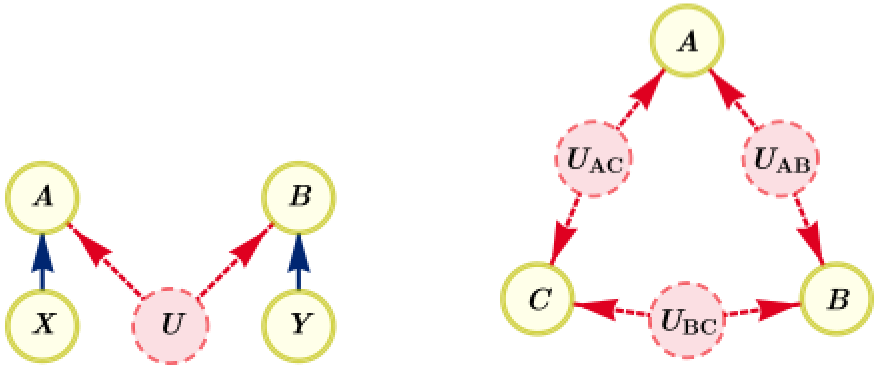
Quantum inflation: a tutorial

Introduction

- Quantum inflation is a tool for the **causal compatibility problem with quantum variables**
- **Causal compatibility** means to decide if a distribution is compatible with a **causal network**
- It can be used to **exclude** distributions that are **not compatible** with a network

Definitions

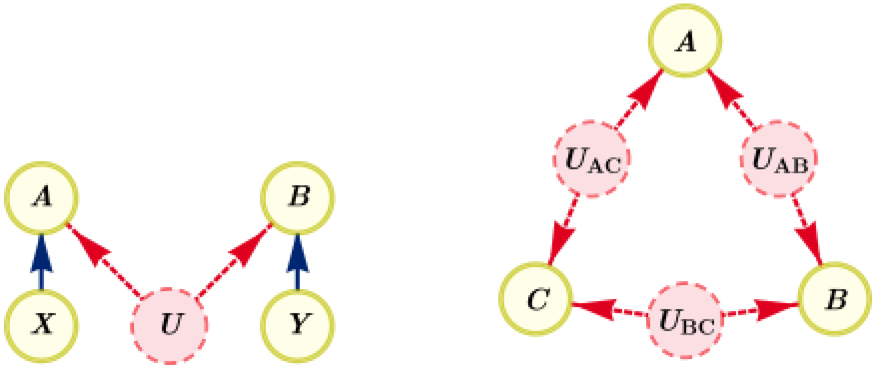
Causal network



Is a directed acyclic graph (DAG) with two types of nodes:

- **Observable nodes:** Represent things that can be read out. Their outcome is classical
- **Latent nodes:** Represent underlying correlations/causes that cannot be directly measured.
    - Can be either classical (e.g. shared randomness) or quantum (e.g. entangled states)

(We assume quantum nodes are always latent: we can only extract information by measuring)



It also has two types of arrows:

- **Classical arrows:** Outgoing from classical nodes; mean the entire variable is copied
- **Quantum arrows:** Outgoing from quantum nodes; represent subsystems of the same variable

Interpretation

- **Observable nodes receiving quantum arrow:** measurement (classical outcome)
- **Latent quantum nodes with incoming quantum arrows:** quantum channel

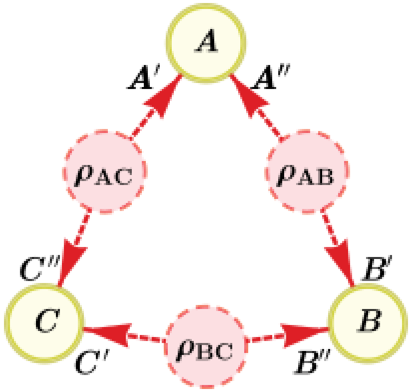
Compatibility

- A distribution P_{obs} is **compatible** with G is there is a realisation over G
- A **realisation** is a set of quantum states/transformations/measurements generating P
- Inflation can be used to exclude compatibility

