Signalbehandling for computer-ingeniører

ComTek-5, E22

2. Discrete-Time Systems and Convolution

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This lecture…

The purpose of this 2’nd lecture is to discuss the notion of Discrete-Time Systems, and to introduce some of the properties and limitations associated with DTS. Basically, we want to design a system (first and foremost a filter) which can modify a discrete-time signal, and therefore it is important to understand how we describe such a system and how it (mathematically) transform an input signal into an output signal.

One very fundamental element in this process is the concept of ”convolution” (på dansk; foldning). Earlier in your study, you might have been introduced to convolution, but it might be (???) that you have not realized how fundamental and important this concept really is. In order to understand how linear time-invariant systems are actually working, we therefore spend some time refreshing our memories and understanding of convolution.

Additionally, we will discuss the terms ”time” and ”frequency”, and briefly introduce how signals and systems can be represented in these two domains. This becomes very important for our continuous studies and understanding of signal processing systems.

Let’s start out with a few fundamental observations regarding signals

The unit step is related to the impulse by

$$u[n] = \sum_{k=-\infty}^{n} \delta[k];$$

More generally, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

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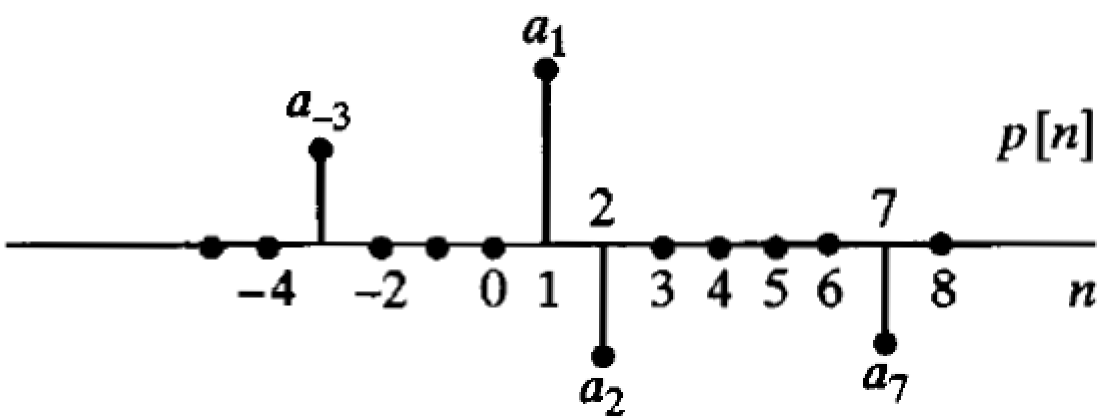
An example – any sequence can be expressed as a linear combination of time-delayed unit samples

$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7].$$



In general, it follows that;

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

OBS…!!

To be used very soon…

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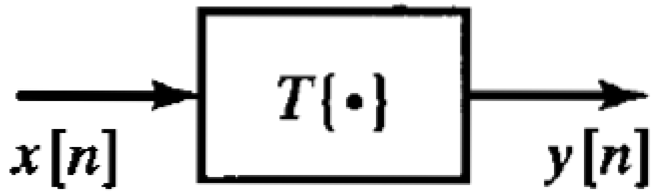
A few words on Discrete-Time Systems

$$y[n] = T\{x[n]\}$$



An input sequence $x[n]$ is transformed into a unique output sequence $y[n]$.

- An example is the ”ideal delay system”;

$$y[n] = x[n - n_d], \qquad -\infty < n < \infty$$

- Another example is the ”Moving Average (MA) system”

$$\begin{aligned} y[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \\ &= \frac{1}{M_1 + M_2 + 1} \{x[n + M_1] + x[n + M_1 - 1] + \cdots + x[n] \\ &\quad + x[n-1] + \cdots + x[n - M_2]\}. \end{aligned}$$

Linear Systems – application of ”Superposition”

Addition property

$$T\{x_1[n] + x_2[n]\} = T\{x_2[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$

Scaling property

$$T\{ax[n]\} = aT\{x[n]\} = ay[n]$$

…and here combined

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

Overall, we can say that…

if $x[n] = \sum_k a_k x_k[n]$,

then the output of a linear system will be

$y[n] = \sum_k a_k y_k[n]$, where

$y_k[n]$ is the system response to the input $x_k[n]$.

