Short $\mathsf{Res}^*(\mathsf{polylog})$ refutations if and only if narrow Res refutations

(to a question of Neil Thapen[1])

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In this note we show that any k-CNF which can be refuted by a quasi-polynomial $\mathsf{Res}^*(\mathsf{polylog})$ refutation has a "narrow" refutation in Res (i.e., of polylogarithmic width). Notice that while $\mathsf{Res}^*(\mathsf{polylog})$ is a complete proof system, this is not the case for Res if we restrict to narrow refutation. In particular is not even possible to express all CNFs with narrow clauses. But even for constant width CNF the former system is complete and the latter is not (see for example [BG01]). We are going to show that the formulas "left out" are the ones which require large $\mathsf{Res}^*(\mathsf{polylog})$ refutations. We also show the converse implication: a narrow Resolution refutation implies a short $\mathsf{Res}^*(\mathsf{polylog})$ refutation.

Preliminaries

We consider a formula F in CNF form, with m clauses and n variables, we fix k to be the width (i.e., number of literals) of the largest clause in k. The i-th clause of F is denoted as C_i, so that we can write F as $\bigwedge_{i=1}^{m} C_i$.

A *refutation* of size S for the formula F is a sequence of formulas $D_1, \ldots, D_S$ such that $D_i = C_i$ for $1 \leq i \leq m$, and any D_i for $i > m$ is either an *axiom* or is *logically inferred* from two previous formulas in the sequence.

Different proof systems are characterized by the types formulas allowed as proof lines, the axioms and logical inference rules. Further constraints on the structure of the refutation may be imposed: while this may cause refutations to be longer, it facilitates the process of searching for the refutation itself. A customary constraint is to impose *tree-like* structure on the refutation: with the exception of the axioms and the initial clauses of F, any formula in the sequence can be used at most once as premise for a logical inference. In this way the refutation looks like a binary tree in which every leaf is either labelled by an axiom or by an initial clause, and every internal vertex correspond to a formula inferred in the refutation. Notice that in a tree-like refutation any formula that is needed more than once must be re-derived from scratch. A proof system is called *dag-like* is no constraint on the structure of the proof is imposed (indeed the graph of the inference looks like a directed acyclic graph).

Res : every line in the refutation is a clause (i.e., disjunction of literals), and there is a single inference rule:

$$\frac{A \vee x \qquad B \vee \neg x}{A \vee B}$$

The *width* of a Res refutation is the maximum number of literals contained in a clause of the refutation. In this note we focus on refutations with polylogarithmic width.

$\mathsf{Res}^*(l)$: the structure of the proof must be tree-like, every line in the refutation is a l-DNF, there is an axion introduction rule:

$$\frac{}{l_1 l_2 \cdots l_s \bigvee \neg l_1 \vee \neg l_2 \vee \ldots \vee \neg l_s} \quad \text{for } 1 \leq s \leq l,$$

and an inference rule

$$\frac{A \vee l_1 l_2 \cdots l_s \qquad B \vee \neg l_1 \vee \neg l_2 \vee \ldots \vee \neg l_s}{A \vee B} \quad \text{for } 1 \leq s \leq l.$$

[1] apparently the present result can be proved by methods of Bounded Arithmetic. He asked whether there exists a simpler and more direct proof.

Notes on the definition: there are different ways to define $\mathsf{Res}^*(l)$, which are all equivalent up to a polynomial size increase in the size of the refutation. Notice that there are neither weakening rule nor AND introduction rules: this allows to control how terms appear in the refutation and this makes the next proofs simpler. The cut rule is defined to require exactly the same number of literals on both sides. This is more rigid than usual, but **since the system is tree-like** this is without loss of generality.

Main statement

In this note we prove that

Theorem 1. *Let F be a k-CNF. F has a quasi-polynomial size $\mathsf{Res}^*(\mathsf{polylog})$ refutation if and only if has a Res refutation of polylogarithmic width.*

Notice that the Res refutations are dag-like, and that any such refutation of polylogarithmic width must have size at most quasi-polynomial.

Proof of narrow Res simulation

The proof is based on the following Lemma, which immediately implies one direction of Theorem 1.

Lemma 1. *Let F be a k-CNF. If F has a refutation in $\mathsf{Res}^*(l)$ which has L leaves, then has a Res refutation of width $l\lceil\log L\rceil + max\{k,l\}$.*

Proof. The proof of the lemma is by induction on the number of leaves L in the refutation. If $L = 1$ the initial CNF contains the empty clause, and the result is trivial.

Let us assume $L > 1$, and fix $w = l\log L + max\{k,l\}$. Consider the last step of the refutation. Since it results in an empty DNF, it must be the result of a cut rule, between formulas $A = \bigwedge_{i=1}^{s} l_i$ and $B = \bigvee_{i=1}^{s} \neg l_i$, for a set of s literals.

Since the refutation is tree-like the two proofs of A and B are disjoint, and the number of leaves in each proofs (L_A and L_B) respectively are such that $L_A + L_B = L$. Thus either L_A or L_B is less than or equal to $\frac{L}{2}$, and both are less than L. The proof is divided in two cases:

($L_A \leq L/2$): Since $F \vdash \bigwedge_{i=1}^{s} l_i$ in $\mathsf{Res}^*(\mathsf{polylog})$ with L_A leaves, by fixing $l_i = 0$ we get that $F|_{l_i=0} \vdash \Box$, with a $\mathsf{Res}^*(l)$ refutation which uses at most L_A leaves. By inductive hypothesis the same refutation can be done in Res with width at most $l\log L_A + max\{k,l\} \leq l\log L + max\{k,l\} - l \leq w - 1$.

