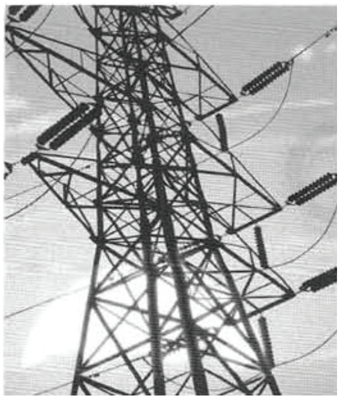

CHAPTER 13

Complex Numbers and Functions

Complex numbers and their geometric representation in the **complex plane** are discussed in Secs. 13.1 and 13.2. Complex analysis is concerned with complex **analytic functions** as defined in Sec. 13.3. Checking for analyticity is done by the **Cauchy–Riemann equations** (Sec. 13.4). These equations are of basic importance, also because of their relation to **Laplace's equation.**

The remaining sections of the chapter are devoted to elementary complex functions (exponential, trigonometric, hyperbolic, and logarithmic functions). These generalize the familiar real functions of calculus. Their detailed knowledge is an absolute necessity in practical work, just as that of their real counterparts is in calculus.

Prerequisite: Elementary calculus.
References and Answers to Problems: App. 1 Part D, App. 2.

13.1 Complex Numbers. Complex Plane

Equations without *real* solutions, such as $x^2 = -1$ or $x^2 - 10x + 40 = 0$, were observed early in history and led to the introduction of complex numbers.[1] By definition, a **complex number** z is an ordered pair (x, y) of real numbers x and y, written

$$z = (x, y).$$

x is called the **real part** and y the **imaginary part** of z, written

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

By definition, two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

(0, 1) is called the **imaginary unit** and is denoted by i,

$$i = (0, 1). \tag{1}$$

[1]First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501–1576), who found the formula for solving cubic equations. The term "complex number" was introduced by CARL FRIEDRICH GAUSS (see the footnote in Sec. 5.4), who also paved the way for a general use of complex numbers.

Addition, Multiplication. Notation $z = x + iy$

Addition of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \quad y_1 + y_2). \tag{2}$$

Multiplication is defined by

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, \quad x_1 y_2 + x_2 y_1). \tag{3}$$

In particular, these two definitions imply that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

and

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

as for real numbers x_1, x_2. Hence the complex numbers ***"extend"*** the real numbers. We can thus write

$$(x, 0) = x. \qquad \text{Similarly,} \qquad (0, y) = iy$$

because by (1) and the definition of multiplication we have

$$iy = (0, 1)y = (0, 1)(y, 0) = (0 \cdot y - 1 \cdot 0, \quad 0 \cdot 0 + 1 \cdot y) = (0, y).$$

Together we have by addition $(x, y) = (x, 0) + (0, y) = x + iy$:

In practice, complex numbers $z = (x, y)$ are written

$$z = x + iy \tag{4}$$

or $z = x + yi$, e.g., $17 + 4i$ (instead of $i4$).

Electrical engineers often write j instead if i because they need i for the current.

If $x = 0$, then $z = iy$ and is called **pure imaginary.** Also, (1) and (3) give

$$i^2 = -1 \tag{5}$$

because by the definition of multiplication, $i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$.

For **addition** the standard notation (4) gives [see (2)]

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

For **multiplication** the standard notation gives the following very simple recipe. Multiply each term by each other term and use $i^2 = -1$ when it occurs [see (3)]:

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).\end{aligned}$$

This agrees with (3). And it shows that $x + iy$ is a more practical notation for complex numbers than (x, y).

If you know vectors, you see that (2) is vector addition, whereas the multiplication (3) has no counterpart in the usual vector algebra.

