Signalbehandling for computer-ingeniører
COMTEK-5, E21
&
Signalbehandling
EIT-5, E22

11. The Discrete Fourier Transform – The Discrete Fourier Series

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The Discrete Fourier Transform, DFT

In this lecture we will initiate our discussion on the Discrete Fourier Transform which is the fundamental transform used for all ”computerized” calculation of the spectral content of a discrete-time signal.

Remaining lectures, overview

1. Introduction to the DFT; Fourier Series

2. More on (application of) the DFT; Convolution

3. Efficient computation of the DFT; The Fast Fourier Transform (FFT)

4. Practical implementation; The Short Time Fourier Transform (STFT)

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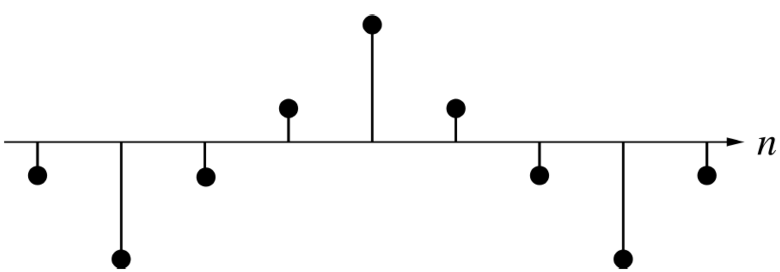
What we know already…
The Discrete Time Fourier Transform (DTFT)

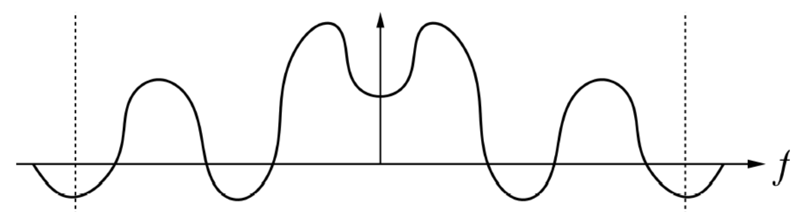
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Time:



Frequency:



Remember that the DTFT is a complex number

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}.$$

rectangular form polar form

Note that $X(e^{j\omega})$ is a continuous and periodic function….!

As we have seen many times, this complex number has a modulus and an argument, also known as the AMPLITUDE and the PHASE.

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DTFT pair

1. $\delta[n]$	1	Impulse <-> Flat spectrum
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$	Time shifted impulse
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$	DC
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1-ae^{-j\omega}}$	Exponential descreasing
5. $u[n]$	$\dfrac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$	
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1-ae^{-j\omega})^2}$	
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$	
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$	Lowpass filter
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$	Rect. window
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$	Complex exponential
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$	

$$\text{DTFT is} \begin{cases} \text{Discrete in time} \\ \text{Continuous / periodic in frequency} \end{cases}$$

$$\text{DFT is} \begin{Bmatrix} \text{Discrete in time (finite length)} \\ \text{Discrete / periodic in frequency} \end{Bmatrix}$$

In other words:

- The DTFT is a continuous function in ω while the DFT is a sequence
- The DTFT as well as the DFT is periodic in frequency
- The DFT works on a finite length input sequence
- The DFT can be implemented by efficient computational schemes

Now, in order to get started, let's first have a look at the Fourier Series (Time-Continuous)

What is a Fourier Series…??

A Fourier Series is a representation of a CONTINUOUS-TIME PERIODIC signal using sinusoids…

- This is EXTREMELY usefull because sinusoids have simple interaction with LTI systems; If we know how the system responds to a sinusoid, and our signal is represented as a sum of sinusoids, then (using the principle of super-position) it is easy to find the response to the signal.

- Fourier Series are also used as an Analysis Tool for finding amplitude- and phase spectrum of the periodic signal.

Fourier Series – general case

A continuous-time periodic signal can be represented as a sum of continuous-time (complex) sinusoids.

The (observable) signal $x(t)$ is periodic with period T.

The sinusoids should also have period T, i.e., they are hamonically related with individual frequencies;

$$\Omega_k = k2\pi f = k\frac{2\pi}{T} = k\Omega_f$$

where $\Omega_f = \dfrac{2\pi}{T}$ is the fundamental (angular) frequency.

