Signalbehandling for computer-ingeniører

COMTEK-5, E22

&

Signalbehandling

EIT-5, E22

7. Digital IIR Filters:
Impulse Invariant Method
Bilinear Transformation

Assoc. Prof. Peter Koch, AAU

Frequency

	Time	Frequency
Continuous Time	$x(t)$	$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$ $= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\Omega)t}\,dt$ $= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ for $\sigma = 0$ $= X(j\Omega)$ Fourier transform
Discrete Time	$x[n]$	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ $= \sum_{n=-\infty}^{\infty} x[n]\,(re^{j\omega})$ $= \sum_{n=-\infty}^{\infty}(r \cdot x[n])e^{j\omega}$ $= \sum_{n=-\infty}^{\infty} x[n]\,e^{j\omega}$ for $r = 1$ $= X\left(e^{j\omega}\right)$ Discrete Time FT

$x(t) \xleftrightarrow{\text{Laplace}} X(s)$

$x[n] \xleftrightarrow{\text{z transform}} X(z)$

Sampling $T = \frac{1}{f_s}$: $x(t) \rightarrow x[n]$

Reconstruction: $x[n] \rightarrow x(t)$

$$\omega = \Omega T = 2\pi f \frac{1}{f_s}$$

AALBORG UNIVERSITY
DENMARK

7TH LECTURE

„SYNTHESIS OF IIR-FILTER"

SPECIFICATION OF THE DIGITAL FILTER

DESIGN THE ANALOG PROTO-TYPE FILTER

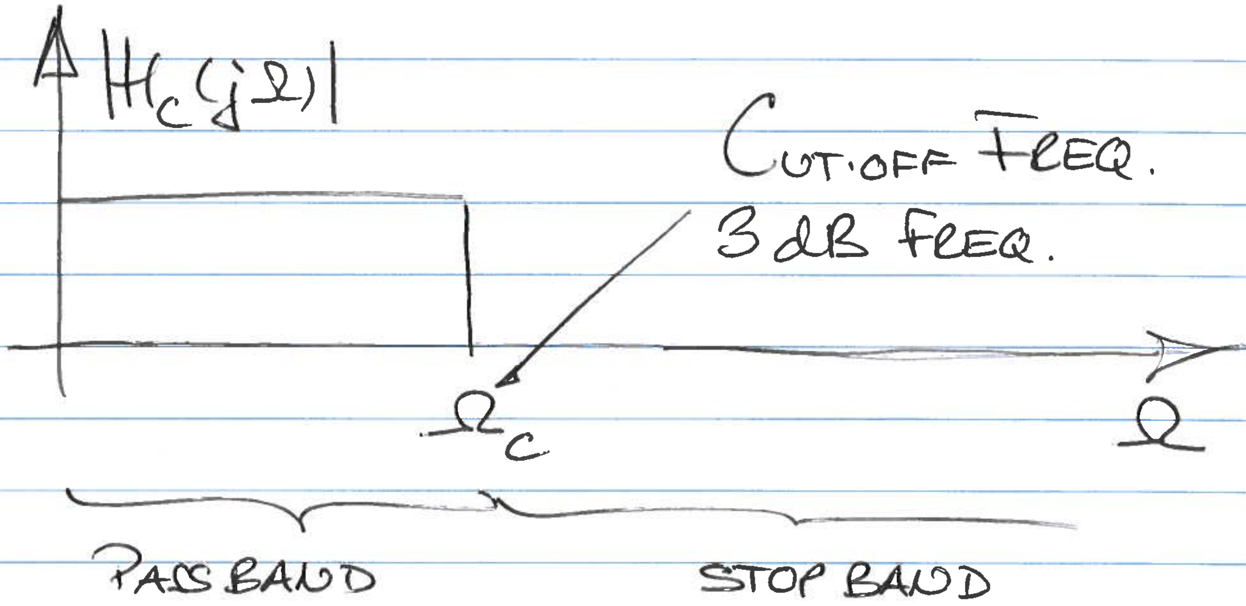
$H(s)$

DO THE TRANSFORMATION

$H(s) \rightarrow H(z)$

- A BRIEF RE-CAP ON ANALOG FILTERS (BUTTERWORTH)
- IMPULSE INVARIANCE METHOD
- THE BILINEAR TRANSFORMATION.

BUTTERWORTH LP. FILTERS



WE CAN DESIGN THIS IDEAL FILTER ONLY MATHEMATICAL.

THUS WE NEED AN APPROXIMATION.

ONE SUCH OPTION IS THE BUTTERWORTH FUNCTION

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \left(j\Omega / j\Omega_c\right)^{2N}}$$