This is a consequence of Superposition…

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Time-invariant systems

A time-invariant system (often referred to equivalently as a shift-invariant system) is a system for which a time shift or delay of the input sequence causes a corresponding shift in the output sequence. Specifically, suppose that a system transforms the input sequence with values $x[n]$ into the output sequence with values $y[n]$. Then the system is said to be time invariant if, for all n_0, the input sequence with values $x_1[n] = x[n - n_0]$ produces the output sequence with values $y_1[n] = y[n - n_0]$.

Or put differently;

A system is time invariant if it, for a given input sequence, produces a specific output sequence which is totally INDEPENDENT OF WHEN the input is applied.

Causality

A system is causal if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \leq n_0$.

Or put differently;

* A system is causal if its output depends only on the past.

* A system is causal if it replies AFTER being asked, and not BEFORE.

Consider the *forward difference system* defined by the relationship

$$y[n] = x[n+1] - x[n].$$

This system is not causal, since the current value of the output depends on a future value of the input.

Stability

A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that

$$|x[n]| \leq B_x < \infty, \qquad \text{for all } n.$$

Stability requires that, for every bounded input, there exist a fixed positive finite value B_y such that

$$|y[n]| \leq B_y < \infty, \qquad \text{for all } n.$$

It is important to emphasize that the properties we have defined in this section are properties of *systems*, not of the inputs to a system. That is, we may be able to find inputs for which the properties hold, but the existence of the property for some inputs does not mean that the system has the property. For the system to have the property, it must hold for *all* inputs.

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Linear Time-Invariant Systems, LTI Systems

From p.4 we remember that an arbitrary sequence can be expressed as a linear combination of delayed unit samples, i.e.;

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

In $x[n]$ is the input to a linear system T, then we can write;

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

Considering $x[k]$ as scalars, then from the superposition principles, we have;

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

where $h_k[n]$ is the impulse responses to the input sequences $\delta[n-k]$.

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Linear Time-Invariant Systems, LTI Systems

So, we have; $y[n] = \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k] h_k[n]$

Assuming now that ”time invariance” also holds, then the impulse responses are identical, no matter when the impulse is applied at the input, i.e., if $h[n]$is the response to $\delta[n]$, then $h[n-k]$ is the response to $\delta[n-k]$, and thus we have;

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

Let’s put it differently: A system which is linear and time-invariant, i.e., an LTI system, is completely characterized by its impulse response $h[n]$, because we can derive the output $y[n]$ for any arbitrary input $x[n]$, if we know $h[n]$.

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Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = x[n] * h[n]$$

How should we interprete this expression..??

An input sample at time $n = k$, i.e., $x[k]\delta[n-k]$ is being transformed by the system into the output sequence $x[k]h[n-k]$, for $-\infty < n < \infty$.

For all values of k, these output sequences are now added, constituting the resulting output sequence $y[n]$ – see the example on figure 8, p.27 (see next slide).

The conclusion therefore is that the convolution sum is a direct consequence of linearity and time invariance…

