FUNZIONE DI DISTRIBUZIONE

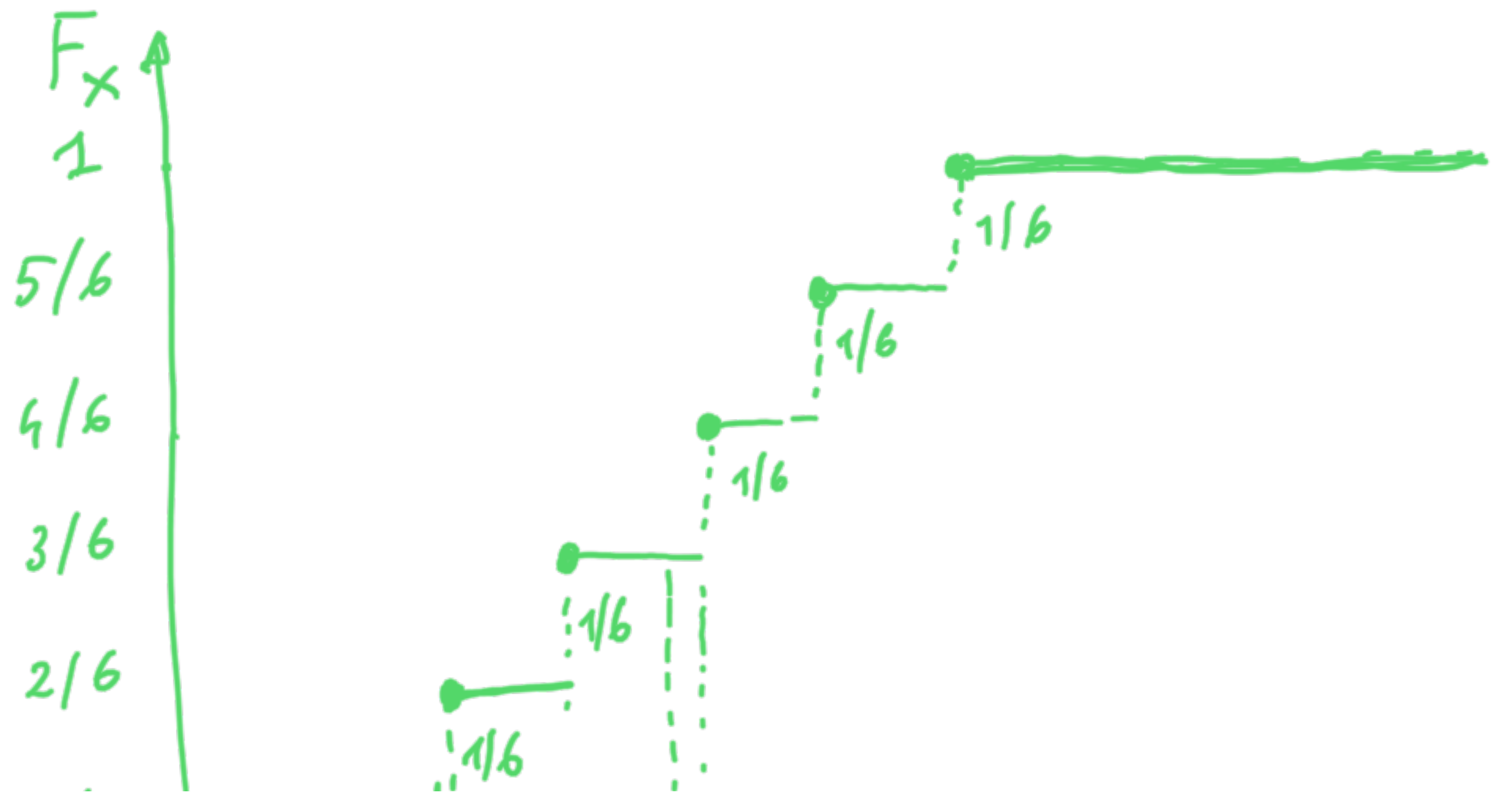
ESEMPIO

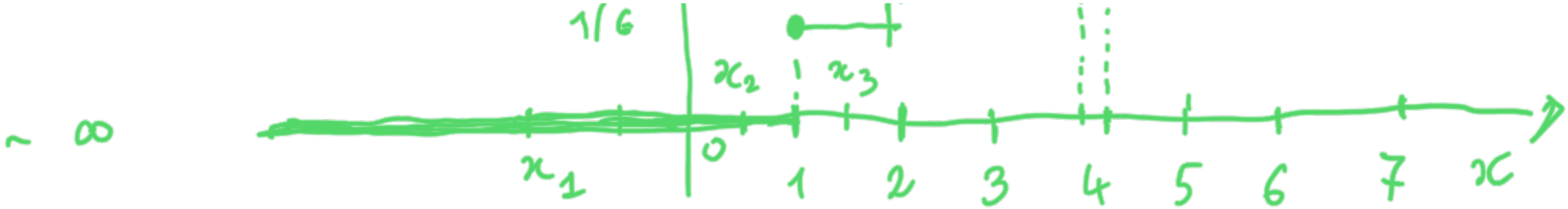
$$X \sim U_d(1,2,3,4,5,6)$$

Spettro	Probabilità	Probabilità accumulata
	f	F
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	6/6
Σ	6/6 = 1	

$$\mathbb{P}(2 \le X \le 4) = \mathbb{P}(X \in \{2,3,4\})$$
$$= \mathbb{P}(X=2) + \mathbb{P}(X=3) + \mathbb{P}(X=4)$$
$$= 1/6 + 1/6 + 1/6$$
$$= 3/6$$

$$F_4 - F_2 = 4/6 - 2/6 = 2/6$$
$$F_4 - F_1 = 4/6 - 1/6 = 3/6$$





$$\mathbb{P}(X \le x_1) = 0 \qquad \mathbb{P}(X \le 2) = 2/6$$

$$\mathbb{P}(X \le x_2) = 0 \qquad \mathbb{P}(X \le 7) = 1$$

$$\mathbb{P}(X \le 1) = 1/6$$

$$\mathbb{P}(X \le x_3) = 1/6$$

Da un punto di vista qualitativo, risulta:

a) F_X è non decrescente,

b) $\lim_{\varepsilon \to 0^+} F_X(x+\varepsilon) = F_X(x)$,

