Signal Processing Terms

Lecture 1

- $\delta[n] = \begin{cases} 1, n = 0 \\ 0, Otherwise \end{cases}$ Unit sample sequence
- $u[n] = \begin{cases} 1, n > 0 \\ 0, Otherwise \end{cases}$ Unit step sequence
- $u[n] = \sum_{k=-\infty}^{n} \delta[n]$
- $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Lecture 2

- Time invariant systems are systems where a shift in the input results in a similar shift in the output. $y[n] = x[n] \Leftrightarrow y[n-n_0] = x[n-n_0]$
- Causal systems are systems that are only dependent on current and previous input values. These systems do not depend on knowing the future.
- Stable systems (in the bounded input, bounded output sense) are systems where every bounded sequence will produce a bounded output when put through the system.
- Linear time-invariant systems:
  - $y[n] = T\{x[n]\} = T\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

  - A system which is linear and time-invariant, i.e., an LTI system, is completely characterized by its impulse response $h[n]$, because we can derive the output $y[n]$ for any arbitrary input $x[n]$, if we know $h[n]$.
-

Convolution is commutative, i.e., $x[n] * h[n] = h[n] * x[n]$

Convolution is distributive, i.e., $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

Convolution is associative, i.e., $(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$

-

LTI System Properties

An LTI system is stable if, and only if the system impulse response is absolute summable, i.e.,

$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$

If the impulse response has a finite number of samples, then $S < \infty$.

Such a system is known as a Finite Impulse Response (FIR) system, and is always stable.

Systems with an infinite number of samples in the impulse response are stable if the infinite sum converges, i.e., $S < \infty$.

Such systems are denoted Infinite Impulse Response (IIR) systems

- The impulse response of the inverse system of an LTI system should fulfill:

$$h[n] * h'[n] = h'[n] * h[n] = \delta[n]$$

- Linear constant-coefficient difference equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- Laplace of the impulse response in a continuous system is called the transfer function

$$H(s) = \frac{X(s)}{Y(s)} \ (In\ LTI\ system)$$

Lecture 3

- Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- Z-transform:

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \ (Two\ sided/unilatelar)$$

$$X(z) = X(re^{j\omega}) = \sum_{n=0}^{\infty} x[n]z^{-n} \ (One\ sided/bilatelar)$$

- Discrete-Time Fourier Transform (DTFT):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

ω is the frequency, and the transform is periodic with period 2π

- The z-transform converges only if:

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

- Sum of a geometric series:

Sum [edit]

The sum of the first n terms of a geometric series, up to and including the r^{n-1} term, is given by the closed-form formula:

$$
\begin{aligned}
s_n &= ar^0 + ar^1 + \cdots + ar^{n-1} \\
&= \sum_{k=0}^{n-1} ar^k = \sum_{k=1}^{n} ar^{k-1} \\
&= \begin{cases} a\left(\frac{1-r^n}{1-r}\right), & \text{for } r \neq 1 \\ an, & \text{for } r = 1 \end{cases}
\end{aligned}
$$

- Properties of Radius of Convergence (ROC) for z-transform are in lecture 3 slide 25
- Stability of an LTI requires that the impulse response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

  Which also means that the DTFT for the impulse response also exists (is within the ROC)
- Since an LTI system needs to be stable and causal, then the poles of the system need to be inside the unit circle, so that the ROC can include the unit circle while including all the area from the last pole to infinity (which is a property of a right sided input).
- When solving questions with ROC; the best method seems to just look at the poles and zeroes and use those to conclude the ROC.

Lecture 4

- Modulated (sampled) impulse train of a continuous signal $x_c(t)$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

- This gets converted to the discrete sequence:

$$x[n] = x_s(nT)$$

- The periodic frequency is related to the normal frequency by $\Omega = 2\pi f$
- Frequency representation of a sampled signal is periodic with period $\Omega_s = 2\pi f_s = \frac{2\pi}{T_s}$
- Slides 13-15 contain important info regarding the relation between ω and Ω_s when sampling
- The rest of the slides talk about bit representation etc.
- Question 5 b and c in exercises fail to make sense

