AALBORG UNIVERSITY
DENMARK

Written exam in
Signal processing, 5 ECTS

Thursday, January 8, 2015
9.00 – 13.00

Solutions

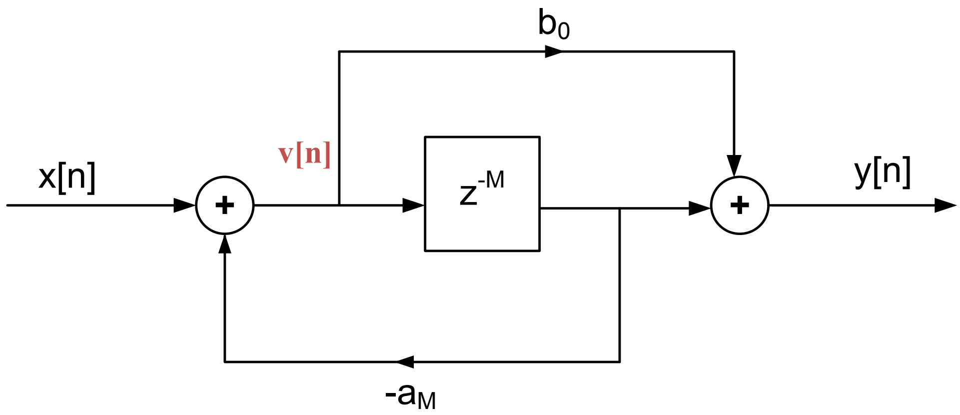
Read carefully :

- Remember to write your **full name on every sheet** you return!
- The answers concerning analog filters, digital filters and spectral estimation have to be written on **separate sheets**.
- Problems are weighted as listed.
- Results **without sufficient explanations will not give full credits**!
- All ordinary tools may be used e.g. books, notes, programmable calculators and laptops.
- All electronic communication devices **must be turned off** at all times.

- **<u>Communication with others is strictly prohibited</u>**.

Problem 1 (weighted with 12% - Digital filters)

A digital filter is has two coefficients a_M, b_0 and a delay block z^{-M} as illustrated by the flow graph below.



Questions:

1. Determine the transfer function H(z).

A new variable v[n] is introduced (see figure). It's seen that:

$$Y(z) = V(z)(b_0 \ + Z^{-M}) \qquad (1)$$
$$V(z) = X(z) \ + (-a_M V(z) Z^{-M}) \qquad (2)$$

⇕ By re-structuring the expression for V(z) we get:

$$X(z) = V(z)(1 + a_M Z^{-M})$$

⇕

$$V(z) = \frac{X(x)}{(1 + a_M Z^{-M})}$$

⇕ Inserting this into (1) gives us:

$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = \frac{(b_0 \ + Z^{-M})}{(1 + a_M Z^{-M})}}$$

2. Determine the difference equation.

From H(z) above we have:

$$Y(z)(1 + a_M Z^{-M}) = X(z)(b_0 \ + Z^{-M})$$

⇕

$$y[n] \ + a_M y[n-M] \ = b_0 x[n] + x[n-M]$$

⇕

$$\boldsymbol{y[n] = \ b_0 x[n] + x[n-M] - a_M y[n-M]}$$

3. Determine the magnitude response $|H(e^{j\omega})|$.

$$\left|H(e^{j\omega})\right| = \left|\frac{(b_0 \ + e^{-j\omega M})}{(1 + a_M e^{-j\omega M})}\right| = \left|\frac{(b_0 \ + e^{-j\omega M})}{e^{-j\omega M}(e^{j\omega M} + a_M)}\right|$$
$$= \left|\frac{b_0 + cos(-\omega M) + jsin(-\omega M)}{(1 + a_M cos(-\omega M) + a_M jsin(-\omega M))}\right|$$

