Signalbehandling for computer-ingeniører

ComTek-5, E22

1. Signals and Complex Numbers

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Uddrag af studieordningen 2021
Computer-teknologi

Formål:
Analyse og filtrering af signaler er en disciplin, der er en forudsætning for alle specialiseringer i elektroniske systemer. Disciplinen anvendes indenfor automation, kommunikation, multimedie systemer, m.m. Kursets formål er at understøtte den studerende i at forstå centrale begreber, teorier og metoder til analyse og filtrering af digitale signaler, samt anvende teorier og metoder til analyse og filtrering af digitale signaler

Viden. Skal have viden om
• fundamentale begreber inden for diskret-tids signaler og systemer, herunder modulus og argument af komplekse tal, den komplekse eksponentialfunktion, samplings-teori, foldning, impulsrespons, overføringsfunktion samt frekvensrespons, differens-ligninger, stabilitet og kausalitet
• Z-transformationen og den inverse Z-transformation
• basale begreber i relation til analoge filtre, herunder filtertyper (LP, HP, BP og BS). Butterworth approksimationen, filter-orden, 3dB-frekvens samt poler og nulpunkter
• teorier og metoder til spektralestimering
• teorier og metoder til design af digitale filtre (IIR/FIR)
• teorierne og metodernes begrænsninger
• sammenhæng mellem analyse af signaler i tids- og frekvensdomænet
• teorier og metoder til transformation mellem forskellige domæner

Færdigheder. Skal kunne
• redegøre for tids- og frekvens-forholdene vedr. sampling af tids-kontinuerte singaler
• redegøre for sammenhængen mellem foldning, impulsrespons og overføringsfunktion
• redegøre for samt anvende Z-transformation og dens inverse
• specificere et analogt Butterworth LP-filter og redegøre for dets overføringsfunktion og frekvensrespons
• anvende teorier og metoder til spektralestimering herunder DFT/FFT
• demonstrere sammenhæng mellem frekvensopløsning, vinduesfunktioner og zero-padding
• anvende teorier og metoder til design af digitale filtre
• redegøre for betydningen af faselinearitet og gruppeløbstid
• redegøre for sammenhæng mellem filtres pol-/nulpunktsdiagrammer og frekvensrespons
• designe, implementere og teste digitale filtre ud fra givne specifikationer
• klargøre digitale filtre til praktisk implementering og herunder kunne gøre brug af hensigtsmæssig filterstruktur, kvantisering og skalering

Course Overview

1. A bit about signals and complex numbers (*)
2. Diskrete time systems and convolution (*)
3. Laplace and z transform (*)
4. Sampling theory (*)
5. Short intro to analog filters
6. Digital IIR filters, the impulse invariant method
7. Digital IIR filters, the bilinear transformation
8. Digital FIR filters, the window method
9. Frequency-transformations and -analysis
10. Implementation of digital filters
11. The Discrete Fourier Transformation (DFT)
12. The Discrete Fourier Transformation, cont.
13. The Fast Fourier Transformation (FFT)
14. Short Time Fourier Transformation (STFT)

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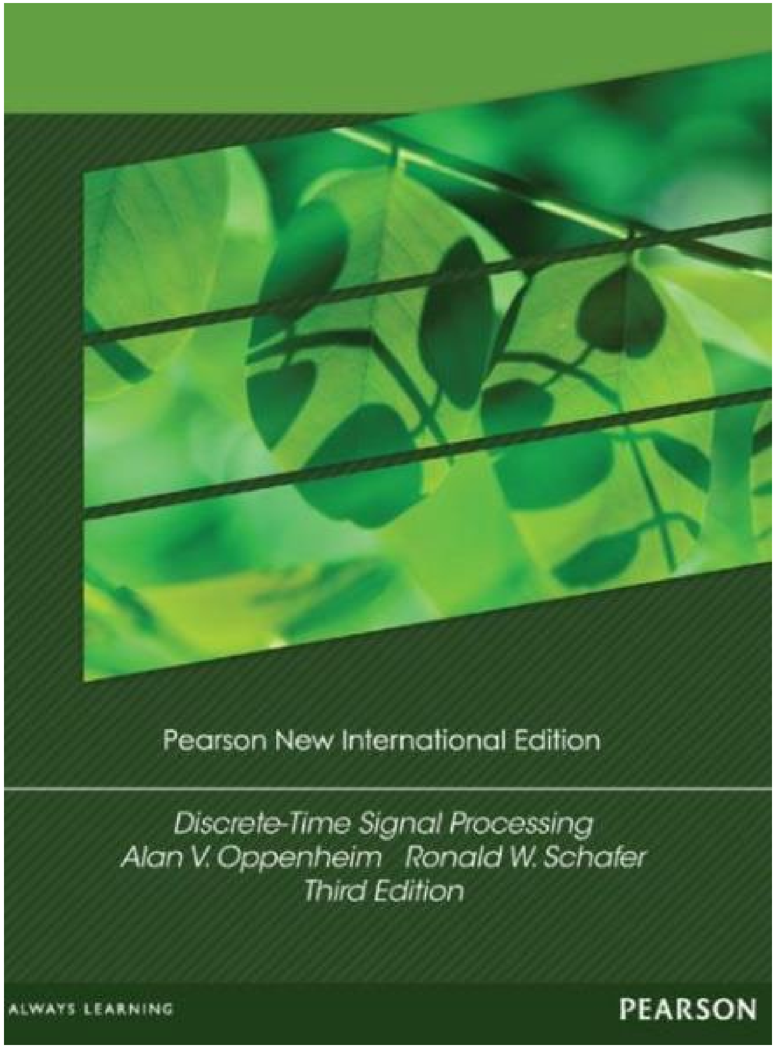
* Probably you have studied these or similar topics earlier.

Our primary text book is Oppenheim & Schafer; ”Discrete-Time Signal Processing”, now in its 3rd edition (1989, 1999, 2014). Ready for you to pick up at Factum Books

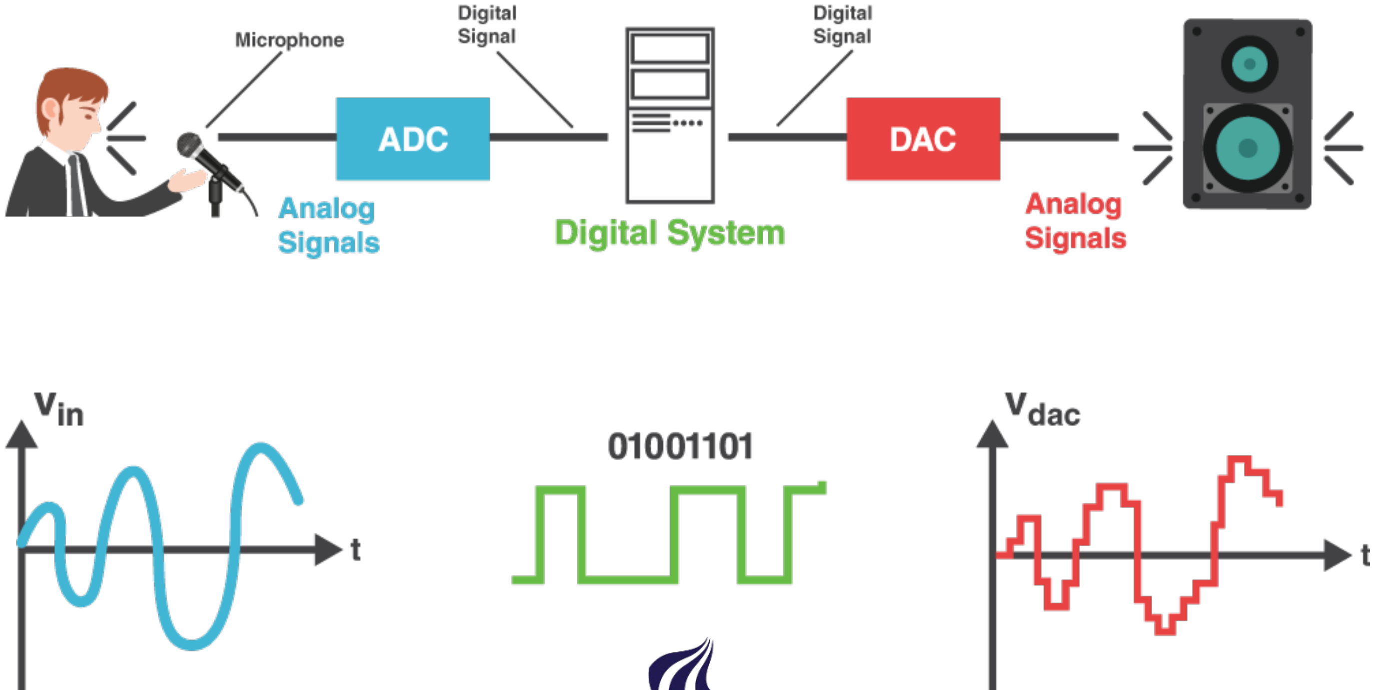


What is Signal Processing..??