EXAMPLE 1 Real Part, Imaginary Part, Sum and Product of Complex Numbers

Let $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$. Then $\text{Re}\, z_1 = 8$, $\text{Im}\, z_1 = 3$, $\text{Re}\, z_2 = 9$, $\text{Im}\, z_2 = -2$ and

$$z_1 + z_2 = (8 + 3i) + (9 - 2i) = 17 + i,$$

$$z_1 z_2 = (8 + 3i)(9 - 2i) = 72 + 6 + i(-16 + 27) = 78 + 11i. \qquad \blacksquare$$

Subtraction, Division

Subtraction and **division** are defined as the inverse operations of addition and multiplication, respectively. Thus the **difference** $z = z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$. Hence by (2),

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \tag{6}$$

The **quotient** $z = z_1/z_2$ $(z_2 \neq 0)$ is the complex number z for which $z_1 = zz_2$. If we equate the real and the imaginary parts on both sides of this equation, setting $z = x + iy$, we obtain $x_1 = x_2 x - y_2 y$, $y_1 = y_2 x + x_2 y$. The solution is

$$z = \frac{z_1}{z_2} = x + iy, \qquad x = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \qquad y = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}. \tag{7*}$$

The ***practical rule*** used to get this is by multiplying numerator and denominator of z_1/z_2 by $x_2 - iy_2$ and simplifiying:

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}. \tag{7}$$

EXAMPLE 2 Difference and Quotient of Complex Numbers

For $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$ we get $z_1 - z_2 = (8 + 3i) - (9 - 2i) = -1 + 5i$ and

$$\frac{z_1}{z_2} = \frac{8 + 3i}{9 - 2i} = \frac{(8 + 3i)(9 + 2i)}{(9 - 2i)(9 + 2i)} = \frac{66 + 43i}{81 + 4} = \frac{66}{85} + \frac{43}{85}\,i.$$

Check the division by multiplication to get $8 + 3i$. $\blacksquare$

Complex numbers satisfy the same commutative, associative, and distributive laws as real numbers (see the problem set).

Complex Plane

This was algebra. Now comes geometry: the geometrical representation of complex numbers as points in the plane. This is of great practical importance. The idea is quite simple and natural. We choose two perpendicular coordinate axes, the horizontal x-axis, called the **real axis**, and the vertical y-axis, called the **imaginary axis.** On both axes we choose the same unit of length (Fig. 315). This is called a **Cartesian coordinate system.**

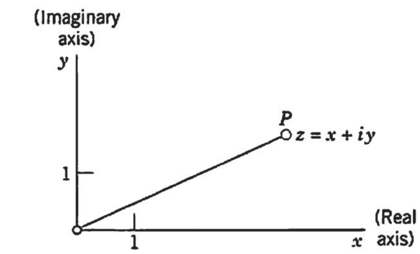

Fig. 315. The complex plane

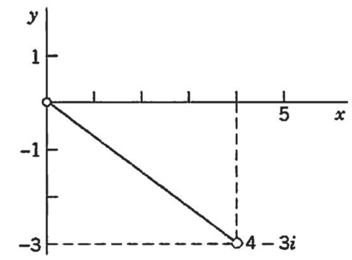

Fig. 316. The number $4 - 3i$ in the complex plane

We now plot a given complex number $z = (x, y) = x + iy$ as the point P with coordinates x, y. The xy-plane in which the complex numbers are represented in this way is called the **complex plane.**[2] Figure 316 shows an example.

Instead of saying "the point represented by z in the complex plane" we say briefly and simply ***"the point z in the complex plane."*** This will cause no misunderstandings.

Addition and subtraction can now be visualized as illustrated in Figs. 317 and 318.

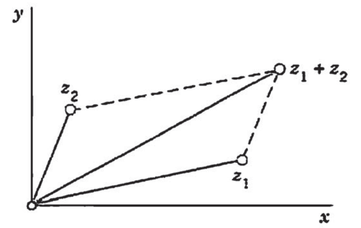

Fig. 317. Addition of complex numbers

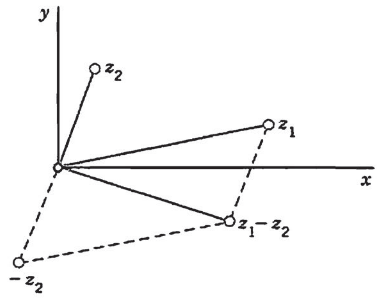

Fig. 318. Subtraction of complex numbers

Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

It is obtained geometrically by reflecting the point z in the real axis. Figure 319 shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

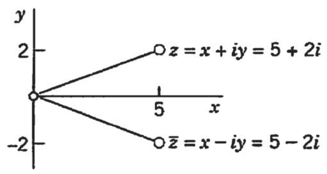

Fig. 319. Complex conjugate numbers

[2]Sometimes called the **Argand diagram,** after the French mathematician JEAN ROBERT ARGAND (1768–1822), born in Geneva and later librarian in Paris. His paper on the complex plane appeared in 1806, nine years after a similar memoir by the Norwegian mathematician CASPAR WESSEL (1745–1818), a surveyor of the Danish Academy of Science.

