Signalbehandling for computer-ingeniører

ComTek-5, E22

4. Sampling Continuous-Time Signals

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Today's Lecture

In this lecture we will study the fundamental operation for any discrete-time signal and system, namely sampling of signals in the time domain.

You may have previously heard briefly about the Nyquist sampling theorem, but it seems (looking into your previous study regulations) that you have not studied the fundamental math behind sampling.

Since sampling in the time domain has some dominant consequences in the frequency domain which impact how we should design LTI systems, we therefore will discuss some of these mathematical topics.

Basically, we will introduce a mathematical model which, in the time domain, describes the sampling process, and we will next use this model to understand how the sampled signal looks like in the frequency domain.

Furthermore, we will also briefly touch upon some of the more practical matters as related to sampling, e.g., amplitude quantization which takes place in the analog-to-digital conversion.

Sampling

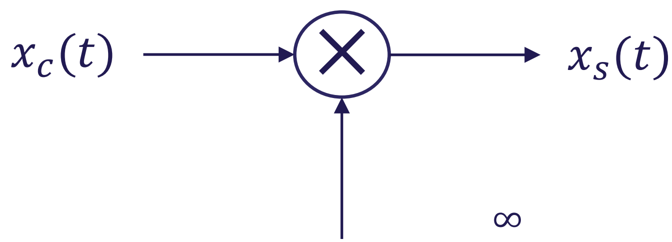
Let's assume a continuous-time signal $x_c(t)$.

We want to know this signal only at equidistant time-instances, i.e.,

$t = nT$, $-\infty < n < \infty$. T is known as the sample period, and equals

$T = \frac{1}{f_s}$, where f_s is the sample frequency.

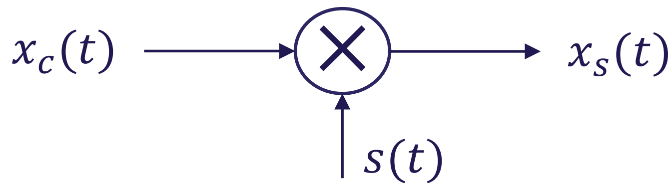
Now, the idea is to modulate $x_c(t)$ with a signal $s(t)$, which is a periodic impulse train;



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

where δ is the Dirac delta function.

Sampling



Using the modulation signal, we now derive an expression for the signal $x_s(t)$.

$$x_s(t) = x_c(t) \cdot s(t)$$

$$x_s(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)$$

Using the relation $x(t)\delta(t) = x(0)\delta(t)$, we can now rewrite $x_c(t)\delta(t - nT)$;

$$x_c(t)\delta(t - nT) = x_c(t|_{t-nT=0})\delta(t - nT)$$

$x_c(t)\delta(t - nT) = x_c(nT)\delta(t - nT)$ and thus

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

Sampling

So, the modulated signal is; $x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$

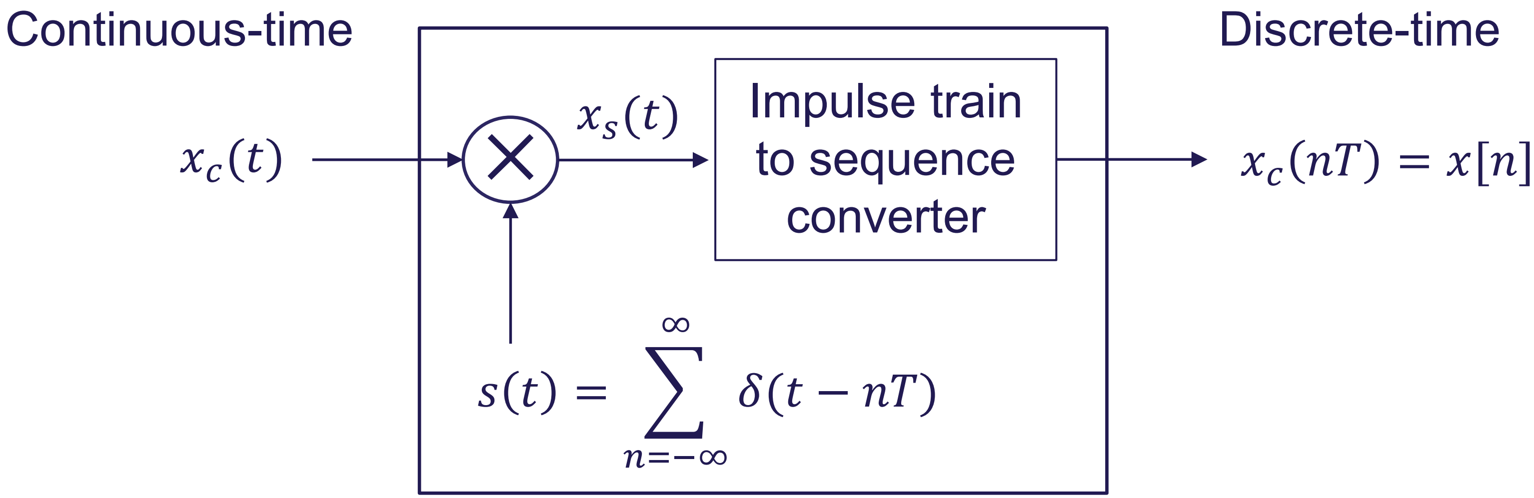
which can be interpreted as an impulse train, where the high of the individual impulses at time $t = nT$ equals the value of the continuous-time signal at the time instances $t = nT$.

Note:

The signal $x_s(t)$ is still a continuous-time signal, which equals 0 in between every single impulse.

Therefore, in order to establish the wanted discrete-time signal, it is necessary to convert the continuous-time signal $x_s(t)$ into a sequence.

Sampling



Continuous- to Discrete-time Converter

Frequency Domain Representation of Sampling

We have that $x_s(t) = x_c(t) \cdot s(t)$, and thus we can conclude that in the frequency domain, $X_s(j\Omega)$ can be derived as the convolution between $X_c(j\Omega)$ and $S(j\Omega)$.