Lecture 5

- H(s) is known as a system's transfer function or system function.
- Ω is the angle frequency, $H(j\Omega)$ is the frequency response, $|H(j\Omega)|$ is amplitude response and $Arg\big(H(j\Omega)\big)$ is phase response.
- **$20\log_{10}(|H(j\Omega)|)$ is the amplitude response in decibels.**
- A brick wall filter is an ideal low pass filter.
- The squared amplitude response for the normalized nth order Butterworth lowpass filter is:

$$|H(j\Omega)|^2 = \frac{A_0}{1 + \Omega^{2n}}, \qquad A_0 \textit{ is DC gain}$$

- Idk might want to reread all this inside the exam lol

Lecture 6

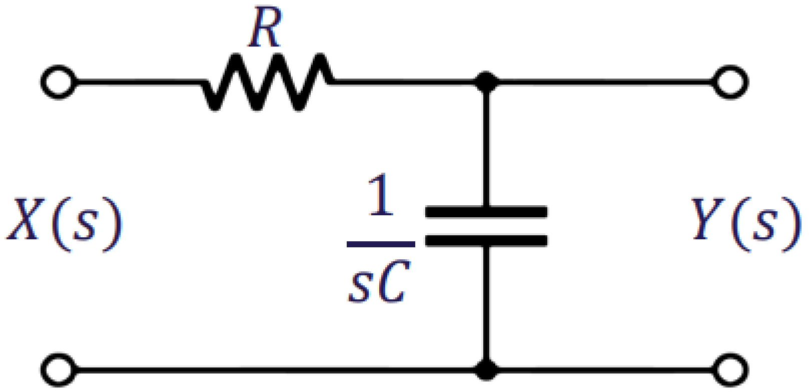
-

Relation between continuous-time and discrete-time frequency; $\omega = \Omega T = 2\pi f T = \frac{2\pi f}{f_s}$

-

In the discrete-time domain, everything has be related to the sample freqency

- Transfer function of an RC filter



$$H(s) = \frac{1}{sRC + 1}$$

- In the impulse invariant method, we take the continuous impulse response, we sample it to get a discrete impulse response, then we use the transfer function of that to get the difference equation of the filter.
- When converting the continuous impulse response to a discrete one using the impulse invariance method then the equation is:

$$h[n] = \frac{1}{f_s} h_c(t)|_{t=nT}$$

Lecture 7

- Idk seems important:

IN ORDER TO MAINTAIN THE SAME PASS-BAND AMPLIFICATION, WE MUST MULTIPLY WITH T;

$$\boxed{h[n] = T \cdot h_c(nT)}$$

- In bilinear transformation:

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

- Relation between Ω and ω:

$$s = \frac{2}{T} \cdot j \cdot \tan\left(\frac{\omega}{2}\right) = \sigma + j\Omega$$

$$\Omega = Im\{s\} = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \Leftrightarrow \omega = 2\arctan\left(\frac{\Omega T}{2}\right)$$

- The bilinear transformation leads to warping where the cutoff frequency moves downward. This is why pre-warping is introduced to the analog filter to be transformed so that the digital filter has the correct cutoff frequency.

$$\boxed{\Omega_{C,new} = \frac{2}{T} \cdot \tan\frac{\omega_c}{2}}$$

PREWARPING

- The solutions to the exercises are great to use as reference
- When finding the transfer function remember to normalize so that DC-gain is 0dB

Lecture 8

- A finite impulse response has a transfer function that can be reduced to only a polynomial in the numerator. (I think)
- We design FIR filters that have a linear phase response $Arg\big(H(z)\big) = -\alpha\omega + \beta$ so that the wave form of the signal passed through is preserved as much as possible.
- When a signal is passed through a filter with non-linear phase response, the signal's different frequency waves will be delayed by different amounts. (Very layman explanation)
- The group delay is a measure of how these frequencies are delayed and is written as:

$$G\left(e^{j\omega}\right) = -\frac{d}{d\omega}Arg\big(H(z)\big)$$

  Which in the case of linear phase gives α.
- Slide 13 has an example of one with a signal being passed through a non-linear phase filter.
- Linear phase is achieved in symmetric and anti-symmetric impulse responses.
- These responses are symmetric around M, where in the symmetric $h[n] = h[M-n]$ and in the anti-symmetric $h[n] = -h[M-n]$.
- Another aspect is that M can either be odd or even.