$\Updownarrow$

$$\left|H(e^{j\omega})\right| = \left|\frac{b_0 + cos(\omega M) - jsin(\omega M)}{(1 + a_M cos(\omega M) - a_M jsin(\omega M))}\right|$$

4. Compute the DC gain (i.e. $|H(e^{j\omega})|$ for ω=0).

$$\left|H(e^{j\omega})\right|_{\omega=0} = \left|\frac{b_0 + 1}{(1 + a_M)}\right|$$

Problem 2 (Weighted with 10% - Digital filters)
Transform the 1. order analog Butterworth filter:

$$H_a(s) = \frac{1}{1 + \frac{s}{\Omega_c}} \quad , \quad \text{where } \Omega_c \text{ is the cut-off frequency}$$

into a digital filter using the bilinear transformation. The sampling frequency is fs= 1000Hz. The digital filter should have -3 dB cutoff frequency at 200 Hz. The DC gain should be 0 dB.

Questions:

2.1 Determine the transfer function H(z).

Initially the cutoff frequency is prewarped:

$$\Omega_{c,pw} = \frac{2}{T_d}\tan\left(\frac{\omega_c}{2}\right)$$

$$\omega_c = \frac{2\pi * 200}{1000} = \frac{2\pi}{5}$$

$$\Omega_{c,pw} = 2 * 1000 \ \tan\left(\frac{\frac{2\pi}{5}}{2}\right) = \ 1453\,rad \sim 231Hz$$

Do bilinear transformation (i.e. insert s=$\frac{2}{T_d}\frac{z-1}{z+1}$) ,

$$H(z) = \frac{\Omega_{c,pw}}{\Omega_{c,pw} + \frac{2}{T_d}\frac{z-1}{z+1}}$$

$\Updownarrow$

$$H(z) = \frac{\Omega_{c,pw}(z+1)}{\Omega_{c,pw}(z+1) + \frac{2}{T_d}(z-1)}$$

$\Updownarrow$

$$H(z) = \frac{(z+1)}{(z+1) + \frac{2(z-1)}{T_d\Omega_{c,pw}}}$$

$\Downarrow$

$$H(z) = \frac{z+1}{(z+1) + 1.376(z-1)}$$

$\Downarrow$

$$H(z) = \frac{z+1}{2.376z - 0.376}$$

$\Downarrow$

$$H(z) = \frac{1+z^{-1}}{2.376 - 0.376z^{-1}} = 0.421\frac{1+z^{-1}}{1-0.159z^{-1}}$$

We have to ensure a gain of 0 dB at DC, i.e.

$$\left|H(e^{j\omega})\right|_{\omega=0} = G\left|0.421\frac{1+e^{-j\omega}}{1-0.159e^{-j\omega}}\right|_{\omega=0} = 0\ dB$$

$\Downarrow$

$$\left|H(e^{j\omega})\right|_{\omega=0} = G(0.421\frac{1+1}{1-0.159}) = 0\ dB$$

$\Downarrow$

$$G = 1$$

I.e. transfer function is given by:

$$H(z) = 0.421\frac{1+z^{-1}}{1-0.159z^{-1}}$$

2.2 Compute the gain in dB at 200 Hz.

$$\left|H(e^{j\omega})\right| = \left|0.421\frac{1+e^{-j\omega}}{1-0.159e^{-j\omega}}\right|$$

$$\omega_c = \frac{2\pi * 200}{1000} = \frac{2\pi}{5}$$

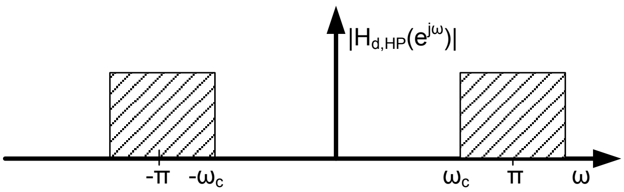
$$\left|H(e^{j\omega})\right| = \left|0.421\frac{1+cos(\frac{2\pi}{5}) + jsin(-\frac{2\pi}{5})}{1-0.159(cos(\frac{2\pi}{5}) + jsin(-\frac{2\pi}{5}))}\right|$$