So, the signal $x(t)$ can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\Omega_f t}$$

where $X[k]$ are weight factors (complex amplitudes), and the exponentials are complex sinusoids with frequency $k\Omega_f$.

Fourier Series

In order to find the $X[k]$'s, we have to integrate the signal $x(t)$ against a complex sinusoid;

$$X[k] = \frac{1}{T}\int_{<T>} x(t)e^{-jk\Omega_f t}dt = \frac{\Omega_f}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\Omega_f t}dt, \text{ where } \frac{1}{T} = \frac{\Omega_f}{2\pi} \text{ cf., p.8}$$

$<T>$ represents a single period, e.g., [*0;T*[or [-*T/2;T/2*[, or whatever is the most convenient interval in order to calculate the integral…

So, knowing $x(t)$, we can find $X[k]$, and vice versa…

$$x(t) \overset{FS,\Omega_f}{\longleftrightarrow} X[k]$$

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Fourier Series of a real signal *x*(*t*)

For a real signal *x*(*t*), we can express *x*(*t*) as a sum of sines and cosines (and not complex exponentials, as we have just seen for the general case…)

To see this, we apply the ”conjugate symmetry” property of the Fourier Series;

$$X[-k] = X^*[k]$$

i.e., the real part of the Fourier Series is an even function of *k* and the imaginary part of the Fourier Series is an odd function of *k*.

To convince ourself about this property, let's write it out…

$$X^*[k] = \frac{1}{T}\int_{<T>} [x(t)e^{-jk\Omega_f t}]^* dt = \frac{1}{T}\int_{<T>} x(t)e^{-j(-k)\Omega_f t} dt = X[-k]$$

where we observe that $x^*(t) = x(t)$ because we have said that $x(t)$ is real…

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So, given the signal $x(t)$ as an infinite sum, we can re-write this expression;

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\Omega_f t} = X[0] + \sum_{k=1}^{\infty}(X[k]e^{jk\Omega_f t} + X[-k]e^{-jk\Omega_f t})$$ cf. p.8

Now, let's define the Fourier Series coffecifients as follows;

$$X[k] = \frac{B[k]}{2} - j\,\frac{A[k]}{2}$$

Using the conjugate symmetry property, we have;

$$X[-k] = \frac{B[k]}{2} + j\,\frac{A[k]}{2}$$

These two expressions are now used in the expression above for $x(t)$;

$$x(t) = X[0] + \sum_{k=1}^{\infty}(X[k]e^{jk\Omega_f t} + X[-k]e^{-jk\Omega_f t})$$

$$x(t) = X[0] + \sum_{k=1}^{\infty}\{(\frac{B[k]}{2} - j\,\frac{A[k]}{2})e^{jk\Omega_f t} + (\frac{B[k]}{2} + j\,\frac{A[k]}{2})e^{-jk\Omega_f t}\}$$

$$x(t) = X[0] + \sum_{k=1}^{\infty}\{(\frac{B[k]}{2} - j\frac{A[k]}{2})e^{jk\Omega_f t} + (\frac{B[k]}{2} + j\frac{A[k]}{2})e^{-jk\Omega_f t}\}$$

$$x(t) = X[0] + \sum_{k=1}^{\infty}\{\ B[k](\frac{e^{jk\Omega_f t} + e^{-jk\Omega_f t}}{2}) + A[k](\frac{e^{jk\Omega_f t} - e^{-jk\Omega_f t}}{2j})\quad\}$$

$$x(t) = X[0] + \sum_{k=1}^{\infty}\{B[k]\cos\left(k\Omega_f t\right) + A[k]\sin\left(k\Omega_f t\right)\}$$ where

$$X[0] = \frac{1}{T}\int_{<T>} x(t)dt$$ which is the average of the signal $x(t)$ over 1 period, and

$$B[k] = \frac{2}{T}\int_{<T>} x(t)\cos\left(k\Omega_f t\right)dt$$

Cf., the definition of $X[k]$, p. 9, and re-formulation, p. 11

$$A[k] = \frac{2}{T}\int_{<T>} x(t)\sin\left(k\Omega_f t\right)dt$$

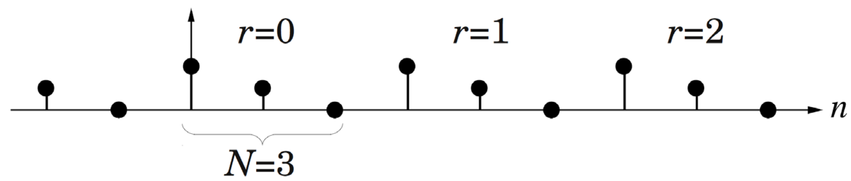
So, when x(t) is real, it can be expressed as a weighted sum of real sinusoids…!