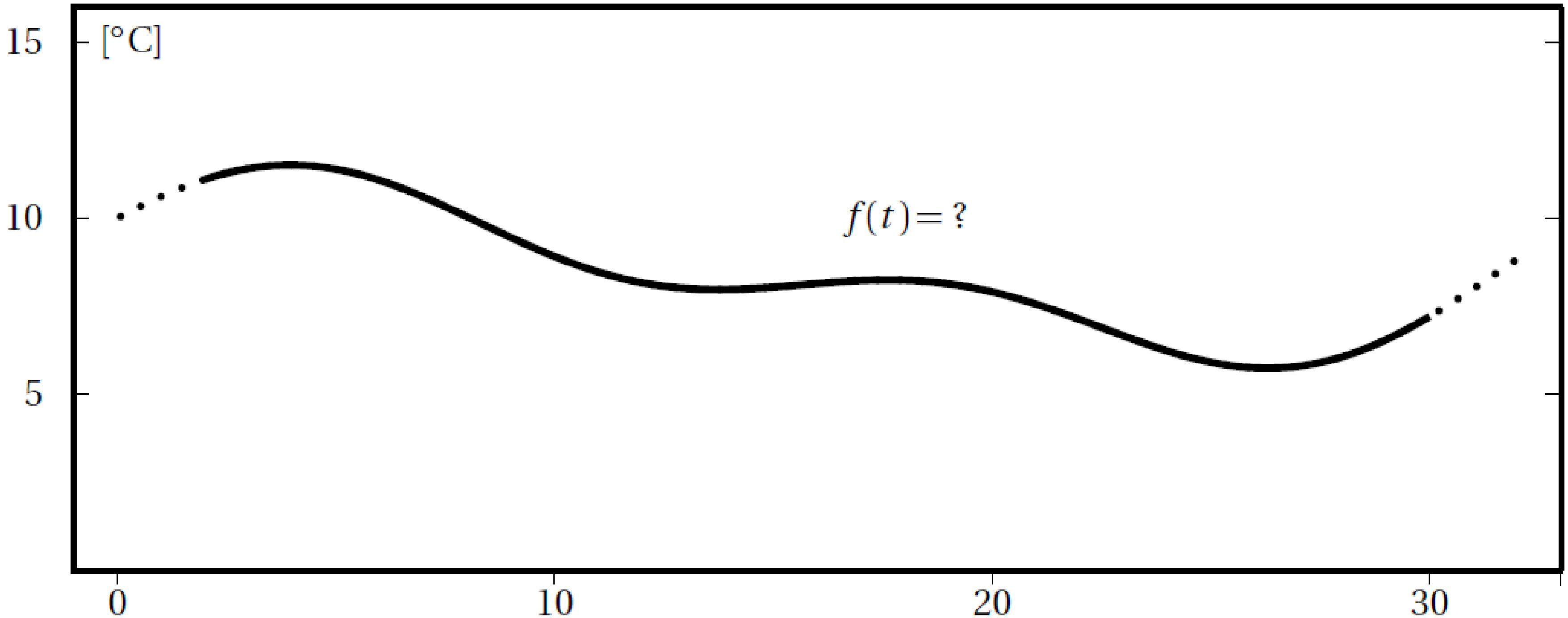
Essentially, the term Signal Processing represents all possible things that we can do to a signal, i.e., modification, analysis, transport and storage.

Signal Processing can be conducted in both the continuous time and the discrete time domain.

In this course we will address mainly Signal Processing in the discrete time domain…



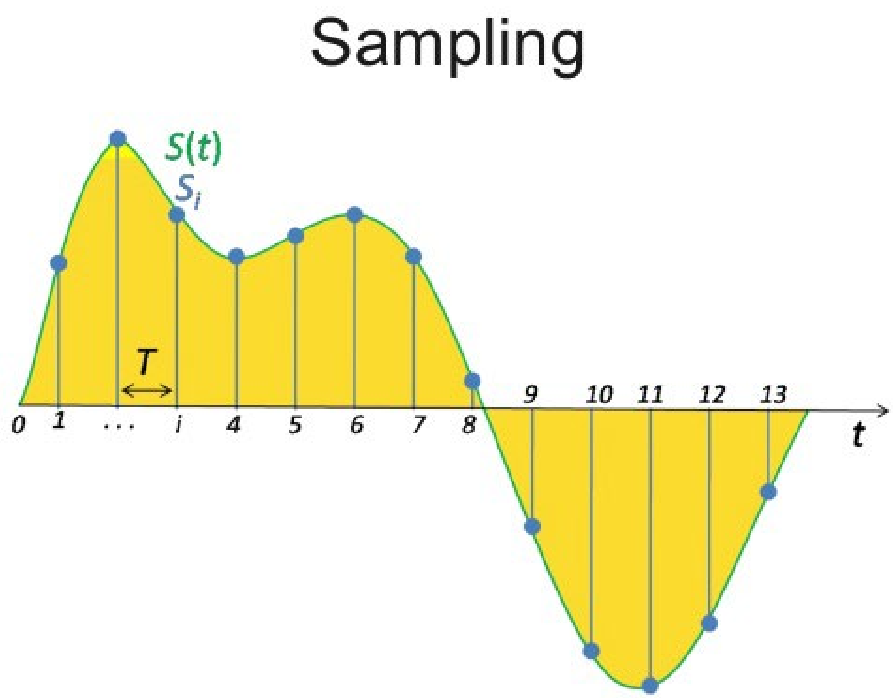
Signal Processing – an example: Temperature observation



The graph shows the temperature as a *continuous function of time* – thus we can tell the temperature at any arbitrary time t. In other words; there exists infinitely many temperature values for any time interval in the overall observation period.

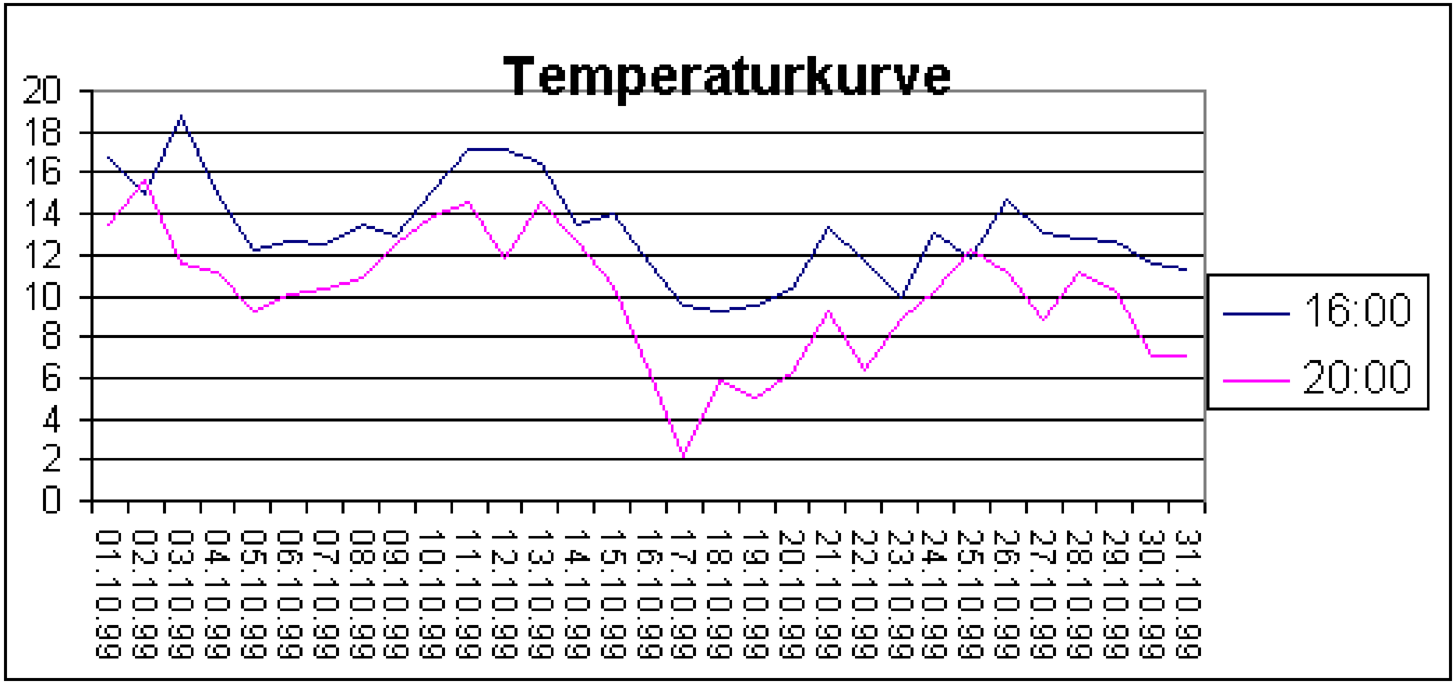
This is not necessarily the most wanted situation – in particular, if we want to represent and process the observation using a computer. And that is what we want..!

We make the observation discrete in time



”**Sampling**” is a proces where the value of the signal is registered at time instances equally spaced on the time axis (in Danish this is called **tids-ækvidistante Tidspunkter)**. For any given time interval, the signal is therefore known at a finite number of time instances, and thus we say that the signal is now ”**discrete in time**”. The time interval between to adjacent ”samples” is know as the **sample period**, T_s.

