Signalbehandling for computer-ingeniører

COMTEK-5, E22

&

Signalbehandling

EIT-5, E22

9. Digital FIR Filters – LP/HP/BP/BS – Analysis of Transfer Functions

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SIGNAL PROCESSING
9th LECTURE

LAST TIME WE TALKED ABOUT SYSTEMS WITH GENERALIZED LINEAR PHASE;

$$(1) \quad H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{-j(\alpha\omega - \beta)}$$

REAL AMPLITUDE FUNCTION

COMPLEX PHASE FUNCTION, EQU. FOR STRAIGHT LINE

FREQUENCY RESPONSE FOR GENERAL SYSTEM;

$$(2) \quad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n}$$

DERIVING EXPRESSIONS FOR $\tan \underline{H(e^{j\omega})}$, BOTH FOR (1) AND (2), AND EQUALIZING THESE TWO, WE GOT;

$$\sum_{n=-\infty}^{\infty} h[n] \cdot \sin(\omega(n-\alpha)+\beta) = 0, \quad \forall \omega$$

WHICH IS A NECESSARY CONDITION FOR LINEAR PHASE.

$$\sum_{n=-\infty}^{\infty} h[n] \cdot \sin(\omega(n-\alpha)+\beta) = 0 \;;\; \forall \omega.$$

$\Downarrow$

A SET OF $h[n]$, α, AND β WHICH FULFILLS THE EQUATION;

$2\alpha = M$, INTEGER $\to$ EVEN / $\to$ ODD (FILTER ORDER)

$\beta = \begin{cases} 0 \\ \pi \end{cases}$

$h[n]$ $\to$ SYMMETRIC ; $h[n] = h[M-n]$

$h[n]$ $\to$ ANTI-SYMMETRIC; $h[n] = -h[M-n]$

α IS KNOWN AS THE SYMMETRY POINT $(\alpha = M/2)$

NOW, IF WE ASSUME THE SYSTEM h TO BE A CAUSAL FIR SYSTEM, THEN

$$h[n] = 0, \quad n < 0 \text{ AND } n > M$$

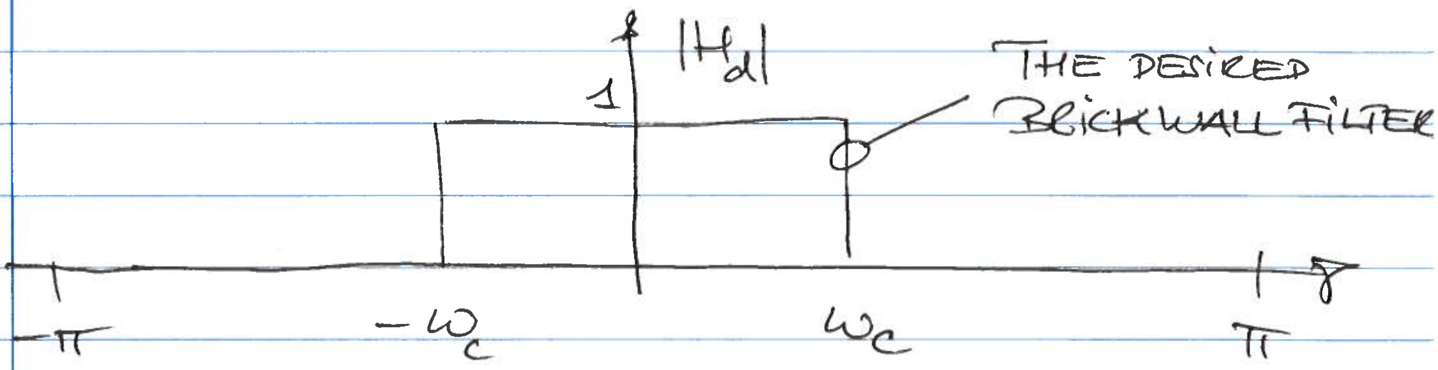
$\Downarrow$

FIR TYPE I, II, III AND IV

M \ $h[n]$	SYMMETRIC	ANTI-SYMMETRIC
EVEN	I	III
ODD	II	IV

DESIGNING AN FIR FILTER

- THE DESIRED AMPLITUDE RESPONSE
- LINEAR PHASE RESPONSE



THE DESIRED FREQUENCY RESPONSE

$$H_d(e^{j\omega}) = \begin{cases} 1 \cdot e^{-j\omega M/2} & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

$\alpha = M/2$ AND $\beta = 0$

USING THE INVERSE DTFT, WE CAN FIND

$$h_d[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega M/2} \cdot e^{j\omega n} \, d\omega$$

$$\Downarrow$$

$$h_d[n] = \frac{\sin(\omega_c(n - M/2))}{\pi(n - M/2)}$$

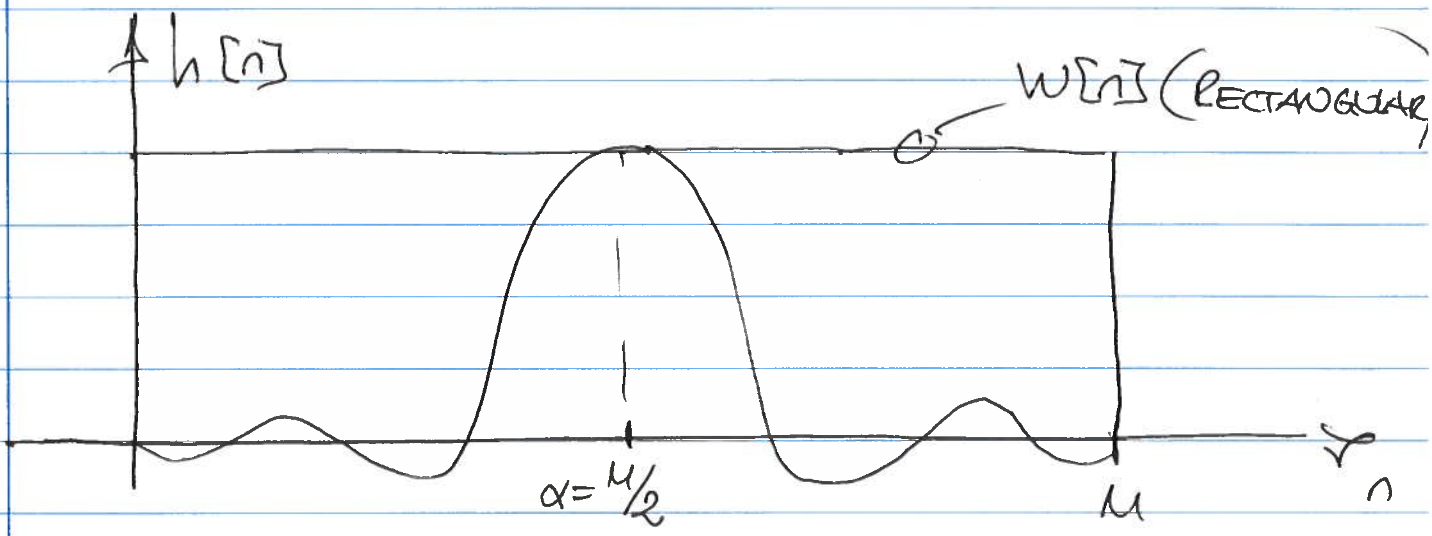
NON-CAUSAL
INFINITE

SYMMETRIC FUNCTION

THE IMPLEMENTABLE IMPULSE RESPONSE;

$$h[n] = h_d[n] \cdot w[n]$$

WINDOW FUNCTION; $w[n] = \begin{cases} f[n] & 0 \le n \le M \\ 0 & \text{OTHERWISE} \end{cases}$



NOTE THAT α „MOVES" AS WE ADJUST THE WINDOW LENGTH.

ALSO NOTE THAT $w[n]$ IS SYMMETRIC, THUS

$h[n] = h_d[n] \cdot w[n]$ IS ALSO SYMMETRIC

AND THEREFORE $H(z)$ HAS LINEAR PHASE

QUESTION IS ; HOW DOES THE AMPLITUDE RESPONSE LOOKS LIKE ??

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cdot\cos(\omega k)\right)$$

WHERE $\begin{cases} a[k] = 2\cdot h[M/2 - k] \text{ for } k = 1 \ldots M/2 \\ a[0] = h[M/2] \end{cases}$

EQU 140a, p.343

$\Downarrow$

$$|H(e^{j\omega})| = \sum_{k=0}^{M/2} a[k]\cdot\cos(\omega k)$$

NOW, ASSUME THAT WE WANT 0dB DC GAIN

$$G\cdot|H(e^{j\omega})| = 1\Big|_{\omega=0}$$

$\Downarrow$

$$G\cdot\sum_{k=0}^{M/2} a[k] = 1$$

$\Downarrow$

$$G = \frac{1}{\sum_{k=0}^{M/2} a[k]}$$

FOR GIVEN WINDOW FUNCTION $w[n]$, OUR TASK NOW IS TO FIND THE SMALLEST POSSIBLE VALUE FOR THE FILTER ORDER M WHICH SATISFIES THE SPECIFICATIONS ;

THE PROBLEM WE ARE FACING HERE IS

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

IDEALY WE WOULD LIKE $W(e^{j\omega})$ TO BE AN IMPULSE TRAIN, WHICH HOWEVER WOULD REQUIRE $w[n] = 1, \forall n$ non-causal infinite.

SO, WE ARE LOOKING FOR A COMPROMISE.

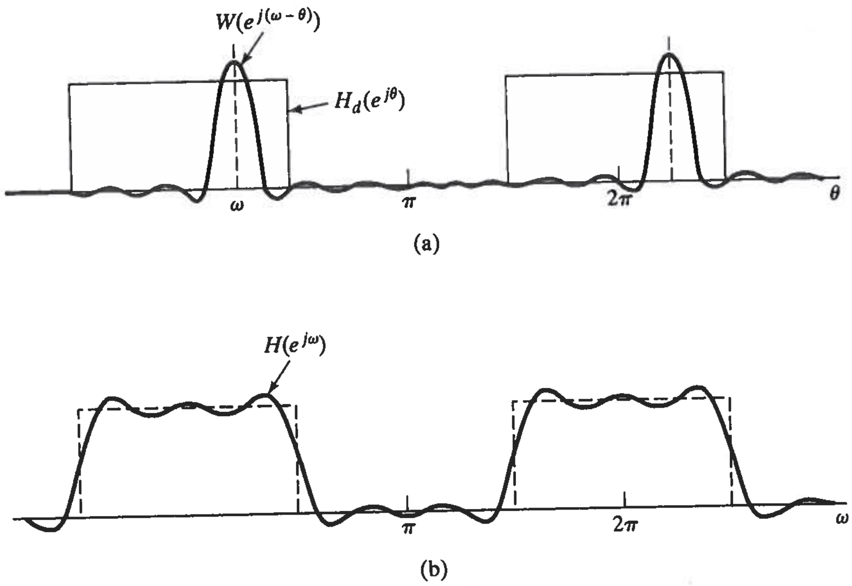


Figure 27 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

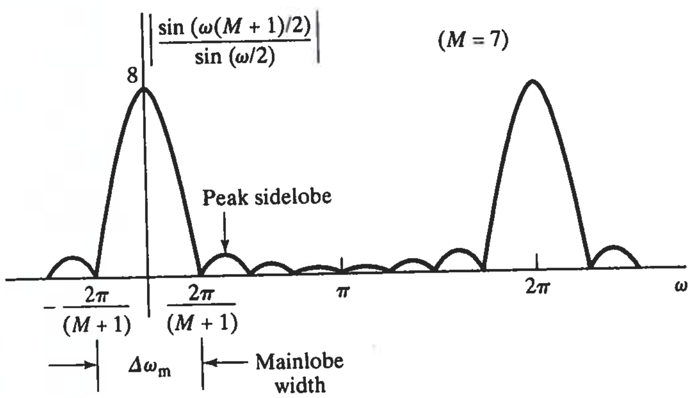


Figure 28 Magnitude of the Fourier transform of a rectangular window ($M = 7$).

5.1 Properties of Commonly Used Windows

Some commonly used windows are shown in Figure 29. These windows are defined by the following equations:

Rectangular

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{60a}$$

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TABLE 2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

OBS! OBS!

- PEAK OF SIDE-LOBE IS CONSTANT
- WIDTH OF MAIN-LOBE IS INVERSE PROPORTIONAL TO M

SO, NO MATTER HOW HIGH A FILTER ORDER WE CHOOSE, WE CANNOT (USING THE SAME TYPE OF WINDOW) REDUCE THE PASS- AND STOP-BAND RIPPLE.

WE CAN HOWEVER, BY INCREASING M, OBTAIN A BETTER APPROXIMATION TO THE TRANSITION FROM PASS- TO STOP-BAND ~ A NARROWER TRANSITION BAND.

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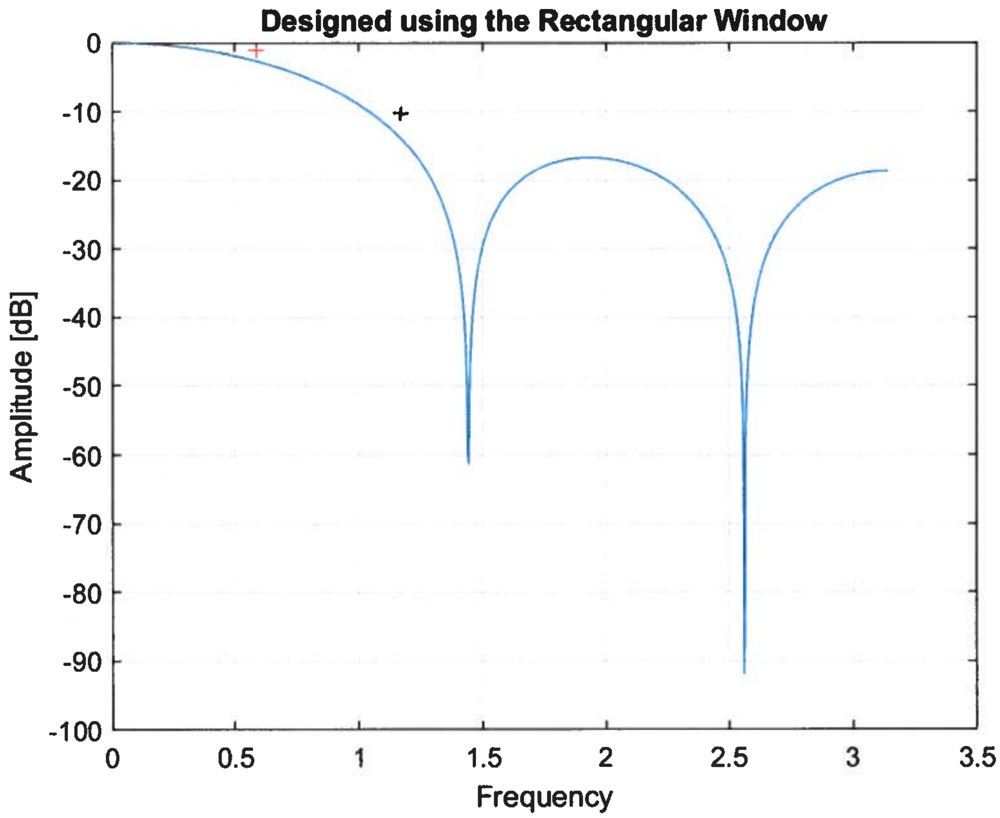
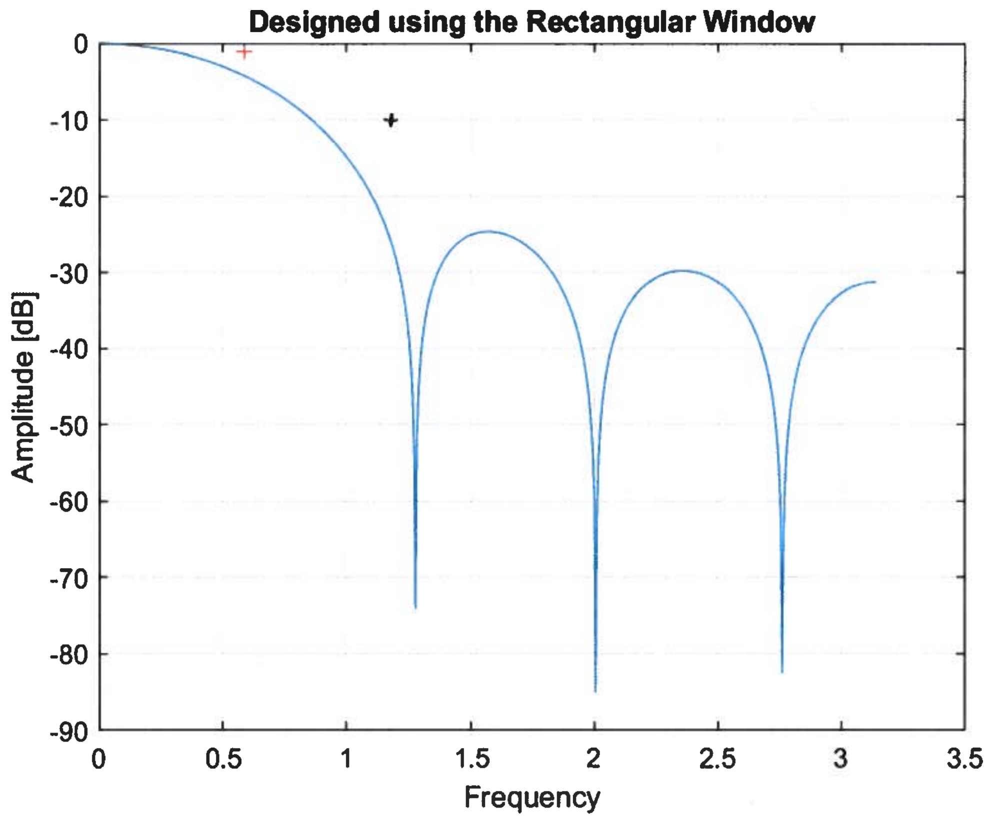

Designed using the Rectangular Window
Amplitude [dB]
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

