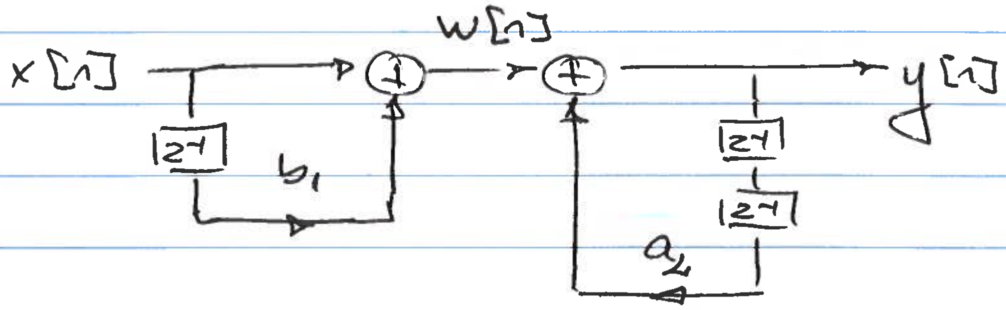
Ordinary Exam, Jan. 3 2022. 1.

COMTEK, EIT

A.1

B.1



A) Difference equation.

$$\begin{cases} w[n] = x[n] + b_1 x[n-1] \\ y[n] = w[n] + a_2 y[n-2] \end{cases}$$

$\Downarrow$

$$y[n] = a_2 y[n-2] + x[n] + b_1 x[n-1]$$

B) Transfer function.

$$Y(z) = \mathcal{Z}\{y[n]\}$$

$\Downarrow$

$$Y(z) = \mathcal{Z}\{a_2 y[n-2] + x[n] + b_1 x[n-1]\}$$

$\Downarrow$

$$Y(z) = a_2 Y(z) z^{-2} + X(z) + b_1 X(z) z^{-1}$$

$\Downarrow$

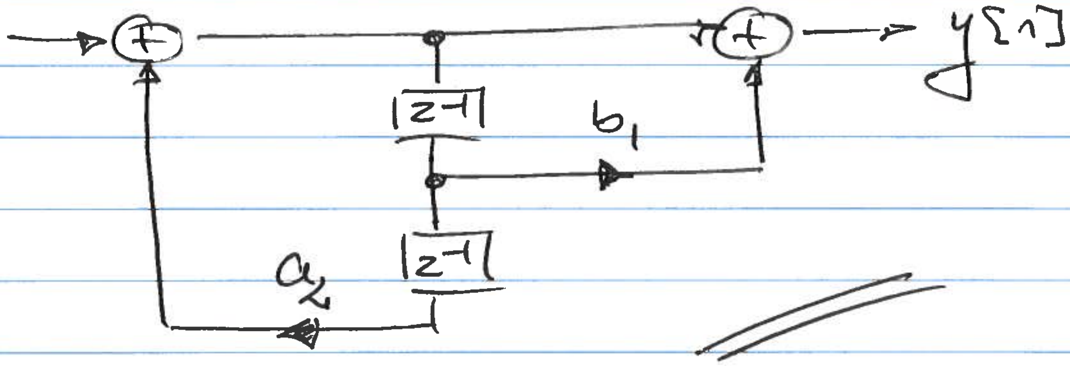
$$Y(z)(1 - a_2 z^{-2}) = X(z)(1 + b_1 z^{-1})$$

$\Downarrow$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + b_1 z^{-1}}{1 - a_2 z^{-2}}$$

c) Direct Form II structure.

$$y[n] = a_2 y[n-2] + x[n] + b_1 x[n-1]$$



D) H(z) should represent a stable filter.

$$H(z) = \frac{1 + b_1 z^{-1}}{1 - a_2 z^{-2}} = \frac{B(z)}{A(z)}$$

Stability: One possibility is to require that the poles are located inside the unit circle.