The sample period could e.g., be 1 day

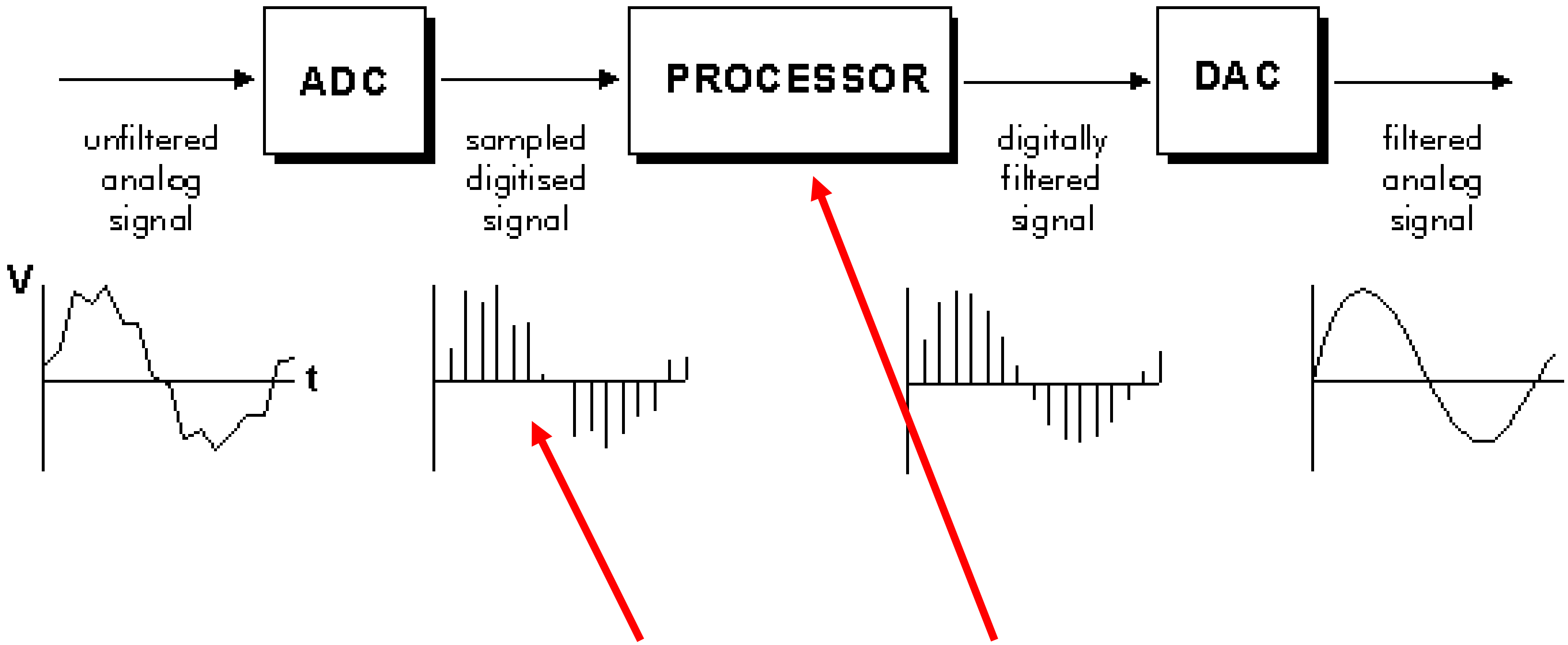


If the sampling instance is delayed (say from 16:00 to 20:00), we will see a different observation.

Note that the two curves are derived by inter-connecting the individual samples, but actually we don’t know the temperature (i.e., the value of the signal) in between two adjacent samples.

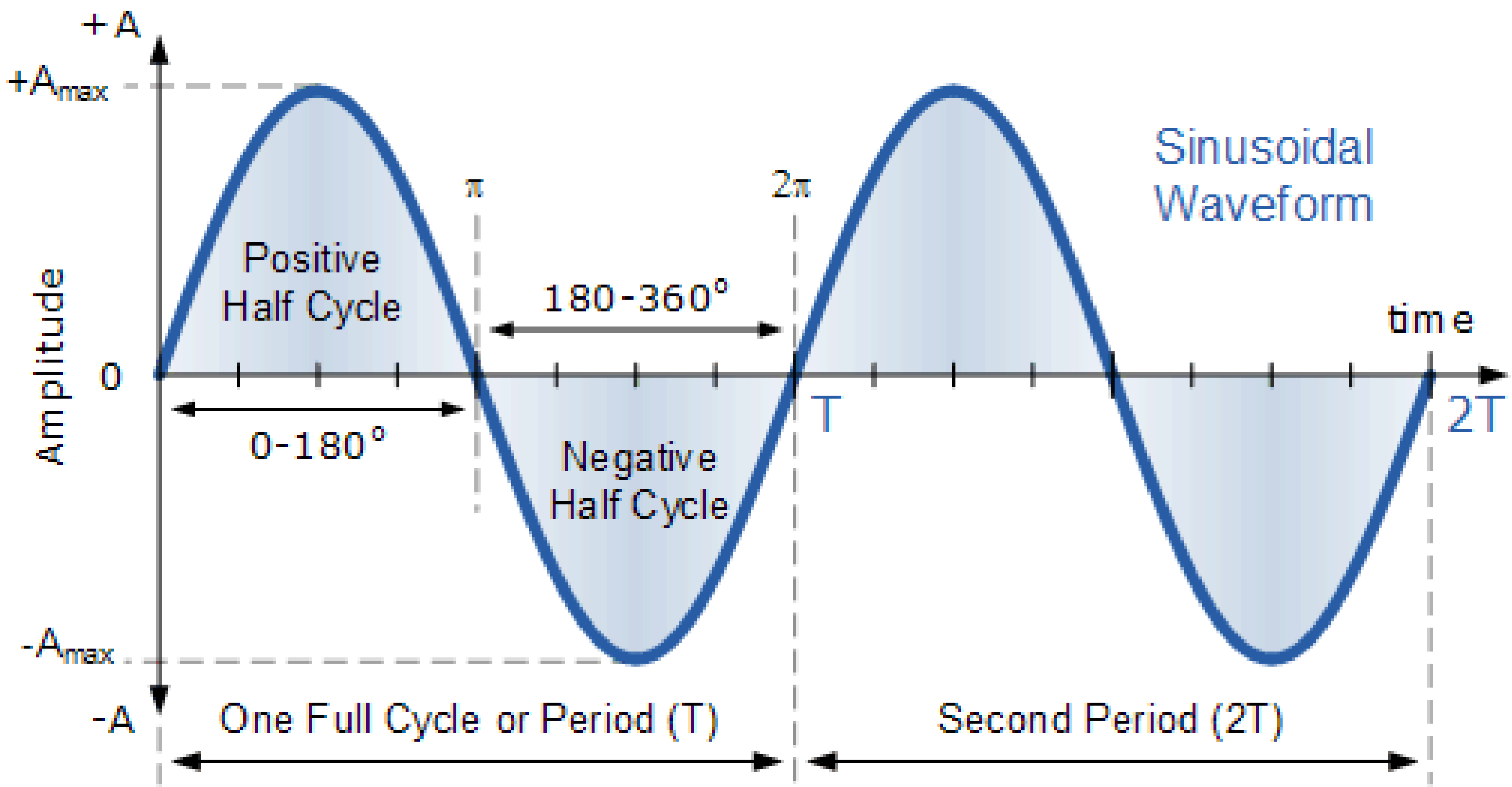
Now we want to ”do something” to the signal…

The fundamental functional blocks in a discrete tiem signal processing systems



Soon we will discover that the SIGNAL and the SYSTEM are best described by using mathematical terms and methods which to a large extent are based on complex numbers – thus we start our discussion with a ”brush up” of this topic…

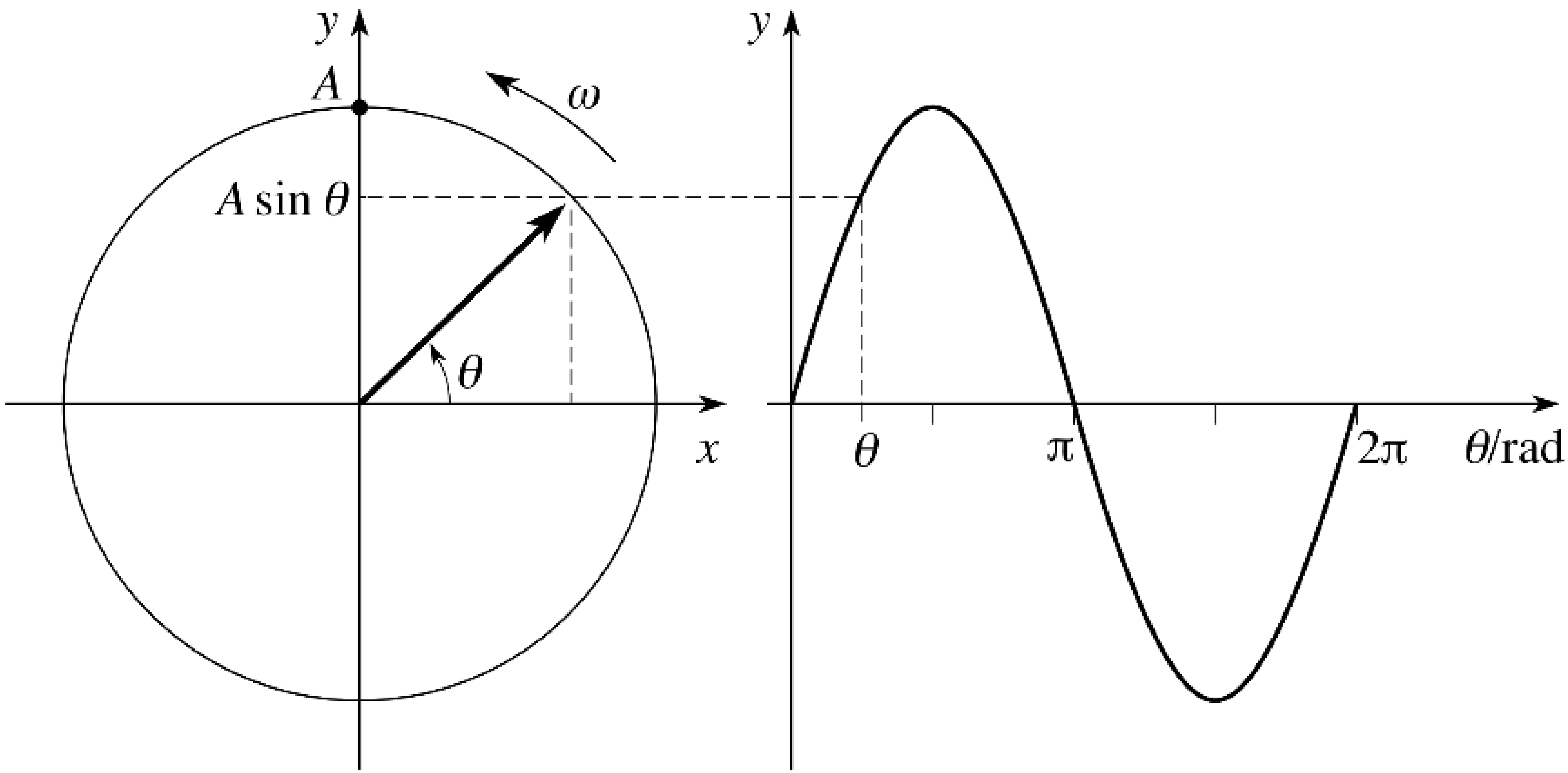
Let's start by looking at a sinusoid signal