(AMPLITUDE RESPONSE)^2

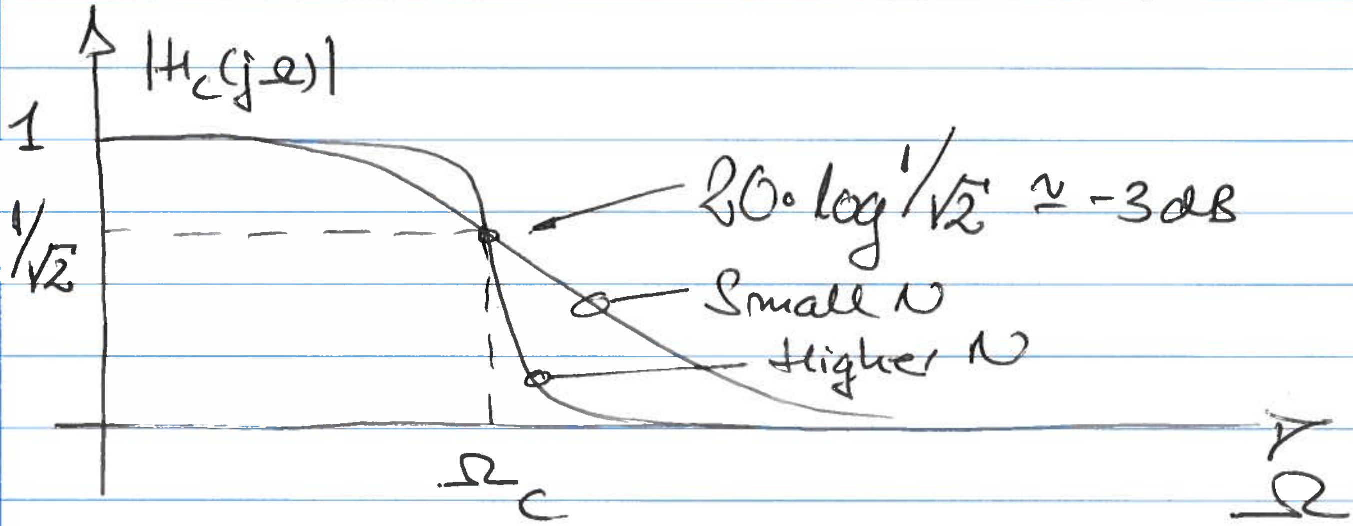
WHERE N IS THE FILTER ORDER

THIS FUNCTION HAS SOME INTERESTING CHARACTERISTICS:

- $|H_c(j\Omega)|^2 = 1 \Big|_{\Omega=0} \quad \forall N$

- $|H_c(j\Omega)|^2 = \frac{1}{2} \Big|_{\Omega=\Omega_c} \quad \forall N$

- MONOTONICALLY DECREASING IN PASS/STOP-BAND.



- THE HIGHER N, THE BETTER APPROXIMATION TO THE IDEAL FILTER.

  ↳ THE NARROWER THE TRANSITION BAND

POLE·LOCATION

$$|H_c(s)|^2 = |H_c(j\Omega)|^2\Big|_{s=j\Omega} = H_c(s)H_c(-s)$$

$\Downarrow$

$$|H_c(s)|^2 = \frac{1}{1+\left(s/j\Omega_c\right)^{2N}}$$

$$1+\left(s/j\Omega_c\right)^{2N} = 0$$

The poles are the values of s where the denominator equals zero

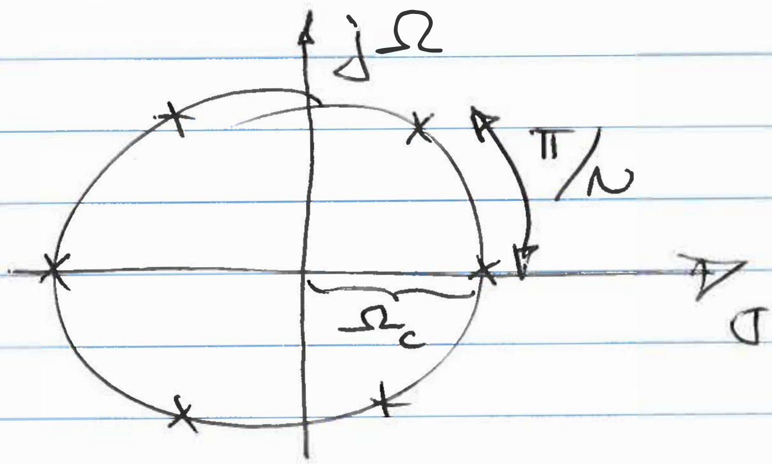
$\Downarrow$

$$s/j\Omega_c = \sqrt[2N]{-1}$$

$\Downarrow$

$$s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c \cdot e^{(j\pi/2N)(2k+N-1)}$$

for $k = 0, 1, \dots 2N-1$



H(s) IS NOW DERIVED FROM THE POLES IN THE LEFT·HAND SIDE OF THE PLANE. (STABILITY)

THUS, WE CAN EXPRESS THE TRANSFER FUNCTION AS;

$$H_c(s) = \frac{G}{\prod_{k=1}^{N} (s - s_k)}$$

THE DENOMINATOR REPRESENTS THE

BUTTERWORTH POLYNOMIAL

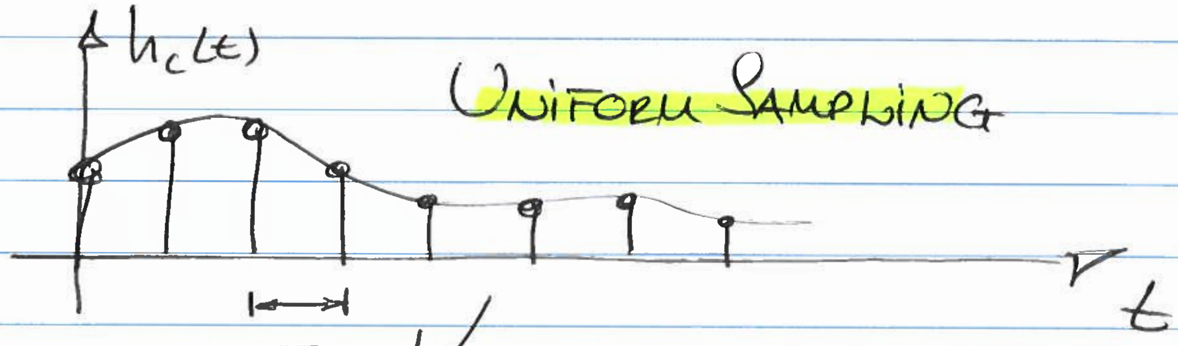
Denominator coefficients for polynomials of the form $s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots\ldots\ldots\ldots + a_1 s + a_0$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1	1									
2	1	1.414								
3	1	2.000	2.000							
4	1	2.613	3.414	2.613						
5	1	3.236	5.236	5.236	3.236					
6	1	3.864	7.464	9.142	7.464	3.864				
7	1	4.494	10.098	14.592	14.592	10.098	4.494			
8	1	5.126	13.137	21.846	25.688	21.846	13.137	5.126		
9	1	5.759	16.582	31.163	41.986	41.986	31.163	16.582	5.759	
10	1	6.392	20.432	42.802	64.882	74.233	64.882	42.802	20.432	6.392