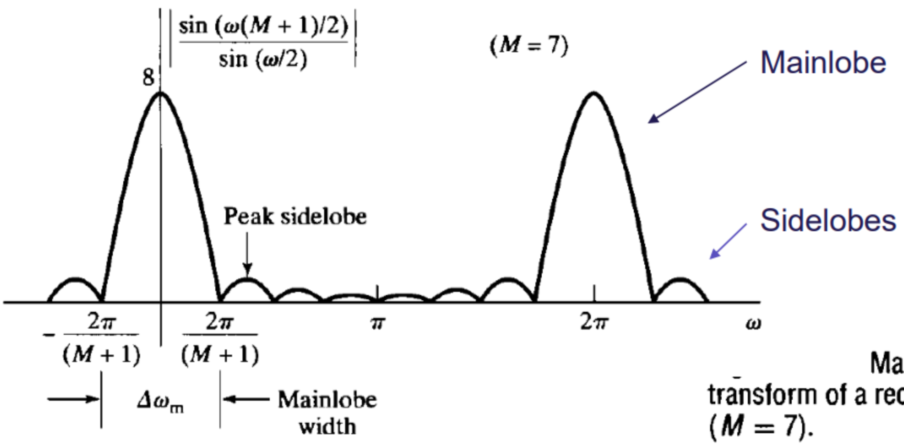
- The four types of these filters are:

$h[n]$ \ M	EVEN	ODD
SYMM.	I	II
ANTI-SYMM	III	IV

- Frequency response of a type I FIR with $\beta = 0$ is:

$$H(e^{j\omega}) = \left(h\left[\frac{M}{2}\right] + 2\sum_{k=1}^{\frac{M}{2}} h\left[\frac{M}{2}+k\right]\cos(\omega k) \right) e^{-\frac{j\omega M}{2}}$$

- The window function is multiplied by the impulse response of the ideal filter to get an FIR. This window function will result in convolution in the frequency domain. Therefore, we want the frequency response of the window function to resemble unit sample (Dirac delta function). The slides call it an impulse train, so I don't know.
- An example of the frequency response of a window function:



Magnitude of the Fourier transform of a rectangular window ($M = 7$).

The side lobes need to be small, while the main love needs to be narrow.

- The different window functions are available in slide 27

- Process of using the window method

DESIGN PROCEDURE

- $|H_d(e^{j\omega})|$ + LINEAR PHASE
- FIND $h_d[n]$, INVERSE FOURIER TRANSFORM
- CHOOSE WINDOW FUNCTION $w[n]$
- $h[n] = h_d[n] \cdot w[n]$ FILTER ORDER
- EXPRESS $H(e^{j\omega})$ AS A FUNCTION OF M
- PLOT $|H(e^{j\omega})|$ AND CHECK SPECIFICATIONS
- OK? YES → DONE, NO → NEW $w[n]$ / → TUNE M

ITERATIVE PROCESS

- When finding the ideal filter it will usually have the following form:

$$H_d(e^{j\omega}) = \begin{cases} e^{j(-\alpha\omega+\beta)}, & |\omega| \leq \omega_{cutoff} \\ 0, & Otherwise \end{cases}$$

Where usually $\alpha = \frac{M}{2}$ and $\beta = 0 \vee \beta = \pi$ resulting in:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\frac{M}{2}}, & |\omega| \leq \omega_{cutoff} \\ 0, & Otherwise \end{cases}$$

- These filters are symmetric around M/2. Meaning that M is where the last point in the response is.
- The impulse response is usually found using the inverse DTFT which is:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

- When finding the frequency response of the truncated impulse response it is smart to use the function already given in the book for frequency response of types I-IV FIRs

-

SO, NO MATTER HOW HIGH A FILTER ORDER WE CHOOSE, WE CANNOT (USING THE SAME TYPE OF WINDOW) REDUCE THE PASS- AND STOP-BAND RIPPLE.

WE CAN HOWEVER, BY INCREASING M, OBTAIN A BETTER APPROXIMATION TO THE TRANSITION FROM PASS- TO STOP-BAND ~ A NARROWER TRANSITION BAND.