$$\text{Frequency} = \frac{1}{\text{Periodic time}} \quad \text{or} \quad f = \frac{1}{T}\ \text{Hz}$$

$$\text{Periodic time} = \frac{1}{\text{Frequency}} \quad \text{or} \quad T = \frac{1}{f}\ \text{sec}$$

The sinusoid can be generated by plotting, as a function of the angle $\theta(t)$, the projection of a rotating point onto the y-axis



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Mathematical description of the sinusoid as a function of the time t. Here $\theta(t) = \omega t$.

amplitude
vertical scaling

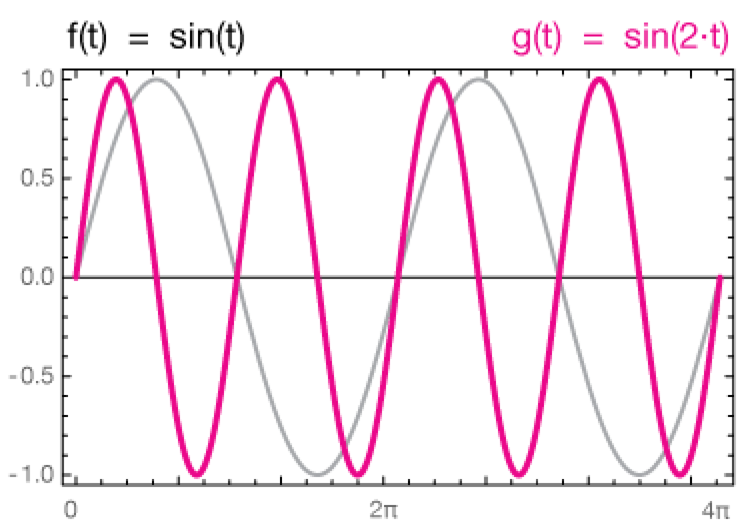
phase shift
horizontal translation

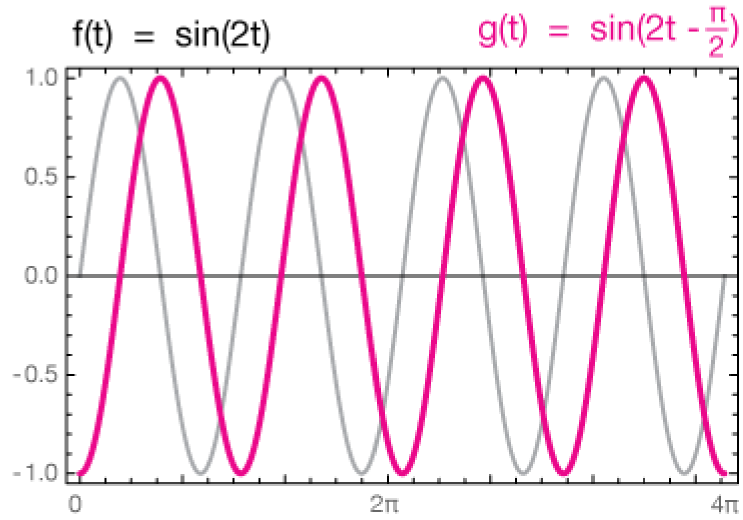
$$f(t) = A \cdot \sin(\omega t - \phi) + k$$

frequency factor
(number of cycles per 2π rad or 360°)
horizontal scaling

vertical offset
vertical translation

Frequency and Phase





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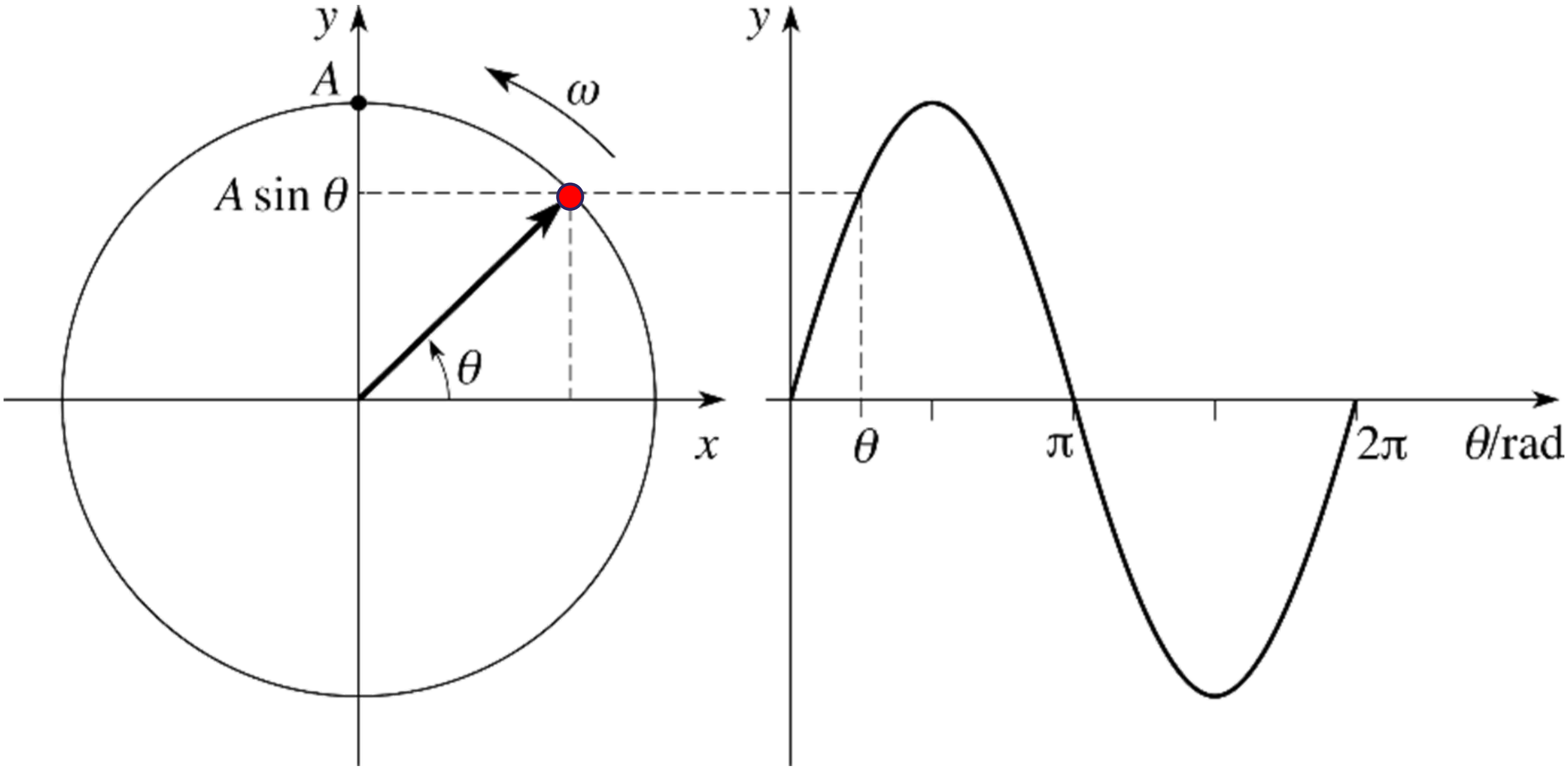
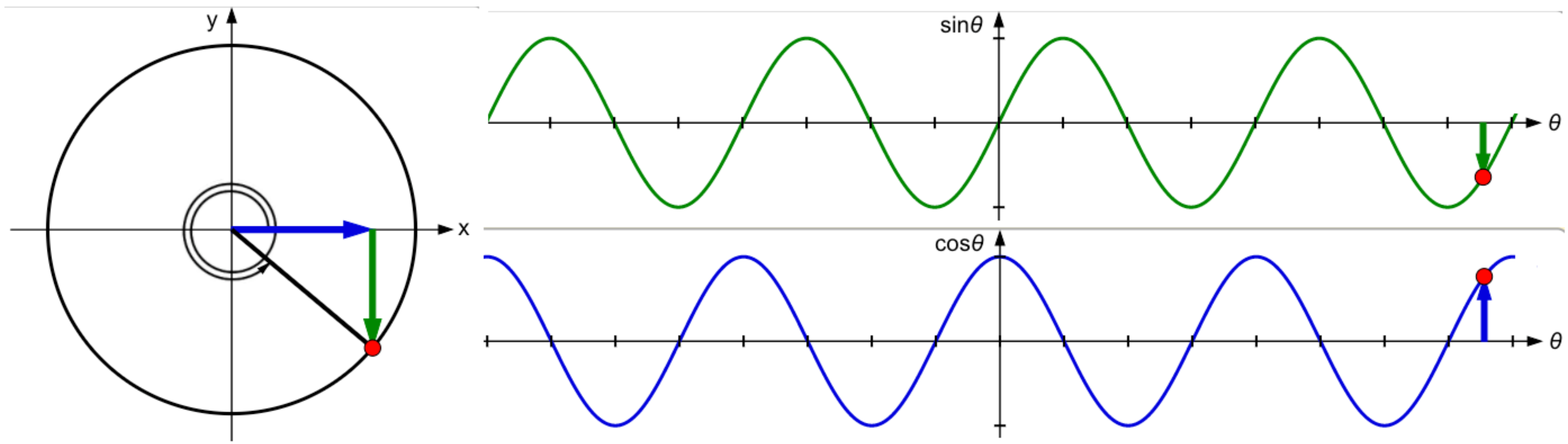
Let's look at the figure once again…