c) $\lim_{x \to -\infty} F_X(x) = 0$; $\lim_{x \to +\infty} F_X(x) = 1$.

Quantitativamene, inoltre, si ha:

$$F_X(2^-) := \lim_{\varepsilon \to 0^+} F_X(2-\varepsilon) \quad \left(\text{limite nell'intorno sinistro di } 2\right)$$

e

$$F_X(4) - F_X(2^-) = 4/6 - 1/6 = 3/6;$$

$$1/6 = \mathbb{P}(X=4) = F_X(4^+) - F_X(4^-)$$

$$= F_X(4) - F_X(4^-)$$

$$= 4/6 - 3/6 = 1/6;$$

$$x \in \mathbb{R}, \qquad F_X(x) := \sum_{x_i \in S_X : x_i \leq x} 1/6 = \sum_{i=1}^{6} 1_{\{i \leq x\}} \cdot 1/6$$

$$F_X(2,5) = 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} = \frac{2}{6};$$

$$F_X(3,4) = 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} = \frac{3}{6};$$

$$F_X(1) = 1\cdot\frac{1}{6} + 0\cdot\frac{1}{6} + 0\cdot\frac{1}{6} + 0\cdot\frac{1}{6} + 0\cdot\frac{1}{6} + 0\cdot\frac{1}{6} = \frac{1}{6}.$$