Poles are the roots of A(z) — thus no requirement on B(z) and b_1.

$$A(z) = 1 - a_2 z^{-2} = 0$$

$$\Downarrow$$

$$z^2 - a_2 = 0$$

$$\Downarrow$$

$$z^2 = a_2$$

$$\Downarrow$$

$$z = \pm\sqrt{a_2}$$

So, double pole in $z=\sqrt{a_2}$ or in $z=-\sqrt{a_2}$

$\Downarrow$

$$\left|\sqrt{a_2}\right| < 1$$

E) $b_1 = 1$ and $a_2 = 0.9$

$$y[n] = 0.9\,y[n-2] + x[n] + x[n-1]$$

$\Downarrow$ and Causal

$$h[n] = 0.9\,h[n-2] + \delta[n] + \delta[n-1]$$

$\Downarrow$

$$h[0] = 0.9\,h[-2] + \delta[0] + \delta[-1] = 1$$
$$h[1] = 0.9\,h[-1] + \delta[1] + \delta[0] = 1$$
$$h[2] = 0.9\,h[0] + \delta[2] + \delta[1] = 0.9$$
$$h[3] = 0.9\,h[1] + \delta[3] + \delta[2] = 0.9$$
$$h[4] = 0.9\,h[2] + \delta[4] + \delta[3] = (0.9)^2$$
$$h[5] = 0.9\,h[3] + \delta[5] + \delta[4] = (0.9)^2$$
$$h[6] = 0.9\,h[4] + \delta[6] + \delta[5] = (0.9)^3$$
$$h[7] = 0.9\,h[5] + \delta[7] + \delta[6] = (0.9)^3$$

Based on this result we conclude that the general expression for $h[n]$ is ;

$$h[n+1] = h[n] = 0.9^{n/2}, \quad n = 0, 2, 4, 6 \ldots$$

F) $f_s = 1\,kHz \Rightarrow T = 10^{-3}$ [sec].

$h[0] = 1$

For which value of n is $h[n] = 0{,}05$?

$$h[n] = 0.9^{n/2}, \quad n = 0, 2, 4, 6, \dots$$

$$\Downarrow$$

$$0.05 = 0.9^{n/2}$$

$$\Downarrow$$

$$n/2 \log(0.9) = \log(0.05)$$

$$\Downarrow$$

$$n = \left\lceil 2 \cdot \frac{\log(0.05)}{\log(0.9)} \right\rceil = \lceil 56.87 \rceil = 57$$

Since n is an integer even number,

$$\underline{\underline{n = 58}}$$

And thus;

$$\Delta t = 58 \cdot 10^{-3} \text{ [sec]} = \underline{\underline{58\,ms}}$$

A.2
B.2

$$H_c(s) = \frac{1}{s+\Omega_c}, \quad \Omega_c = 2\pi 600 \text{ rad/sec.}$$

a) $H(z)$ by using the impulse invariance method. $f_s = 3600$ Hz and DC gain equals 0 dB.

$$h(t) = \mathcal{L}^{-1}\{H_c(s)\}$$

$$\Downarrow$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+\Omega_c}\right\}$$

$$\Downarrow$$

$$h(t) = e^{-t\cdot\Omega_c} \cdot u(t)$$

Now, sample $h(t)$;

$$h[n] = T \cdot h(t)\Big|_{t=nT}$$

$$\Downarrow$$

$$h[n] = \frac{1}{3600} \cdot e^{-n\cdot\frac{1}{3600}\cdot 2\cdot\pi\cdot 600} \cdot u[n]$$

$$\Downarrow$$

$$h[n] = \frac{1}{3600} \cdot e^{-\frac{\pi}{3}n} \cdot u[n]$$

$$H'(z) = \mathcal{Z}\{h[n]\}$$

$$= \sum_{n=-\infty}^{\infty} h[n] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3600} \cdot e^{-\frac{\pi}{3} n} \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3600} \cdot \left(e^{-\frac{\pi}{3}} \cdot z^{-1}\right)^{n}$$

$$= \frac{1}{3600} \sum_{n=0}^{\infty} c^{n} \; ; \quad c = e^{-\frac{\pi}{3}} \cdot z^{-1}$$

