Applications of quantum nonlocality, review on related open problems in quantum information theory

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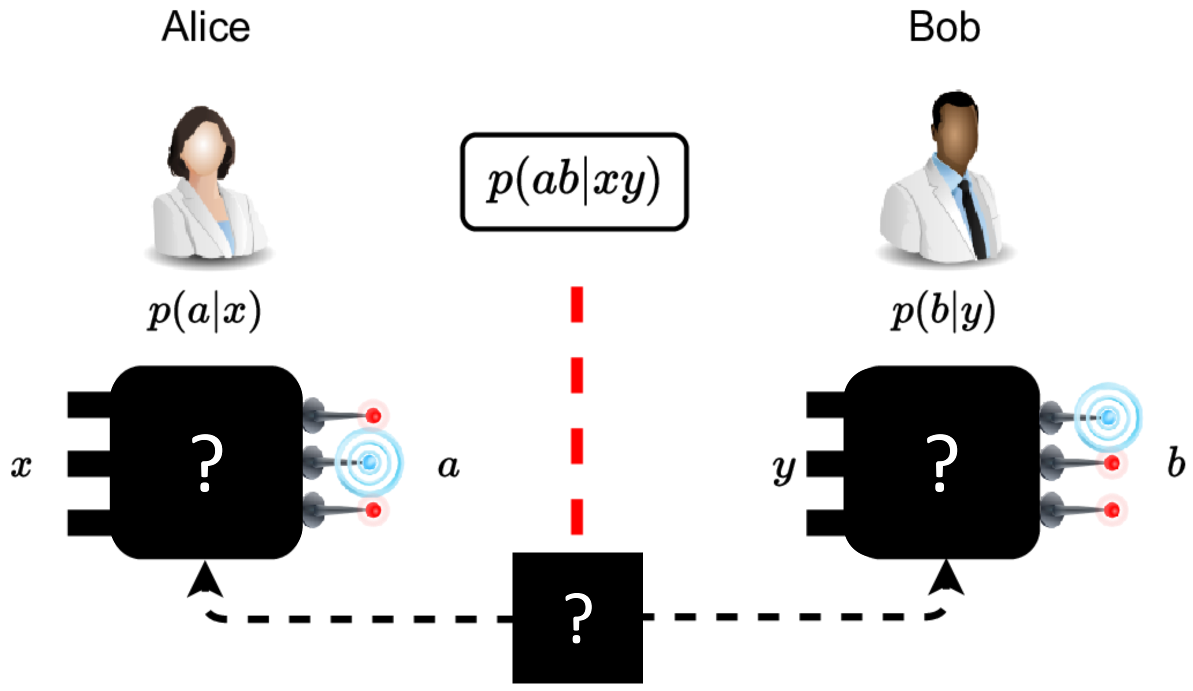
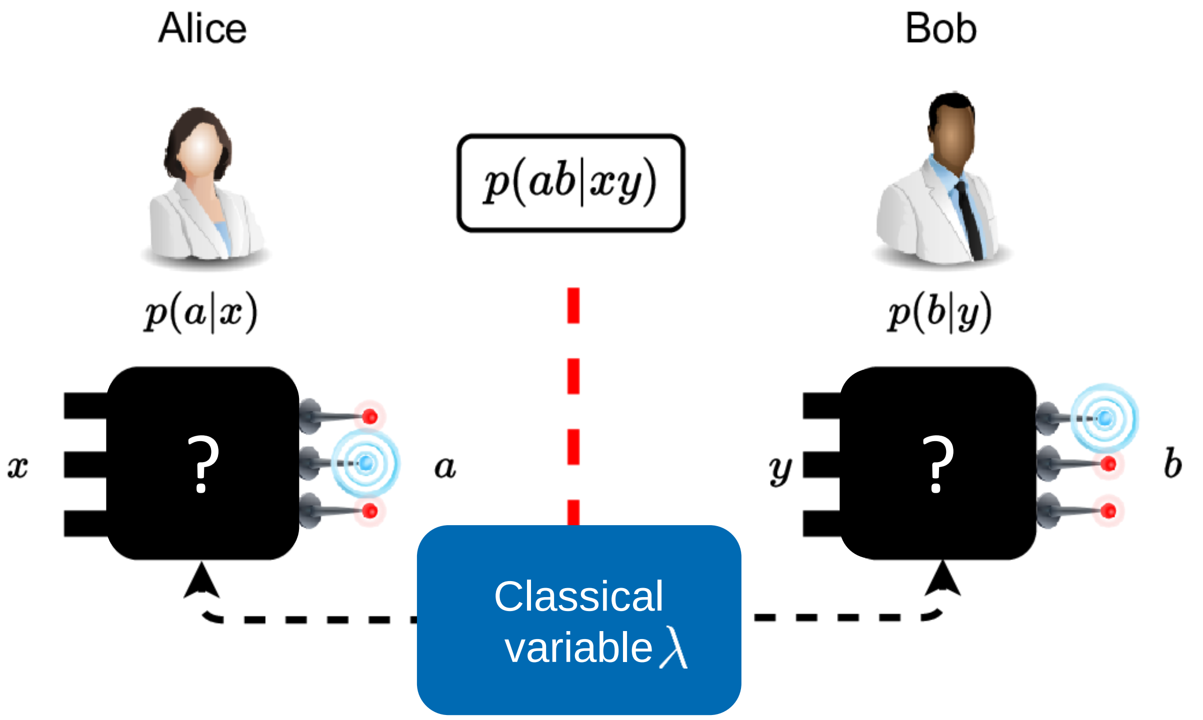