Quantum inflation by example

1. Choose a network structure

For simplicity, consider the triangle network without inputs



2. Choose the observable probability distribution

- In the triangle, we want something like $P_{obs}(a, b, c)$
- **Example:** in vertex colouring $P_{obs}(a = b) = P_{obs}(b = c) = 0$ for every label a, b, c

3. Assume P_{obs} is compatible with the network structure

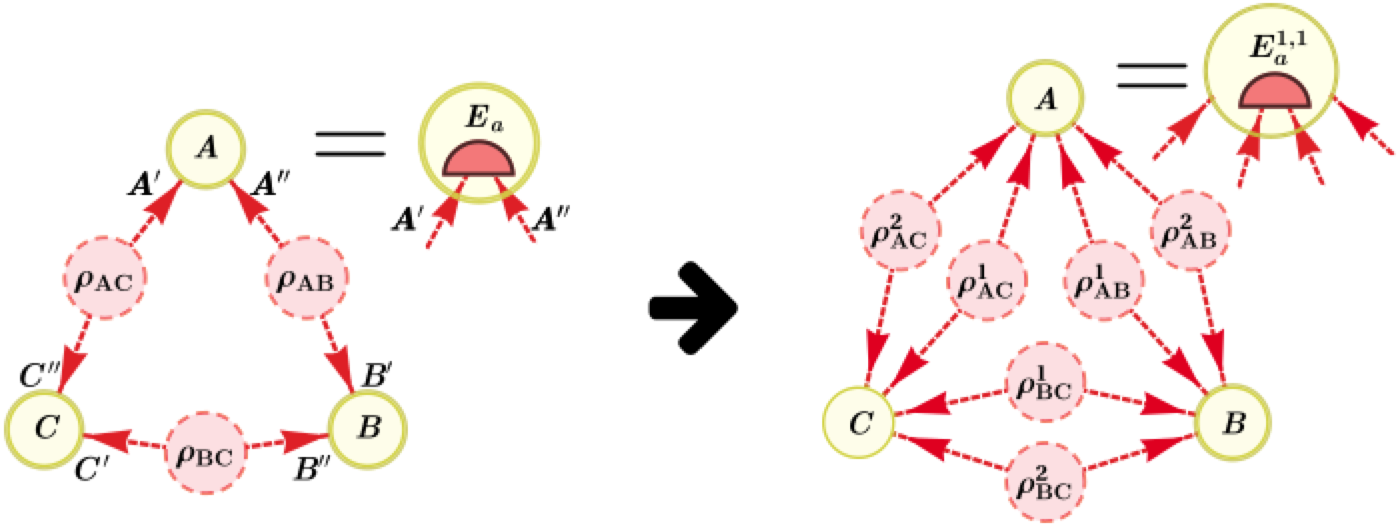
This means there are:

- Quantum states ρ_{AB}, ρ_{BC}, ρ_{AC}
- Quantum measurements $M_A = \{E_a\}_a$, $M_B = \{F_b\}_b$, $M_C = \{G_c\}_c$
    - (acting on the appropriate subsystems)
- They generate P_{obs}, for instance $\langle E_a F_b G_c \rangle = P_{obs}(a, b, c)$