…and in the general case, x(t) is a weighted sum of complex sinusoids…!

Now, let's turn our attention to the Fourier Series of a periodic sequence - The Discrete Fourier Series

Tilde denotes "periodicity"

$$\tilde{x}[n]=\tilde{x}[n+rN]$$



N is the period and r represents the period number.

Similarly to the continuous time case, the periodic sequence can be constructed as a sum of complex sinusoids…

Also periodic with period N

$$e_k[n]=e^{j\underbrace{(2\pi/N)}_{\text{"Base frequency"}}kn}=e_k[n+rN]$$

k denotes the frequency index which is multiplied onto the "Base frequency", i.e., the "fundamental frequency".

Note the immediate similarity with the continuous time case...

The periodic sequence is therefore gives as;

$$\tilde{x}[n] = \frac{1}{N} \sum_{k} \tilde{X}[k] e^{j(2\pi/N)kn}$$

Question now is; how many FS coefficients $\tilde{X}[k]$ are needed…?

Since $e_{k+lN}[n] = e_k[n]$ only *N* complex exponentials are needed to describe $\tilde{x}[n]$.

Remember that everything is 2π periodic in the discrete-time domain…

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

Similarly, we can obtain the FS coefficients (see p. 653)

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn} \Leftrightarrow$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn}$$

which is periodic in N, i.e.

So again, discrete time leads to periodic frequency...!

$$\tilde{X}[0] = \tilde{X}[N], \tilde{X}[1] = \tilde{X}[N+1] \text{ etc.}$$

Fourier Series coefficients can be considered as either a

1) Finite sequence from 0 to N-1, or a
2) Periodic sequence for all k.

DFS of a periodic sequence

DFS analysis: $$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

DFS synthesis: $$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

where

$W_N = \mathrm{e}^{-j(2\pi/N)}$ ("Twiddle-factor")

Example 8.1 Discrete Fourier Series of a Periodic Impulse Train

DFS analysis

We consider the periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1, & n = rN, \quad r \text{ any integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{8.14}$$

Since $\tilde{x}[n] = \delta[n]$ for $0 \leq n \leq N-1$, the DFS coefficients are found, using Eq. (8.11), to be

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = W_N^0 = 1. \tag{8.15}$$

In this case, $\tilde{X}[k]$ is the same for all k. Thus, substituting Eq. (8.15) into Eq. (8.12) leads to the representation

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}. \tag{8.16}$$

DFS synthesis

So, what we see here is that the impulse (in time) gives a flat spectrum (in freq.), and that the impulse train can be expressed as a sum of N complex exponentials

Duality in the Discrete Fourier Series

Here we let the discrete Fourier series coefficients be the periodic impulse train

$$\tilde{Y}[k] = \sum_{r=-\infty}^{\infty} N\delta[k - rN].$$

Substituting $\tilde{Y}[k]$ into Eq. (8.12) gives

DFS synthesis

$$\tilde{y}[n] = \frac{1}{N}\sum_{k=0}^{N-1} N\delta[k] W_N^{-kn} = W_N^{-0} = 1.$$

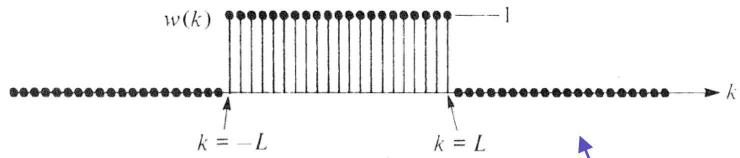
Note that we need to look over one period only and thus $\delta[k - rN]$ reduces to $\delta[k]$, i.e., $r = 0$.