$$= \frac{1}{3600} \cdot \frac{1 - c^{\infty}}{1 - c}$$

$$= \frac{1}{3600} \cdot \frac{1}{1 - e^{-\pi/3} z^{-1}}$$

Now, 0 dB DC gain (DC ~ $\omega = 0$ rad)

$$H(e^{j\omega}) = \frac{1}{3600} \cdot \frac{1}{1 - e^{-\pi/3} \cdot e^{-j\omega}} \quad (z = e^{j\omega})$$

$$G \cdot \left| H(e^{j\omega}) \right| = 1 \Big|_{\omega=0} \quad (\sim 0dB)$$

$\Downarrow$

$$G \cdot \frac{\frac{1}{3600}}{1-e^{-\pi/3}} = 1$$

$\Downarrow$

$$G = 3600 \cdot (1-e^{-\pi/3}) = 2336{,}7$$

$$H(z) = G \cdot H'(z)$$

$\Downarrow$

$$H(z) = \frac{2336{,}7}{3600} \cdot \frac{1}{1-e^{-\pi/3}z^{-1}}$$

$\Downarrow$

$$H(z) = \frac{0{,}6491}{1-0{,}3509z^{-1}}$$

B) 3 dB cut-off frequency.

$$\left| H(e^{j\omega}) \right| = \left| \frac{0{,}6491}{1-0{,}3509e^{-j\omega}} \right| = -3dB$$

$\Downarrow$

$$\left| \frac{0.6491}{1-0.3509e^{-j\omega}} \right| = 0{,}707946 \quad \text{find } \omega.$$

$$\frac{0.6491}{0.7079} = \left| 1 - 0.3509\, e^{-j\omega} \right|$$

$$= \left| 1 - 0.3509 \cos\omega + j\, 0.3509 \sin\omega \right|$$

$$= \sqrt{(1 - 0.3509\cos\omega)^2 + (0.3509\sin\omega)^2}$$

$$= \sqrt{1 + (0.3509\cos\omega)^2 - 2\cdot 0.3509\cos\omega + (0.3509\sin\omega)^2}$$

$$= \sqrt{1{,}1231 - 0{,}7018\cos\omega}$$

$$\Downarrow$$

$$\left(\frac{0.6491}{0.7079}\right)^2 = 1{,}1231 - 0{,}7018\cos\omega$$

$$\Downarrow$$

$$\cos\omega = \frac{1{,}1231 - \left(\frac{0.6491}{0.7079}\right)^2}{0.7018}$$

$$\Downarrow$$

$$\cos\omega = 0.4026$$

$$\Downarrow$$

$$\omega = 1.1565 \text{ rad}$$

c) Phase response @ $\omega = \pi/2$.

$$H(e^{j\omega}) = \frac{0.6491}{1 - 0.3509e^{-j\omega}}$$

$$\angle H(e^{j\omega}) = \angle 0.6491 - \angle 1 - 0.3509 \cdot e^{-j\omega}$$

$\Downarrow$

$$\angle H(e^{j\omega}) = -\angle (1 - 0.3509\cos\omega) + j(0.3509\sin\omega)$$

$\Downarrow$

$$\angle H(e^{j\omega}) = -\arctan\left\{\frac{0.3509\sin\omega}{1 - 0.3509\cos\omega}\right\}$$

