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Option INGÉNIERIE FINANCIÈRE. Option ACTUARIAT.

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MODEL CALIBRATION AND SIMULATION

Calibration of Vasicek model

Practical Lab 3

Term structures of interest rates describe the relation between interest rates and bonds with different maturity times. The Vasicek model - is so called short rate model, as it models the dynamics of the instantaneous short rate r_t directly. The model elaborates also the dynamic evolution of the yield curve. The short rate is the annualized interest rate for an infinitesimally short period of time. The short rate r_t is defined as

$$r_t = lim_{t \to T} Y(t, T)$$

where t denotes a moment in time. Let $P(t, T)$ denote the value of a zero coupon bond at time t that pays one at maturity time T and $Y(t, T)$ the corresponding Yield curve. In continuous time we then find:

$$P(t, T) = e^{-Y(t,T)(T-t)}$$

Rewriting this formula gives a way to describe the interest rate as a function of the value of a bond:

$$Y(t, T) = -\frac{\log(P(t, T))}{(T - t)}$$

Since $P(t,T)$ is a simple discount factor, it is clear that $Y(t,T)$ may not be negative to make sure the discount factor lies between zero and one. Let us discuss the model that builds on this introduction, the Vasicek model. Vasicek defines the short rate process as the stochastic differential equation

$$dr_t = (\eta - \gamma r_t)dt + \sigma dW_t$$

where W_t is a standard Brownian motion, with solution

$$r_t = r_0 e^{-\gamma t} + \frac{\eta}{\gamma}(1 - e^{-\gamma t}) + \sigma \int_0^t e^{-\gamma(t-s)} dW_s.$$

The law of r_t is Gaussian at all times t, with mean

$$E[r_t] = r_0 e^{-\gamma t} + \frac{\eta}{\gamma}(1 - e^{-\gamma t})$$

and variance

$$Var[r_t] = \sigma^2 \int_0^t (e^{-\gamma(t-s)})^2 ds = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}).$$

In case the short term interest rate process r_t is a deterministic function of time, a standard arbitrage argument shows that the price $P(t,T)$ of the bond is given by

$$P(t,T) = e^{-\int_t^T r_s ds}$$

In case r_t is an F_t -adapted random process we replace the formula with

$$P(t,T) = E_Q[e^{-\int_t^T r_s ds} | F_t]$$

where Q is the risk neutral measure. The discounted bond price process

$$t \to e^{-\int_0^t r_s ds} P(t,T)$$

is a martingale under Q. Since all solutions of stochastic differential equations have the Markov property the arbitrage price $P(t,T)$ can be rewritten as a function $P(t,r)$ of r and t. It can be shown that the pricing function of a zero-coupon paying bond in the Vasicek model verifies the following partial differential equation:

$$\begin{cases} \frac{\partial P(r,t;T)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} + (\eta - \gamma r)\frac{\partial P}{\partial r} - rP = 0 \\ P(r,T;T) = 1 \end{cases}$$

This equation has the exact solution which can be expressed in the exponential form:

$$\begin{cases} P(r,t;T) = e^{A(t,T) - r \cdot B(t,T)} \\ \tau = T - t \\ B(t,T) = \frac{1 - e^{-\gamma\tau}}{\gamma} \\ A(t,T) = (B(t,T) - \tau)\frac{\eta\gamma - \sigma^2/2}{\gamma^2} - \frac{\sigma^2 B^2(t,T)}{4\gamma} \end{cases}$$

Define the Yield curve which is the measure of future values of interest rate

$$Y(t,T) = -\frac{\log P(r,t;T)}{T - t}$$

The time interval $T - t$ is called the time to maturity.

Part I

Plot the graph

$$T \to Y(0,T) \quad for \quad \gamma = 0.25, \eta = 0.25 \cdot 0.03, \sigma = 0.02$$

$$Y(0,T) = -\frac{\log P(r,0;T)}{T} = -\frac{A(0,T) - r_0 \cdot B(0,T)}{T}$$

Show that the future term structure produced by Vasicek model may assume three different curve shapes:

a) Asymptotically upward sloping (typical) $r_0 = 0.01$

b) Slightly humped $r_0 = 0.027$

c) Inverted $r_0 = 0.05$

Verify que

$$\lim_{T\to 0} Y(0,T) = r_0, \lim_{T\to\infty} Y(0,T) = \frac{\eta}{\gamma} - \frac{1}{2}(\frac{\sigma}{\gamma})^2$$

