Signalbehandling for computer-ingeniører

ComTek-5, E22

3. z-transform

Assoc. Prof. Peter Koch, AAU

Today's lecture

Since our aim is to study theories and methods related to modification and analysis of discrete-time signals, we need to have a certain knowledge of the important z-transform.

The z-transform is a general form of the Fourier transform which you may have studied in some detail earlier. You may recall that the Fourier transform converges only on the unit circle, but now we will introduce the z-transform which may converge in the complete complex plane.

Later in the course we will see that the design of a discrete-time (i.e., a digital) filter may take as an outset a similar continuous-time (i.e., analog) filter which can be described in the complex frequency domain using the Laplace transform. Our purpose is to get a very brief insight into analog filters, and for that reason we also need to understand the working of the Laplace transformation, which is therefore also subject for a short discussion in this lecture.

A quick look into the Study Regulation
”Beregningsteknik for computer-ingeniører”, 3. sem.

Formål:
Meningen med dette kursus er at give de studerende viden om, og at hjælpe de studerende i deres forståelse af, matematiske teorier og metoder der kan benyttes i forskellige domæner relateret til computersystemer, komplekse systemer og computernetværk. Denne generelle teori er nødvendig i modellering og performance evaluering af computersystemer og netværk og er et krav for mange andre kurser.

Læringsmål

Viden
Demonstrere en forståelse af koncepter, teorier og metoder anvendt indenfor Fourier analyse og lineære systemer:
- Grundliggende egenskaber af diskret og kontinuer Fourier transformering,
- Convolution teori,
- Impuls respons og transfer funktioner i Fourier domænet,
- Forbindelser mellem Z-transformation, Laplace og Fourier transformationer

Demonstrere en forståelse af koncepter, teorier og metoder anvendt indenfor diskret matematik:
- rekursive algoritmer og funktioner. Tidskompleksitet
- kombinatorik
- logiske notationer

Færdigheder
Kunne gøre brug af de fornødne skriftlige færdigheder i disse sammenhænge.
Anvende concepter, teorier og metoder brugt indenfor Fourier analyse
- Udvikling af Fourier serier
- Udvikling af Fourier transformationer for reelle og komplekse funktioner
- Udvikling af Fourier transformation for et produkt af funktioner og convolution af funktioner

Anvende koncepter, teorier og metoder brugt indenfor diskret matematik
- Operationer indenfor endelige felter
- Algebraiske strukturer

Kompetencer
Den studerende skal kunne anvende begreber og teknikker fra de følgende områder: Fourier analyse, diskret matematik

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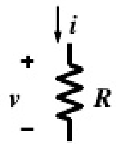
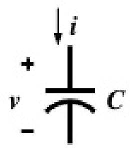
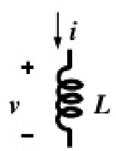
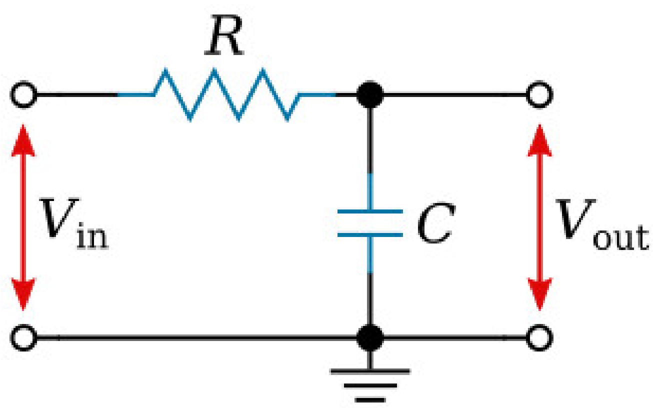
Laplace – why bother about that…??

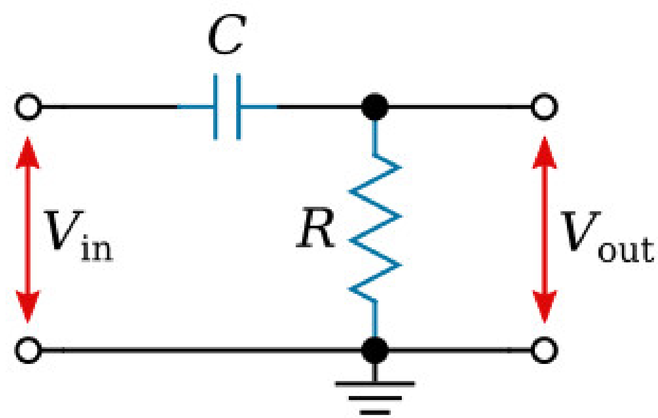
	Time Domain	s-Domain
	$v = iR$	$V(s) = I(s)R$
	$i = C\dfrac{dv}{dt}$ $v(t) = \dfrac{1}{C}\displaystyle\int_0^t i\,dt + v(0^-)$	$I(s) = sCV(s) - Cv(0^-)$ $V(s) = \dfrac{1}{sC}I(s) + \dfrac{v(0^-)}{s}$
	$v = L\dfrac{di}{dt}$ $i(t) = \dfrac{1}{L}\displaystyle\int_0^t v\,dt + i(0^-)$	$V(s) = sLI(s) - Li(0^-)$ $I(s) = \dfrac{1}{sL}V(s) + \dfrac{i(0^-)}{s}$

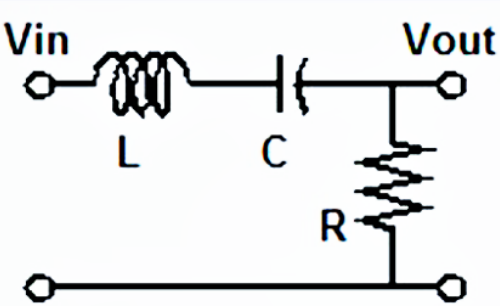
Note how initial conditions are incorporated

…and these circuit elements are used in analog filters.

