
<u>Example of an open set:</u>
1. if $a < b$ are real numbers then the interval

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}$$

is an open set.
Proof: take $r = min\{x-a, b-x\}$ (both a and b are strictly positive), the minimum is positive and the ball $B(x,r) = \{y \in \mathbb{R} : |x-y| < r\}$ is a subset of (a,b). As $\mathbf{x}$ is arbitrary, that works $\forall \mathbf{x} \in (a,b)$ and so it satisfies the definition of an open set.
2. The infinite interval (a,∞) and $(-\infty,b)$ are also open but the intervals

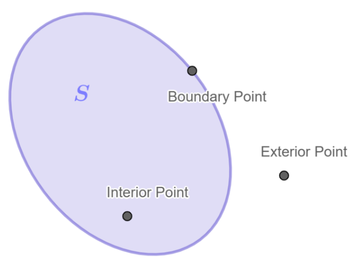
$$(a,b] = \{x \in \mathbb{R} : a < x \leqslant b\} \quad \text{and} \quad [a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

[1]An open set is a set with the property that if $\mathbf{x}$ is a point in the set then all points that are sufficiently near to it also belong to the set.

are not open sets.
3. the rectangle $(a,b)\times(c,d)=\left\{(x,y)\in\mathbb{R}^2 : a<x<b, c<y<d\right\}$ is an open set.
Definition: Closed Set
A subset $C\subseteq\mathbb{R}^n$ is closed if its complement $\mathbb{R}^n\ C$ is open.
Convention: The empty set and the space $\mathbb{R}^n$ are the only two spaces both open and closed at the same time.
Definition: Closed Ball
Let $\mathbf{a}\in\mathbb{R}^n$ and $r>0$. The set

$$\overline{B(\mathbf{a},r)}=\{\mathbf{x}\in\mathbb{R}^n : d(\mathbf{x},\mathbf{a})\leq r\}$$

is called the closed ball of radius r centered at **a** and it is a closed set. Example of a closed set:
1. The closed interval $[a,b]=\{x\in\mathbb{R} : a\leq x\leq b\}$ is a closed set and its complementary $\mathbb{R}\ [a,b]=(-\infty,a)\cup(b,\infty)$ is an open set.
2. Infinite intervals with closed boundary $[a,\infty)$ and $(-\infty,b]$ are closed sets.
3. Halfopen intervals such as $[a,b)$ or $(a,b]$ are neither closed nor open sets.
4. Any set consisting of only finitely many points is a closed set.
Propositions:
· if $U\subseteq\mathbb{R}^n$ is open and $C\subseteq\mathbb{R}^n$ is closed then $U\ C$ is open. (Open Closed = Open)
· if $U\subseteq\mathbb{R}^n$ is closed and $C\subseteq\mathbb{R}^n$ is open then $U\ C$ is closed. (Closed Open = Closed)
· if $U_1,\cdots U_k\subseteq\mathbb{R}^n$ are open then $U_1\cup\cdots\cup U_k$ and $U_1\cap\cdots\cap U_k$ are open.
· if $C_1,\cdots C_k\subseteq\mathbb{R}^n$ are open then $C_1\cup\cdots\cup C_k$ and $C_1\cap\cdots\cap C_k$ are closed[2].
Definition: Let S be a subset of $\mathbb{R}^n$ and **x** a point in $\mathbb{R}^n$
· We call **x** an interior point of S if $\exists r>0$: the ball $B(x,r)$ is contained in S.
· We call **x** an exterior point of S if $\exists r>0$: the ball $B(x,r)$ has empty intersection with S
· We call **x** an boundary point of S if $\exists r>0$: the ball $B(x,r)$ if it is neither an interior point neither an exterior point. **x** is a boundary point if $\forall r>0$ the ball $B(x,r)$ has non empty intersection with S without being entirely contained in S.



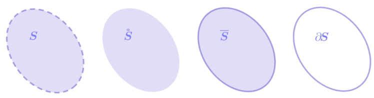
(a) Schematic for the 3 types of points

[2] intersezione e unione di sottoinsiemi chiusi (rispettivamente aperti) danno un sottoinsieme ancora chiuso (rispettivamente aperto).