$\Downarrow$

$$\angle H(e^{j\omega}) = -\arctan\{0.3509\}, \quad \omega = \pi/2$$

$\Downarrow$

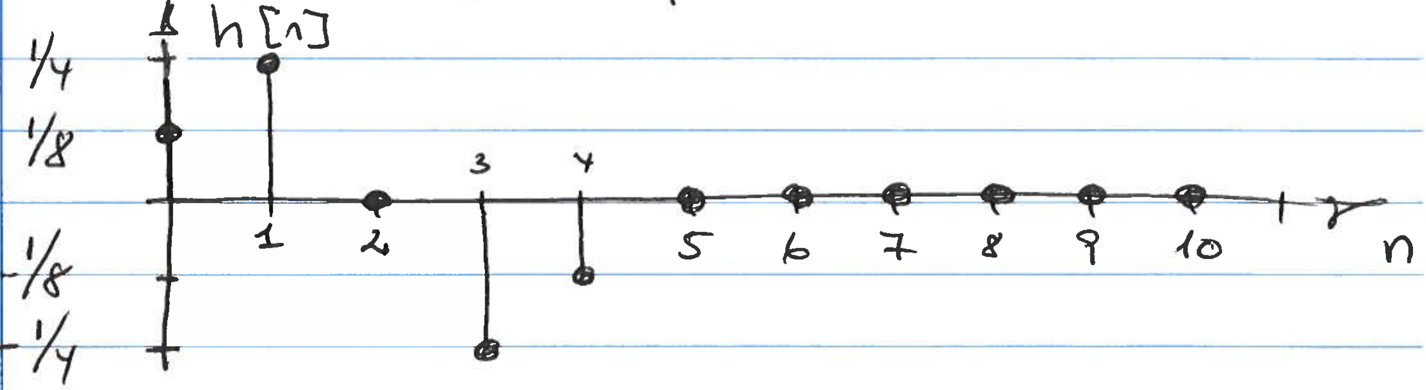
$$\angle H(e^{j\pi/2}) = -0.3375 \text{ rad}$$

A.3
B.3

Type III Linear Phase FIR filter.

$h[0] = -h[4] = 1/8$
$h[1] = -h[3] = 1/4$
$h[2] = 0$

A) impulse response for $0 \leq n \leq 10$



B) amplitude response @ $\omega = \pi/2$.

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$$H(e^{j\omega}) = j e^{-j\omega M/2} \left\{ \sum_{k=1}^{M/2} c[k] \cdot \sin(\omega k) \right\}$$

$$c[k] = 2h[M/2 - k], \quad k = 1, 2, \ldots M/2$$

$$\left| H(e^{j\omega}) \right| = \left| j e^{-j\omega M/2} \left\{ \sum_{k=1}^{M/2} c[k] \cdot \sin(\omega k) \right\} \right|$$

$\Downarrow$

$$\left| H(e^{j\omega}) \right| = \left| \sum_{k=1}^{2} c[k] \cdot \sin(\omega k) \right| \Bigg|_{M=4}$$

$\omega = \pi/2$

$$\left|H(e^{j\pi/2})\right| = \left|\sum_{k=1}^{2} c[k]\cdot\sin(k\cdot\pi/2)\right|$$

$\Downarrow$

$$\left|H(e^{j\pi/2})\right| = \left|c[1]\cdot\sin(\pi/2) + c[2]\cdot\sin(\pi)\right|$$

$$c[1] = 2h[1] = 2\cdot\frac{1}{4} = \frac{1}{2}$$

$$c[2] = 2h[0] = 2\cdot\frac{1}{8} = \frac{1}{4}$$

$$\left|H(e^{j\pi/2})\right| = \left|\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 0\right| = \frac{1}{2}$$

c) Transfer function $H(z)$

$$H(z) = \frac{B(z)}{A(z)} = B(z) \text{ for FIR}$$

$\Downarrow$

$$H(z) = \sum_{k=0}^{M} b_k\cdot z^{-k} \qquad \text{and } b_k = h[k]$$

$\Downarrow$

$$H(z) = \sum_{k=0}^{4} h[k]\cdot z^{-k}$$

$\Downarrow$

$$H(z) = 1/8 + 1/4\, z^{-1} - 1/4\, z^{-3} - 1/8\, z^{-4}$$

D) Pole-Zero diagram.