Analog filters – examples

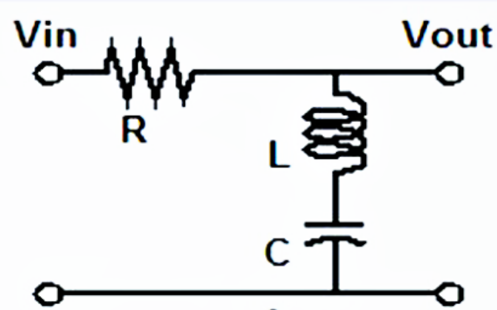






$$G(s) = \frac{s \cdot R/L}{s^2 + s \cdot R/L + 1/LC}$$

Band-Pass Series RLC Filter



$$G(s) = \frac{s^2 + 1/LC}{s^2 + s \cdot R/L + 1/LC}$$

Band-Stop series RLC Filter

A brief re-cap of Laplace…

$$\mathscr{L}\left(\mathrm{x}(t)\right) = \mathrm{X}(s) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-st}\, dt$$

$$\mathrm{x}(t) \xleftrightarrow{\mathscr{L}} \mathrm{X}(s)$$

This is the bilaterale (or two-sided) Laplace transform, where $x(t)$ is defined in the interval $-\infty < t < \infty$.

s is a complex variabel; $s = \sigma + j\Omega$

Note that for $\sigma = 0$, the Laplace transform reduces to the Fourier transform.

Complex Exponential Excitation

If a continuous-time LTI system is excited by a complex exponential $\mathrm{x}(t) = Ae^{st}$, where A and s can each be any complex number, the system response is also a complex exponential of the same functional form except multiplied by a complex constant. The response is the convolution of the excitation with the impulse response and that is

$$\mathrm{y}(t) = \int_{-\infty}^{\infty} \mathrm{h}(\tau)\mathrm{x}(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \mathrm{h}(\tau) A e^{s(t-\tau)}\,d\tau = \underbrace{Ae^{st}}_{\mathrm{x}(t)} \int_{-\infty}^{\infty} \mathrm{h}(\tau)e^{-s\tau}\,d\tau$$

The quantity $\mathrm{H}(s) = \int_{-\infty}^{\infty} \mathrm{h}(\tau)e^{-s\tau}\,d\tau$ is called the **Laplace transform** of $\mathrm{h}(t)$.

The Transfer Function

Let $\mathrm{x}(t)$ be the excitation and let $\mathrm{y}(t)$ be the response of a system with impulse response $\mathrm{h}(t)$. The Laplace transform of $\mathrm{y}(t)$ is

$$\mathrm{Y}(s)=\int_{-\infty}^{\infty}\mathrm{y}(t)e^{-st}dt=\int_{-\infty}^{\infty}\left[\mathrm{h}(t)*\mathrm{x}(t)\right]e^{-st}dt$$

$$\mathrm{Y}(s)=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\mathrm{h}(\tau)\mathrm{x}(t-\tau)d\tau\right)e^{-st}dt$$

$$\mathrm{Y}(s)=\int_{-\infty}^{\infty}\mathrm{h}(\tau)d\tau\int_{-\infty}^{\infty}\mathrm{x}(t-\tau)e^{-st}dt$$

Move this integral into $x(t-\tau)$

The Transfer Function

$$\mathrm{Y}(s) = \int_{-\infty}^{\infty} \mathrm{h}(\tau) d\tau \int_{-\infty}^{\infty} \mathrm{x}(t-\tau) e^{-st} dt$$

New variable

Let $\lambda = t - \tau \Rightarrow d\lambda = dt$. Then

$$\mathrm{Y}(s) = \int_{-\infty}^{\infty} \mathrm{h}(\tau) d\tau \int_{-\infty}^{\infty} \mathrm{x}(\lambda) e^{-s(\lambda+\tau)} d\lambda = \underbrace{\int_{-\infty}^{\infty} \mathrm{h}(\tau) e^{-s\tau} d\tau}_{\mathrm{H}(s)} \underbrace{\int_{-\infty}^{\infty} \mathrm{x}(\lambda) e^{-s\lambda} d\lambda}_{\mathrm{X}(s)}$$

$$\mathrm{Y}(s) = \mathrm{H}(s)\mathrm{X}(s)$$

$\mathrm{H}(s)$ is called the **transfer function**.