n (order)	Normalized Denominator Polynomials in Factored Form
1	$(1+s)$
2	$(1+1.414s+s^2)$
3	$(1+s)(1+s+s^2)$
4	$(1+0.765s+s^2)(1+1.848s+s^2)$
5	$(1+s)(1+0.618s+s^2)(1+1.618s+s^2)$
6	$(1+0.518s+s^2)(1+1.414s+s^2)(1+1.932s+s^2)$
7	$(1+s)(1+0.445s+s^2)(1+1.247s+s^2)(1+1.802s+s^2)$
8	$(1+0.390s+s^2)(1+1.111s+s^2)(1+1.663s+s^2)(1+1.962s+s^2)$
9	$(1+s)(1+0.347s+s^2)(1+s+s^2)(1+1.532s+s^2)(1+1.879s+s^2)$
10	$(1+0.313s+s^2)(1+0.908s+s^2)(1+1.414s+s^2)(1+1.782s+s^2)(1+1.975s+s^2)$

NOW THAT WE HAVE $H_c(s)$, WE ALSO HAVE $h_c(t)$.

DERIVE $H(z)$ USING THE IMPULSE INVARIANCE METHOD



$T = 1/f_{sample}$

$h[n] = h_c(nT)$ WHERE n IS THE SAMPLE NUMBER.

NOW, LET'S SEE HOW IT LOOKS LIKE IN THE FREQUENCY DOMAIN;

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\}$$

$$= \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h_c(nT) \cdot e^{-j\omega n}$$

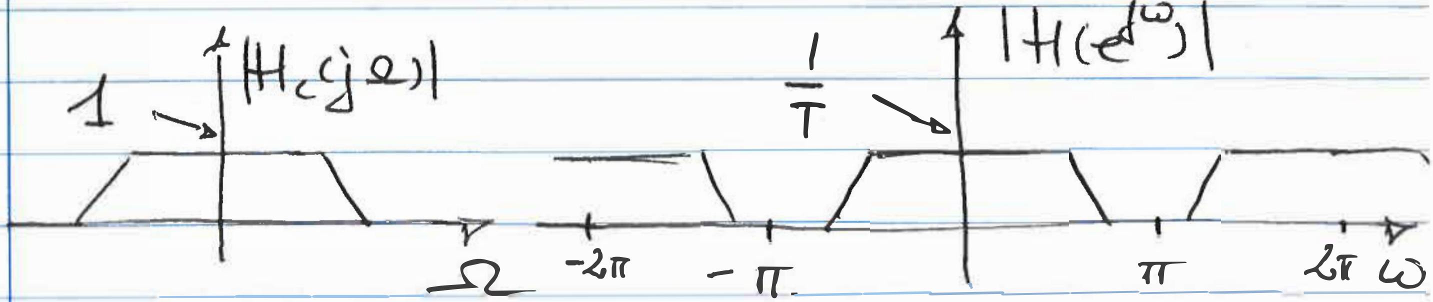
$h[n]$ IS DISCRETE IN TIME

$\Downarrow$

$H(e^{j\omega})$ IS PERIODIC IN FREQUENCY

This is a consequence of sampling

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T} + j\frac{2\pi}{T}\cdot k\right), \quad \omega = T\cdot\Omega$$



ANALOG DIGITAL

IN ORDER TO MAINTAIN THE SAME PASS·BAND AMPLIFICATION, WE MUST MULTIPLY WITH T;

$$\boxed{h[n] = T\cdot h_c(nT)}$$

NOW, LET'S ASSUME THAT WE HAVE $H_c(s)$, WHICH GENARALLY HAS THE FORM;

$$H_c(s) = \frac{\sum_{k=0}^{M}\beta_k s^k}{\sum_{l=0}^{N}\alpha_l\cdot s^l}$$

FOR BUTTERWORTH LOW·PASS, $M=0$

FIND $H(z)$?

1) FIRST, FIND $h_c(t)$

WE USE PARTIAL FRACTION EXPANSION OF $H_c(s)$

$$H_c(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_N}{s-s_N} = \sum_{k=1}^{N} \frac{A_k}{s-s_k}$$

NEXT WE DO INVERSE LAPLACE;

$$h_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots + A_N e^{s_N t} = \sum_{k=1}^{N} A_k \cdot e^{s_k \cdot t}$$

(EQU. 8 p. 523)

2) NEXT, WE SAMPLE $h_c(t)$

$$h[n] = T \cdot h_c(nT)$$

$$\Downarrow$$

$$h[n] = T \cdot \sum_{k=1}^{N} A_k \cdot e^{s_k n T} \cdot u[n]$$

Causal System.

3) FINALLY, WE CONDUCT Z-TRANSFORM

$$H(z) = Z\{h[n]\} = \sum_{n=0}^{\infty} \left\{ T \cdot \sum_{k=1}^{N} A_k \cdot e^{s_k n T} \right\} \cdot z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} \left\{ T \cdot \sum_{k=1}^{N} A_k \cdot e^{s_k n T} \right\} \cdot z^{-n}$$

$\Downarrow$

$$H(z) = T \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} \left(e^{s_k T} \cdot z^{-1} \right)^n$$

$\Downarrow$

$$\boxed{H(z) = \sum_{k=1}^{N} \frac{T A_k}{1 - e^{s_k T} z^{-1}}}$$

ROC: $|z| > e^{s_k T}$

THIS IS THE GENERAL TRANSFER FUNCTION $H(z)$ FOUND BY TRANSFORMING $H(s)$, USING THE IMPULSE INVARIANCE METHOD

NOW, SINCE $H(s)$ IS STABLE, WE COULD HOPE FOR $H(z)$ ALSO BEING STABLE — LET'S FIND OUT;

POLES IN $H(s)$; s_k

POLES IN $H(z)$; $z_k = e^{s_k \cdot T}$

WE WOULD LIKE $H(s)_{\text{STABLE}} \longrightarrow H(z)_{\text{STABLE}}$

$$\begin{cases} s_k = \sigma_k + j\Omega_k \\ z_k = e^{s_k \cdot T} \end{cases}$$