…and let's observe the projection onto BOTH the y-axis AND the x-axis

In that case, we discover two sinusoids which are shifted 90° against eachother – thus a sine and a cosine…



It therefore makes good sense to define the red dot on the circle as a signal which consists of two sinusois which are shifted 90° against each other.

In general, the red-dot-signal therefore has an x- as well as a y-component. We are now inspired to conclude that the signal can be seen as a point in the complex plane, rotating with a given angular velocity on a circle with a given radius.

Fundamentals of Complex Numbers

Normally, we say that the equation $x^2 = -1$ has no solutions.

However, let's assume that a solution exists, and this solution is denoted j.

The number j is known as *the imaginary unit.*

So, we can conclude that $j^2 = -1$.

Mathematicians normally use i for the imaginary unit – and engineers use j.

A Complex Number is a number having the form $a + jb$ where both a and b are real numbers.

Two complex numbers $a + jb$ and $c + jd$ are identical if and only if $a = c$ and $b = d$.

The set of complex numbers is denoted $\mathbb{C}$.

Now, let $a + jb$ be a complex number. We denote a as the real part and b as the imaginary part of the number.

Nornally, we use the short form z for the complex number, i.e.,

$z = \text{Re } z + j \text{ Im } z$.

If Im $z = 0$, the number z is real.

If Re $z = 0$, the number z is (purely) imaginary.

Addition of complex numbers:

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

Subtraction of complex numbers:

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

Multiplication of complex numbers:

$$(a + jb) \cdot (c + jd) = (ac - bd) + j(bc + ad)$$

So, multiplication is done by using traditional multiplication rules, and then utilize that $j^2 = -1$.

Division of complex numbers:

$$\frac{a+jb}{c+jd}$$

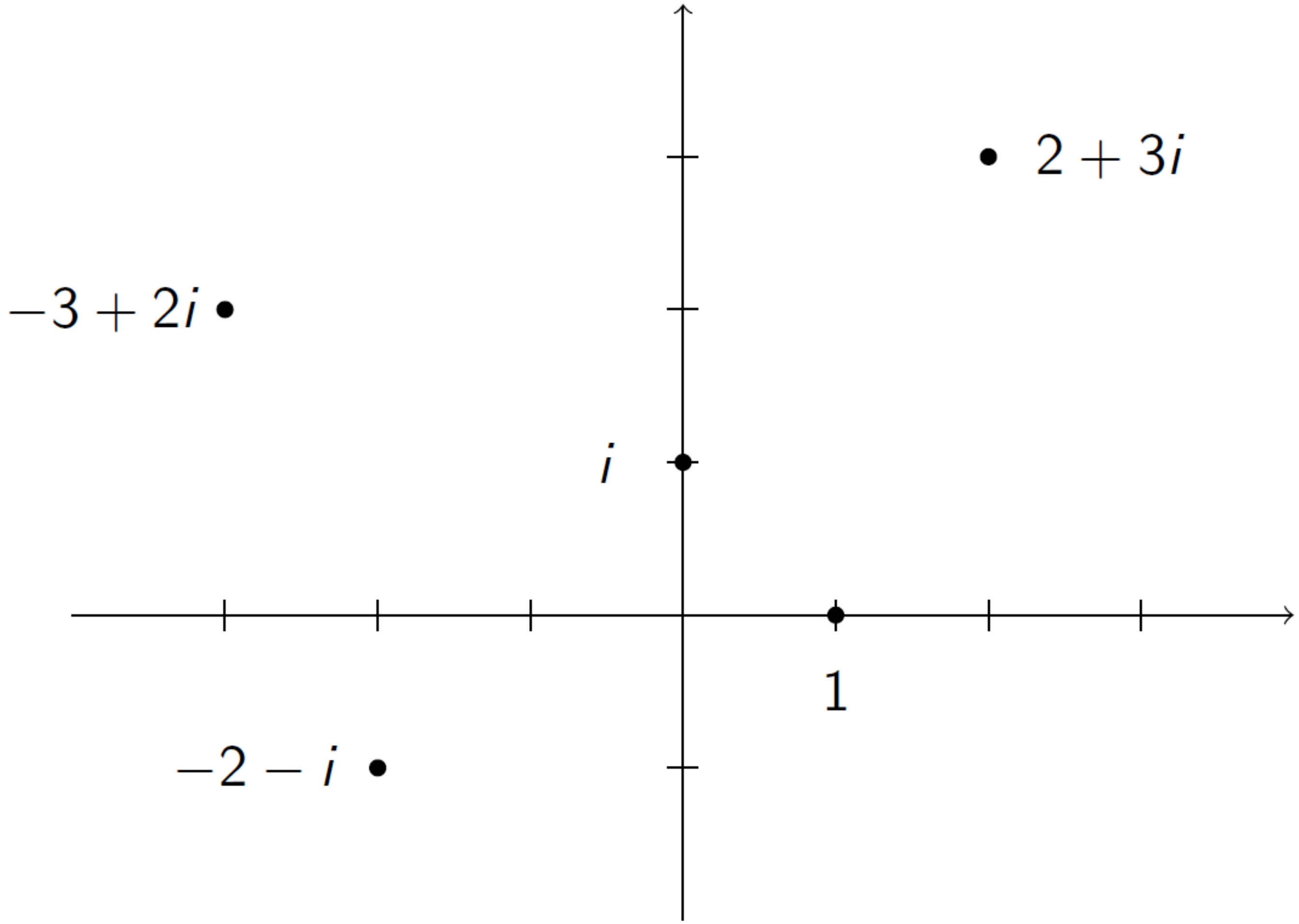
First we extend the fraction with $(c-jd)$, i.e.,

$$\frac{a+jb}{c+jd}=\frac{(a+jb)(c-jd)}{(c+jd)(c-jd)}$$

$$=\frac{(ac+bd)+j(bc-ad)}{c^2+d^2}$$

$$=\frac{ac+bd}{c^2+d^2}+j\frac{bc-ad}{c^2+d^2}$$

A complex number $a + jb$ is often identified as a point (a, b) in the plane.



The absolute value, normally also known as the modulus, of a complex number $z = a + jb$ is given as

$|z| = \sqrt{a^2 + b^2}$.

Geometrically, this is the distance from z to Origo.

If $z = a + j0$, i.e., a real number, then $|z| = |a|$.

If $z = 0 + jb$, i.e., an imaginary number, then $|z| = |b|$.

In the special case where $z = 0$, we have $|z| = 0$.

Now, given two complex numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, then let's calculate the number $|z_1 - z_2|$;

$$|z_1 - z_2| = |\ (a_1 + jb_1) - (a_2 + jb_2)| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

This is the formula for the distance between the points (a_1, b_1) and (a_2, b_2) in the complex plane.

Therefore, $|z_1 - z_2|$ is the distance between the two complex numbers z_1 and z_2 when they are considered as points in the complex plane.

Example: Let z_0 be a complex number, then the equation

$$|z - z_0| = r$$

specifies all the points in the plane, which have a distance r to z_0.

Therefore, the equation specifies a circle with radius r and centre z_0.

This will become very essential for our discussion of LTI systems…

Given a complex number $z = a + jb$.