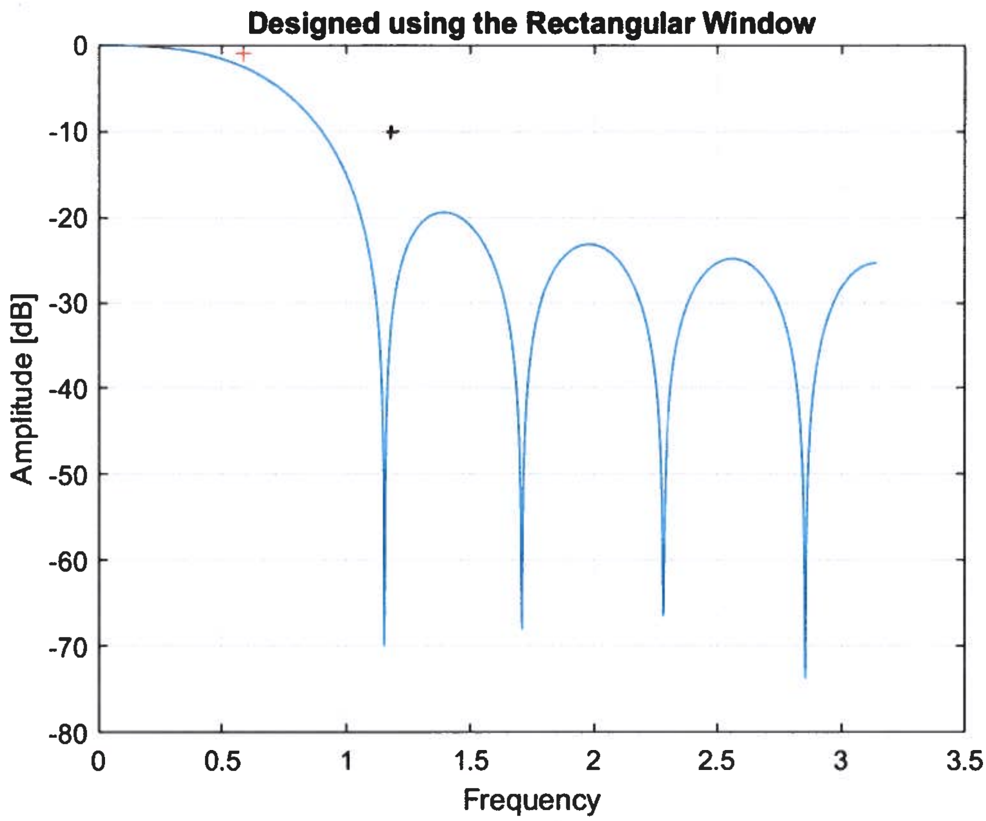
$M=4$

10

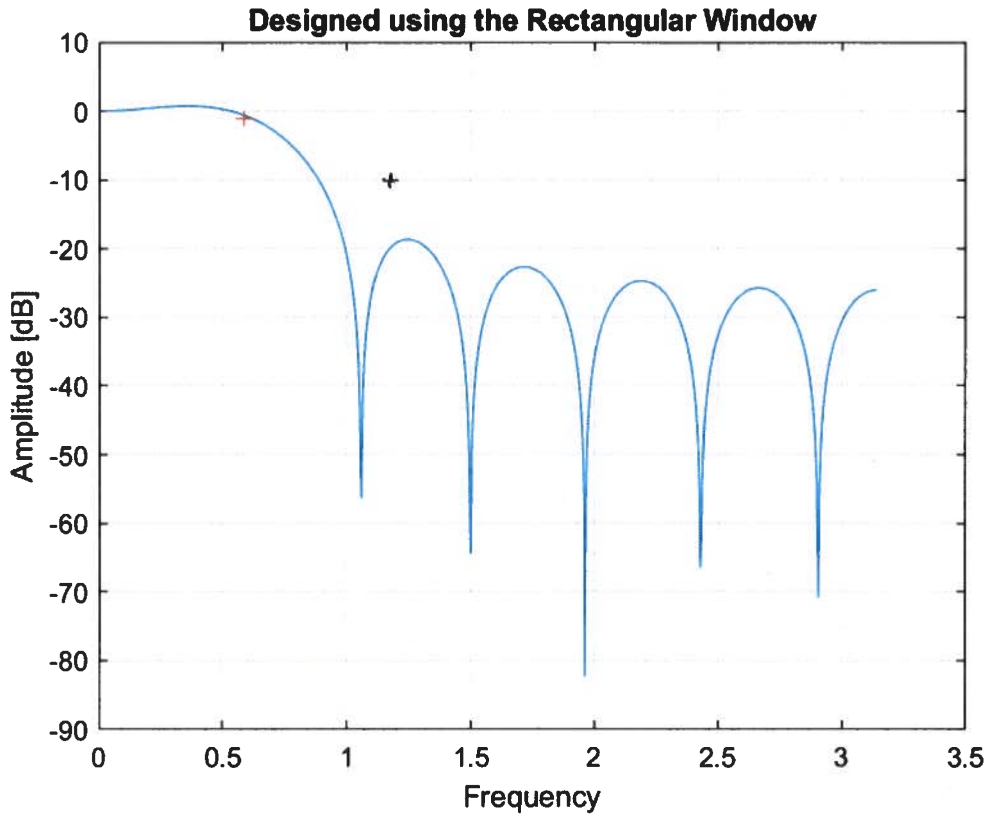


Designed using the Rectangular Window
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
Amplitude [dB]
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

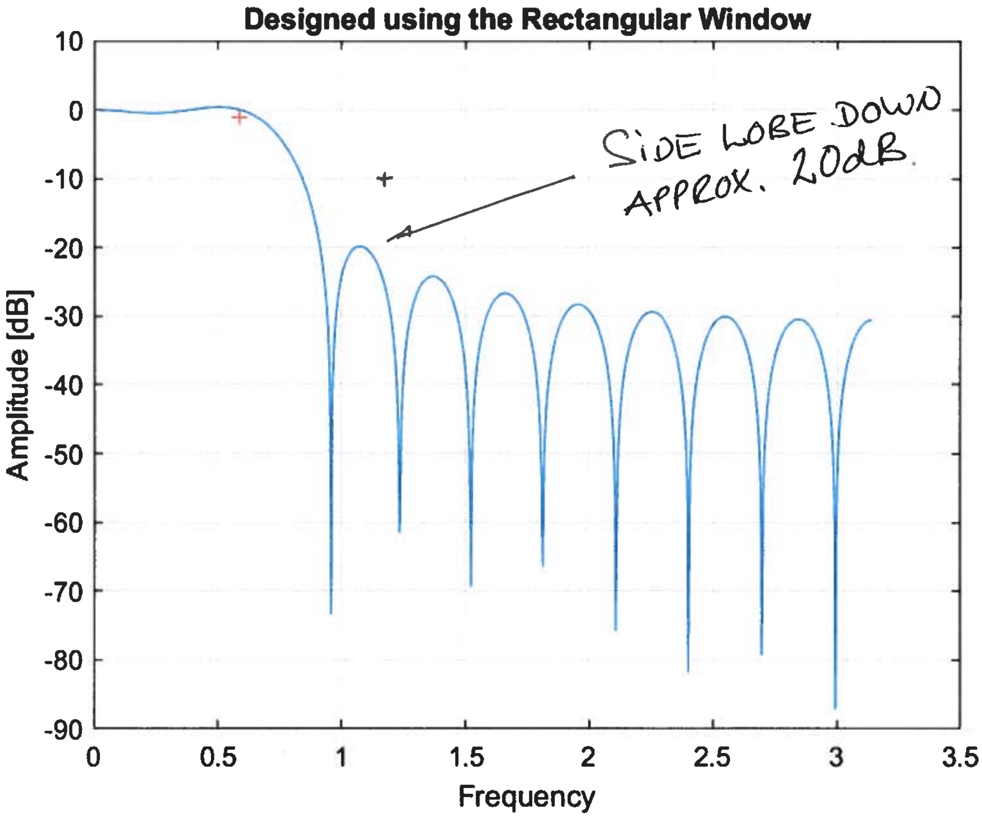
$M=8$


Designed using the Rectangular Window
0
-10
-20
-30
-40
-50
-60
-70
-80
Amplitude [dB]
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

$M=10$


Designed using the Rectangular Window
Amplitude [dB]
Frequency
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
0
0.5
1
1.5
2
2.5
3
3.5

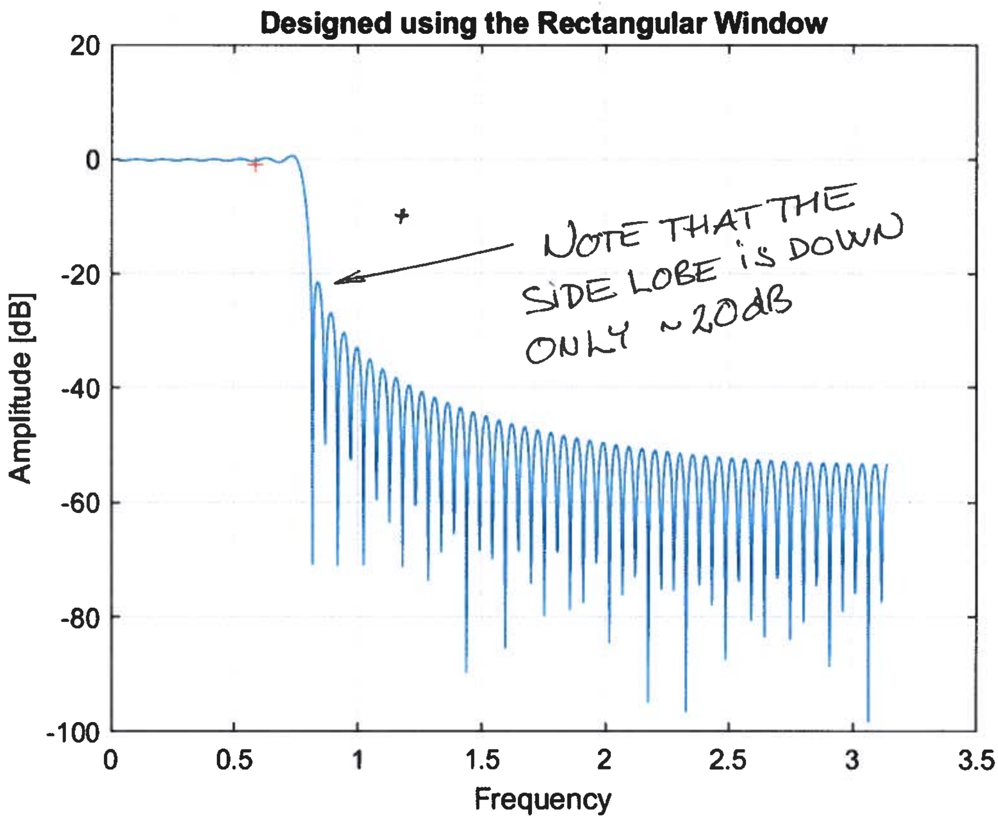
M=12



Designed using the Rectangular Window
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
Amplitude [dB]
0
0.5
1
1.5
2
2.5
3
3.5
Frequency
SIDE LOBE DOWN APPROX. 20dB.

$M=20$

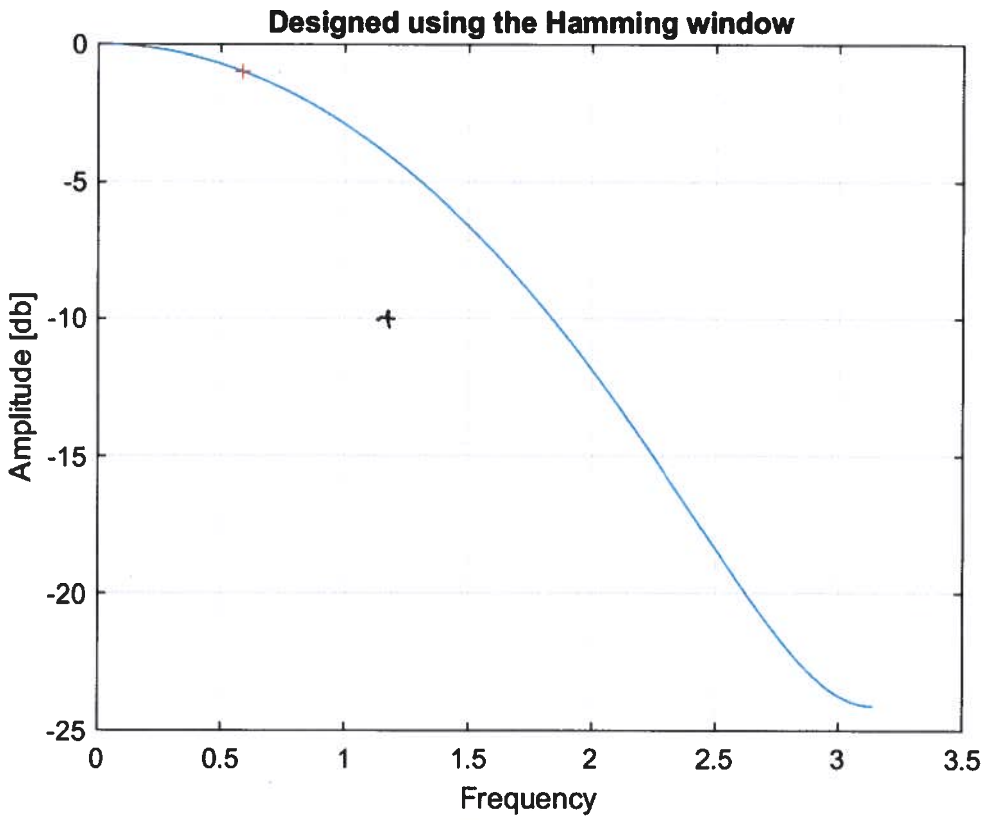
14



Designed using the Rectangular Window
20
0
-20
-40
-60
-80
-100
0
0.5
1
1.5
2
2.5
3
3.5
Frequency
Amplitude [dB]
NOTE THAT THE SIDE LOBE IS DOWN ONLY ~20dB

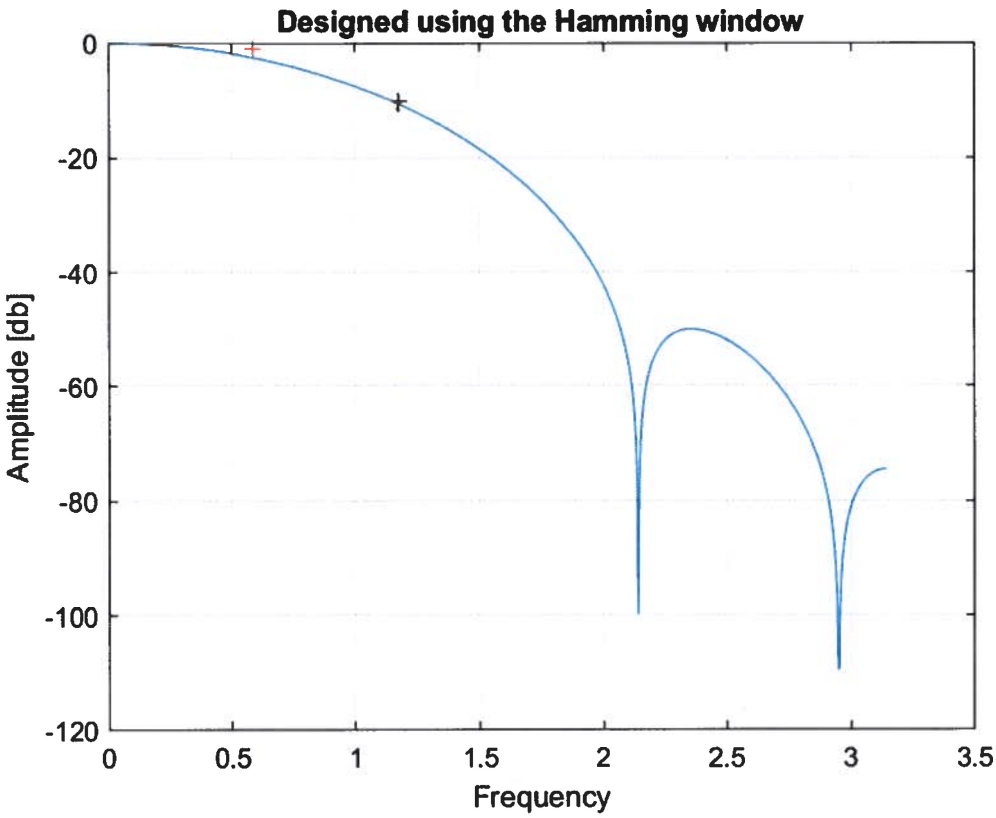
M=120

LET'S TRY WITH THE HAMMING WINDOW



$N=4$

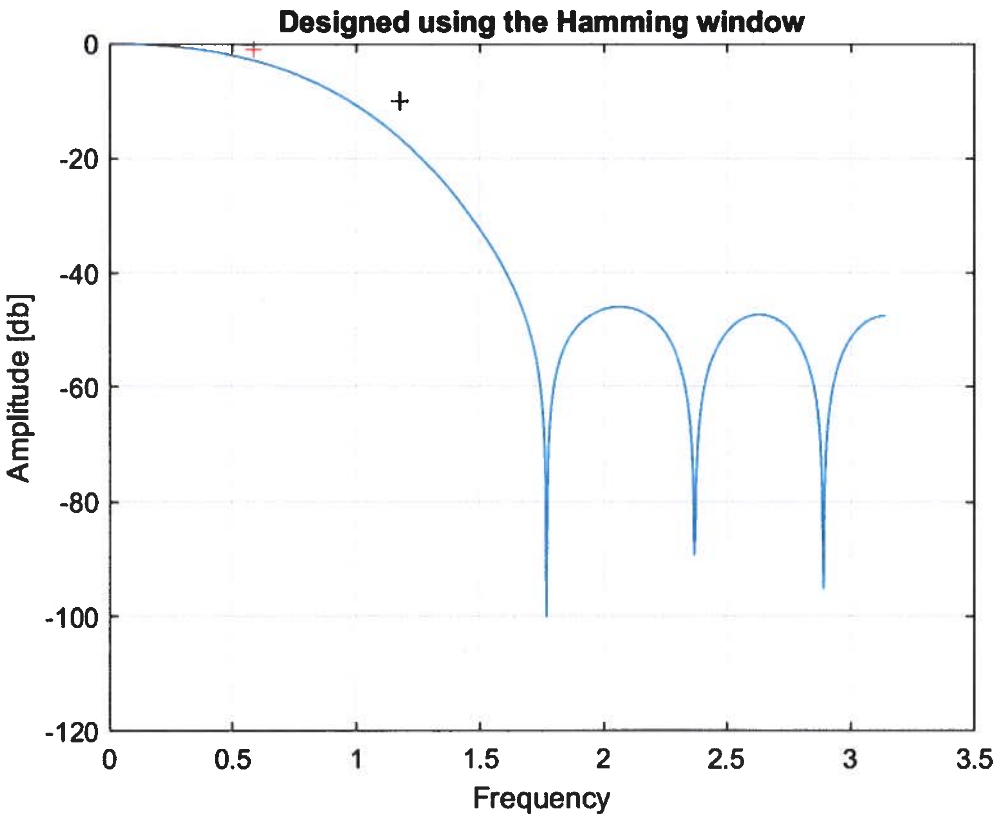
16



Designed using the Hamming window
0
-20
-40
-60
-80
-100
-120
Amplitude [db]
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

M=8

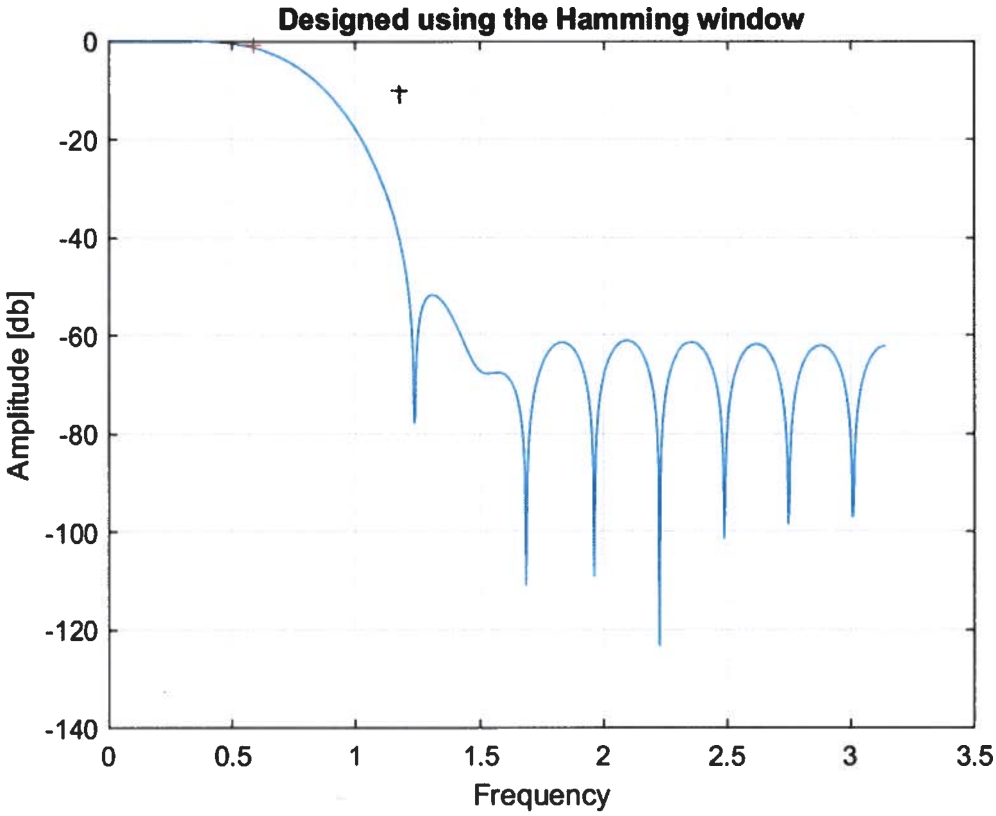
17



Designed using the Hamming window
Amplitude [db]
0
-20
-40
-60
-80
-100
-120
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