Lecture 9

- Transfer function of an ideal high pass filter can be written as:

$$H_{d(HP)} = e^{-j\omega\frac{M}{2}} - H_{d(LP)}\left(e^{j\omega}\right)$$

Where the ide;a low pass transfer function has the same cutoff frequency as the high pass.

-

$$h_{d_{HP}}[n] = \frac{\sin(\pi(n-M/2))}{\pi(n-M/2)} - \frac{\sin(\omega_c(n-M/2))}{\pi(n-M/2)}$$

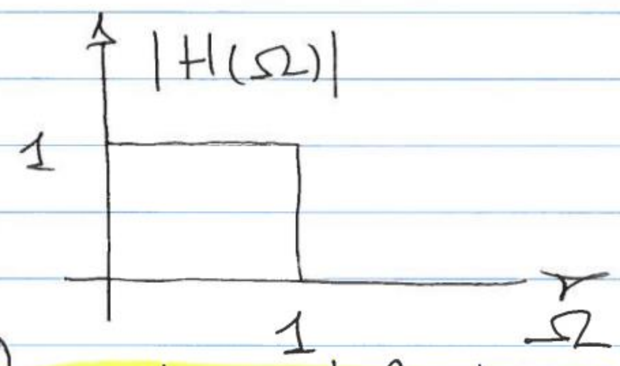
$$= \begin{cases} 1 & \text{for } n = M/2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{IDENTICAL TO LP}$$

$\Downarrow$

THE IMPULSE RESPONSE (AND THUS THE FILTER COEFFICIENT) FOR THE HP. FILTER CAN BE EXPRESSED IN TERMS OF THE LP. FILTER.

- $b_{i_{HP}} = -b_{i_{LP}} \qquad i = 0, \ldots, M/2-1, M/2+1, \ldots, M$
- $b_{M/2_{HP}} = 1 - b_{M/2_{LP}}$

- All other type of filters can be constructed by doing a frequency transformation on a normalized LP filter:



LP → LP
LP → HP
LP → BP
LP → BS

NORMALIZED LP. FILTER

-

TABLE 1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY μ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

- It is possible to find the amplitude and phase response if the positions of the zeroes and poles are known in the z-plane.
  The amplitude response is found using $\frac{\sum Distance\ to\ zeroes}{\sum Distance\ to\ poles}$
  The phase response is found using $\sum Angle\ with\ zeroes - \sum Angle\ with\ poles$
- Using this it can be concluded that at the angle of the poles a peak will occur, while on the angle of the zeroes a dip will occur. Check slide 45 for more info.
- The closer the poles are to the unit circle the higher the peaks are (approaching infinity), while the closer the zeroes are to the unit circle, the lower the dips are (approaching 0).

Lecture 10

- If any question needs quantization or scaling. I will leave that question to the end and will look throughout the slides again to see what I can do with it.
- The suggested solution for the exercise provides some insight on how to do scaling.
- Direct form II representation does not require writing down the $w[n]$ that comes in the middle, as it is a dummy sequence that is not needed.

- Direct form I takes the zeroes first (x[n] calculations) then the poles (y[n] calculations), while in direct form II the poles are done before the zeroes. This is why in direct form II the y[n] coefficients are at the right side of the diagram.
- Direct Form I