$\square$

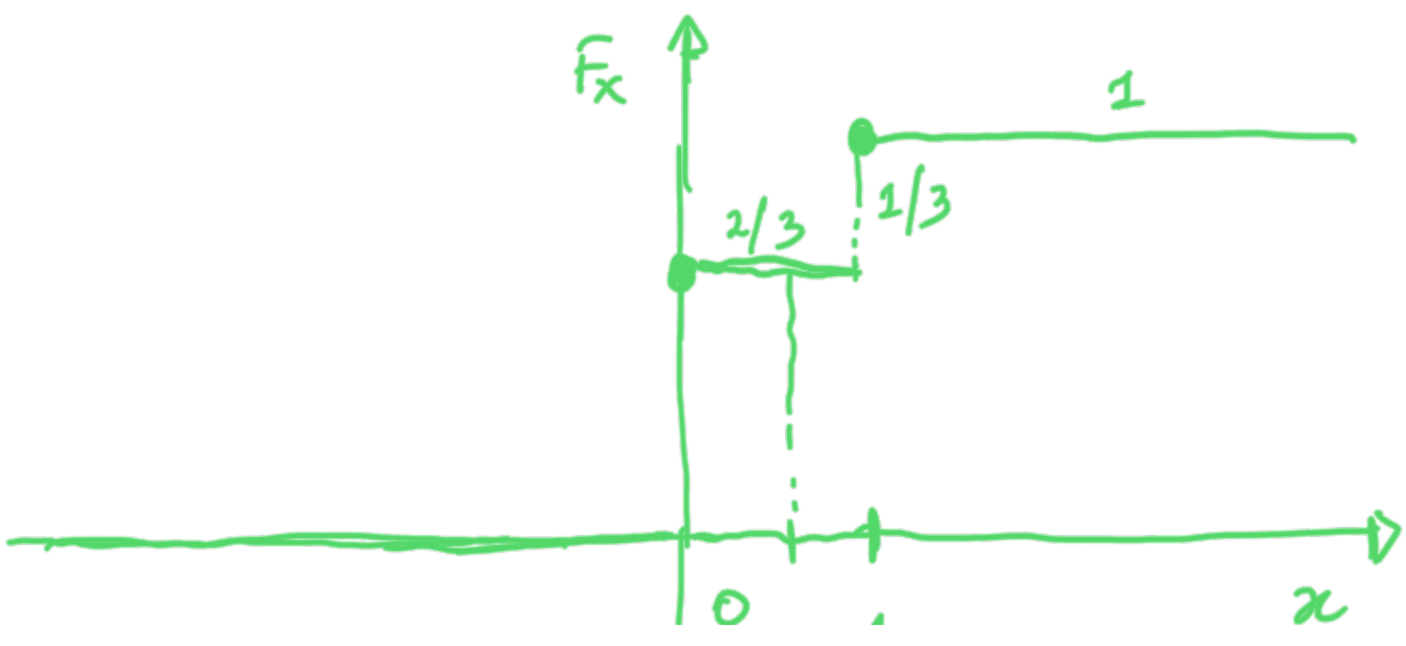
<u>ESEMPIO</u>

$X \sim B(1, 1/3)$ (legge di Bernoulli di parametro $p = 1/3$)

$S_X = \{0, 1\}$

$$\mathbb{P}(X=0) = q = 1-p = 2/3,$$

$$\mathbb{P}(X=1) = p = 1/3$$



$$x \in \mathbb{R}, \qquad F_X(x) = \sum_{i=0}^{1} 1_{\{i \le x\}} \cdot \mathbb{P}(X = i)$$