PhiQus

Basics of the Bell scenario



$$p(ab|xy) \neq p(a|x)p(b|y)$$

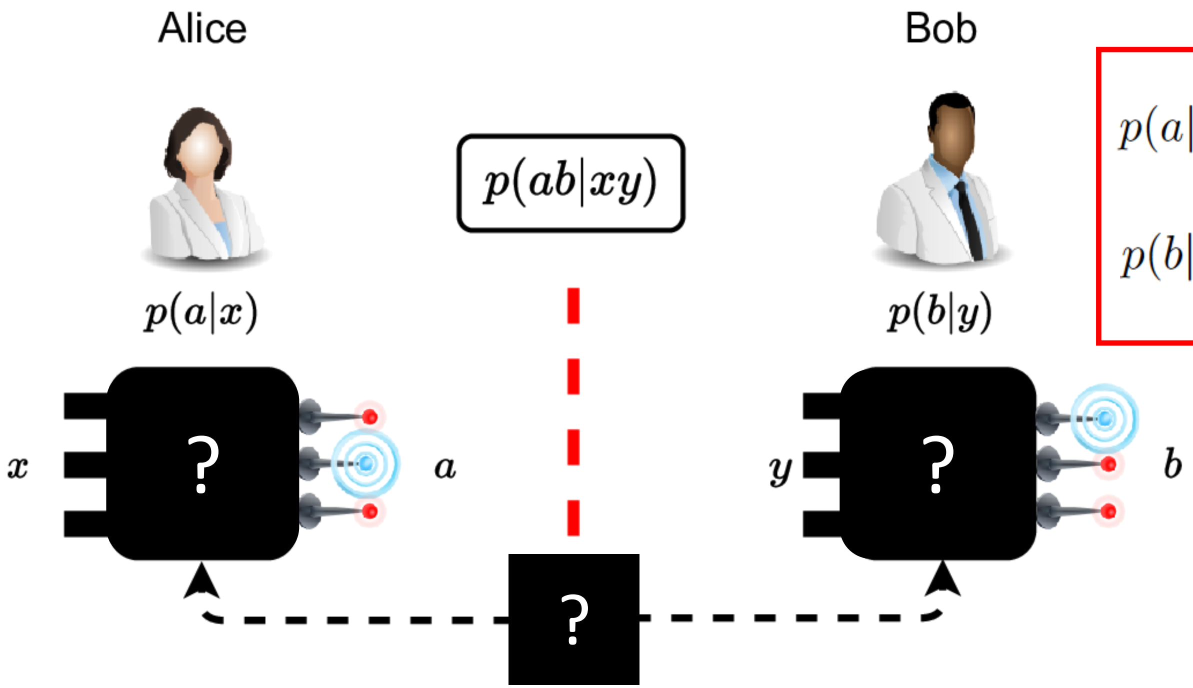
- **No-communication scenario**
- **Prior agreement to a strategy is possible**
- **Questions/Inputs are chosen randomly (often also independently)**
- **No reference to inner functioning of the box**
- **Run of an experiment over many rounds**

Classical Ansatz: $p(ab|xy,\lambda) = p(a|x,\lambda)p(b|y,\lambda)$

How to explain the correlations?

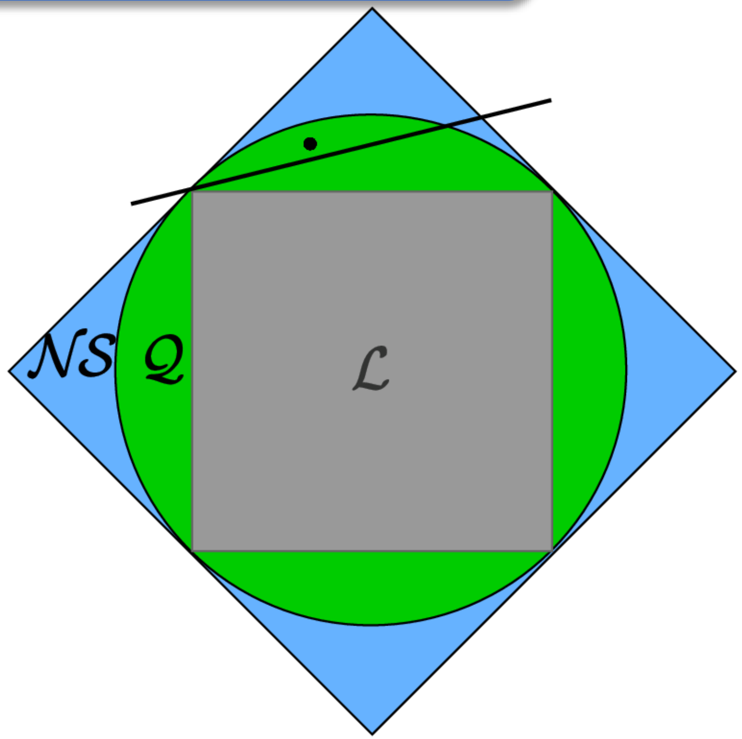
Local: $\boxed{p(ab|xy) \overset{\mathcal{L}}{=} \sum_{\lambda} p_{\lambda} p(a|x,\lambda)p(b|y,\lambda)}$

Basic Quantum Advantages



$$p(a|x) = p(a|xy) = \sum_b p(ab|xy)$$

$$p(b|y) = p(b|xy) = \sum_a p(ab|xy)$$



Bell inequality: $$\sum_{a,b,x,y} C_{ab|xy} p(ab|xy) \overset{\mathcal{L}}{\leq} L \overset{\mathcal{Q}}{\leq} Q \overset{\mathcal{NS}}{\leq} \mathrm{NS}$$

CHSH inequality: $$\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \overset{\mathcal{L}}{\leq} 2 \overset{\mathcal{Q}}{\leq} 2\sqrt{2} \overset{\mathcal{NS}}{\leq} 4$$

with $\langle A_x B_y \rangle = p(a = b|xy) - p(a \neq b|xy)$

CHSH Game:

$$a \oplus b = x \cdot y$$

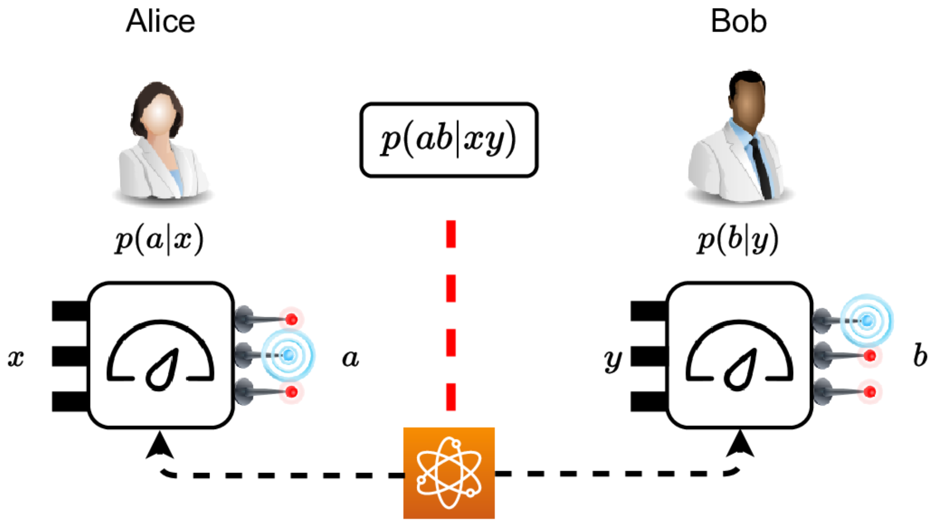
$$L = \frac{3}{4} \qquad Q \approx 0.85 \quad \mathrm{NS} = 1$$