$M = 12$

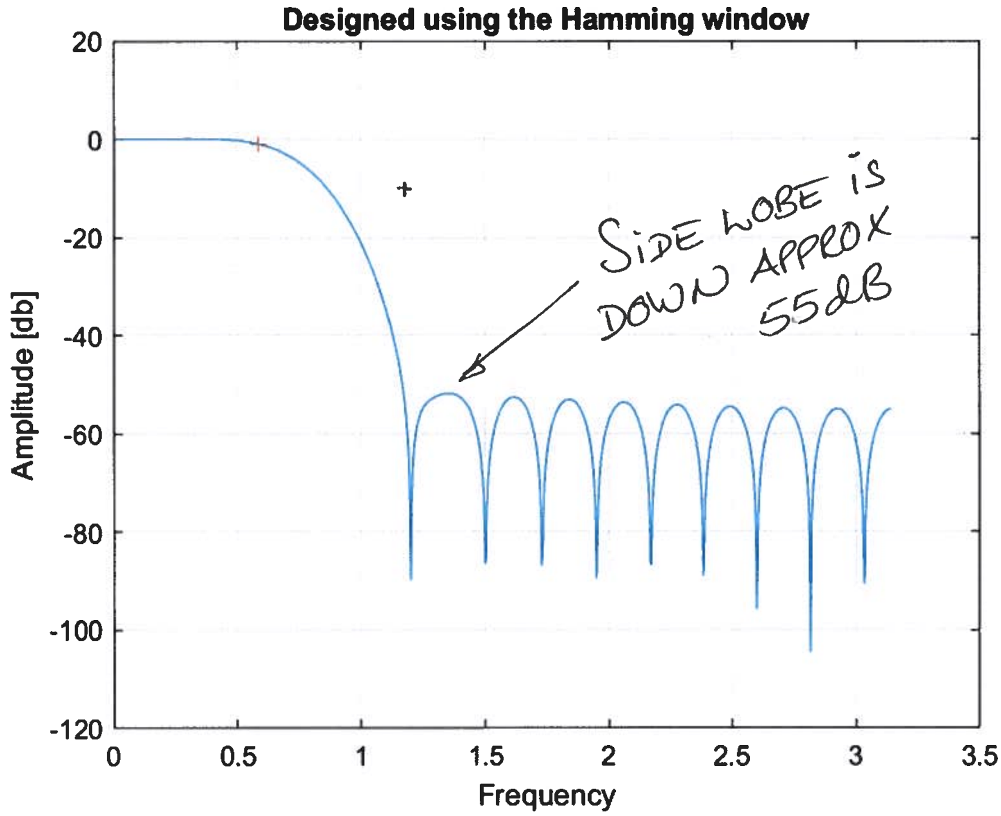
18



Designed using the Hamming window
Amplitude [db]
Frequency
0
-20
-40
-60
-80
-100
-120
-140
0.5
1
1.5
2
2.5
3
3.5

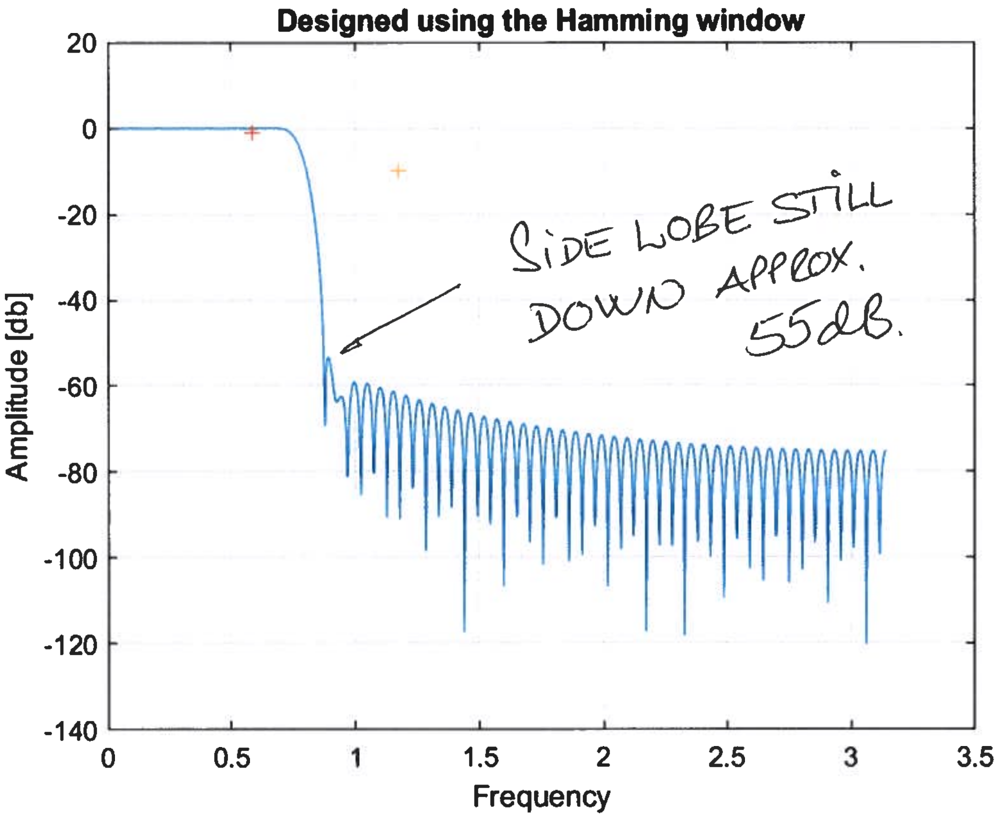
M=24

19


Designed using the Hamming window
Amplitude [db]
Frequency
SIDE LOBE IS DOWN APPROX 55dB

M=28

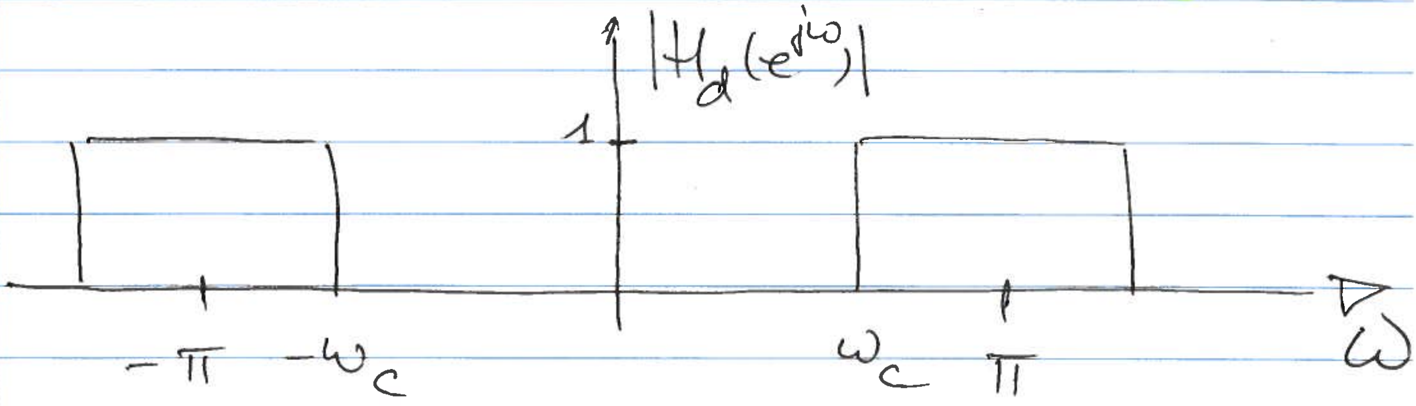
20



Designed using the Hamming window
20
0
-20
-40
-60
-80
-100
-120
-140
0
0.5
1
1.5
2
2.5
3
3.5
Frequency
Amplitude [db]
SIDE LOBE STILL DOWN APPROX. 55dB.

$M = 120$

Other Filter Types – High-Pass



$$H_d(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ e^{-j\omega M/2} & \omega_c \le |\omega| \le \pi \end{cases}$$

The desired frequency response — which we can now rewrite;

$$H_{d_{HP}}(e^{j\omega}) = e^{-j\omega M/2} - H_{d_{LP}}(e^{j\omega})$$

Using this expression, we can now find the desired impulse response;

$$h_{d_{HP}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{d_{HP}} \cdot e^{j\omega n}\, d\omega$$

$\Downarrow$

$$h_{d_{HP}}[n] = \frac{\sin(\pi(n - M/2))}{\pi(n - M/2)} - \frac{\sin(\omega_c(n - M/2))}{\pi(n - M/2)}$$

$-\infty < n < \infty$

Equ. 80, p. 571.

$$h_{d_{HP}}[n] = \frac{\sin(\pi(n-M/2))}{\pi(n-M/2)} - \frac{\sin(\omega_c(n-M/2))}{\pi(n-M/2)}$$

$$= \begin{cases} 1 & \text{for } n = M/2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{IDENTICAL TO LP}$$

$\Downarrow$

THE IMPULSE RESPONSE (AND THUS THE FILTER COEFFICIENT) FOR THE HP-FILTER CAN BE EXPRESSED IN TERMS OF THE LP-FILTER.

- $b_{i_{HP}} = -b_{i_{LP}} \qquad i = 0, \dots, M/2-1, M/2+1, \dots, M$
- $b_{M/2_{HP}} = 1 - b_{M/2_{LP}}$

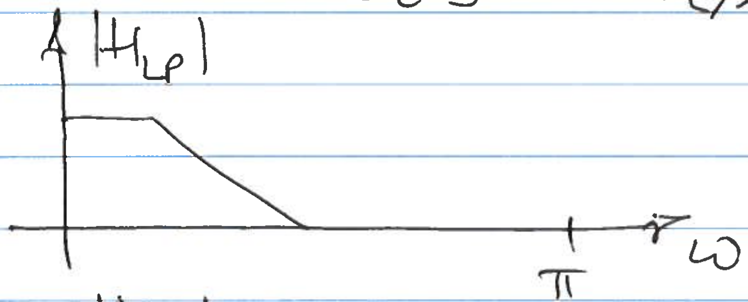
THE IMPULSE RESPONSE IS STILL SYMMETRIC AND THUS STILL LINEAR PHASE.

SO, DESIGNING A HIGH-PASS FILTER CAN BE DONE BY DESIGNING THE „CORRESPONDING" LOW-PASS FILTER, AND THEN SIMPLY JUST NEGATING THE COEFFICIENTS (AND SPECIAL $i = M/2$)

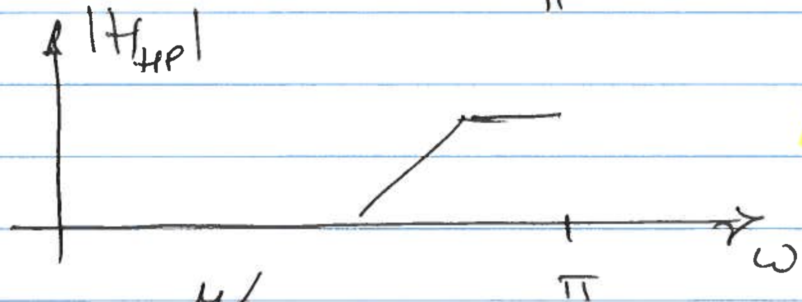
... AND THE PASS-BAND GAIN FOR TYPE-I:

$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=0}^{M/2} a[k]\cdot\cos(\omega k)\right\}$$

$$a[0] = h[M/2]$$
$$a[k] = 2\cdot h[M/2-k] \quad \text{for } k=1..M/2$$



$$G\cdot|H_{LP}| = 1\Big|_{\omega=0}$$



$$G\cdot|H_{HP}| = 1\Big|_{\omega=\pi}$$

$$G\cdot\sum_{k=0}^{M/2} a[k]\cdot\cos(\pi\cdot k) = 1$$

$\Downarrow$

$$G = \frac{1}{a[0]-a[1]+a[2]-a[3]+\dots-a[M/2]}$$

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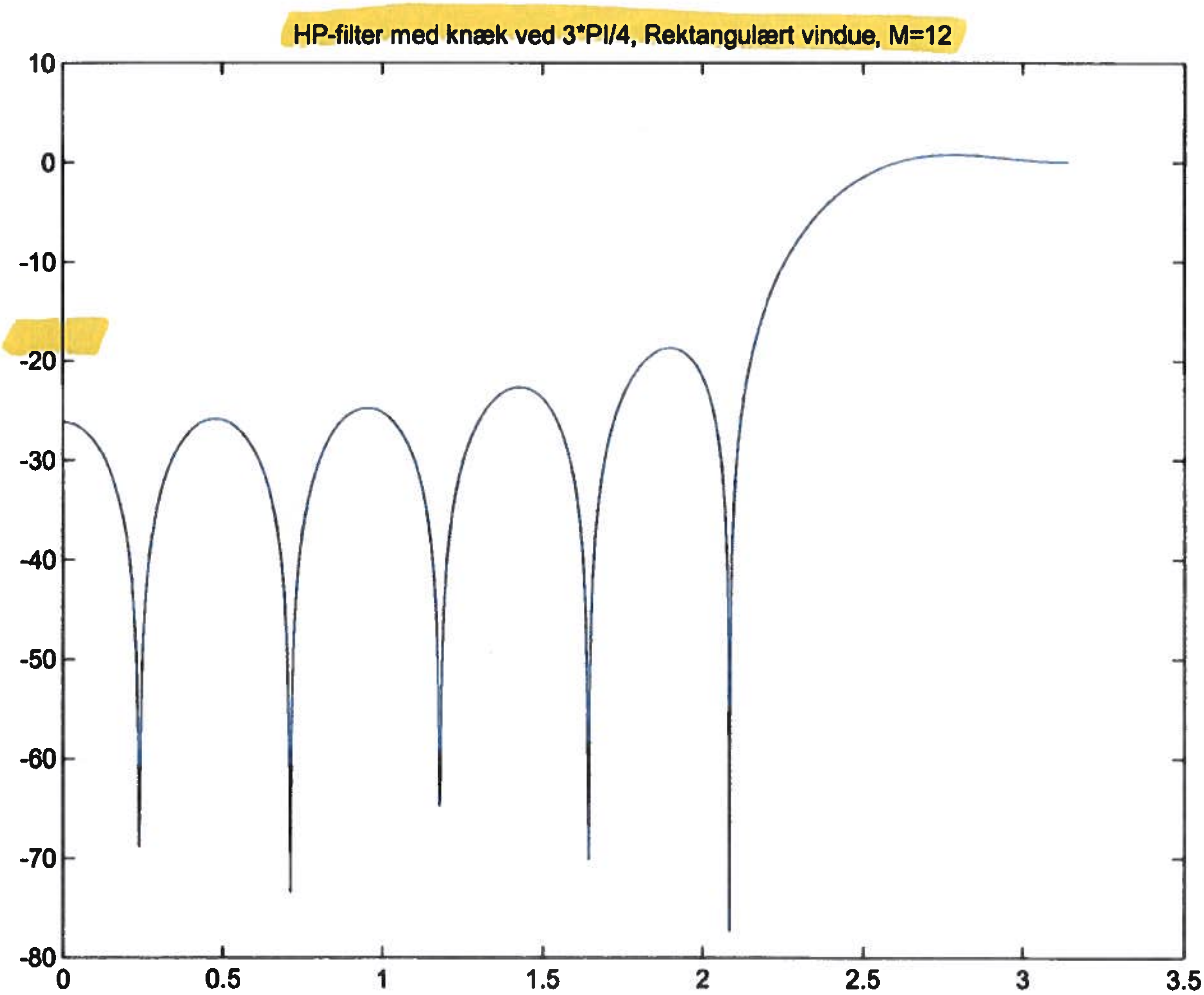
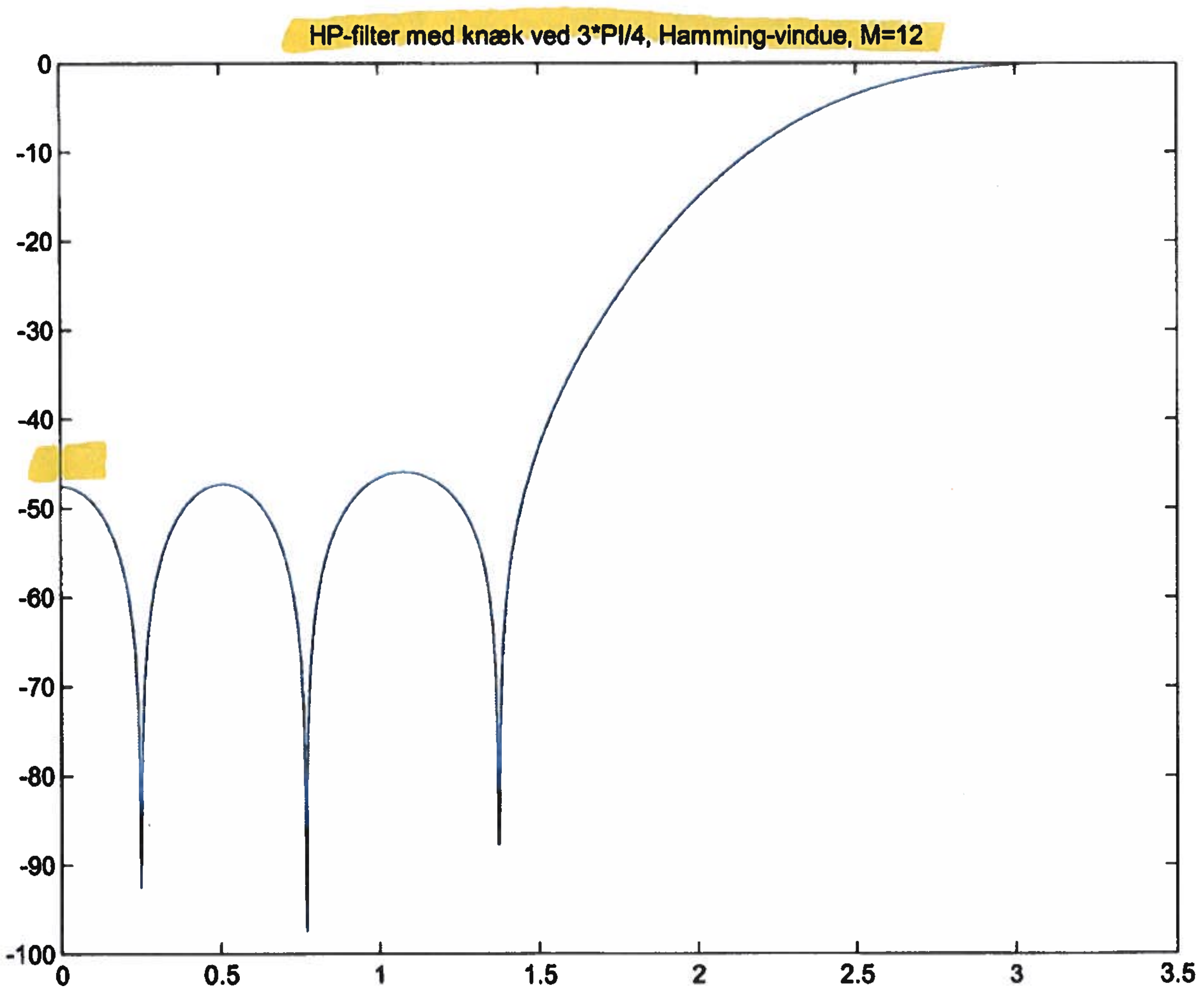
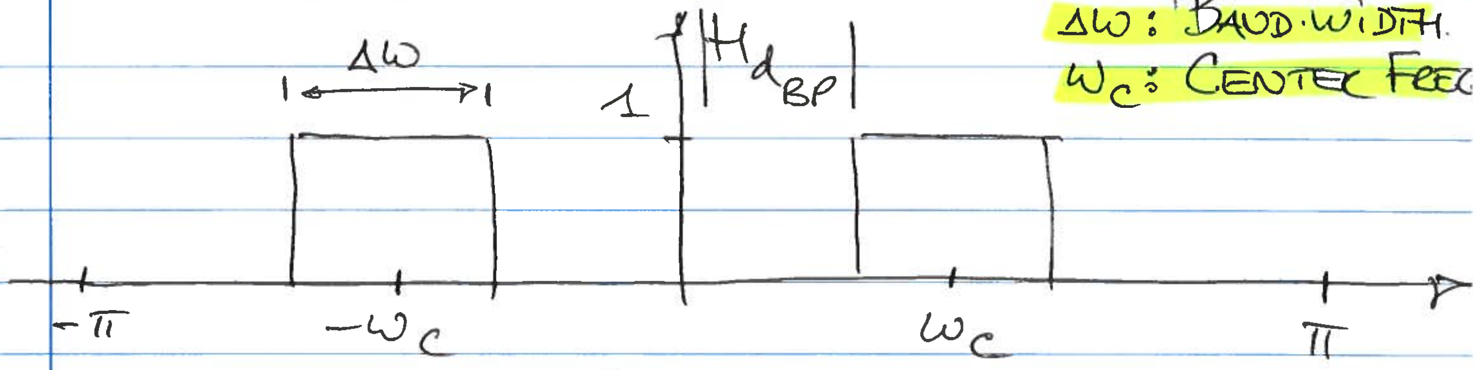

