CHAPTER 9

THE FINITE ELEMENT METHOD

This chapter has the aim of providing, through some beam examples, an introductory overview of the finite element method to obtain approximate discretized dynamic models of continuous structures.

9.1 The finite element method as a Ritz-Galerkin approximation

Probably the most popular and powerful method for the approximation of the dynamic response of continuous elastic systems is the well known finite element method (FEM). Within our context, the finite element method can be considered as a special case of the Ritz-Galerkin method. In the above discussion, the Ritz base functions, like polynomials or trigonometric functions, were assumed to cover the entire domain of interest and to be non-zero over this region. This choice suffer from the disadvantage of the difficulty associated with proper construction of global approximating functions for arbitrary complex domains with complex geometric boundary conditions. In order to overcome such problems, an alternative is given by using *piecewise continuous functions*, such as piecewise linear polynomials, as Ritz functions for the approximation of the solution and its virtual variation. Base functions of this type are often called local functions, or functions with compact support, because they are non-zero only in local regions of the domain. They form the basis of the finite element method. Note that, contrary to a Ritz-Galerkin approximation based on global functions which generally leads to equations in which no banding occurs and the mass and stiffness matrices are full, in the finite element process the adoption of piecewise functions, in which each nodal parameter influences only adjacent elements, leads to sparse and usually banded matrices. A further difference in kind is the usual association in the finite element method of each unknown generalized coordinate with a particular node displacement. This allows a simple physical interpretation invaluable to an engineer.

In the present notes, it is presumed that the reader is already familiar with the theoretical basics of the finite element method for the analysis of static problems. Therefore, we will not go into much details in the presentation. Furthermore, only a simple outline of the method as applied to vibration of rods, bars and beams is presented, leaving a complete presentation and more advanced topics to textbooks specifically devoted to FEM (see References at the end of the chapter).

9.2 Finite element model of a vibrating rod

Let consider a rod of length ℓ, axial stiffness $EA(x)$ and mass per unit length $m(x)$. The rod is subjected to a longitudinal distributed load per unit length $p_x(x,t)$, which has spatial distribution $p_0(x)$ and temporal evolution $p(t)$. We would like to develop a finite element model for vibration analysis of the rod.

The dynamic equilibrium of the system can be written in weak form through the principle of virtual work as

$$\int_\ell \delta u_{/x}^T EA u_{/x} dx = -\int_\ell \delta u^T m\ddot{u} dx + \int_\ell \delta u^T p_x dx \tag{9.1}$$

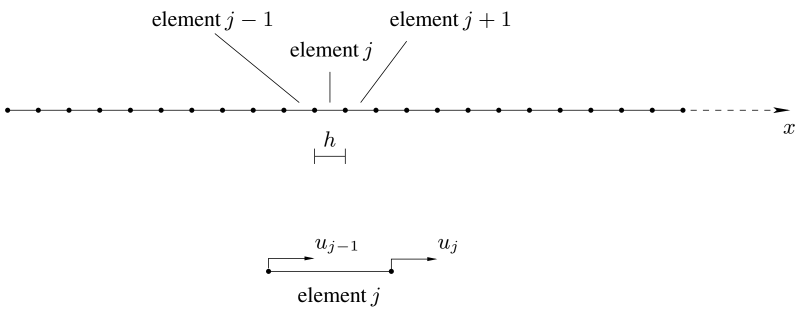

Figure 9.1 Finite element model of the rod.

where $u = u(x,t)$ is the longitudinal displacement at any location x along the rod. A finite element formulation can be obtained by dividing the rod length into n elements of length h (see Figure 9.1) such that the above equation can be expressed in the form of a sum over the individual elements as follows

$$\sum_{j=1}^{n}\int_{(j-1)h}^{h}\delta u_{/x}^{T} E A u_{/x} dx = -\sum_{j=1}^{n}\int_{(j-1)h}^{h}\delta u^{T} m\ddot{u} dx + \sum_{j=1}^{n}\int_{(j-1)h}^{h}\delta u^{T} p_0 dx p(t) \tag{9.2}$$

Note that, for the sake of simplicity and ease of notation, we have assumed elements of the same length h. However, the following derivation is still valid in the general case of elements of different length.