The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

$$\text{(8)} \qquad \operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z}).$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely.
Working with conjugates is easy, since we have

$$\text{(9)} \qquad \begin{aligned} &\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad &&\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2, \\ &\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad &&\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

EXAMPLE 3 Illustration of (8) and (9)

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\operatorname{Im} z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$
$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i.$$

■

PROBLEM SET 13.1

1. **(Powers of i)** Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.
2. **(Rotation)** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 2 + 2i$, $z = -1 - 5i$, $z = 4 - 3i$.
3. **(Division)** Verify the calculation in (7).
4. **(Multiplication)** If the product of two complex numbers is zero, show that at least one factor must be zero.
5. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.
6. **(Laws for conjugates)** Verify (9) for $z_1 = 24 + 10i$, $z_2 = 4 + 6i$.

7–15 COMPLEX ARITHMETIC

Let $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$. Showing the details of your work, find (in the form $x + iy$):

7. $(5z_1 + 3z_2)^2$
8. $\bar{z}_1 \bar{z}_2$
9. $\operatorname{Re}(1/z_1^{\,2})$
10. $\operatorname{Re}(z_2^{\,2})$, $(\operatorname{Re} z_2)^2$
11. z_2/z_1
12. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$
13. $(4z_1 - z_2)^2$
14. $\bar{z}_1/z_1$, $z_1/\bar{z}_1$
15. $(z_1 + z_2)/(z_1 - z_2)$

16–19 Let $z = x + iy$. Find:

16. $\operatorname{Im} z^3$, $(\operatorname{Im} z)^3$
17. $\operatorname{Re}(1/\bar{z})$
18. $\operatorname{Im}[(1 + i)^8 z^2]$
19. $\operatorname{Re}(1/\bar{z}^2)$
20. **(Laws of addition and multiplication)** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \quad \textit{(Commutative laws)}$$
$$\left.\begin{aligned}(z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3),\\ (z_1 z_2) z_3 &= z_1 (z_2 z_3)\end{aligned}\right\} \quad \textit{(Associative laws)}$$
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \quad \textit{(Distributive law)}$$
$$0 + z = z + 0 = z,$$
$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

13.2 Polar Form of Complex Numbers. Powers and Roots

The complex plane becomes even more useful and gives further insight into the arithmetic operations for complex numbers if besides the xy-coordinates we also employ the usual polar coordinates r, θ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

We see that then $z = x + iy$ takes the so-called **polar form**

$$z = r(\cos\theta + i\sin\theta). \tag{2}$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}\,. \tag{3}$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 320). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 321).

θ is called the **argument** of z and is denoted by $\arg z$. Thus (Fig. 320)

$$\theta = \arg z = \arctan\frac{y}{x} \qquad (z \neq 0). \tag{4}$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 320. Here, as in calculus, all ***angles are measured in radians and positive in the counterclockwise sense.***

For $z = 0$ this angle θ is undefined. (Why?) For a given $z \neq 0$ it is determined only up to integer multiples of 2π since cosine and sine are periodic with period 2π. But one often wants to specify a unique value of $\arg z$ of a given $z \neq 0$. For this reason one defines the **principal value** $\operatorname{Arg} z$ (with capital A!) of $\arg z$ by the double inequality

$$-\pi < \operatorname{Arg} z \leqq \pi. \tag{5}$$

Then we have $\operatorname{Arg} z = 0$ for positive real $z = x$, which is practical, and $\operatorname{Arg} z = \pi$ (not $-\pi$!) for negative real z, e.g., for $z = -4$. The principal value (5) will be important in connection with roots, the complex logarithm (Sec. 13.7), and certain integrals. Obviously, for a given $z \neq 0$ the other values of $\arg z$ are $\arg z = \operatorname{Arg} z \pm 2n\pi$ ($n = \pm 1, \pm 2, \cdots$).