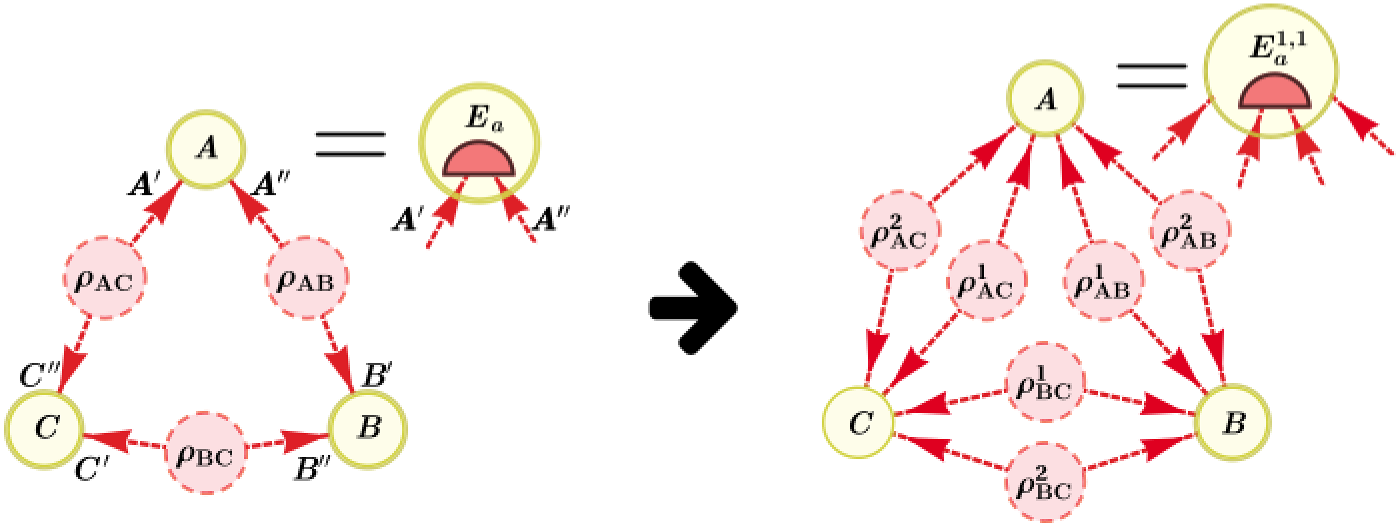
(Finding these states and measurements is a difficult problem, even in the simplest cases)

4. Derive the constraints

- Suppose a quantum realisation exists
- Then there is also many other realisations using n "copies" of the sources.
  - (Notice this is not "cloning a system", this is "pushing the button that creates the state")
- These copies can then be "wired" in many different ways
- **But all these ways must lead to the same observed probability distribution!**



This **reveals constraints that may exclude compatibility** of P_{obs} with the network