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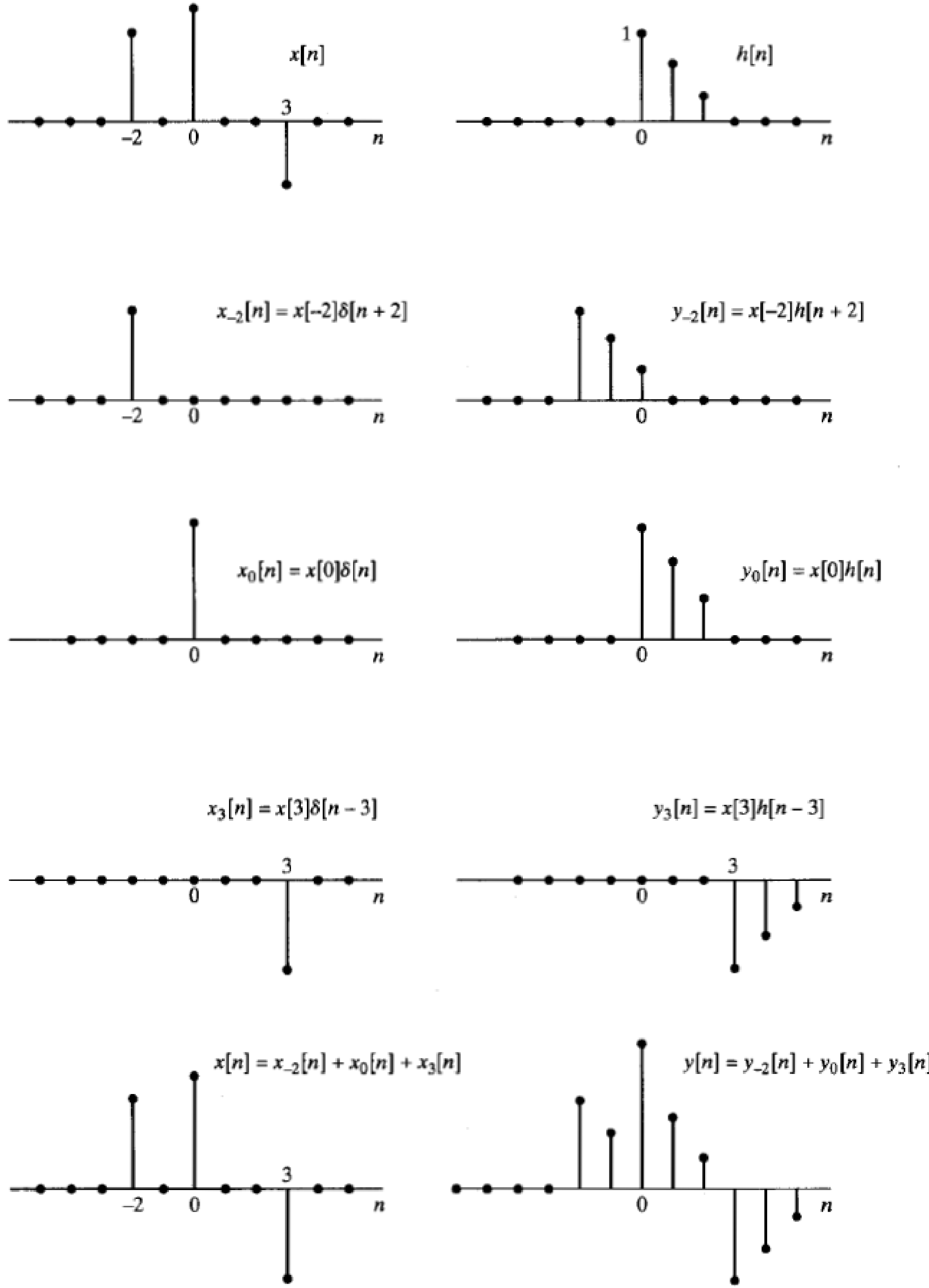
x[n]
h[n]
1
3
-2
0
n
x_{-2}[n] = x[-2]δ[n + 2]
y_{-2}[n] = x[-2]h[n + 2]
x_0[n] = x[0]δ[n]
y_0[n] = x[0]h[n]
x_3[n] = x[3]δ[n - 3]
y_3[n] = x[3]h[n - 3]
x[n] = x_{-2}[n] + x_0[n] + x_3[n]
y[n] = y_{-2}[n] + y_0[n] + y_3[n]

Calculating the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

How do we generate the sequence $h[n-k]$, $-\infty < k < \infty$??

In order to answer this question, let's realize that $h[n-k] = h[-(k-n)]$.