HP-filter med knæk ved 3*PI/4, Rektangulært vindue, M=12
10
0
-10
-20
-30
-40
-50
-60
-70
-80
0
0.5
1
1.5
2
2.5
3
3.5

25


HP-filter med knæk ved 3*PI/4, Hamming-vindue, M=12
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
0
0.5
1
1.5
2
2.5
3
3.5

BAND-PASS FILTERS

$\Delta\omega$: BANDWIDTH
ω_c : CENTER FREQ



$$H_{d_{BP}}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & \begin{cases} -\omega_c - \frac{\Delta\omega}{2} \le \omega \le -\omega_c + \frac{\Delta\omega}{2} \\ \omega_c - \frac{\Delta\omega}{2} \le \omega \le \omega_c + \frac{\Delta\omega}{2} \end{cases} \\ 0 & \text{OTHERWISE} \end{cases}$$

$\Downarrow$

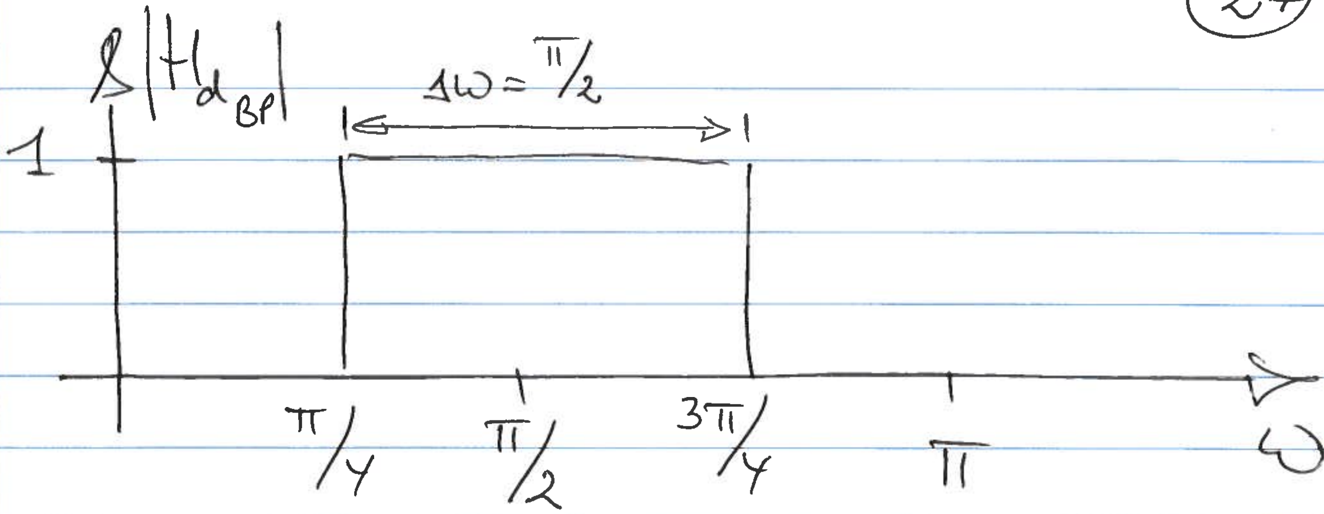
$$h_{d_{BP}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{d_{BP}}(e^{j\omega}) \cdot e^{j\omega n}\, d\omega$$

$\Downarrow$

$$h_{d_{BP}}[n] = \frac{1}{2\pi}\left\{\int_{-\omega_c-\frac{\Delta\omega}{2}}^{-\omega_c+\frac{\Delta\omega}{2}} e^{-j\omega M/2} \cdot e^{j\omega n}\, d\omega + \int_{\omega_c-\frac{\Delta\omega}{2}}^{\omega_c+\frac{\Delta\omega}{2}} e^{-j\omega M/2} \cdot e^{j\omega n}\, d\omega\right\}$$

$\Downarrow$

$$h_{d_{BP}}[n] = \frac{\sin\left((\omega_c + \frac{\Delta\omega}{2})(n - M/2)\right) - \sin\left((\omega_c - \frac{\Delta\omega}{2})(n - M/2)\right)}{\pi(n - M/2)}$$



CENTER FREQUENCY : $\omega_c = \pi/2$

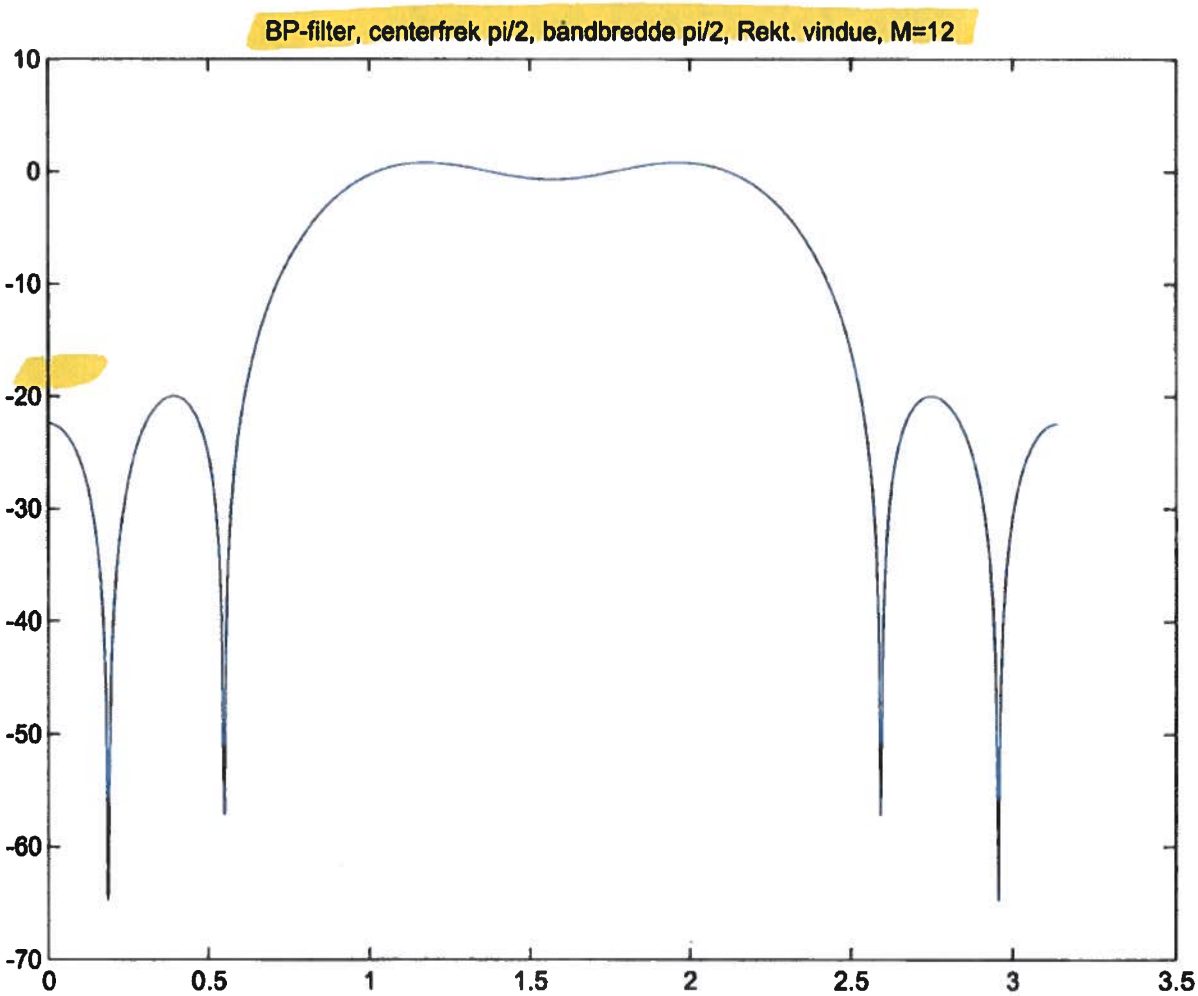
BANDWIDTH : $\Delta\omega = \pi/2$

$$h_{d_{BP}}[n] = \frac{\sin\left(\frac{3\pi}{4}(n - M/2)\right) - \sin\left(\frac{\pi}{4}(n - M/2)\right)}{\pi(n - M/2)}$$

$$h[n] = h_{d_{BP}}[n] \cdot w[n]$$

AGAIN, LET'S EXPERIMENT WITH M AND THE WINDOW FUNCTION.

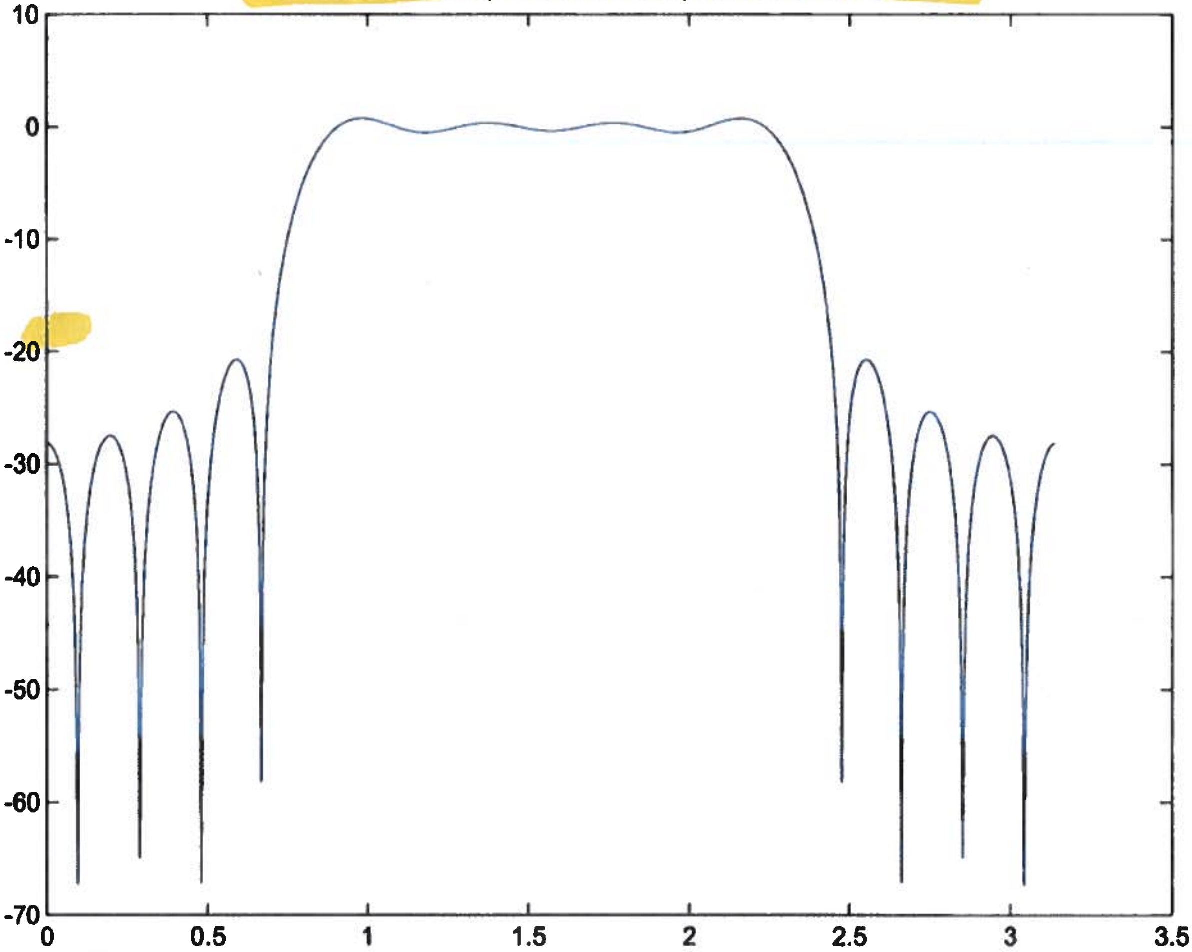
(28)



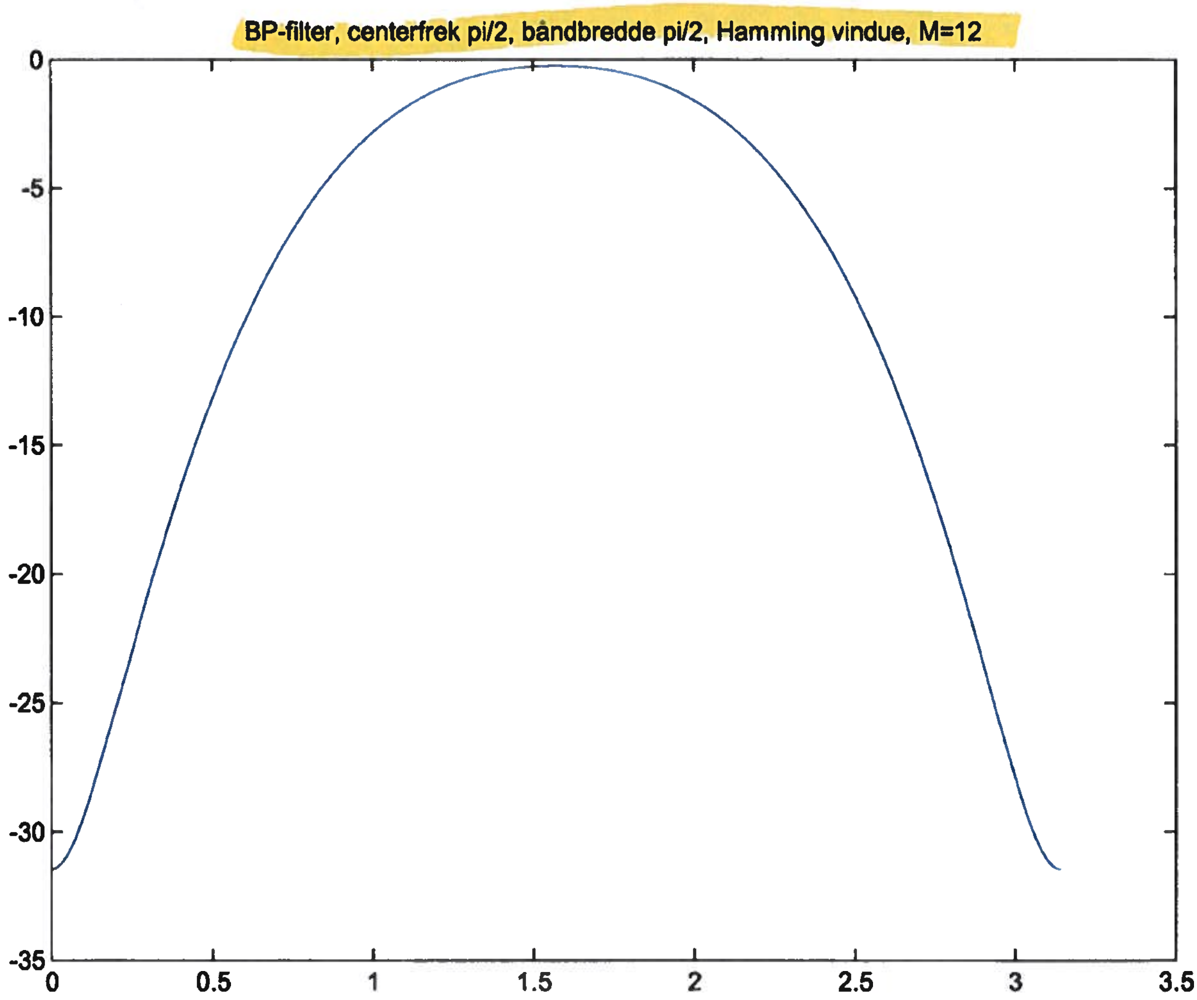
BP-filter, centerfrek pi/2, båndbredde pi/2, Rekt. vindue, M=12
10
0
-10
-20
-30
-40
-50
-60
-70
0
0.5
1
1.5
2
2.5
3
3.5

(29)