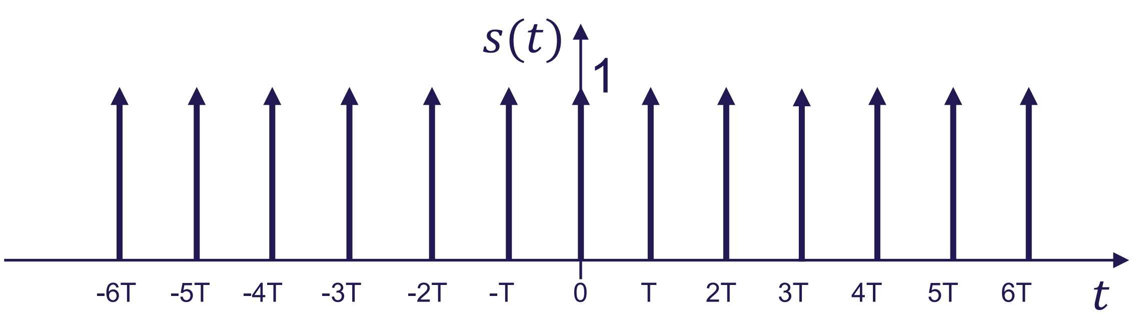
[Remember that an important property of the Fourier transform is that multiplication in the time domain leads to convolution in the frequency domain]

Therefore, we now need expressions for $S(j\Omega)$ and $X_c(j\Omega)$ – the latter of the two simply being the Fourier transform of the observed signal $x_c(t)$. We therefore seek an expression for $S(j\Omega)\ldots$

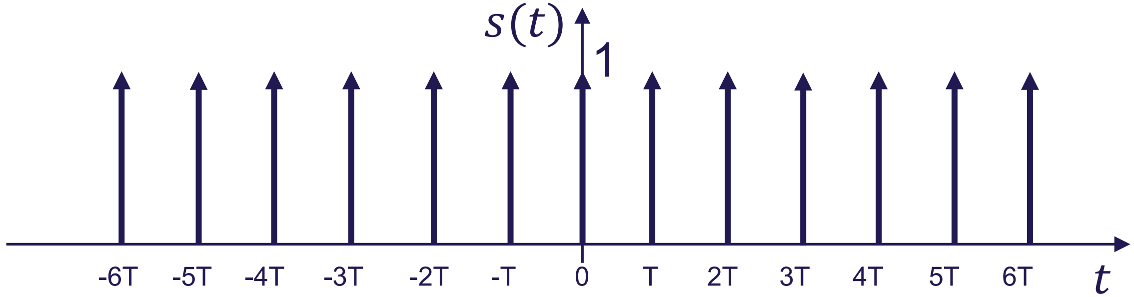
$$S(j\Omega) = \mathfrak{F}\{s(t)\}$$

$$S(j\Omega) = \mathfrak{F}\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\}$$

We know that $s(t)$ is periodic with the period T.



Frequency Domain Representation of Sampling



We thus conclude that $s(t)$ is periodic with the fundamental frequency $\Omega_s = 2\pi f_s = \frac{2\pi}{T}$

Now, the Fourier series for a periodic function with fundamental frequency Ω_s is $s(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\Omega_s t}$ where a_k is the Fourier coefficients.

These coefficients can be determined by calculating 1 period of the Fourier integral;

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} s(t) e^{-jk\Omega_s t}\, dt$$

From previous, we have that $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ which we now insert;

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-jk\Omega_s t}\, dt$$

Since we evaluate over only one period, this reduces to

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\Omega_s t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, dt = \frac{1}{T}$$

So, all the Fourier coefficients $a_k = \frac{1}{T}$ and thus we can write $s(t)$ as;

$s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\Omega_s t}$ …another way of writing the infinite impulse train.

Consulting a Fourier transform table, we find that a function of this kind has the following Fourier transform;

$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$, which is an impulse train, equ. 5 on p.166.

We had that $x_s(t) = x_c(t) \cdot s(t)$. From a Fourier transform table, we next realize

$$\mathcal{F}\{x_s(t)\} = \mathcal{F}\{x_c(t) \cdot s(t)\} = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

Conducting convolution of X_c with an impulse train, we obtain a periodic frequency response where X_c appears at the frequencies where the impulses are located. Thus;

$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$, equ. 6 on p. 166.