So, what we see here is that an impulse located in $k = 0$ (in frequency) is a ”flat” sequence (in time), i.e., a DC signal.

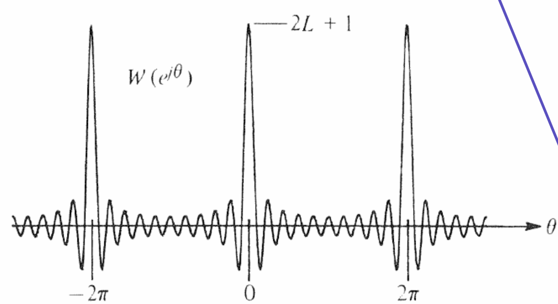
This is ”opposite” to the DFS of an impulse train – thus called a DUALITY…

Duality – we have seen something quite similar previously (FIR filter design)

Window function in time

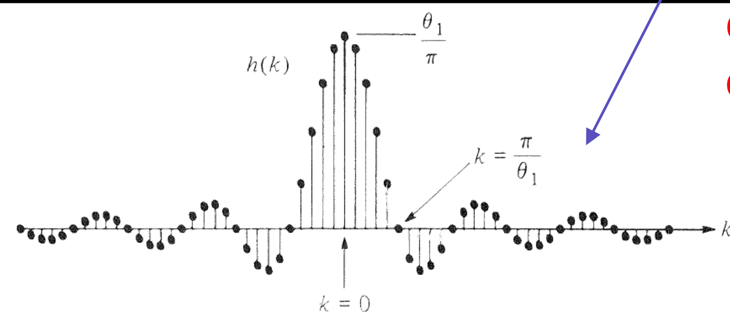


Window function in freq.

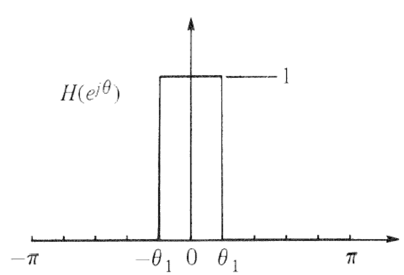


Note however, that these are NOT periodic sequences and thus the freq.-responses are continuous functions of Θ

Impulse response



Freq./ Amplitude response



Duality is a consequense of the similarity between DFS analysis and synthesis – let's see it once more…

DFS analysis: $$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

DFS synthesis: $$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

Other important properties of the DFS

Linearity:

$$\left.\begin{array}{l}\tilde{x}_1[n] \xrightarrow{DFS} \tilde{X}_1[k] \\ \\ \tilde{x}_2[n] \xrightarrow{DFS} \tilde{X}_2[k]\end{array}\right\} \text{yields}$$

$$a\,\tilde{x}_1[n] + b\,\tilde{x}_2[n] \xrightarrow{DFS} a\,\tilde{X}_1[k] + b\,\tilde{X}_2[k]$$

Time-shift:

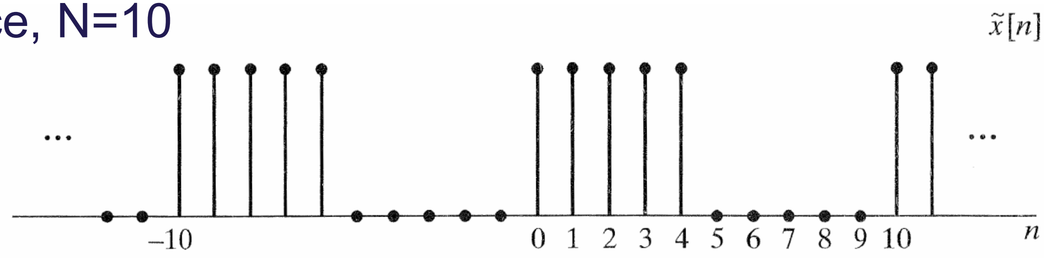
$$\tilde{x}[n-m] \xrightarrow{DFS} W_N^{km}\,\tilde{X}[k]$$