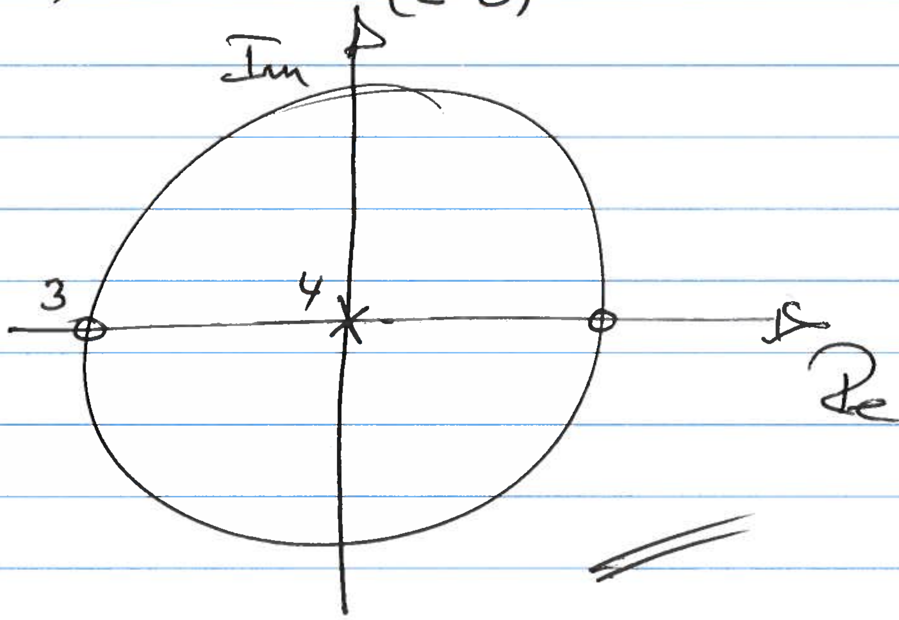
$$H(z) = \frac{B(z)}{A(z)} = \frac{1/8 + 1/4 z^{-1} - 1/4 z^{-3} - 1/8 z^{-4}}{1}$$

$\Downarrow$

$$H(z) = \frac{1/8 z^4 + 1/4 z^3 - 1/4 z + 1/8}{z^4}$$

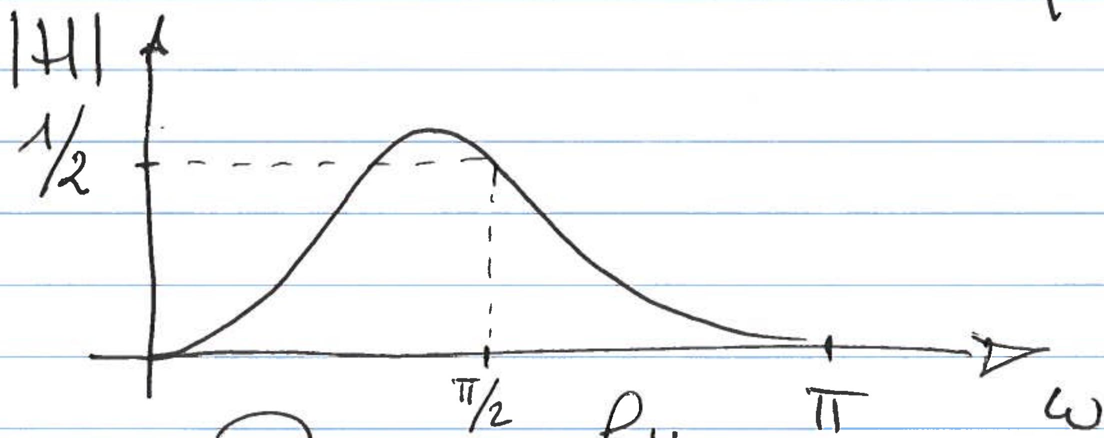
$\Downarrow$

$$H(z) = \frac{(z-1)(z+1)(z+1)(z+1)}{(z-0)^4}$$



E) From B we know that $|H| = 1/2$ for $\omega = \pi/2$

From D we know that $|H| = 0$ for $\begin{cases} \omega = 0 \\ \omega = \pi \end{cases}$