According to a Ritz-Galerkin approach, the axial displacement and the corresponding virtual variation can be approximated inside each element by the following expression

$$\begin{aligned} u(x,t) &= N_1(x)u_{j-1}(t) + N_2(x)u_j(t) \\ &= \mathbf{N}(x)\mathbf{u}_j(t) \qquad\qquad (j-1)h < x < jh \end{aligned} \tag{9.3}$$

and

$$\begin{aligned} \delta u(x,t) &= N_1(x)\delta u_{j-1}(t) + N_2(x)\delta u_j(t) \\ &= \mathbf{N}(x)\delta\mathbf{u}_j(t) \qquad\qquad (j-1)h < x < jh \end{aligned} \tag{9.4}$$

where N_1 and N_2 are called interpolation or shape functions, and

$$\mathbf{u}_j(t) = \begin{Bmatrix} u_{j-1}(t) \\ u_j(t) \end{Bmatrix} \tag{9.5}$$

is the vector of nodal degrees of freedom of the jth element. The above formulation is referred to a two-node rod element. In this case, each node is associated with the nodal displacement u. Therefore, the jth element has a total of two degrees of freedom. Note that higher-order approximation can be introduced. For example, instead of using two terms, one may expand the displacement u using three terms. A different element formulation is then derived where the rod element has three nodes and therefore three degrees of freedom. We will limit our analysis to the common two-node rod element.

For convenience, the global coordinate x may be replaced by a local coordinate ξ defined as

$$\xi = \frac{x-(j-1)h}{h} \tag{9.6}$$

Accordingly, we write

$$\begin{aligned} \frac{d}{dx} &= \frac{d}{d\xi}\frac{d\xi}{dx} = \frac{1}{h}\frac{d}{d\xi} \\ dx &= h d\xi \end{aligned} \tag{9.7}$$

and the longitudinal displacement $u(\xi,t)$ and its virtual variation $\delta u(\xi,t)$ are approximated by

$$u(\xi,t) = N_1(\xi)u_{j-1}(t) + N_2(\xi)u_j(t) = \mathbf{N}(\xi)\mathbf{u}_j(t) \qquad 0 < \xi < 1 \tag{9.8}$$

and

$$\delta u(\xi, t) = N_1(\xi)\delta u_{j-1}(t) + N_2(\xi)\delta u_j(t) = \mathbf{N}(\xi)\delta\mathbf{u}_j(t) \qquad 0 < \xi < 1 \tag{9.9}$$

Since we are dealing with elements having two degrees of freedom, the displacement function can be represented by a linear polynomial, namely

$$u(\xi) = a_1 + a_2\xi \tag{9.10}$$

The two coefficients a_i are determined by imposing the following end conditions

$$\begin{aligned} u\left[(j-1)h\right] &= u_{j-1} & \rightarrow \quad u(0) &= u_{j-1} \\ u\left[jh\right] &= u_j & \rightarrow \quad u(1) &= u_j \end{aligned} \tag{9.11}$$

It follows that

$$\begin{aligned} a_1 &= u_{j-1} \\ a_1 + a_2 &= u_j \end{aligned} \tag{9.12}$$

Therefore, the displacement is expressed as

$$u(\xi, t) = (1-\xi)\, u_{j-1}(t) + \xi u_j(t) \tag{9.13}$$

and the interpolation functions, usually called *hat functions*, are given by

$$\begin{aligned} N_1(\xi) &= 1 - \xi \\ N_2(\xi) &= \xi \end{aligned} \tag{9.14}$$

Using what just derived, each stiffness term in the principle of virtual work is approximated as

$$\begin{aligned} \int_{(j-1)h}^{jh} \delta u_{/x}^T EA u_{/x} dx &= \int_0^1 \delta u_{/\xi}^T \frac{1}{h} EA_j(\xi) u_{/\xi} \frac{1}{h} h d\xi \\ &= \delta\mathbf{u}_j^T(t) \frac{1}{h} \int_0^1 \mathbf{N}_{/\xi}^T EA_j \mathbf{N}_{/\xi} d\xi\, \mathbf{u}_j(t) = \delta\mathbf{u}_j^T(t)\mathbf{K}_j\mathbf{u}_j(t) \end{aligned} \tag{9.15}$$

where

$$\mathbf{K}_j = \frac{1}{h}\int_0^1 \mathbf{N}_{/\xi}^T EA_j \mathbf{N}_{/\xi} d\xi \tag{9.16}$$

and

$$EA_j(\xi) = EA\left[h\xi + (j-1)h\right] \tag{9.17}$$

The matrix $\mathbf{K}_j$ is called rod element stiffness matrix.

Similarly, each contribution to the inertial work is expressed as

$$\int_{(j-1)h}^{jh} \delta u^T m(x)\ddot{u} dx = \delta\mathbf{u}_j^T(t) h \int_0^1 \mathbf{N}^T m_j \mathbf{N} d\xi \ddot{\mathbf{u}}_j(t) = \delta\mathbf{u}_j^T(t)\mathbf{M}_j\ddot{\mathbf{u}}_j(t) \tag{9.18}$$

where

$$\mathbf{M}_j = h\int_0^1 \mathbf{N}^T m_j \mathbf{N} d\xi \tag{9.19}$$

and

$$m_j(\xi) = m\left[h\xi + (j-1)h\right] \tag{9.20}$$

The matrix $\mathbf{M}_j$ is called rod element mass matrix.

Finally, the contribution of each element j due to the external distributed load is given by

$$\int_{(j-1)h}^{h} \delta u^T p_0 dx p(t) = \delta\mathbf{u}_j^T(t) h \int_0^1 p_{0j}(\xi)\mathbf{N}^T d\xi\, p(t) = \delta\mathbf{u}_j^T(t)\mathbf{L}_j^p p(t) \tag{9.21}$$

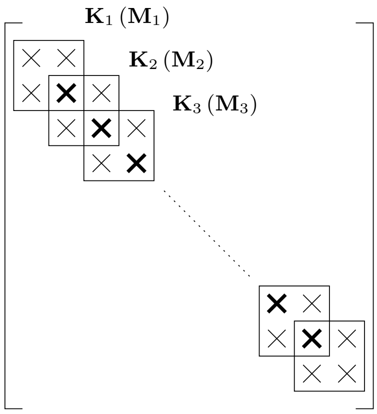

Figure 9.2 Scheme for the assembly of the $\mathbf{K}$ $(\mathbf{M})$ matrix (rod case).

where

$$\mathbf{L}_j^p = h\int_0^1 p_{0j}(\xi)\mathbf{N}^T d\xi \tag{9.22}$$

and

$$p_{0j}(\xi) = p_0\left[h\xi + (j-1)h\right] \tag{9.23}$$

The vector $\mathbf{L}_j^p$ is called element load vector.

It is common practice in the finite element method to approximate the stiffness, mass and load distributions by assuming them to be sectionally constant, i.e., constant over each finite element. This procedure may not be as critical as it may appear, since finite element models typically use a large number n of elements, in which case the sectionally constant parameter distributions become nearly exact. For sectionally constant axial rigidity, i.e, $EA_j(\xi) = EA_j =$ constant, the stiffness matrix of any element j reduces to

$$\mathbf{K}_j = \frac{EA_j}{h}\int_0^1 \mathbf{N}_{/\xi}^T\mathbf{N}_{/\xi}d\xi = \frac{EA_j}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{9.24}$$

When $m_j(\xi) = m_j =$ constant, the element mass matrix is written as

$$\mathbf{M}_j = m_j h\int_0^1 \mathbf{N}^T\mathbf{N}d\xi = \frac{m_j h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{9.25}$$

The element load vector for $p_{0j} =$ constant is explicitly given by

$$\mathbf{L}_j^p = hp_{0j}\int_0^1 \mathbf{N}^T d\xi = \frac{p_{0j}h}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{9.26}$$

According to the above derivation, the overall dynamic equilibrium of the rod can be expressed as follows

$$\sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{M}_j\ddot{\mathbf{u}}_j(t) + \sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{K}_j\mathbf{u}_j(t) = \sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{L}_j^p p(t) \tag{9.27}$$

Since the nodal variable u_j appears twice in the above summation, the assembly procedure yields

$$\sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{M}_j\ddot{\mathbf{u}}_j(t) = \delta\mathbf{u}^T(t)\mathbf{M}_{rod}\ddot{\mathbf{u}}(t) \tag{9.28}$$

and

$$\sum_{j=1}^{n} \delta \mathbf{u}_j^T(t) \mathbf{K}_j \mathbf{u}_j(t) = \delta \mathbf{u}^T(t) \mathbf{K}_{rod} \mathbf{u}(t) \tag{9.29}$$

where $\mathbf{u}(t)$ is the vector of nodal displacements

$$\mathbf{u}(t) = \begin{Bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_n(t) \end{Bmatrix} \tag{9.30}$$

and the stiffness (mass) matrix is assembled as shown in Figure 9.2, where the right bottom entry $(2, 2)$ of the element matrix $\mathbf{K}_j$ ($\mathbf{M}_j$) and the left top entry $(1, 1)$ of $\mathbf{K}_{j+1}$ ($\mathbf{M}_{j+1}$) add up. The previous assembly procedure applied to loading terms yields

$$\sum_{j=1}^{n} \delta \mathbf{u}_j^T(t) \mathbf{L}_j^p p(t) = \delta \mathbf{u}^T(t) \mathbf{L}_{rod}^p p(t) \tag{9.31}$$

where the resulting $\mathbf{L}_{rod}^p$ vector is obtained by adding the bottom component of the jth element vector to the top component of the $(j+1)$th element vector. Exploiting the arbitrariness of virtual variation $\delta \mathbf{u}^T$, the finite element model of the present rod is written as

$$\mathbf{M}_{rod} \ddot{\mathbf{u}}(t) + \mathbf{K}_{rod} \mathbf{u}(t) = \mathbf{L}_{rod}^p p(t) \tag{9.32}$$

The final step in the analysis is to ensure that the geometric boundary conditions are satisfied. As it stands, the previous analysis refers to a free-free rod. If the rod is clamped at the left end, then the nodal displacement u_0 is zero. This condition can be introduced by omitting u_0 from the set of degrees of freedom of the complete rod, and at the same time omitting the first row and column from the mass and stiffness matrices $\mathbf{M}_{rod}$ and $\mathbf{K}_{rod}$, and also the first row of the load vector $\mathbf{L}_{rod}^p$. Similarly, if the rod is fixed at both ends, both u_0 and u_n are omitted from the set of degrees of freedom. Accordingly, one must also omits the first and last rows and columns of the rod matrices, and the first and last row of the load vector.

As an illustrative example, let consider an uniform fixed-free rod of length ℓ, axial rigidity EA and mass per unit length m. We would like to compute the approximate eigenvalues and eigenvectors for FE models of increasing number of elements and compare the results with the exact solutions.

After dividing the rod into n elements of length $h = \ell/n$ and before applying the geometric boundary conditions, the mass and stiffness matrices of the complete rod are given by

$$\mathbf{M}_{rod} = \frac{mh}{6} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \tag{9.33}$$

$$\mathbf{K}_{rod} = \frac{EA}{h} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{9.34}$$

Table 9.1 Finite element approximation of a uniform fixed-free rod using n two-node elements. Comparison with exact non-dimensional values $\lambda_n^e = \frac{m\ell^2\omega_n^2}{EA} = \frac{(2n-1)^2\pi^2}{4}$.

	Mode				
n	1	2	3	4	5
1	3.0000	–	–	–	–
2	2.5967	31.6891	–	–	–
3	2.5243	27.0000	88.8603	–	–
4	2.4993	24.8721	82.0727	171.6280	–
5	2.4878	23.8939	75.0000	168.6484	279.0031
Exact	2.4674	22.2066	61.6850	120.9027	199.8595

Since the rod if fixed at the left end, the first row and column of the above matrices are omitted. Therefore, the mass and stiffness matrices of the fixed-free rod are expressed, respectively, by

$$\mathbf{M}_{rod}^{fixed-free} = \frac{mh}{6}\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \tag{9.35}$$

and

$$\mathbf{K}_{rod}^{fixed-free} = \frac{EA}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{9.36}$$

The natural frequencies are the solutions of the eigenvalue problem

$$\left(\mathbf{K}_{rod}^{fixed-free} - \omega^2\mathbf{M}_{rod}^{fixed-free}\right)\mathbf{u} = \mathbf{0} \tag{9.37}$$

or, in non-dimensional form,

$$\left(\mathbf{K}^\star - \lambda^\star\mathbf{M}^\star\right)\mathbf{u} = \mathbf{0} \tag{9.38}$$

where

$$\mathbf{K}^\star = (h/EA)\mathbf{K}_{rod}^{fixed-free} \qquad \mathbf{M}^\star = (1/mh)\mathbf{M}_{rod}^{fixed-free} \tag{9.39}$$

and

$$\lambda^\star = \frac{\lambda}{n^2} \tag{9.40}$$

$$\lambda = \frac{\omega^2 m\ell}{EA} \tag{9.41}$$

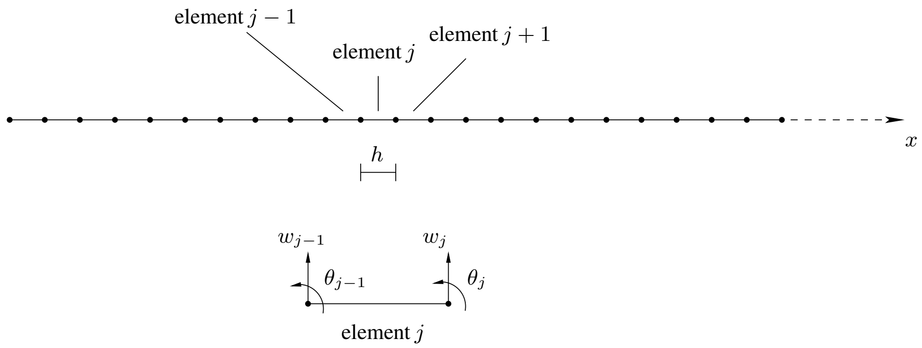

Figure 9.3 Finite element model of a beam.

9.3 Finite element model of a vibrating bar

Let consider a bar of length ℓ, torsional stiffness $GJ(x)$ and mass polar moment of inertia per unit length $I_p(x)$. The bar is subjected to a distributed torque per unit length $m_x(x,t)$, which has spatial distribution $m_0(x)$ and temporal evolution $m(t)$. We would like to develop a finite element model for vibration analysis of the bar.

The dynamic equilibrium of the system can be written in weak form through the principle of virtual work as

$$\int_\ell \delta\theta^T_{/x} GJ \theta_{/x} dx = -\int_\ell \delta\theta^T I_p \ddot{\theta} dx + \int_\ell \delta\theta^T m_x dx \tag{9.42}$$

where $\theta = \theta(x,t)$ is the twist of the cross section at any location x along the bar. As previously outlined, the bar problem is completely equivalent to the rod. Therefore, the corresponding finite element model can be obtained using the same procedure adopted for the rod. The element stiffness matrix is given by

$$\mathbf{K}_j = \frac{1}{h}\int_0^1 \mathbf{N}^T_{/\xi} GJ_j \mathbf{N}_{/\xi} d\xi \tag{9.43}$$

where ξ is the element local coordinate, h is the element length, $\mathbf{N}(\xi)$ is the row vector of hat interpolating functions and

$$GJ_j(\xi) = GJ\left[h\xi + (j-1)h\right] \tag{9.44}$$

The element mass matrix is given by

$$\mathbf{M}_j = h\int_0^1 \mathbf{N}^T I_{pj} \mathbf{N} d\xi \tag{9.45}$$

where

$$I_{pj}(\xi) = I_p\left[h\xi + (j-1)h\right] \tag{9.46}$$

The element load vector is expressed as

$$\mathbf{L}^m_j = h\int_0^1 m_{0j}(\xi)\mathbf{N}^T d\xi \tag{9.47}$$

where

$$m_{0j}(\xi) = m_0\left[h\xi + (j-1)h\right] \tag{9.48}$$

9.4 Finite element model of a vibrating beam (without prestress)

Let consider a slender beam of length ℓ, flexural stiffness $EJ(x)$ and mass per unit length $m(x)$. The beam is subjected to a transverse distributed load per unit length $p_z(x,t)$, which has spatial distribution $p_0(x)$ and temporal evolution $p(t)$. We would like to develop a finite element model for vibration analysis of the beam.

The dynamic equilibrium of the system can be written in weak form through the principle of virtual work as

$$\int_\ell \delta w^T_{/xx} EJ(x) w_{/xx} dx + \int_\ell \delta w^T m(x)\ddot{w} dx = \int_\ell \delta w^T p_0(x) dx\, p(t) \tag{9.49}$$

where $w = w(x,t)$ is the transverse displacement of a generic point of the beam. According to a finite element formulation, the beam length is divided into elements so that each term in the above equation is expressed in the form of a sum over the individual elements to yield

$$\sum_{j=1}^{n} \int_{(j-1)h}^{jh} \delta w^T_{/xx} EJ(x) w_{/xx} dx + \sum_{j=1}^{n} \int_{(j-1)h}^{jh} \delta w^T m(x)\ddot{w} dx = \sum_{j=1}^{n} \int_{(j-1)h}^{jh} \delta w^T p_0 dx\, p(t) \tag{9.50}$$

where n is the total number of elements and h is the length of each element (see Figure 9.3). Note that the beam has been divided by elements of the same length h. Inside each element j, the transverse displacement can be approximated by the following expression

$$\begin{aligned} w(x,t) &= N_1(x)w_{j-1}(t) + N_2(x)\theta_{j-1}(t) + N_3(x)w_j(t) + N_4(x)\theta_j(t) \\ &= \mathbf{N}(x)\mathbf{u}_j(t) \qquad\qquad (j-1)h < x < jh \end{aligned} \tag{9.51}$$

where N_1, N_2, N_3 and N_4 are the interpolation functions, and

$$\mathbf{u}_j(t) = \begin{Bmatrix} w_{j-1}(t) \\ \theta_{j-1}(t) \\ w_j(t) \\ \theta_j(t) \end{Bmatrix} \tag{9.52}$$

is the vector of nodal degrees of freedom of the jth element. The above formulation is referred to a two-node beam element. In this case, each node is associated with a displacement w and a rotation θ. Therefore, the jth element has a total of four degrees of freedom.

For convenience, we replace the global coordinate x with the local coordinate

$$\xi = \frac{x-(j-1)h}{h} \tag{9.53}$$

Accordingly, we write

$$\begin{aligned} &\frac{d}{dx} = \frac{d}{d\xi}\frac{d\xi}{dx} = \frac{1}{h}\frac{d}{d\xi} \\ &\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \frac{1}{h^2}\frac{d^2}{d\xi^2} \\ &dx = hd\xi \end{aligned} \tag{9.54}$$

and the transverse displacement $w(\xi,t)$ is approximated by

$$w(\xi,t) = N_1(\xi)w_{j-1}(t) + N_2(\xi)\theta_{j-1}(t) + N_3(\xi)w_j(t) + N_4(\xi)\theta_j(t) = \mathbf{N}(\xi)\mathbf{u}_j(t) \qquad 0 < \xi < 1 \tag{9.55}$$

Since each element has four degrees of freedom, two nodal displacements and two nodal rotations, the displacement function can be represented by a polynomial having four constants, namely

$$w(\xi) = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3 \tag{9.56}$$

The four coefficients a_i are determined by imposing the following end conditions

$$\begin{aligned} &w\left[(j-1)h\right] = w_{j-1} && \rightarrow \quad w(0) = w_{j-1} \\ &\theta\left[(j-1)h\right] = w_{/x}\left[(j-1)h\right] = \theta_{j-1} && \rightarrow \quad w_{/\xi}(0) = h\theta_{j-1} \\ &w\left[jh\right] = w_j && \rightarrow \quad w(1) = w_j \\ &\theta\left[jh\right] = w_{/x}\left[jh\right] = \theta_j && \rightarrow \quad w_{/\xi}(1) = h\theta_j \end{aligned} \tag{9.57}$$

It follows that

$$
\begin{aligned}
a_1 &= w_{j-1} \\
a_2 &= h\theta_{j-1} \\
a_3 &= -3w_{j-1} - 2h\theta_{j-1} + 3w_j - h\theta_j \\
a_4 &= 2w_{j-1} + h\theta_{j-1} - 2w_j + h\theta_j
\end{aligned}
\tag{9.58}
$$

Therefore, the displacement is expressed as

$$
\begin{aligned}
w(\xi, t) = &\left(1 - 3\xi^2 + 2\xi^3\right) w_{j-1}(t) + h\left(\xi - 2\xi^2 + \xi^3\right)\theta_{j-1}(t) \\
&+ \left(3\xi^2 - 2\xi^3\right) w_j(t) + h\left(\xi^3 - \xi^2\right)\theta_j(t)
\end{aligned}
\tag{9.59}
$$

and the interpolation functions, usually called *Hermitian functions*, are given by

$$
\begin{aligned}
N_1(\xi) &= \left(1 - 3\xi^2 + 2\xi^3\right) \\
N_2(\xi) &= h\left(\xi - 2\xi^2 + \xi^3\right) \\
N_3(\xi) &= \left(3\xi^2 - 2\xi^3\right) \\
N_4(\xi) &= h\left(\xi^3 - \xi^2\right)
\end{aligned}
\tag{9.60}
$$

According to what derived so far, the stiffness term of any element j is given by

$$
\begin{aligned}
\int_{(j-1)h}^{jh} \delta w_{/xx}^T EJ(x) w_{/xx} dx &= \int_0^1 \delta w_{/\xi\xi}^T \frac{1}{h^2} EJ_j(\xi) w_{/\xi\xi} \frac{1}{h^2} h d\xi \\
&= \delta \mathbf{u}_j^T(t) \frac{1}{h^3} \int_0^1 EJ_j(\xi) \mathbf{N}_{/\xi\xi}^T \mathbf{N}_{/\xi\xi} d\xi \, \mathbf{u}_j(t) = \delta \mathbf{u}_j^T(t) \mathbf{K}_j \mathbf{u}_j(t)
\end{aligned}
\tag{9.61}
$$

where

$$
\mathbf{K}_j = \frac{1}{h^3} \int_0^1 EJ_j(\xi) \mathbf{N}_{/\xi\xi}^T \mathbf{N}_{/\xi\xi} d\xi \tag{9.62}
$$

is the beam element stiffness matrix and

$$
EJ_j(\xi) = EJ\left[h\xi + (j-1)h\right] \tag{9.63}
$$

The mass term of any element j is

$$
\int_{(j-1)h}^{jh} \delta w^T m(x) \ddot{w} dx = \delta \mathbf{u}_j^T(t) h \int_0^1 m_j(\xi) \mathbf{N}^T \mathbf{N} d\xi = \delta \mathbf{u}_j^T(t) \mathbf{M}_j \ddot{\mathbf{u}}_j(t) \tag{9.64}
$$

where

$$
\mathbf{M}_j = h \int_0^1 m_j(\xi) \mathbf{N}^T \mathbf{N} d\xi \tag{9.65}
$$

is the beam element mass matrix and

$$
m_j(\xi) = m\left[h\xi + (j-1)h\right] \tag{9.66}
$$

The contribution of each element j due to the external distributed load is given by

$$
\int_{(j-1)h}^{jh} \delta w^T p_0(x) dx \, p(t) = \delta \mathbf{u}_j^T(t) h \int_0^1 p_{0j}(\xi) \mathbf{N}^T d\xi \, p(t) = \delta \mathbf{u}_j^T(t) \mathbf{L}_j^p p(t) \tag{9.67}
$$

where

$$
\mathbf{L}_j^p = h \int_0^1 p_{0j}(\xi) \mathbf{N}^T d\xi \tag{9.68}
$$

is the element load vector and

$$
p_{0j}(\xi) = p_0\left[h\xi + (j-1)h\right] \tag{9.69}
$$

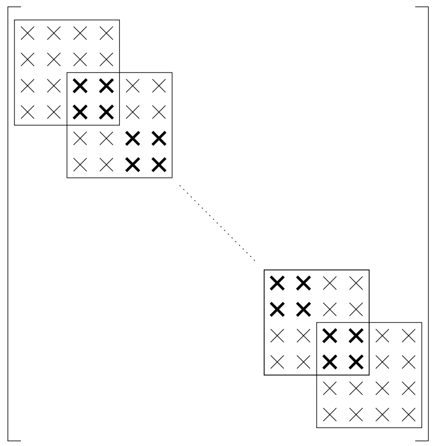

Figure 9.4 Scheme for the assembly of the $\mathbf{K}$ ($\mathbf{M}$) matrix (beam case).

For sectionally constant flexural rigidity ($EJ_j(\xi) = EJ_j =$ constant) and mass per unit length ($m_j(\xi) = m_j =$ constant), the element stiffness and matrices are written as

$$\mathbf{K}_j = \frac{EJ_j}{h^3}\int_0^1 \mathbf{N}^T_{/\xi\xi}\mathbf{N}_{/\xi\xi}d\xi = \frac{EJ_j}{h^3}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \tag{9.70}$$

and

$$\mathbf{M}_j = h\int_0^1 m_j(\xi)\mathbf{N}^T\mathbf{N}d\xi = \frac{m_j h}{420}\begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix} \tag{9.71}$$

The element load vector for $p_{0j} =$ constant is explicitly given by

$$\mathbf{L}^p_j = \frac{p_{0j}h}{2}\begin{bmatrix} 1 \\ h/6 \\ 1 \\ -h/6 \end{bmatrix} \tag{9.72}$$

Referring back to the principle of virtual work and using what derived so far, the dynamic equilibrium of the beam can be written as

$$\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{M}_j\ddot{\mathbf{u}}_j(t) + \sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{K}_j\mathbf{u}_j(t) = \sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{L}_j^p p(t) \tag{9.73}$$

It is noted that, expect for the first and the last elements, the nodal coordinates w_j and θ_j appear twice in the above summation, once as the bottom components in the element nodal vector referred to element j, and once as the top components in the element nodal vector referred to element $j+1$. Therefore, the mass and stiffness terms can be written as

$$\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{M}_j\ddot{\mathbf{u}}_j(t) = \delta\mathbf{u}^T(t)\mathbf{M}_{beam}\ddot{\mathbf{u}}(t) \tag{9.74}$$

and

$$\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{K}_j\mathbf{u}_j(t) = \delta\mathbf{u}^T(t)\mathbf{K}_{beam}\mathbf{u}(t) \tag{9.75}$$

Table 9.2 Finite element approximation of a uniform simply-supported beam. Comparison with exact non-dimensional frequencies $\beta_n \ell = n\pi$.

	Mode				
n	1	2	3	4	5
1	3.3098	7.0852	–	–	–
2	3.1478	6.6195	10.4947	14.1703	–
3	3.1429	6.3202	9.9293	13.5396	18.1112
4	3.1420	6.2956	9.5105	13.2390	16.6911
5	3.1418	6.2884	9.4621	12.7103	16.5488
$\vdots$					
50	3.1416	6.2832	9.4248	12.5664	15.7080
Exact	π	2π	3π	4π	5π

where $\mathbf{u}(t)$ contains the nodal degrees of freedom

$$\mathbf{u}(t) = \begin{Bmatrix} w_0(t) \\ \theta_0(t) \\ w_1(t) \\ \theta_1(t) \\ \vdots \\ w_n(t) \\ \theta_n(t) \end{Bmatrix} \tag{9.76}$$

and the mass (stiffness) matrix is assembled as in Figure 9.4, where the right bottom 2×2 block of the element matrix $\mathbf{M}_j$ ($\mathbf{K}_j$) and the left top 2×2 block of the element matrix $\mathbf{M}_{j+1}$ ($\mathbf{K}_{j+1}$) add up. The same can be done for the loading term. We can write

$$\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{L}_j^p p(t) = \delta\mathbf{u}^T(t)\mathbf{L}_{beam}^p p(t) \tag{9.77}$$

where $\mathbf{L}_{beam}^p$ is obtained by adding the two bottom components of $\mathbf{L}_j^p$ to the two top components of $\mathbf{L}_{j+1}^p$.

