Department of Electronic Systems

AALBORG UNIVERSITY
DENMARK

Written exam in
Signal processing, 5 ECTS

SOLUTIONS

Wednesday, January 9, 2013
9.00 – 13.00

Read carefully :

- Remember to write your **full name on every sheet** you return!
- The answers concerning analog filters, digital filters and spectral estimation have to be written on **separate sheets**.
- Problems are weighted as listed.
- Results **without sufficient explanations will not give full credits**!
- All ordinary tools may be used e.g. books, notes, programmable calculators and laptops.
- All electronic communication devices **must be turned off** at all times.
- **Communication with others is strictly prohibited**.

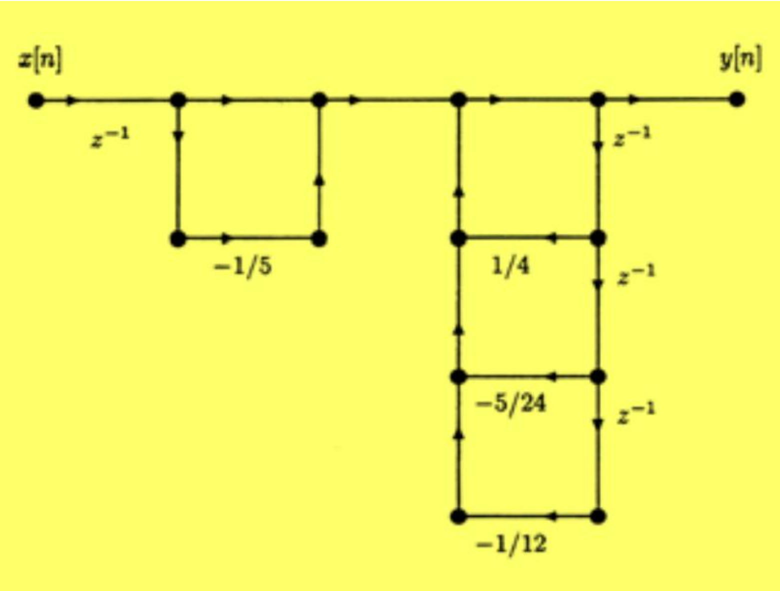
Problem 1 (weighted with 10% - Digital filters)
A discrete-time causal system has a z-transform function:

$$G(z) = \frac{(1 - \frac{1}{5}z^{-1})}{(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})(1 + \frac{1}{4}z^{-1})}$$

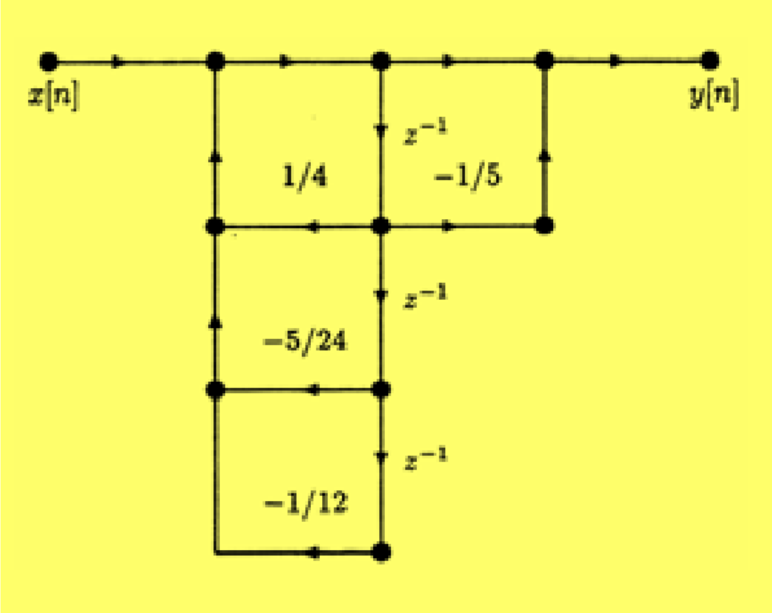
Questions:

1) Draw the filter as a Direct form I implementation

$$G(z) = \frac{\left(1 - \frac{1}{5}z^{-1}\right)}{\left(1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}\right)}$$

