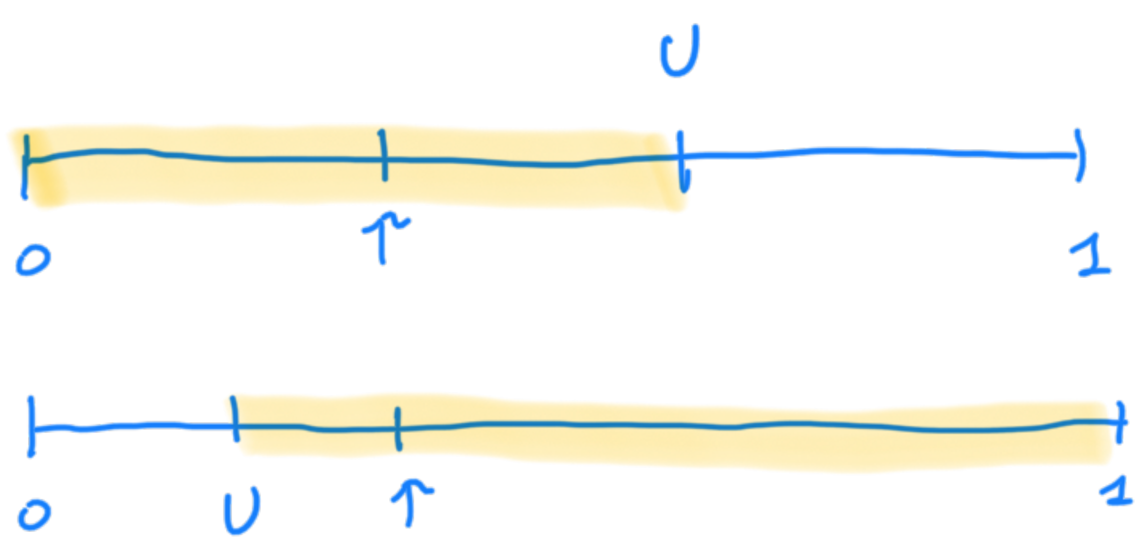
Si può procedere per altra via utilizzando la seguente formula:

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) \cdot f_X(x)\, dx.$$

ESEMPIO 5.2 e

$$U : \qquad f_U(u) = \begin{cases} 1, & 0 < u < 1 \\ 0, & \text{altrimenti} \end{cases}$$



$$g \equiv L_r : (0,1) \longrightarrow (0,1)$$

$$u \longrightarrow L_r(u) = \begin{cases} u, & u \geq r, \\ 1-u, & u < r. \end{cases}$$

$$\mathbb{E}\left[L_r(U)\right] = \int_{-\infty}^{+\infty} L_r(u)\, f_U(u)\, du$$

$$= \int_0^1 L_r(u)\, du = \int_0^r (1-u)\, du + \int_r^1 u\, du$$

$$= \int_0^r du - \int_0^r u\, du + \int_r^1 u\, du = u\Big|_0^r - \frac{u^2}{2}\Big|_0^r + \frac{u^2}{2}\Big|_r^1$$

$$= r - \frac{r^2}{2} + \left(\frac{1}{2} - \frac{r^2}{2}\right) = \frac{1}{2} + r - r^2$$