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) \xleftrightarrow{\mathscr{L}} \mathrm{Y}(s) = \mathrm{H}(s)\mathrm{X}(s)$$

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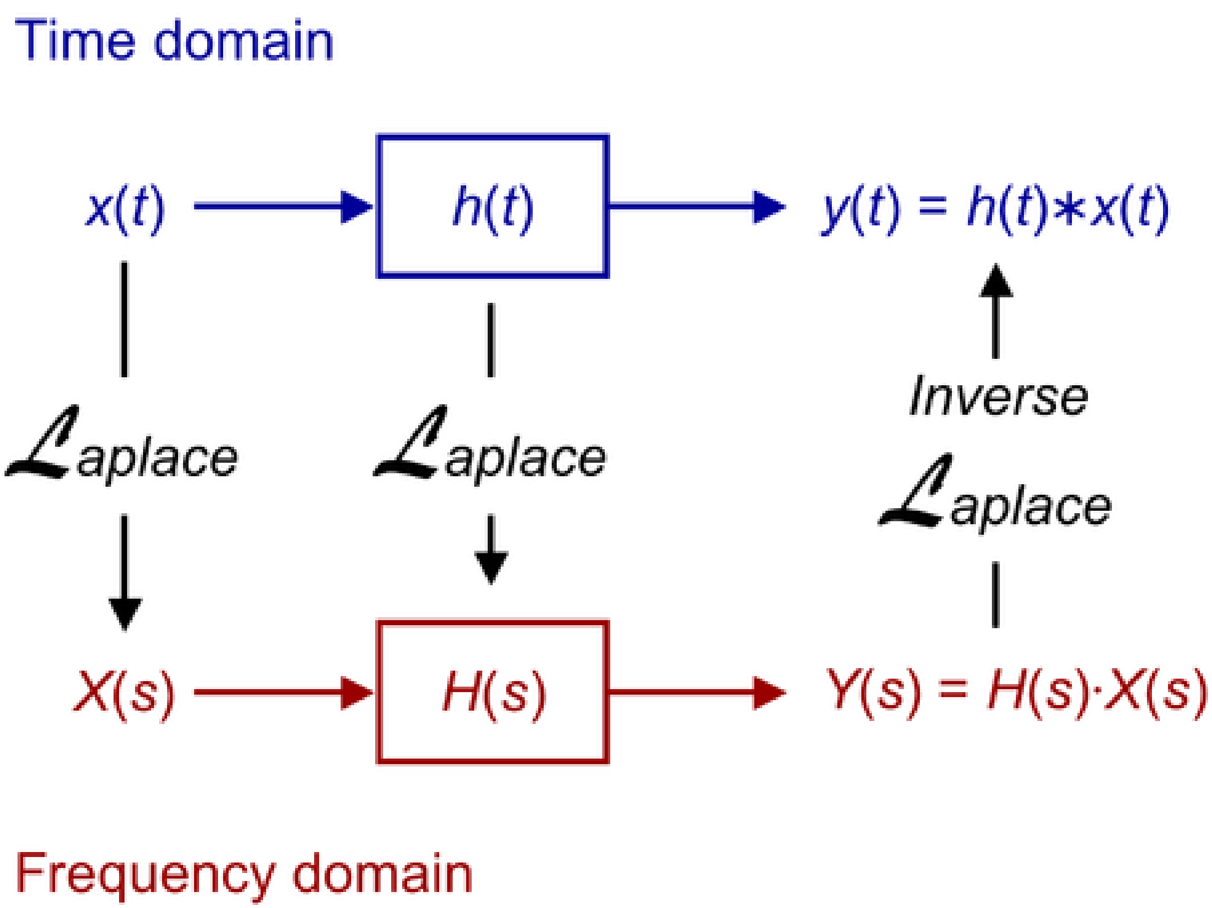
The Time- and the Frequency Domain



Laplace transform Properties

1. Linearity	$\sum_{n=1}^{N} \alpha_n x_n(t)$	$\sum_{n=1}^{N} \alpha_n X_n(s)$
2. Time shift	$x(t - t_0)u(t - t_0)$	$X(s)\exp(-st_0)$
3. Frequency shift	$\exp(s_0 t)x(t)$	$X(s - s_0)$
4. Time scaling	$x(\alpha t), \alpha > 0$	$1/\alpha\, X(s/\alpha)$
5. Differentiation	$dx(t)/dt$	$s\,X(s) - x(0^-)$
6. Integration	$\int_0^t x(\tau)\, d\tau$	$\frac{1}{s} X(s)$
7. Multiplication by t	$t\,x(t)$	$-\frac{dX(s)}{ds}$
8. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}[X(s - j\omega_0) + X(s + j\omega_0)]$
	$x(t)\sin\omega_0 t$	$\frac{1}{2j}[X(s - j\omega_0) - X(s + j\omega_0)]$
9. Convolution	$x(t) * h(t)$	$X(s)H(s)$
10. Initial value	$x(0^+)$	$\lim_{s\to\infty} s\,X(s)$
11. Final value	$\lim_{t\to\infty} x(t)$	$\lim_{s\to 0} s\,X(s)$

