$\underline{y} = (y_1, y_2, \ldots, y_n)$ rilevazione dei dati

$$x_1, x_2, \ldots, x_k$$

$$n_1, n_2, \ldots, n_k$$

$$\frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{n} (n_1 \cdot x_1 + n_2 \cdot x_2 + \cdots + n_k x_k)$$

$$= x_1 \cdot \frac{n_1}{n} + x_2 \frac{n_2}{n} + \cdots + x_k \frac{n_k}{n}$$

Definizione di probabilità frequentista

$$\mathbb{P}_f(A) = \text{'}\lim_n\text{'} \frac{n_A}{n}$$

$$\mathbb{P}_f(A) \approx \frac{n_A}{n}.$$

Sia ora X una v.a. con spettro

$$S_X = \{x_1, x_2, \dots, x_k\}$$

e

$$x_j \in S_X, \quad \mathbb{P}(X = x_j) = p_j$$

$$\frac{n_1}{n} \approx p_1, \dots, \frac{n_k}{n} \approx p_k$$

Allora,

$$\mathbb{E}(X) = x_1 \cdot p_1 + x_2 p_2 + \dots + x_k p_k$$

$$= \sum_{j=1}^{k} x_j \cdot p_j = \sum_{j=1}^{k} x_j \, \mathbb{P}(X = x_j).$$

<u>ESEMPIO</u>

$$X \sim B(1, p)$$

$$\boxed{\mathbb{E}(X) = p}$$

$$S_X = \{0, 1\}$$

$$p_1 = \mathbb{P}(X=0) = 1-p, \quad p_2 = \mathbb{P}(X=1) = p.$$

Risulta

$$\mathbb{E}(X) = 0 \cdot (1-p) + 1 \cdot p = p$$

$$\mathbb{E}(X^2) = 0^2 \cdot (1-p) + 1^2 \cdot p = p. \qquad \square$$

ESEMPIO

$S_n \sim X \sim B(n, p)$ $\qquad \boxed{\mathbb{E}(X) = n \cdot p}$

$$S_X = \{0, 1, \dots, n\}$$

$$J = (0, 1, \dots, n), \quad \mathbb{P}(X = J) = \underbrace{\binom{n}{J} p^J (1-p)^{n-J}}$$

$$\mathbb{E}(X) = \sum_{J=0}^{n} J \binom{n}{J} p^J (1-p)^{n-J}$$

$$= \sum^{n} J \frac{n!}{J!(n-J)!} p^J (1-p)^{n-J}$$

$$j=1$$

$$= \sum_{j=1}^{n} \frac{\cancel{j}\; n!}{\cancel{j}\,(j-1)!\,(n-j)!}\, p^{j} (1-p)^{n-j}$$

$$= n \sum_{j=1}^{n} \frac{(n-1)!}{(j-1)!\,(n-j)!}\, p^{j} (1-p)^{n-j}$$

$$= np \sum_{j=1}^{n} \binom{n-1}{j-1} p^{j-1} (1-p)^{n-j}$$

$\ell = j-1 \Rightarrow j = \ell+1 \,, \quad n-j = n-(\ell+1) = (n-1)-\ell$

$$= np \underbrace{\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} p^{\ell} (1-p)^{(n-1)-\ell}}_{=1}$$

si riferisce a una

legge $B(n-1, p)$

$$= n\,p\left[p + (1-p)\right]^{n-1}$$

$$= n\,p \cdot 1^{n-1} = n \cdot p$$

In alternativa,

$$S_n = X_1 + X_2 + \dots + X_n \quad \text{con } \underbrace{X_i \sim B(1,p)}$$

$$\mathbb{E}(S_n) = \mathbb{E}(X_1 + X_2 + \dots + X_n)$$

$$= \mathbb{E}(X_1) + \mathbb{E}(X_2) + \dots + \mathbb{E}(X_n)$$

$$= p + p + \dots + p$$

$$= n\,p.$$

Utilizzando il simbolo di sommatoria

$$\mathbb{E}(S_n) = \mathbb{E}\left(\sum^{n} X_i\right) = \sum^{n} \mathbb{E}(X_i) = \sum^{n} p$$

$$\mathbb{E}(S_n) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i) = \sum_{i=1}^{n} p$$

$$= p \sum_{i=1}^{n} 1 = n\, p.$$

PROPRIETÀ

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

$a \in \mathbb{R}$,
$$\mathbb{E}(X+a) = \mathbb{E}(X) + \mathbb{E}(a)$$
$$= \mathbb{E}(X) + a$$