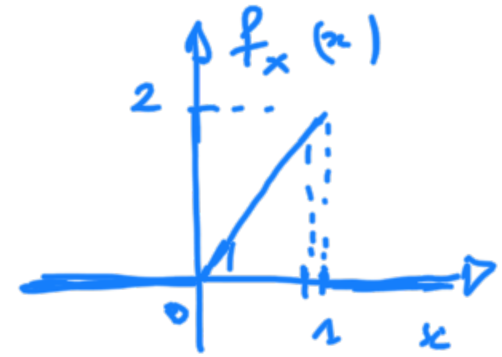
$$= \frac{1}{2} + r(1-r).$$

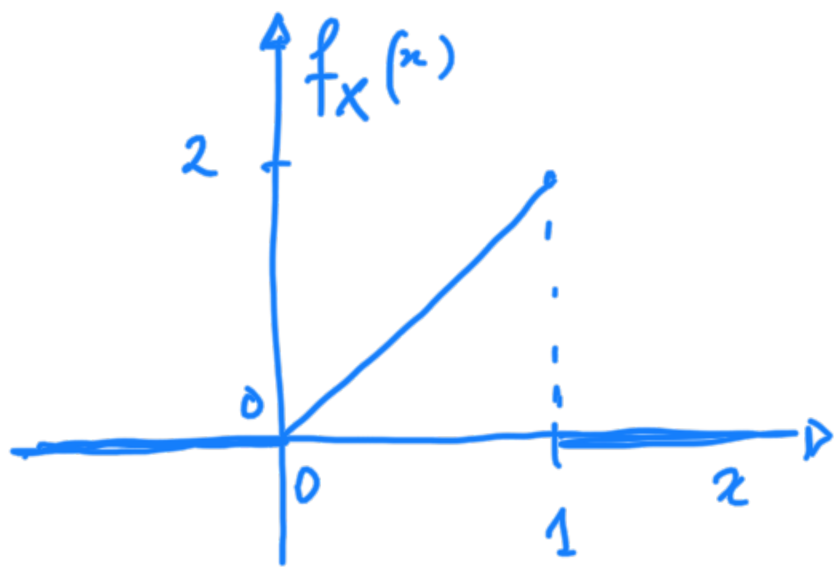
$\square$

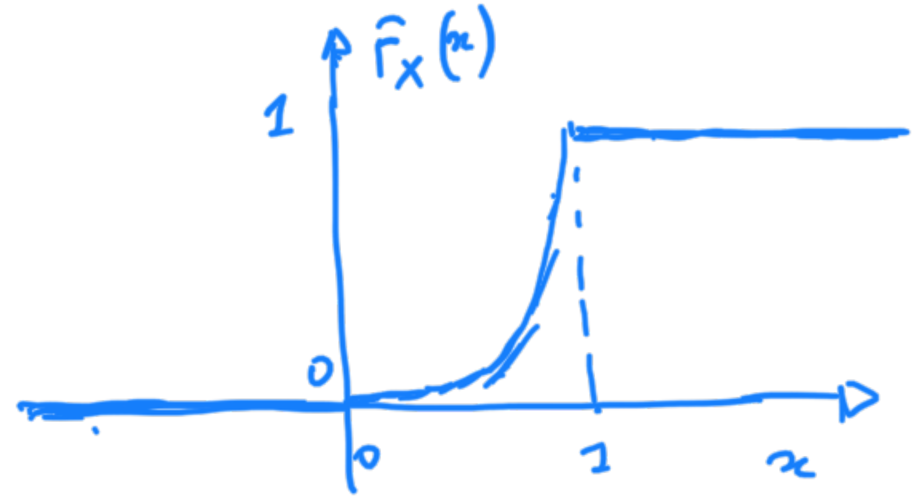
ESEMPIO 5.2e (cont. 5.2a)

X continua e avente come f.d.p.

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1; \\ 0, & \text{altrimenti.} \end{cases}$$







$f_X(x) \geq 0$ in $\mathbb{R}$; $\displaystyle\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_0^1 2x = x^2\Big|_0^1 = 1.$

$\mu_X = \mathbb{E}(X) = \displaystyle\int_{-\infty}^{+\infty} x \cdot f_X(x) = \int_0^1 x \cdot 2x \, dx$

$$= 2\int_0^1 x^2\,dx = 2\,\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}(1-0)$$

$$= \frac{2}{3}.$$

$$\mu_2' = \mathbb{E}(X^2) = \int_{-\infty}^{+\infty} x^2 \cdot f_X(x)\,dx = \int_0^1 x^2\, 2x\,dx$$

$$= 2\int_0^1 x^3\,dx = 2\,\frac{x^4}{4}\Big|_0^1 = \frac{2}{4} = \frac{1}{2}.$$

$$\mathbb{D}(X^2) = \mathbb{E}\left[(X-\mu_X)^2\right] = \mathbb{E}(X^2) - \mathbb{E}^2(X)$$

$$= \mu_2' - \mu_X^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}.$$

PROPOSIZIONE

$$\mathbb{E}(aX+b) = a\,\mathbb{E}(X)+b.$$

$$\mu_{aX+b} = a\,\mu_X + b.$$

PROPOSIZIONE

$$\mathbb{D}^2(aX+b) = \mathbb{D}^2(aX) = a^2\,\mathbb{D}^2(X).$$

DIM

$$\mathbb{D}^2(aX+b) = \mathbb{E}\left\{\left[(aX+b)-\mu_{aX+b}\right]^2\right\}$$

$$= \mathbb{E}\left\{\left[(aX+b)-(a\mu_X+b)\right]^2\right\}$$

$$= \mathbb{E}\left[(aX+b-a\mu_X-b)^2\right]$$

$$= \mathbb{E}\left[(aX-a\mu_X)^2\right] = \mathbb{E}\left[a^2(X-\mu_X)^2\right]$$

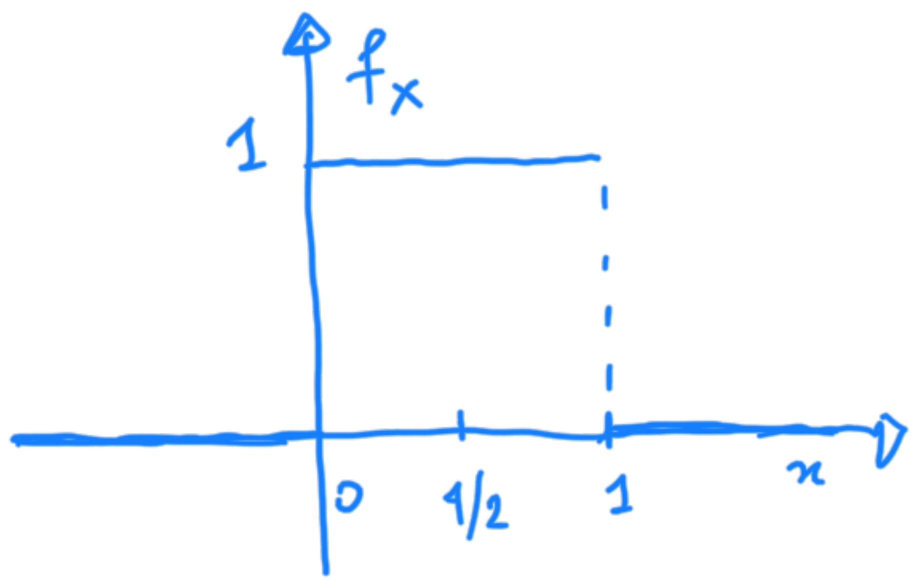
$$= a^2 \ \mathbb{E}\left[(X-\mu_X)^2\right] = a^2 \ \mathbb{D}^2(X).$$

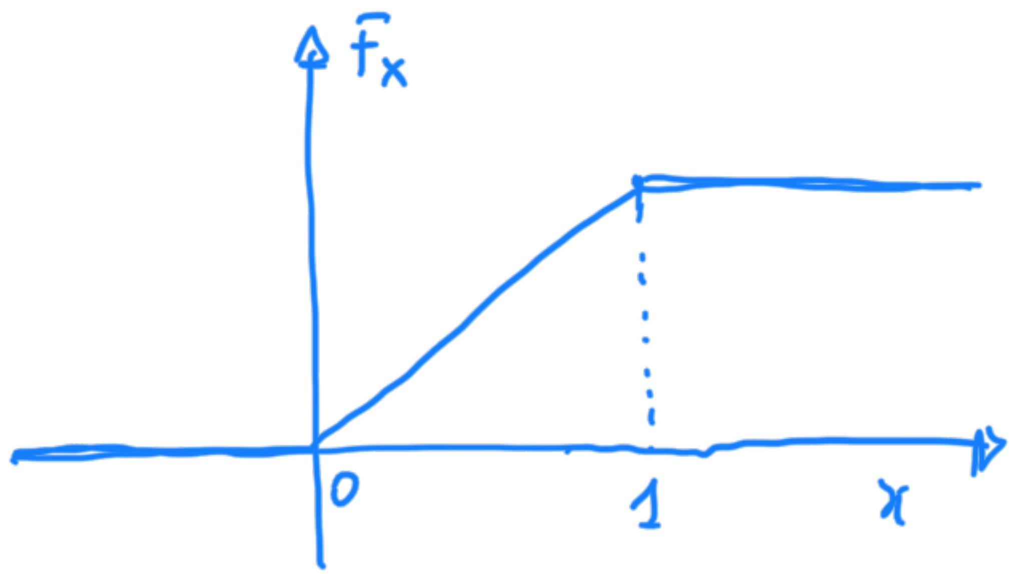
$\square$

LEGGI UNIFORMI

Una v. a. continua X la cui legge ha la seguente f. d. p.

$$f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{altrimenti}, \end{cases}$$





si dice uniforme nell' intervallo $(0,1)$ e si scrive:

$$X \sim U(0,1).$$

Risulta:

$$\eta_X = \mathbb{E}(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \int_0^1 x\, dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}.$$

Risulta:

$$\eta_2' = \mathbb{E}(X^2) = \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

Allora:

$$\mathbb{D}^2(X) = \eta_2' - \eta_X^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

<u>PROPOSIZIONE</u>

$X \sim U(0,1)$ e

$a < b$ $\qquad Y = a + (b-a)X$

allora

$$f_Y(y) = \begin{cases} \frac{1}{b-a}, & a < y < b, \\ 0, & \text{altrimenti.} \end{cases}$$

La v.a. Y si dice che è distribuita uniformemente nell'intervallo (a, b).