BP-filter, centerfrek pi/2, båndbredde pi/2, Rekt. vindue, M=28

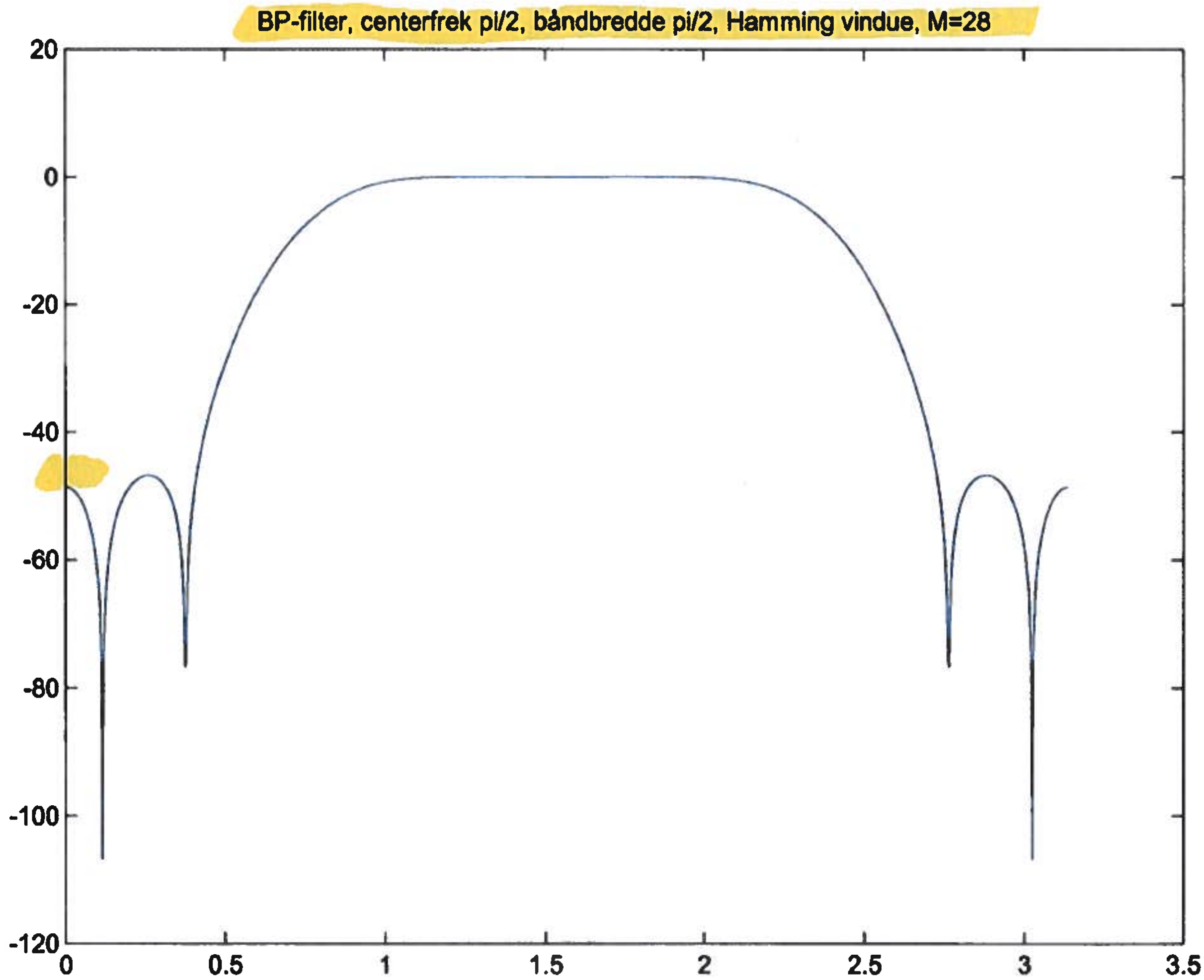


30



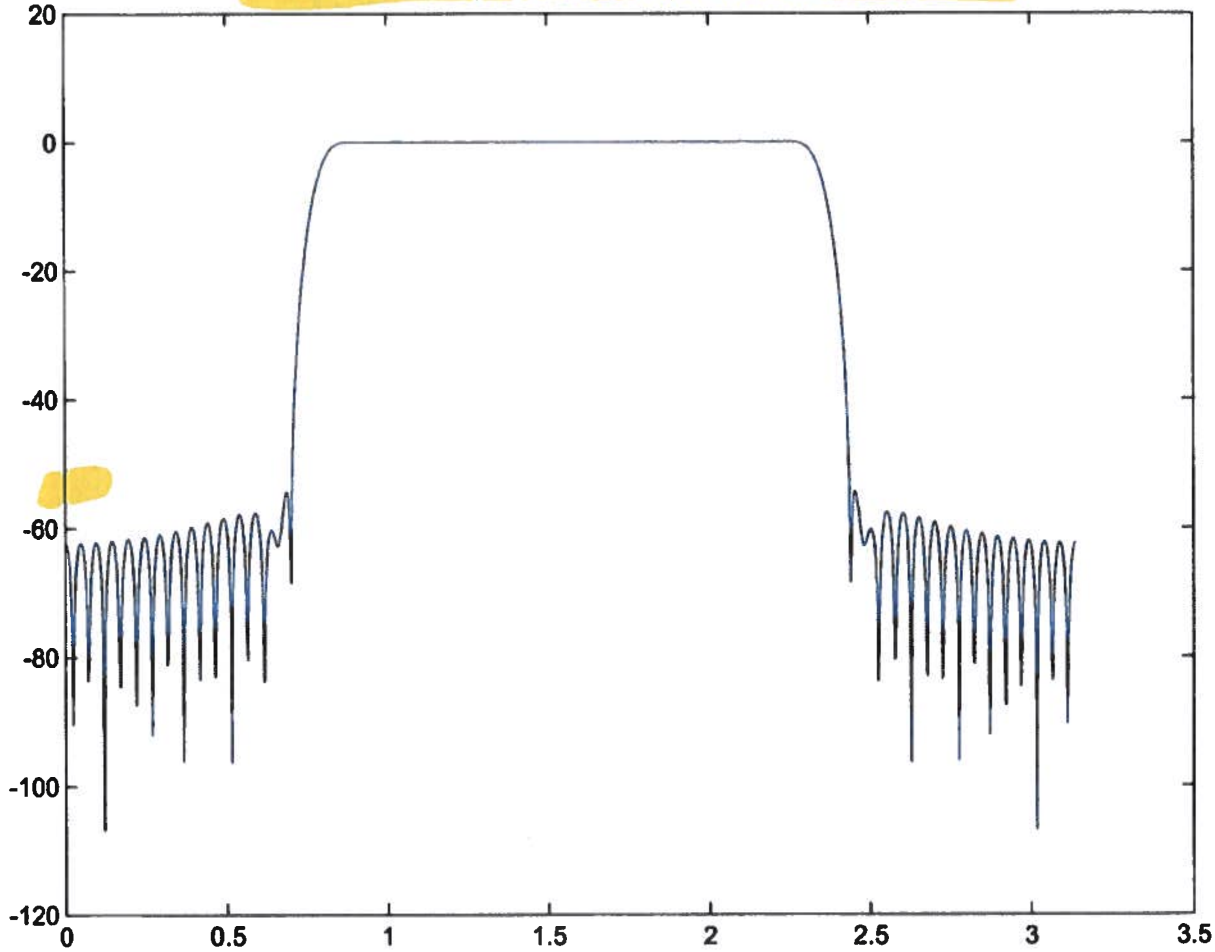
BP-filter, centerfrek pi/2, båndbredde pi/2, Hamming vindue, M=12
0
-5
-10
-15
-20
-25
-30
-35
0
0.5
1
1.5
2
2.5
3
3.5

(31)

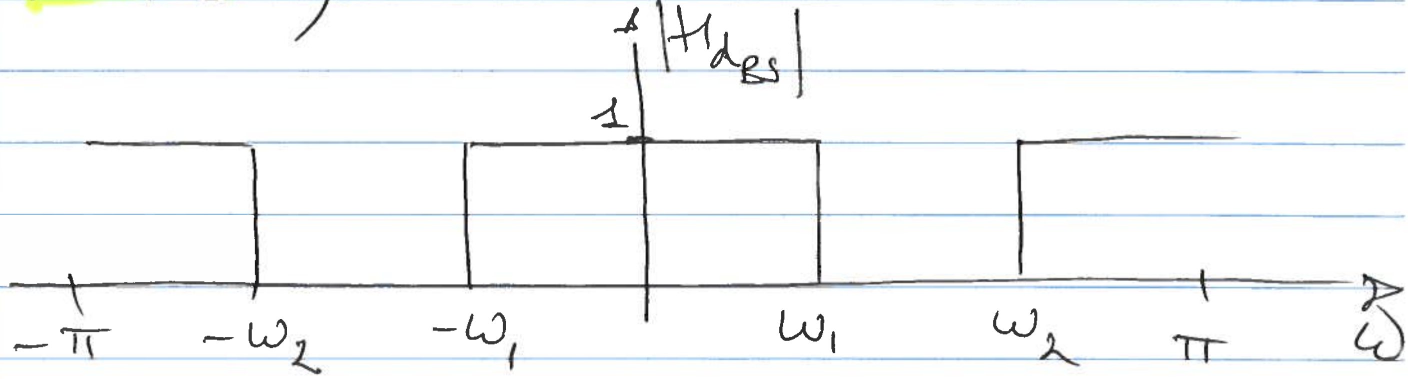

BP-filter, centerfrek pi/2, båndbredde pi/2, Hamming vindue, M=28
20
0
-20
-40
-60
-80
-100
-120
0
0.5
1
1.5
2
2.5
3
3.5

32

BP-filter, centerfrek pi/2, båndbredde pi/2, Hamming vindue, M=128



... AND, OF COURSE, SIMILARLY FOR BAND STOP FILTERS;



$$H_{d_{BS}}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & \begin{cases} -\pi \le \omega \le -\omega_2 \\ -\omega_1 \le \omega \le \omega_1 \\ \omega_2 \le \omega \le \pi \end{cases} \\ 0 & \text{OTHERWISE} \end{cases}$$

$\Downarrow$

$$h_{d_{BS}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{d_{BS}}(e^{j\omega}) \cdot e^{j\omega n}\, d\omega$$

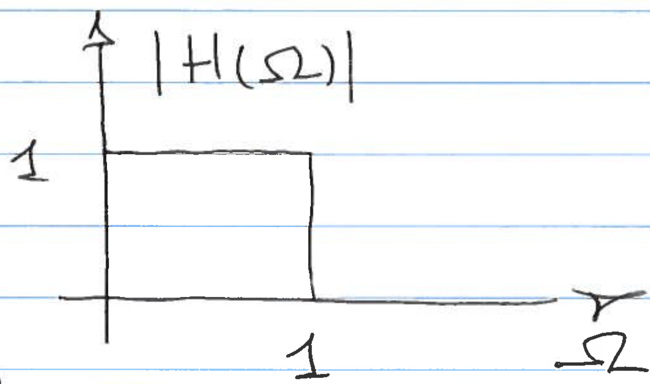
etc ...

Now, a few words on design of HP IIR filter;

SPECs

Analog Prototype LP filter ↓ Bilinear ✓ / impulse invariance ✓ ↓ Frequency Transformation ↓ Digital HP filter

Analog Prototype HP filter ↓ Bilinear ✓ / impulse invariance % ↓ Digital HP filter

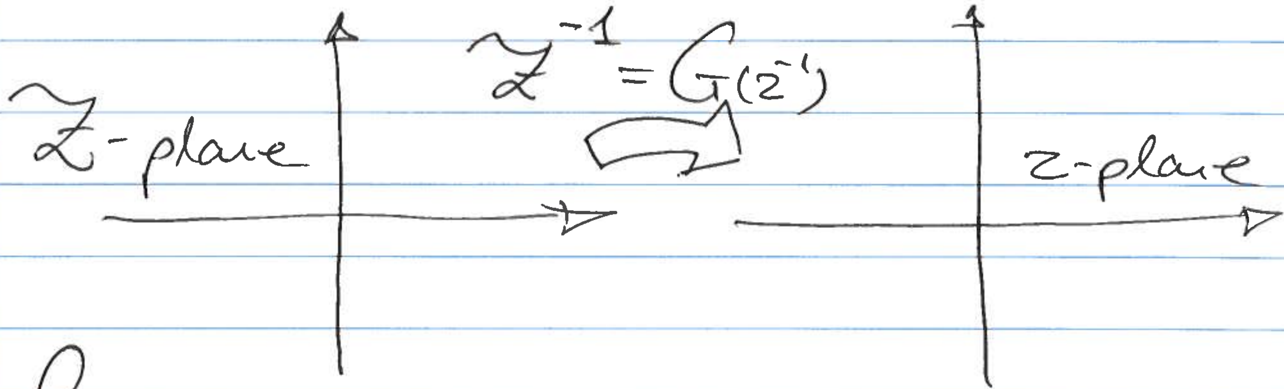


Normalized LP filter

LP → LP
LP → HP
LP → BP
LP → BS

FREQUENCY TRANSFORMATION

DIGITAL PROTOTYPE LP-FILTER $\longrightarrow H_{LP}(Z)$

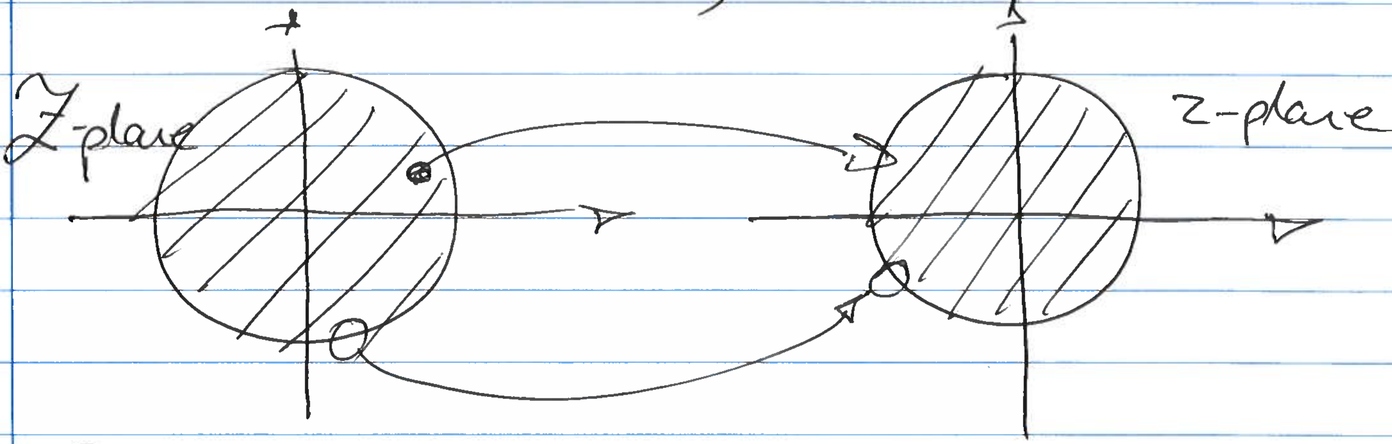


SO, THE IDEA IS TO APPLY AN ALGEBRAIC TRANSFORMATION, WHICH MAPS FROM THE Z-plane TO THE z-plane.

$$\boxed{Z^{-1} = \pm \prod_{k=1}^{N} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}}$$

$$H(z) = H_{LP}(Z)\Big|_{Z^{-1} = G(z^{-1})}$$

$$\mathcal{Z}^{-1} = G(z^{-1})$$



So, STABLE AND CAUSAL FILTERS ARE MAPPED TO STABLE AND CAUSAL FILTERS.

HOWEVER: THE MAPPING IS NON-LINEAR

Ex. LP ⟶ LP

$$\mathcal{Z}^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$ EQU. 44 p.552

WHERE $$\alpha = \frac{\sin\{(\theta_c - \omega_c)/2\}}{\sin\{(\theta_c + \omega_c)/2\}}$$ EQU 47 p.554.

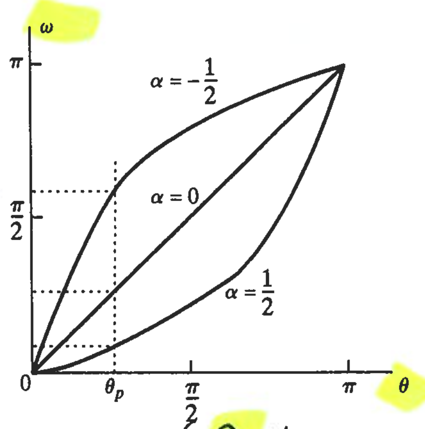
CRITICAL FREQ. IN THE PROTOTYPE FILTER ($\mathcal{Z}$)

CRITICAL FREQUENCY IN THE RESULTING FILTER (z).

$$\omega = \arctan\left\{\frac{(1-\alpha^2)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right\}$$ Equ 46 p. 552

37

(z)



(Z)

Figure 24 Warping of the frequency scale in lowpass-to-lowpass transformation.

TABLE 1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY μ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \dfrac{z^{-1}-\alpha}{1-\alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\theta_p-\omega_p}{2}\right)}{\sin\left(\frac{\theta_p+\omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\frac{\theta_p+\omega_p}{2}\right)}{\cos\left(\frac{\theta_p-\omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\dfrac{z^{-2}-\frac{2\alpha k}{k+1}z^{-1}+\frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2}-\frac{2\alpha k}{k+1}z^{-1}+1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2}+\omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)}$ $k = \cot\left(\dfrac{\omega_{p2}-\omega_{p1}}{2}\right)\tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \dfrac{z^{-2}-\frac{2\alpha}{1+k}z^{-1}+\frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2}-\frac{2\alpha}{1+k}z^{-1}+1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2}+\omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)}$ $k = \tan\left(\dfrac{\omega_{p2}-\omega_{p1}}{2}\right)\tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

ANALYSIS OF DISCRETE-TIME SYSTEMS

(ARE THE SPEC^S FULFILLED ??)

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

FIRST WE NEED TO FIND THE ROOTS IN THE $B(z)$- AND THE $A(z)$ POLYNOMIAL $\Rightarrow$ THE ZERO^S AND THE POLE^S OF $H(z)$

NOW, THREE POSSIBLE SCENARIOS ;

1. $M=N$, EXTEND $B(z)$ AND $A(z)$ WITH z^M

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{M-k}}{\sum_{k=0}^{M} a_k z^{M-k}}$$

THE PURPOSE IS TO OBTAIN POSITIVE EXPONENTS ONLY.

(2) $M > N$, EXTEND $B(z)$ AND $A(z)$ WITH z^M

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{M-k}}{z^{M-N} \cdot \sum_{k=0}^{N} a_k z^{N-k}}$$

WE GET $M-N$ EXTRA ROOTS IN $A(z)$, I.E.

$M-N$ EXTRA POLES IN $z=0$ IN $H(z)$

(3) $M < N$, EXTEND $B(z)$ AND $A(z)$ WITH z^N

$$H(z) = \frac{z^{N-M} \sum_{k=0}^{M} b_k z^{M-k}}{\sum_{k=0}^{N} a_k z^{N-k}}$$

WE GET $N-M$ EXTRA ROOTS IN $B(z)$, I.E.,

$N-M$ EXTRA ZEROS IN $z=0$ IN $H(z)$

It is now possible to FACTORIZED $H(z)$;

$$H(z) = \frac{b_0}{a_0} \cdot \frac{\prod_{k=1}^{L} (z - c_k)}{\prod_{k=1}^{L} (z - d_k)} \qquad L = \max\{N, M\}$$

ZEROS

POLES

Using this expression, we can now write the FREQUENCY RESPONSE;

$$H(e^{j\omega}) = \frac{b_0}{a_0} \cdot \frac{\prod_{k=1}^{L} (e^{j\omega} - c_k)}{\prod_{k=1}^{L} (e^{j\omega} - d_k)}$$

Which is now used to derive the AMPLITUDE- AND THE PHASE-RESPONSE;

AMPLITUDE

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \cdot \frac{\prod_{k=1}^{L} |e^{j\omega} - c_k|}{\prod_{k=1}^{L} |e^{j\omega} - d_k|}$$

LENGTH OF VECTORS in the Z-plane

PHASE

$$\arg\{H(e^{j\omega})\} = \arg\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{L} \arg\{e^{j\omega} - c_k\} - \sum_{k=1}^{L} \arg\{e^{j\omega} - d_k\}$$

ANGLES TO VECTORS IN THE Z-plane

WE CAN NOW GIVE THE AMP AND PHASE A GEOMETRIC INTERPRETATION

LET'S HAVE A LOOK AT AN EXAMPLE ;

GIVEN $H(z) = \dfrac{1+z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$ $\left.\begin{matrix} M=1 \\ N=2 \end{matrix}\right\} M<N$

SO, EXTEND $B(z)$ AND $A(z)$ WITH z^2

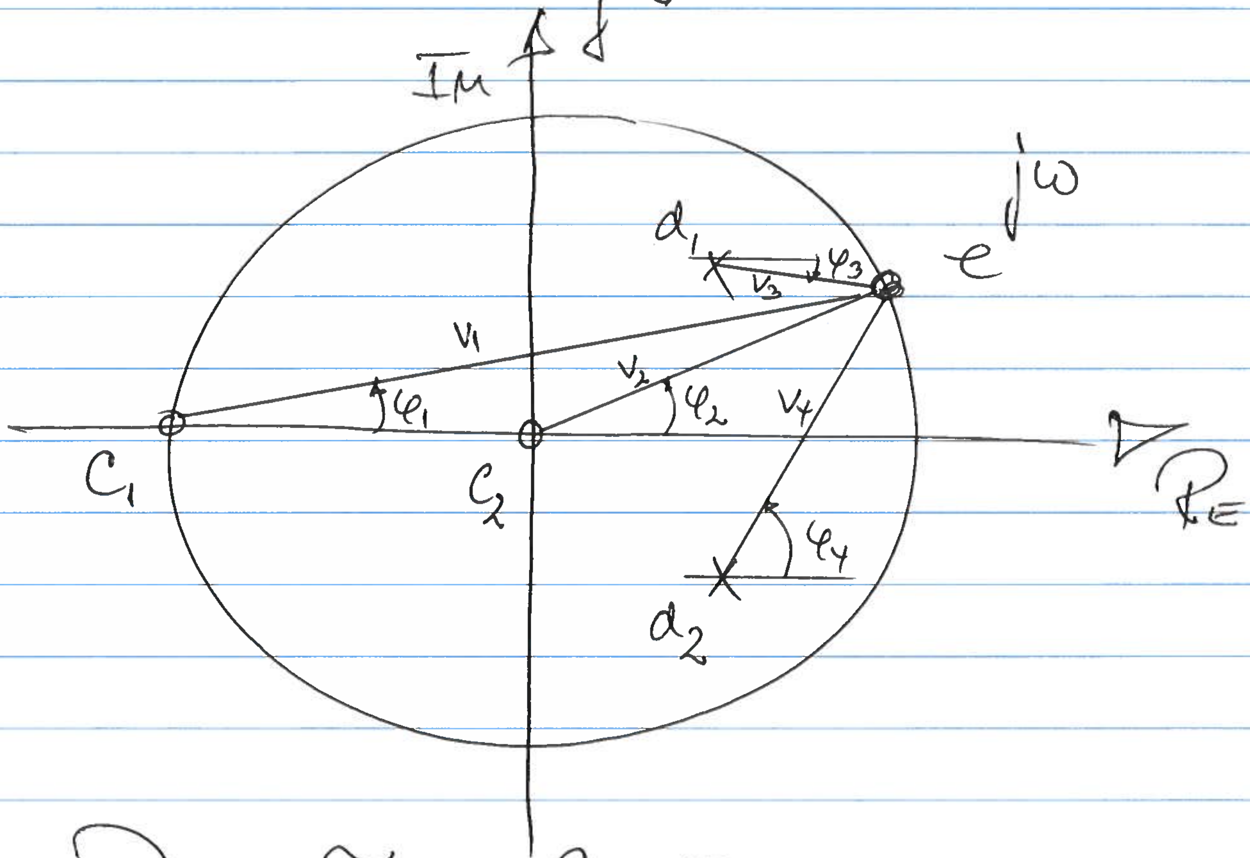
$$H(z) = \frac{z^2+z}{z^2-z+\frac{1}{2}}$$

$\Downarrow$

$$H(z) = \frac{1}{1} \cdot \frac{(z-0)(z-(-1))}{\left(z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right)\left(z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right)}$$

ZEROS: $z=0$, $z=-1$

POLES: $z = \frac{1}{2} \pm j\frac{1}{2}$



POLE-ZERO PLOT

AMPLITUDE RESPONSE

$$\left|H(e^{j\omega})\right| = \cancel{\left|\frac{b_0}{a_0}\right|}^{\,1} \cdot \frac{|V_1|\cdot|V_2|}{|V_3|\cdot|V_4|}$$

WE NOTE THAT $|V_k|$ ARE ALL FUNCTIONS OF ω.

