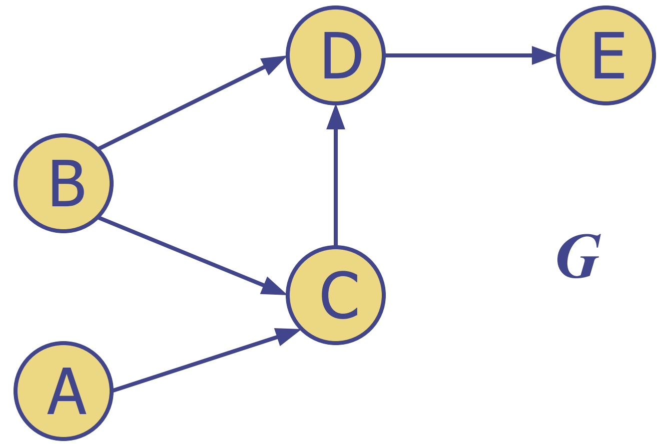
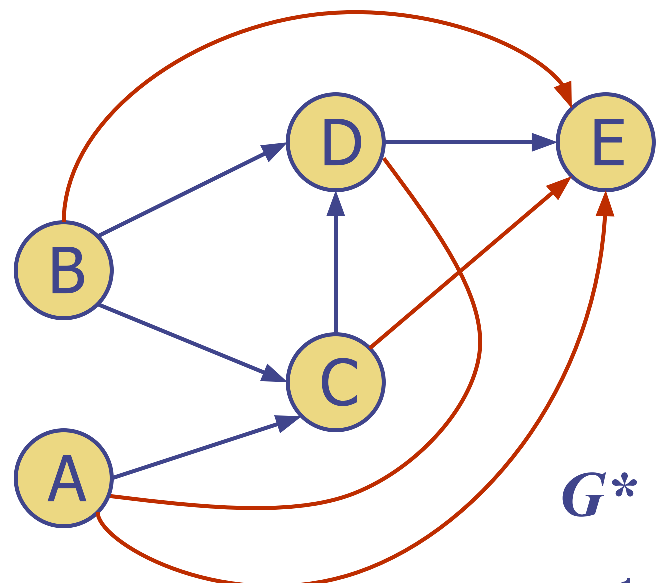
# Transitive Closure

- Given a digraph $G$, the transitive closure of $G$ is the digraph $G^*$ such that
  - $G^*$ has the same vertices as $G$
  - if $G$ has a directed path from $u$ to $v$ ($u \neq v$), $G^*$ has a directed edge from $u$ to $v$
- The transitive closure provides reachability information about a digraph

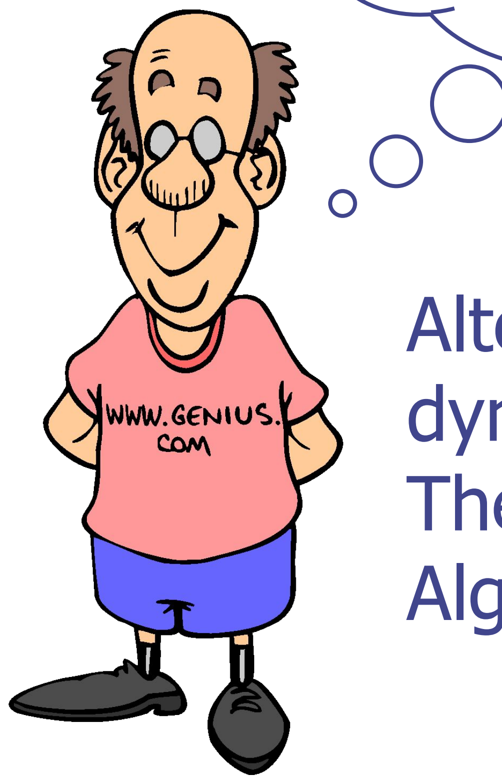


# Computing the Transitive Closure

- We can perform DFS starting at each vertex
  - O(n(n+m))