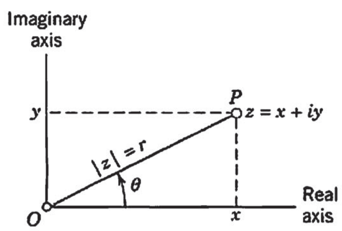

Fig. 320. Complex plane, polar form of a complex number

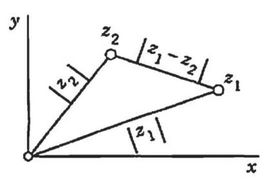

Fig. 321. Distance between two points in the complex plane

EXAMPLE 1 Polar Form of Complex Numbers. Principal Value Arg z

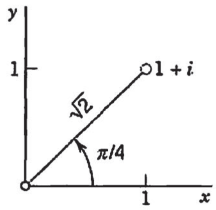

Fig. 322. Example 1

$z = 1 + i$ (Fig. 322) has the polar form $z = \sqrt{2}\,(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$. Hence we obtain

$$|z| = \sqrt{2}, \qquad \arg z = \tfrac{1}{4}\pi \pm 2n\pi \ (n = 0, 1, \cdots), \qquad \text{and} \qquad \operatorname{Arg} z = \tfrac{1}{4}\pi \ \text{(the principal value).}$$

Similarly, $z = 3 + 3\sqrt{3}i = 6\,(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$, $|z| = 6$, and $\operatorname{Arg} z = \frac{1}{3}\pi$. ■

CAUTION! In using (4), we must pay attention to the quadrant in which z lies, since $\tan \theta$ has period π, so that the arguments of z and $-z$ have the same tangent. *Example:* for $\theta_1 = \arg(1 + i)$ and $\theta_2 = \arg(-1 - i)$ we have $\tan \theta_1 = \tan \theta_2 = 1$.

Triangle Inequality

Inequalities such as $x_1 < x_2$ make sense for *real* numbers, but not in complex because *there is no natural way of ordering complex numbers.* However, inequalities between absolute values (which are real!), such as $|z_1| < |z_2|$ (meaning that z_1 is closer to the origin than z_2) are of great importance. The daily bread of the complex analyst is the **triangle inequality**

$$|z_1 + z_2| \leqq |z_1| + |z_2| \qquad \text{(Fig. 323)} \tag{6}$$

which we shall use quite frequently. This inequality follows by noting that the three points 0, z_1, and $z_1 + z_2$ are the vertices of a triangle (Fig. 323) with sides $|z_1|$, $|z_2|$, and $|z_1 + z_2|$, and one side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob. 35). (The triangle degenerates if z_1 and z_2 lie on the same straight line through the origin.)

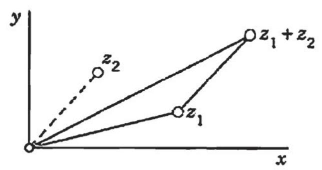

Fig. 323. Triangle inequality

By induction we obtain from (6) the **generalized triangle inequality**

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|; \tag{6*}$$

that is, *the absolute value of a sum cannot exceed the sum of the absolute values of the terms.*

EXAMPLE 2 Triangle Inequality

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then (sketch a figure!)

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.020.$$ ■

Multiplication and Division in Polar Form

This will give us a "geometrical" understanding of multiplication and division. Let

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos \theta_2 + i \sin \theta_2).$$

Multiplication. By (3) in Sec. 13.1 the product is at first

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

The addition rules for the sine and cosine [(6) in App. A3.1] now yield

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \tag{7}$$

Taking absolute values on both sides of (7), we see that *the absolute value of a product equals the* ***product*** *of the absolute values of the factors,*

$$|z_1z_2| = |z_1||z_2|. \tag{8}$$

Taking arguments in (7) shows that *the argument of a product equals the* ***sum*** *of the arguments of the factors,*

$$\arg(z_1z_2) = \arg z_1 + \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{9}$$

Division. We have $z_1 = (z_1/z_2)z_2$. Hence $|z_1| = |(z_1/z_2)z_2| = |z_1/z_2||z_2|$ and by division by $|z_2|$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0). \tag{10}$$

Similarly, $\arg z_1 = \arg[(z_1/z_2)z_2] = \arg(z_1/z_2) + \arg z_2$ and by subtraction of $\arg z_2$

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 \qquad \text{(up to multiples of } 2\pi). \tag{11}$$

Combining (10) and (11) we also have the analog of (7),

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]. \tag{12}$$

To comprehend this formula, note that it is the polar form of a complex number of absolute value r_1/r_2 and argument $\theta_1 - \theta_2$. But these are the absolute value and argument of z_1/z_2, as we can see from (10), (11), and the polar forms of z_1 and z_2.