$$F_X(1/2) = \mathbb{P}(X=0) = 2/3;$$

$$F_X(-1) = 0;$$

$$F_X(4) = 1 \cdot \mathbb{P}(X=0) + 1 \cdot \mathbb{P}(X=1) = 1;$$

$$F_X(1{,}2) = 1 \cdot \mathbb{P}(X=0) + 1 \cdot \mathbb{P}(X=1) = 1;$$

$$\mathbb{P}(X=1) = F_X(1) - F_X(1^-) = 1 - 2/3 = 1/3;$$

$$\mathbb{P}(X=0{,}3) = F_X(0{,}3) - F_X(0{,}3^-) = 2/3 - 2/3 = 0.$$

$\square$

ESEMPIO

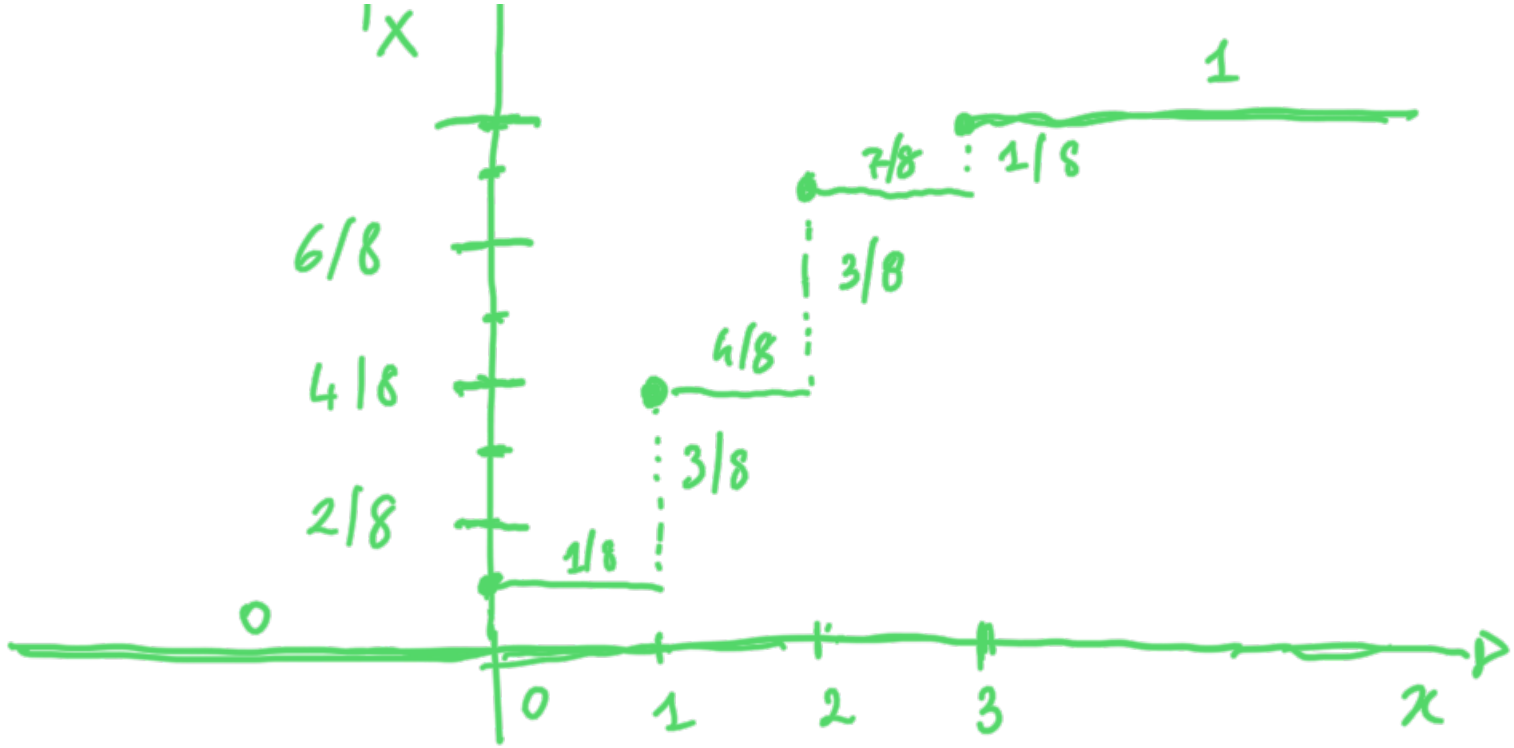
$$X \sim B\left(3, 1/2\right)$$

(legge binomiale di parametri $p = 1/2$ e $n = 3$)