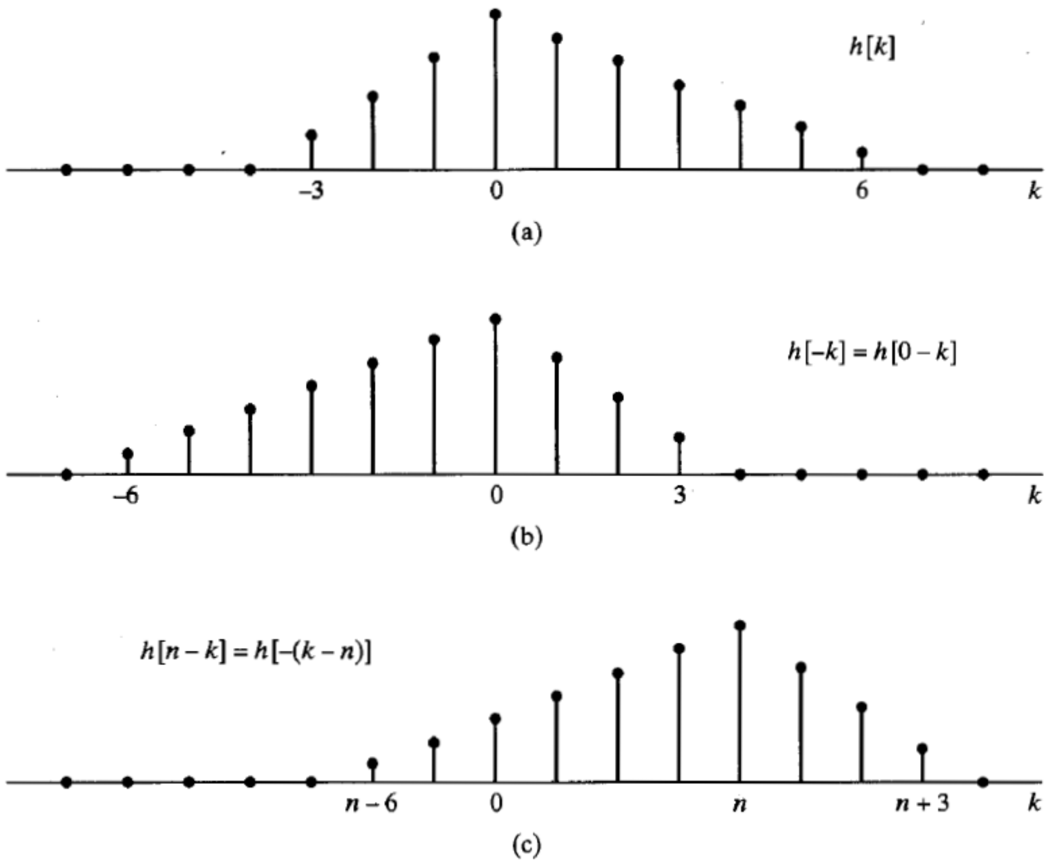

Figure 2.9 Forming the sequence $h[n-k]$. (a) The sequence $h[k]$ as a function of k. (b) The sequence $h[-k]$ as a function of k. (c) The sequence $h[n-k] = h[-(k-n)]$ as a function of k for $n = 4$.

Calculating the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

1. reflecting $h[k]$ about the origin to obtain $h[-k]$;
2. shifting the origin of the reflected sequence to $k = n$.

To implement discrete-time convolution, the two sequences $x[k]$ and $h[n-k]$ are multiplied together for $-\infty < k < \infty$, and the products are summed to compute the output sample $y[n]$. To obtain another output sample, the origin of the sequence $h[-k]$ is shifted to the new sample position, and the process is repeated.

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LTI System Properties

Convolution is commutative, i.e., $x[n] * h[n] = h[n] * x[n]$

Convolution is distributive, i.e., $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

Convolution is associative, i.e., $(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$

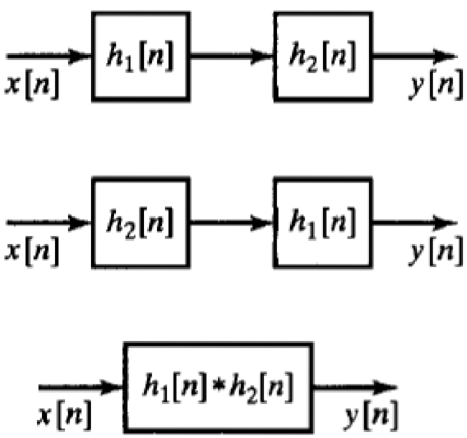

Figure 2.11 Three linear time-invariant systems with identical impulse responses.

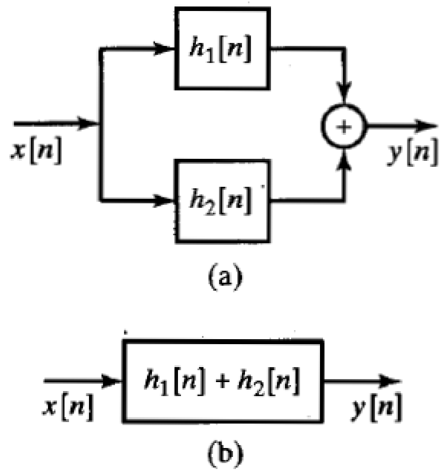

Figure 2.12 (a) Parallel combination of linear time-invariant systems. (b) An equivalent system.

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LTI System Properties

An LTI system is stable if, and only if the system impulse response is absolute summable, i.e.,

$$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

If the impulse response has a finite number of samples, then $S < \infty$.