Part II Calibration to Yield curve

We will calibrate the model according the Yield curve came from the market (from the prices of Bonds) at the moment $t = 0$.

i	1	2	3	4	5	6	7	8	9	10
maturité T_i	3	6	9	12	15	18	21	24	27	30
yield Y_i^{market}	0.035	0.041	0.0439	0.046	0.0484	0.0494	0.0507	0.0514	0.052	0.0523

Our aim is to fit the data to the curve of the form:

$$Y(t,T)|_{t=0} = -\frac{\log P(t,T)}{T-t}|_{t=0} = -\frac{A(0,T) - r_0 \cdot B(0,T)}{T}$$

We are looking for 3 parameters:

$$\beta^1 \equiv \eta, \quad \beta^2 \equiv \sigma^2, \quad \beta^3 = \gamma$$

We apply the non-linear least squares Levenberg-Marquart's algorithm to find β_{min} such as

$$\left\{ \min_\beta \Phi = \sum_{p=1}^{m=10} (Y_p^{market} - Y(\beta, T_p))^2, \qquad Y(T_p, \beta) = -\frac{A(0,T_p,\beta) - r_0 \cdot B(0,T_p,\beta)}{T_p} \right.$$

We need the following derivatives of the Yield curve in respect of the parameters of the model:

$$\frac{\partial B(t,T)}{\partial \gamma} = \frac{\tau e^{-\gamma\tau} - B}{\gamma}$$

$$\frac{\partial A(t,T)}{\partial \gamma} = \frac{1}{\gamma^2}\{\eta(\frac{\partial B}{\partial \gamma}\gamma - B) + \tau\eta - \frac{\sigma^2}{2}(\frac{\partial B}{\partial \gamma} - 2\frac{B}{\gamma}) - \frac{\tau\sigma^2}{\gamma} - \frac{\sigma^2 B}{4}(2\gamma\frac{\partial B}{\partial \gamma} - B)\}$$

Levenberg-Marquart

- Choose $\beta_0^i, \quad i = 1, 2, 3, \qquad \varepsilon.$

$\beta^1 \equiv \eta = 1, \qquad \beta^2 \equiv (\sigma)^2 = 1, \qquad \beta^3 \equiv \gamma = 1, \qquad r_0 = 0.023, \quad \varepsilon = 10^{-9}, \quad \lambda = 0.01$

- While $||\beta_{k+1} - \beta_k|| > \varepsilon$ do
- Set $(Res)_p = Y_p^{market} - Y(T_p, \beta)$
- Calculate $J_{pj} = \frac{\partial (Res)_p}{\partial \beta^j}$,
- Calculate

$$d_k = -(J^T\, J + \lambda \cdot I)^{-1}(\beta_k) \cdot\ J^T(\beta_k) \cdot\ Res(\beta_k)$$

$$\beta_{k+1} = \beta_k + d_k$$

- End while

We calculate a Jacobien $J_{pj}, \quad p = 1...10, \quad j = 1...3$, the matrix 10×3

$$J_{pj} = \frac{\partial (Res)_p}{\partial \beta^j} \equiv -\frac{\partial Y(T_p, \beta)}{\partial \beta^j}$$

$$J(p,1) = \frac{\partial (Res)_p}{\partial \beta^1} \equiv -\frac{\partial Y(T_p, \beta)}{\partial \eta} = \frac{B - \tau}{\tau\gamma}$$

$$J(p,2) = \frac{\partial (Res)_p}{\partial \beta^2} \equiv -\frac{\partial Y(T_p, \beta)}{\partial (\sigma^2)} = -\frac{1}{\tau\gamma}(\frac{(B-\tau)}{2\gamma} + \frac{B^2}{4})$$

$$J(p,3) = \frac{\partial (Res)_p}{\partial \beta^3} \equiv -\frac{\partial Y(T_p, \beta)}{\partial \gamma} = \frac{1}{\tau}(\frac{\partial A(t,T_p)}{\partial \gamma} - r_0\frac{\partial B(t,T_p)}{\partial \gamma})$$