$$S_X = \{0, 1, 2, 3\}$$

$$\begin{cases} \mathbb{P}(X=0) = 1/8 = \binom{3}{0} \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^3, \\ \mathbb{P}(X=1) = 3/8 = \binom{3}{1} \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^2, \\ \mathbb{P}(X=2) = 3/8 = \binom{3}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right), \\ \mathbb{P}(X=3) = 1/8 = \binom{3}{3} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^0, \end{cases}$$

$$x \in \mathbb{R}, \quad F_X(x) = \sum_{i=0}^{3} 1_{\{i \le x\}} \qquad \mathbb{P}(X=i)$$



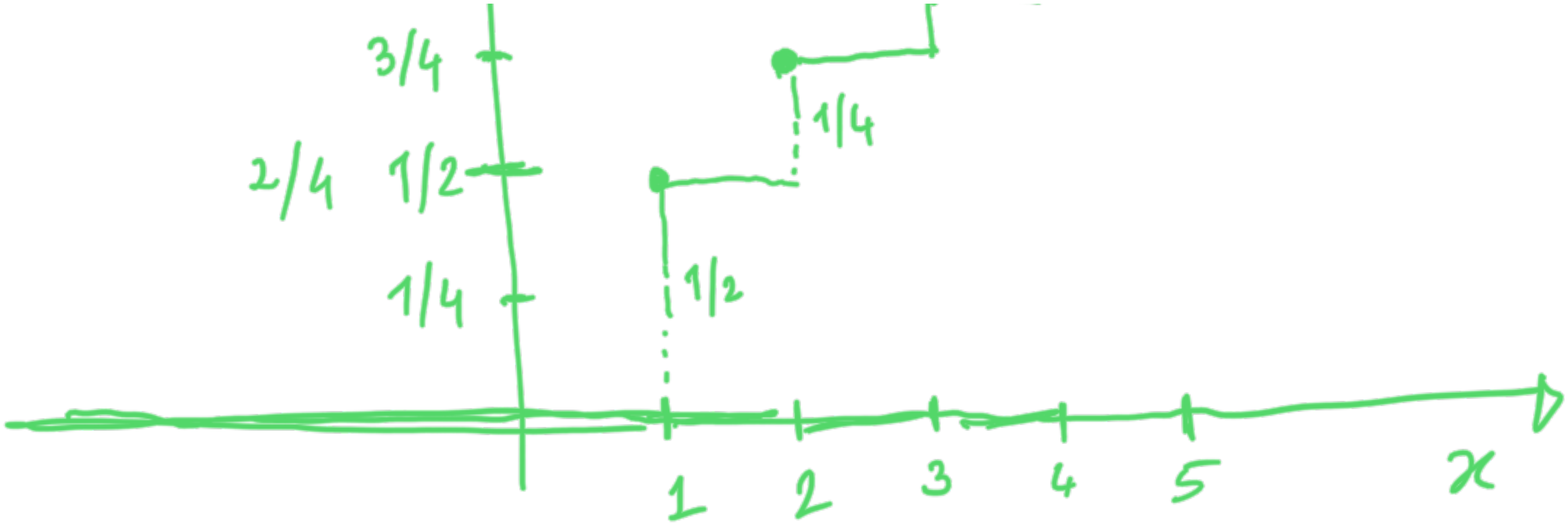
$\square$

ESEMPIO

$X \sim G(1/2)$ (legge geometrica di parametro $p = 1/2$)

$$S_X = \mathbb{N} = \{1, 2, 3, \dots\}$$

$n \in \mathbb{N}$, $\mathbb{P}(X=n) = q^{n-1} \cdot p = p\,(1-p)^{n-1}$



$$x \in \mathbb{R}, \qquad F_X(x) = \sum_{i=1}^{\infty} 1_{\{i \le x\}} \mathbb{P}(X=i).$$

$\square$

Sia $(\Omega, \mathcal{F}, \mathbb{P})$ uno spazio di probabilità e sia

$$X : \Omega \xrightarrow{\mathcal{F}\text{-mis}} \mathbb{R}$$
$$\omega \longmapsto X(\omega)$$

un numero aleatorio.

