Analysis II

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1 The vector space $\mathbb{R}^n$

1.1 Operations

Addition and scalar multiplications are defined as follows:

$$\mathbf{x}+\mathbf{y}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}+\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}=\begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

$$\lambda\mathbf{x}=\lambda\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

$$\lambda\mathbf{x}+\mathbf{y}=\lambda\mathbf{x}+\lambda\mathbf{y}$$

Scalar product, $\langle\cdot,\cdot\rangle:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, in the vector space $\mathbb{R}^n$ is defined as:

$$\langle\mathbf{x},\mathbf{y}\rangle=\sum_{k=1}^{n}x_k y_k=\mathbf{x}^{\mathbf{T}}\mathbf{y}=\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The scalar product satisfies the following properties:
1. Positive definiteness: $\langle\mathbf{x},\mathbf{x}\rangle\geq 0\forall x$ with $\langle\mathbf{x},\mathbf{x}\rangle=0\xleftrightarrow{\text{iff}}x=0$
2. Simmetry: $\langle x,y\rangle=\langle y,x\rangle$
3. Bilinearity: $\langle\alpha x+\beta y,z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle\ \forall x,y,z\in\mathbb{R}^n$ and $\forall\alpha,\beta\in\mathbb{R}$

1.2 Euclidean Norm on $\mathbb{R}^n$

The function $\|\cdot\|_2:\mathbb{R}^n\to\mathbb{R}$ is defined by

$$\|x\|_2=\sqrt{\langle x,x\rangle}=\left(\sum_{k=1}^{n}x_k^2\right)^{\frac{1}{2}}$$

The Euclidean norm on $\mathbb{R}^n$ has the following properties:
1. Non-negativity: $\|\mathrm{x}\|_2\geqslant 0$ for all $\mathrm{x}\in\mathbb{R}^n$, with eequality $\xleftrightarrow{\text{iff}}\mathbf{x}=\mathbf{0}$.
2. Homogeneity: $\|\lambda\cdot\mathbf{x}\|_2=|\lambda|\cdot\|\mathbf{x}\|_2$ for all $\lambda\in\mathbb{R}$ and $\mathbf{x}\in\mathbb{R}^n$
3. Triangle inequality: $\|\mathbf{x}+\mathbf{y}\|_2\leqslant\|\mathbf{x}\|_2+\|\mathbf{y}\|_2\forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ (also called Cauchy Schwartz inequality)

$$\left|\langle x,y\rangle\right|\leq\|x\|_2\left\|y\right\|_2 \text{(Angle Formula)}$$

if **x** and **y** are orthogonal then:

$$\left\|x+y\right\|_2^2=\|x\|_2^2+\left\|y\right\|_2^2 \text{(Pythagoras)}$$

<u>Definition:</u>
The Euclidean distance on $\mathbb{R}^n$ is the function $d(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ given by:

$$d(\mathbf{x},\mathbf{y}) := \|x-y\|_2 = \sqrt{(x_1-y_1)^2 + \ldots + (x_n-y_n)^2}$$

This function outputs the distance between two points in $\mathbb{R}^n$ and it satisfies the following 3 properties:
1. Non-negativity: $d(\mathbf{x},\mathbf{y}) \geqslant 0$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, with equality $\overset{\text{iff}}{\longleftrightarrow}$ x = y.
2. Symmetry: $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$.
3. Triangle inequality: $d(\mathbf{x},\mathbf{y}) \leqslant d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y})$.

1.3 Topology on $\mathbb{R}^n$

<u>Definition:</u> Open Ball
Let $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$. The set

$$B(a,r) = \{x \in \mathbb{R}^n : d(x,a) < r\}$$

is called the open ball of radius r centered at **a**.
If and are two distinct points then: if $\mathbf{x},\mathbf{y} \in \mathbb{R}^n and \mathbf{x} \neq \mathbf{y}$ then we can find a sufficiently small open ball centered in **x** and another centered in **y** such that the two balls don't touch. Open balls are open sets[1]
<u>Definition:</u> Open set
A subset $U \subseteq \mathbb{R}^n$ is open if $\forall \mathbf{x} \in U \ \exists \varepsilon > 0 :$ the open ball $B(x,\varepsilon)$ is contained in U.
<u>Example of an open set:</u>
1. if $a < b$ are real numbers then the interval

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}$$

is an open set.
Proof: take $r = min\{x-a, b-x\}$ (both a and b are strictly positive), the minimum is positive and the ball $B(x,r) = \{y \in \mathbb{R} : |x-y| < r\}$ is a subset of (a,b). As **x** is arbitrary, that works $\forall \mathbf{x} \in (a,b)$ and so it satisfies the definition of an open set.
2. The infinite interval (a,∞) and $(-\infty,b)$ are also open but the intervals

$$(a,b] = \{x \in \mathbb{R} : a < x \leqslant b\} \quad \text{and} \quad [a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

[1] An open set is a set with the property that if **x** is a point in the set then all points that are sufficiently near to it also belong to the set.

are not open sets.
3. the rectangle $(a,b)\times(c,d)=\big\{(x,y)\in\mathbb{R}^2 : a<x<b, c<y<d\big\}$ is an open set.
Definition: Closed Set
A subset $C\subseteq\mathbb{R}^n$ is closed if its complement $\mathbb{R}^n\ C$ is open.
Convention: The empty set and the space $\mathbb{R}^n$ are the only two spaces both open and closed at the same time.
Definition: Closed Ball
Let $\mathbf{a}\in\mathbb{R}^n$ and $r>0$. The set

$$\overline{B(\mathbf{a},r)}=\{\mathbf{x}\in\mathbb{R}^n : d(\mathbf{x},\mathbf{a})\le r\}$$

is called the closed ball of radius r centered at **a** and it is a closed set. Example of a closed set:
1. The closed interval $[a,b]=\{x\in\mathbb{R} : a\le x\le b\}$ is a closed set and its complementary $\mathbb{R}\ [a,b]=(-\infty,a)\cup(b,\infty)$ is an open set.
2. Infinite intervals with closed boundary $[a,\infty)$ and $(-\infty,b]$ are closed sets.
3. Halfopen intervals such as $[a,b)$ or $(a,b]$ are neither closed nor open sets.
4. Any set consisting of only finitely many points is a closed set.
Propositions:
· if $U\subseteq\mathbb{R}^n$ is open and $C\subseteq\mathbb{R}^n$ is closed then $U\ C$ is open. (Open Closed = Open)
· if $U\subseteq\mathbb{R}^n$ is closed and $C\subseteq\mathbb{R}^n$ is open then $U\ C$ is closed. (Closed Open = Closed)
· if $U_1,\cdots U_k\subseteq\mathbb{R}^n$ are open then $U_1\cup\cdots\cup U_k$ and $U_1\cap\cdots\cap U_k$ are open.
· if $C_1,\cdots C_k\subseteq\mathbb{R}^n$ are open then $C_1\cup\cdots\cup C_k$ and $C_1\cap\cdots\cap C_k$ are closed[2].

[2] intersezione e unione di sottoinsiemi chiusi (rispettivamente aperti) danno un sottoinsieme ancora chiuso (rispettivamente aperto).

2 Allegati

2.1 Dimostrazione 1