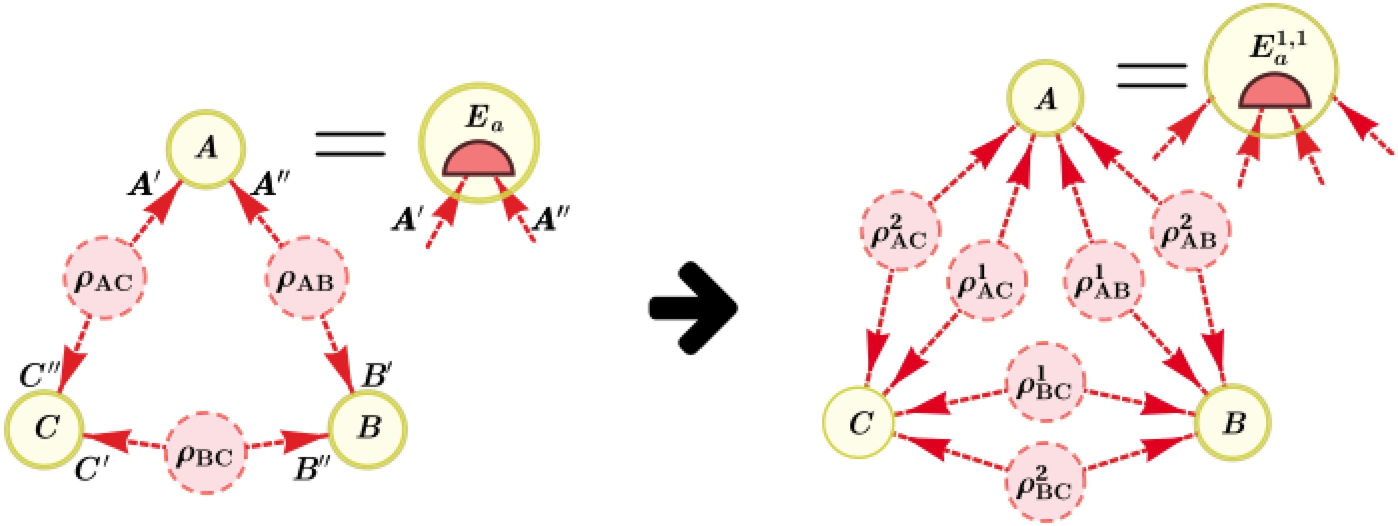


Commutation rules

- $E_a^{1,1}$ and $E_a^{2,2}$ commute because they act on different subspaces

$$E_a^{1,1} E_a^{2,2} = E_a^{2,2} E_a^{1,1}$$

(The same holds for any $E_a^{i,j}, E_a^{k,l}$ with $i \neq k$ and $j \neq l$, also for F_b and G_c)



Source permutation

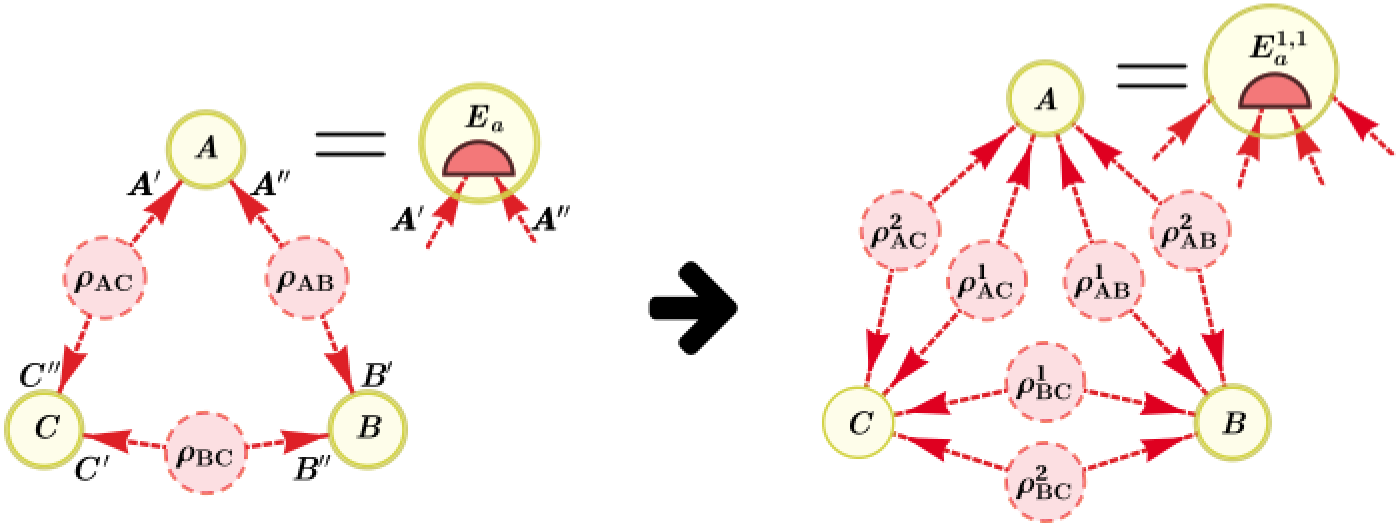
- States such as ρ^1_{AB} and ρ^2_{AB} can be relabelled, leading e.g. to

$$\langle E_a^{1,1} E_{a'}^{1,2} F_b^{2,2} \rangle = \langle E_a^{1,2} E_{a'}^{1,1} F_b^{1,2} \rangle$$