Basic Laplace Transform Pairs

Impulse	$\delta(t)$	1
Step	$u(t) = 1, \quad t \geq 0$	$\frac{1}{s}$
Ramp	$r(t) = t, \quad t \geq 0$	$\frac{1}{s^2}$
Exponential	$e^{-\alpha t} \quad e^{-\alpha t} u(t)$	$\frac{1}{s + \alpha}$
Damped Ramp	$te^{-\alpha t}$	$\frac{1}{(s + \alpha)^2}$
Sine	$\sin(\beta t)$	$\frac{\beta}{s^2 + \beta^2}$
Cosine	$\cos(\beta t)$	$\frac{s}{s^2 + \beta^2}$
Damped Sine	$e^{-\alpha t} \sin(\beta t)$	$\frac{\beta}{(s + \alpha)^2 + \beta^2}$
Damped Cosine	$e^{-\alpha t} \cos(\beta t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \beta^2}$

Definition of the z-transform

The Fourier transform of a sequence $x[n]$ was defined in Chapter 2 as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

The DTFT..!!

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

The is a geometric series (på dansk: en uendelig potens-række), where z is a continuous complex variable.

Often, we use the notation;

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

Bilateral vs. unilateral *z*-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Two-sided/bilateral *Z*-transform

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

One-sided/unilateral *Z*-transform

Clearly, the bilateral and unilateral transforms are equivalent only if $x[n] = 0$ for $n < 0$. In this book, we focus on the bilateral transform exclusively.

Relation to the Fourier transform

Fourier transform; when it exists, the Fourier transform is simply $X(z)$ with $z = e^{j\omega}$. This corresponds to restricting z to have unity magnitude; i.e., for $|z| = 1$, the z-transform corresponds to the Fourier transform. More generally, we can express the complex variable z in polar form as

$$z = re^{j\omega}.$$

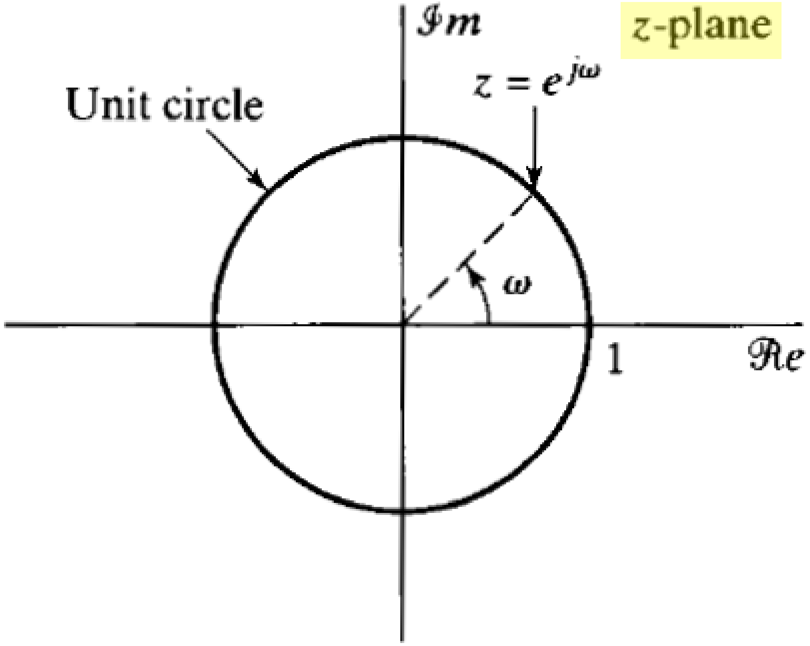
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n},$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

This is the Fourier transform of the sequence $x[n]$ multiplied with the exponential sequence r^n. The value $r = 1$ is of special interest…

z is a complex variable, and therefore it makes sense to consider the complex plane (now denoted the z-plane) – and in particular the unit circle on which the z-transform equals the Fourier transform



From this figure, we realize that the Discrete Time Fourier Transform (DTFT) is periodic with the period 2π.

The infinite linear frequency axis is therefore represented as infinitely many iterations on the unit circle.

Q; What is the value of z at DC..?

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Some thoughts on Convergence

First of all; an infinite sum does not always converge, i.e., for the Fourier-transform the sum is not always a finite value for a given sequence $x[n]$.

Similarly, we cannot guarantee that the z-transform converges for all sequences $x[n]$, or for all values of z.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

For a given sequence, the set of values of z for which the z-transform converges, is denoted the ”Region of Convergence”, ROC.

The Fourier transform converges if the sequence $x[n]$ is absolutly summable. Similarly, the z-transform also converges if

$$\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty$$

As a consequence, the z-transform can converge for a sequence $x[n]$ for which the Fourier-transformation does not converges.

For example, the sequence $x[n] = u[n]$ is not absolutely summable, and therefore, the Fourier transform does not converge absolutely. However, $r^{-n}u[n]$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with a region of convergence $|z| > 1$.

Some thoughts on Convergence

Now, given z-transform $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

for which we just concluded that it converges if $$\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty$$

Since a complex z number can be expressed as $z = re^{j\omega}$, we now conclude that for a given sequence $x[n]$, convergence **does not** depend on the frequency ω, but only on the value r, i.e., the modulus of z.

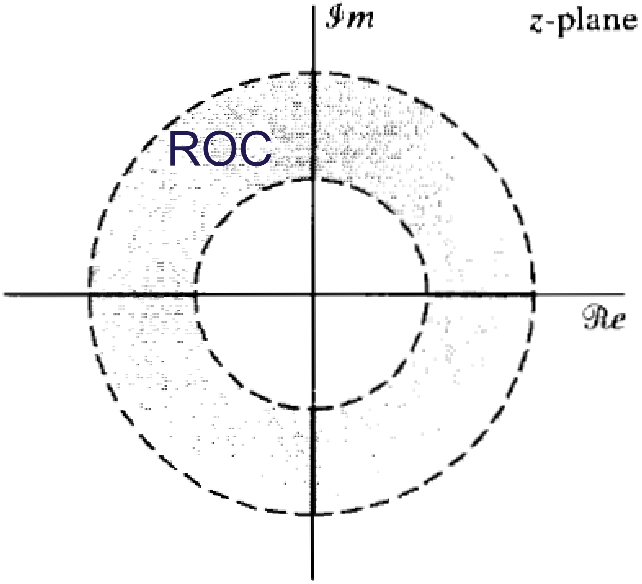
Therefore, if $$\sum_{n=-\infty}^{\infty} |x[n]| \, |z|^{-n} < \infty$$

then the ROC consist of all the values of z for which this inequality holds.

Similarly, since the ROC is independent of the frequency, then if a value of z is in the ROC, then all other values of z having the same modulus, are also in the ROC, i.e., values located on a circle with radius $|z|$.

In general, the z-values defining the ROC for a given sequence $x[n]$, thus represent a ring in the z-plane.

Some thoughts on Convergence



The inner circle can move towards (and enclose) origo, while the outer circle can move towards infinity…!

If the ROC includes the unit circle, this of course implies convergence of the z-transform for $|z| = 1$, or equivalently, the Fourier transform of the sequence converges. Conversely, if the ROC does not include the unit circle, the Fourier transform does not converge absolutely.

Application of the z-transform

The z-transform is most useful when the infinite sum can be expressed in closed form, i.e., when it can be "summed" and expressed as a simple mathematical formula. Among the most important and useful z-transforms are those for which $X(z)$ is a rational function inside the region of convergence, i.e.,

$$X(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials in z.

The roots in $P(z)$ are denoted the ”zeros” of $X(z)$.
The roots in $Q(z)$ are denoted the ”poles” of $X(z)$.

These roots and their location in the z-plane determine essential properties of $X(z)$.

IMPORTANT…!!!

Illustrative examples

Consider the signal $x[n] = a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence.

$$X(z) = \sum_{n=-\infty}^{\infty} = a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the region of convergence is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Inside the region of convergence, the infinite series converges to

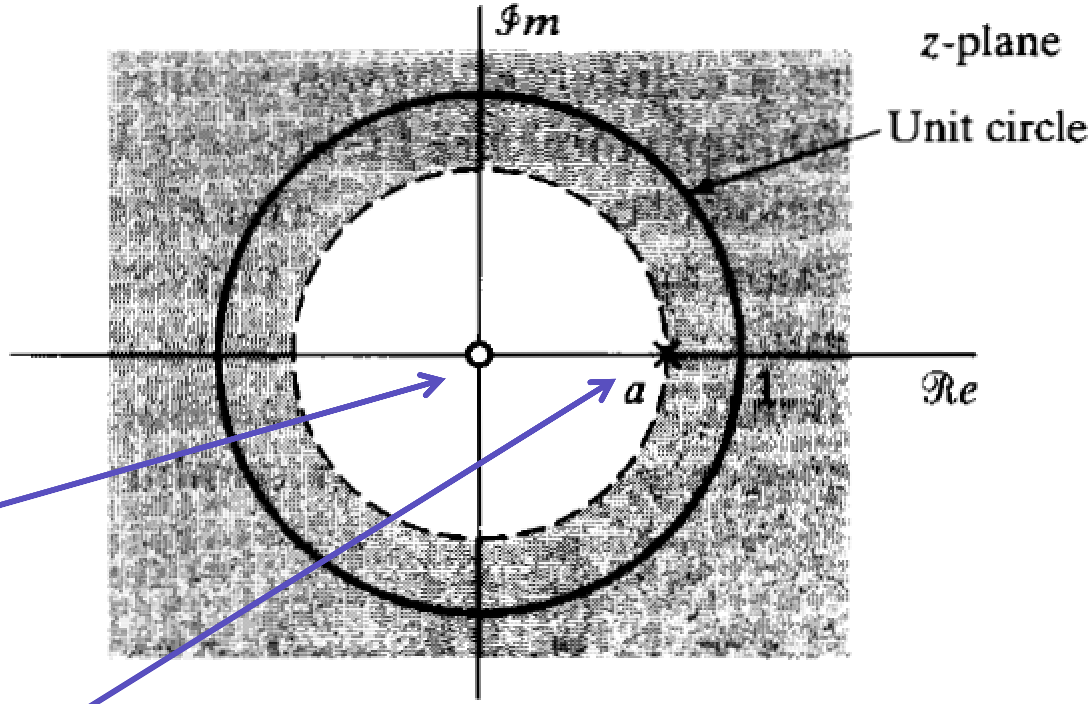
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|.$$