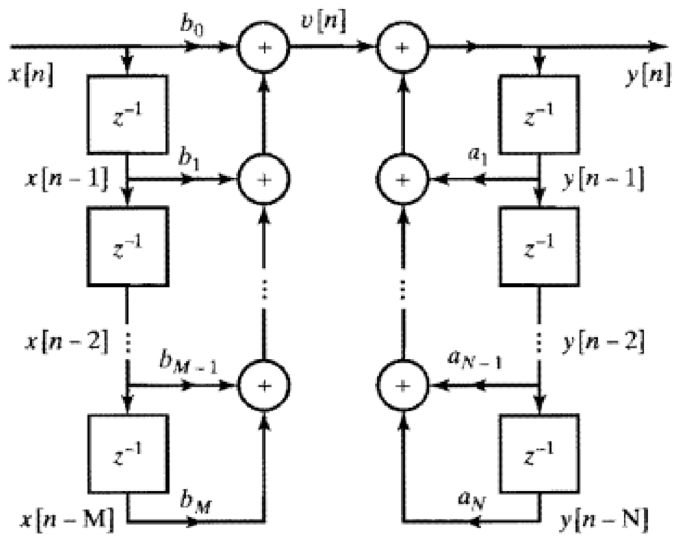
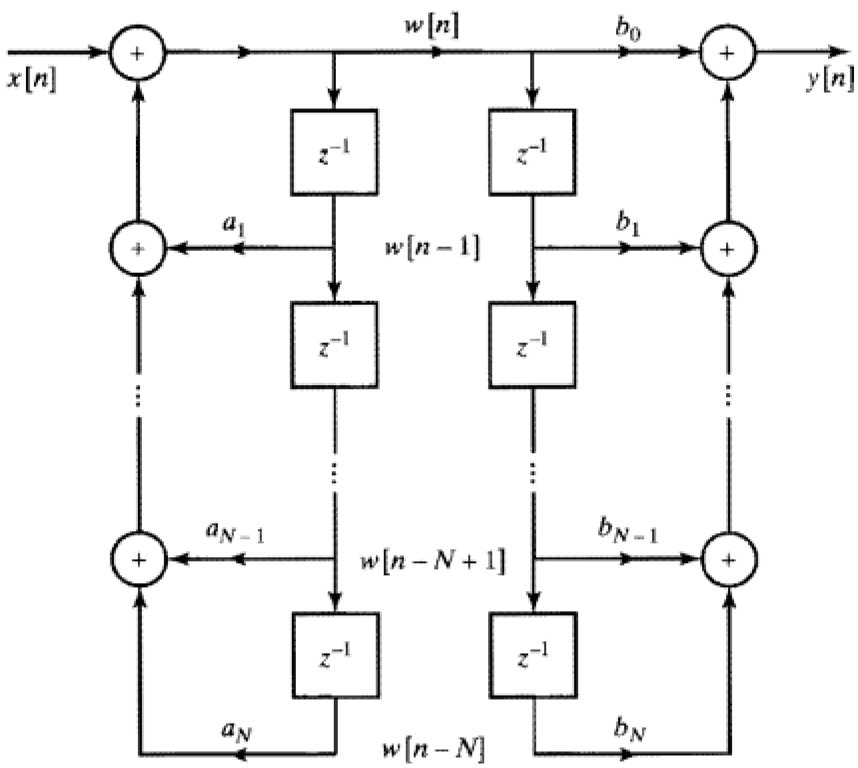
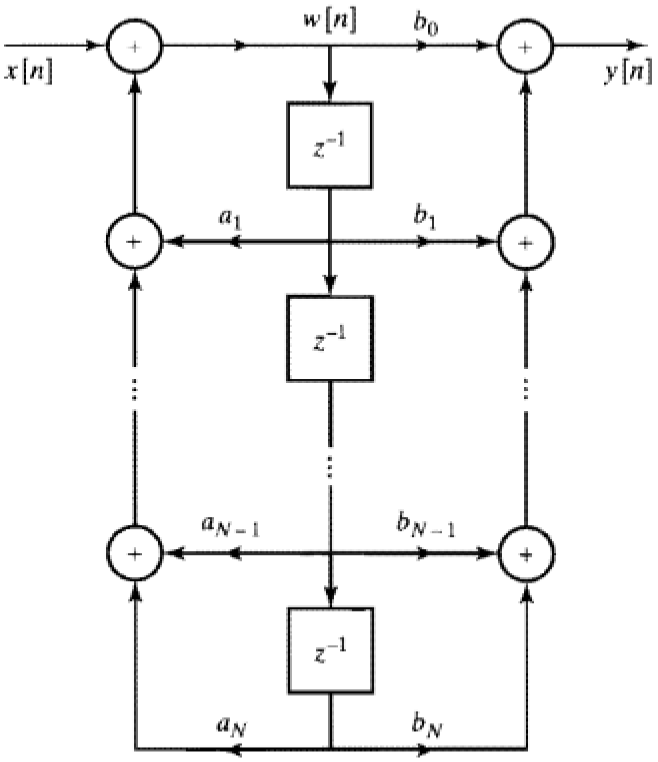


Figure 6.3 Block diagram representation for a general N^{th}-order difference equation.