Part III. Work to do

1. Programmer the algorithm.
2. Find $\beta_1, \beta_2, \beta_3$
3. Plot Y_p^{market}, $Y(T_p, \beta)$ and comparer.
4. Recalibrate the yield curve if in one Year $(t = 1)$ the interest rate $r_t = 0.04$ and we know the Yield curves came from the market at the moment $t = 1$. Take in to account that the time to maturity is now $T - t = T - 1$.

i	1	2	3	4	5	6	7	8	9	10
maturité T_i	3	6	9	12	15	18	21	24	27	30
yield Y_i^{market}	0.056	0.064	0.074	0.081	0.082	0.09	0.087	0.092	0.0895	0.091

Part IV. Calibration to historical dates

We know that

$$r_t = r_0 e^{-\gamma t} + \frac{\eta}{\gamma}(1 - e^{-\gamma t}) + \sigma \int_0^t e^{-\gamma(t-s)} dW_s$$

The law of r_t is Gaussian at all times t, with mean

$$E[r_t] = r_0 e^{-\gamma t} + \frac{\eta}{\gamma}(1 - e^{-\gamma t})$$

and variance

$$Var[r_t] = \sigma^2 \int_0^t (e^{-\gamma(t-s)})^2 ds = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t})$$

It can be shown that for each t the random variable r_t follows normal distribution:

$$r_t = N(r_0 e^{-\gamma t} + \frac{\eta}{\gamma}(1 - e^{-\gamma t}), \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}))$$

It means one can simulate the interest rate using the following formula:

$$r_{i+1} = r_i e^{-\gamma \Delta t} + \frac{\eta}{\gamma}(1 - e^{-\gamma \Delta t}) + \sigma \sqrt{\frac{(1 - e^{-2\gamma \Delta t})}{2\gamma}} N(0,1)$$

Let us suppose that the market is governed by Vasicek process.

1. **Simulate yourself the market dates using the theoretical formula:**

$$r_{i+1} = r_i e^{-\gamma \Delta t} + \frac{\eta}{\gamma}(1 - e^{-\gamma \Delta t}) + \sigma \sqrt{\frac{(1 - e^{-2\gamma \Delta t})}{2\gamma}} N(0,1)$$

and trace the dates and the graphe: $t_i \to r_i$.

Identify :

$$x_i = r_i, \quad y_i = r_{i+1}$$

and plot the function $x_i \to y_i = f(x_i)$.

2. Fit the market dates to the linear function:

$$y = ax + b$$

Using the Levenberg - Marquart algorithm calculate the parameters a et b.

After that calculate the variance D^2 of random variable $r_{i+1} - (ar_i + b)$

$$D^2 = \frac{1}{N} \sum_{i=1}^{N} (r_{i+1} - (ar_i + b))^2.$$

3. We know that if the market follows the Vasicek model that

$$a = e^{-\gamma t}, \quad b = \frac{\eta}{\gamma}(1 - e^{-\gamma t}),$$

$$D = \sigma\sqrt{\frac{(1-e^{-2\gamma t})}{2\gamma}}.$$

Calculate

$$\gamma = -\frac{\ln a}{\Delta t}$$

$$\eta = \gamma\frac{b}{1-a}$$

$$\sigma = D \cdot \sqrt{\frac{-2\ln a}{\Delta t(1-a^2)}}$$

and compare with the parameters which you have used for simulation.

To begin take $T = 5, \eta = 0.6, \gamma = 4, \sigma = 0.08$.

Annexe.

- The forward rate $f(t,T,S)$ at time t for a loan on $[T,S]$ is given by

$$f(t,T,S) = \frac{lnP(t,T) - lnP(t,S)}{S-T},$$

$P(t,T)$ is the price of zero-coupon bond.

- Le taux zero-coupon $Y(t,T)$ s'appelle aussi "Spot forward rate" ou "Yield curve" à l'instant t

$$Y(t,T) = f(t,t,T)$$

- Le prix de zero-coupon bond à l'instant t est donné par la formule:

$$P(t,T) = e^{-(T-t)Y(t,T)}$$

- Le taux zero-coupon $Y(t,T)$ est lié avec "Instantaneous forward rate" $f(t,s)$

$$f(t,T) = lim_{S\to T}f(t,S,T)$$

$$Y(t,T) = \frac{\int_t^T f(t,s)ds}{T-t}$$

- "Instantaneous interest rate" $r_t = lim_{t\to T}f(t,T)$