Here we have used the familiar formula for the sum of terms of a geometric series. The z-transform has a region of convergence for any finite value of $|a|$. The Fourier transform of $x[n]$, on the other hand, converges only if $|a| < 1$. For $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

Illustrative examples

$$X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|.$$



A zero in $z=0$

A pole in $z=a$

Almost the same sequence – now left-sided and negative

Now let $x[n] = -a^n u[-n-1]$. Since the sequence is nonzero only for $n \le -1$, this is a *left-sided* sequence. Then

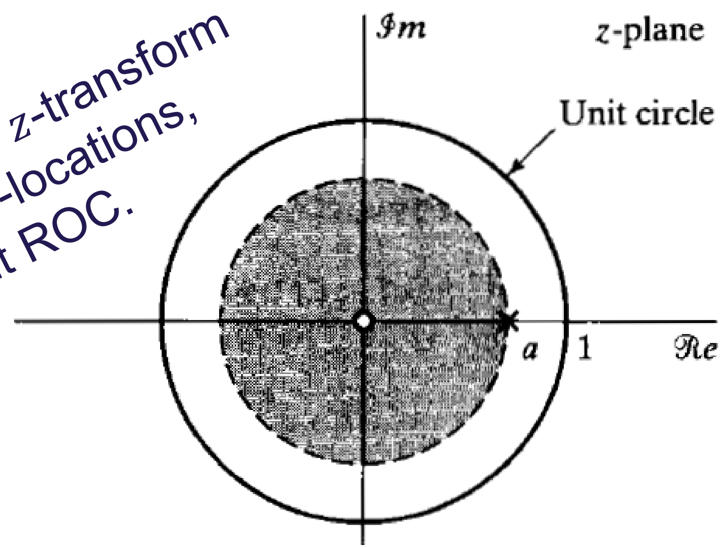
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n.$$

If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the sum in Eq. (3.12) converges, and

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|.$$

OBS..!! Exactly the same z-transform with the same pole/zero-locations, but now with a different ROC.



How do we represent the z-transform..??

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|.$$

As seen from this example, the z-transform can be expressed both in terms of positive and negative powers of z.

When it comes to determination of poles and zeros, then it is beneficial to express the z-transform in terms of positive powers of z – why..?

Some properties of the ROC

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \leq r_R < |z| < r_L \leq \infty$.

This is a direct consequence of the fact, that for a given sequence $x[n]$, the convergence of

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

depends only on the modulus $|z|$ and not the argument $\arg(z)$.

Some properties of the ROC

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

This is because the z-transform is identical to the Fourier transform for $|z|=1$.

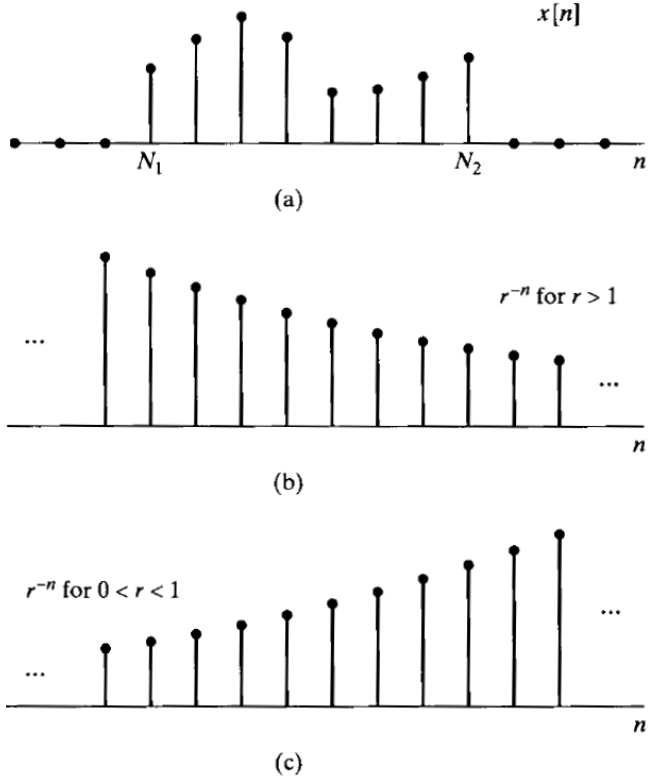
Some properties of the ROC

PROPERTY 3: The ROC cannot contain any poles.

Because $X(z)$ is infinite for $z = z_{pole}$.

Some properties of the ROC

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.



Finite-length sequence and weighting sequences implicit in convergence of the z-transform. (a) The finite-length sequence $x[n]$. (b) Weighting sequence r^{-n} for $1 < r$. (c) Weighting sequence r^{-n} for $0 < r < 1$.

The z-transform converges, if $x[n]|z|^{-n}$ (i.e., $x[n]r^{-n}$) is absolutly summable.

$x[n]$ consists of finite-valued samples, which is not affected if $x[n]$ is being multiplied with a finite-valued exponential sequence.

For a finite length sequence, $x[n]r^{-n}$ will thus be absolutely summable.

Be aware though when $N_2 > 0$ and $r=0$ (i.e.,. $|z|=0$), for $0<n\le N_2$, $x[n]r^{-n}$ becomes infinite, and thus does $z=0$ not belong to the ROC.