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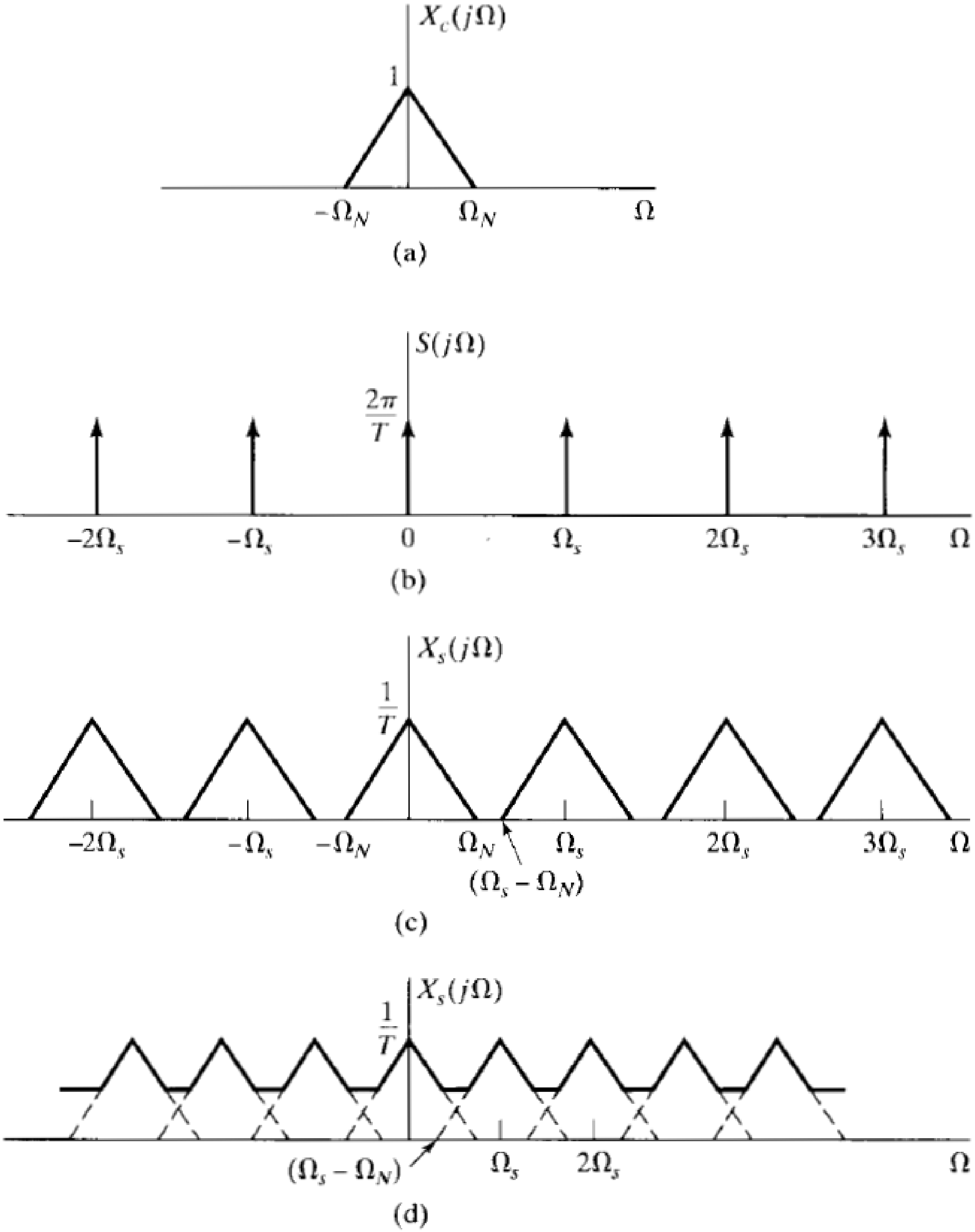
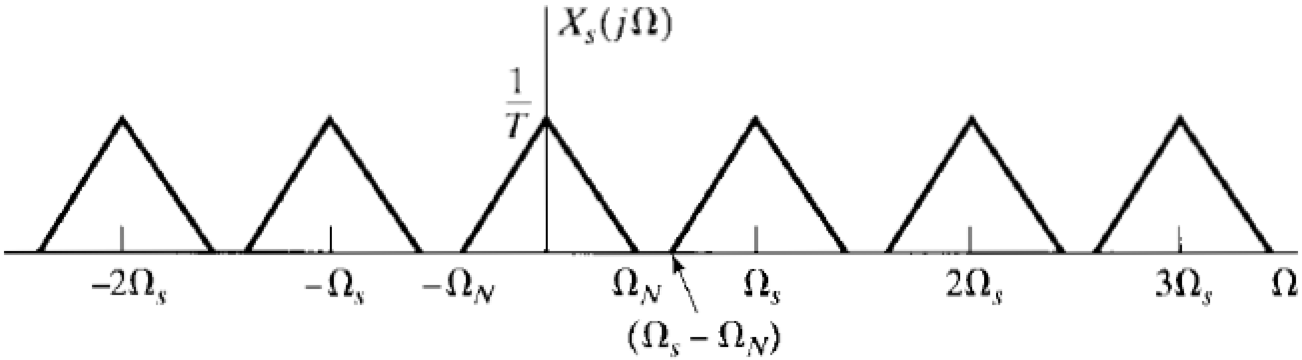


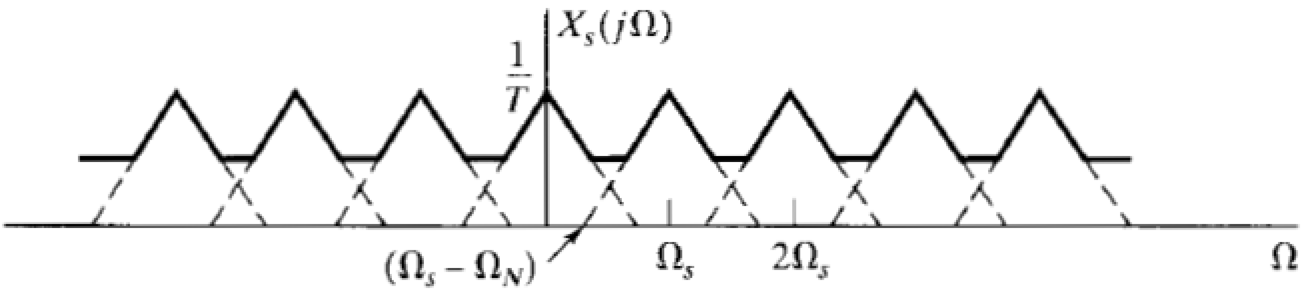
Figure 4.3 Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$.

Aliasing



(c)

Here $X_c(j\Omega)$ can be reconstructed by using an ideal LP filter.



(d)

Here $X_c(j\Omega)$ cannot be reconstructed by LP filtering.

Nyquist Sampling Theorem

Let $x_c(t)$ be a bandlimited signal with

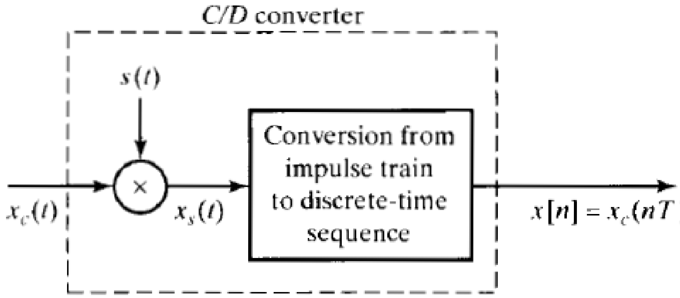
$$X_c(j\Omega) = 0 \qquad \text{for } |\Omega| \geq \Omega_N.$$

Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $n = 0, \pm 1, \pm 2, \ldots,$ if

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N.$$

The frequency Ω_N is commonly referred to as the *Nyquist frequency*, and the frequency $2\Omega_N$ that must be exceeded by the sampling frequency is called the *Nyquist rate*.

We have looked at $X_s(j\Omega)$ but not at $X(j\Omega)$…



Let's apply the continuous-time FT on $x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$

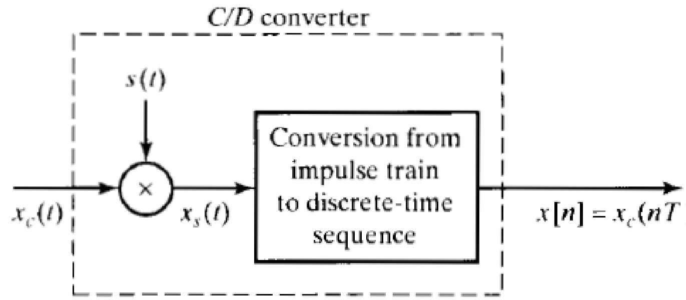
$$X_s(j\Omega) = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t-nT)\right) e^{-j\Omega t} dt$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT) \int_{-\infty}^{\infty} \delta(t-nT) e^{-j\Omega t} dt$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega nT}$$

The delta-function is 0 except for $t = nT$. Therefore, the integral equals 0 for all values of t, except for $t = nT$, which is therefore substituted into the exponential function.