EXAMPLE 3 Illustration of Formulas (8)–(11)

Let $z_1 = -2 + 2i$ and $z_2 = 3i$. Then $z_1z_2 = -6 - 6i$, $z_1/z_2 = 2/3 + (2/3)i$. Hence (make a sketch)

$$|z_1z_2| = 6\sqrt{2} = 3\sqrt{8} = |z_1||z_2|, \qquad |z_1/z_2| = 2\sqrt{2}/3 = |z_1|/|z_2|,$$

and for the arguments we obtain $\operatorname{Arg} z_1 = 3\pi/4$, $\operatorname{Arg} z_2 = \pi/2$,

$$\operatorname{Arg}(z_1z_2) = -\frac{3\pi}{4} = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi, \qquad \operatorname{Arg}(z_1/z_2) = \frac{\pi}{4} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2. \qquad \blacksquare$$

EXAMPLE 4 Integer Powers of z. De Moivre's Formula

From (8) and (9) with $z_1 = z_2 = z$ we obtain by induction for $n = 0, 1, 2, \cdots$

$$z^n = r^n (\cos n\theta + i \sin n\theta). \tag{13}$$

Similarly, (12) with $z_1 = 1$ and $z_2 = z^n$ gives (13) for $n = -1, -2, \cdots$. For $|z| = r = 1$, formula (13) becomes **De Moivre's formula**[3]

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \tag{13*}$$

We can use this to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos \theta$ and $\sin \theta$. For instance, for $n = 2$ we have on the left $\cos^2 \theta + 2i \cos \theta \sin \theta - \sin^2 \theta$. Taking the real and imaginary parts on both sides of (13*) with $n = 2$ gives the familiar formulas

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \qquad \sin 2\theta = 2 \cos \theta \sin \theta.$$

This shows that *complex* methods often simplify the derivation of *real* formulas. Try $n = 3$. ■

Roots

If $z = w^n$ $(n = 1, 2, \cdots)$, then to each value of w there corresponds *one* value of z. We shall immediately see that, conversely, to a given $z \neq 0$ there correspond precisely n distinct values of w. Each of these values is called an **nth root** of z, and we write

$$w = \sqrt[n]{z}. \tag{14}$$

Hence this symbol is ***multivalued,*** namely, *n-valued.* The n values of $\sqrt[n]{z}$ can be obtained as follows. We write z and w in polar form

$$z = r(\cos \theta + i \sin \theta) \qquad \text{and} \qquad w = R(\cos \phi + i \sin \phi).$$

Then the equation $w^n = z$ becomes, by De Moivre's formula (with ϕ instead of θ)

$$w^n = R^n(\cos n\phi + i \sin n\phi) = z = r(\cos \theta + i \sin \theta).$$

The absolute values on both sides must be equal; thus, $R^n = r$, so that $R = \sqrt[n]{r}$, where $\sqrt[n]{r}$ is positive real (an absolute value must be nonnegative!) and thus uniquely determined. Equating the arguments $n\phi$ and θ and recalling that θ is determined only up to integer multiples of 2π, we obtain

$$n\phi = \theta + 2k\pi, \qquad \text{thus} \qquad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

where k is an integer. For $k = 0, 1, \cdots, n - 1$ we get n *distinct* values of w. Further integers of k would give values already obtained. For instance, $k = n$ gives $2k\pi/n = 2\pi$,

[3]ABRAHAM DE MOIVRE (1667–1754), French mathematician, who pioneered the use of complex numbers in trigonometry and also contributed to probability theory (see Sec. 24.8).

hence the w corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

$$(15)\qquad \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i \sin\frac{\theta + 2k\pi}{n}\right)$$

where $k = 0, 1, \cdots, n - 1$. These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides. The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k = 0$ in (15) is called the **principal value** of $w = \sqrt[n]{z}$.

Taking $z = 1$ in (15), we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

$$(16)\qquad \sqrt[n]{1} = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n - 1.$$

These n values are called the ***n*th roots of unity.** They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 324–326 show $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$.