<u>DIM</u>

$x \in \mathbb{R}$, $\quad F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$

$y \in \mathbb{R}$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}[a + (b-a)X \leq y]$$

$$= \mathbb{P}[(b-a)X \leq (y-a)]$$

$$= \mathbb{P}\left(X \leq \frac{y-a}{b-a}\right)$$

$$= F_X\left(\frac{y-a}{b-a}\right) \quad \Rightarrow$$

per derivazione

$$f_Y(y) = f_X\left(\frac{y-a}{b-a}\right) \cdot D_y\left(\frac{y-a}{b-a}\right)$$

$$= f_X\left(\frac{y-a}{b-a}\right)\frac{1}{b-a}.$$

Se

$$y \leq a \iff \frac{y-a}{b-a} < 0 \quad \Rightarrow \quad f_Y(y) = 0$$

$$y \geq b \iff \frac{y-a}{b-a} > 1 \quad \Rightarrow \quad f_Y(y) = 0$$

$$a < y < b \iff 0 < \frac{y-a}{} < 1 \quad \Rightarrow \quad f_Y(y) = \frac{1}{}.$$

$b - a$ $\qquad\qquad\qquad\qquad$ y $\qquad\qquad$ $b-a$

$\square$

PROPOSIZIONE

Se $\quad Y \sim U(a,b)$ $\quad$ allora

$$\mathbb{E}(Y) = \frac{a+b}{2}$$

$$\mathbb{D}^2(Y) = \frac{(b-a)^2}{12}.$$

DIM

$$\mathbb{E}(Y) = \mathbb{E}\left[a + (b-a)X\right] = a + (b-a)\,\mathbb{E}(X)$$

$$= a + \frac{b-a}{2} = \frac{2a+b-a}{2} = \frac{a+b}{2}.$$

$$\mathbb{D}^2(Y) = \mathbb{D}^2\left[a + (b-a)X\right] = \mathbb{D}^2\left[(b-a)X\right]$$

$$= (b-a)^2\,\mathbb{D}^2(X) = \frac{(b-a)^2}{12}.$$

$\square$

ESEMPIO 5.3.e

Una linea di autobus arriva sulla fermata di un utente alle

7:00 7:15 7:30 7:45

L' utente in questione arriva alla fermata tra le 7:00 e le 7:30 a caso.

Determinare la probabilità che egli aspetti un autobus:

(a) meno di 5 minuti,

(b) più di 10 minuti.

Soluzione

X il tempo dopo le ore 7:00 ha

f.d.p.

$$f_X(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30. \\ 0, & \text{altrimenti.} \end{cases}$$

(a) $\mathbb{P}(\{10 < X < 15\} \,\dot\cup\, \{25 < X < 30\})$

$$= \mathbb{P}(10 < X < 15) + \mathbb{P}(25 < X < 30)$$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$$

$$= \frac{1}{30}\left[(15-10) + (30-25)\right] = \frac{10}{30} = \frac{1}{3}.$$

(b) $\mathbb{P}(\{0 < X < 5\} \,\dot\cup\, \{15 < X < 20\})$

$$= \frac{5}{30} + \frac{5}{30} = \frac{10}{30} = \frac{1}{3}.$$

$\square$

<u>ESEMPIO 5.3b</u>

$$X \sim U(0, 10)$$

(a) $$\mathbb{P}(X < 3) = \int_0^3 \frac{1}{10}\, dx = \frac{1}{10}(3-0) = \frac{3}{10}.$$

(b) $$\mathbb{P}(X > 6) = \int_6^{10} \frac{1}{10}\, dx = \frac{4}{10}.$$

(c) $$\mathbb{P}(3 < X < 8) = \int_3^8 \frac{1}{10}\, dx = \frac{8-3}{10} = \frac{5}{10}.$$

$\square$