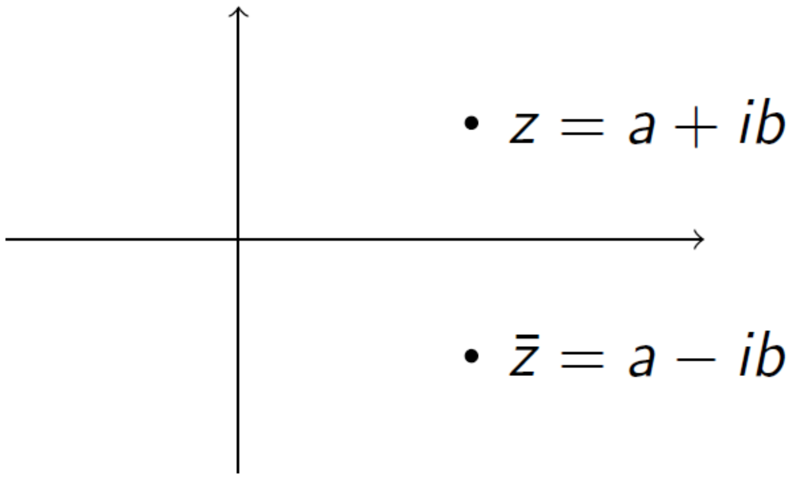
Then the complex conjugated number $\bar{z}$ is given as; $\bar{z} = a - jb$.

Examples:

$$\overline{(-5 - j)} = -5 + j$$

$$\bar{8} = 8$$

From a geometrical point of view, complex conjugation corresponds to a mirroring around the x-axis.



The complex conjugated number, $\bar{z}$, can be used to find modulus squared of z;

$z\bar{z} = (a + jb)(a - jb) = a^2 + b^2 = |z|^2$.

Also, recall how we conducted division of complex numbers;

$$\frac{z_1}{z_2} = \frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)}$$

See p.20

$$= \frac{(ac+bd)+j(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + j\frac{bc-ad}{c^2+d^2}$$

$$= \frac{ac+bd}{|z_2|^2} + j\frac{bc-ad}{|z_2|^2} = \frac{z_1\bar{z}_2}{|z_2|^2}$$

Other relevant and useful formulas…

The conjugated of a sum: $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

The conjugated of a product: $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

The conjugated of a fraction: $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

The conjugated of a conjugated: $\overline{(\bar{z})} = z$

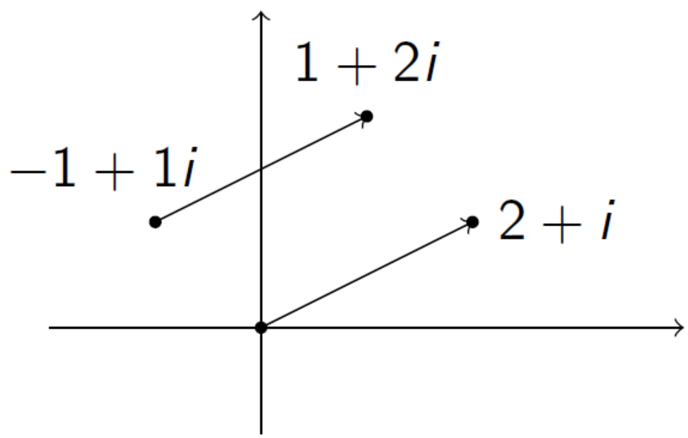
The modulus of a conjugated: $|\bar{z}| = |z|$

A complex number expressed as a vector

A vector is an arrow with a certain length and a certain direction. A vector is characterized by two coordinates (a, b) which specify the length of the vector in the direction of the x-axis and the y-axis, respectively.

The complex number $a + jb$ can be identified by the vector with the coordinates (a, b).

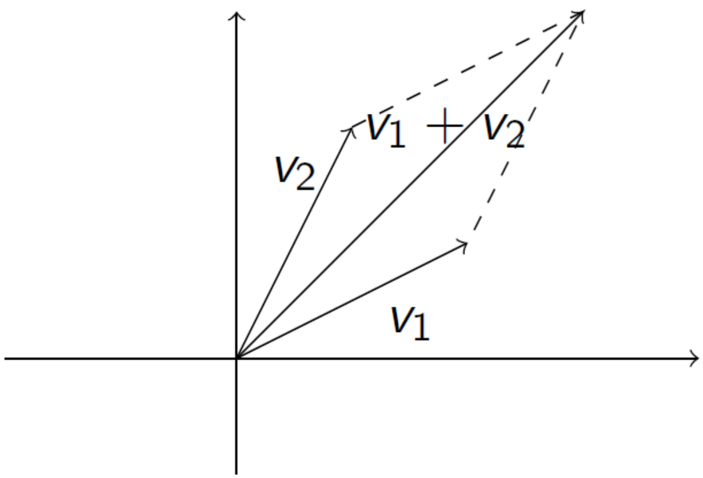
Below is shown two different positions of the vector corresponding to $2 + j$.

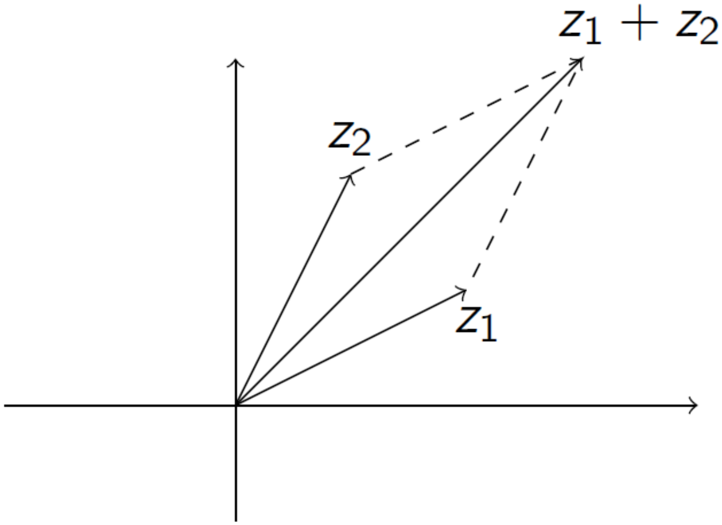


Given two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. The corresponding vectors are denoted (x_1, y_1) and (x_2, y_2).

Conducting the vector addition $(x_1, y_1) + (x_2, y_2)$ is equivalent to conducting the addition of the two complex numbers;

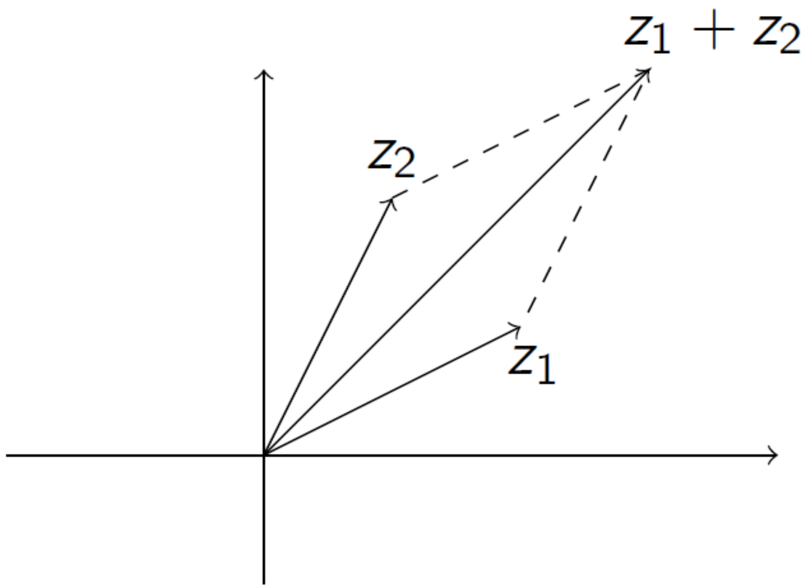
$z_1 + z_2 = (x_1 + x_2) + j\ (y_1 + y_2)$





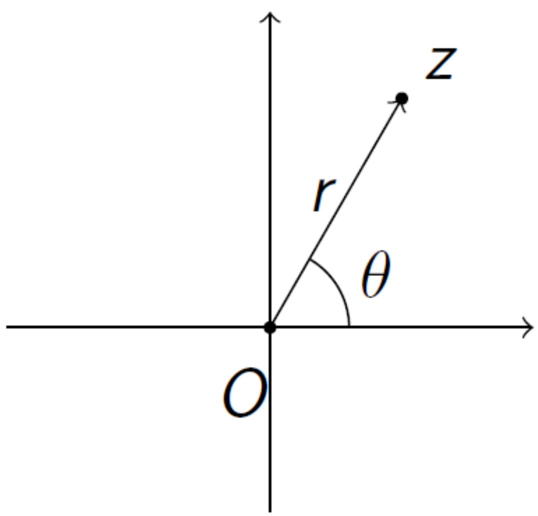
Now, this geometric interpretation of adding two complex numbers leads to ”trekants-uligheden” (derived from Cauchy-Schwarz' inequality);

$$|z_1 + z_2| \leq |z_1| + |z_2|$$



Similarly, we have that $|z_1| - |z_2| \leq |z_1 - z_2|$

The complex number z can also be defined by its polar coordinates (r, θ), where $r = |z|$ is the distance from Origo. For complex number, $r \geq 0$.



The angle θ, against the x-axis, is denoted the *argument* (or the phase) of z.

The argument is given up to an interger multiple of 2π.