$\Downarrow$

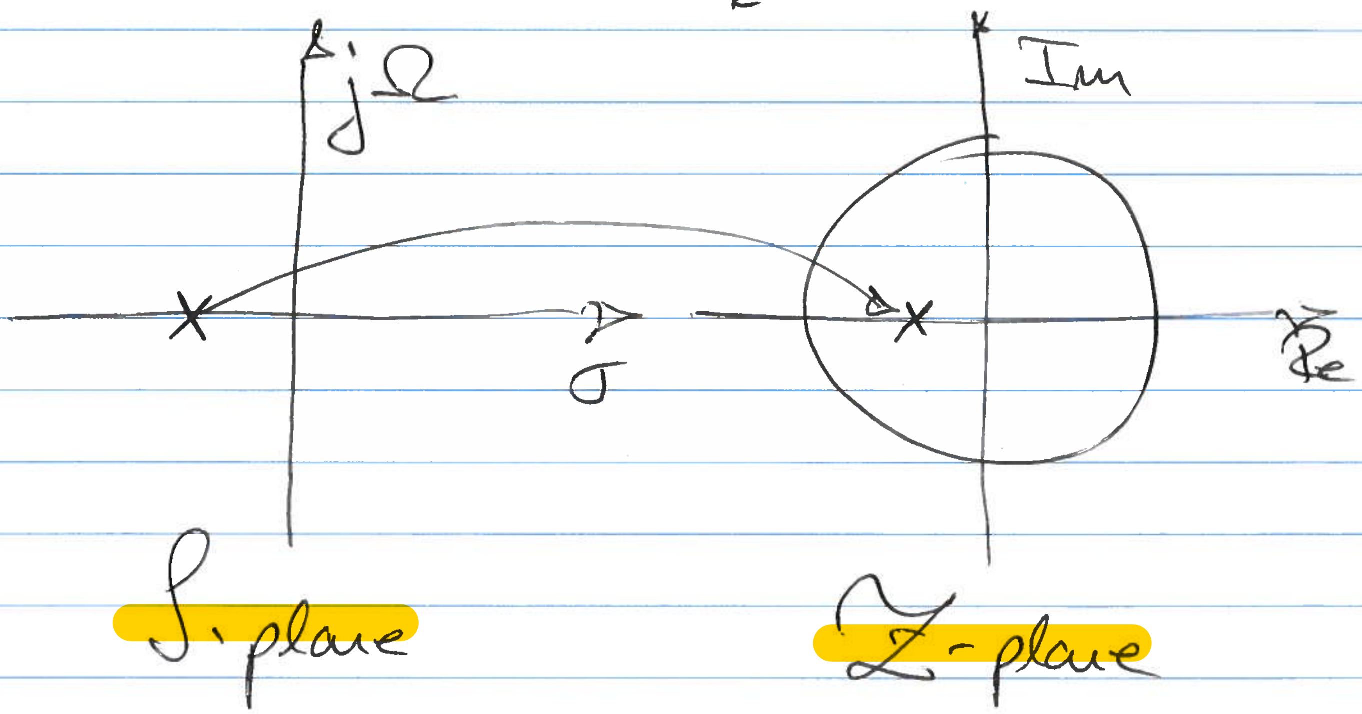
$$z_k = e^{\sigma_k \cdot T} \cdot e^{j\Omega_k \cdot T}$$

$\Downarrow$

$$|z_k| = e^{\sigma_k \cdot T}$$

$\Downarrow$

$$|z_k| < 1 \Big|_{\sigma_k < 0}$$



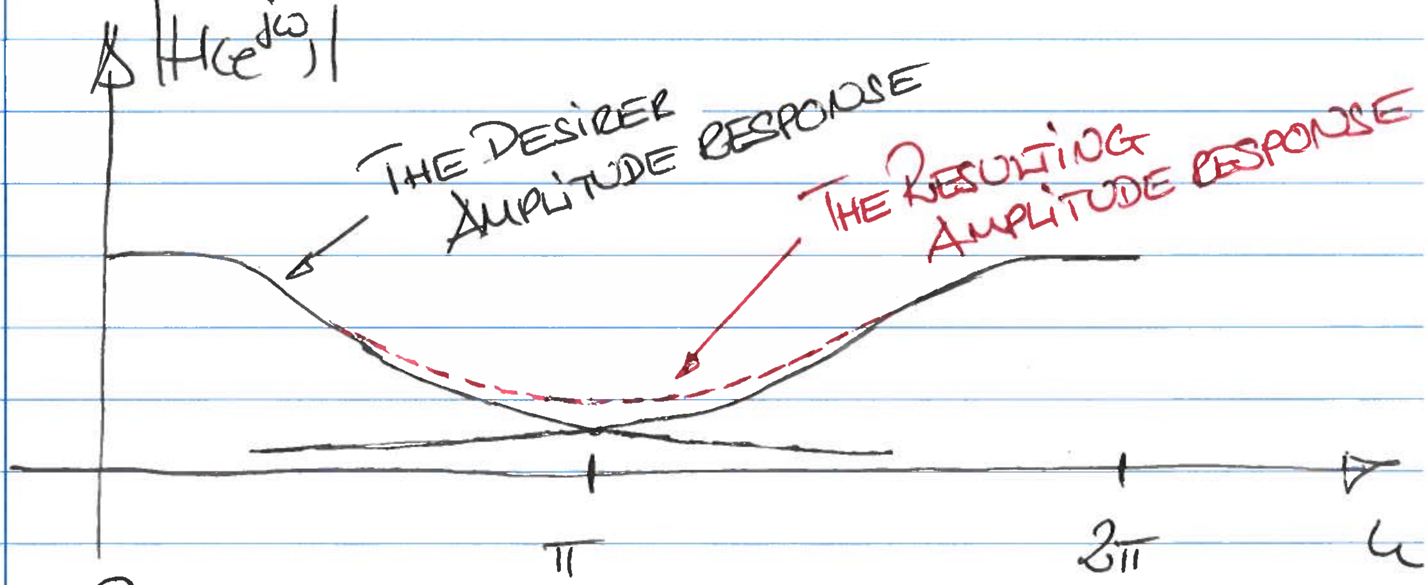
- THERE IS NO SIMPLE MATHEMATICAL DESCRIPTION OF THE S-plane TO Z-plane mapping.