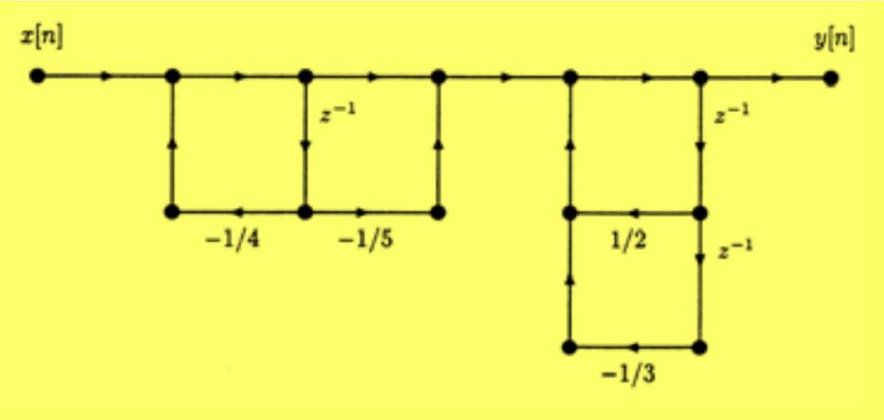


2) Draw the filter as a Direct form II implementation



3) Draw the filter as a cascade implementation

$$G(z) = \frac{1 - \frac{1}{5}z^{-1}}{(1 + \frac{1}{4}z^{-1})} \cdot \frac{1}{(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})}$$



(Other combinations of zeros and poles are possible)

Problem 2 (weighted with 14% - Digital filters)
A 2. order digital IIR filter has to be designed from an analog Butterworth filter via the Bilinear Transformation method. The sampling frequency should be f_s=1kHz. The 3 dB cut-off frequency f_c has to be 125 Hz.

The transfer function $H_a(s)$ for the analog Butterworth filter is given by:

$$H_a(s) = \frac{1}{1 + \frac{\sqrt{2}s}{\Omega_c} + \left(\frac{s}{\Omega_c}\right)^2}$$

Where Ω_c is the cut-off frequency.

Questions:
1) Determine the transfer function for the digital filter using Bilinear Transformation.

The cut-off frequency is pre-warped:

$$\Omega_{c,pre} = 2/T_d \tan\left(\frac{\omega_c}{2}\right)$$

Inserting the values, we get:

$$\Omega_{c,pre} = \frac{2}{T_d}\tan\left(\frac{2\pi\left(\frac{125}{1000}\right)}{2}\right) = \frac{2}{T_d}\tan\left(\frac{\pi}{8}\right) = 2000\tan\left(\frac{\pi}{8}\right) = 828.42 \text{ rads/sample (~132 Hz)}$$

The bilinear transformation is then applied i.e.

$$s = \frac{2}{T_d}\frac{z-1}{z+1}$$

This is then used in $H_a(s)$

$$H(z) = \frac{1}{1 + a\frac{z-1}{z+1} + b\left(\frac{z-1}{z+1}\right)^2}$$

Where

$$a = \frac{\sqrt{2}\cdot\frac{2}{T_d}}{\frac{2}{T_d}\tan\left(\frac{\pi}{8}\right)} = \frac{\sqrt{2}}{\tan\left(\frac{\pi}{8}\right)} = 3.4142\ ;\ b = \left(\frac{\left(\cdot\frac{2}{T_d}\right)}{\left(\frac{2}{T_d}\tan\left(\frac{\pi}{8}\right)\right)}\right)^2 = \frac{1}{\left(\tan\left(\frac{\pi}{8}\right)\right)^2} = 5.8284$$

$$H(z) = \frac{(z+1)^2}{(z+1)^2 + a(z-1)(z+1) + b(z-1)^2}$$

$$H(z) = \frac{z^2 + 2z + 1}{(z^2 + 2z + 1) + a(z^2 - 1) + b(z^2 - 2z + 1)}$$

$$H(z) = \frac{z^2 + 2z + 1}{(1 + a + b)z^2 + (2 - 2b)z + (1 - a + b)}$$

Inserting the values for a and b we get:

$$H(z) = \frac{z^2 + 2z + 1}{10.2426\,z^2 - 9.6568\,z + 3.4142}$$

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{10.2426 \ - 9.6568\, z^{-1} + 3.4142 z^{-2}}$$

$$H(z) = 0.0976\ \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.9428\, z^{-1} + 0.3333 z^{-2}}$$

2) Determine the difference equation.

$$H(z) = \frac{Y(z)}{X(z)} = 0.0976\ \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.9428\, z^{-1} + 0.3333 z^{-2}}$$

$$Y(z) - 0.9428\, Y(z) z^{-1} + 0.3333 Y(z) z^{-2} = 0.0976 (X(z) + 2X(z) z^{-1} + z^{-2})$$

$$\mathrm{y[n]} = 0.0976\mathrm{x[n]} + 0.1953\mathrm{x[n-1]} + 0.0976\mathrm{x[n-2]} + 0.9428 y[n-1] - 0.3333 y[n-2$$