- Direct Form I before becoming Direct Form II



- Direct Form II

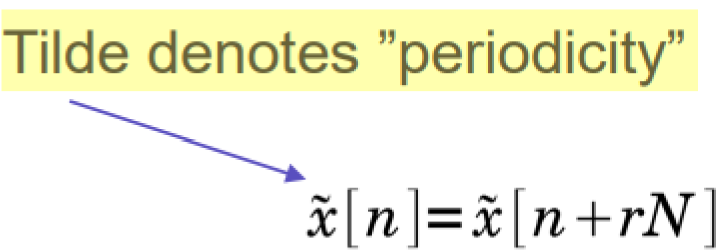


- Transfer function:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

Lecture 11

- In a Fourier series the sinusoids used should all be periodic in period T (Which is the period of the initial function). Therefore, the frequencies of the sinusoids are multiples of the angular frequency $\Omega_k = k\frac{2\pi}{T} = k\Omega_f$
-



- Continuous Fourier series (Period T) vs Discrete Fourier series (Period N):

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\Omega_f t}, \qquad \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

- Continuous Fourier series weights/coefficients:

$$X[k] = \frac{1}{T}\int_{<T>} x(t)e^{-jk\Omega_f t}dt = \frac{\Omega_f}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\Omega_f t}dt, \text{ where } \frac{1}{T} = \frac{\Omega_f}{2\pi} \text{ cf., p.8}$$

<*T*> represents a single period, e.g., [*0;T*[or [*-T/2;T/2*[, or whatever is the most convenient interval in order to calculate the integral…

- Discrete Fourier series weights/coefficients:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{j\frac{2\pi}{N}kn}$$

-

DFS analysis: $$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

DFS synthesis: $$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

- In these examples $W_N = e^{-j\frac{2\pi}{N}}$ which is called the Twiddle Factor.
- Duality is explained in slides 17-20
- DFS is linear
-

Time-shift:

$$\tilde{x}[n-m] \xrightarrow{DFS} W_N^{km}\tilde{X}[k]$$

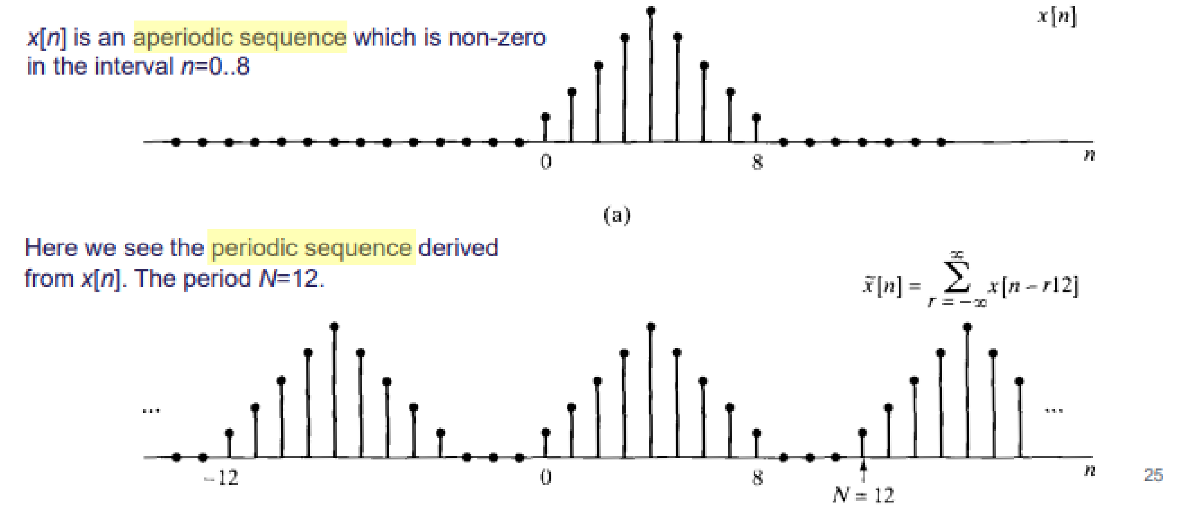
Frequency-shift:

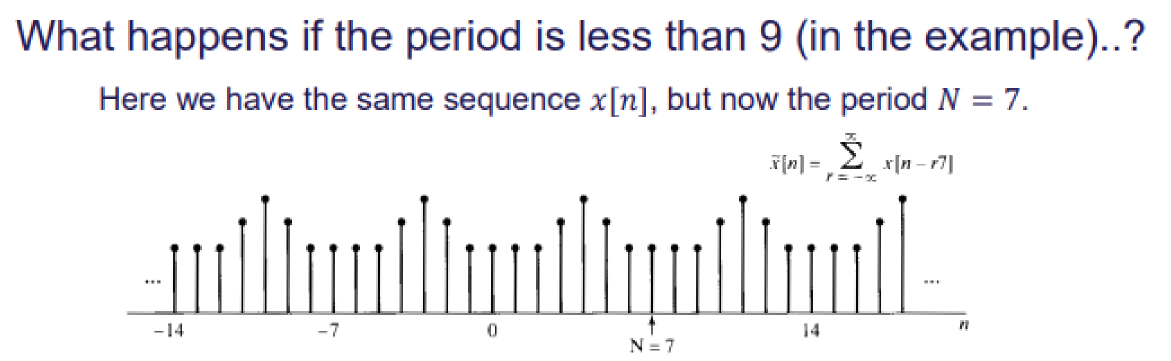
$$\tilde{X}[k-l] \xrightarrow{DFS} W_N^{-nl}\tilde{x}[n]$$

- The DFS can be obtained by sampling the DTFT along the unit circle:

$$\tilde{X}[k] = X\left(e^{j\frac{2\pi}{N}k}\right)$$

- If this sampled DFS is synthesized the resulting sequence $\tilde{x}[n]$ will be the result of adding infinitely many shifter replicase of the original sequence $x[n]$. If the original sequence is not finite, this will lead to a time aliased periodic sequence.
- Furthermore, by choosing a period N lower than the number of points in the finite sequence, then the periodic sequence will also be time aliased:





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Relation between the DFS and the DFT

$$X[k]=\begin{cases}\tilde{X}[k], & 0\leq k\leq N-1\\ 0, & \text{otherwise}\end{cases}$$

In other words: The Discrete Fourier Transform can be written as

$$X[k]=\sum_{n=0}^{N-1} x[n]W_N^{kn} \quad \text{and}$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$$

But in that case both X and x must be 0 outside $0\rightarrow N-1$

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Lecture 12

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Periodic Convolution, DFS

$$\left.\begin{aligned}\tilde{X}_3[k] &= \tilde{X}_1[k]\tilde{X}_2[k]\\ \tilde{x}_3[n] &= \sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]\end{aligned}\right\} \quad \sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k]$$

The periodic convolution of periodic sequences thus corresponds to multiplication of the corresponding periodic sequences of Fourier series coefficients.

- The many ways to writing the periodic sequence of a certain finite sequence of length N:

$$\tilde{x}[n]=\sum_{r=-\infty}^{\infty}x[n-rN]=x[mod_N(n)]=x\left[((n))_N\right]$$

- The DFT is the DFS but only in one period, where values outside this period are 0:

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{X}[k] = X[(k \text{ modulo } N)] = X[((k))_N].$$

- DFT analysis and synthesis equations:

Analysis equation: $$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn},$$

Synthesis equation: $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}.$$

- This lecture talks about circular convolution and circular time-shift.
- The frequency resolution of a DFT sampled out of a DTFT can be calculated using:

$$f_k = k\frac{f_s}{N}$$

$$\Omega_k = k\frac{\Omega_s}{N} = \frac{2\pi k f_s}{N} = \frac{2\pi k}{NT}$$

Lecture 14

- When windowing a sequence using non-sufficient N value might lead to leakage, where some peaks in the amplitude response start merging together.

-

The effective frequency resolution depends on the window's main-lobe width and thus the window length.

The leakage depends on the ratio between main-lobe amplitude and side-lobe amplitudes.

The rectangular window gives the highest possible frequency resolution but also has the largest side-lobes.