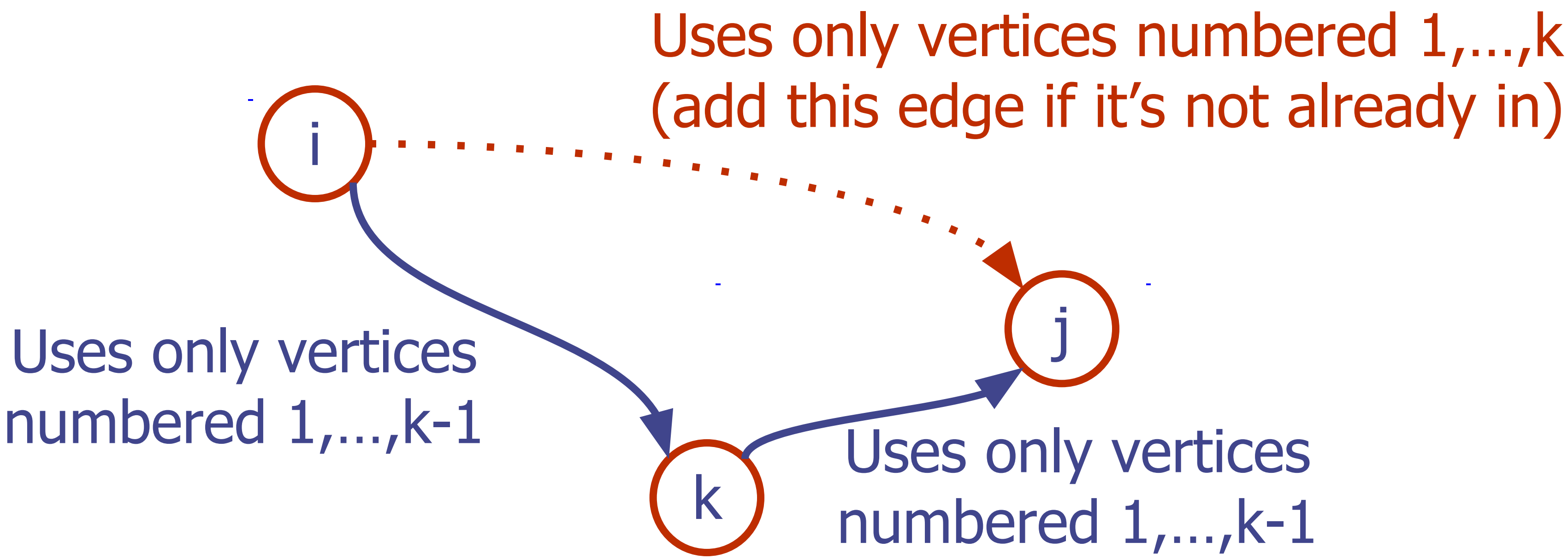
If there's a way to get from A to B and from B to C, then there's a way to get from A to C.



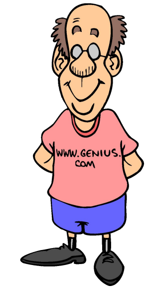

Alternatively ... Use dynamic programming: The Floyd-Warshall Algorithm

# Floyd-Warshall Transitive Closure


- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:



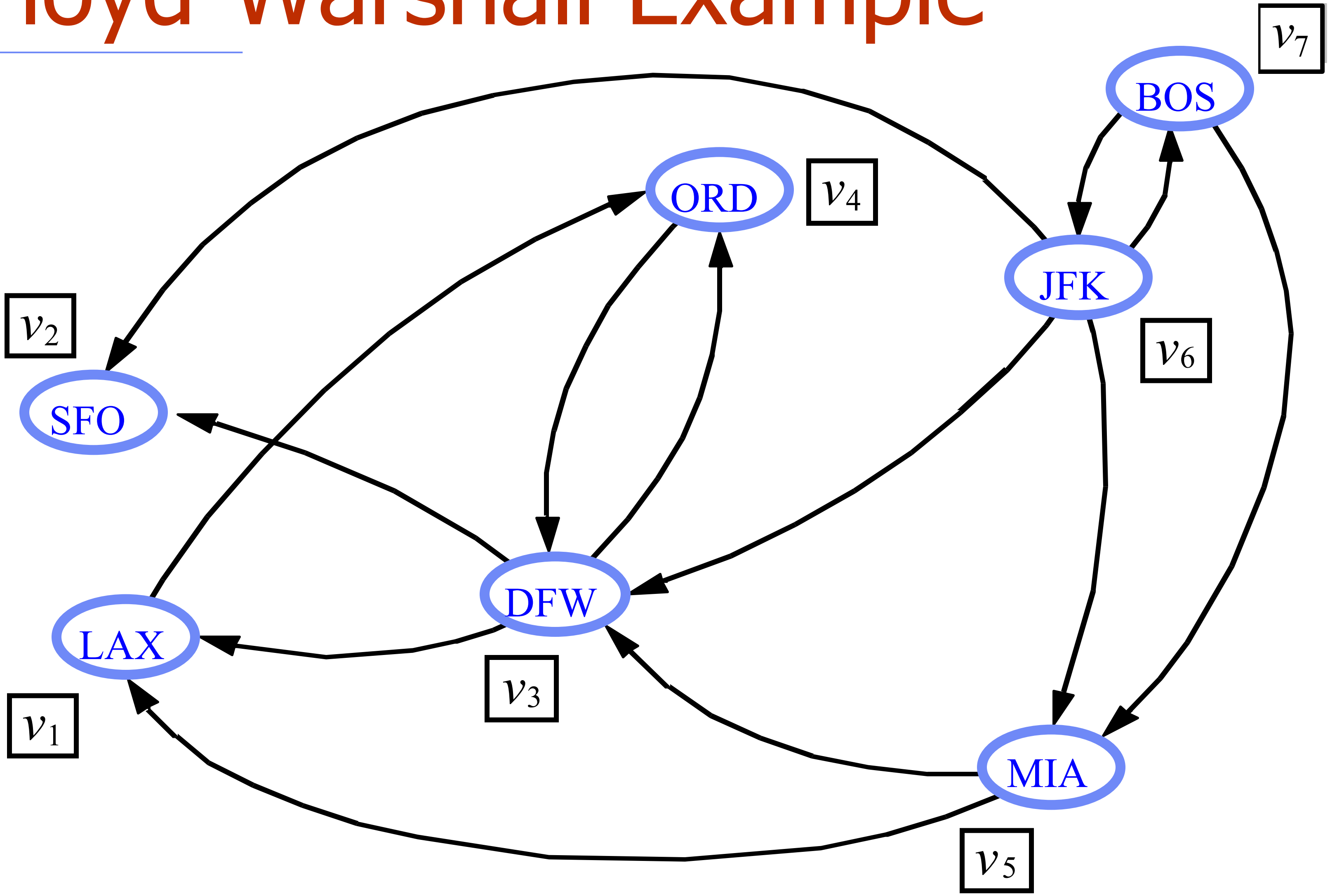
# Floyd-Warshall's Algorithm



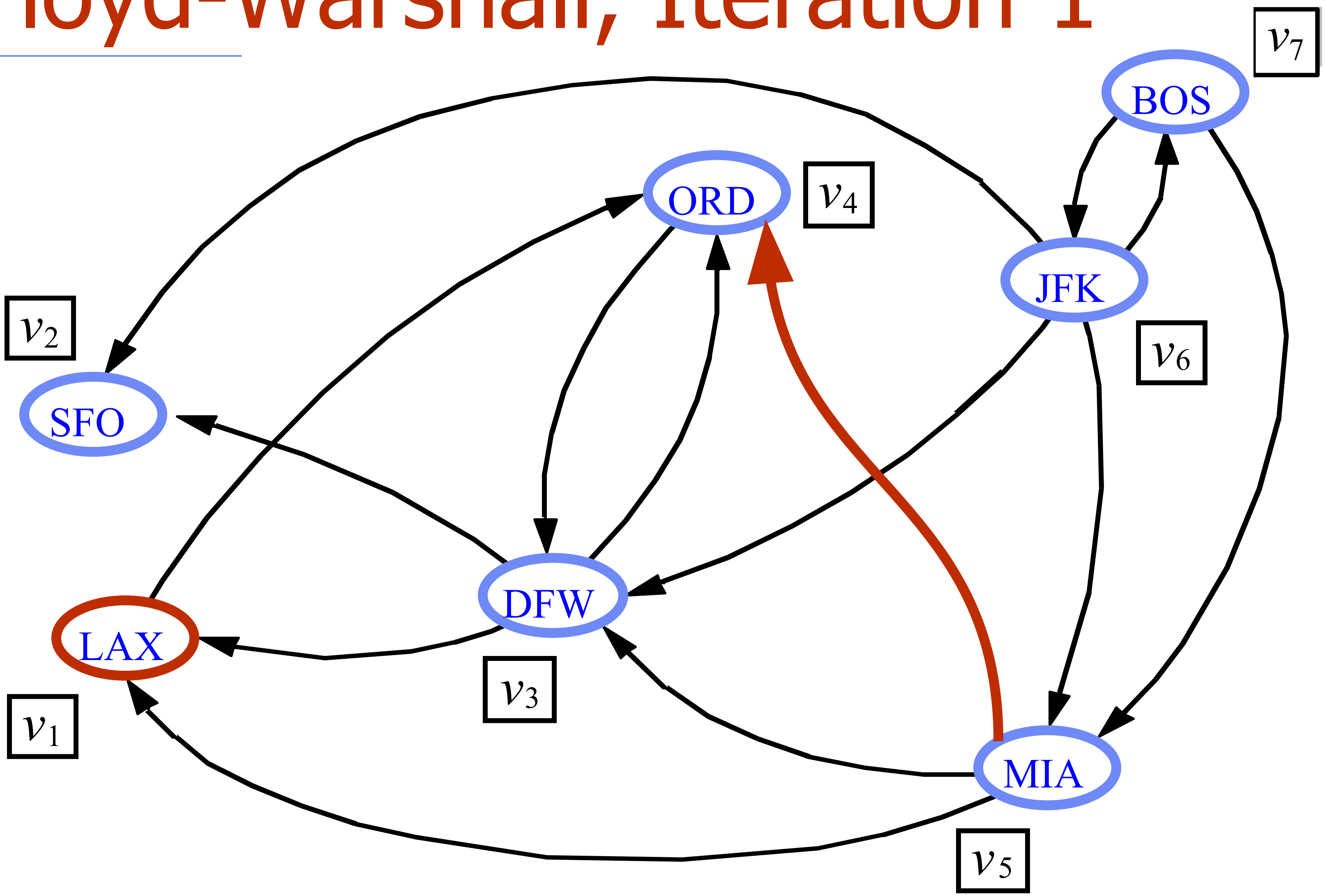

- Number vertices $v_1, \ldots, v_n$
- Compute digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - $G_k$ has directed edge $(v_i, v_j)$ if $G$ has a directed path from $v_i$ to $v_j$ with intermediate vertices in $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase $k$, digraph $G_k$ is computed from $G_{k-1}$
- Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

**Algorithm** *FloydWarshall*($G$)
  **Input** digraph $G$
  **Output** transitive closure $G^*$ of $G$
  $i \leftarrow 1$
  **for all** $v \in G.vertices()$
    denote $v$ as $v_i$
    $i \leftarrow i + 1$
  $G_0 \leftarrow G$
  **for** $k \leftarrow 1$ **to** $n$ **do**
    $G_k \leftarrow G_{k-1}$
    **for** $i \leftarrow 1$ **to** $n$ $(i \neq k)$ **do**
      **for** $j \leftarrow 1$ **to** $n$ $(j \neq i, k)$ **do**
        **if** $G_{k-1}.areAdjacent(v_i, v_k)$ $\wedge$ $G_{k-1}.areAdjacent(v_k, v_j)$
          **if** $\neg G_k.areAdjacent(v_i, v_j)$
            $G_k.insertDirectedEdge(v_i, v_j, k)$

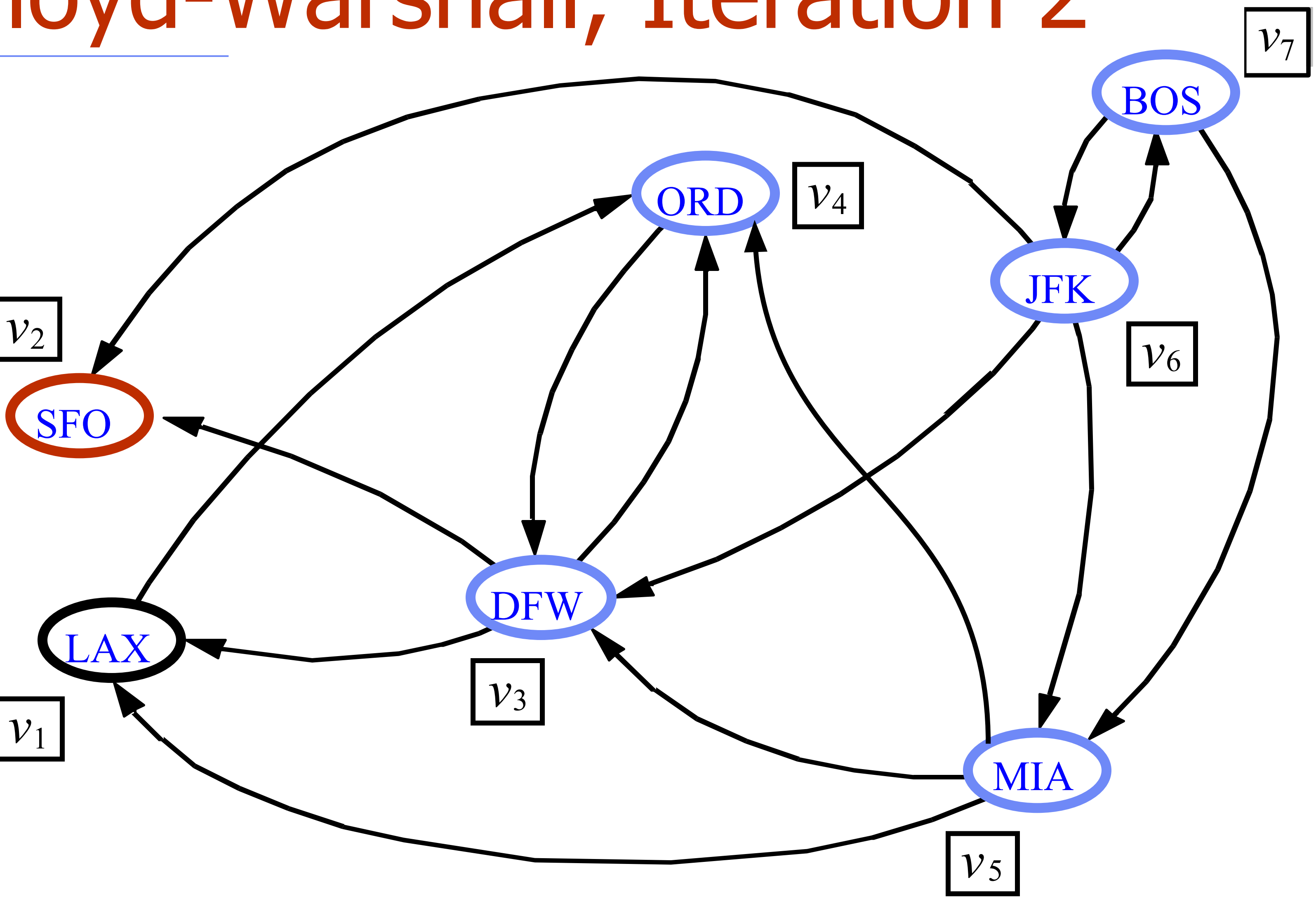
# Floyd-Warshall Example



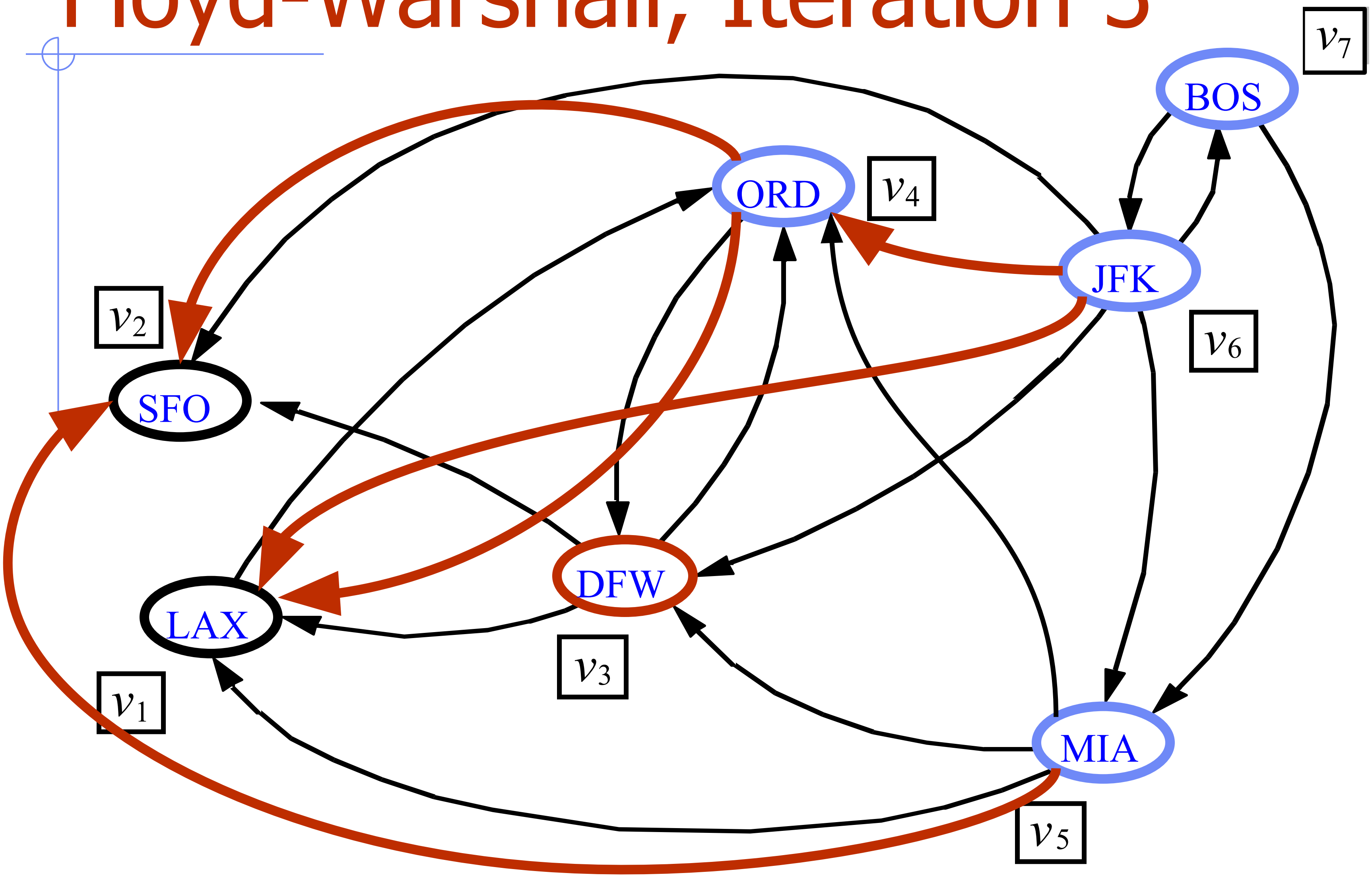
# Floyd-Warshall, Iteration 1



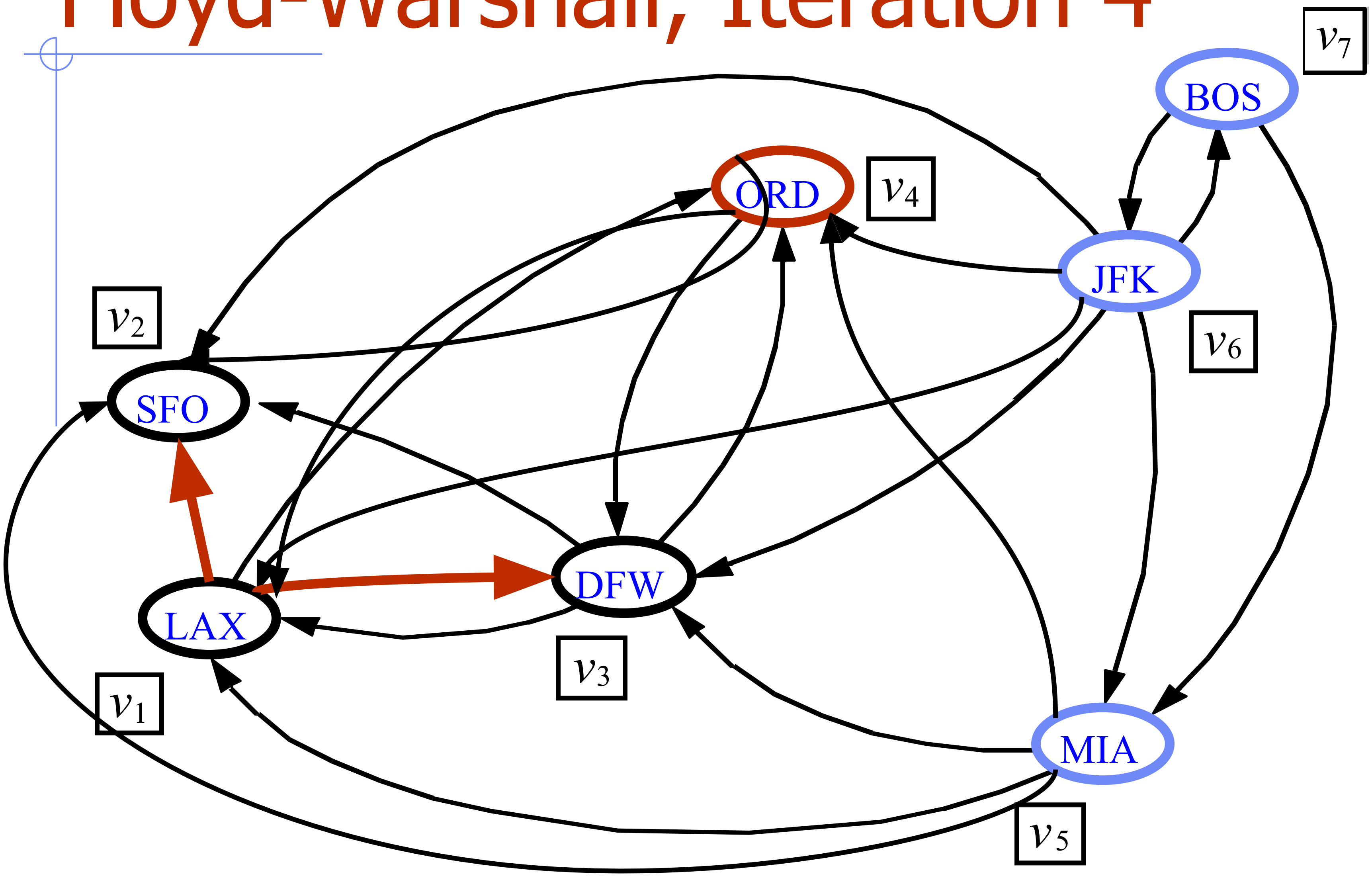
# Floyd-Warshall, Iteration 2