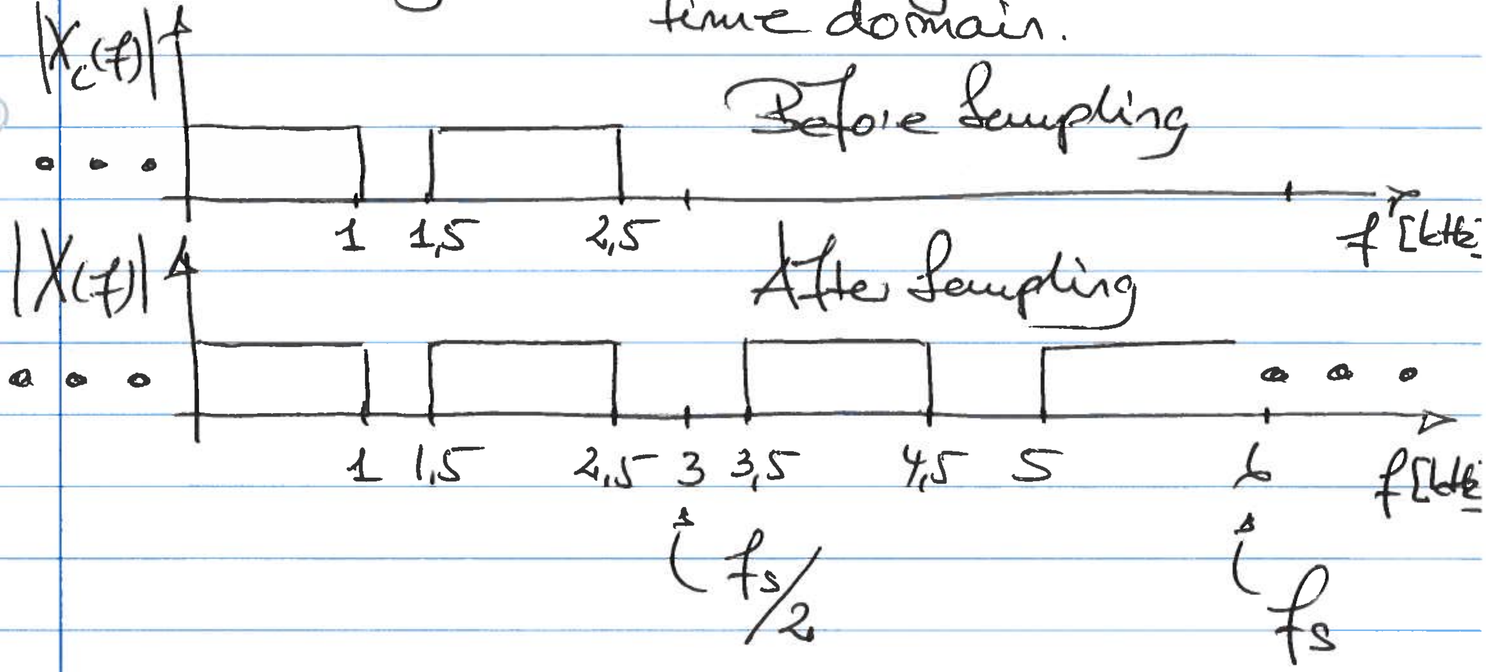
Band-pass filter

B1 C1

$$|X_c(\Omega)| = \begin{cases} 1 & |\Omega| \leq 2\pi 1000 \text{ rad/sec} \\ 1 & 2\pi \cdot 1500 \text{ rad/sec} \leq |\Omega| \leq 2\pi 2500 \text{ rad/sec} \\ 0 & \text{otherwise.} \end{cases}$$

$f_s = 6000\,Hz$.

A) Sampling $\Rightarrow$ Periodicity in the discrete time domain.



B) 20 ms segment sampled at 6000 Hz.

$$N \cdot T = 20 \cdot 10^{-3} \text{ [sec]}$$

$$\Downarrow$$

$$N = f_s \cdot 20 \cdot 10^{-3} = 6 \cdot 10^{3} \cdot 20 \cdot 10^{-3} = 120 \text{ samples}$$

These 120 samples are equaly distributed in $[0; 2\pi]$, i.e.,

$$\Delta f = \frac{6000\,Hz}{120} = 50\,Hz$$

c) ° increasing the spectral resolution, i.e., make Δf smaller;

$$\Delta f = \frac{f_s}{N}$$

For constant sample frequency f_s, N can be increased. Done by increasing the window length (in the time domain).