$b \in \mathbb{R}$
$$\mathbb{E}(bX) = \sum_{j=1}^{k} b\, x_j\, \mathbb{P}(X = x_j)$$
$$= b \sum_{j=1}^{k} x_j\, \mathbb{P}(X = x_j)$$
$$= b\, \mathbb{E}(X).$$

$a, b \in \mathbb{R}$
$$\mathbb{E}(aX + bY) = \mathbb{E}(aX) + \mathbb{E}(bY)$$

$$= a\,\mathbb{E}(X) + b\,\mathbb{E}(Y)$$

$a_1, \dots, a_n \in \mathbb{R}$, $X_1, \dots, X_n$ v.a.,

$$\mathbb{E}\left(\sum_{m=1}^{n} a_m X_m\right) = \sum_{m=1}^{n} a_m\,\mathbb{E}(X_m).$$

$\square$

ESEMPIO

$$\boxed{\mathbb{E}(X) = 1/p}$$

$$T_1 \sim X \sim G(p)$$

$$S_X = \{1, 2, \dots\} = \mathbb{N}$$

$j \in \mathbb{N}$, $\quad \mathbb{P}(X = j) = (1-p)^{j-1} \cdot p.$

Allora

$$E(T) = \mathbb{E}(X) = \sum_{j=1}^{\infty} j \cdot (1-p)^{j-1} \cdot p$$

$$= p \sum_{j=1} j\,(1-p)^{j-1} = \dots = \frac{1}{p}.$$

dimostrazione rimandata $\square$

<u>ESEMPIO</u>

$$\mathbb{E}(X) = \lambda$$

$$X \sim \Pi(\lambda)$$

$$S_X = \{0, 1, \dots\} = \mathbb{N}_0$$

$$j \in \mathbb{N}_0, \quad \mathbb{P}(X=j) = e^{-\lambda}\frac{\lambda^j}{j!}$$

$$\mathbb{E}(X) = \sum_{j=0}^{\infty} j\, e^{-\lambda}\frac{\lambda^j}{j!} = e^{-\lambda}\sum_{j=0}^{\infty} j\,\frac{\lambda^j}{j!}$$

$$= e^{-\lambda} \sum_{j=1}^{\infty} j \frac{\lambda^j}{j!} = e^{-\lambda} \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!} \cdot \lambda$$

$\ell = j - 1$

$$= e^{-\lambda} \lambda \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} = e^{-\lambda} \lambda \, e^{\lambda}$$

$$= \lambda \underbrace{\sum_{\ell=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^\ell}{\ell!}}_{=1} = \lambda .$$

si riferisce alla stessa legge $\pi(\lambda)$

$\square$

Sviluppo in serie di e^{λ}

$$\sum^{\infty} \frac{\lambda^\ell}{\ldots} = 1 + \frac{\lambda}{\ldots} + \frac{\lambda^2}{\ldots} + \frac{\lambda^3}{3!} + \ldots = e^{\lambda}$$

$$\ell=0 \quad \ell! \qquad 0! \qquad 1! \qquad 2! \qquad 3!$$

In particolare

$$e^{-1} = \underbrace{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{40!}} + \dots$$

$$e^{2} \simeq 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots$$

In generale per ottenere la somma A della serie di termine generale a_i

(a) si sommano i primi n addendi per ottenere la somma parziale

$$A_n = \sum_{i=1} a_i ,$$

(b) si determina il limite delle successione $(A_n)_{n\in\mathbb{N}}$

$$\lim_{n\to\infty} A_n$$

(c) se il limite in (b) appartiene a $\mathbb{R}$ allora $A = \lim_{n\to\infty} A_n$,

(d) se $\lim_n A_n = +\infty$ oppure

$$\lim_n A_n = -\infty$$

allora si dice che la serie è divergente

(e) se $\lim A_n$ non esiste allora

$n \to \infty$

si dice che la serie è non regolare.