-

Exam set notes

- Might want to focus a bit on direct form II diagrams
- $H(z) = \frac{Y(z)}{X(z)}$
- When getting a transfer function from a direct form diagram it is best to get the filter equation first and then z-transform it to get the transfer function.
- Using the FFT library is better than making my own DFT function

Tables

- Laplace properties

Laplace transform Properties

1. Linearity	$\sum_{n=1}^{N} \alpha_n x_n(t)$	$\sum_{n=1}^{N} \alpha_n X_n(s)$
2. Time shift	$x(t-t_0)u(t-t_0)$	$X(s)\exp(-st_0)$
3. Frequency shift	$\exp(s_0 t)x(t)$	$X(s-s_0)$
4. Time scaling	$x(\alpha t), \alpha > 0$	$1/\alpha\, X(s/\alpha)$
5. Differentiation	$dx(t)/dt$	$s\,X(s) - x(0^-)$
6. Integration	$\int_0^t x(\tau)\,d\tau$	$\frac{1}{s} X(s)$
7. Multiplication by t	$t\,x(t)$	$-\frac{dX(s)}{ds}$
8. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}[X(s-j\omega_0) + X(s+j\omega_0)]$
	$x(t)\sin\omega_0 t$	$\frac{1}{2j}[X(s-j\omega_0) - X(s+j\omega_0)]$
9. Convolution	$x(t) * h(t)$	$X(s)H(s)$
10. Initial value	$x(0^+)$	$\lim_{s\to\infty} s\,X(s)$
11. Final value	$\lim_{t\to\infty} x(t)$	$\lim_{s\to 0} s\,X(s)$

- Laplace pairs

Basic Laplace Transform Pairs

Impulse	$\delta(t)$	1
Step	$u(t)=1, \quad t\geq 0$	$\frac{1}{s}$
Ramp	$r(t)=t, \quad t\geq 0$	$\frac{1}{s^2}$
Exponential	$e^{-\alpha t} \quad e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
Damped Ramp	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$
Sine	$\sin(\beta t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$\cos(\beta t)$	$\frac{s}{s^2+\beta^2}$
Damped Sine	$e^{-\alpha t}\sin(\beta t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped Cosine	$e^{-\alpha t}\cos(\beta t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

- Z-transform pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

- Z-transform properties

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{Jm}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Initial-value theorem:

$$x[n] = 0, \quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$$

- The different window methods

Rectangular

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2, \\ 2 - 2n/M, & M/2 < n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Hanning

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Blackman

$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

- Properties of different window functions

TABLE 2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

- Frequency transformation for frequency warping

TABLE 1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY μ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \dfrac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

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- DTFT pairs

DTFT pair

1. $\delta[n]$	1	Impulse <-> Flat spectrum
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$	Time shifted impulse
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$	DC
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1 - ae^{-j\omega}}$	Exponential descreasing
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$	
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$	
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$	
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$	Lowpass filter
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$	Rect. window
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$	Complex exponential
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$	

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- Properties of DFS

SUMMARY OF PROPERTIES OF THE DFS

Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{J}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{J}m\{\tilde{X}[k]\} = -\mathcal{J}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \sphericalangle\tilde{X}[k] = -\sphericalangle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$

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- DFT

Finite-Length Sequence (Length N)	N-point DFT (Length N)	
1. $x[n]$	$X[k]$	
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$	
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$	Linear
4. $X[n]$	$Nx[((-k))_N]$	Duality
5. $x[((n-m))_N]$	$W_N^{km} X[k]$	Time shift
6. $W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$	Frequency shift
7. $\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k] X_2[k]$	Circular convolution in time
8. $x_1[n] x_2[n]$	$\frac{1}{N} \sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$	Circular convolution in frequency
9. $x^*[n]$	$X^*[((-k))_N]$	
10. $x^*[((-n))_N]$	$X^*[k]$	
11. $\mathcal{Re}\{x[n]\}$	$X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$	
12. $j\mathcal{Im}\{x[n]\}$	$X_{op}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$	
13. $x_{ep}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{Re}\{X[k]\}$	
14. $x_{op}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{Im}\{X[k]\}$	
Properties 15–17 apply only when $x[n]$ is real.		
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{Re}\{X[k]\} = \mathcal{Re}\{X[((-k))_N]\} \\ \mathcal{Im}\{X[k]\} = -\mathcal{Im}\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$	
16. $x_{ep}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{Re}\{X[k]\}$	
17. $x_{op}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{Im}\{X[k]\}$	

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