The consequence is that we get a reduced accuracy in the time domain

$\sim$ Heisenberg ! Can be discussed in some more details.

D) $x[n] = x_c[n] \cdot w[n]$ where $w[n]$ is a window function.

$\Downarrow$

$$X(e^{j\omega}) = X_c(e^{j\omega}) * W(e^{j\omega})$$

where $W(e^{j\omega})$ is the Fourier transform of the window function.

In order to match the discontinuities of $X_c(e^{j\omega})$ as good as possible, we want the main-lobe of $W(e^{j\omega})$ to be as narrow as possible $\Longrightarrow$ Rectangular window

B.2
C.2

$x[n] = \{1, 1, 0, 0\}, \ N = 4.$

Calculate $X[k]$ for $k = 0, 1, \ldots, N-1$

A) $$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}nk}, \quad k = 0, 1, 2, 3$$

$\Downarrow$

$$X[k] = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{\pi}{2}nk}$$

$$X[0] = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{\pi}{2}n \cdot 0} = 1 + 1 + 0 + 0 = 2$$

$$X[1] = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{\pi}{2}n \cdot 1} = 1 + e^{-j\frac{\pi}{2}} = 1 - j$$

$$X[2] = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{\pi}{2} \cdot n \cdot 2} = 1 + e^{-j\pi} = 0$$

$$X[3] = \sum_{n=0}^{3} x[n] \cdot e^{-j\frac{\pi}{2} \cdot n \cdot 3} = 1 + e^{-j\frac{3\pi}{2}} = 1 + j$$

$$X[k] = \{2, 1-j, 0, 1+j\}$$

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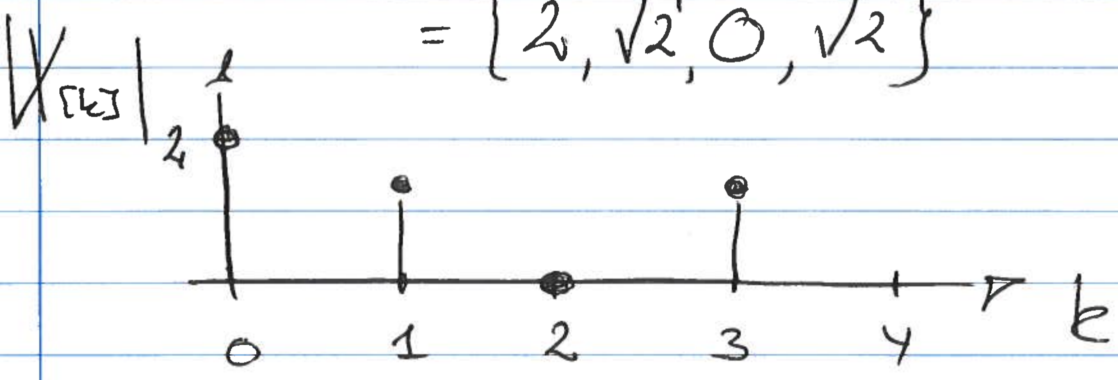
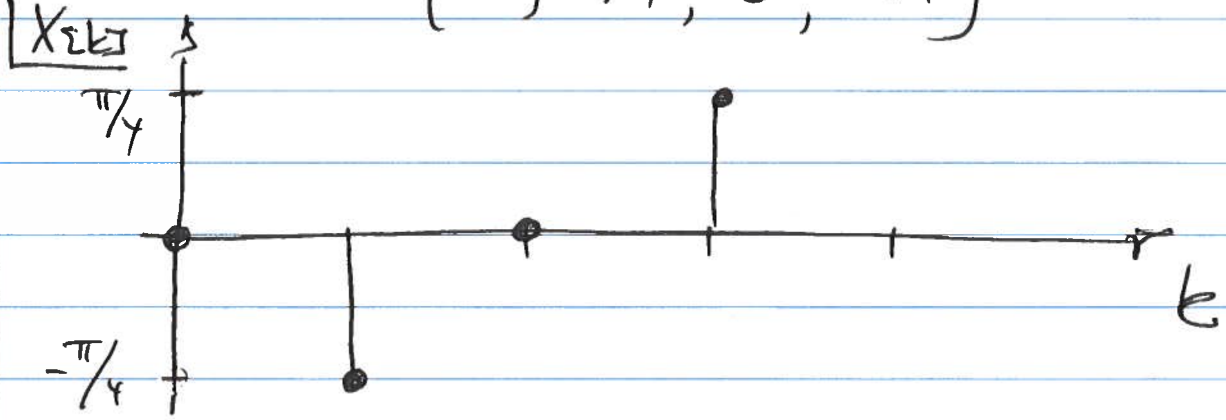
B) $|X[k]| = \{|2|, |1-j|, |0|, |1+j|\}$