If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number z ($\neq 0$), then the n values of $\sqrt[n]{z}$ in (15) are

$$(17)\qquad w_1, \qquad w_1\omega, \qquad w_1\omega^2, \qquad \cdots, \qquad w_1\omega^{n-1}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

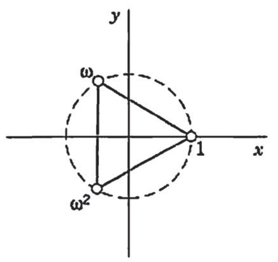

Fig. 324. $\sqrt[3]{1}$

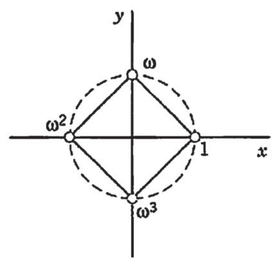

Fig. 325. $\sqrt[4]{1}$

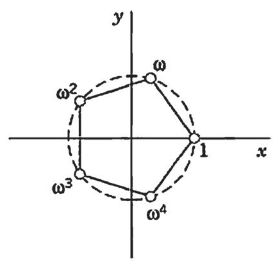

Fig. 326. $\sqrt[5]{1}$

PROBLEM SET 13.2

1–8 POLAR FORM

Do these problems very carefully since polar forms will be needed frequently. Represent in polar form and graph in the complex plane as in Fig. 322 on p. 608. (Show the details of your work.)

1. $3 - 3i$

2. $2i, -2i$

3. -5

4. $\frac{1}{2} + \frac{1}{4}\pi i$

5. $\dfrac{1 + i}{1 - i}$

6. $\dfrac{3\sqrt{2} + 2i}{-\sqrt{2} - (2/3)i}$

7. $\dfrac{-6+5i}{3i}$

8. $\dfrac{2+3i}{5+4i}$

9–15 PRINCIPAL ARGUMENT

Determine the principal value of the argument.

9. $-1-i$
10. $-20+i,\ -20-i$
11. $4 \pm 3i$
12. $-\pi^2$
13. $7 \pm 7i$
14. $(1+i)^{12}$
15. $(9+9i)^3$

16–20 CONVERSION TO $x + iy$

Represent in the form $x + iy$ and graph it in the complex plane.

16. $\cos\frac{1}{2}\pi + i\sin(\pm\frac{1}{2}\pi)$
17. $3(\cos 0.2 + i \sin 0.2)$
18. $4(\cos\frac{1}{3}\pi \pm i\sin\frac{1}{3}\pi)$
19. $\cos(-1) + i\sin(-1)$
20. $12(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi)$

21–25 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt{-i}$
22. $\sqrt[8]{1}$
23. $\sqrt[4]{-1}$
24. $\sqrt[3]{3+4i}$
25. $\sqrt[5]{-1}$

26. **TEAM PROJECT. Square Root. (a)** Show that $w = \sqrt{z}$ has the values

$$w_1 = \sqrt{r}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right],$$

$$(18)\quad w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2}+\pi\right) + i\sin\left(\frac{\theta}{2}+\pi\right)\right] = -w_1.$$

(b) Obtain from (18) the often more practical formula

$$(19)\quad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z|+x)} + (\operatorname{sign} y)\, i\sqrt{\tfrac{1}{2}(|z|-x)}\right]$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $4i$, $16 - 30i$, and $9 + 8\sqrt{7}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

27–30 EQUATIONS

Solve and graph all solutions, showing the details:

27. $z^2 - (8-5i)z + 40 - 20i = 0$ (Use (19).)
28. $z^4 + (5-14i)z^2 - (24+10i) = 0$
29. $8z^2 - (36-6i)z + 42 - 11i = 0$
30. $z^4 + 16 = 0$. Then use the solutions to factor $z^4 + 16$ into quadratic factors with *real* coefficients.

31. **CAS PROJECT. Roots of Unity and Their Graphs.** Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

32–35 INEQUALITIES AND AN EQUATION

Verify or prove as indicated.

32. **(Re and Im)** Prove $|\text{Re } z| \leqq |z|$, $|\text{Im } z| \leqq |z|$.
33. **(Parallelogram equality)** Prove

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Explain the name.

34. **(Triangle inequality)** Verify (6) for $z_1 = 4 + 7i$, $z_2 = 5 + 2i$.
35. **(Triangle inequality)** Prove (6).

13.3 Derivative. Analytic Function

Our study of complex functions will involve point sets in the complex plane. Most important will be the following ones.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 327) has already occurred in Sec. 13.2. Figure 328 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$