Quantum Theory

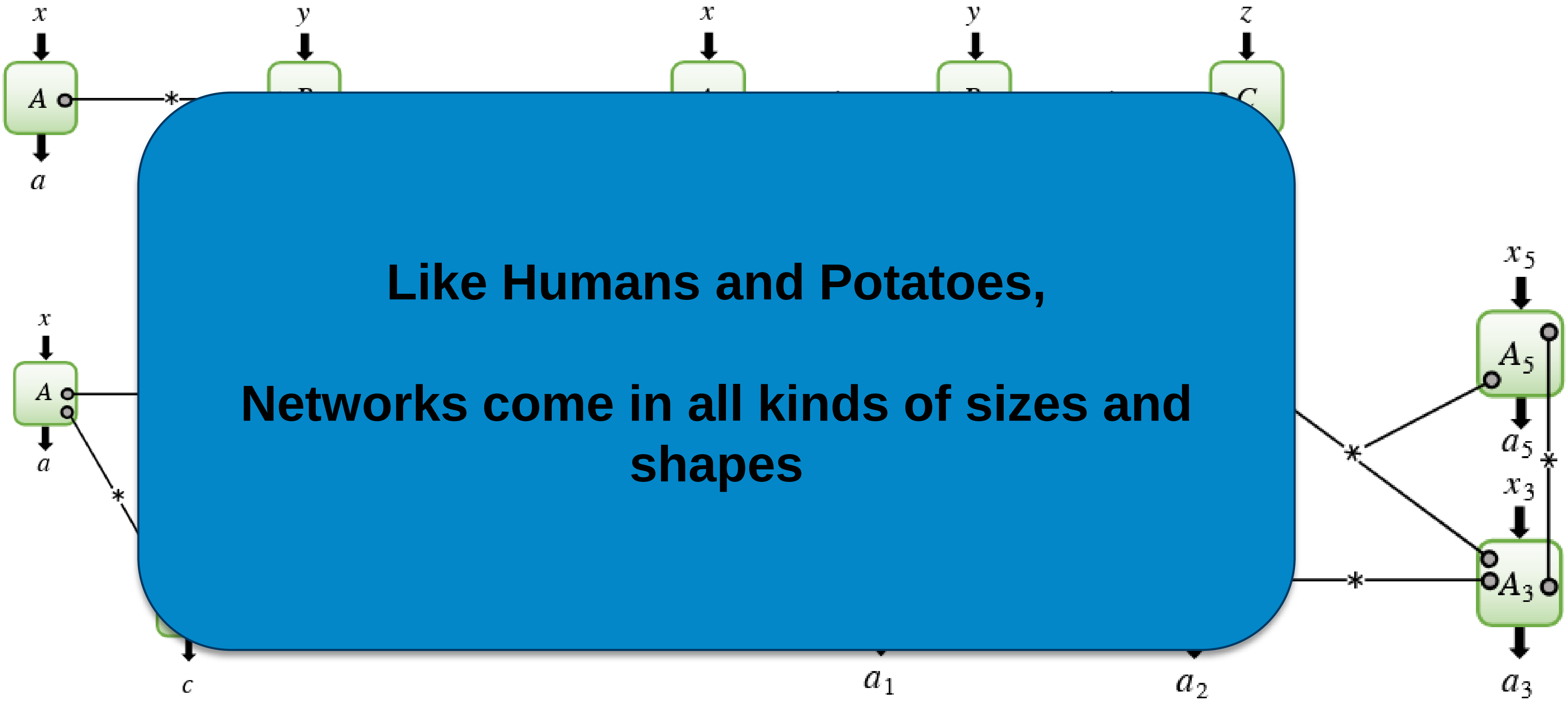
Quantum theory 101:

1. **States in complex Hilbert space** $|\Psi\rangle \in \mathcal{H}$ $\longrightarrow$ $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \iff \rho \geq 0, \mathrm{Tr}[\rho] = 1$
2. **Evolution of systems** $\rho \longmapsto \rho' = U\rho U^\dagger$
3. **Measurements** $\{M_a\}_a$ s.t. $0 \leq M_a \leq \mathbb{1}, \sum_a M_a = \mathbb{1} \longrightarrow p(a) = \mathrm{Tr}[M_a \rho]$
4. **Composite systems** $\mathcal{H}_{ST} := \mathcal{H}_S \otimes \mathcal{H}_T$ $\longrightarrow$ $\rho_{ST} := \rho_S \otimes \rho_T$

Quantum correlations: $\mathbf{p(ab|xy) = \mathrm{Tr}[(A_{a|x} \otimes B_{b|y})\rho]}$



Network Nonlocality

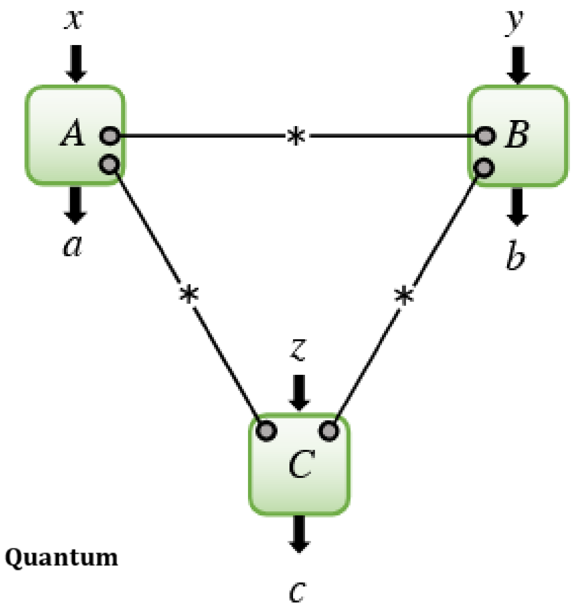


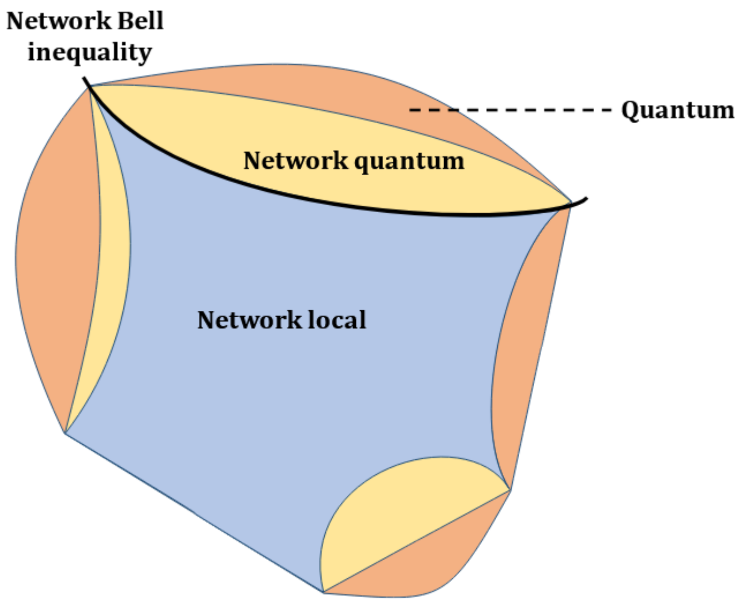
Like Humans and Potatoes,

Networks come in all kinds of sizes and shapes

Basics of Network Nonlocality

- Independent sources (could be assisted by shared randomness)
- No settings required for quantum advantages
- Usually thought of as a set-up with sources and measurements
- Typically no transformations in between
- Sets of correlations are not convex anymore
- Less generic tools available so far