$= \{2, \sqrt{2}, 0, \sqrt{2}\}$



$\angle X[k] = \{\arg(2), \arg(1-j), \arg(0), \arg(1+j)\}$

$= \{0, -\pi/4, 0, \pi/4\}$



B.3 $x[n] = 0.5^n$

C.3 One period for $n = 0,..N-1$, $N = 100$.

Start by calculating the DTFT;

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} 0.5^n \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{99} \underbrace{(0.5 \cdot e^{-j\omega})}_{q}{}^n \quad \text{Geometric Series}$$

$$= \frac{1-q^N}{1-q} = \frac{1-(0.5 \cdot e^{-j\omega})^{100}}{1-0.5 \cdot e^{-j\omega}}$$

$$= \frac{1}{1-0.5 \cdot e^{-j\omega}}$$

Next, sample $X(e^{j\omega})$ @ $\omega = \frac{3\pi}{2}$

$$\mathrm{DFT}\{x[n]\} = \mathrm{DTFT}\{x[n]\}\Big|_{\omega = \frac{2\pi}{N} \cdot k}$$

$$\omega = \frac{3\pi}{2} = \frac{2\pi}{100} \cdot k \Rightarrow k = 75.$$

$$X[75] = X(e^{j\frac{3\pi}{2}}) = \frac{1}{1-0.5\cdot e^{-j\frac{3\pi}{2}}}$$

$$= \frac{1}{1-0.5\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)}$$

$$= \frac{1}{1-0.5(0+j)}$$

$$= \frac{1}{1-j0.5}$$

Or;

$$= \frac{1+j0.5}{1.25}$$

$$= 0.8+j0.4$$

B.4

C.4

$$W_N = e^{-j\frac{2\pi}{N}} \text{ — twiddle factor}$$

A)
$$W_N^{k(N-n)} = e^{-j\frac{2\pi}{N}(k(N-n))}$$
$$= e^{-j\frac{2\pi}{N}(kN-kn)}$$
$$= e^{j\frac{2\pi}{N}kN} \cdot e^{j\frac{2\pi}{N}kn}$$
$$= e^{j\frac{2\pi}{N}kn}$$
$$= W_N^{-kn}$$

$$W_N^{-kn} = e^{j\frac{2\pi}{N}kn}$$
$$= \cos\left(\frac{2\pi}{N}kn\right) + j\sin\left(\frac{2\pi}{N}\cdot kn\right)$$
$$= \left(\cos\left(-\frac{2\pi}{N}\cdot k\cdot n\right) - j\sin\left(-\frac{2\pi}{N}kn\right)\right)^*$$
$$= \left(e^{-j\frac{2\pi}{N}kn}\right)^*$$
$$= \left(W_N^{kn}\right)^*$$

3) W_8^1 in polar coordinates

$$W_8^1 = e^{-j\frac{2\pi}{8}\cdot 1}$$

$$= e^{-j\frac{\pi}{4}}$$

$$= \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$