Problem 3 (weighted with 10% - Digital filters)
Determine the impulse response h[k] of a linear-phase FIR filter for which M=3 (i.e. length of impulse response is 4). The amplitude response is specified as:

$$\left|H\left(\frac{\pi}{2}\right)\right| = \sqrt{2} \text{ and } \ |H(\pi)| = 2$$

(Hints: it may prove helpful to consider the requirements and expressions for the frequency responses for FIR filters Type I - IV in the text book. Notice: $sin\left(\frac{\pi}{4}\right) = sin\left(\frac{3\pi}{4}\right) = \sqrt{\frac{1}{2}}$ *)*

Questions:

1) Why has it to be a filter Type IV?

As M is odd then it must be either a Type II or a Type IV filter.

The amplitude response is (according to the text book – O&S, 2nd ed. p. 299):

Type II:

$$|H(e^{j\omega})| = \sum_{k=1}^{\frac{M+1}{2}} b[k] \cos\left(\omega\left(k - \frac{1}{2}\right)\right),$$

Where

$$b[k] = 2h[((M+1))/2 - k], \qquad k = 1,2, \dots, (M+1)/2$$

Type IV:

$$|H(e^{j\omega})| = \sum_{k=1}^{\frac{M+1}{2}} d[k]\sin\left(\omega\left(k-\frac{1}{2}\right)\right)$$

Where

$$d[k] = 2h[((M+1))/2-k], \quad k = 1,2,\dots,(M+1)/2$$

For Type II the amplitude response at π will be zero. This is in conflict with the requirement $|H(\pi)| = 2$. This is not the case for Type IV!

2) Determine the impulse response

It's know that it is a Type IV filter and that:

$$\left|H\left(\frac{\pi}{2}\right)\right| = \sqrt{2} \text{ and}$$
$$|H(\pi)| = 2$$

The amplitude response is:

$$|H(e^{j\omega})| = \sum_{k=1}^{\frac{M+1}{2}} b[k]\cos\left(\omega\left(k-\frac{1}{2}\right)\right),$$

The known values are inserted

$$|H(e^{j\omega})|_{\frac{\pi}{2}} = \sum_{k=1}^{\frac{3+1}{2}} d[k]\sin\left(\frac{\pi}{2}\left(k-\frac{1}{2}\right)\right) = \sqrt{2} \wedge |H(e^{j\omega})|_{\pi} = \sum_{k=1}^{\frac{3+1}{2}} d[k]\sin\left(\pi\left(k-\frac{1}{2}\right)\right) = 2$$

$$d[1]\sin\left(\frac{\pi}{2}\left(1-\frac{1}{2}\right)\right) + d[2]\sin\left(\frac{\pi}{2}\left(2-\frac{1}{2}\right)\right) = \sqrt{2} \;\wedge$$
$$d[1]\sin\left(\pi\left(1-\frac{1}{2}\right)\right) + d[2]\sin\left(\pi\left(2-\frac{1}{2}\right)\right) = 2$$

$$d[1]\sin\left(\frac{\pi}{4}\right) + d[2]\sin\left(\frac{3\pi}{4}\right) = \sqrt{2} \;\wedge$$
$$d[1]\sin\left(\frac{\pi}{2}\right) + d[2]\sin\left(\frac{3\pi}{2}\right) = 2$$

Notice that $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \sqrt{\frac{1}{2}}$

$$d[1]\sqrt{\frac{1}{2}} + d[2]\sqrt{\frac{1}{2}} = \sqrt{2} \;\wedge\; d[1] + d[2](-1) = 2$$

$$d[1]\sqrt{\frac{1}{2}}+d[2]\sqrt{\frac{1}{2}}=\sqrt{2}\ \wedge\ d[1]=2+d[2]$$

$$(2+d[2])\sqrt{\frac{1}{2}}+d[2]\sqrt{\frac{1}{2}}=\sqrt{2}\ \wedge\ d[1]=2+d[2]$$

$$d[2]=0\ \wedge\ d[1]=2$$

$$d[1]=2h[1],\quad d[2]=2h[0]$$

For the Type IV the impulse response is anti-symmetric. The impulse response is then:

$$h[0]=0,\quad h[1]=1, h[2]=-1,\quad h[3]=0$$