Frequency-shift:

$$\tilde{X}[k-l] \xrightarrow{DFS} W_N^{-nl}\,\tilde{x}[n]$$

Example of a DFS

Periodic sequence, N=10

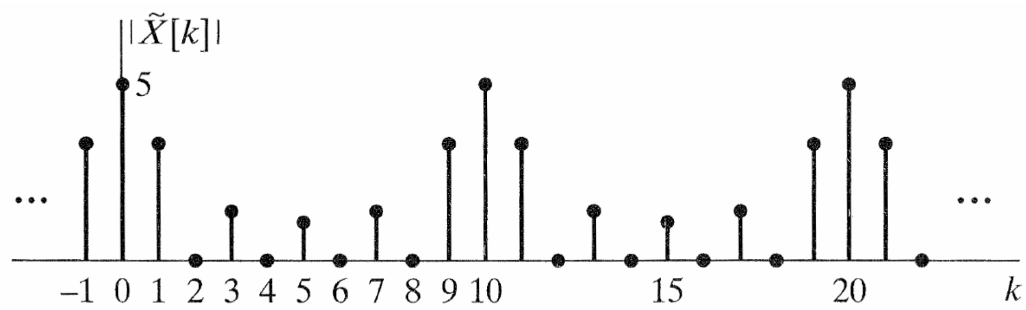


$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

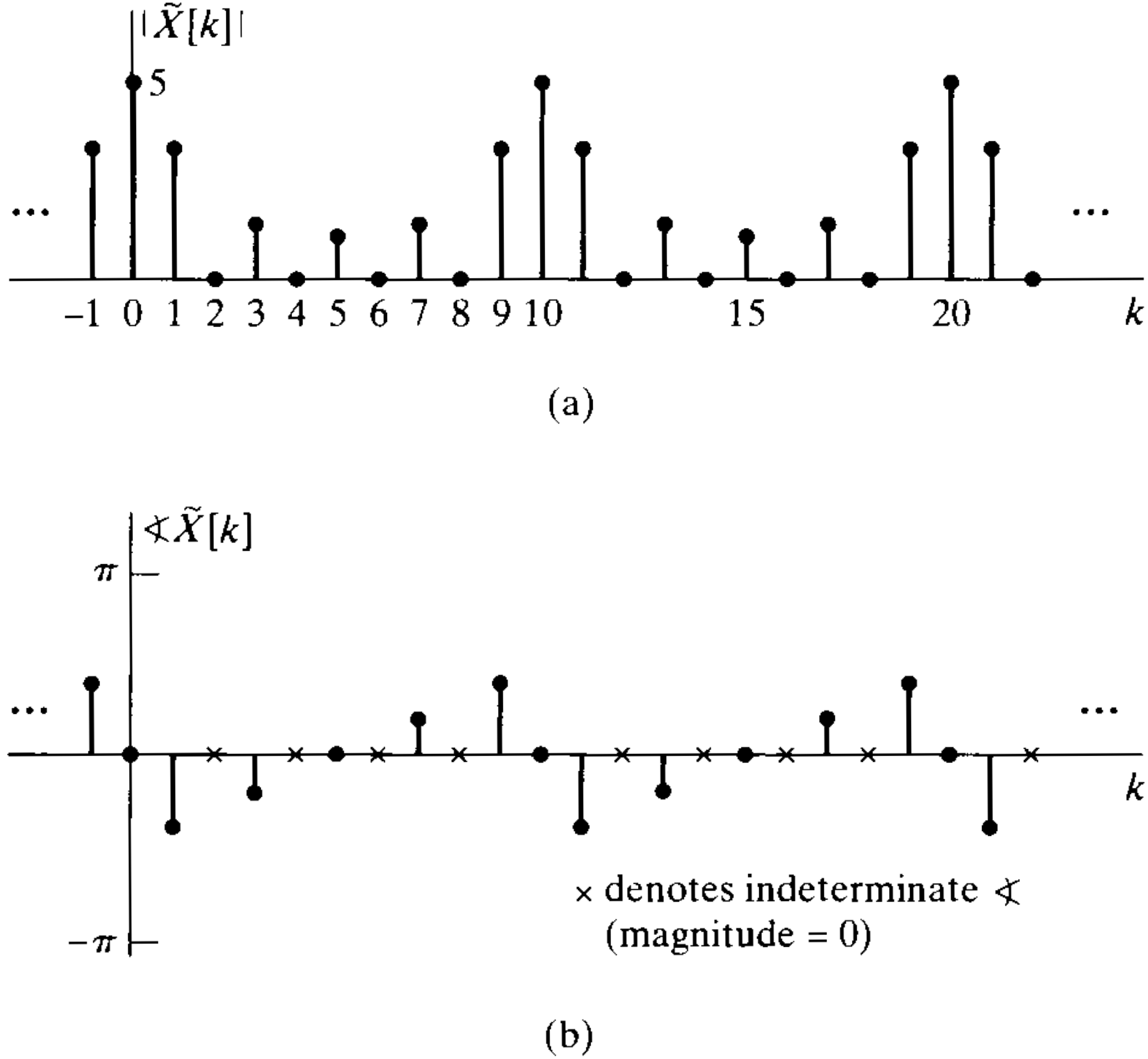
Geometric series where $q = W_{10}^{k}$

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j(2\pi/10)kn} \Rightarrow$$

$$\tilde{X}[k] = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$



…an interesting observation related to the phase response

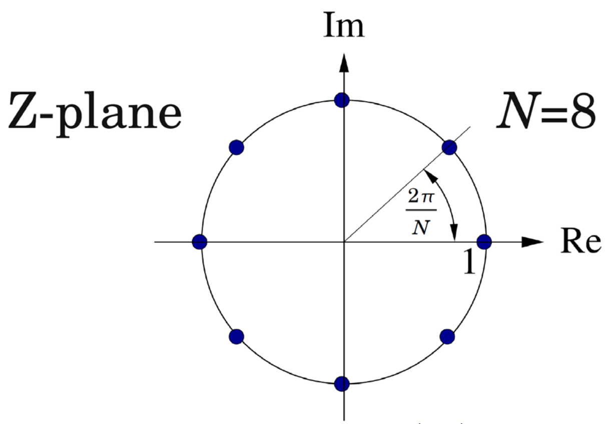


Relation between periodic and aperiodic sequences

Remember that the DTFT of a sequence $x[n]$, i.e., $X(e^{j\omega})$, is identically equal to the z-transform $X(z)$ on the unit circle; $z = e^{j\omega}$.

Note that $x[n]$ is not necessarily a periodic sequence – typically, it is aperiodic..!

