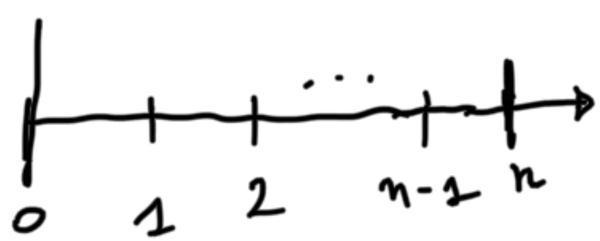
$X_n \sim B(1,p)$

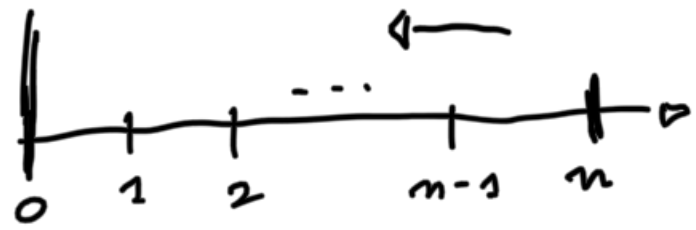
prova di ordine n



$S_n \sim B(n,p)$

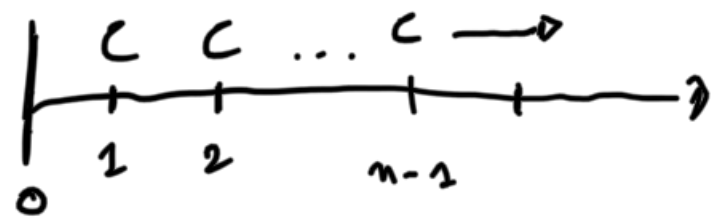
somme delle prime n prove

$S_n = X_1 + X_2 + \cdots + X_n$



$T_1 \sim G(p)$

numero delle prove per osservare le prime " testa



$S_{T_1} = \mathbb{N} = \{1, 2, \dots\}$

$$S_{S_n} = \{0, 1, \dots, n\}$$

$m \in \{0, 1, \dots, n\}$

$$\boxed{P(S_n = m) = \binom{n}{m} p^m (1-p)^{n-m}}$$

$n \in \mathbb{N}$, $P(T_1 = n) = p(1-p)^{n-1}$

$C_1 C_2 \cdots C_{n-1} | T_n$

$\{S_{n-1} = 0\} \cap \{X_n = 1\}$

si consideri $k \in \mathbb{N}$

DOMANDA

DETERMINARE IL NUMERO DELLE PROVE NECESSARIE PER OSSERVARE LA k-ma "TESTA" PER LA PRIMA VOLTA

RISPOSTA

$$W_K : \begin{cases} S_{W_k} = \{ K, K+1, K+2, \dots \} \\ \{S_{n-1} = K-1\} \cap \{X_n = 1\} \\ n \in S_{W_k}, \ \mathbb{P}(W_k = n) = \binom{n-1}{k-1} p^K (1-p)^{n-K} \end{cases}$$

$$\mathbb{P}(\{S_{n-1} = K-1\} \cap \{X_n = 1\}) = \mathbb{P}(S_{n-1} = K-1) \cdot \mathbb{P}(X_n = 1)$$

$$= \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \cdot p$$

$W_k \sim \text{Pascal}(k, p)$; $T_1 \sim G(p) \sim \text{Pascal}(1, p)$

$$W_k = T_1^{(1)} + T_1^{(2)} + \dots + T_1^{(k)}$$

$$T_1^{(1)} \sim T_1^{(2)} \sim \dots \sim T_1^{(k)} \sim T_1 \sim G(p)$$

<u>Esempio</u>

$K = 3$

$$\begin{array}{cccc|cccccc|ccc}
 & & & & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 \\
C & C & C & T & C & C & C & C & C & T & C & T & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13
\end{array}$$

$t_1^{(1)} = 4$ $\quad t_1^{(2)} = 6$ $\quad t_1^{(3)} = 2$

Quindi:

W_k è la somma di K numeri aleatori "geometrici" "indipendenti" e "somiglianti"

ESEMPI

$k = 4$

$$\begin{array}{c|ccc|cc|cccc|ccc}
 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 3 & 4 & & & \\
T & C & C & T & C & T & C & C & C & T & C & C & T \; \dots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13
\end{array}$$

$$10 = 1 + 3 + 2 + 4 = t_1^{(1)} + t_1^{(2)} + t_1^{(3)} + t_1^{(4)}$$

$k = 3$

$$\begin{array}{ccccc|c|cccccccc|ccccc}
 & & & & & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & & & & & \\
C & C & C & C & T & T & C & C & C & C & C & C & C & T & C & T & C & C & - \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19
\end{array}$$

$$14 = 5 + 1 + 8$$

ESEMPI

$$\mathbb{P}(W_k = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}.$$

$W \sim \text{Pascal}(5, 1/4)$

$S_W = \{5, 6, 7, \dots\}$

$k = 5$,

1) $\mathbb{P}(W = 3) = \mathbb{P}(\emptyset) = 0$

$n = 7$, 2) $\mathbb{P}(W = 7) = \binom{6}{4} \underbrace{p^5}_{} \underbrace{(1-p)^2}_{}$

$n = 5$, 3) $\mathbb{P}(W = 5) = \binom{4}{4} p^5 (1-p)^0 = p^5$

4) $\mathbb{P}(W = \pi) = \mathbb{P}(\emptyset) = 0.$

$A, B \in \mathcal{F} \Rightarrow \mathbb{P}(A \,\dot\cup\, B) = \mathbb{P}(A) + \mathbb{P}(B)$

$A \cap B = \emptyset$

finita additività

TEOREMA (esclusione - inclusione, 2 eventi)

$$A, B \in \mathcal{F} \Rightarrow \mathbb{P}(A \cup B) = [\mathbb{P}(A) + \mathbb{P}(B)] - \mathbb{P}(A \cap B)$$

DIM

(1) $$B = B \cap \Omega = B \cap (A \dot\cup A^c) =$$

$$= (B \cap A) \dot\cup (B \cap A^c)$$

$$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c)$$

(2) $$A \cup B = A \dot\cup (B \cap A^c)$$

$$\mathbb{P}(A \cup B) \overset{(2)}{=} \mathbb{P}[A \dot\cup (B \cap A^c)]$$

$$= \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$$

$$\overset{(1)}{=} \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B). \quad \square$$

$\square$

<u>TEOREMA</u> (inclusione - esclusione, 3 eventi)

$A, B, C \in \mathcal{F}$

$$\mathbb{P}(A \cup B \cup C) = [\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)] + \\ - [\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)] + \\ + \mathbb{P}(A \cap B \cap C)$$

<u>DIM</u>

Si lascia allo studente.

<u>TEOREMA</u> (inclusione - esclusione, n eventi)

$A_1, A_2, \ldots, A_n \in \mathcal{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{\substack{i,j=1 \\ i<j}}^{n} \mathbb{P}(A_i A_j) + \cdots + (-1)^{n-1}\,\mathbb{P}(A_1 A_2 \cdots A_n).$$