V_1: $$|V_1| = \sqrt{(\cos\omega + 1)^2 + \sin^2\omega}$$

EVALUATE $\left|H(e^{j\omega})\right|$ for $\omega \in [0;\pi]$

PHASE RESPONSE

$$\arg\{H(e^{j\omega})\} = \arg\{1\} + \sum_{k=1}^{2} \arg\{e^{j\omega} - c_k\} - \sum_{k=1}^{2} \arg\{e^{j\omega} - d_k\}$$

$$= \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4$$

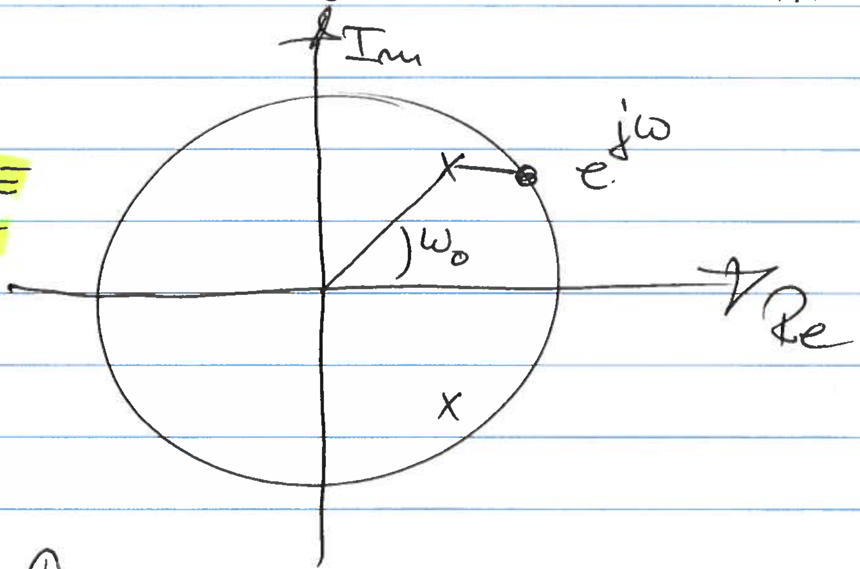
WE NOTE THAT φ_k ARE ALL FUNCTIONS OF ω.

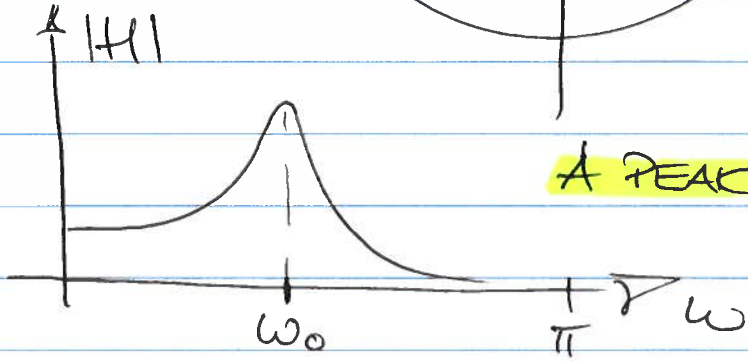
φ_1: $$\tan(\varphi_1) = \frac{\sin\omega}{\cos\omega + 1} \Rightarrow \varphi_1 = \text{ARCTAN}\left\{\frac{\sin\omega}{\cos\omega + 1}\right\}$$

EVALUATE $\arg\{H(e^{j\omega})\}$ for $\omega \in [0;\pi]$

Interpretation of Pole-Zero Locations

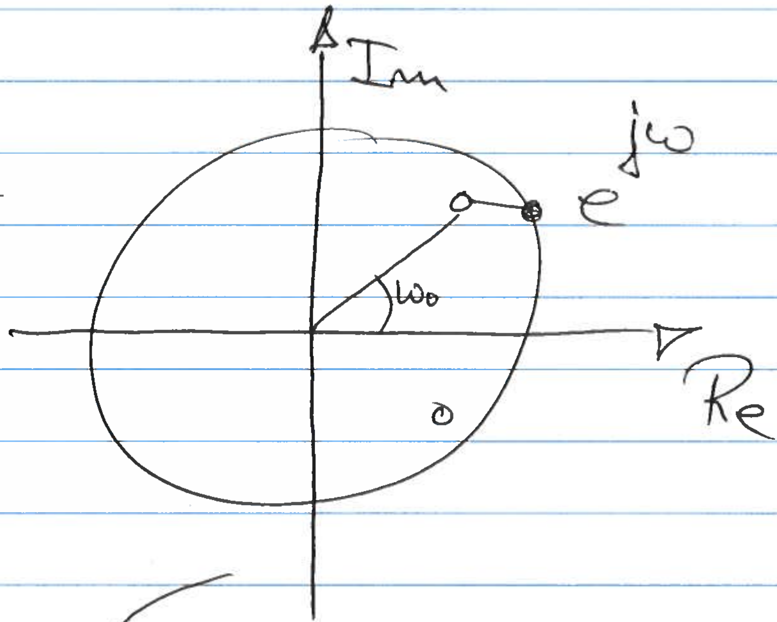
A pole close to the unit circle

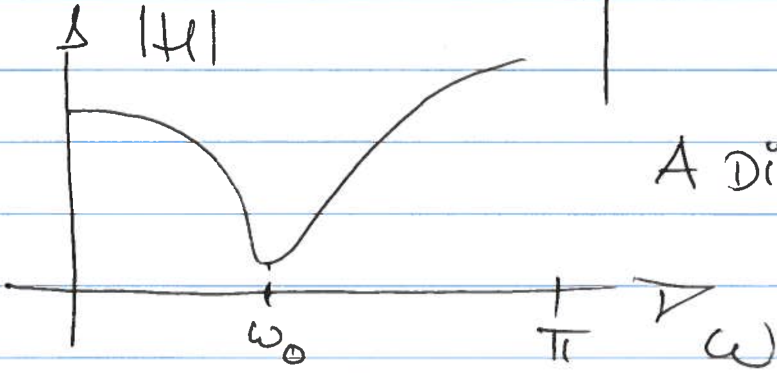




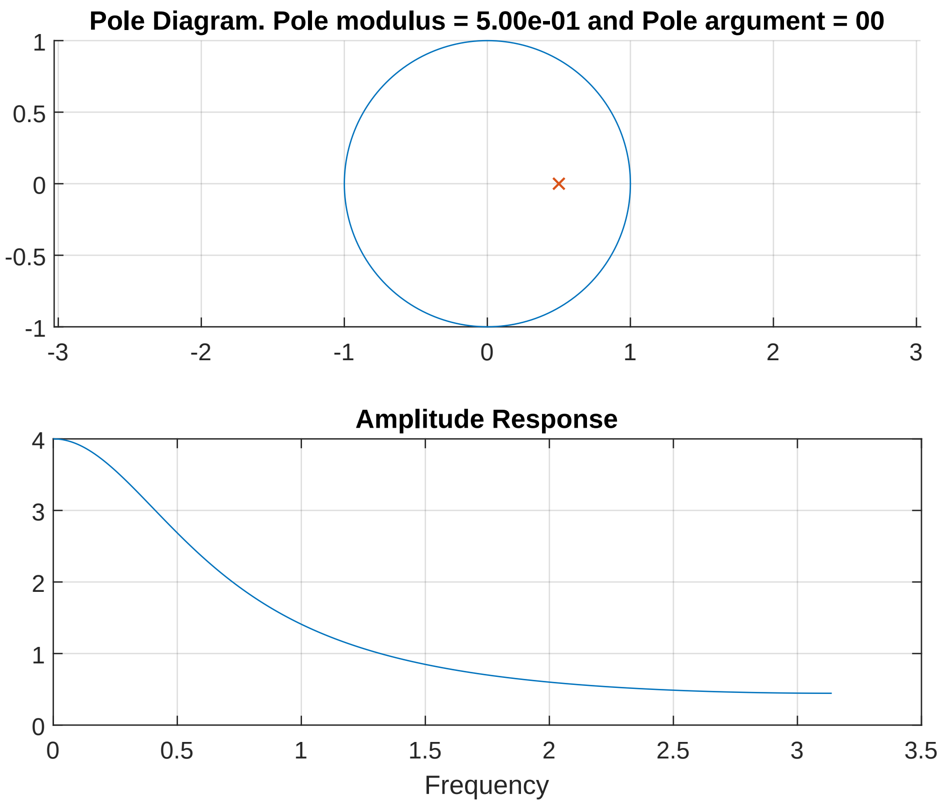
A peak in |H| at ω_0

A zero close to the unit circle

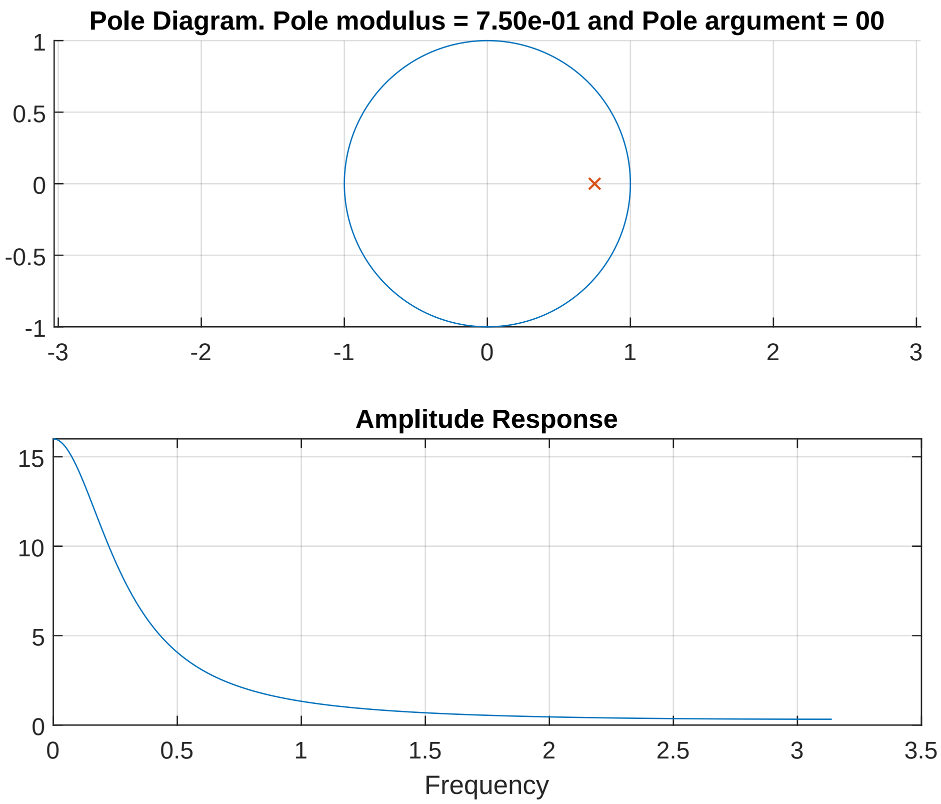


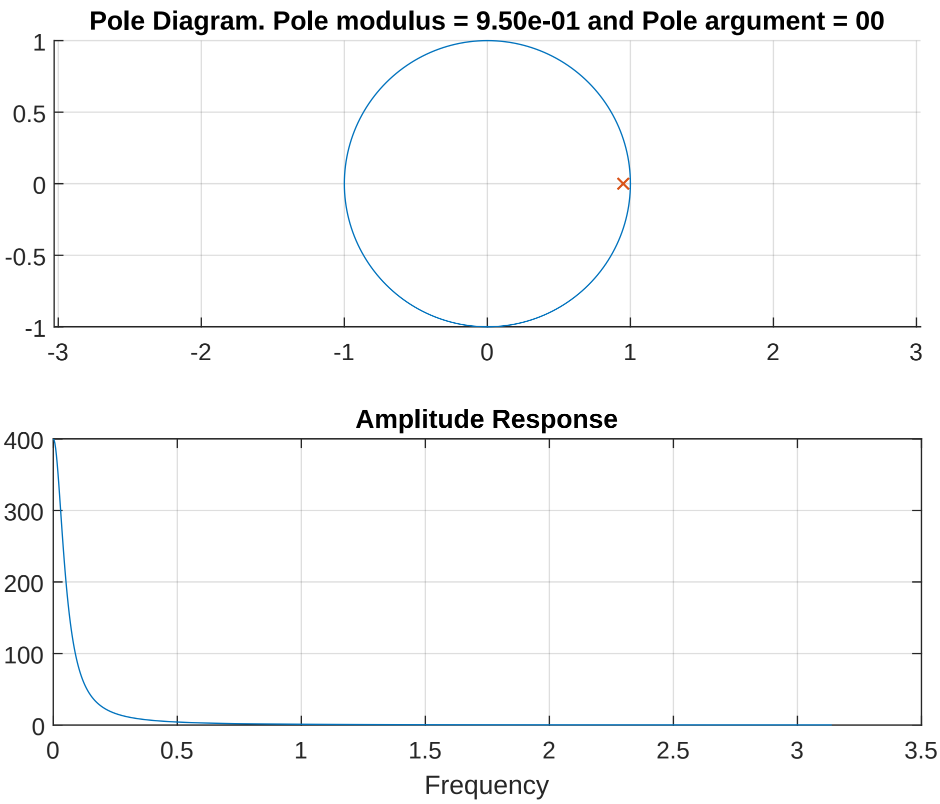


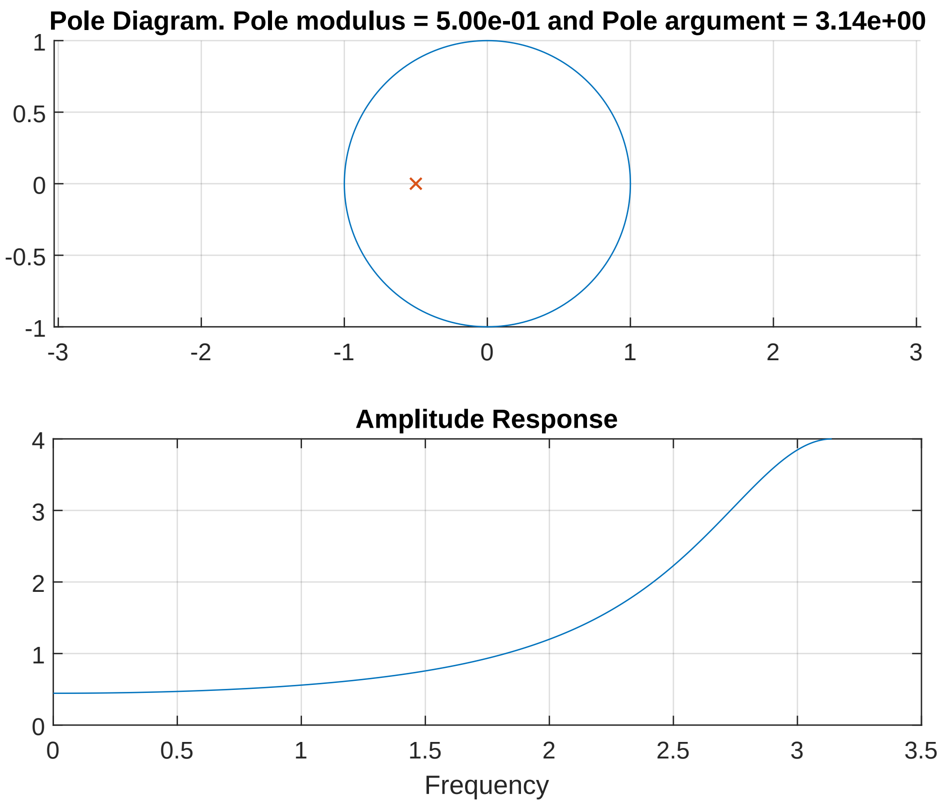
A dip in |H| at ω_0



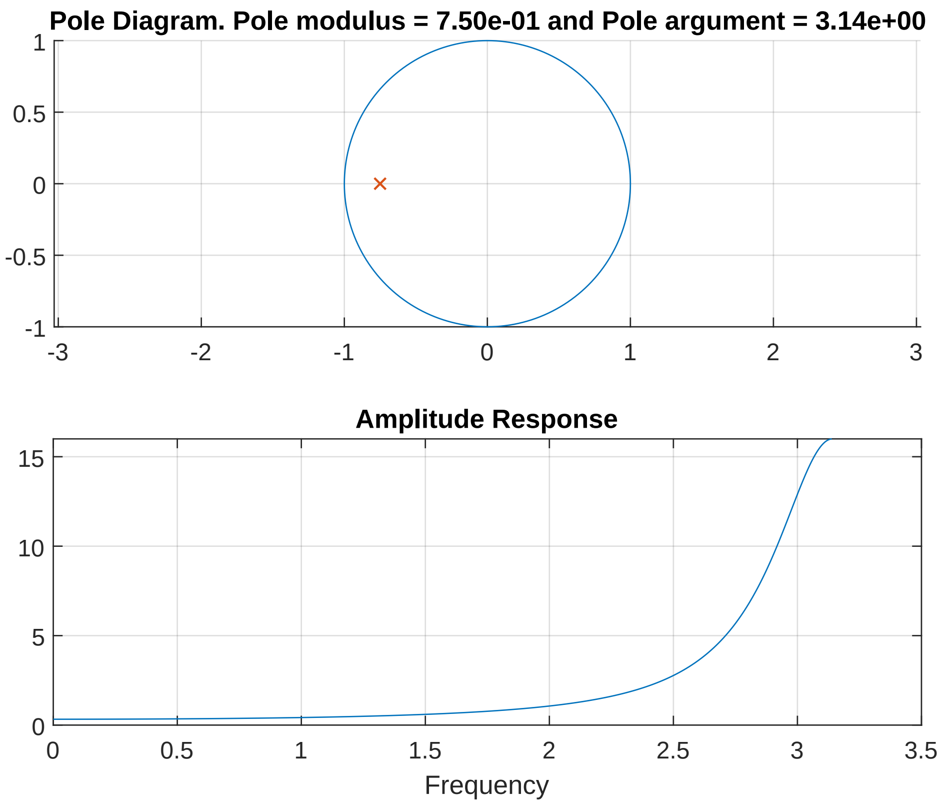
Pole Diagram. Pole modulus = 5.00e-01 and Pole argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Pole Diagram. Pole modulus = 7.50e-01 and Pole argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
15
10
5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