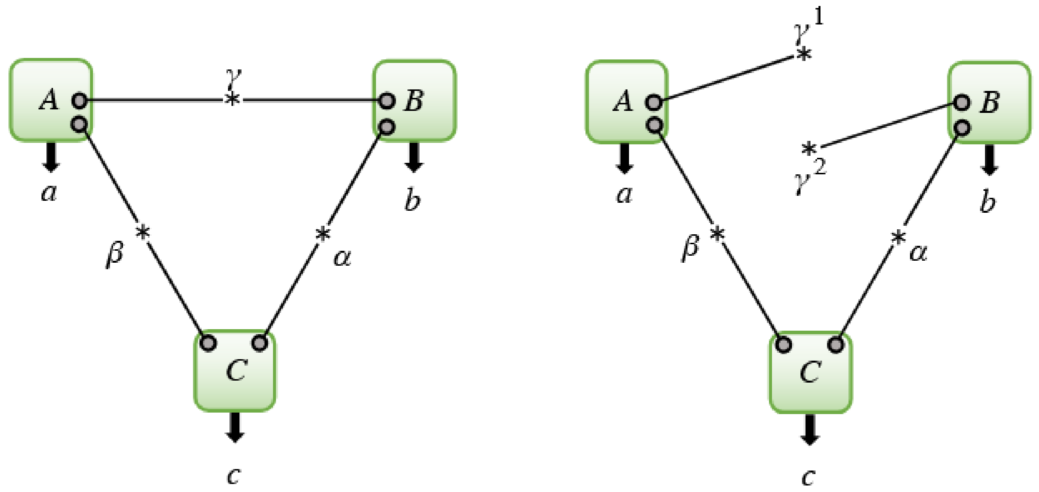




No-go Results in Networks

Can we achieve a shared random bit in the triangle network?

$$P_{\text{GHZ}} := \frac{1}{2}([000] + [111])$$



Proof by contradiction

$$P^{\text{AC}}_{\text{inf}} = P^{\text{AC}}_{\text{GHZ}} = \tfrac{1}{2}\left([00] + [11]\right)$$

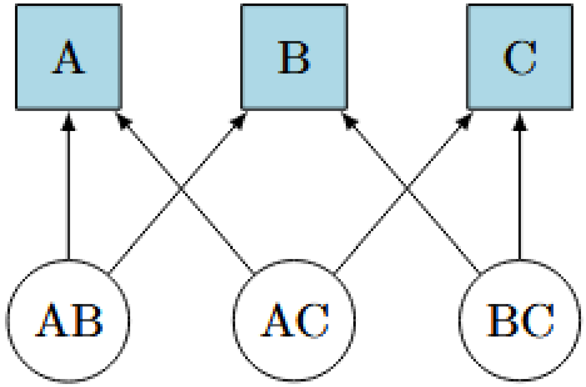
$$P^{\text{BC}}_{\text{inf}} = P^{\text{BC}}_{\text{GHZ}} = \tfrac{1}{2}\left([00] + [11]\right)$$

$$P^{\text{AB}}_{\text{inf}} = P^{\text{A}}_{\text{inf}} P^{\text{B}}_{\text{inf}}$$

The Proof does not rely on properties of a specific theory (classical, quantum,…) but on device replication and independence

Shared Randomness vs. Common Cause

Is a shared random bit possible without a common cause?



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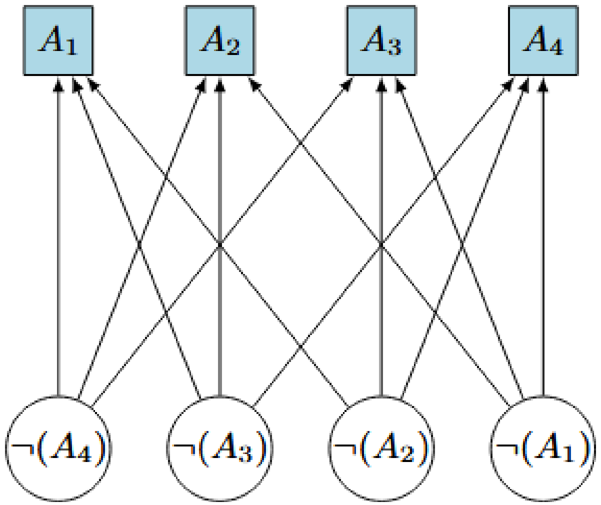
Triangle scenario



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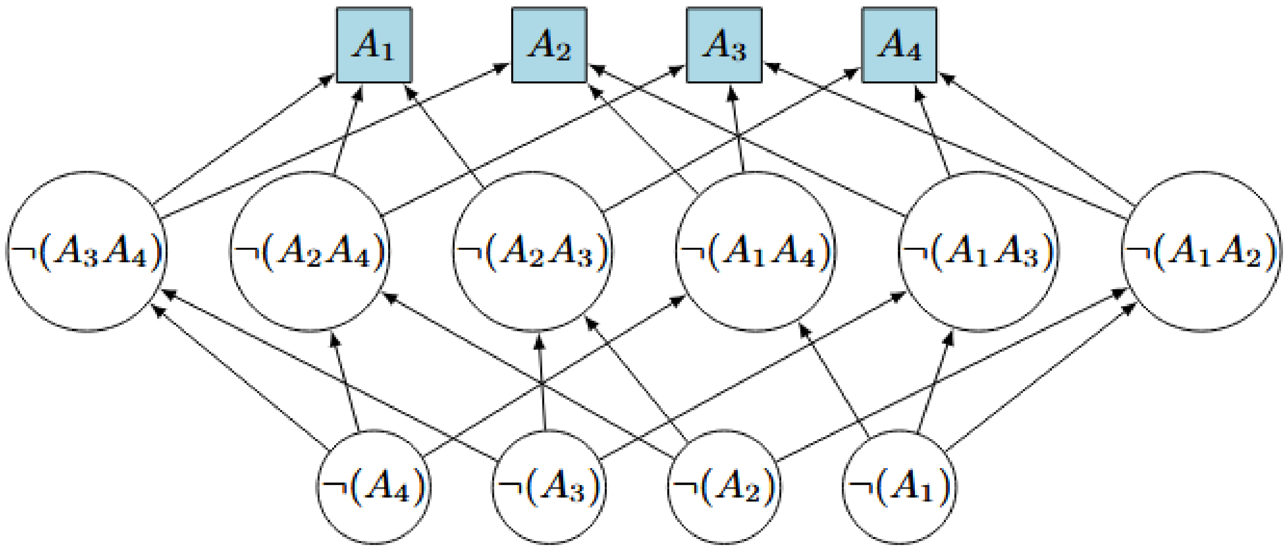
One-layer tetrahedron scenario

In networks without intermediate layers, the answer is no!