By weakening we have that $F \vdash l_i$ in Res in width w, for any $1 \leq i \leq s$. Using such literals we can deduce $F|_{l_1=1,\ldots,l_s=1}$ from F in width k by removing any occurrences of literals $\neg l_i$. Since $F \vdash B$ in $\mathsf{Res}^*(\mathsf{polylog})$, we can prove $F|_{l_1=1,\ldots,l_s=1} \vdash \Box$ in $\mathsf{Res}^*(\mathsf{polylog})$ with at most $L_B < L$ leaves. By inductive hypothesis this refutation can be done in Res in width w. Composing the resolution proofs $F \vdash l_i$ for $1 \leq i \leq s$, the proof of $F, l_1, \ldots l_s \vdash F|_{l_1=1,\ldots,l_s=1}$ and the proof $F|_{l_1=1,\ldots,l_s=1} \vdash \Box$, we get a Res refutation of width w of $F \vdash \Box$.

($L_B \leq L/2$): we may assume that $s > 1$ because otherwise formulas A and B can be swapped and the reasoning for the previous case applies. Since $F \vdash \bigvee_{i=1}^{s} \neg l_i$ in $\mathsf{Res}^*(\mathsf{polylog})$ with L_B leaves we get that $F|_{l_1=1,\ldots,l_s=1} \vdash \Box$, with a $\mathsf{Res}^*(l)$ refutation with $L_B \leq \frac{L}{2}$ leaves. By inductive hypothesis the same refutation can be done in Res with width at most $l\log L_B + max\{k,l\} \leq l\log L + max\{k,l\} - l \leq w - l$. By weakening Res proves $F \vdash \bigvee_{i=1}^{s} \neg l_i$ in width w.

We now conclude the argument arguing that Res proves $F, B \vdash \square$ with width w. To see that observe the $\mathsf{Res}^*(\mathsf{polylog})$ proof of $F \vdash A$: each occurrence of A is introduced in such proof using the axiom $A \vee B$. Substitute such axiom with the new initial clause B. By an easy induction along the tree-like derivation, such transformation produces a $\mathsf{Res}^*(\mathsf{polylog})$ proof of $F, B \vdash \square$ with $L_B < L$ leaves. By inductive hypothesis this implies a Res refutation of width w (notice that the initial width of the formula increases, but that is accounted in the definition of w). $\square$

Proof of the short $\mathsf{Res}^*(l)$ simulation

The following lemma gives the other direction of theorem 1.

Lemma 2. *Let F be any CNF. If F has a Res refutation of width w and size S, then has a $\mathsf{Res}^*(w)$ refutation of size $O(S)$.*

Proof. Consider the Res refutation $D_1, D_2, \ldots D_S$ of F. We define the sequence of w-DNFs $E'_t = \bigvee_{i=1}^{t} \neg D_i$. By backward induction on t from $S-1$ to 0 we are going to derive a w-DNF E_t such that the terms of E_t are a subset of the terms of $E'(t)$. Since E_0 is the empty DNF that would conclude the proof.

For $t = S-1$ notice that E'_{S-1} contains $x \vee \neg x$ for some variable x in F, which is an axiom in $\mathsf{Res}^*(\mathsf{polylog})$.

Fix $t < S-1$ and consider D_a and D_b which has been used to derive D_{t+1}. For convenience write as follows

$$D_a \equiv A \vee x \quad D_b \equiv B \vee \neg x \quad D_{t+1} \equiv A \vee B \quad E_{t+1} \equiv \Delta \vee (\neg A \wedge \neg B)$$

for some w-DNF Δ, some clauses A, B and some variable x. Terms of Δ are all contained in E'_t. Employ the following tree-like deduction

$$\begin{array}{rrr}
(\neg A \wedge \neg x) \vee A \vee x & \text{As an axiom of } \mathsf{Res}^*(w) & (1) \\
(\neg B \wedge x) \vee B \vee \neg x & \text{As an axiom of } \mathsf{Res}^*(w) & (2) \\
(\neg A \wedge \neg x) \vee (\neg B \wedge x) \vee A \vee B & \text{Cut on term } x \text{ between (1) and (2)} & (3) \\
\Delta \vee (\neg A \wedge \neg B) & E_{t+1} \text{ deduced by induction hypothesis} & (4) \\
(\neg A \wedge \neg x) \vee (\neg B \wedge x) \vee \Delta & \text{Cut on term } \neg A \wedge \neg B \text{ between (3) and (4)} & (5)
\end{array}$$

Notice that formula (5) is a w-DNF which terms are contained in E'_t.

For $t = 0$ we get the empty DNF. At each step E_t is derived in $\mathsf{Res}^*(w)$ using a single occurrence of formula E_{t+1}. That means that the whole refutation is tree-like and has size $O(S)$. $\square$

References

[BG01] Maria Luisa Bonet and Nicola Galesi. Optimality of size-width tradeoffs for resolution. *Computational Complexity*, 10(4):261–276, 2001.

[BSW01] Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow - resolution made simple. *J. ACM*, 48(2):149–169, 2001.