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Option INGÉNIERIE FINANCIÈRE. Option ACTUARIAT.

MODEL CALIBRATION AND SIMULATION

TP2-b Simulation of Delta-hedging with constant transaction cost.

Part I. The Procedure of Dynamic Hedging

Consider a market in which a security is traded with a proportional transaction cost rate k_0. Assume that an agent sells a derivative security for V_0 with a payoff V_T depending only on the value of the underlying security at the expiration date T. The agent can use the amount V_0 to buy or sell the underlying asset in any amount to hedge such a financial contract without incurring any loss in any possible state at T. Let S_t be the price dynamics of the underlying security and assume that it follows the lognormal process

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S(t=0) = S_0 \tag{1}$$

where W_t is a standard Brownian motion and r is the interest rate. Hence, we confine our discussions of any statistical quantity in the risk neutral probability. The initial value of the hedging portfolio is

$$P_0 = V_0 - \kappa \cdot |A_0| S_0$$

. where A_0 is the initial holding of the underlying stock. Let A_0 is the initial holding of the underlying stock then the portfolio is decomposed in the following way:

$$P_0 = A_0 S_0 + (V_0 - A_0 S_0 - \kappa \cdot |A_0| S_0), B_0 = V_0 - A_0 S_0 - \kappa |A_0| S_0$$

At time t_i, the agents portfolio is

$$P_{t_i} = A_{t_i} S_{t_i} + B_{t_i}$$

where B_{t_i} is the amount in the bank account and A_{t_i} is the number of shares held at time t_i.

Assume the bank account earns a continuously compound rate of r per annum for both borrowing and lending. The agents goal is to maintain a portfolio that replicates the derivatives payoff V_T with a dynamic trading strategy. In any time t_{i+1}, the value of the portfolio is

$$P_{t_{i+1}} = A_{t_i} S_{t_{i+1}} + B_{t_i}(1 + r\Delta t) - \kappa \cdot |A_{t_{i+1}} - A_{t_i}| S_{t_{i+1}}.$$

In this expression the first term on the right hand side represents the profit/loss due to the change in the value of the underlying security, the second is the interest paid or received from the bank account, and the third is the transaction cost of trading.

The portfolio P_t is required to replicate the value of the derivative security $V(S_t, t)$ which is a function of the value of the underlying stock and time. When $\kappa = 0$ (no transaction costs), we can derive the Black-Scholes partial differential equation by letting the length of the rebalance interval approach zero and applying Ito's formula. The partial differential equation that $V(S, t)$ satisfies and the number of shares, A_t, held in the stock are

$$\begin{cases} \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \\ V(T, S) = \max(S - K, 0) \\ A(t, S) = \frac{\partial V}{\partial S} \end{cases} \tag{2}$$

For a call option struck at K and the solution of Black Scholes equation is given by the Black-Scholes formula

$$V^{BS}(S_i, t_i) = S_i N(d_1(S_i, t_i)) - Ke^{-r(T-t_i)} N(d_2(S_i, t_i))$$

$$d_1(S_i, t_i) = \frac{ln(S_i/K) + (r + \sigma^2/2)(T - t_i)}{\sigma\sqrt{T - t_i}} \qquad d_2(S_i, t_i) = \frac{ln(S_i/K) + (r - \sigma^2/2)(T - t_i)}{\sigma\sqrt{T - t_i}}$$

where $N(\Delta)$ is the standard normal cumulative function.

However, replication can become chaotic when transactions costs are present. Leland (1985) suggested an augmented volatility in the Black-Scholes formula to offset the hedging errors caused by the transaction costs. In his paper, the adjusted volatility was defined as

$$\widehat{\sigma} = \sigma\sqrt{1 + \frac{\kappa}{\sigma}\sqrt{\frac{2}{\pi\Delta t}}}$$

On compute the asset holding and the value of the option using modified volatility $\widehat{\sigma}$.

$$V^{BS}(S_i, t_i) = S_i N(d_1(S_i, t_i)) - Ke^{-r(T-t_i)} N(d_2(S_i, t_i))$$

$$A(S_i, t_i) = N(d_1(S_i, t_i))$$

$$d_1(S_i, t_i) = \frac{ln(S_i/K) + (r + \widehat{\sigma}^2/2)(T - t_i)}{\widehat{\sigma}\sqrt{T - t_i}} \qquad d_2(S_i, t_i) = \frac{ln(S_i/K) + (r - \widehat{\sigma}^2/2)(T - t_i)}{\widehat{\sigma}\sqrt{T - t_i}}$$

You are asked to present some simulation results to help understand the effect of transaction costs in option hedging. Suppose the underlying stock has a volatility 25% and the interest rate is 5% per annum. Let the rebalance intervals

$$\Delta t = 1/260, 1/520, 1/1040, 1/4160, 1/8320,$$

with initial value $S_0 = 100$. We carry out replications of call options with strike prices $K = 80, 90, 100, 110, 120$ for the case of transaction cost. You study the profit/loss

$$P\&L(S_0, T) = P(S_T, T) - V^{BS}(S_T, T)$$

We introduce the stoping times for Simulations 2.

Part II. Simulations 1

Perform the simulations for 4 differente cases:
1.$\Delta t = 1/260$ - rebalance on a daily base,$K = 100, \kappa = 0.001$
2.$\Delta t = 1/1040$- it means rebalance 4 times daily,$K = 100, \kappa = 0.001$
3.$\Delta t = 1/260$ - rebalance on a daily base, $K = 100, \kappa = 0.01$
4.$\Delta t = 1/1040$- it means rebalance 4 times daily, $K = 100, \kappa = 0.01$

1. Simulate and plot the evolution of portfolio and the evolution of the option value in the same coordinate system.
2. Plot the distribution and density functions of the profit/loss $P\&L$
3. Plot the distribution and density functions of the profit/loss $P\&L$ without of transaction cost to comparer.

Part II. Simulations 2

The optimality of $\widehat{\sigma}$ depends not only on the rate of the transaction cost k but also on the way of triggering hedging trades. In the deterministic case, the optimal $\widehat{\sigma}$ depends on the length of the rebalance interval. The finer the rebalance interval, the smaller the hedging error, but the larger the transactions costs. On the other hand, a move-based oriented approach can avoid enormous trading. Investors can choose triggering limits in the change of the underlying stock returns for portfolio rebalancing. The following setting is used in our study.

For $i = 0, 1, ..., n$

$$t_{i+1} = \min(T, t_i + \inf\{\tau > 0, \ln(\frac{S_{\tau+t_i}}{S_{t_i}}) \geq u \ or \ \leq -l, \ u = d = 0.01)$$

where $t_0 = 0$ and $(l, u) \geq 0$ are the decrement and increment thresholds for rebalancing in terms of the rate of return in the underlying asset. The hedging trade is triggered when the growth of the underlying asset is found outside of the interval $[-l, u]$. The advantage of using a move-based hedging strategy is that the hedging strategy is "continuously" implemented. The agent can make hedging trades by observing the movements of the underlying security without missing any "jumps" that may result in any catastrophes as in the case of discretely rebalancing strategies.

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