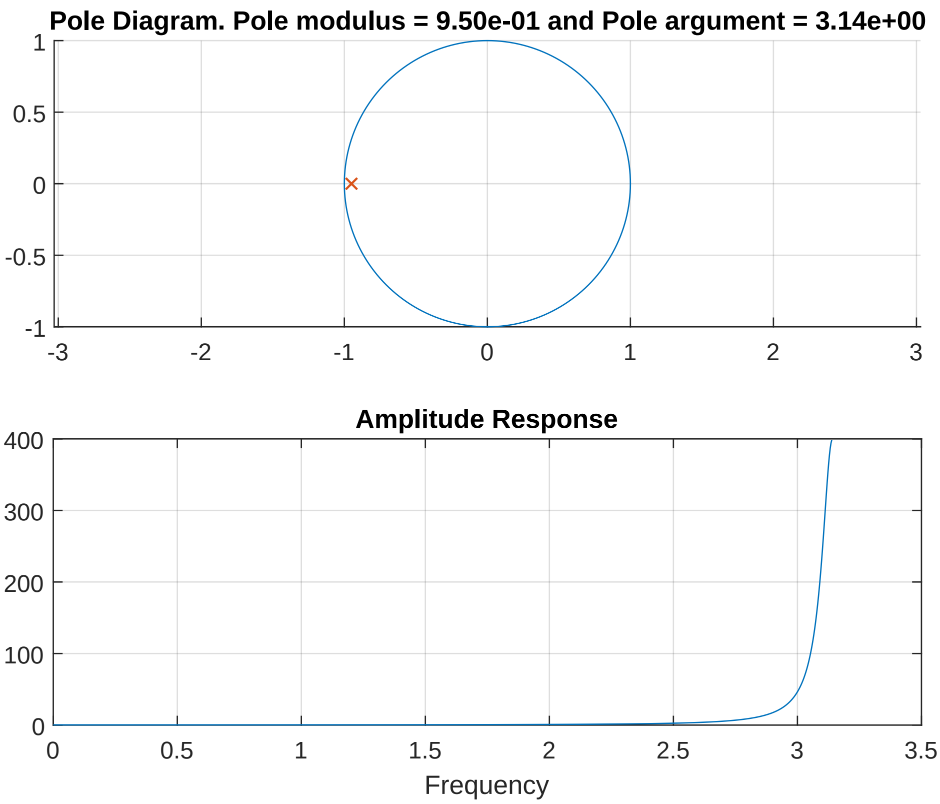

Pole Diagram. Pole modulus = 9.50e-01 and Pole argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
400
300
200
100
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

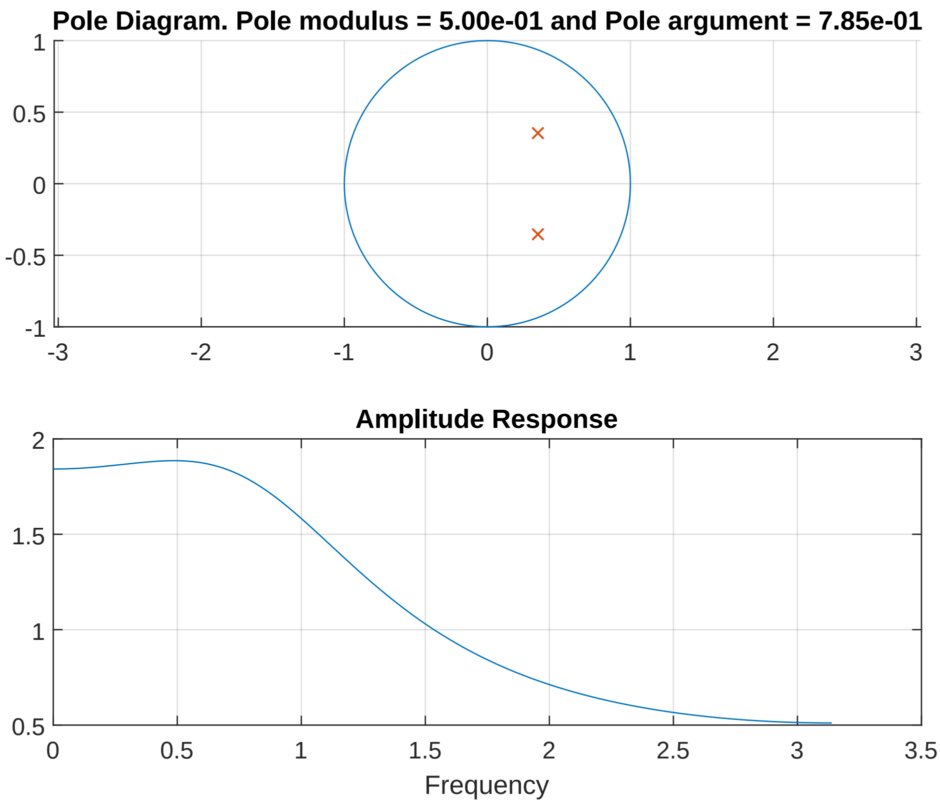


Pole Diagram. Pole modulus = 5.00e-01 and Pole argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

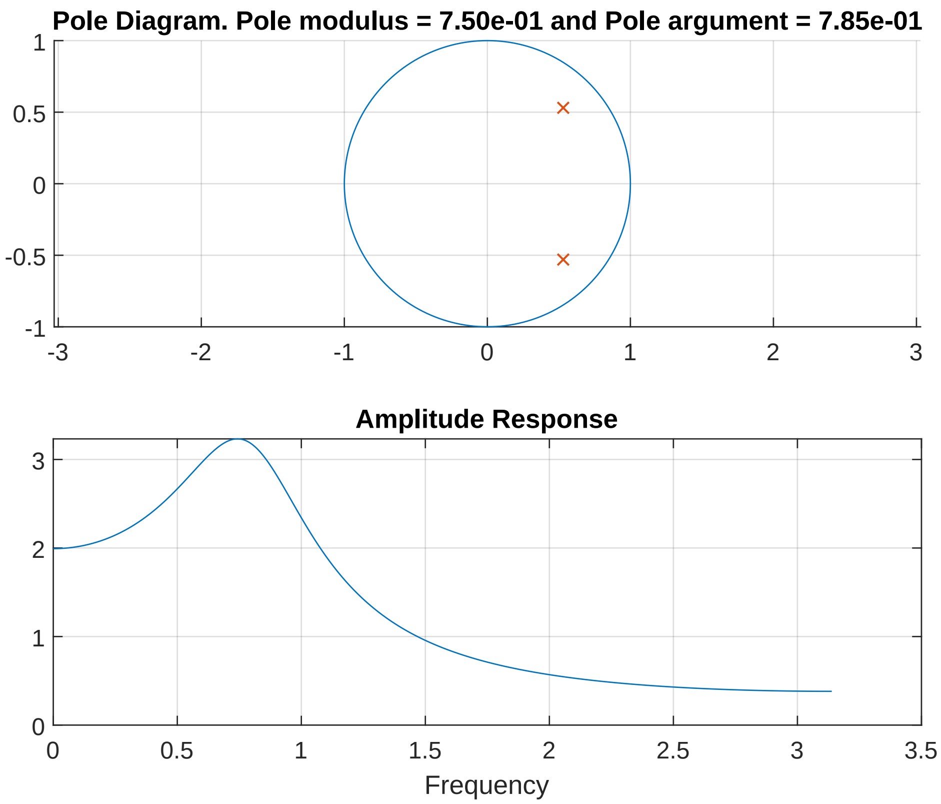


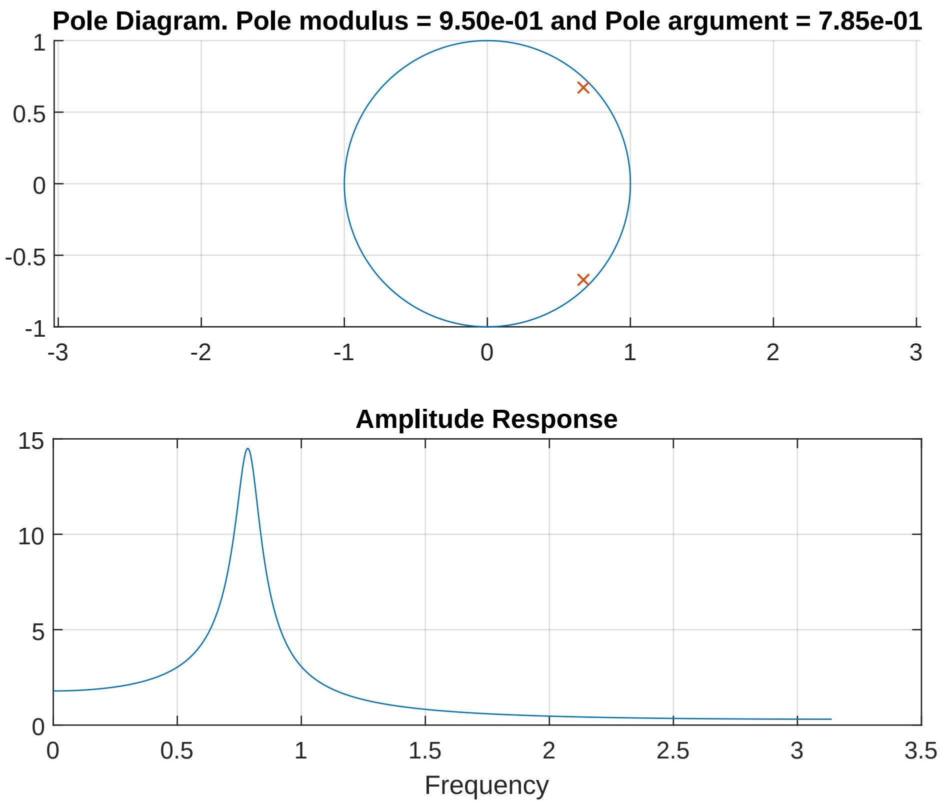
Pole Diagram. Pole modulus = 7.50e-01 and Pole argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
15
10
5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

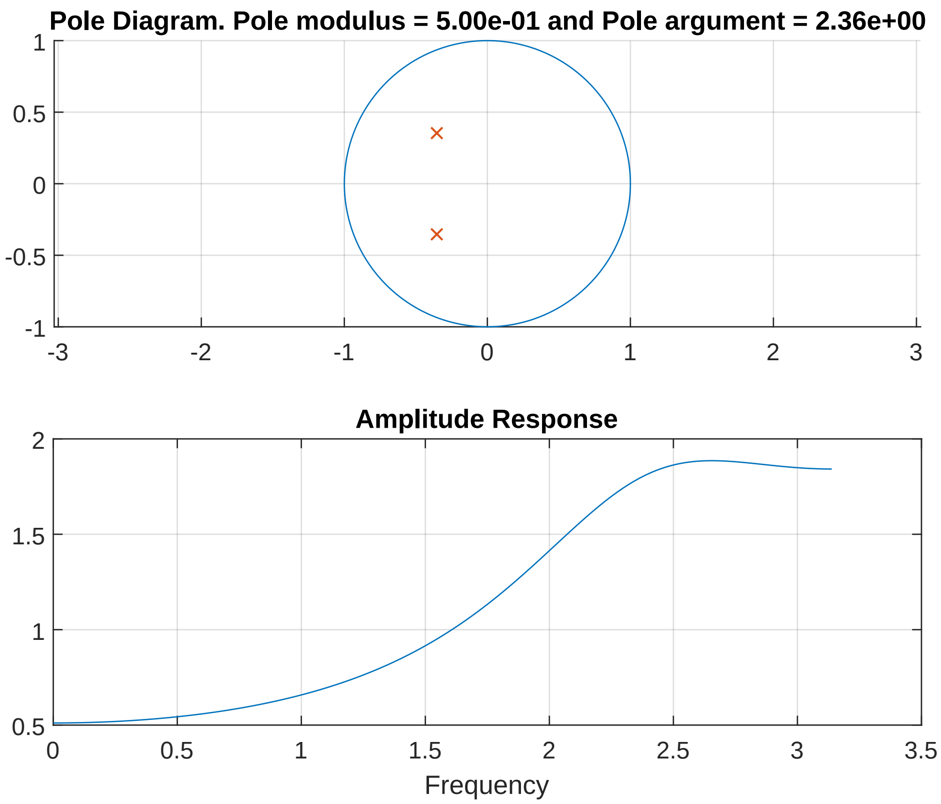

Pole Diagram. Pole modulus = 9.50e-01 and Pole argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
400
300
200
100
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency



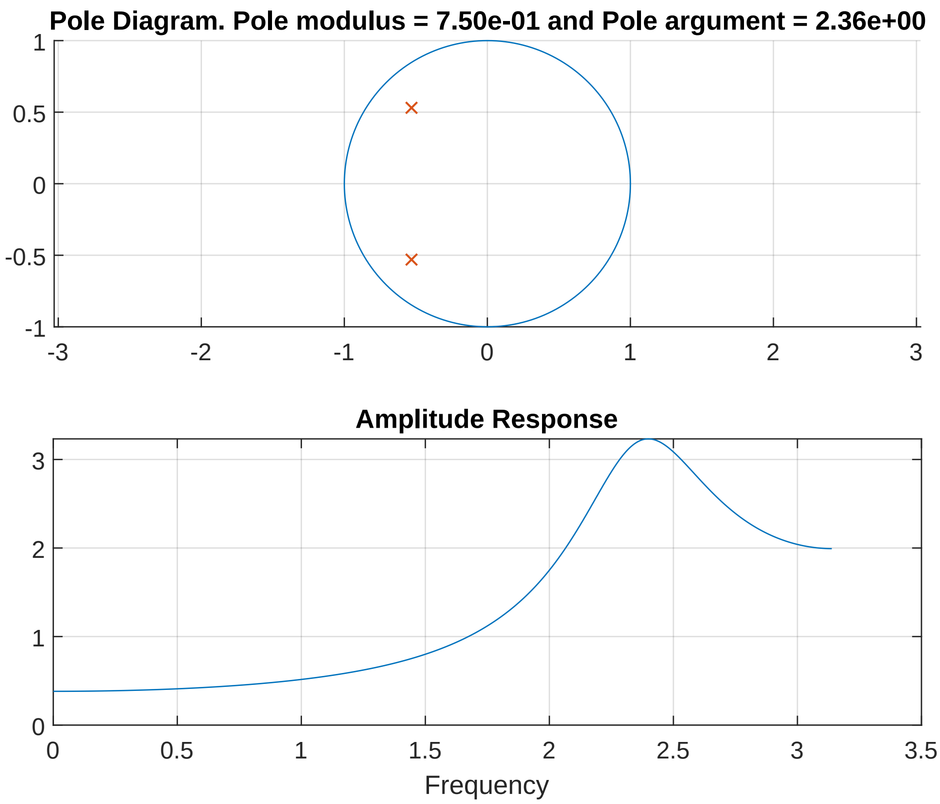
Pole Diagram. Pole modulus = 5.00e-01 and Pole argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Pole Diagram. Pole modulus = 7.50e-01 and Pole argument = 7.85e-01
Amplitude Response
Frequency

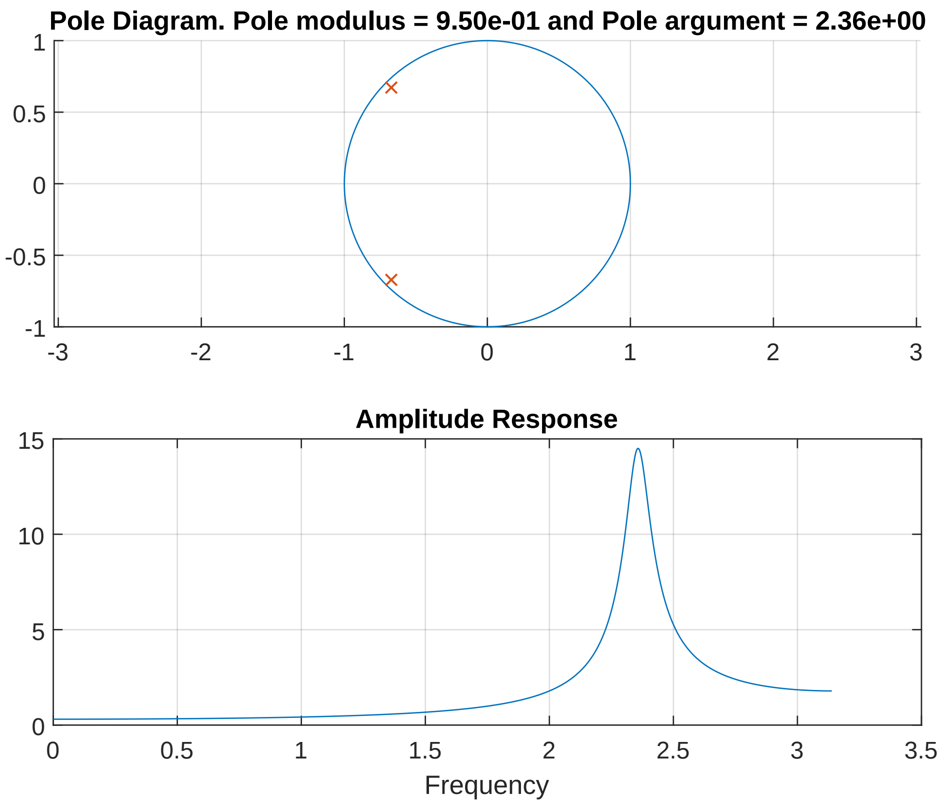

Pole Diagram. Pole modulus = 9.50e-01 and Pole argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
15
10
5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

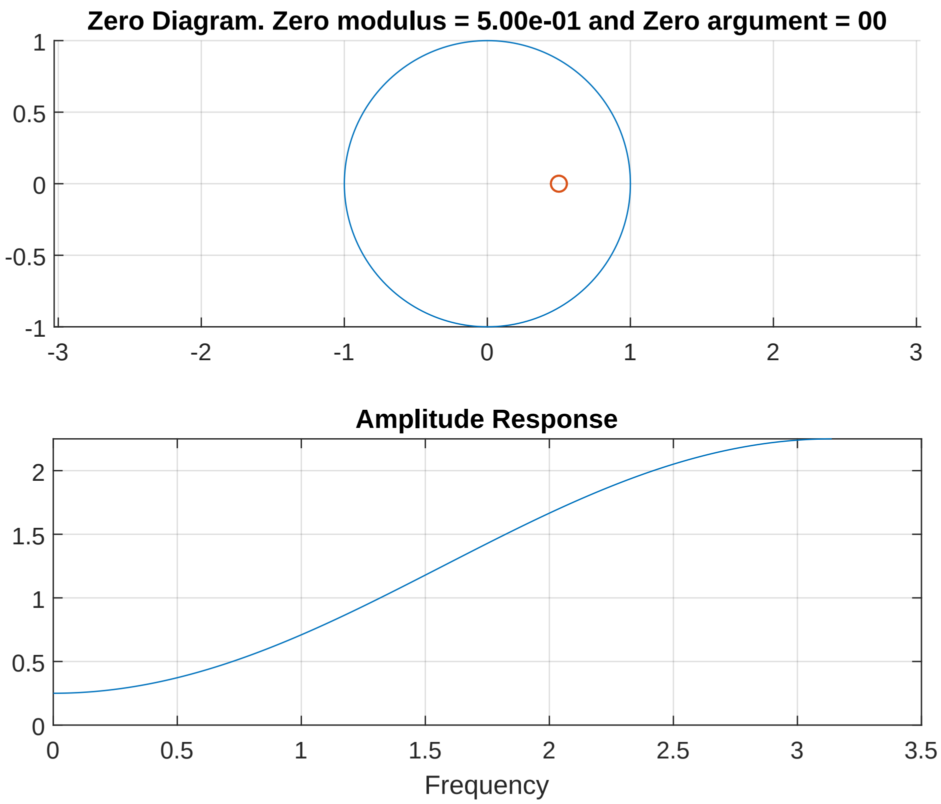


Pole Diagram. Pole modulus = 5.00e-01 and Pole argument = 2.36e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