Definition: Interior
The set of all interior points of a set S is called the interior of S and it is denoted by $\mathring{\mathbf{S}}$ and it's the largest open set contained inside of S
Definition: Boundary
The set of all boundary points of a set S is called the boundary of S and we use $\partial\mathbf{S}$ to denote it
Definition: Closure
The closure of S is denoted by $\overline{S}$ and it's the set of points $\mathbf{x} \in \mathbb{R}^n$ with the property that $\forall r > 0$, $B(x,r) \cap S \neq \emptyset$. The closure of S is the union of all its interior points and boundary points. $\mathring{S}$ is the smallest closed set that contains S.

$$\mathring{S} \subseteq \overline{S} \subseteq S$$

$$\mathring{S} = S \cap \partial S \qquad \overline{S} = S \cup \partial S \qquad \partial S = \overline{S} \cap \mathring{S}$$



(a) Schematic for the 3 types of points

Corollary:
· A set S is open $\overset{\text{iff}}{\longleftrightarrow}$ $S = \mathring{S}$
· A set S is closed $\overset{\text{iff}}{\longleftrightarrow}$ $S = \overline{S}$

Examples:

1.4 Convergence of sequences in $\mathbb{R}^n$

Definition: sequences in $\mathbb{R}^n$
A sequence of elements of $\mathbb{R}^n$ is a function $k \mapsto {}_k$ that associates to every $k \in \mathbb{N}$ an element $\mathbf{x}_i \in \mathbb{R}^n$
Convention: We denote $({}_k)_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}^n$ and we can consider each coordinate as an individual sequence.

$$(\mathbf{x}_k)_{k\in\mathbb{N}} = \begin{pmatrix} (x_{1,k})_{k\in\mathbb{N}} \\ \vdots \\ (x_{n,k})_{k\in\mathbb{N}} \end{pmatrix}$$

Definition: Convergent sequence
A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{x} \in \mathbb{R}^n$ if $\forall\varepsilon > 0 \exists N > 1$: when $k \geq N$, then $d(\mathbf{x}_k, \mathbf{x}) < \varepsilon$.

Convention: We call $\mathbf{x}$ the limit of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and write:

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$$

· If the limit exists, then it's unique.
· A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{x} \overset{\text{iff}}{\longleftrightarrow}$ the distance $d(\mathbf{x}_k, \mathbf{x})$ converges to 0.

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x} \longleftrightarrow \lim_{k\to\infty} d(\mathbf{x}_k, \mathbf{x}) = 0$$

· A sequence converges to $\mathbf{x} \overset{\text{iff}}{\longleftrightarrow}$ each coordinate of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges to the respective coordinate of $\mathbf{x}$

$$(\mathbf{x}_k)_{k\in\mathbb{N}} = \begin{pmatrix} (x_{1,k})_{k\in\mathbb{N}} \\ \vdots \\ (x_{n,k})_{k\in\mathbb{N}} \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{x} \Longleftrightarrow \lim_{k\to+\infty} x_{i,k} = x_i \quad \forall i = 1, \dots, n$$

Theorem: Vector space arithmetic of limits of sequences
Let $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $\left(\mathbf{y}_k\right)_{k\in\mathbb{N}}$ be sequences in $\mathbb{R}^n$ and let $(\lambda_k)_{k\in\mathbb{R}}$ be a sequence in $\mathbb{R}$. (i) If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $\left(\mathbf{y}_k\right)_{k\in\mathbb{N}}$ both converge then so does $\left(\mathbf{x}_k + \mathbf{y}_k\right)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \mathbf{x}_k + \mathbf{y}_k = \lim_{k\to+\infty} \mathbf{x}_k + \lim_{k\to+\infty} \mathbf{y}_k.$$

(ii) If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $(\lambda_k)_{k\in\mathbb{N}}$ both converge then so does $(\lambda_k \mathbf{x}_k)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \lambda_k \mathbf{x}_k = \left(\lim_{k\to+\infty} \lambda_k\right) \cdot \left(\lim_{k\to+\infty} \mathbf{y}_k\right).$$

(iii) If $(\mathbf{x}_k)_{k\in\mathbb{N}}$ and $\left(\mathbf{y}_k\right)_{k\in\mathbb{N}}$ both converge then so does $\left(\langle \mathbf{x}_k, \mathbf{y}_k \rangle\right)_{k\in\mathbb{N}}$ and

$$\lim_{k\to+\infty} \langle \mathbf{x}_k, \mathbf{y}_k \rangle = \left\langle \lim_{k\to+\infty} \mathbf{x}_k, \lim_{k\to+\infty} \mathbf{y}_k \right\rangle$$

Definition: Cauchy sequences
A sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence if $\forall \varepsilon > 0 \exists N > 1 :\ k, l \geq N$ implies $d(\mathbf{x}_k, \mathbf{x}_l) < \varepsilon$ and so it means that evey Cauchy sequence is a convergent sequence and viceversa.
Proposition: Let $S \subseteq \mathbb{R}^n$ be a non empty set and suppose $\mathbf{x} \in \partial S$ (Boundary set), then $\exists$ a sequence of elements $\in S, \mathbf{x}_1, \mathbf{x}_2 \cdots \in S,$:

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$$

Proposition:
Let $C \subseteq \mathbb{R}^n$ be a closed set and let $(\mathbf{x}_k)_{k\in\mathbb{N}}$ be a sequence of elements in C. if $(\mathbf{x}_k)_{k\in\mathbb{N}}$ converges then the limit $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$ must belong to C.
Definition: Bounded set
A subset $E \subseteq \mathbb{R}^n$ is bounded if it is contained in a ball of finite radius centered at the origin:

$$E \subseteq B(0,r) \text{for some} R < \infty$$

A closed set don't need to be bounded, $[0,\infty)$ by convention is closed but not bounded.
Definition: Compact set:
A subset $C \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.
Definition: Subsequence
A subsequence $(\mathbf{x}_k)_{k\in\mathbb{R}}$ is any sequence of the form $\left(\mathbf{x}_{k_i}\right)_{i\in\mathbb{R}}$ where $(k_i)_{i\in\mathbb{R}}$ is a strictly increasing sequence of positive integers.
If a sequence converges then any subsequence of it converges to the same limit.
Theorem: Bolzano-Weierstrass theorem in $\mathbb{R}^n$
Let $C \subseteq \mathbb{R}^n$ be compact. Any sequence $(\mathbf{x}_k)_{k\in\mathbb{R}}$ of elements in C posseses a convergent subsequence $\left(\mathbf{x}_{k_i}\right)_{i\in\mathbb{R}}$ whose limit is in C.
Definition: Bounded sequence in $\mathbb{R}^n$
A sequence $(\mathbf{x}_k)_{k\in\mathbb{R}}$ is bounded if $\exists C > 0 : \|\mathbf{x}_k\|_2 \le C \ \forall k \in \mathbb{N}$ (C a constant).
Every convergent sequence is bounded but non viceversa, for example $x_k = (-1)^k$ is bounded but not convergent.
Corollary (Consequence of the previous definition and bolzano-Weierstrass theorem)
Each bounded sequence $(\mathbf{x}_k)_{k\in\mathbb{R}}$ in $\mathbb{R}^n$ has a convergent susbequence $\left(\mathbf{x}_{k_i}\right)_{i\in\mathbb{R}}$

1.5 Paths and Path-Connected Sets

Definition: Path
Let $I \subseteq \mathbb{R}^n$ be an interval. A path (or curve) in $\mathbb{R}^n$ is a function $f : I \to \mathbb{R}^n$ with:

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

where $f_i : I \to \mathbb{R}$ is continuos $\forall i = 1, \cdots, n$. A path is a vector with every component as a continuos function from an interval I to $\mathbb{R}$
Definition: Path connected sets
Let $E \subseteq \mathbb{R}^n$. We say E is path-connected if $\forall \mathbf{x}, \mathbf{y} \in E \ \exists$ a path $f : [0,1] \to E$ with $f(0) = \mathbf{x}$ and $f(1) = \mathbf{y}$.
$f : [0,1] \to E$ means that $Im(f) \subseteq E$.

2 Allegati

2.1 Dimostrazione 1