- THE ZEROS IN $H(z)$ ARE FUNCTIONS OF THE POLES AND $T \cdot A_k$, AND THUS THE ZEROS ARE BEING MAPPED DIFFERENTLY THAN THE POLES.

THERE IS A PROBLEM THOUGH....

- $H(e^{j\omega})$ IS A PERIODIC FUNCTION.

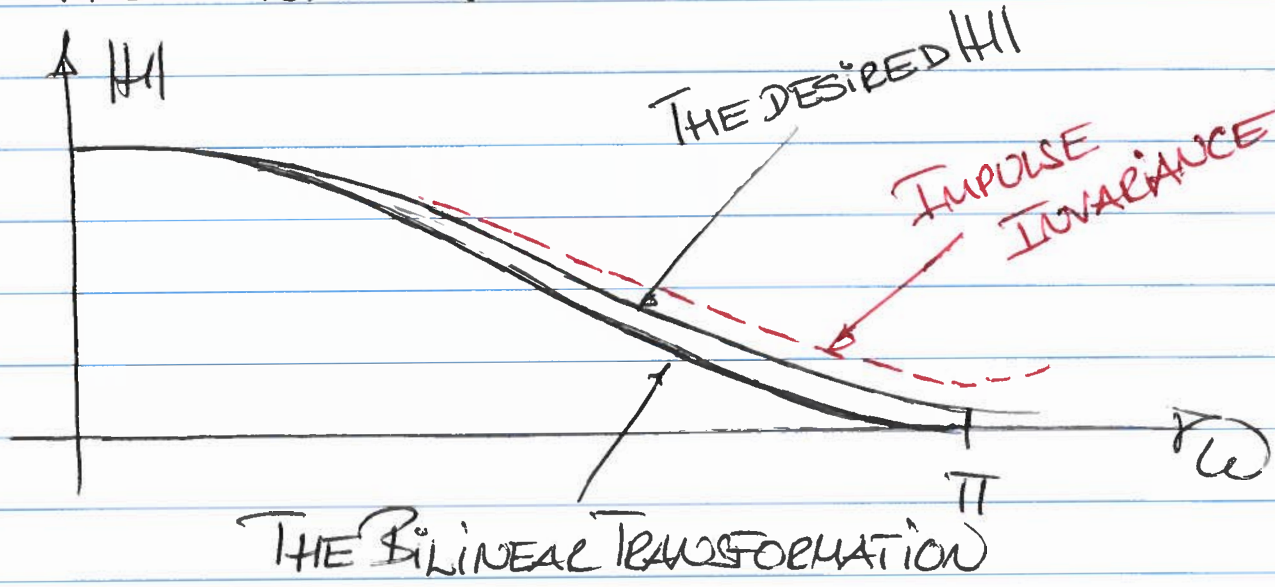
- NO ANALOG FILTER IS BAND-LIMITED.



SO, THERE IS AN OVERLAP AMONG CONSECUTIVE VERSIONS OF THE DESIRED AMPLITUDE RESPONSE

ALIASING

HOW CAN WE POSSIBLY ELIMINATE THE ALIASING PROBLEM ?



- IDEA :

$$\boxed{-\infty < \Omega < \infty \quad \rightarrow \quad -\pi \leq \omega \leq \pi}$$

THIS NON-LINEAR TRANSFORMATION MAPS THE ENTIER FREQUENCY AXIS Ω ONTO ONE ITERATION ON THE UNIT CIRCLE.

THE IDEA IS THAT s IS SUBSTITUTED BY A FUNCTION ;

$$H(z) = H(s)\Big|_{s=f(z)}$$

- $z = e^{j\omega}$ (ON THE UNIT · CIRCLE).
- $\omega = T \cdot \Omega$ (RELATION BETWEEN ANALOG AND DIGITAL FREQUENCY)
- $s = j\,\Omega$ (THE ANALOG FREQ.-AXIS)

$$\Downarrow$$

$$z = e^{jT \cdot \frac{s}{j}} = e^{sT}$$

$$\Downarrow$$

$$\ln(z) = s \cdot T$$

$$\Downarrow$$

TAYLOR SERIES FOR $\ln(z)$

$$s = \frac{1}{T} \cdot \ln(z) = \frac{1}{T} \cdot 2 \left\{ \frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^2 + \cdots \right\}$$

$$\Downarrow$$

$$s \simeq \frac{1}{T} 2 \cdot \frac{z-1}{z+1}$$

$$\Downarrow$$

$$\boxed{s = \frac{2}{T} \cdot \frac{z-1}{z+1}}$$

Equ. 18 p. 528

THIS IS THE BILINEAR TRANSFORMATION

Now, A MAJOR QUESTION is; How is THE s-plane MAPPED TO THE z-plane ??

$$S = \frac{2}{T} \cdot \frac{Z-1}{Z+1}$$

$$\Downarrow$$

$$Z = \frac{1 + \frac{T}{2}S}{1 - \frac{T}{2}S} = \frac{1 + \frac{T}{s}(\sigma + j\Omega)}{1 - \frac{T}{s}(\sigma + j\Omega)}$$

USING THIS EQUATION, WE CAN NOW INVESTIGATE THREE IMPORTANT THINGS;

1) ORIGIN IN THE S-plane

$S = 0 \rightarrow z = 1$ i.e.; DC → DC

2) LEFT-HAND SIDE OF THE S-plane

$$\mathrm{Re}\{s\} = \sigma < 0$$

$$|z| = \left. \frac{\sqrt{(1+\frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}}{\sqrt{(1-\frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}} < 1 \right|_{\sigma < 0}$$

$$\Downarrow$$

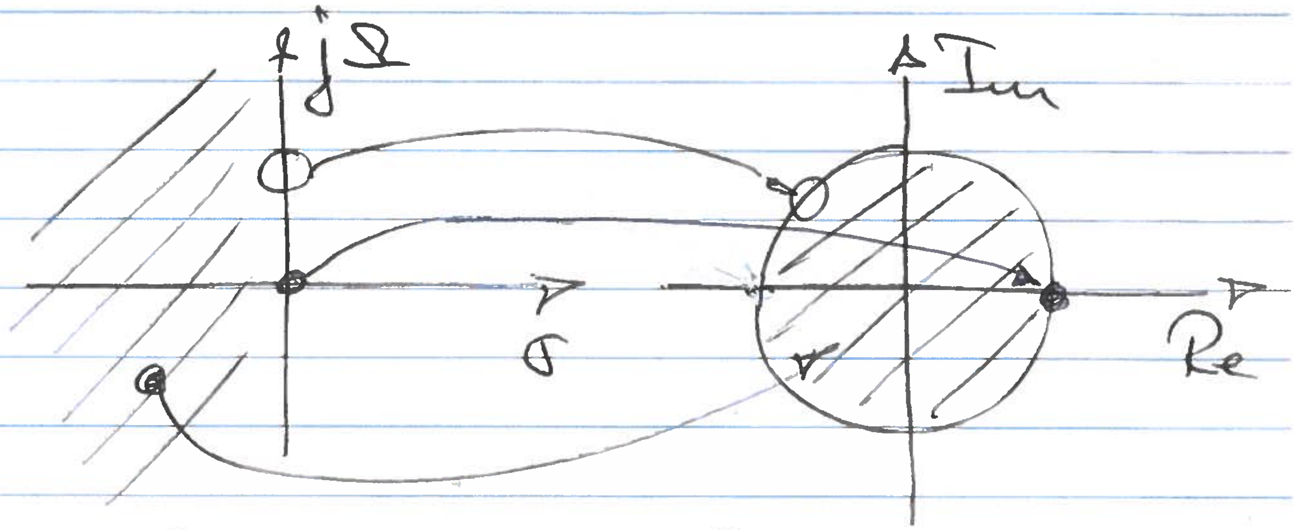
LEFT-HAND SIDE OF THE S-plane is MAPPED TO THE Z-plane INSIDE THE UNIT-CIRCLE

3) $j\Omega$ · AXIS IN THE S-plane

$\sigma = 0 \implies |z| = 1 \quad \forall \Omega$

$\Downarrow$

AND THUS WE CAN CONCLUDE THAT THE $j\Omega$ · AXIS MAPS TO THE UNIT · CIRCLE.



STABLE $H(s) \implies$ STABLE $H(z)$

NOW, WHAT IS THE RELATION BETWEEN Ω AND ω USING THE BILINEAR TRANSF. ??

$$s = \sigma + j\Omega = \frac{2}{T}\,\frac{e^{j\omega} - 1}{e^{j\omega} + 1}\Bigg|_{z = e^{j\omega}}$$

IF WE UTILIZE THAT $e^{j\omega} = e^{j\frac{\omega}{2}} \cdot e^{j\frac{\omega}{2}}$ AND EULER'S IDENTITY FOR COS AND SIN

$$\Downarrow$$

$$s = \underbrace{\frac{2}{T} \cdot j \cdot \tan\frac{\omega}{2}}$$

THE RIGHT-HAND SIDE IS IMAGINARY

$$\Downarrow$$

$$\Omega = \frac{2}{T} \cdot \tan\frac{\omega}{2} \qquad \text{EQU. 26}$$

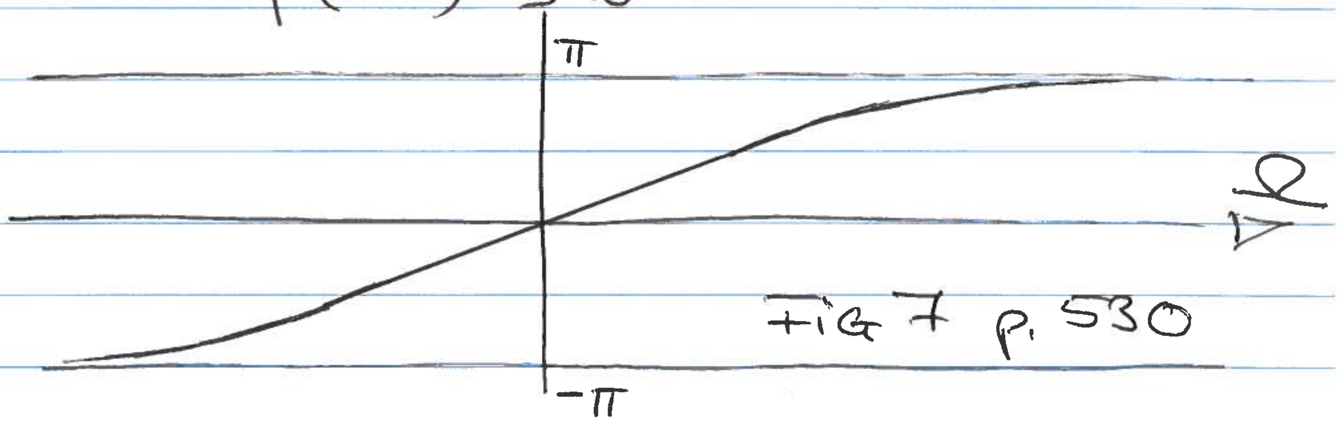
AND

$$\omega = 2 \cdot \arctan\left(\frac{\Omega \cdot T}{2}\right) \qquad \text{EQU. 27}$$

p. 529.

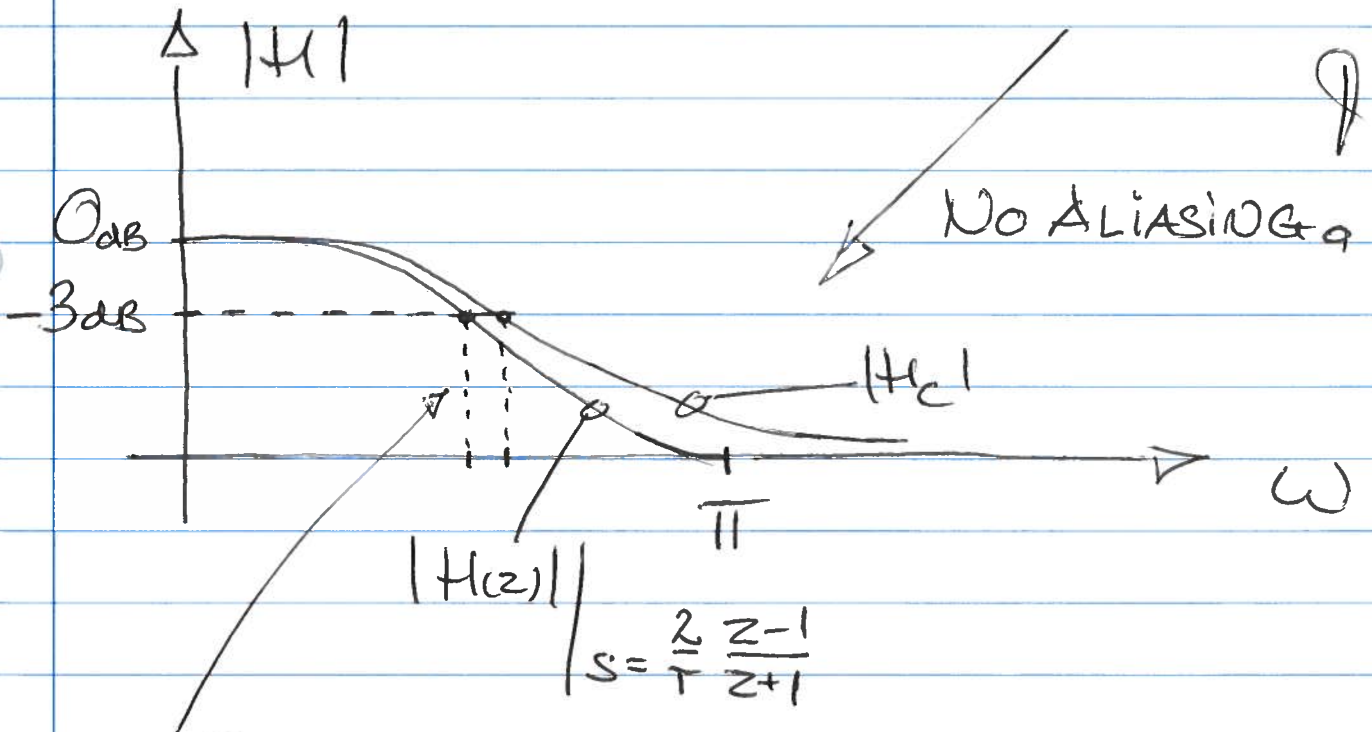
So, MAPPING FROM THE s-plane TO THE z-plane, IS AN ARCTAN - FUNCTION.

$\omega = f(\Omega)$



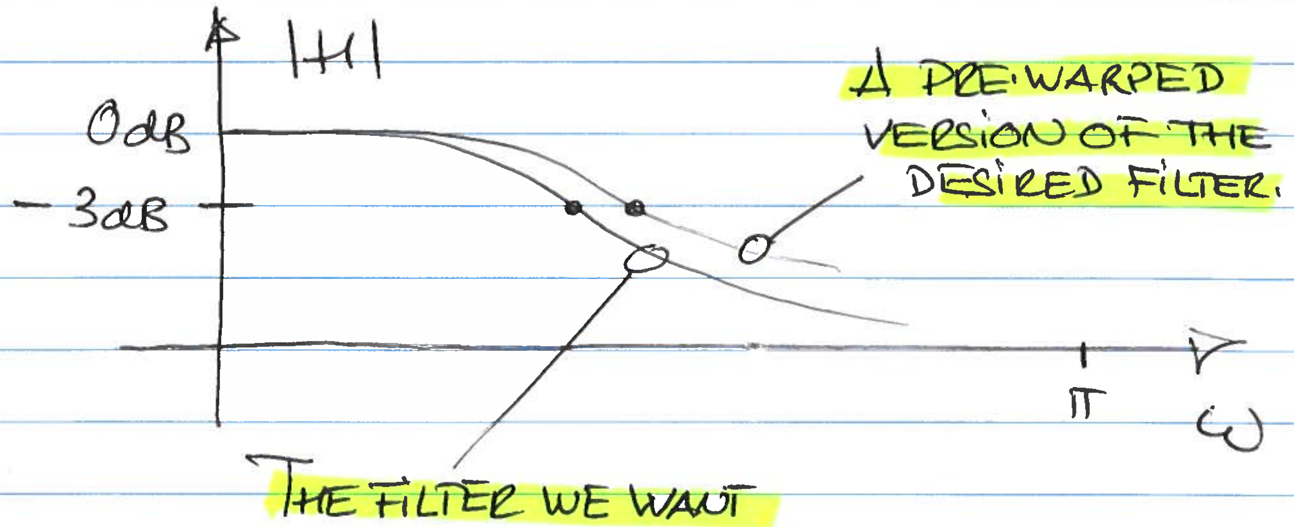
- ARCTAN is „ALMOST" LINEAR FOR VALUES OF Ω CLOSE TO 0, BUT BECOMES MORE NON-LINEAR AS Ω INCREASES. BAD NEWS

- $\omega = 2 \cdot \text{ARCTAN}\left(\Omega \cdot T/2\right)$ IS DEFINED IN $]-\pi ; \pi[$ GOOD NEWS



BUT NOW THE 3dB FREQ MOVES DOWNWARDS !