(Notice that the relabelling must reflect on E, F, G together, so not any relabelling is valid)

- In general: Given three permutations $\pi, \pi', \pi'' \in S_n$ and any function Q:

$$\langle Q(\{E_a^{i,j}, F_b^{k,l}, G_c^{m,n}\}) \rangle = \langle Q(\{E_a^{\pi(i),\pi'(j)}, F_b^{\pi'(k),\pi''(l)}, G_c^{\pi''(m),\pi(n)}\}) \rangle$$



Consistency with observed distribution

- For the original network, we wanted that $\langle E_a F_b G_c \rangle = P_{obs}(a,b,c)$
- This implies further constraints in the inflated network, for example one could say

$$\langle \Pi_{i=1} E_{a_i}^{i,i} F_{b_i}^{i,i} G_{c_i}^{i,i} \rangle = \Pi_{i=1} P_{obs}(a_i, b_i, c_i)$$

(Collectively, these constraints approximate the factorisation of spaces imposed by the network)

5. Check feasibility

- After drawing the inflation and deriving the constraints, we want to know if they are feasible
- Feasibility can be investigated using either analytical on numerical methods
- Alternatively, formulates optimisation instead of feasibility
    - Get lower bound on the "success probability" compatible with the network

Remarks

- Infeasibility immediately excludes that distribution from the network
- Feasibility does not give guarantees (convergence of quantum inflation is an open problem)

Numerically

1. Construct the inflation at a given level
2. Solve the associated polynomial optimisation problem (NPA)

Some tools that can help:

- **NPA:** ncpol2sdpa (Python), QuantumNPA (Julia), moment (C++)
- **Inflation:** inflation (Python)

Examples:

- **[arXiv:2105.09325]** "Full network nonlocality"
- **[arXiv:1709.06242]** "Causal Compatibility Ineqs. Admitting Quantum Violations in the Triangle"

Analytically

Many ways to proceed, for example:

- Search for an inflation that gives rise to inequalities that are already known for your problem
- Try to find a violation of the inequality within the constraints imposed by the inflation

Examples:

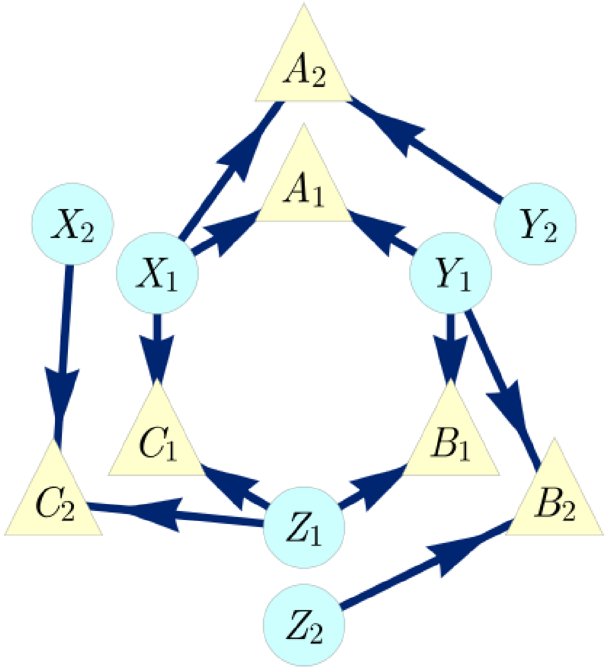
- **[arXiv:2108.02732]** "Symmetries in quantum nets. lead to no-go theorems for entanglement"
- **[arXiv:1906.06495]** "Constraints on nonlocality in nets. from no-signaling and independence"
- **[arXiv:1901.08287]** "Limits on correlations in nets. for quantum and no-signaling resources"

Remarks

Classical quantum and nonsignalling inflations

Inflation can be used for realisations in classical and in general probabilistic theories

- **Classical inflation:** Sources can be broadcast at will (fan-out inflations)



- **Quantum inflation:** Non-fanout inflations + Non-commuting quantities
- **Nonsignalling inflations:** Non-fanout inflations ("no-broadcasting theorems" for GPTs)

Note

Classical inflation is a hierarchy (asymptotically) sufficient tests to decide compatibility

We do not know in which cases *quantum* is convergent, and we know that some quantum compatibility problems are undecidable [arXiv:2001.04383, arXiv:1703.08618]

Partial inflations

- We do not need to consider n copies of every source, nor all permutations and constraints.
- Sometimes a specific permutation gives the result and makes the analysis simpler

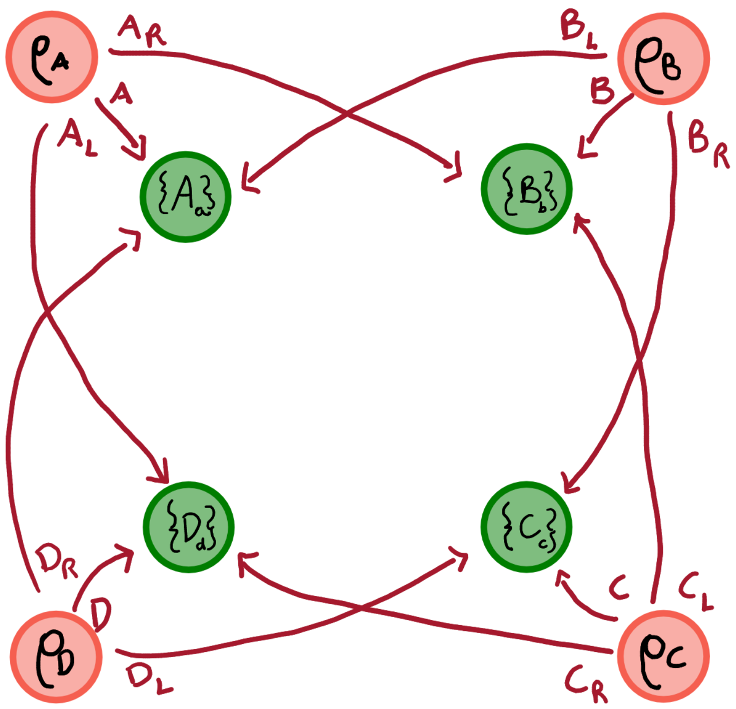
Other kinds of nodes

In our example we only have relations of the form (quantum) ➞ (classical), but others are possible

- **Classical nodes with classical inputs** can be used to represent measurement choices
- **Quantum nodes with quantum inputs** can be used to represent quantum channels
- ... combinations of these

Example: Vertex colouring on a 4-cycle

One round of communication



- Each party generates a 3-partite state, two parts to their neighbours
- After this round, each party holds three parts (their own, their left and their right neighbours')
- They then measure locally using their decoding function (# effects are the available colours)
- $p(a,b,c,d) = \mathrm{tr}(A_a B_b C_c D_d \cdot \rho)$ must be such that

$$0 = p(a=b) = p(b=c) = p(c=d) = p(d=a)$$

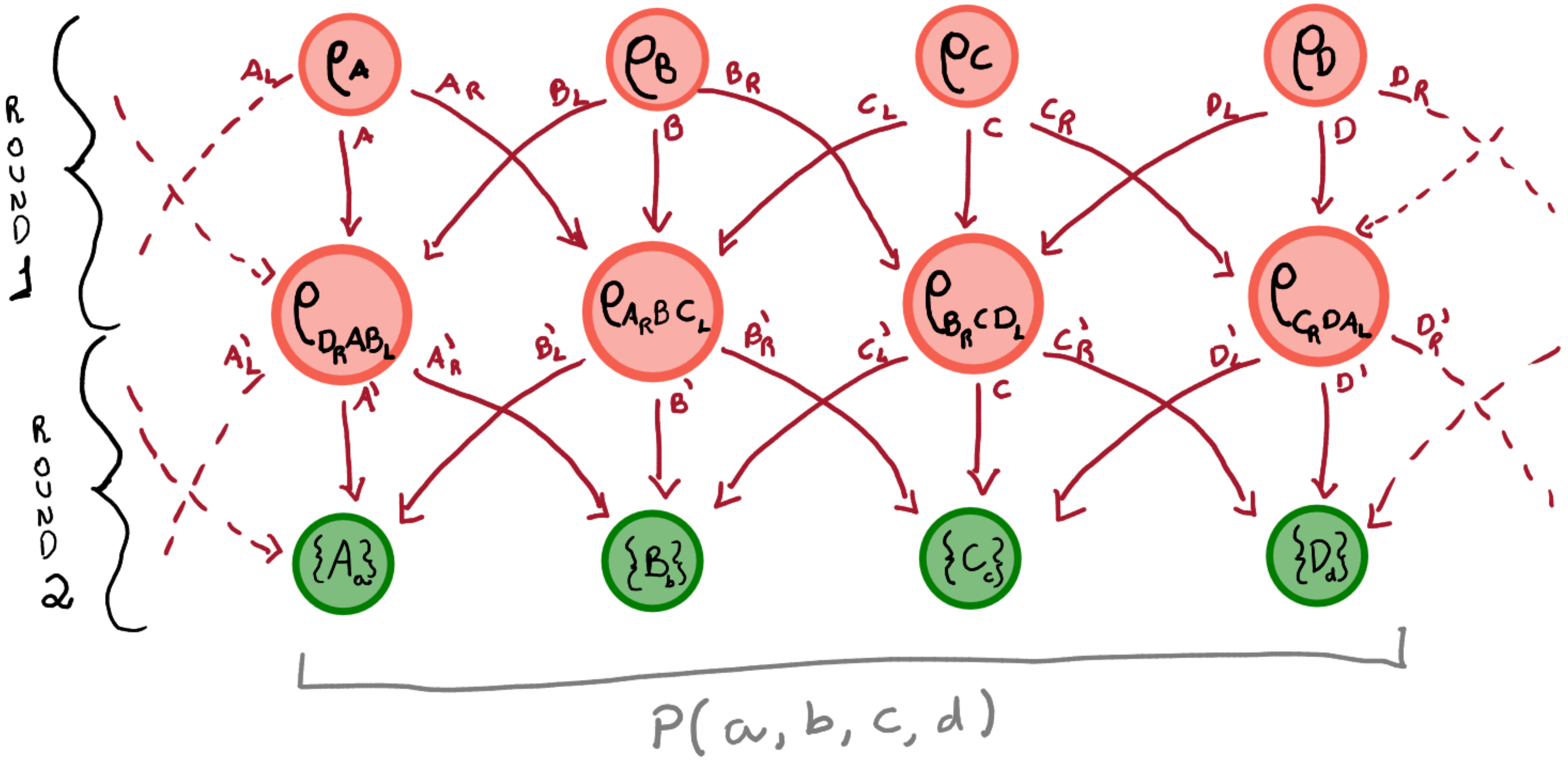
Goal: propose an inflation, derive the constraints and show they are infeasible

Work in progress

This is a concrete way to show it cannot be solved in quantum-LOCAL in a single round, but:

- The inflations get large, so we must be smart to choose partial constraints/use symmetries
- We also want same measurements everywhere (this allows more inflations)
- For bounded dimension we need different methods (preliminary p_{suc} 's)

Two rounds of communication



- Differs from the previous discussions because it has **nodes with both parents and children**!
- Inflation for these DAGs is also possible with extra subtleties (see [Sec. V, arXiv:1909.10519])

Thanks!