Similarly, if $N_1<0$ and $N_1\le n<0$ then $x[n]r^{-n}$ is infinite if $r=\infty$ and thus does $z=\infty$ not belong to the ROC.

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The remaining 4 properties for the ROC are given here without further comments

See eventually pp. 117-119

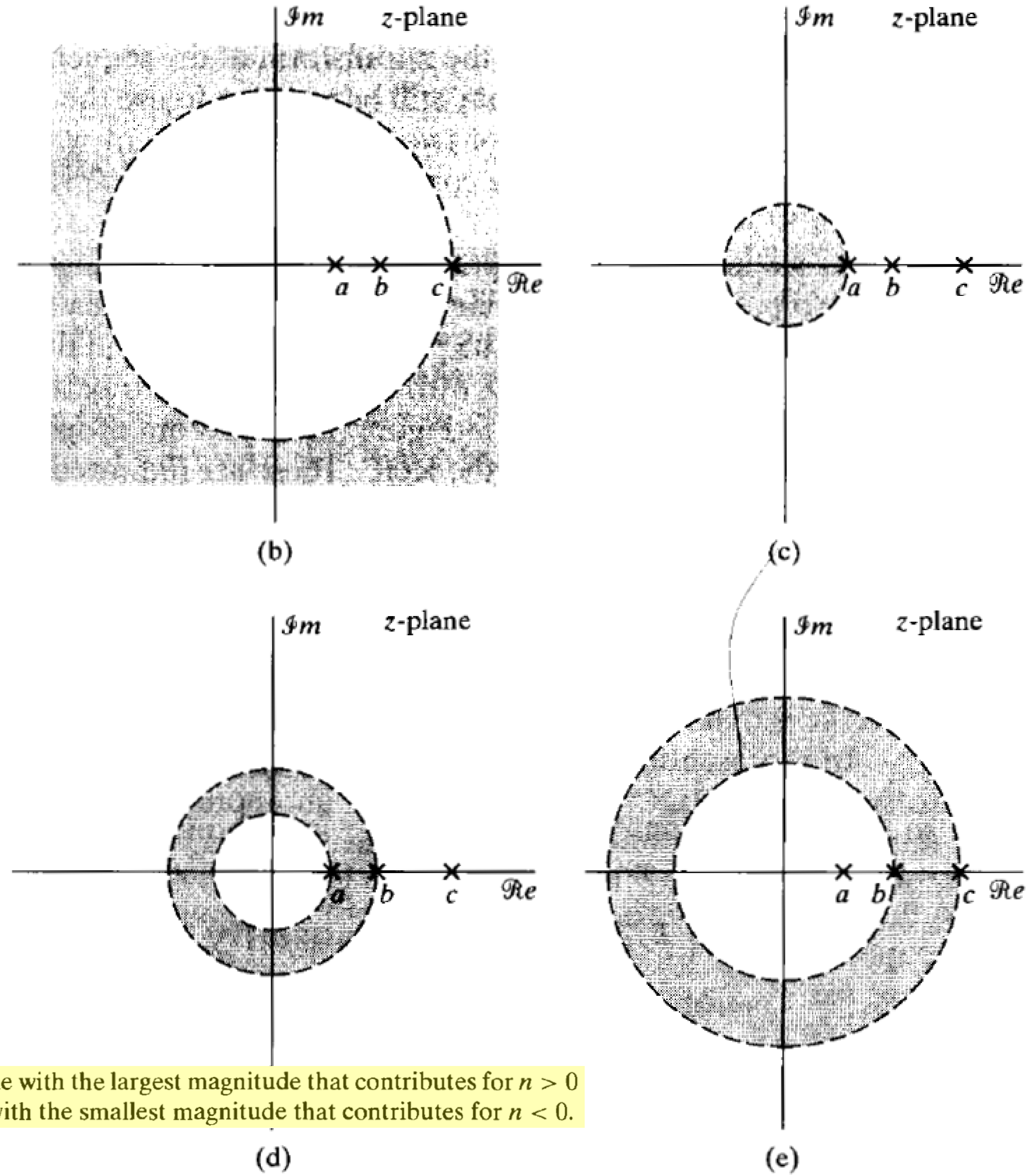
PROPERTY 5: If $x[n]$ is a *right-sided sequence,* i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence,* i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

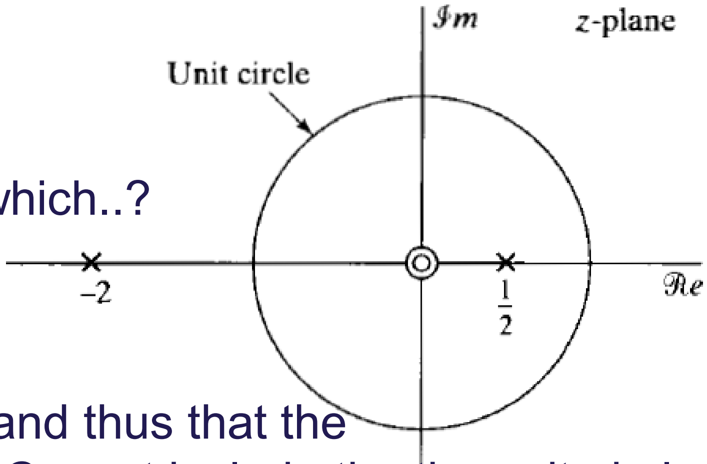
ROC for right-(b), left-(c), and two-sided (d,e) sequences



The region of convergence is bounded on the inside by the pole with the largest magnitude that contributes for $n > 0$ and on the outside by the pole with the smallest magnitude that contributes for $n < 0$.