We'll let arg(z) denote the set of angles corresponding to the point z.

Assuming that z is given by the polar coordinates (r, θ_0), then

arg$(z) = \{\theta_0 + 2k\pi, k = 0, \pm 1, \pm 2, \dots\}$.

Example:

arg$(1 + j) = = \{ \frac{\pi}{4} + 2k\pi, k = 0, \pm 1, \pm 2, \dots\}$.

If you want to specify an unambiguous representation of z using polar coordinates, you must choose one particular branch of the argument.

Normally, one would choose $\theta \in]-\pi, \pi]$. The argument for z located in this interal is denoted Arg z. In most signal processing literature, this value is know as the Principal Value of the argument.

If instead one would choose the branch of the argument which is in the interval $]\tau, \tau + 2\pi]$, then it is denoted $\arg_\tau z$.

Using polar coordinates, we can now write $z = x + jy$ in polar form:

$$z = r(\cos\theta + j\sin\theta)$$

Example: Write $1 + j\sqrt{3}$ in polar form.

We find that
$r = \sqrt{1^2 + \sqrt{3}^2} = 2$, and
$\tan\theta = \frac{\sqrt{3}}{1}$.

The last equation has two solutions in $]-\pi, \pi]$; $\theta = \frac{\pi}{3}$ and $\theta = -\frac{2\pi}{3}$, where the first one corresponds to the point. Thus we can write

$$1 + j\sqrt{3} = 2(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3})$$

Two complex numbers are given in polar form;

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$$

Then we have

$$\begin{aligned} z_1 z_2 &= r_1 r_2((\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + j(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)) \\ &= r_1 r_2((\cos(\theta_1 + \theta_2) + j(\sin\theta_1 + \theta_2)) \end{aligned}$$

From this, we can derive two important observations;

$$|z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

OBS…!!!!

Similarly, for

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2) \neq 0$$

we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + j(\sin\theta_1 - \theta_2)).$$

…can be shown by using equ. on p.20

In particular we note that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

OBS…!!!!

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Given a complex number $z = r(\cos\theta + j\sin\theta)$.

The complex conjugated is then $\bar{z} = r(\cos(\theta) - j\sin(\theta)) = r(\cos(-\theta) + j\sin(-\theta))$.

Beside, we have that

$$\frac{1}{z} = \frac{1}{r(\cos\theta + j\sin\theta)}$$

$$= \frac{1}{r}\frac{1}{\cos\theta + j\sin\theta}$$

$$= \frac{1}{r}\frac{\cos\theta - j\sin\theta}{(\cos\theta + j\sin\theta)(\cos\theta - j\sin\theta)}$$

$$= \frac{1}{r}(\cos\theta - j\sin\theta)$$

$$= \frac{1}{r}(\cos(-\theta) + j\sin(-\theta))$$

So, in conclusion, $\frac{1}{z}$ is parallel to $\bar{z}$.

A few words about the complex exponential function

$$z = x + iy$$

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of e^z in terms of the real functions e^x, $\cos y$, and $\sin y$ is

(1)
$$e^z = e^x(\cos y + i \sin y).$$

This definition is motivated by the fact the e^z ***extends*** the real exponential function e^x of calculus in a natural fashion. Namely:

(A) $e^z = e^x$ for real $z = x$ because $\cos y = 1$ and $\sin y = 0$ when $y = 0$.

(B) e^z is analytic for all z. (Proved in Example 2 of Sec. 13.4.)

(C) The derivative of e^z is e^z, that is,

(2)
$$(e^z)' = e^z.$$

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A few words about the complex exponential function

Further Properties. A function $f(z)$ that is analytic for all z is called an **entire function.** Thus, e^z is entire. Just as in calculus the ***functional relation***

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \tag{3}$$

holds for any $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Indeed, by (1),

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2).$$

Since $e^{x_1}e^{x_2} = e^{x_1+x_2}$ for these *real* functions, by an application of the addition formulas for the cosine and sine functions (similar to that in Sec. 13.2) we see that

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos (y_1 + y_2) + i \sin (y_1 + y_2)] = e^{z_1+z_2}$$

as asserted. An interesting special case of (3) is $z_1 = x$, $z_2 = iy$; then

$$e^z = e^x e^{iy}. \tag{4}$$

Furthermore, for $z = iy$ we have from (1) the so-called **Euler formula**

$$e^{iy} = \cos y + i \sin y. \tag{5}$$

OBS…!!!!!!!

Hence the **polar form** of a complex number, $z = r(\cos \theta + i \sin \theta)$, may now be written

$$z = re^{i\theta}. \tag{6}$$

Signals in the discrete time domain

The *unit sample sequence* (Figure 2.3a) is defined as the sequence

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases}$$

In the discrete time domain, we denote signals as ”sequences”…

The *unit step sequence* (Figure 2.3b) is given by

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Exponential sequences are extremely important in representing and analyzing linear time-invariant discrete-time systems. The general form of an exponential sequence is

$$x[n] = A\alpha^n. \qquad (2.11)$$

If A and α are real numbers, then the sequence is real. If $0 < \alpha < 1$ and A is positive, then the sequence values are positive and decrease with increasing n, as in Figure 2.3(c). For $-1 < \alpha < 0$, the sequence values alternate in sign, but again decrease in magnitude with increasing n. If $|\alpha| > 1$, then the sequence grows in magnitude as n increases.

Sinusoidal sequences are also very important. A sinusoidal sequence has the general form

$$x[n] = A\cos(\omega_0 n + \phi), \qquad \text{for all } n, \qquad (2.13)$$

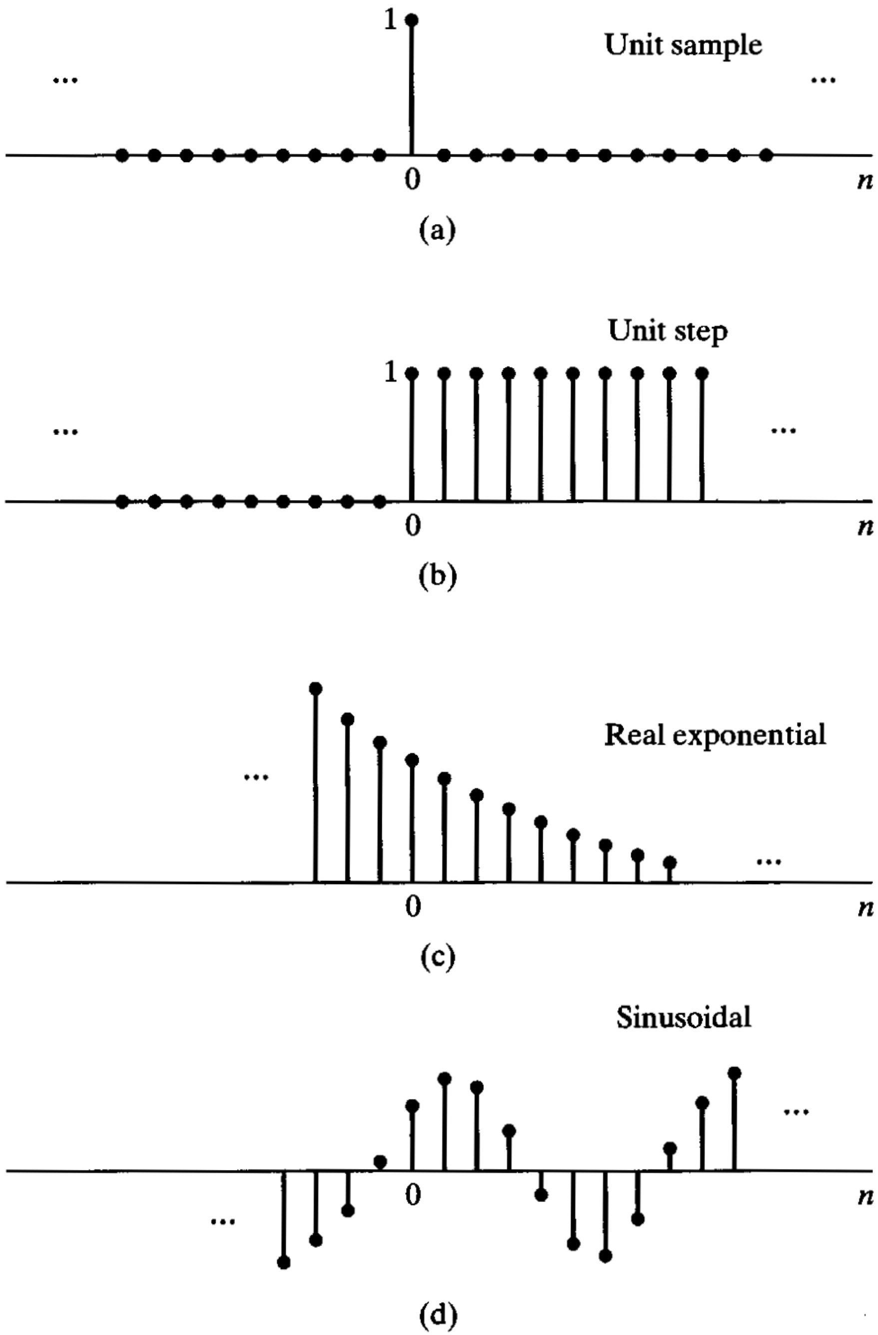


Figure 2.3 Some basic sequences. The sequences shown play important roles in the analysis and representation of discrete-time signals and systems.