We now consider N equally spaces point on the unit circle;



We next sample $X(z)$ in the points $\mathrm{z} = e^{j\left(\frac{2\pi}{N}\right)k}$ which leads to a sampled frequency response;

$$\tilde{X}[k] = X(z)\big|_{z=e^{j\left(\frac{2\pi}{N}\right)k}} = X\left(e^{j\left(\frac{2\pi}{N}\right)k}\right) = X(e^{j\omega_k}) \qquad 0 \le k \le N-1$$

Tilde denotes ”periodicity” in k with period N.

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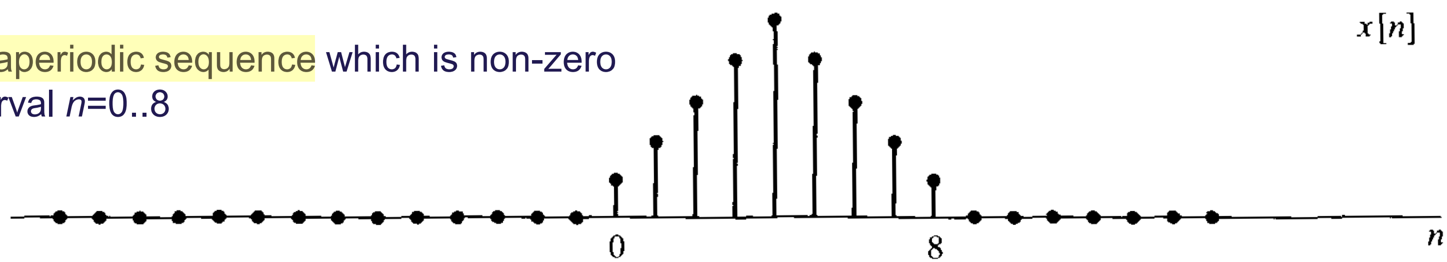
$$\tilde{X}[k] = X(z)\Big|_{z=e^{j\left(\frac{2\pi}{N}\right)k}} = X\left(e^{j\left(\frac{2\pi}{N}\right)k}\right) = X(e^{j\omega_k}) \qquad 0 \le k \le N-1$$

This expression represents an *N*-periodic sequence of samples which could be the sequence of Discrete Fourier Series coefficients of a sequence $\tilde{x}[n]$.

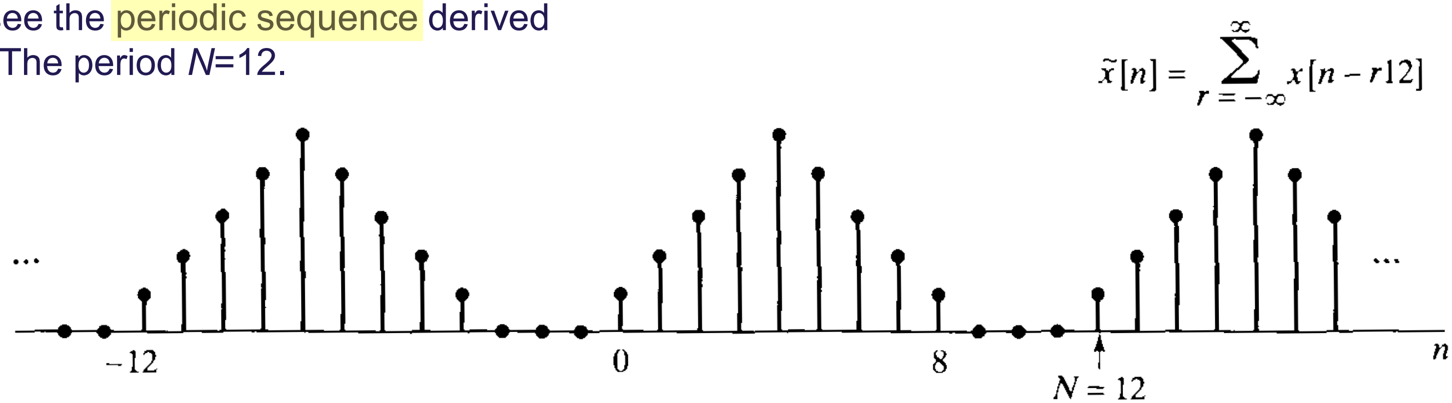
On p. 667 you'll find the math leading to the conclusion that $\tilde{x}[n]$, which corresponds to $\tilde{X}[k]$ obtained by sampling $X(z)$, is formed from $x[n]$ by adding together an infinite number of shifted replicas of $x[n]$. The shifts are all positive and negative integer multiples of *N.*

x[*n*] is an aperiodic sequence which is non-zero in the interval *n*=0..8



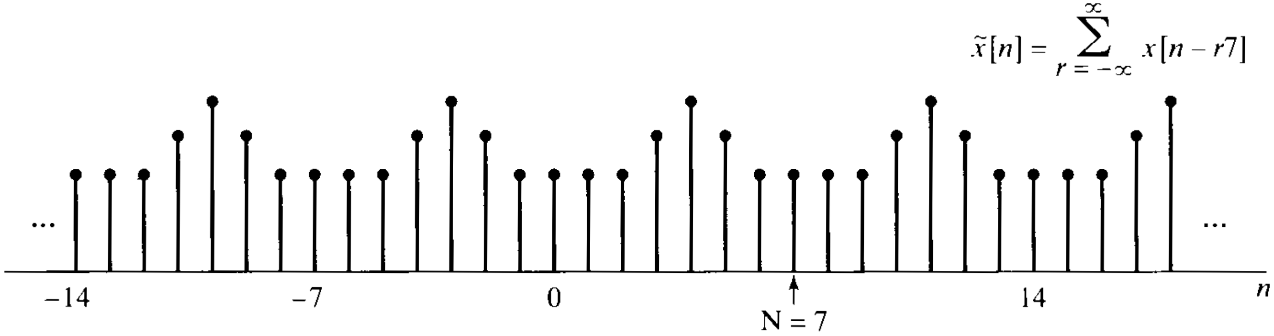
(a)

Here we see the periodic sequence derived from *x*[*n*]. The period *N*=12.