WE MAY ELIMINATE THIS DISTORTION BY INTRODUCING A PRE-DISTORTION, CALLED A PRE-WARPING.



SO, THE IDEA HERE IS, THAT WE MOVE UPWARDS ONE CRITICAL FREQUENCY WHICH AFTER BILINEAR TRANSFORMATION THEN IS LOCATED EXACTLY WHERE WE WAN

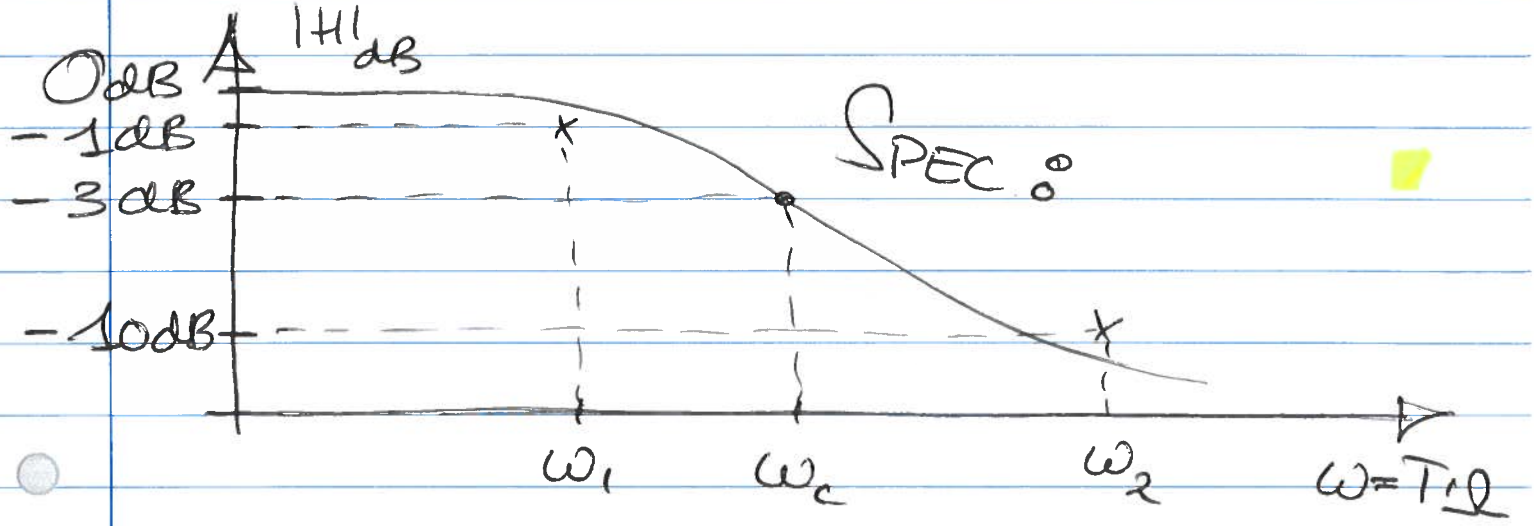
$$\boxed{\Omega_{c,\,new} = \frac{2}{T} \cdot \tan \frac{\omega_c}{2}}$$

PREWARPING

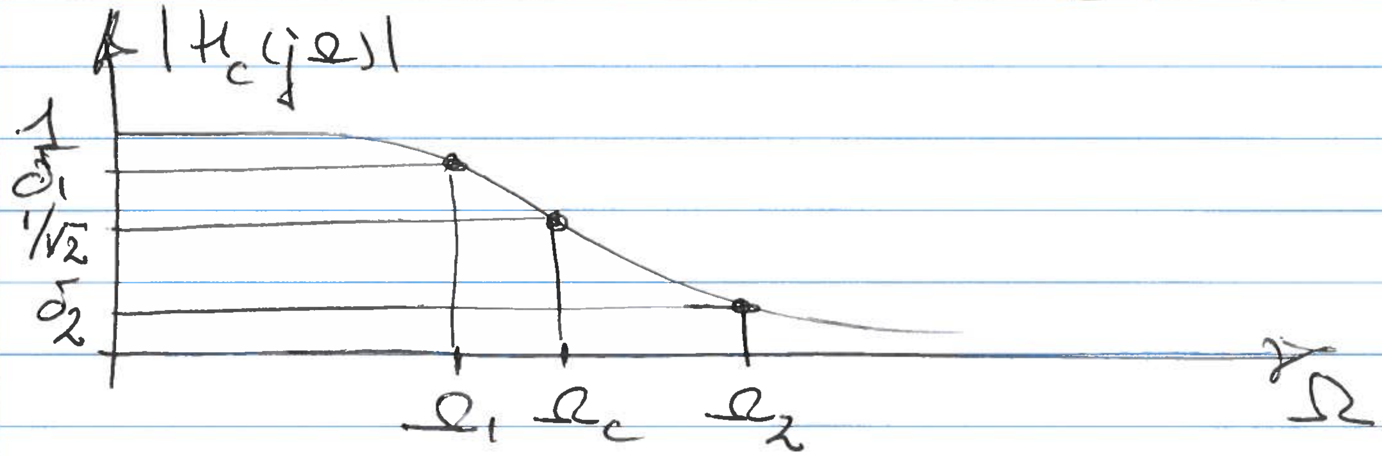
THIS IS DONE ON $H(s)$ PRIOR TO TRANSFORMATION TO $H(z)$.

Hint for today's Exercise

Max flatness ⇒ Butterworth.



First — Find the necessary order of the analog filter $H_c(s)$



1) Find δ_1 and δ_2
2) Two equations ; $\delta_1^2 \leq |H_c(j\Omega)|$
3) Prewarp all critical frequencies
4) Find $N \Rightarrow H_c(s) = ??$
5) Find $H(z) = H_c(s)\Big|_{s=\frac{2}{T}\cdot\frac{z-1}{z+1}}$