The complex exponential sequence

The exponential sequence $A\alpha^n$ with complex α has real and imaginary parts that are exponentially weighted sinusoids. Specifically, if $\alpha = |\alpha|e^{j\omega_0}$ and $A = |A|e^{j\phi}$, the sequence $A\alpha^n$ can be expressed in any of the following ways:

$$
\begin{aligned}
x[n] = A\alpha^n &= |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} \\
&= |A|\,|\alpha|^n e^{j(\omega_0 n+\phi)} \qquad (2.14)\\
&= |A|\,|\alpha|^n \cos(\omega_0 n + \phi) + j|A|\,|\alpha|^n \sin(\omega_0 n + \phi).
\end{aligned}
$$

When $|\alpha| = 1$, the sequence is referred to as a *complex exponential sequence* and has the form

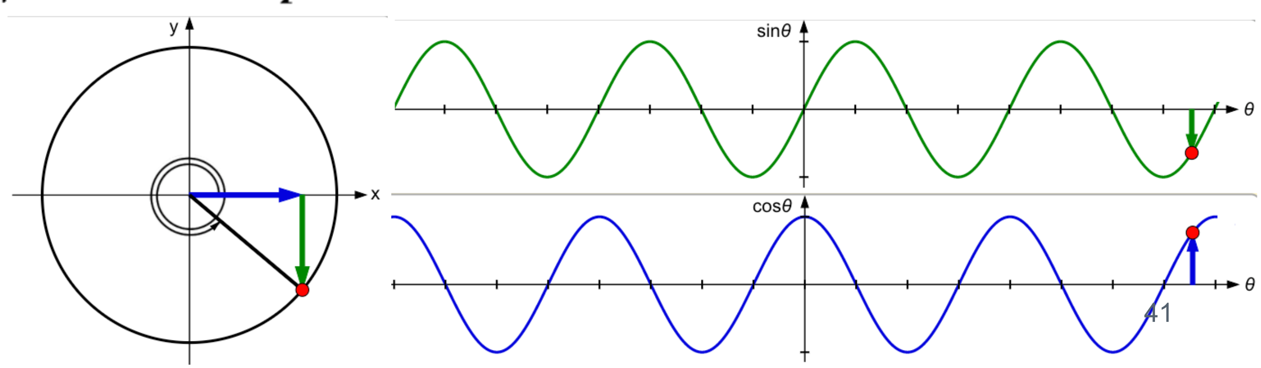
$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi); \qquad (2.15)$$

that is, the real and imaginary parts of $e^{j\omega_0 n}$ vary sinusoidally with n. By analogy with the continuous-time case, the quantity ω_0 is called the *frequency* of the complex sinusoid or complex exponential, and ϕ is called the *phase*.

The "red dot" is therefore

$$\alpha = |\alpha|e^{j\omega_0}$$

and it represents the Unit Circle when $|\alpha| = 1$.



A signal with the frequency $(\omega_0 + 2\pi)$ cannot be distinguished from a signal with the frequency ω_0

An important difference between continuous-time and discrete-time complex sinusoids is seen when we consider a frequency $(\omega_0 + 2\pi)$. In this case,

$$\begin{aligned} x[n] &= Ae^{j(\omega_0+2\pi)n} \\ &= Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n}. \end{aligned} \tag{2.16}$$

More generally, we can easily see that complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$, where r is an integer, are indistinguishable from one another. An identical statement holds for sinusoidal sequences. Specifically, it is easily verified that

$$\begin{aligned} x[n] &= A\cos[(\omega_0 + 2\pi r)n + \phi] \\ &= A\cos(\omega_0 n + \phi). \end{aligned} \tag{2.17}$$

The implications of this property for sequences obtained by sampling sinusoids and other signals will be discussed in Chapter 4. For now, we simply conclude that, when discussing complex exponential signals of the form $x[n] = Ae^{j\omega_0 n}$ or real sinusoidal signals of the form $x[n] = A\cos(\omega_0 n + \phi)$, we need only consider frequencies in an interval of length 2π, such as $-\pi < \omega_0 \leq \pi$ or $0 \leq \omega_0 < 2\pi$.

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OBS…!!!!!!

Periodicity

Another important difference between continuous-time and discrete-time complex exponentials and sinusoids concerns their periodicity. In the continuous-time case, a sinusoidal signal and a complex exponential signal are both periodic, with the period equal to 2π divided by the frequency. In the discrete-time case, a periodic sequence is a sequence for which

$$x[n] = x[n+N], \qquad \text{for all } n, \tag{2.18}$$

where the period N is necessarily an integer. If this condition for periodicity is tested for the discrete-time sinusoid, then

$$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi), \tag{2.19}$$

which requires that

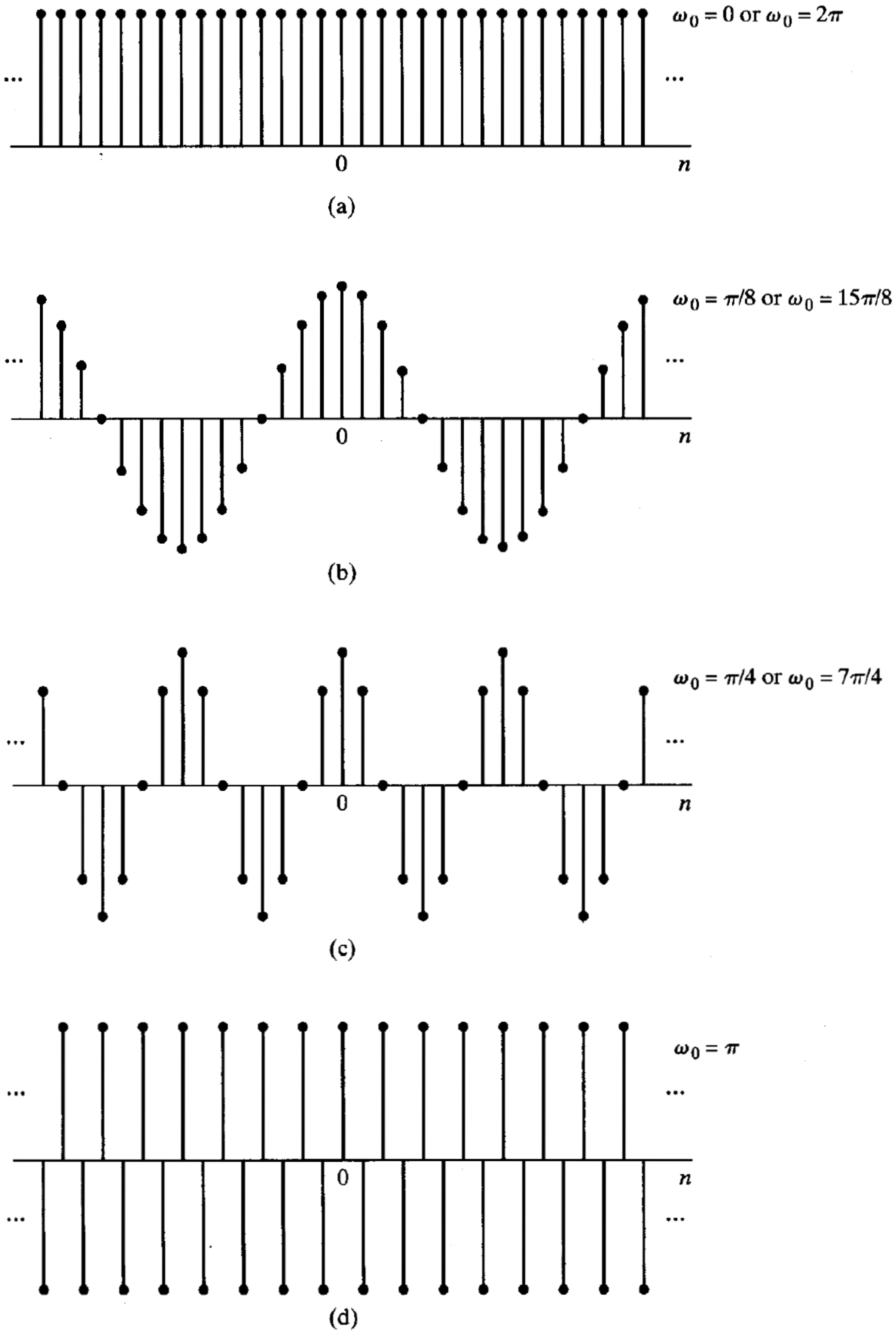
$$\omega_0 N = 2\pi k, \tag{2.20}$$

where k is an integer. A similar statement holds for the complex exponential sequence $Ce^{j\omega_0 n}$; that is, periodicity with period N requires that

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}, \tag{2.21}$$

which is true only for $\omega_0 N = 2\pi k$, as in Eq. (2.20). Consequently, complex exponential and sinusoidal sequences are not necessarily periodic in n with period $(2\pi/\omega_0)$ and, depending on the value of ω_0, may not be periodic at all.

OBS…!!!



Discrete time signals (and systems) have the property that they should always be considered as related to the sample frequency $f_s = 2\pi$.

This leads to the following noteworthy observation…

Figure 2.5 $\cos \omega_0 n$ for several different values of ω_0. As ω_0 increases from zero toward π (parts a–d), the sequence oscillates more rapidly. As ω_0 increases from π to 2π (parts d–a), the oscillations become slower.