Stability, Causality and ROC

Given the pole/zero-diagram for a system with the Impulse response $h[n]$ and the associated z-transform $H(z)$.
There are three possible locations for the ROC – which..?



Stability requires that $h[n]$ is absolutly summable and thus that the Fourier transform of $h[n]$ exists. If so, then the ROC must include the the unit-circle and thus ROC -2<|z|<½.

Cf. Property 7 this means that $h[n]$ is two-sided, and thus not causal – problem..!!

If, on the other hand, we require that the system should be causal, then $h[n]$ needs to be a right-sided sequence, and thus cf. Property 5, the ROC should be the portion of the z-plane where |z|>2. In such a case, the system is not stable, because the ROC does not contain the unit-circle – problem..!!

Which IMPORTANT conclusion can be drawn from these arguments..??

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

The inverse z-transformation

$$x[n] = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz \qquad \{z : |z| = r\} \subset \text{ ROC}$$

In most case it would be (almost) impossible to calculate this contour integral in closed form…

Thus we are looking of other ways to obtain $x[n]$, given $X(z)$.

Calculating the invers Z-transform by the "Inspection method"

If we need to find the inverse z-transform of

$$X(z) = \left(\frac{1}{1 - \frac{1}{2} z^{-1}} \right), \qquad \begin{array}{l} |z| > \frac{1}{2}, \\ |z| < \frac{1}{2}, \end{array}$$

then inspect your z-transform tabel, and find the appropriate pair

5. $a^n u[n]$	$\dfrac{1}{1 - az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1 - az^{-1}}$	$\|z\| < \|a\|$

Extremely useful transformation pair..!!

$$a^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$$

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Calculating the invers z-transform by ”Partial Fraction Expansion”

Often, the z-transform is expressed as the ratio between two polynomias in z…

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

We now want to bring this expression into a more simple form where the individual sub-expressions can be found as z-transform pairs in a tabel.

Initially, we therefore determine zeros and the poles, i.e., the roots in the two polynomias.

Calculating the invers z-transform by ”Partial Fraction Expansion”

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}. \tag{3.38}$$

Equation (3.38) explicitly shows that for such functions, there will be M zeros and N poles at nonzero locations in the z-plane. In addition, there will be either $M - N$ poles at $z = 0$ if $M > N$ or $N - M$ zeros at $z = 0$ if $N > M$.

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Calculating the invers z-transform by ”Partial Fraction Expansion”

The expression $X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$. is re-expressed into $X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$

where c_k and d_k are the zeros and poles, respectively. Then *X(z)* can be re-structured using partial fraction expansion;

If $M < N$ and the poles are all first order, then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

$$A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k}$$

Normally the case for the type of systems we look into in this course…

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z-transform properties (without proof)

The linearity property states that

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2},$$

IMPORTANT..!!

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z-transform properties (without proof)

According to the time-shifting property,

$$x[n-n_0] \stackrel{z}{\longleftrightarrow} z^{-n_0}X(z), \qquad \text{ROC} = R_x \text{(except for the possible addition or deletion of } z=0 \text{ or } z=\infty\text{)}.$$

The derivation of this property follows directly from the z-transform expression. Specifically, if $y[n] = x[n-n_0]$, the corresponding z-transform is

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}.$$

With the substitution of variables $m = n - n_0$,

$$Y(z) = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)}$$

$$= z^{-n_0}\sum_{m=-\infty}^{\infty} x[m]z^{-m},$$

or

IMPORTANT..!!

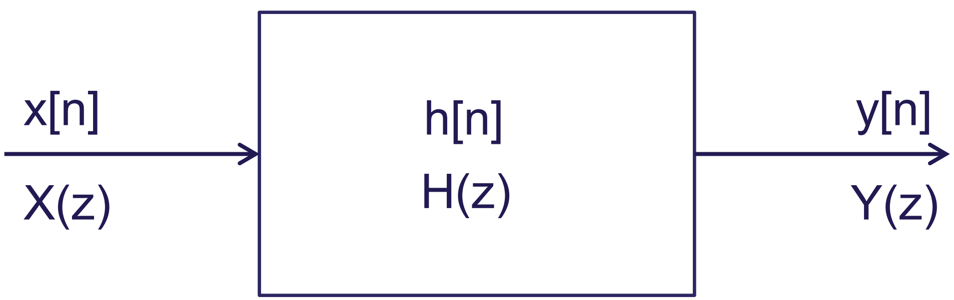
$$Y(z) = z^{-n_0}X(z).$$

z-transform properties (without proof)

According to the convolution property,

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2}.$$

IMPORTANT..!!



y[n] = ?

Y(z) = ?

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
$nx[n]$	$-z \dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Initial-value theorem:

$x[n] = 0, \quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$