Introduction to Probability Theory and Statistics
Exam, August 2022

Probability

Problem 1
X and Y are random variables that can take on values 0 and 1. It is known that X is Bernoulli r.v. It is also known that $P(X = 0, Y = 0) = 3/8$ and $P(X = 1, Y = 1) = 1/3$.

(a) Write down the pmf of X.

(b) Find the joint pmf of X and Y.

(c) Write down the pmf of Y.

(d) Are X and Y independent?

(e) Calculate the mean and variance for r.v. X.

(f) Calculate the mean and variance for r.v. Y.

(g) Find $Cov(X, Y)$.

(h) Find $P(X > 1/2)$.

Problem 2
You are at an airport waiting for a plane. The plane can be delayed and the delay time for a plane at this airport is a continuous random variable X, which is known to be exponentially distributed with parameter $\lambda = 1/2$ (measured in 1/hour).

(a) Find the probability that your plane is delayed for more than 2 hours.

(b) How long on average people are waiting for their planes?

(c) If you are already waiting for an hour, what is the probability that you have to wait for another 3 hours?

Problem 3
There are two construction companies in the town that are building houses: company A and company B. Company A is known for building solid and high-quality houses and their customers are satisfied in 95% of cases. Company B does not have good reputation and their customer satisfaction rate is only 50%. People in the town are happy with company A and they often ask them to build their houses. 4/5 of houses are constructed by this company; while 1/5 of houses are built by company B.

You are speaking with a person who has recently had his house built and he is dissatisfied with the result. What is the probability that this house is built by company B?

Statistics

Problem 1

A new setup in a factory builds products automatically, but the process is not reliable, and a large number of products are defective. The engineers take a random sample of 100 products from the factory, and see that 35 are defective.

1. Determine the 95% confidence interval for the success probability.
2. We need the probability of success to be greater than 60% for the factory to keep working. How can we verify this assertion with 95% significance, starting from the result of the first test?

Problem 2

A communication system is redesigned with a new antenna, which helps increase the received signal power and the download speed. The system is tested on the same 10 files before and after the antenna is changed, with the following results:

	File 1	File 2	File 3	File 4	File 5	File 6	File 7	File 8	File 9	File 10
Time - old (s)	4.69	19.13	16.63	28.6	13.94	12.16	21.13	29.64	14.69	25.82
Time - new (s)	3.48	17.58	15.04	25.11	14.99	10.33	17.84	31.52	12.63	22.85

1. Can you show, with 95% significance, that the antenna actually improves the performance of the system?
2. The antenna is only economically worth it if it decreases download times by at least 1 second. Can you show it with 90% significance?
3. Supposing the same sample mean and sample variance, how many samples would we need to show that performance improves by at least 1 second with 95% significance? *Hint: use the Central Limit Theorem*

Problem 3

We consider a video application, in which the quality of the video depends on the bitrate, with a linear relationship.

1. Estimate the parameters of the linear regression and the 95% confidence interval of the slope parameter.
2. What is the 90% confidence interval of the quality if we transmit the video at a rate $R = 10$ Mb/s?
3. Can we show that the slope parameter β is smaller than 0.07 with 99% significance?

R (Mb/s)	q
13.11	0.38
5.75	0.135
16.98	0.552
18.68	0.662
13.57	0.413
15.15	0.531
14.86	0.531
7.84	0.299
13.11	0.415
3.42	0.065