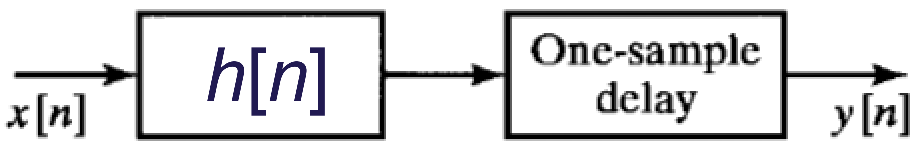
Such a system is known as a Finite Impulse Response (FIR) system, and is always stable.

Systems with an infinite number of samples in the impulse response are stable if the infinite sum converges, i.e., $S < \infty$.

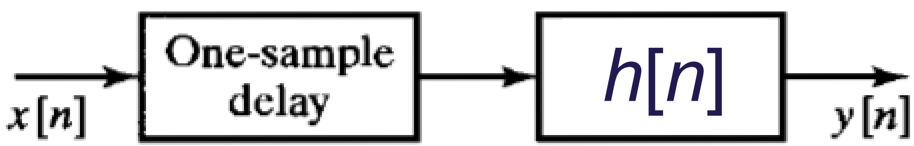
Such systems are denoted Infinite Impulse Response (IIR) systems

LTI System Properties

If the output from an LTI system $h[n]$ is delayed 1 sample (in general N samples) then the resulting output $y[n]$ can also be generated by applying a corresponding delay to the input $x[n]$, and next applying this delayed input to the system $h[n]$.



(a)



(b)

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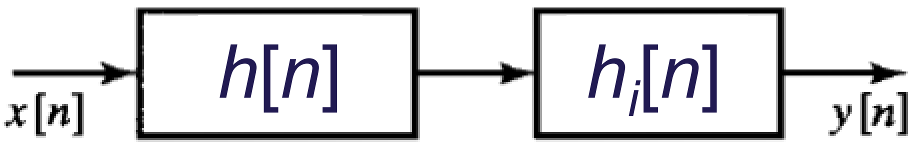
The Inverse System

a linear time-invariant system has impulse response $h[n]$, then its inverse system, if it exists, has impulse response $h_i[n]$ defined by the relation

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]. \tag{2.81}$$



Linear Constant-Coefficent Difference Equations

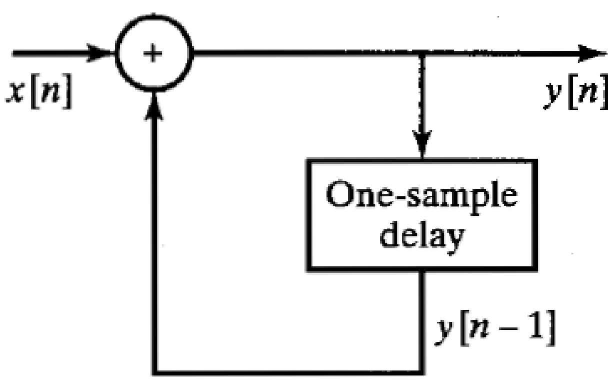
…don't mix up with differential equations.

An important subclass of linear time-invariant systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an Nth-order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m].$$

$$y[n] - y[n-1] = x[n]$$

$$y[n] = x[n] + y[n-1]$$



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Linear Constant-Coefficent Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

Here we note that output $y[n]$ not necessarily is uniquely defined. The reason being that output $y[n]$ depends on previous output $y[n-k]$, i.e., it is a recursive computation.

Only in the situation where we know the values of these previous outputs, we can state uniquely the sequence $y[n]$.

Also note that we assume that $x[n]$ is causal, i.e., $x[n] = 0$ for $n < 0$.

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A Short Note on Frequency Representation

Assume an LTI system with impulse response $h[n]$ onto which we apply the input;

$$x[n] = e^{j\omega n} \text{ for } -\infty < n < \infty$$

which is the complex exponential function.

Using the expression for convolution, we can now write the output sequence $y[n]$;

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right)$$

We now introduce the term Frequency Response;

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$ …and thus we have; $$y[n] = H(e^{j\omega}) e^{j\omega n}$$

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A Short Note on Frequency Representation

Interpretation of the equation; $y[n] = H(e^{j\omega})e^{j\omega n}$

we see that $H(e^{j\omega})$ describes the change in complex amplitude of a complex exponential input signal as a function of the frequency ω. $H(e^{j\omega})$ is called the *frequency response* of the system. In general, $H(e^{j\omega})$ is complex and can be expressed in terms of its real and imaginary parts as

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

or in terms of magnitude and phase as

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\sphericalangle H(e^{j\omega})}$$

…and based on this, we conclude that an LTI system, in general, will impact the amplitude as well as the phase of the input signal.

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