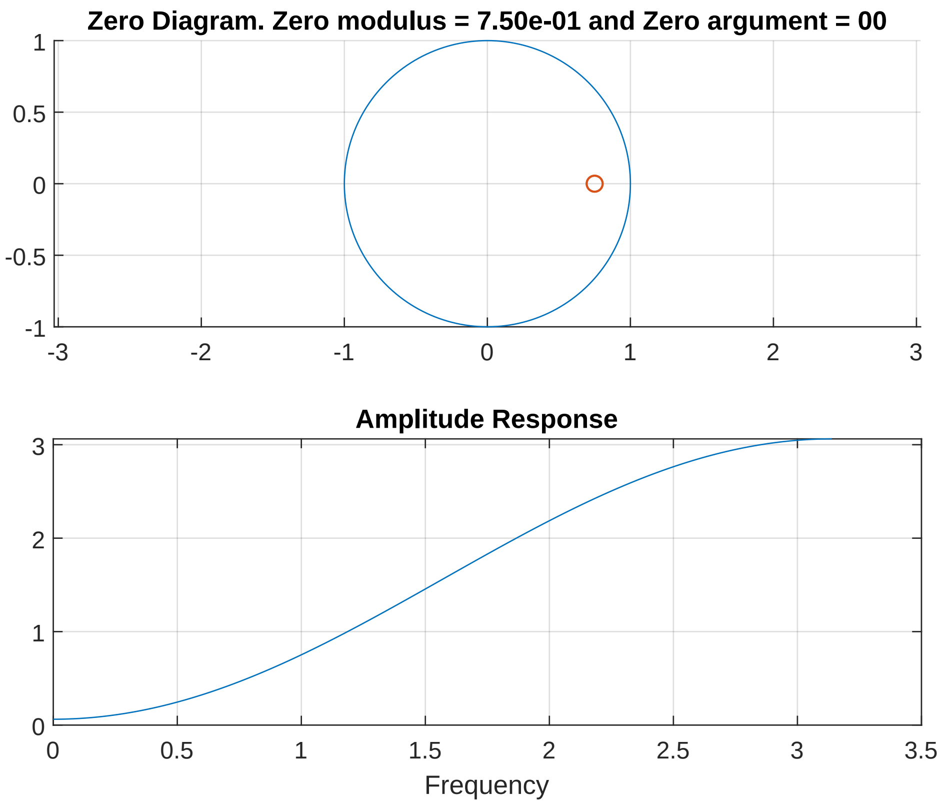


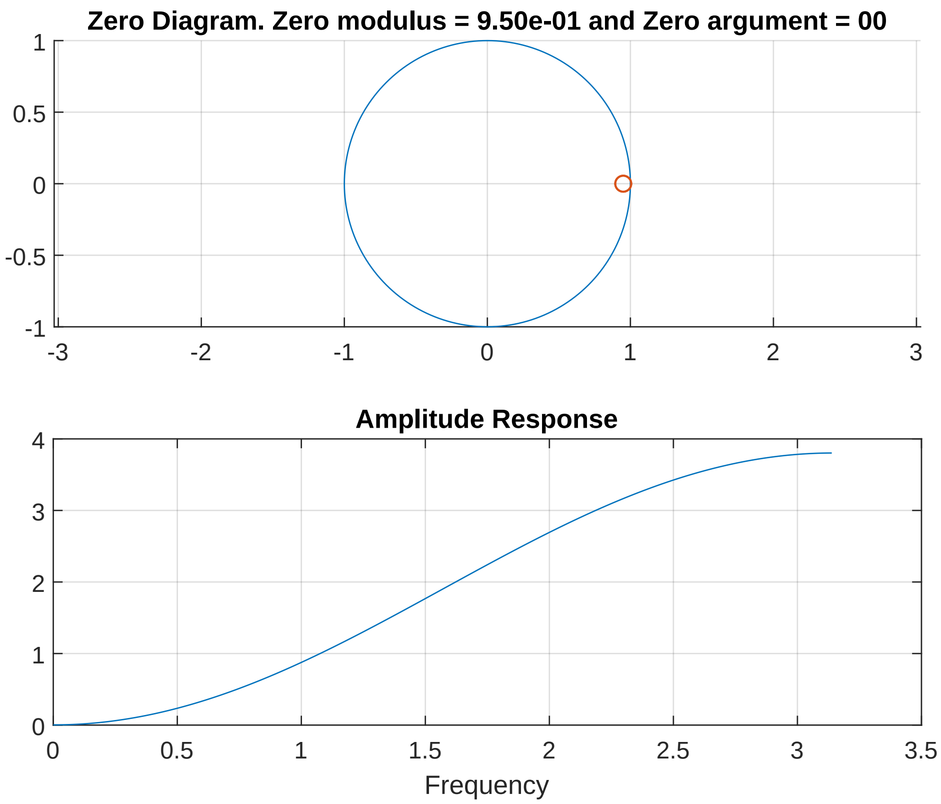
Pole Diagram. Pole modulus = 7.50e-01 and Pole argument = 2.36e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

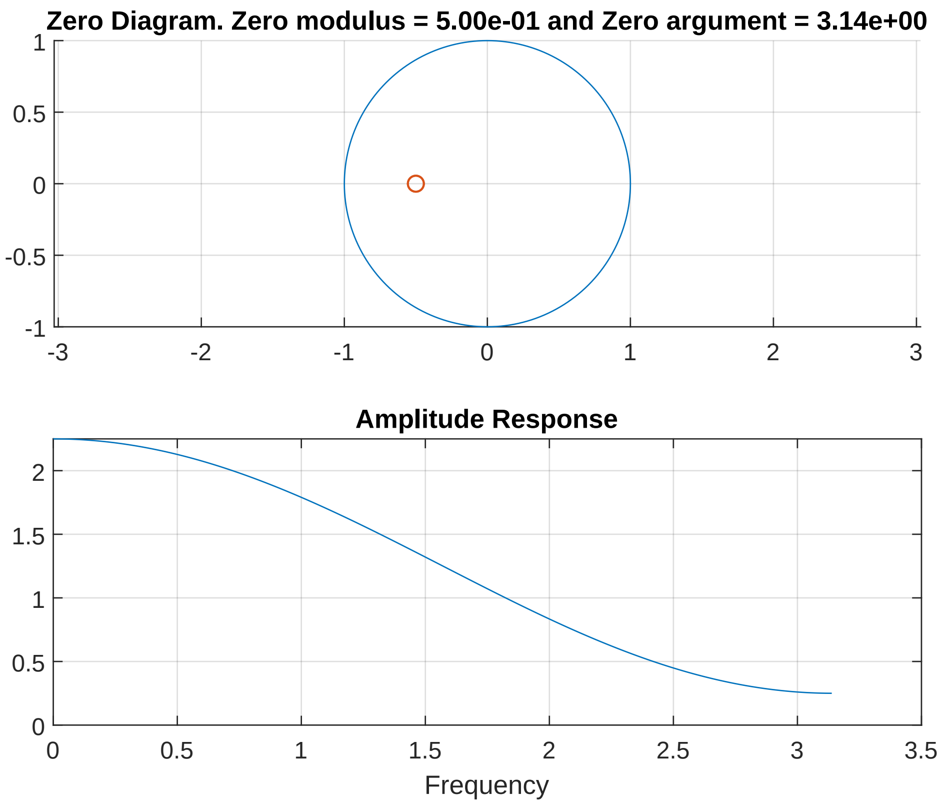

Pole Diagram. Pole modulus = 9.50e-01 and Pole argument = 2.36e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
15
10
5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

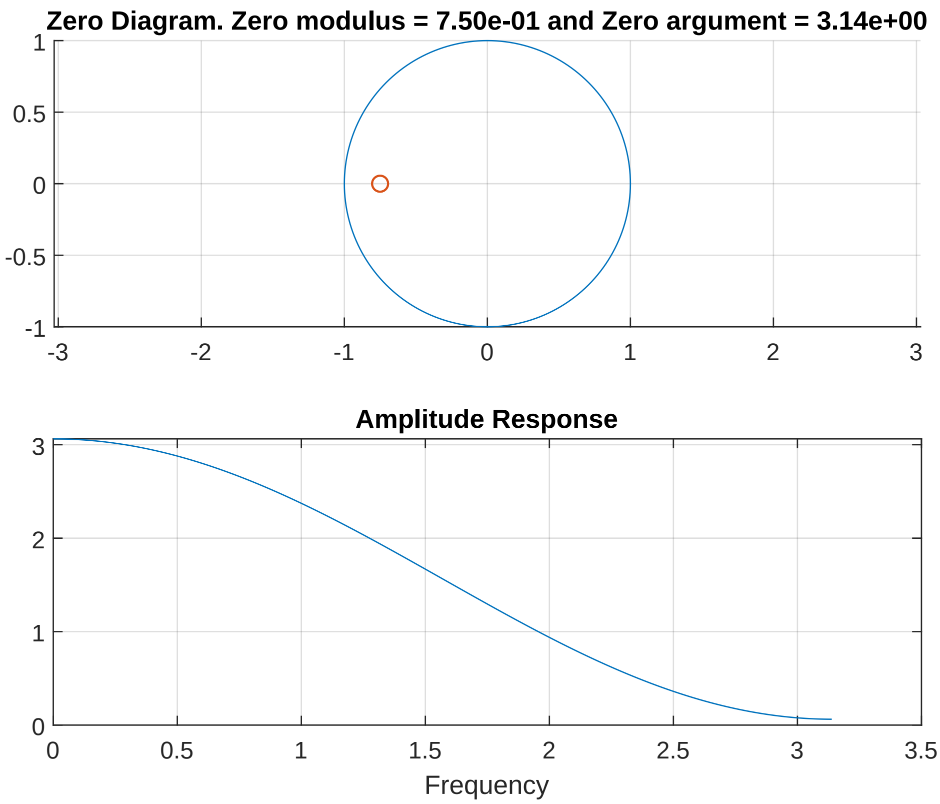


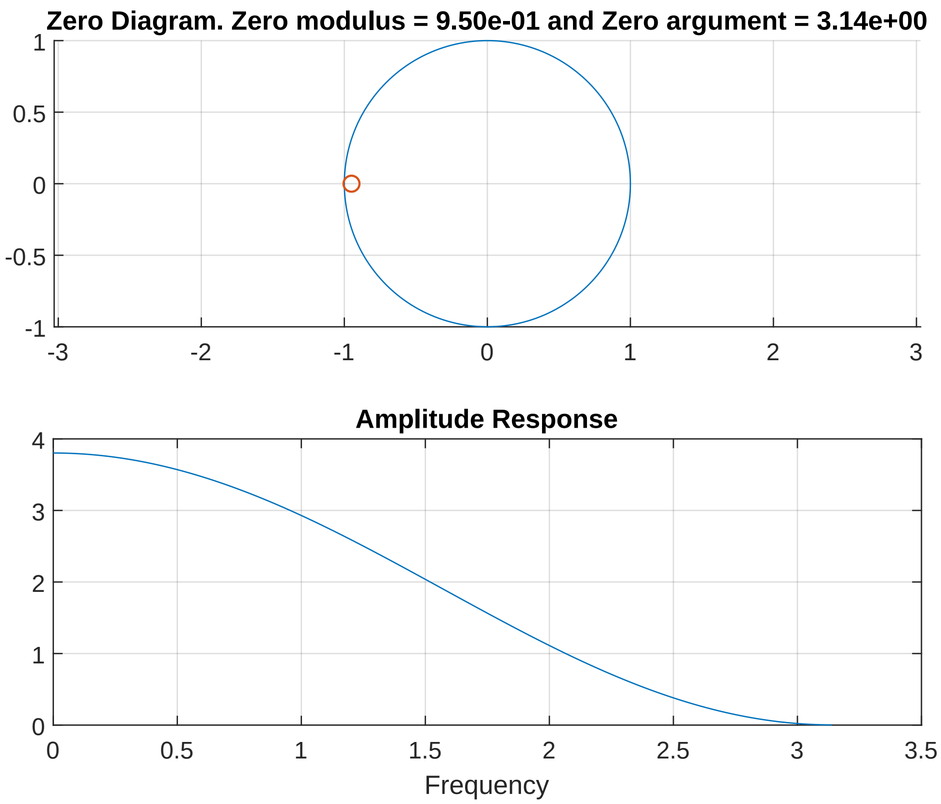
Zero Diagram. Zero modulus = 5.00e-01 and Zero argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

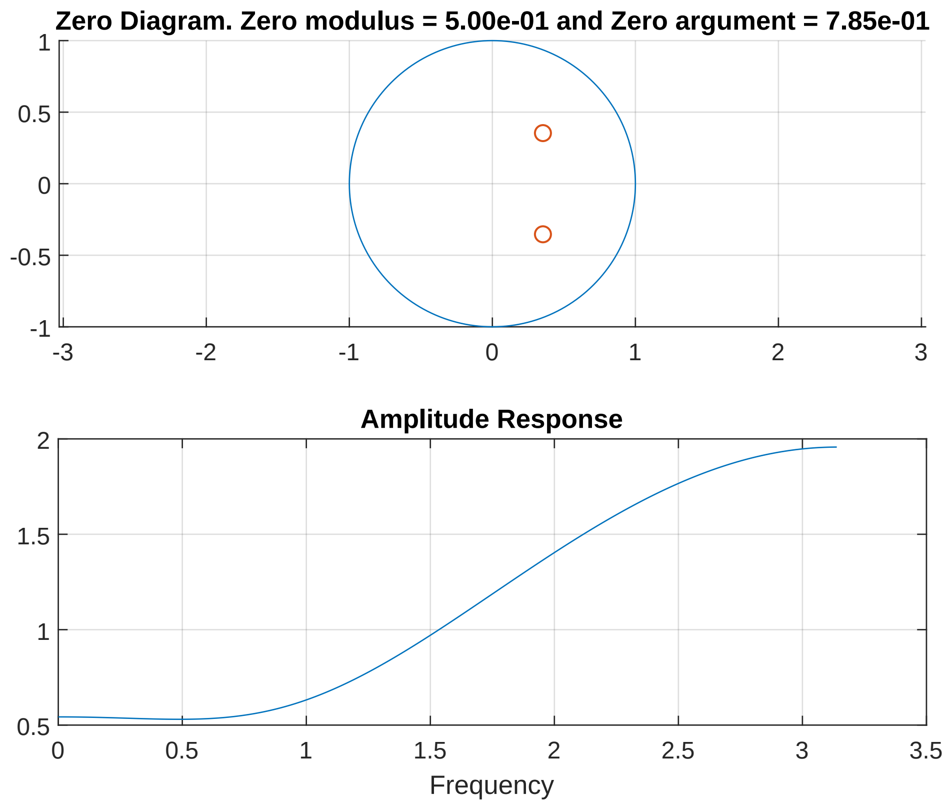

Zero Diagram. Zero modulus = 7.50e-01 and Zero argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Zero Diagram. Zero modulus = 9.50e-01 and Zero argument = 00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

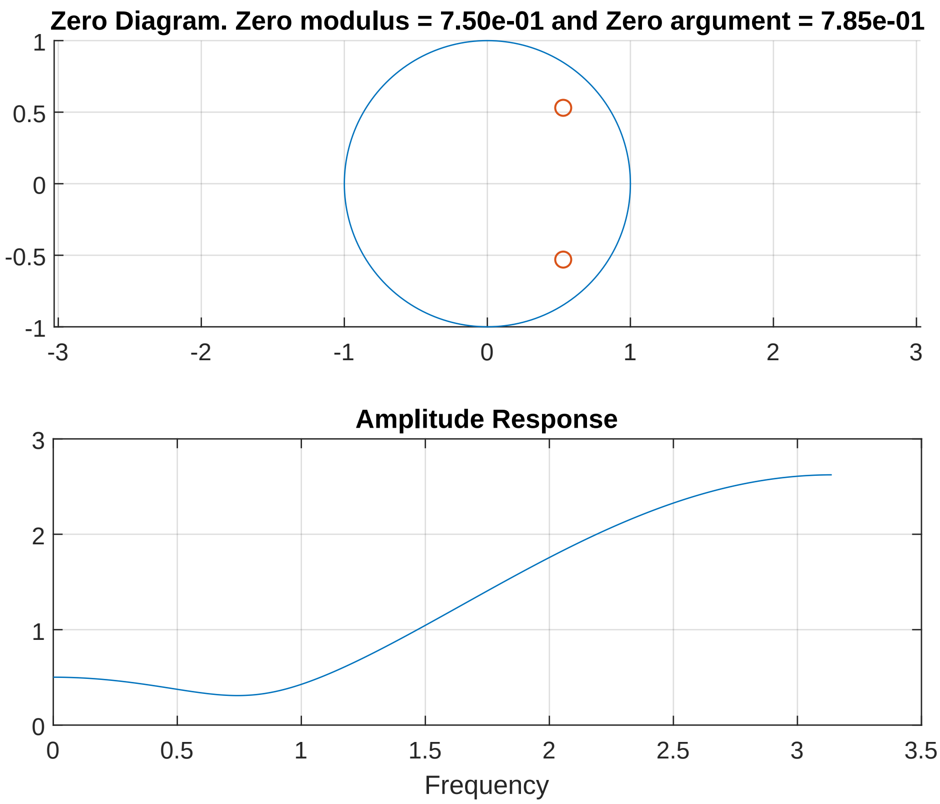

Zero Diagram. Zero modulus = 5.00e-01 and Zero argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency

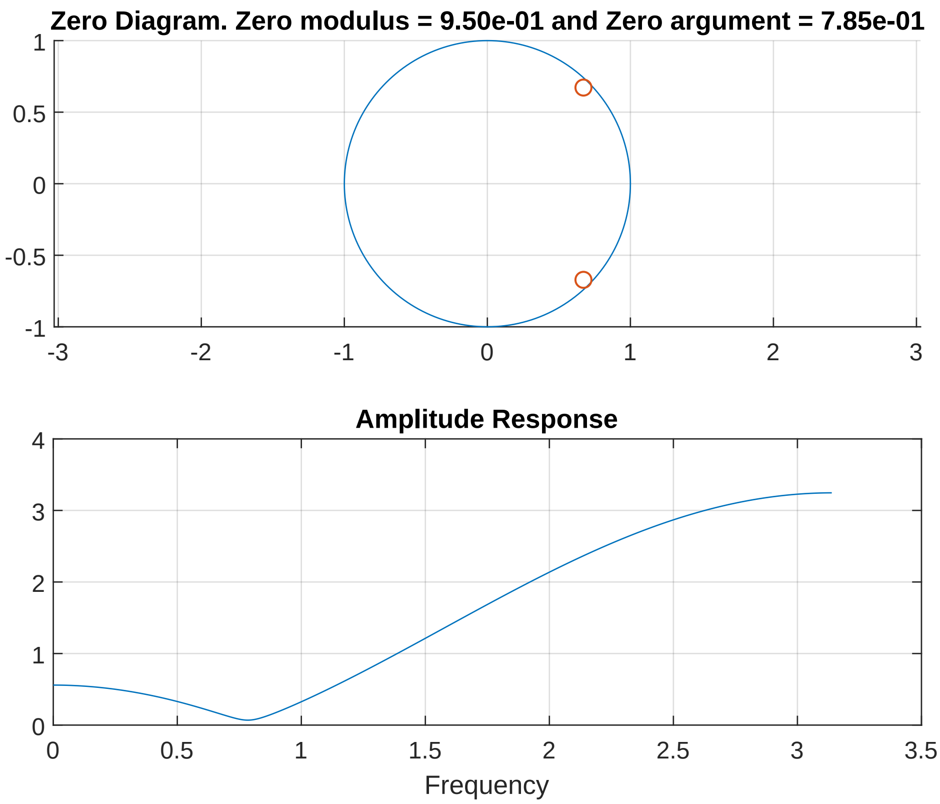

Zero Diagram. Zero modulus = 7.50e-01 and Zero argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Zero Diagram. Zero modulus = 9.50e-01 and Zero argument = 3.14e+00
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency



Zero Diagram. Zero modulus = 5.00e-01 and Zero argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Zero Diagram. Zero modulus = 7.50e-01 and Zero argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency


Zero Diagram. Zero modulus = 9.50e-01 and Zero argument = 7.85e-01
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Amplitude Response
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency