PK

Signal Processing for Computer-Engineers

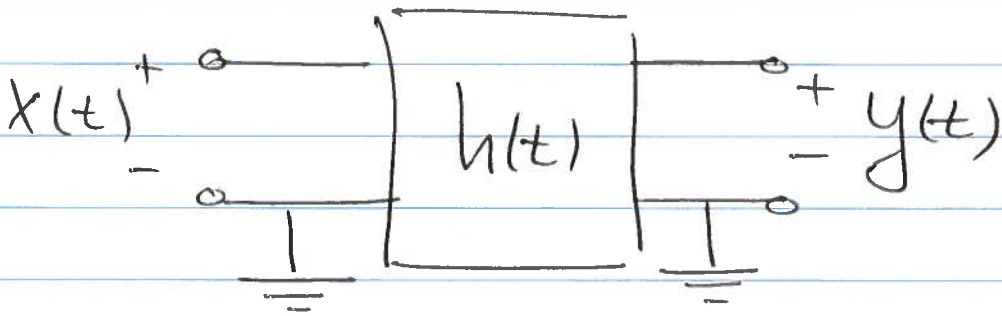
5th Lecture

A short introduction to Analog Filters

A „Filter" is a System which can eliminate specific parts of a signal while other parts are left (almost) unaffected

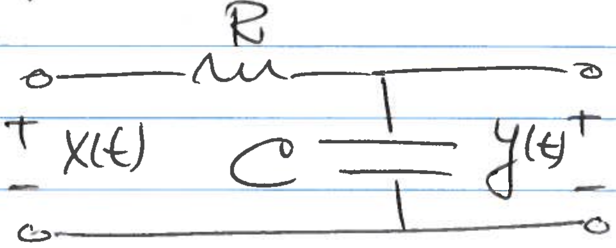
In our context, this system consists of an electrical circuit with resistors, capacitors, inductors, and eventually active components such as OpAmps.

The input is an electrical signal denoted $x(t)$. The system $h(t)$ modify $x(t)$ into an output $y(t) = f(x(t)) = h(t) * x(t)$.

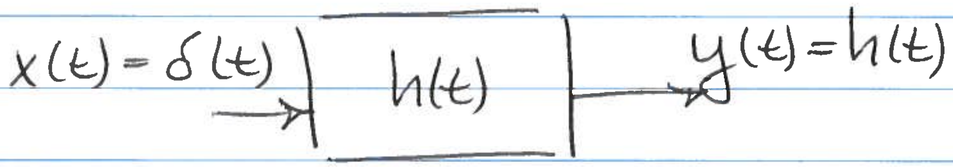


Since $x(t)$, $y(t)$, and $h(t)$ are all continuous-time functions, the filter is denoted a Continuous-time filter, or an analog filter, i.e., a specific circuit topology with an input- and an output terminal pair.

An example :



We know from previous discussions that a continuous-time system $h(t)$ which is excited with an impulse $\delta(t)$, responds with its impulse response



For any given input $x(t)$, the response can be calculated by the Convolution integral

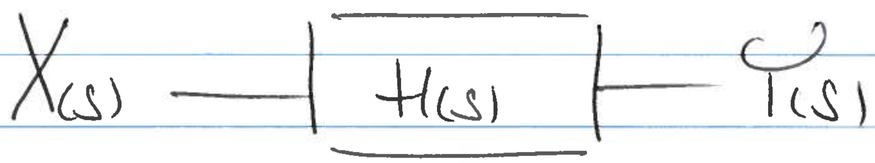
$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

Therefore, given $h(t)$ the system is completely defined/described.

However, $h(t)$ in itself does not tell us the complete story — if we want to know the system's characteristics, we need to take $h(t)$ to the frequency domain.

Keyword here is the Laplace transform.

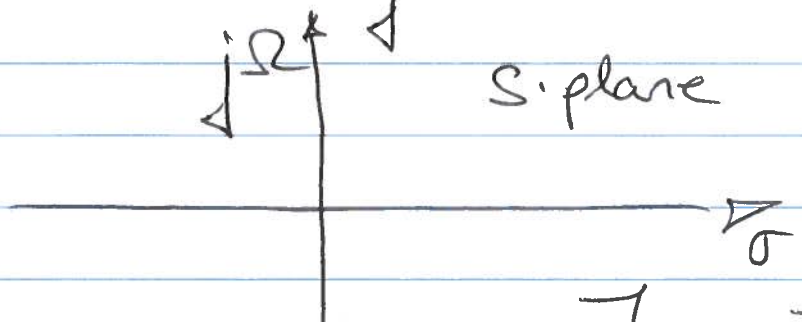
$$X(s) = \mathcal{L}\{x(t)\}, \quad Y(s) = \mathcal{L}\{y(t)\} \text{ and } H(s) = \mathcal{L}\{h(t)\}$$



$$Y(s) = H(s) \cdot X(s)$$

$H(s)$ is known as the system's Transfer function, or System function, where

$$s = \sigma + j\Omega$$



For $\sigma = 0$, $s = j\Omega$ i.e. the infinite freq. axis

$$H(s) = H(\sigma + j\Omega) = H(j\Omega)\Big|_{\sigma=0}$$

$\Omega = 2\pi f$ is the continuous angle-frequency (f is the frequency), and $H(j\Omega)$ is known as the FREQUENCY RESPONSE

Since the frequency response is a complex-valued function, it makes sense defining its modulus and argument;

$$H(j\Omega) = |H(j\Omega)| \cdot Arg(H(j\Omega))$$

$|H(j\Omega)|$: AMPLITUDE RESPONSE

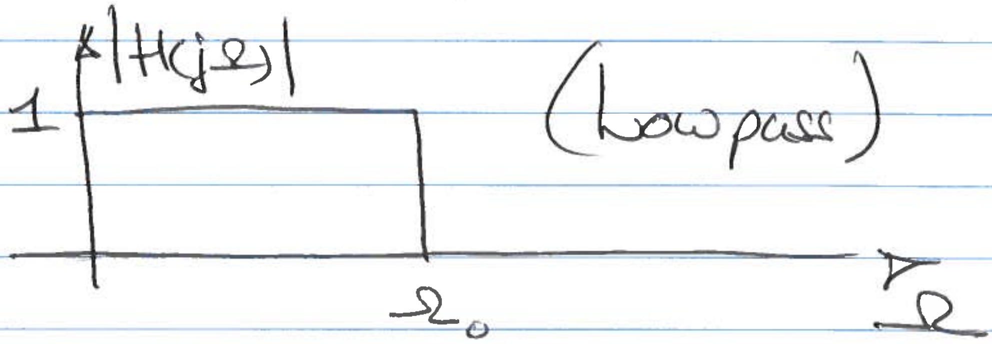
$Arg(H(j\Omega))$: PHASE RESPONSE

Often, the amplitude response is expressed as

$$20 \cdot \log_{10} |H(j\Omega)| \quad [dB]$$

BUTTERWORTH APPROXIMATION

In the ideal situation, we would like to design a „Brickwall Filter", i.e., a filter which has an amplitude response such as;



(Lowpass)

This is: $|H(j\Omega)| = \begin{cases} 1 & |\Omega| \leq \Omega_0 \\ 0 & \text{otherwise} \end{cases}$

Unfortunately, such a filter exists only as a mathematical definition — it cannot be implemented physically.

Therefore, we want to find an approximation to the ideal amplitude response, $|H(j\Omega)|$

Even better though, we may use $|H(j\Omega)|^2$ because this is rational function in Ω^2, which may be an advantage when we have to find the poles in $H(j\Omega)$.

Therefore, let's define the squared amplitude response ;

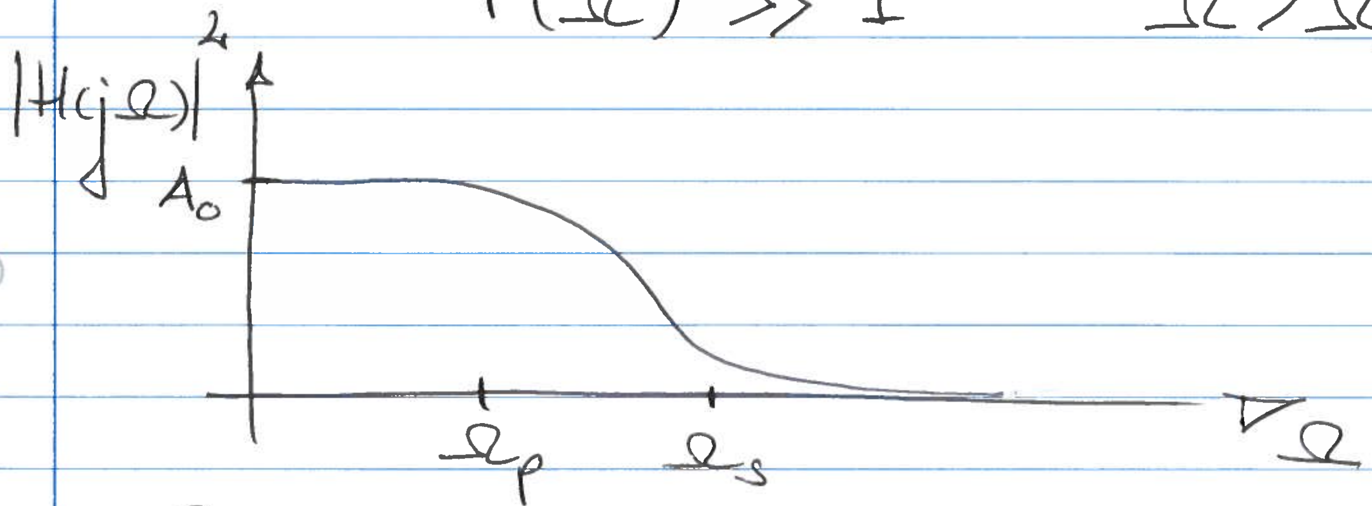
$$|H(j\Omega)|^2 = \frac{A_o}{1+F(\Omega^2)}$$

where A_o is the DC gain $(\Omega=0)$.

Find an expression for $F(\Omega^2)$ such that

$$F(\Omega^2) \ll 1 \qquad 0<\Omega<\Omega_p$$

$$F(\Omega^2) \gg 1 \qquad \Omega>\Omega_s$$



One possible solution could be

$F(\Omega^2) = \Omega^{2n}$, where n denotes the FILTER ORDER

$$\left|H(j\Omega)\right|^2 = \frac{1}{1+\Omega^{2n}}\Bigg|_{A_0=1}.$$

This function is known as the squared amplitude response for the normalized n'th order Butterworth Lowpass Filter.

it has the following characteristics;

- $\left|H(j0)\right|^2 = 1\ (0\,dB)$ for all n.
- $\left|H(j\cdot 1)\right|^2 = 1/2$ for all n.

$$\Downarrow$$

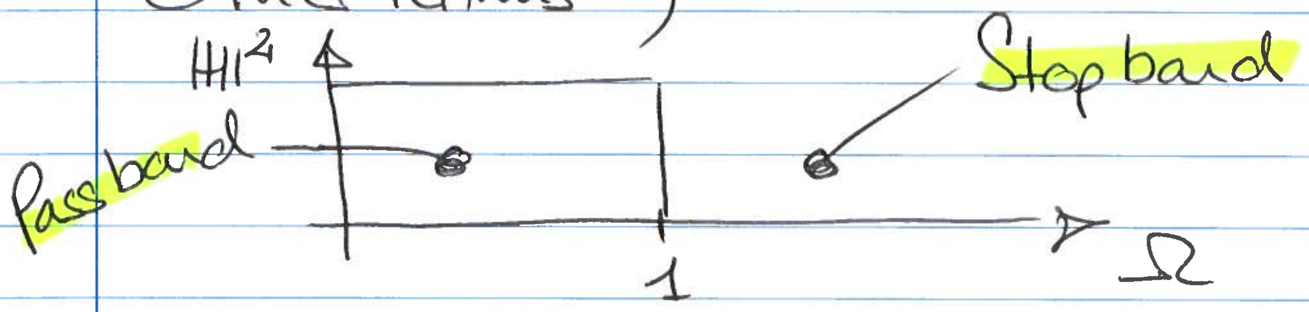
$$\left|H(j\cdot 1)\right| = 1/\sqrt{2}$$

$$\Downarrow$$

$$\left|H(j\cdot 1)\right|_{dB} = 20\cdot \log_{10}\left(\frac{1}{\sqrt{2}}\right) \simeq -3\,dB$$

The frequency $\Omega = 1$ is known as the cut-off frequency (på dansk „knæk-frekvens") or just the „3 dB frequency".

Other terms;



The approximation to the Brickwall filter becomes better as n increases;

Assume $n_1 \gg n_2$

- In the passband; $\Omega^{2n_1} \ll \Omega^{2n_2}$
- In the stopband; $\Omega^{2n_1} \gg \Omega^{2n_2}$

Applying this in $|H(j\Omega)| = \dfrac{1}{\sqrt{1+\Omega^{2n}}}$ we clearly see that the approximation to the Brickwall filter becomes better the larger the filter order n is.

Similarly, we may rewrite $|H(j\Omega)|$ in terms of a Taylor series;

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\Omega^{2n}}} = 1 - \frac{1}{2}\Omega^{2n} + \frac{3}{8}\Omega^{4n} - \frac{5}{16}\Omega^{6n} \dots$$

$\Downarrow$

$$\left.\frac{\partial^k |H(j\Omega)|}{\partial^k \Omega} = 0 \right|_{\Omega=0}, \quad k = 1, 2, \dots 2n-1$$

and

$$\left.\frac{\partial^k |H(j\Omega)|}{\partial^k \Omega} = 1/2 \right|_{\Omega=0} \quad k = 2n$$

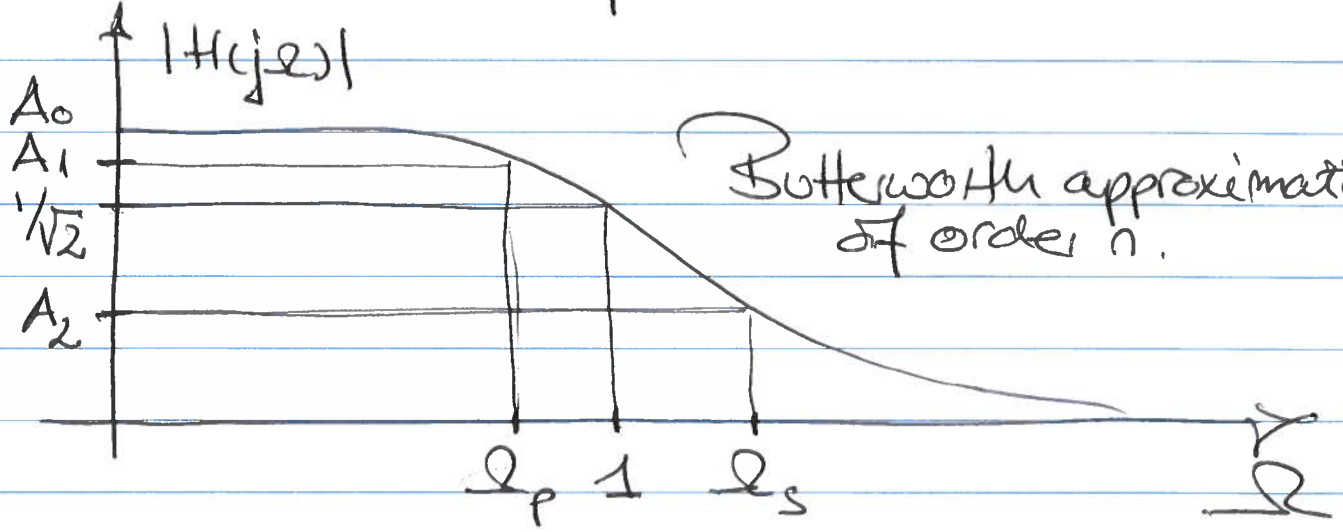
So, all derivatives for $k = 1 \dots 2n$ are equal to 0 (for $\Omega = 0$), except for one, which is a constant.

$\Downarrow$

MAXIMAL FLAT AMPLITUDE RESPONSE

Choosing the Filter order n.

Let's take as an outset the exampel in the textbook p. 29.



First we realize that both Ω_p and Ω_s will approach 1 as n increases.

Therefore, our task is to find the smallest value of n, for which given design specifications are met.

Specifications:

(Ω_p, α_p), (Ω_s, α_s) and Ω_s/Ω_p

where $\alpha_p = 20 \cdot \log_{10}(A_0/A_1)$

$\alpha_s = 20 \cdot \log_{10}(A_0/A_2)$

express the attenuation i dB.

Note that Ω_p and Ω_s so far are unknown.

Let's find an expression for the gain [dB]

$$\text{Amplification}(\Omega) = 10 \cdot \log_{10} |H(j\Omega)|^2$$

$$= 10 \cdot \log_{10} \left(\frac{\frac{A_0}{1+\Omega^{2n}}}{A_0} \right)$$

Here the amplification (gain) is normalized as compared to DC (A_0).

$\Downarrow$

$$\text{Amp}(\Omega) = -10 \cdot \log_{10} (1+\Omega^{2n}) \quad [dB]$$

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from the example we have that

- passband attenuation $\alpha_p = 1\,dB$
- stopband attenuation $\alpha_s = 25\,dB$

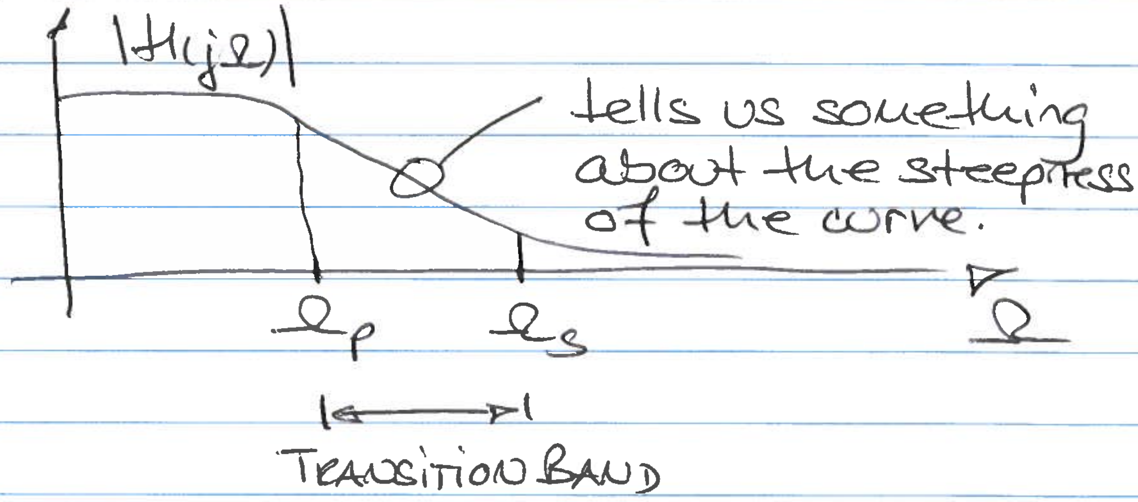
$\Downarrow$

$$\begin{cases} -10 \cdot \log_{10} (1+\Omega_p^{2n}) = -1 \\ -10 \cdot \log_{10} (1+\Omega_s^{2n}) = -25 \end{cases}$$

$\Downarrow$

$$\left\{ \begin{aligned} \Omega_p^{2n} &= 10^{0.1} - 1 \\ \Omega_s^{2n} &= 10^{2.5} - 1 \end{aligned} \right\}$$ 2 equations with 3 unknown.

In order to solve this issue, we use the „TRANSITION BAND RELATION".



$$\frac{\Omega_s}{\Omega_p} = 1.5 \quad \text{(according to the example)}$$

$$\Downarrow$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2n} = \frac{\Omega_s^{2n}}{\Omega_p^{2n}} = (1.5)^{2n}$$

$$\Downarrow$$

$$\frac{10^{2.5} - 1}{10^{0.1} - 1} = 1.5^{2n}$$

$$\Downarrow$$

$$n = 8.761\ldots$$

The order however, must be an integer

$$\Downarrow$$

$$n = \lceil 8.761\ldots \rceil = 9$$

For this value of n, we can now find Ω_p and Ω_s.

$$\Omega_p^{2n} = 10^{0.1} - 1 \qquad \text{Specs fulfilled at } \Omega_p!$$

$$\Downarrow$$

$$\Omega_p = 10^{\frac{\log_{10}(10^{0.1}-1)}{2n}} = \underline{\underline{0.928\ldots}}\Big|_{n=9}$$

and since $\frac{\Omega_s}{\Omega_p} = 1.5$, we find that

$$\Omega_s = 1.5 \cdot 0{,}928\ldots = \underline{\underline{1.392\ldots}}$$

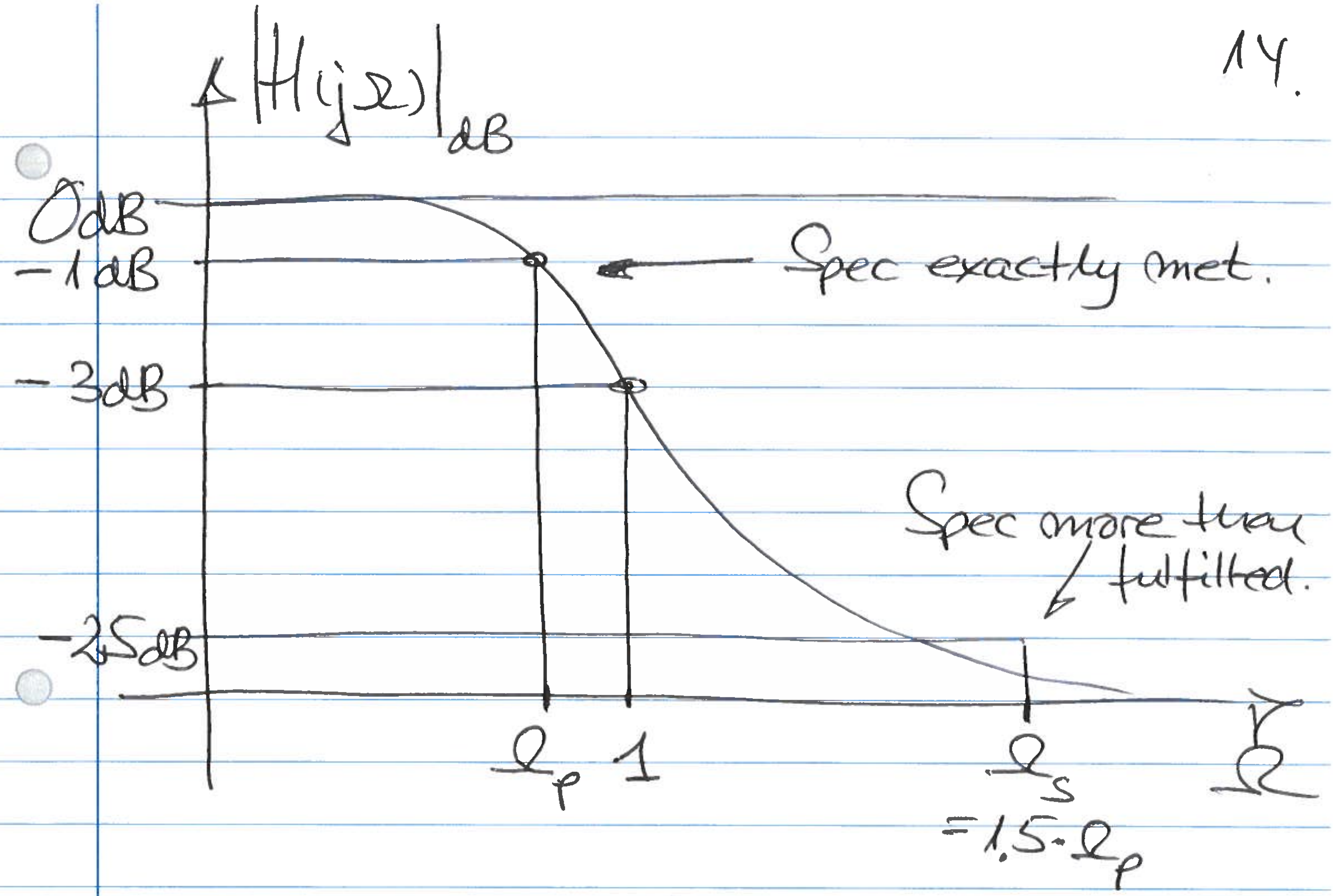
We need however, to check whether the specifications are fulfilled at Ω_s for $n=9$.

$$\alpha_s = 20 \cdot \log_{10}\left(\frac{A_0}{A_2}\right)$$

$$= 20 \cdot \log_{10} \frac{1}{\left|\frac{1}{1+\Omega_s^{2n}}\right|^{1/2}}$$

$$= 10 \cdot \log_{10}\left(1+\Omega_s^{2n}\right)$$

$$= 25{,}84 \text{ dB}\Big|_{\substack{n=9 \\ \Omega_s = 1.5\,\Omega_p}}$$



$$|H(j\Omega)|^2 = \frac{1}{1+\Omega^{18}}$$

Finally, we need to discuss how the filter should be realized.

„NETWORK FUNCTION"

Basically, the purpose is to derive a transfer function $H(s)$

Amplitude response $\rightarrow$ Transfer function.

$$|H(j\Omega)| \rightarrow H(s)$$

known!

For a complex number, we have;

$$|x|^2 = x \cdot \bar{x} \qquad (\sim x \cdot x^*)$$

$$|H(j\Omega)|^2 = H(j\Omega) \cdot H^*(j\Omega)$$

Assuming now that we evaluate the transfer function $H(s)$ on the frequency axis $s = j\Omega$, then we have;

$$|H(j\Omega)|^2 = H(s) \cdot H^*(s)\Big|_{s=j\Omega}$$

$$= H(s) \cdot H(-s)$$

Since we know $|H(j\Omega)|^2$, we can now find $H(s)$.

$$\left|H(j\Omega)\right|^2 = \frac{1}{1+\Omega^{2n}} = H(s)\cdot H(-s)\Big|_{s=j\Omega}$$

$$\Downarrow$$

$$H(s)\cdot H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2n}}\Bigg|_{s=j\Omega} \quad \left(\Omega=\frac{s}{j}\right)$$

$$\Downarrow$$

$$\begin{cases} H(s)\cdot H(-s) = \dfrac{1}{1+(-1)^n s^{2n}} \\ \\ H(s) = \dfrac{P(s)}{Q(s)} \end{cases}$$

$$\Downarrow$$

$$Q(s)\cdot Q(-s) = 1+(-1)^n\cdot s^{2n}$$

We now search for the roots in this polynomia ;

$$1+(-1)^n\cdot s^{2n} = 0$$

$$\Downarrow$$

$$(-1)^n\cdot(s^2)^n = -1$$

$$\Downarrow$$

$$(-s^2)^n = -1 = e^{j(\pi+2k\pi)}$$

Euler.

k is an integer.

$$\Downarrow$$

$$(-s^2)^n = e^{j(2k+1)\pi}$$

$$\Downarrow$$

$$n \cdot \ln(-s^2) = \ln\left\{ e^{j(2k+1)\pi} \right\}$$

$$\Downarrow$$

$$\ln(-s^2) = j\,\frac{2k+1}{n}\cdot\pi$$

$$\Downarrow$$

$$-s^2 = e^{j\frac{2k+1}{n}\cdot\pi}$$

$$\Downarrow$$

$$s^2 = e^{j\frac{2k+1}{n}\pi - \pi}$$

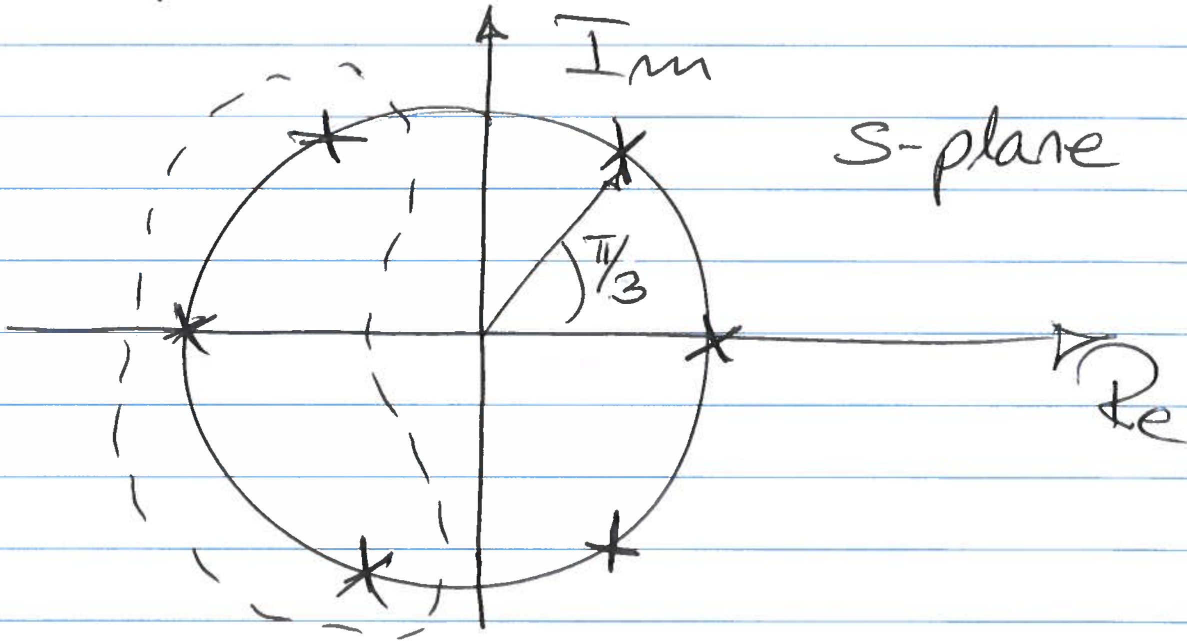
... and then we redo everything once more in order to isolate s;

$$s_k = e^{j\frac{2k+1}{2n}\cdot\pi - \pi/2} \qquad k = 1, 2, \ldots 2n$$

Now we know the roots in $Q(s)\cdot Q(-s)$ and thus we have the poles in $H(s)\cdot H(-s)$.

These $2n$ poles are equaly distributed on a circle with center in origo and radius 1.

Example: $n=3$



We now select the poles located in the left half of the s-plane, since these poles represent a STABLE system $H(s)$!