We have looked at $X_s(j\Omega)$ but not at $X(j\Omega)$…



$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn}.$$

insert

compare

Since

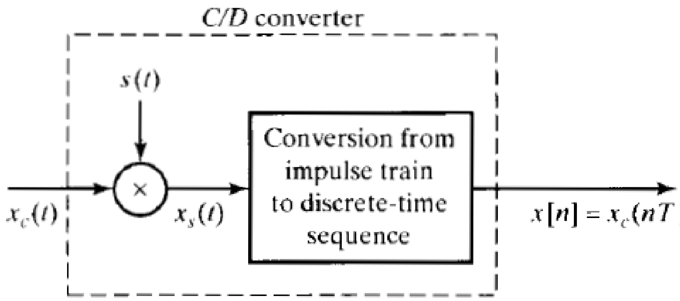
$$x[n] = x_c(nT)$$

and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

it follows that

$$\boxed{X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T}).}$$



So; $X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T})$

Further, we have; $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$ Result from slide no. 9

…and thus; $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$

we see that $X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$ with the frequency scaling specified by $\omega = \Omega T$. This scaling can alternatively be thought of as a normalization of the frequency axis so that the frequency $\Omega = \Omega_s$ in $X_s(j\Omega)$ is normalized to $\omega = 2\pi$ for $X(e^{j\omega})$.

…and thus we can conclude that $\omega = 2\pi$ corresponds to the sample frequency and everything is normalized to the samplefrequency.

VIGTIGT…!!!!

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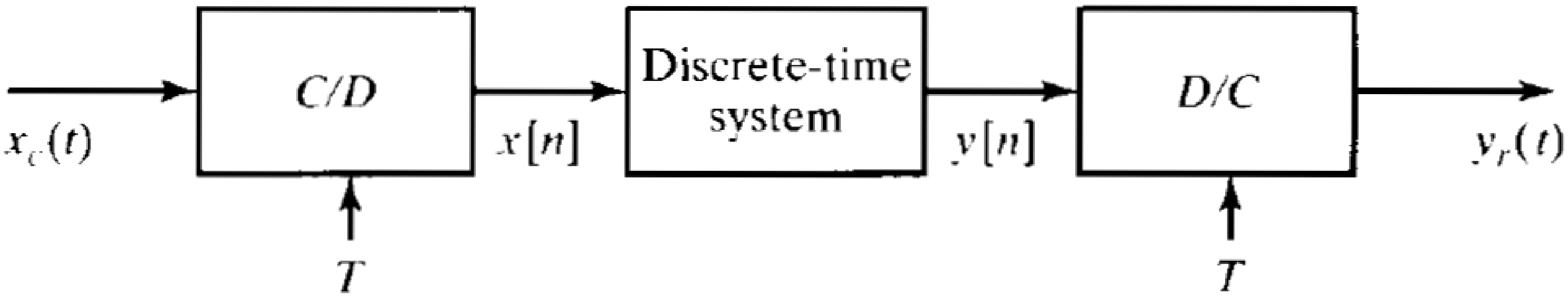
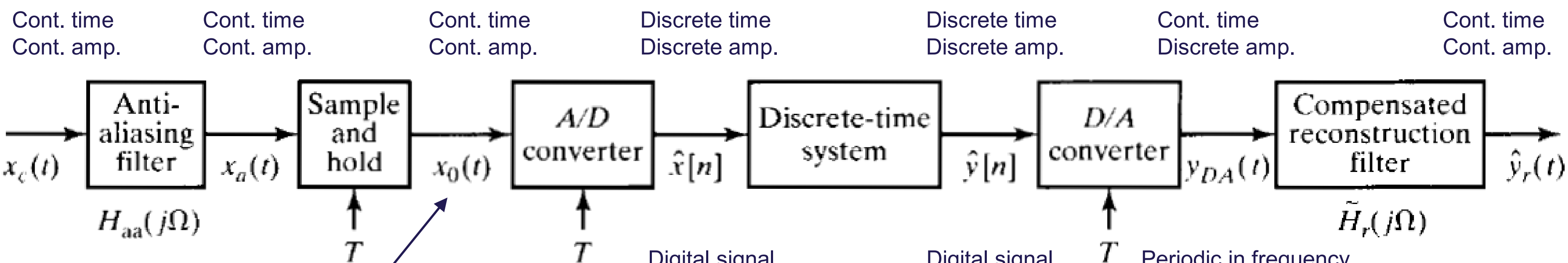
A few words on practical sampling

In the ideal situation, a discrete-time signal processing system should consist of three fundamental functions (or blocks).



The problem is however, that it is not possible to construct an ideal C/D (as we have just discussed), nor an ideal D/C. We must rely on an approximate realization…



”Sampling” means that the signal becomes discrete in time, and thus being known only at $t = nT$. However, since the signal is being held at the output of the S/H unit, then from a strict mathematical Sence, the signal is still time-continuous – thus the notation $x_0(t)$.

Sample and Hold

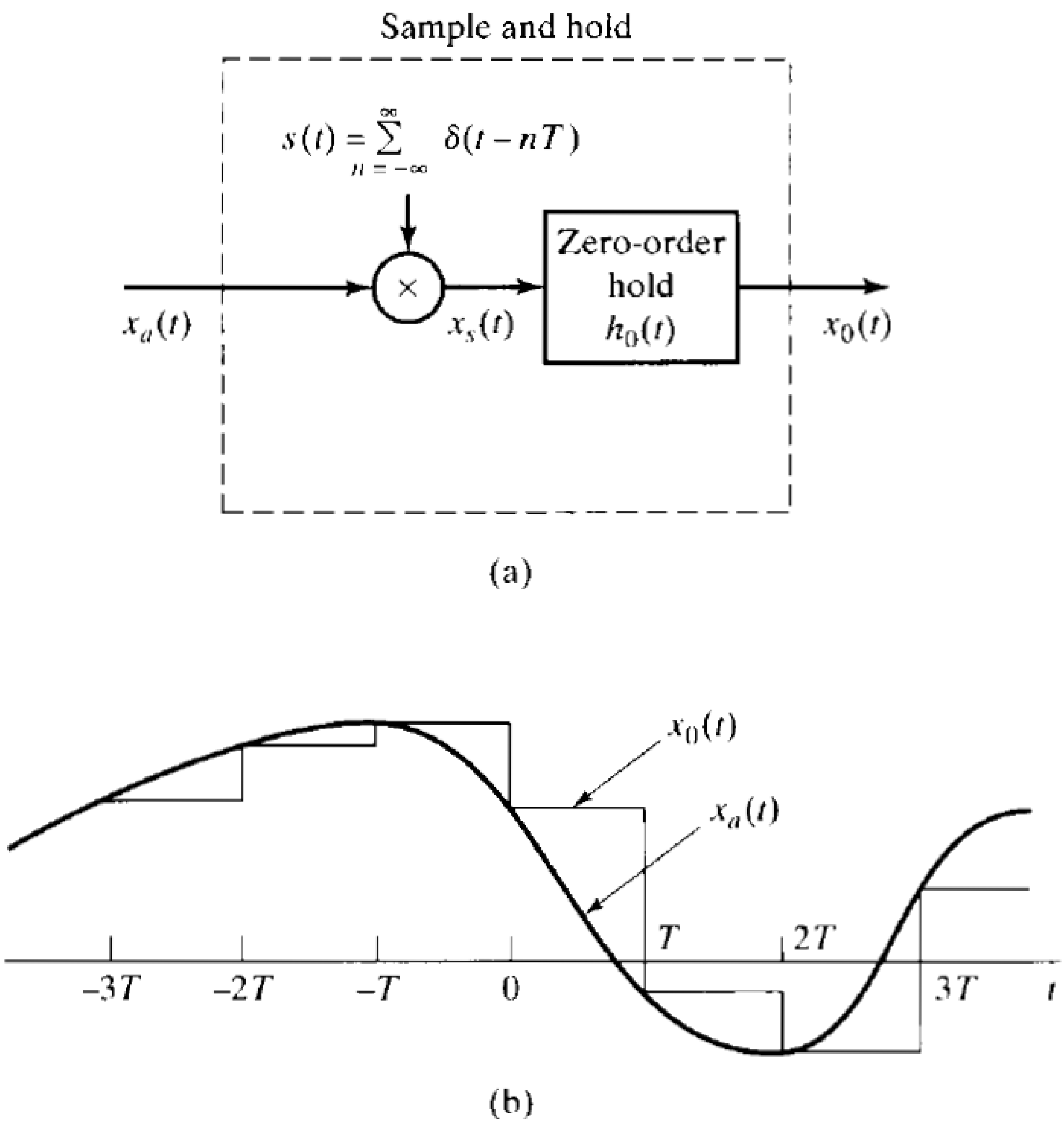
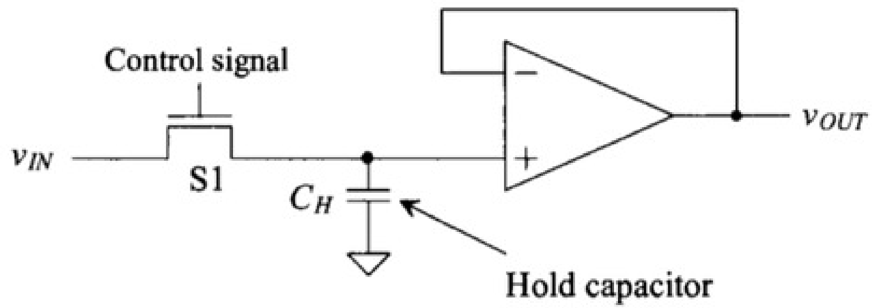
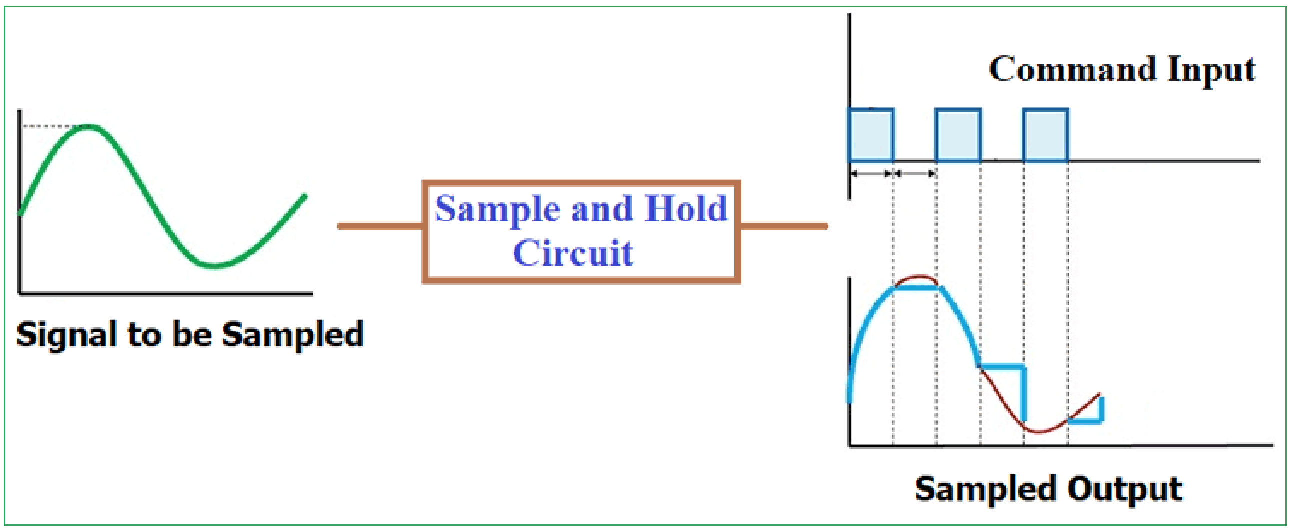


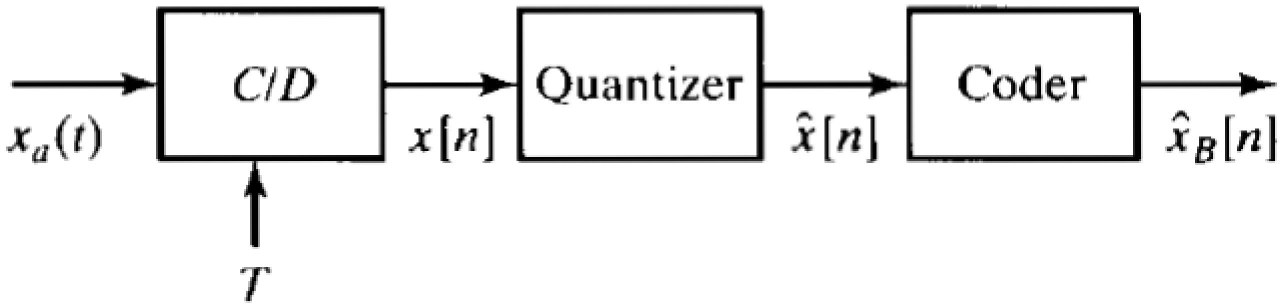
Figure 4.46 (a) Representation of an ideal sample-and-hold.
(b) Representative input and output signals for the sample-and-hold.

Practical Sample/Hold





Quantization

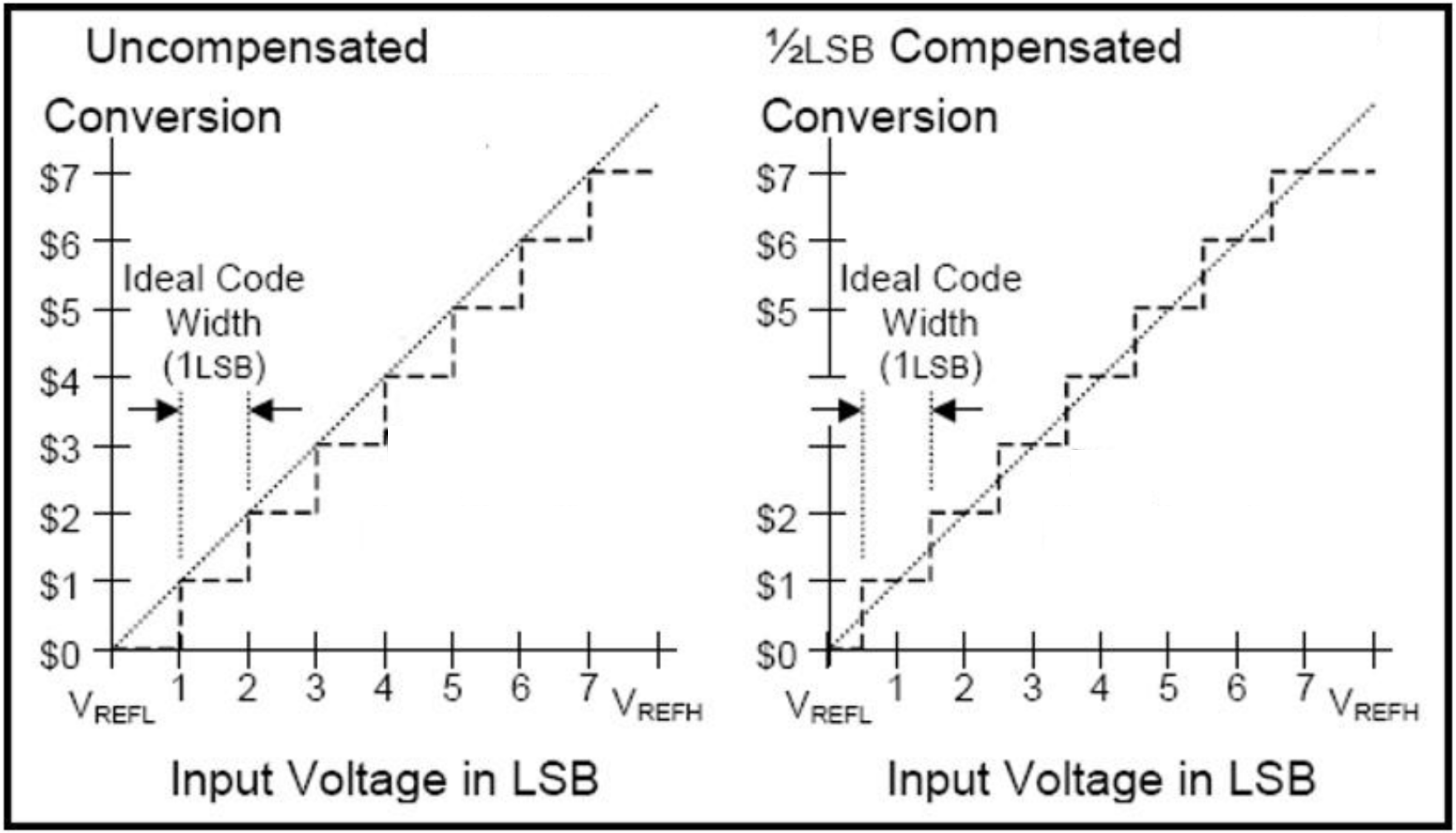


$x[n]$ The discrete-time version of $x_a(t)$ – continuous amplitude

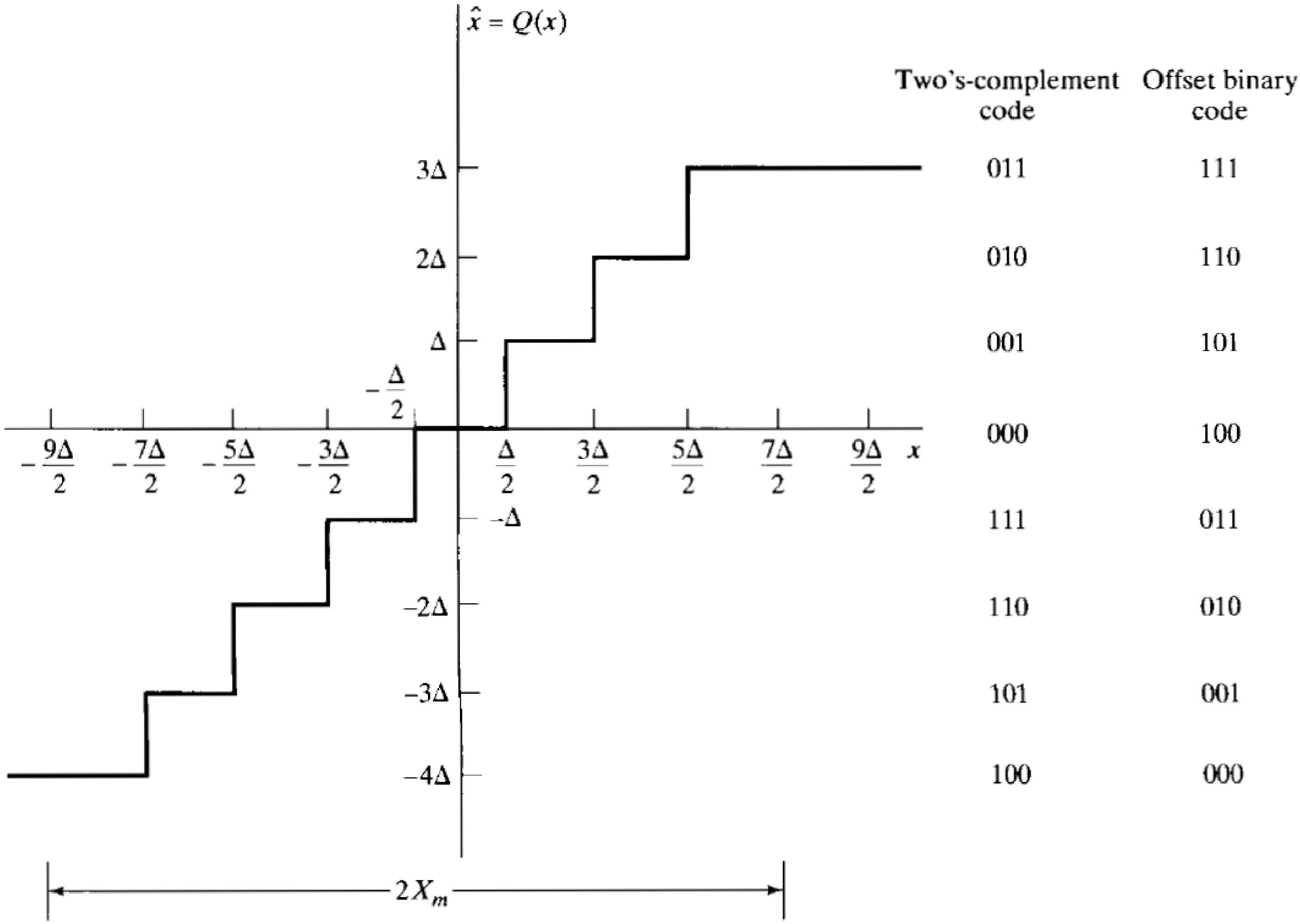
$\hat{x}[n]$ The discrete-time version of $x_a(t)$ – but now also discrete in amplitude

$\hat{x}_B[n]$ The time- and amplitude-discrete version of $x_a(t)$. Furthermore, this signal is also coded into a format which fits the arithmetic units in the computer.

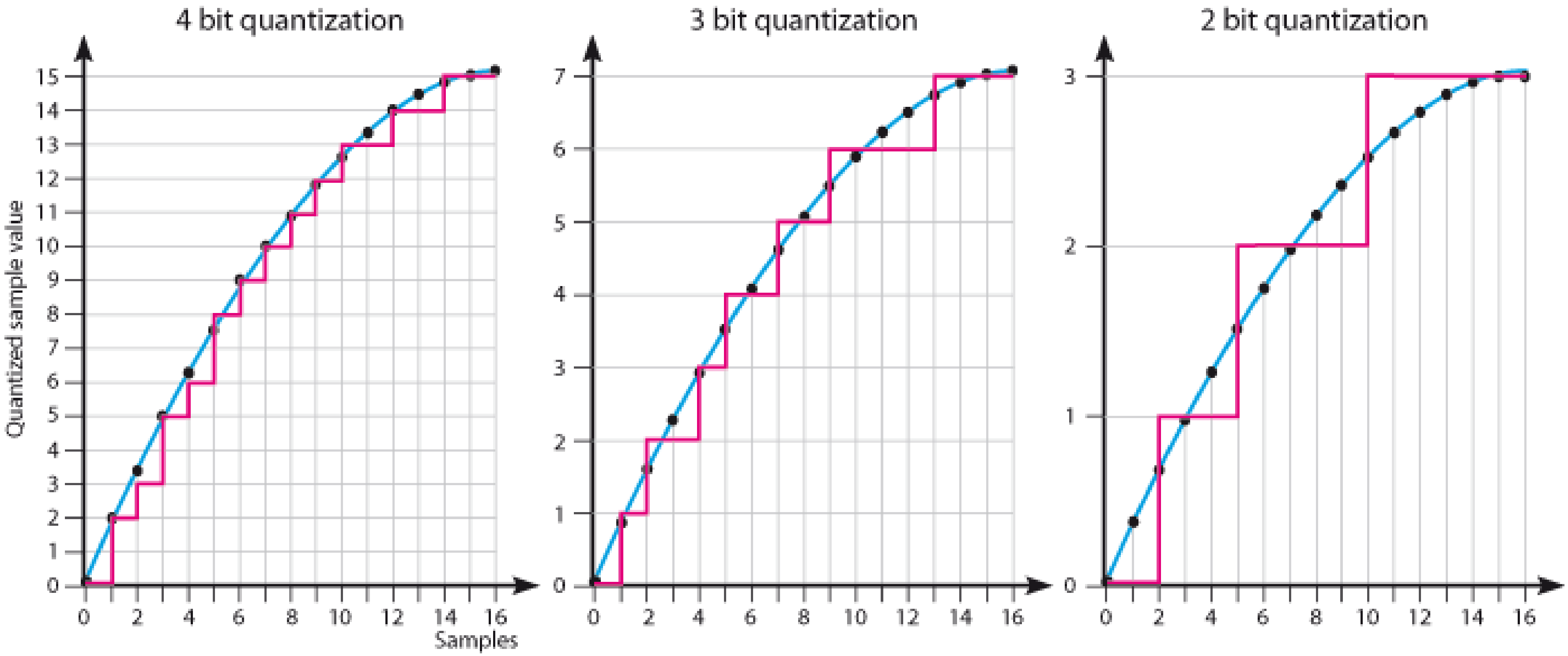
Quantization
- uncompensated and compensated



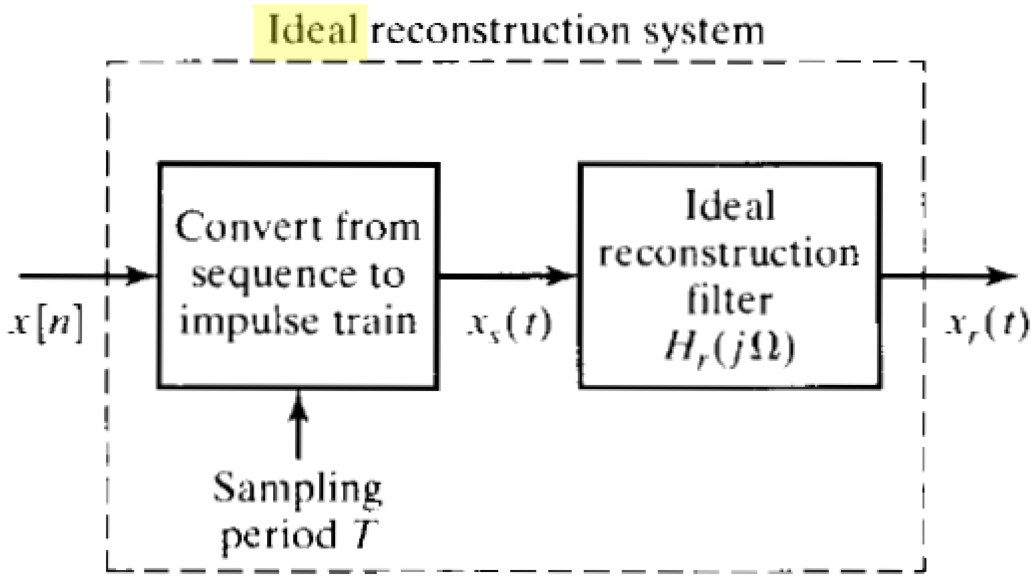
Quantization
½ LSB compensated and next coded



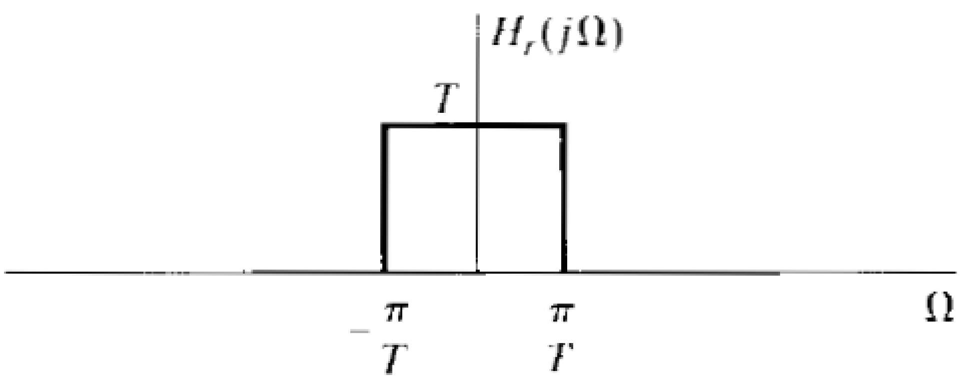
Quantization
- the more bits, the better signal representation



Digital to Analog Conversion – Reconstruction

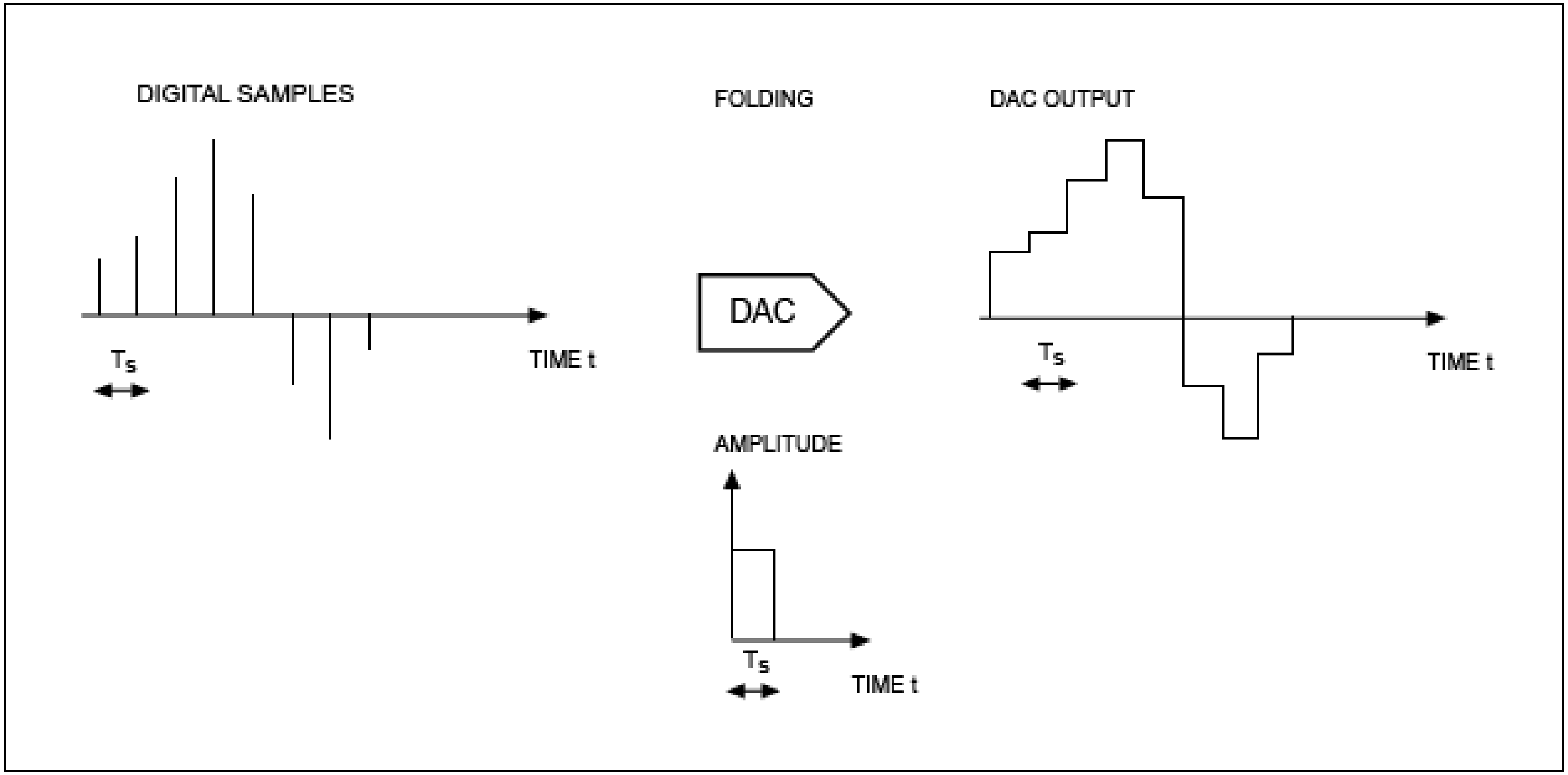


(a)



(b)

Output from the DAC'en, which is next "smoothed" using a reconstruction filter with LP characteristic



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