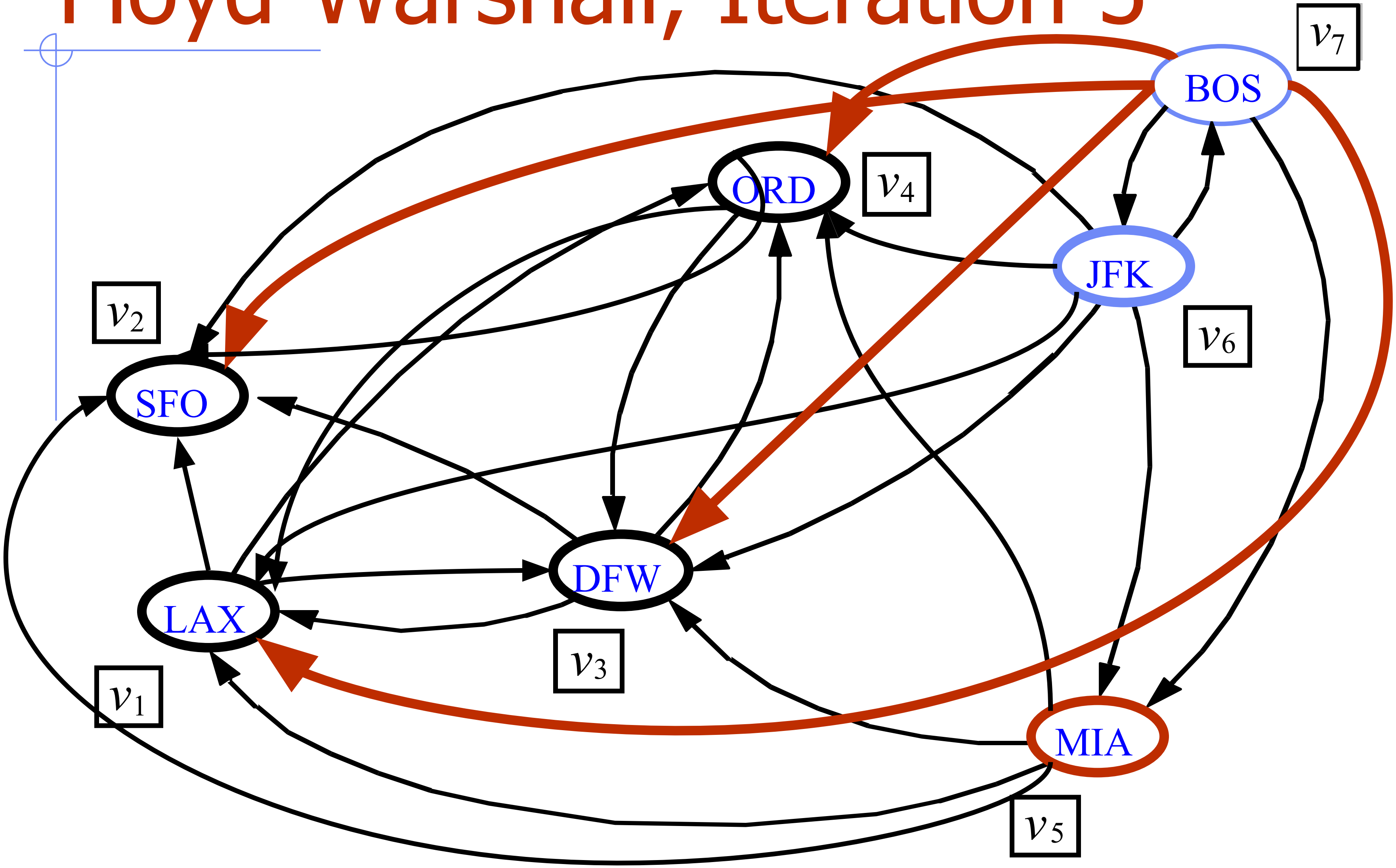
# Floyd-Warshall, Iteration 3



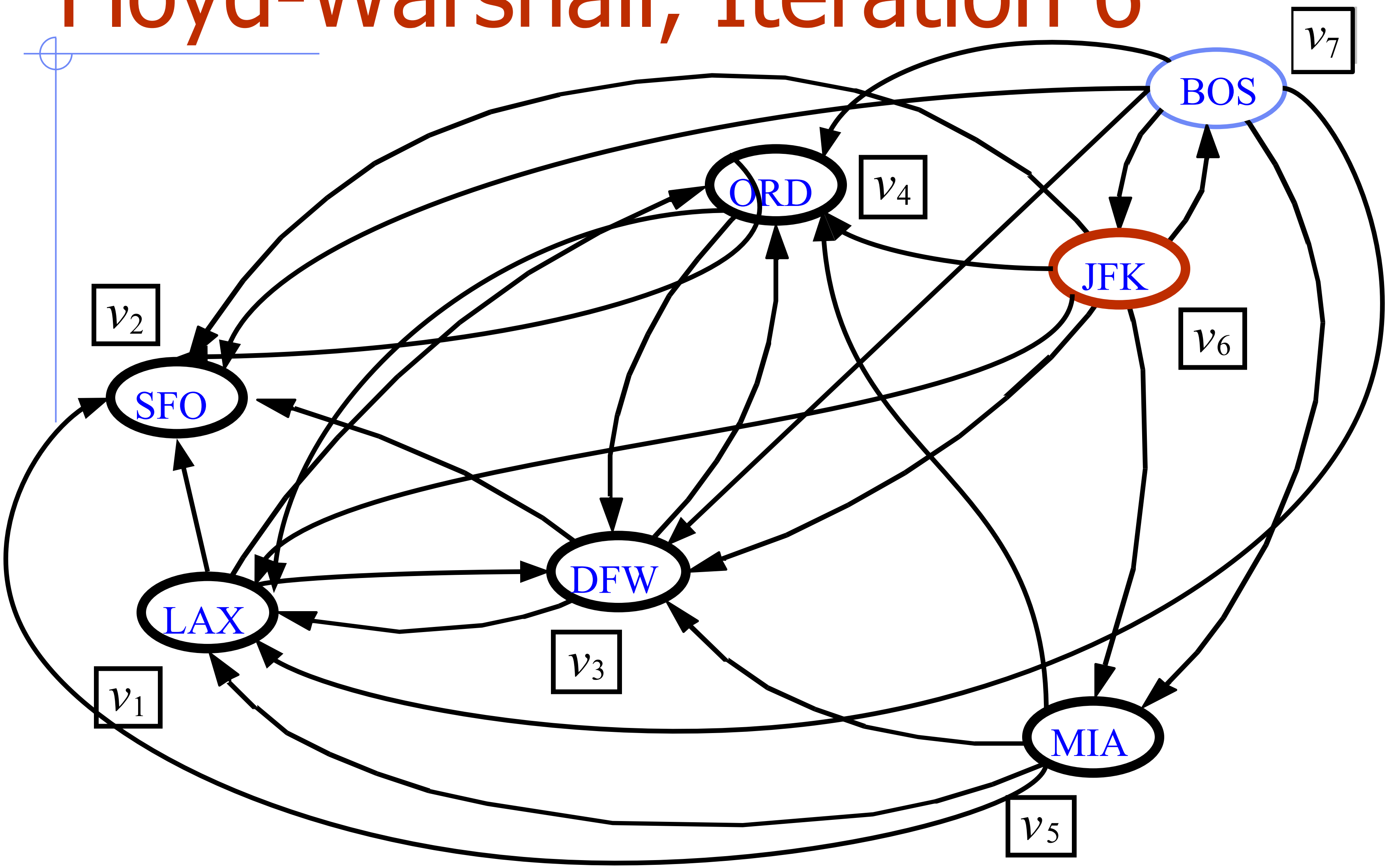
# Floyd-Warshall, Iteration 4



# Floyd-Warshall, Iteration 5



# Floyd-Warshall, Iteration 6



# Floyd-Warshall, Conclusion