$\Downarrow$

$$\left|H(e^{j\omega})\right| = 0.708 = -3dB$$

Problem 3 (weighted with 12% - Digital filters)

Design a high-pass FIR filter using the window method. The ideal amplitude response is illustrated in the figure below.



$$H_{d,HP}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & \omega_c \le |\omega| \le \pi \\ 0 & 0 \le |\omega| \le \omega_c \end{cases}$$

Use a rectangular window, a sampling frequency of f_s=20 kHz, a cut-off frequency of f_c=5 kHz and a filter order of M=4.

Questions:

3.1 Determine the filter coefficients.

$$h_{d,HP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{d,HP}(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{\sin \pi(n-\frac{M}{2})}{\pi(n-\frac{M}{2})} - \frac{\sin \omega_c(n-\frac{M}{2})}{\pi(n-\frac{M}{2})}, \qquad -\infty < n < \infty$$

$$\omega_c = 2\pi 5000/20000 = \pi/2$$

The filter coefficients are:

h[0]=0
h[1]=-0.318
h[2]=0.5
h[3]=-0.318
h[4]=0

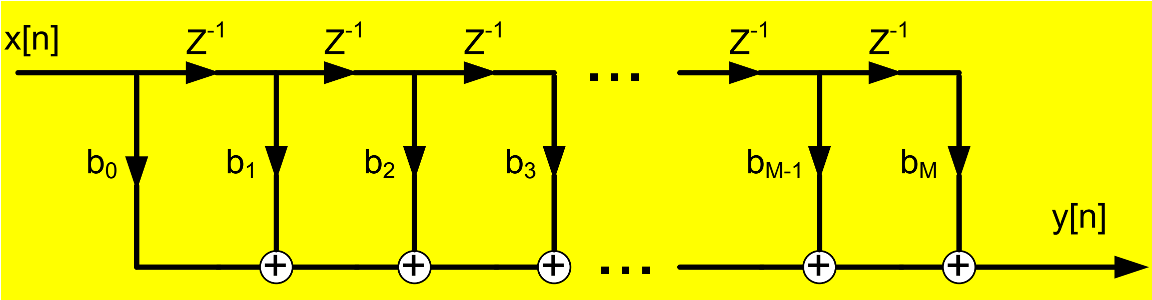
3.2 Determine the transfer function H(z).

As we have the filter coefficients, then we immediately know the transfer function:

$$H(z) = -0.318\, z^{-1} + 0.5\, z^{-2} - 0.318\, z^{-3}$$

3.3 Draw a signal flow graph for the filter.

see below, M=4



3.4 Determine the gain in dB at 5 kHz

$$\omega = \pi/2$$

H(z) = -0.318 z^{-1} + 0.5 z^{-2} - 0.318 z^{-3}

⇓

$$|H(e^{j\omega})| = |-0.318e^{-j\omega} + 0.5e^{-2j\omega} - 0.318e^{-3j\omega}|$$

⇓

$$|H(e^{j\omega})| = |-0.318(cos(-\omega) + jsin(-\omega)) + 0.5(cos(-2\omega) + jsin(-2\omega)) - 0.318(cos(-3\omega) + jsin(-3\omega))|$$

⇓

$$|H(e^{j\omega})| = |-0.318sin(-\pi/2) + 0.5cos(-\pi) \quad - 0.318\, sin(-3\pi/2)|$$

⇓

$$|H(e^{j\omega})| = |0.318 + 0.5 \quad - 0.318\,)| = 0.5 = -6dB$$

3.5 Describe the characteristics of the phase response?

The impulse response for the FIR filter is symmetric, hence the phase is linear

3.6 Compute the group delay.

The group delay is given by:

M/2 => 2 samples