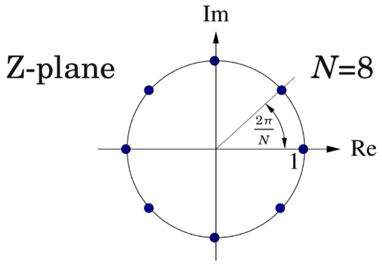


What happens if the period is less than 9 (in the example)..?

Here we have the same sequence $x[n]$, but now the period $N = 7$.



Basically what we see here is "an overlap in the time domain" which can be considered as "aliasing" – N is too small, i.e., we have chosen too few samples in the frequency domain when we sample $X(z)$ on the unit circle.



Consequently, time domain aliasing can be avoided only if $x[n]$ has finite length, just as frequency domain aliasing can be avoided only for signals that have bandlimited Fourier transforms.

OBS…!!

Relation between the DFS and the DFT

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \leqslant k \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

In other words: The Discrete Fourier Transform can be written as

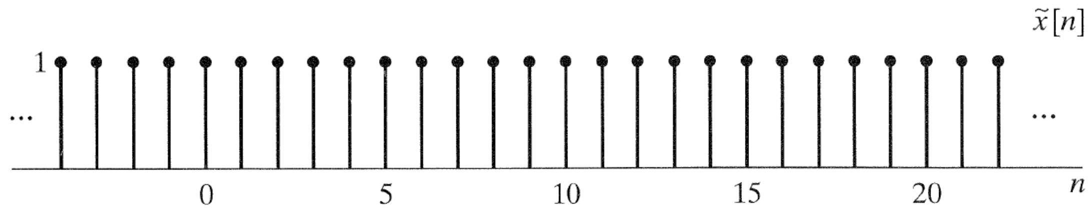
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{and}$$

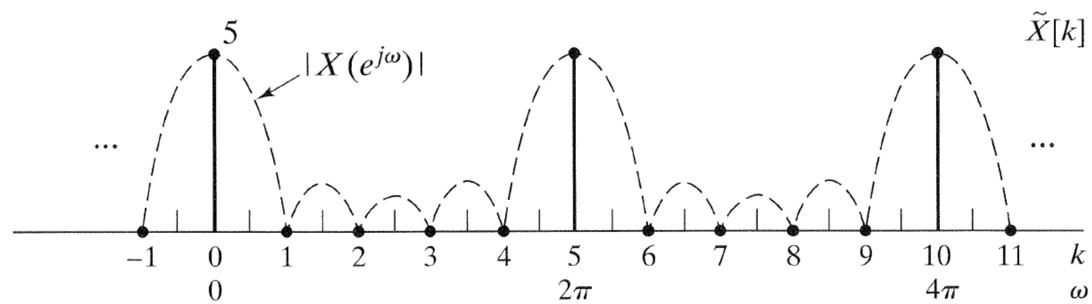
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

But in that case both X and x <u>must</u> be 0 outside $0 \rightarrow N-1$

DFT of a pulse…

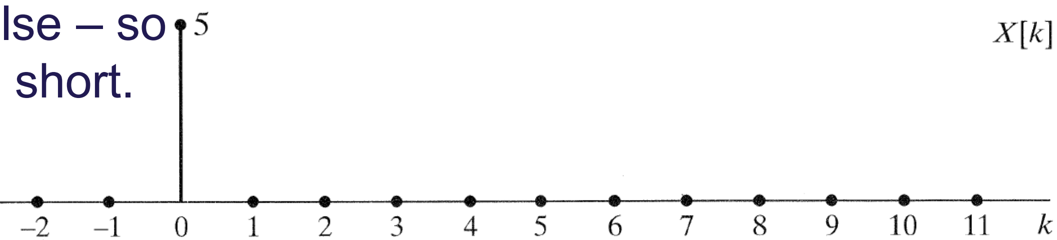
Equal sequence length and period





$X[k]$ must be 0 outside k=*0..N-1*

We expect to see a Sinc function but rather we see an impulse – so the period obviously is too short.



DFT of a pulse…

Period now longer than sequence length

Yes, now it looks better – and the larger we choose N, the more pronounced the Sinc-function will appear…