Shared Randomness vs. Common Cause

Is a shared random bit possible without a common cause?



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Full two-layer tetrahedron scenario

More complicated structure, cannot be reduced to a single-layer scenario in general

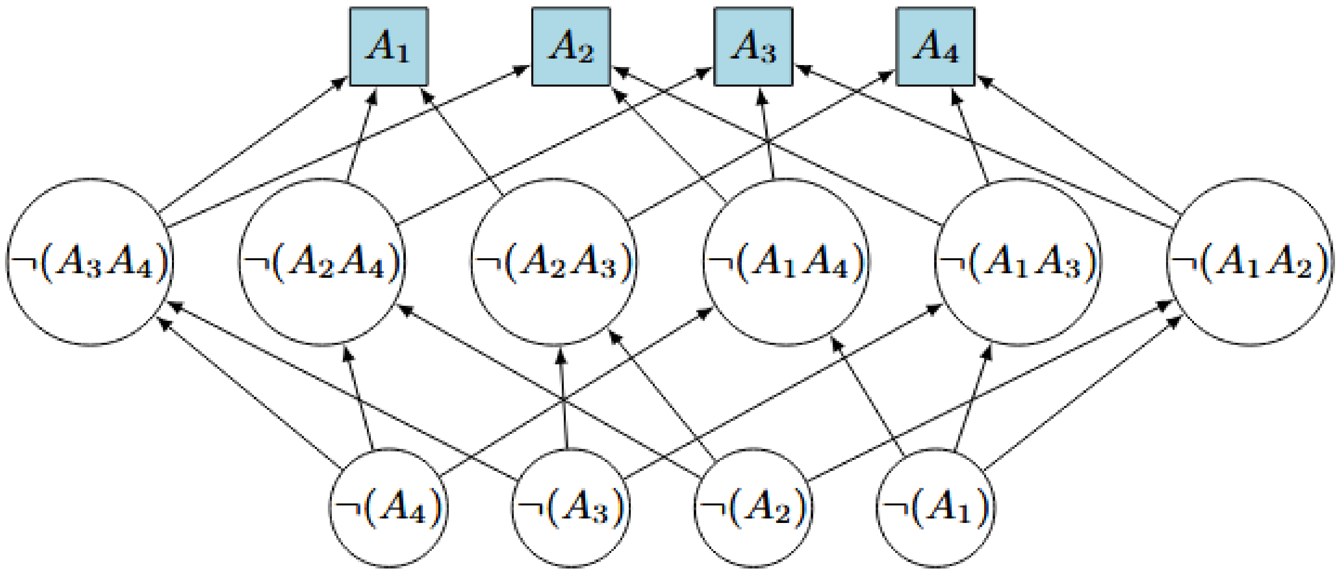
No cloning of information

Shared Randomness vs. Common Cause

Just use inflation again?!

No non-trivial inflation without copying information here!

Classical infeasibility but what about quantum and no-signaling?



d

Full two-layer tetrahedron scenario

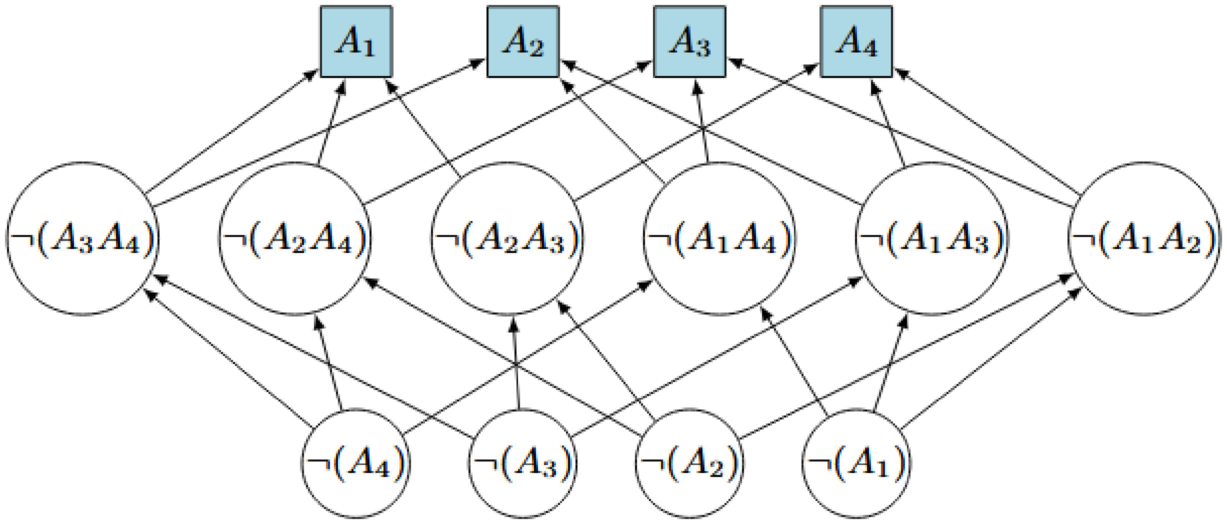
Shared randomness beyond Quantum

No non-trivial inflation, so what can we say?

No constraints, so SR-bit should be possible?

We find a toy theory that implements the SR-bit

It is causal, no-signaling, allows replication of devices

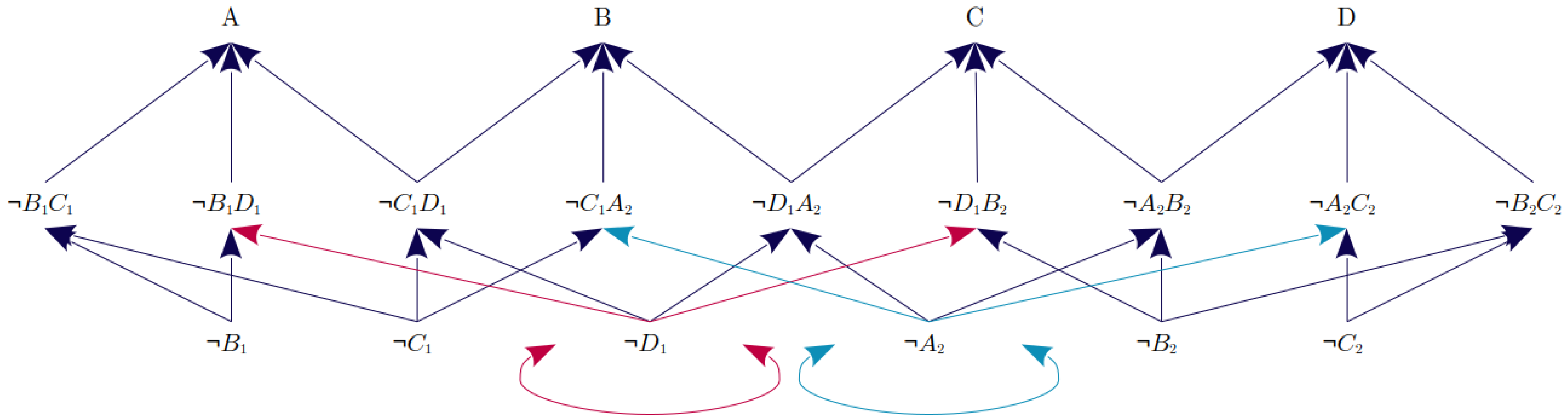


d
Full two-layer tetrahedron scenario

Shared randomness in Quantum Theory

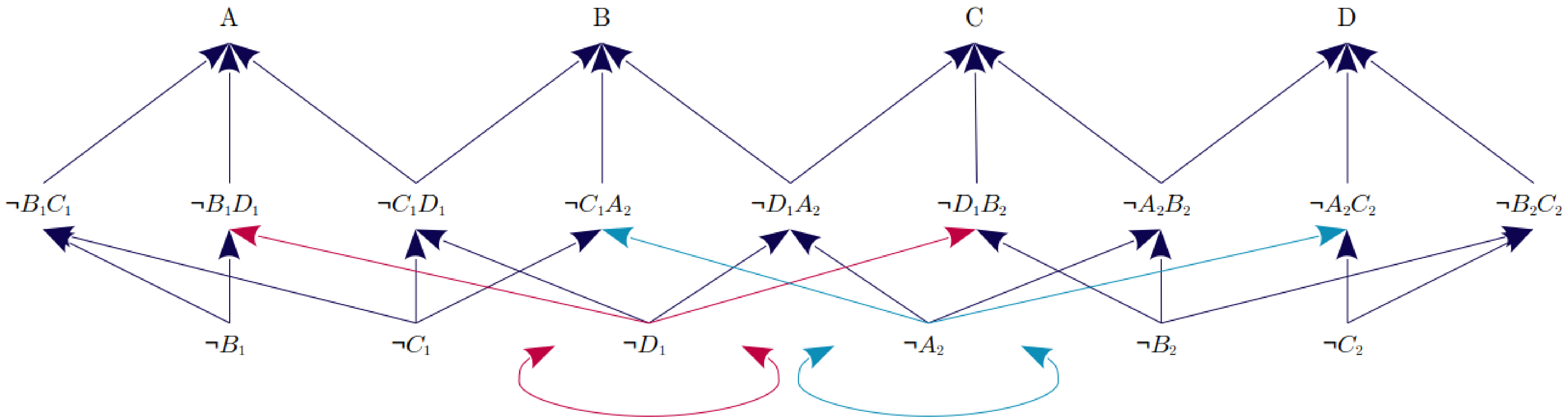
No non-trivial inflation, so what can we say?

We have the quantum formalism available



Unphysical scenario but it leads to mathematical constraints

Shared randomness in Quantum Theory



Define:

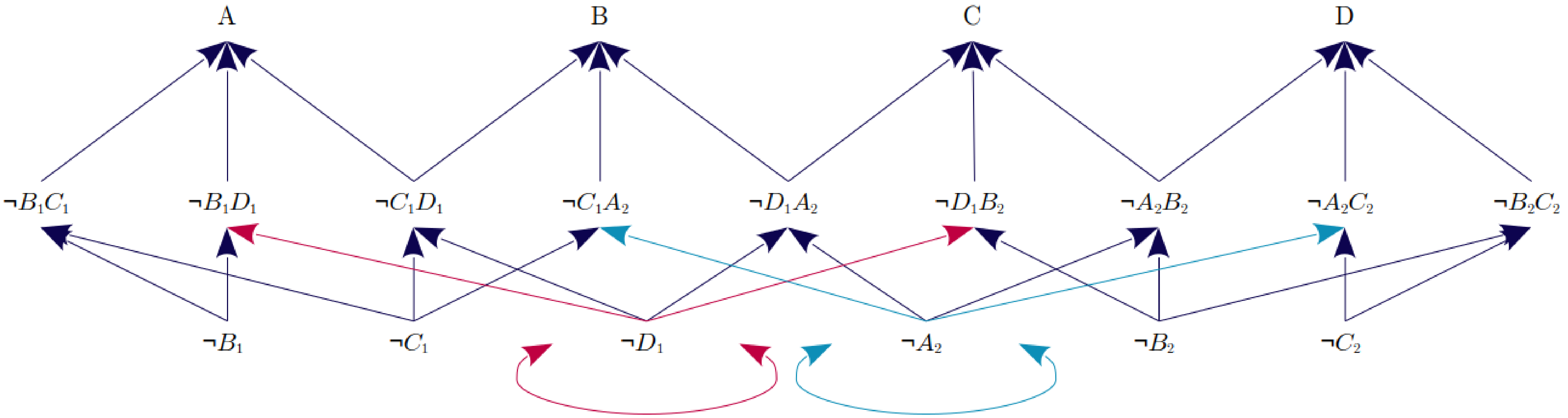
$$|\psi_{\mathcal{I}}\rangle := |\neg B_1\rangle\,|\neg C_1\rangle\,|\neg D_1\rangle\,|\neg A_2\rangle\,|\neg B_2\rangle\,|\neg C_2\rangle$$

$$\tilde{A}_0 := U^{\dagger}_{\neg B_1 C_1} U^{\dagger}_{\neg B_1 D_1} U^{\dagger}_{\neg C_1 D_1} A_0 U_{\neg B_1 C_1} U_{\neg B_1 D_1} U_{\neg C_1 D_1}$$

$$\left[\tilde{A}_0, \widetilde{B}_0\right] = 0,$$
$$\left[\widetilde{B}_0, \widetilde{C}_0\right] = 0,$$
$$\left[\widetilde{C}_0, \widetilde{D}_0\right] = 0,$$

But no commutation between A and C, and B and D

Shared randomness in Quantum Theory



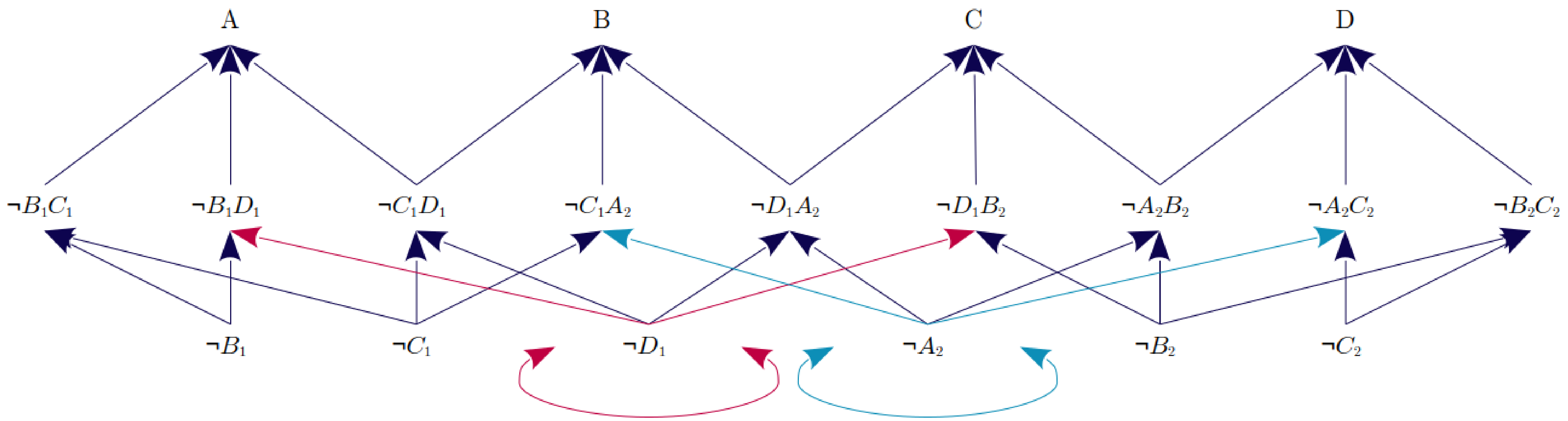
From inflation:

$$\langle \psi_{\mathcal{I}} | \tilde{A}_0 \widetilde{B}_0 | \psi_{\mathcal{I}} \rangle = p(A = 0, B = 0)$$
$$\langle \psi_{\mathcal{I}} | \tilde{A}_0 | \psi_{\mathcal{I}} \rangle = p(A = 0)$$
$$\langle \psi_{\mathcal{I}} | \widetilde{B}_0 | \psi_{\mathcal{I}} \rangle = p(B = 0)$$

Assume:

$$p(A = 0, B = 0) = p(A = 0) = p(B = 0) = p(D = 0) = 1/2$$

Contradiciton via SOS



$$\begin{aligned}
\langle\psi_{\mathcal{I}}|\left(\tilde{A}_0-\widetilde{B}_0\right)^2|\psi_{\mathcal{I}}\rangle &= \langle\psi_{\mathcal{I}}|\,\tilde{A}_0^2+\widetilde{B}_0^2-2\tilde{A}_0\widetilde{B}_0\,|\psi_{\mathcal{I}}\rangle \\
&= \langle\psi_{\mathcal{I}}|\,\tilde{A}_0+\widetilde{B}_0-2\tilde{A}_0\widetilde{B}_0\,|\psi_{\mathcal{I}}\rangle \\
&= \langle\psi_{\mathcal{I}}|\,\tilde{A}_0\,|\psi_{\mathcal{I}}\rangle+\langle\psi_{\mathcal{I}}|\,\widetilde{B}_0\,|\psi_{\mathcal{I}}\rangle-2\,\langle\psi_{\mathcal{I}}|\,\tilde{A}_0\widetilde{B}_0\,|\psi_{\mathcal{I}}\rangle \\
&= p(A=0)+p(B=0)-2p(A=0,B=0) \\
&= 0
\end{aligned}$$

Hence

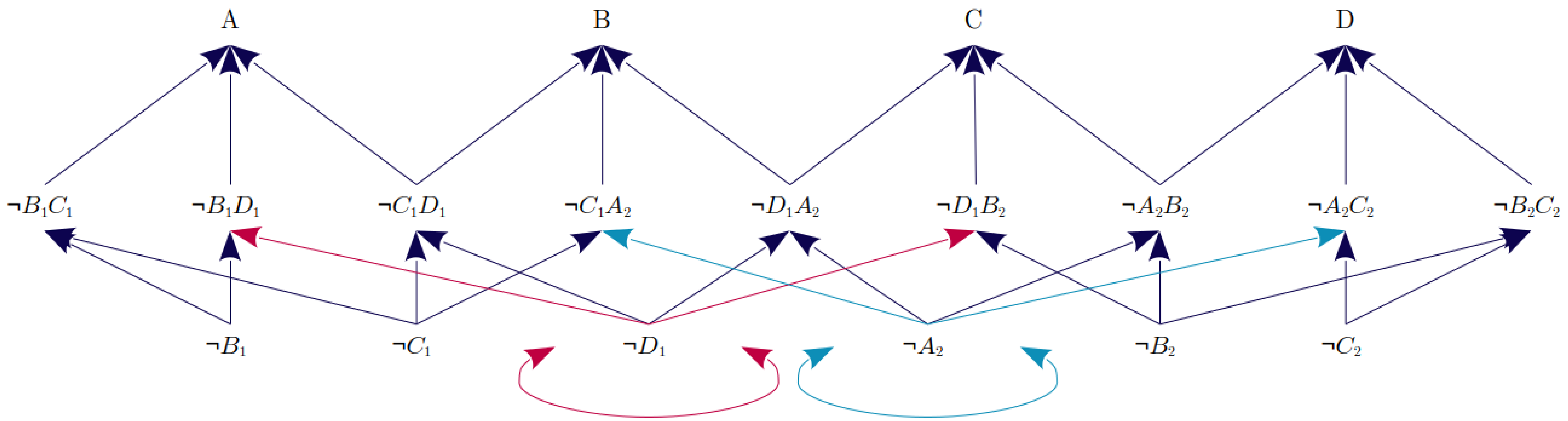
$$\tilde{A}_0\,|\psi_{\mathcal{I}}\rangle=\widetilde{B}_0\,|\psi_{\mathcal{I}}\rangle$$

$$\widetilde{B}_0\,|\psi_{\mathcal{I}}\rangle=\widetilde{C}_0\,|\psi_{\mathcal{I}}\rangle$$

$$\widetilde{C}_0\,|\psi_{\mathcal{I}}\rangle=\widetilde{D}_0\,|\psi_{\mathcal{I}}\rangle$$

$$\boxed{\tilde{A}_0\,|\psi_{\mathcal{I}}\rangle=\widetilde{D}_0\,|\psi_{\mathcal{I}}\rangle}$$

Contradiciton via SOS



$$\boxed{\tilde{A}_0 |\psi_{\mathcal{I}}\rangle = \widetilde{D}_0 |\psi_{\mathcal{I}}\rangle}$$

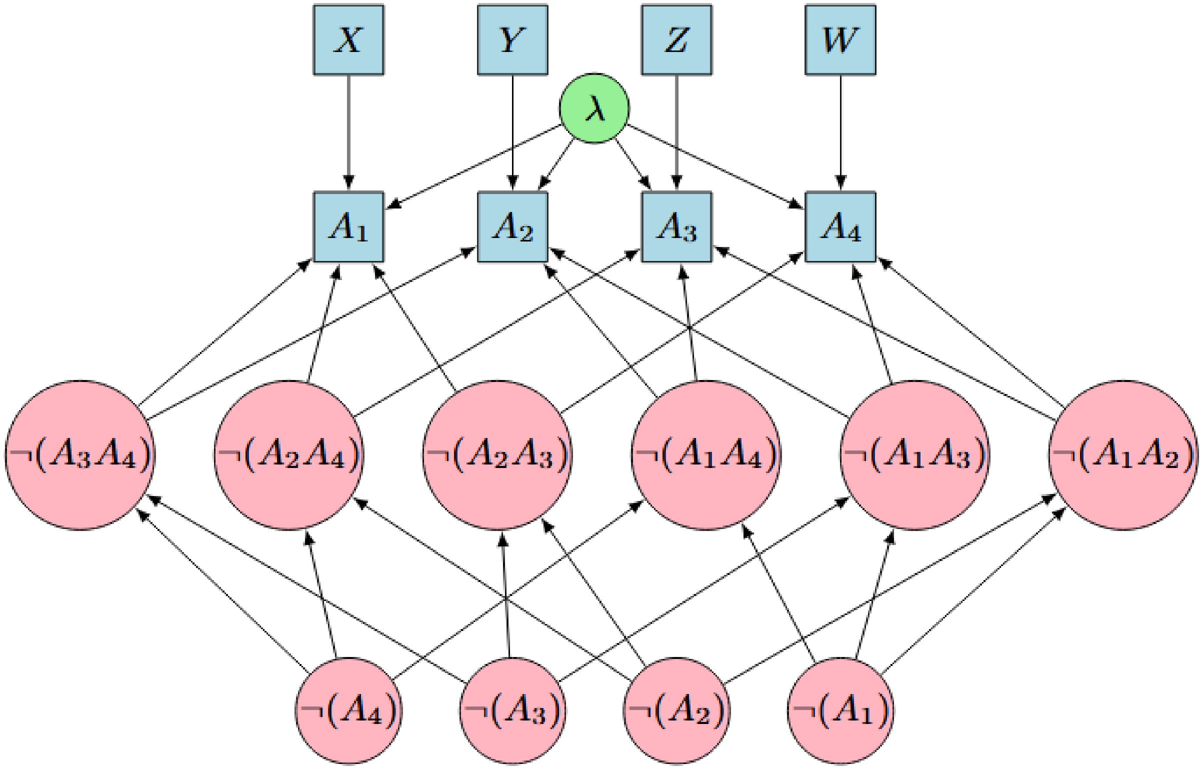
$$\begin{aligned}\langle\psi_{\mathcal{I}}| \tilde{A}_0\widetilde{D}_0 |\psi_{\mathcal{I}}\rangle &= \langle\psi_{\mathcal{I}}| \tilde{A}_0^2 |\psi_{\mathcal{I}}\rangle \\ &= \langle\psi_{\mathcal{I}}| \tilde{A}_0 |\psi_{\mathcal{I}}\rangle \\ &= p(A=0) \\ &= 1/2\end{aligned}$$

But A and D are independent

$$\begin{aligned}\langle\psi_{\mathcal{I}}| \tilde{A}_0\widetilde{D}_0 |\psi_{\mathcal{I}}\rangle &= \langle\psi_{\mathcal{I}}| \tilde{A}_0 |\psi_{\mathcal{I}}\rangle \langle\psi_{\mathcal{I}}| \widetilde{D}_0 |\psi_{\mathcal{I}}\rangle \\ &= p(A=0)p(D=0) \\ &= 1/4\end{aligned}$$

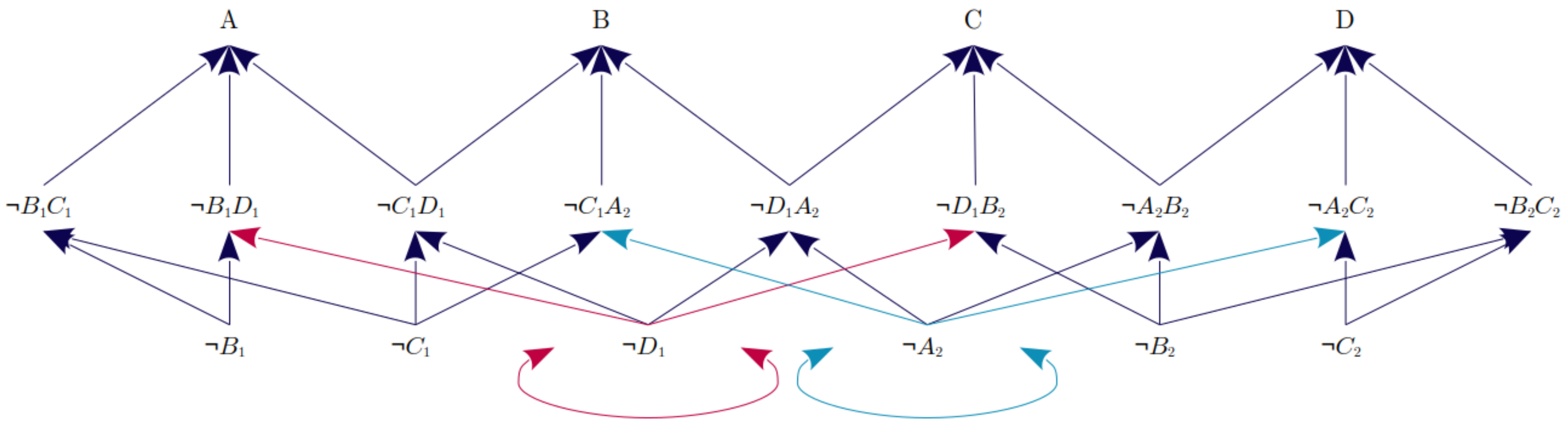
Extensions

- We show this Quantum vs. Post-Quantum separation for every n >= 4 (with more layers)
- We provide a quantitative version of our proof (inequality/game)
- We extend the reasoning to the GHZ state



Summary

- Separation between Quantum and post-quantum theory in a multi-layer circuits
- Analytical proof that is scalable for every n >= 4
- Uses SOS and inflation arguments
- Leads to certificates in terms of games/inequalities



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