$(\mathbb{R}, \mathcal{B}, \mathbb{P}_X)$

$\sigma\left(\{]-\infty, x]\}_{x\in\mathbb{R}}\right)$

Si definisce la funzione di distribuzione di X ponendo:

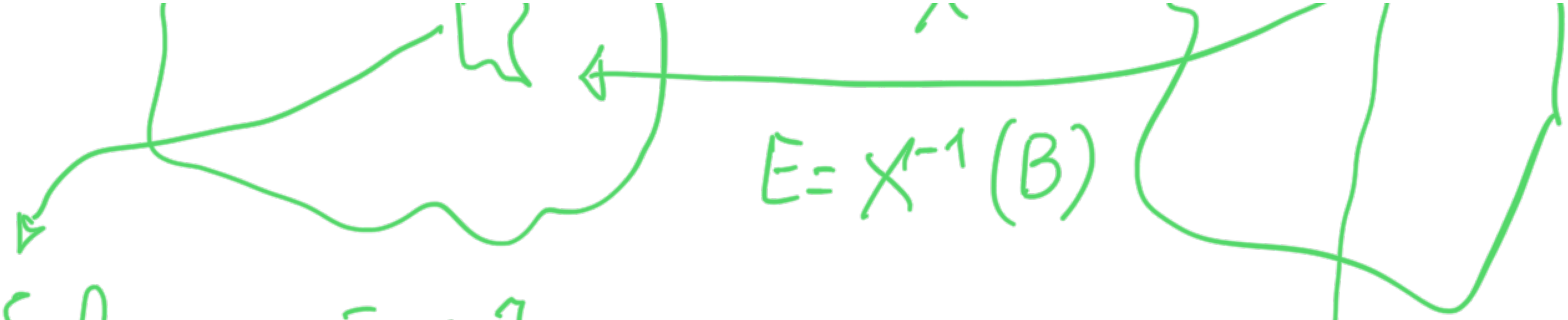
$$
\begin{aligned}
F_X : \ \mathbb{R} &\longrightarrow [0,1] \\
x &\longrightarrow F_X(x) = \mathbb{P}(X \leq x)
\end{aligned}
$$

$$\{X \leq x\} = \{\omega \in \Omega : X(\omega) \leq x\} \subseteq \Omega$$

$\{X \leq x\} \in \mathcal{F}$ in quanto:

$$\{X \leq x\} = X^{-1}(]-\infty, x]) \in \mathcal{F}$$

per la $\mathcal{F}$-misurabilità di X.



$E \subseteq \Omega \,, \quad E \in \mathcal{F}$

B è una semiretta sinistra chiusa qualsiasi

DIMOSTRAZIONE DELLE PROPRIETÀ

a) F_X è non decrescente

$x_1 < x_2$

$$F_X(x_2) = \mathbb{P}(X \leq x_2) = \mathbb{P}\left[\{X \leq x_1\} \cup \{x_1 < X \leq x_2\}\right]$$

$$= \mathbb{P}(X \leq x_1) + \mathbb{P}(x_1 < X \leq x_2)$$

$$\doteq \mathbb{P}(\underbrace{X \le x_1}_{\ge 0}) = \underbrace{F_X(x_1)}_{\ge 0}.$$

In definitiva

$$x_1 < x_2 \Rightarrow F_X(x_1) \le F_X(x_2).$$

$\square$