PK

Signalbehandling f. Computer-Ing. 1.

~~Løsningsforslag~~, 3. forelæsning.

1. Find z-transform, incl. ROC.

a) $x[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$

$$\Downarrow \quad X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$\Downarrow \quad X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{z}\right)^n$$

$$\Downarrow \quad X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n$$

We can bring this expression onto closed form by using the equation

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \; ; \quad N_2 \geq N_1$$

for a geometric series, equ 55 p.31 in ODS.

$$X(z) = \frac{1-0}{1-\frac{1}{2z}} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

So, we have a pole in $z = \frac{1}{2}$.

Now, since $x[n]$ is a right-sided sequence, the cf. Property 5 p. 117, we have that ROC: $|z| > \frac{1}{2}$

b) $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

$$\Downarrow$$

$$X(z) = \sum_{n=-\infty}^{\infty} -\left(\frac{1}{2}\right)^n u[-n-1] \cdot z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{-n} z^{n}$$

$$= -\sum_{n=1}^{\infty} (2z)^n$$

$$\Downarrow$$

$$X(z) = -\frac{(2z)^1 - (2z)^\infty}{1-2z}$$

$$\Downarrow$$

$$X(z) = -\frac{2z}{1-2z} \quad \text{given that } |2z| < 1$$

$$\Downarrow$$

$$X(z) = \frac{2z}{2z-1} = \frac{z}{z-1/2}$$

So, we have a pole in $z = 1/2$, and since $x[n]$ is a left-sided sequence (cf. property 6), then ROC: $|z| < 1/2$

c) $x[n] = \left(\frac{1}{2}\right)^n u[-n]$

$$\Downarrow \quad X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[-n] \cdot z^{-n}$$

$$\Downarrow \quad X(z) = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{-n} z^{n}$$

$$= \sum_{n=0}^{\infty} (2z)^n$$

$$= \frac{(2z)^0 - (2z)^\infty}{1-2z}$$

$$= \frac{1}{1-2z} \quad \text{given that } |2z| < 1$$

Pole in $z = 1/2$ and since $x[n]$ is left-sided, then ROC: $|z| < 1/2$

d) $x[n] = \delta[n]$

$$\Downarrow$$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] \cdot z^{-n}$$

$$\Downarrow$$

$$X(z) = z^{-0} = \underline{\underline{1}}$$

Since $X(z)$ has no poles, $X(z)$ converges in the complete z-plane, and thus ROC: $\underline{\underline{\text{All } z}}$

e) $x[n] = \delta[n-1]$

$$\Downarrow$$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-1] \cdot z^{-n}$$

$$\Downarrow$$

$$X(z) = z^{-1} = \underline{\underline{\frac{1}{z}}}$$

$X(z)$ has a pole in $z=0$ and thus ROC: $\underline{\underline{|z| > 0}}$

f) $x[n] = \delta[n+1]$

$\Downarrow$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n}$$

$\Downarrow$

$$X(z) = z^{-(-1)} = \underline{\underline{z}}$$

Therefore; $X(z)$ has no poles and thus converges in the complete z-plane, the exception being $z=\infty$ cf. property 4 on p. 117, ROC: $\underline{\underline{0 \leq |z| < \infty}}$

g) $x[n] = \left(\frac{1}{2}\right)^n \left(u[n] - u[n-10]\right)$

$\Downarrow$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \left(u[n] - u[n-10]\right) z^{-n}$$

$\Downarrow$

$$X(z) = \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{9} \left(\frac{1}{2z}\right)^n$$

$\Downarrow$

$$X(z) = \frac{\left(\frac{1}{2z}\right)^0 - \left(\frac{1}{2z}\right)^{9+1}}{1 - \frac{1}{2z}} = \frac{1 - \left(\frac{1}{2z}\right)^{10}}{1 - \frac{1}{2z}}$$

$\Downarrow$

$$X(z) = \underline{\underline{\frac{1 - (2z)^{-10}}{1 - (2z)^{-1}}}}$$

Now, $x[n]$ is a finite length sequence cf property 4, with $N_1 = 0$ and $N_2 = 9$

Therefore $N_2 > 0$ and thus $X(z)$ converges in the complete z-plane, except in $z=0$

ROC: $|z| > 0$

5) Given the z-transform

$$X(z) = (1+2z)(1+3z^{-1})(1-z^{-1})$$

Find $x[n]$.

First we rewrite $X(z)$

$$X(z) = 2z + 5 - 4z^{-1} - 3z^{-2}$$

Using pair no. 4, tabel 1, p. 116, we now transform to the time-domain

$$x[n] = 2\delta[n+1] + 5\delta[n] - 4\delta[n-1] - 3\delta[n-2]$$

7) Given the sequence

$$x[n] = u[-n-1] + \left(\tfrac{1}{2}\right)^n u[n]$$

Furthermore, we have the z-transform

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}$$

Find $H(z) = Y(z)/X(z)$?

First we find $X(z)$ from tabel 1, p. 116

$$X(z) = \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$$

which has two poles; $z=1$ and $z=\frac{1}{2}$

We know that $x[n]$ consists of a left- and a right-sides sequence, and thus ROC: $\frac{1}{2} < |z| < 1$

We now rewrite $X(z)$;

$$X(z) = \frac{-(1-\frac{1}{2}z^{-1})(1-z^{-1})}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{-\frac{1}{2}z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}$$

Which is now used to find $H(z) = Y(z)/X(z)$

$$H(z) = Y(z) \cdot \frac{1}{X(z)}$$

$$= \frac{\cancel{-\frac{1}{2}z^{-1}}}{\cancel{(1-\frac{1}{2}z^{-1})}(1+z^{-1})} \cdot \frac{(1-z^{-1})\cdot\cancel{(1-\frac{1}{2}z^{-1})}}{\cancel{-\frac{1}{2}z^{-1}}}$$

$$= \frac{1-z^{-1}}{1+z^{-1}}$$ pole in $z=-1$.

It is given that $H(z)$ is a Causal system, and therefor we can conclude that $h[n]$ is a right-sided sequence, and thus; ROC: $|z|>1$

b) Find ROC for $Y(z) = H(z)\cdot X(z)$

$$Y(z) = \frac{-\frac{1}{2}\cdot z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}$$

Here's a set of arguments;

Since one of the poles in $X(z)$ — the one which limits ROC for $X(z)$ to $|z|<1$ — is eliminated by a corresponding zero in $H(z)$, then ROC for $Y(z)$ is the region in the z-plane which satisfies the two remaining conditions; $|z|>\frac{1}{2}$ and $|z|>1$.

Therefore, $Y(z)$ converges for $|z|>1$

c) Find $y[n] = \mathcal{Z}^{-1}\{Y(z)\}$

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}, \quad |z| > 1.$$

In order to determine $y[n]$, we need first of all to bring $Y(z)$ ~~on~~to a more usefull form, such that we can use the inspection method, i.e., tabel lookup.

Rewriting of $Y(z)$ is done be Partial Fraction Expansion.

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1+z^{-1}}$$

Find A_1 and A_2.

$$A_1 = (1-\frac{1}{2}z^{-1})Y(z)\Big|_{z=\frac{1}{2}} = \frac{(1-\frac{1}{2}z^{-1})(-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}\Bigg|_{z=\frac{1}{2}}$$

$$= \frac{-\frac{1}{2}\cdot 2}{1+2} = -\frac{1}{3}$$

$$A_2 = (1+z^{-1})Y(z)\Big|_{z=-1} = \frac{(1+z^{-1})(-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}\Bigg|_{z=-1}$$

$$= \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$$

Using pair no. 5 from the tabel on p. 116

$$Y(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1+z^{-1}}$$

$$= -\frac{1}{3}\frac{1}{(1-\frac{1}{2}z^{-1})} + \frac{1}{3}\frac{1}{(1+z^{-1})}$$

Can be found in the tabel

$\Downarrow$

$$y[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}(-1)^n u[n]$$

$\Downarrow$

$$y[n] = \frac{1}{3}\left((-1)^n - \left(\frac{1}{2}\right)^n\right), \quad n \geq 0$$