Exploiting the arbitrariness of virtual variations $\delta\mathbf{u}_j^T$, the finite element model of the present beam is written as

$$\mathbf{M}_{beam}\ddot{\mathbf{u}}(t) + \mathbf{K}_{beam}\mathbf{u}(t) = \mathbf{L}_{beam}^p p(t) \tag{9.78}$$

The final step in the analysis is to ensure that the geometric boundary conditions are satisfied. As it stands, the previous analysis refers to a free-free beam.

For example, if the beam is clamped at $x = 0$ (*cantilever beam*), the corresponding nodal variables are zero, i.e., $w_0 = \theta_0 = 0$. Therefore, the vector $\mathbf{u}(t)$ is actually given by

$$\mathbf{u}(t) = \begin{Bmatrix} w_1(t) \\ \theta_1(t) \\ \vdots \\ w_n(t) \\ \theta_n(t) \end{Bmatrix} \tag{9.79}$$

and the first two rows and columns of the mass and stiffness matrices and the first two components of $\mathbf{L}_{beam}^p$ are omitted.

If the beam is *simply supported* at both ends, the nodal variables w_0 and w_n are equal to zero. The vector of nodal degrees of freedom is then given in this case by

$$\mathbf{u}(t) = \begin{Bmatrix} \theta_0(t) \\ w_1(t) \\ \theta_1(t) \\ \vdots \\ \theta_n(t) \end{Bmatrix} \tag{9.80}$$

Accordingly, the first and the second to last rows and columns of the mass and stiffness matrices are omitted, and the first and the second to last elements of the load vector are omitted.

Two numerical examples are presented in Tables 9.2 and 9.3. They refer to the computation of natural frequencies of uniform simply-supported and cantilever beams using a finite element model with increasing number of elements n.

Table 9.3 Finite element approximation of a uniform cantilever beam. Comparison with exact non-dimensional frequencies $\beta_n \ell$.

	Mode				
n	1	2	3	4	5
1	1.8796	5.8997	–	–	–
2	1.8756	4.7140	8.6693	14.7695	–
3	1.8752	4.7018	7.9035	11.8605	16.2709
4	1.8751	4.6968	7.8851	11.0751	15.1042
5	1.8751	4.6953	7.8689	11.0598	14.2485
$\vdots$					
50	1.8751	4.6941	7.8548	10.9955	14.1372
Exact	1.8751	4.6941	7.8547	10.9955	14.1372

9.5 Finite element model of a vibrating beam (with prestress)

Let consider the slender beam of the previous case. Now the beam is also subjected to a prescribed initial stress σ^0_{xx}, which can have an arbitrary variation along the beam length. In order to derive a finite element model of the prestressed beam, let's first write the dynamic equilibrium of the problem using the principle of virtual work as follows

$$\int_\ell \delta w^T_{/xx} EJ(x) w_{/xx} dx + \int_\ell \delta w^T_{/x} N_0(x) w_{/x} dx + \int_\ell \delta w^T m(x) \ddot{w} dx = \int_\ell \delta w^T p_0(x) dx\, p(t) \tag{9.81}$$

where $N_0(x) = \int_A \sigma^0_{xx} dA$ is the axial force corresponding to the prestress condition. The above equation is discretized as follows

$$\begin{aligned} &\sum_{j=1}^{n} \int_{(j-1)h}^{jh} \delta w^T_{/xx} EJ(x) w_{/xx} dx + \sum_{j=1}^{n} \int_{(j-1)h}^{jh} \delta w^T_{/x} N_0(x) w_{/x} dx \\ &+ \sum_{j=1}^{n} \int_{(j-1)h}^{jh} \delta w^T m(x) \ddot{w} dx = \sum_{j=1}^{n} \int_{(j-1)h}^{jh} \delta w^T p_0 dx\, p(t) \end{aligned} \tag{9.82}$$

where n is the total number of elements and h is the length of each element. Therefore, the finite element model of the beam with prestress can be obtained using the procedure adopted for the beam without prestress by including the geometric stiffness matrix arising from the prestress term. This is given for any element j by

$$\begin{aligned}\int_{(j-1)h}^{jh} \delta w_{/x}^T N_0(x) w_{/x} dx &= \int_0^1 \delta w_{/\xi}^T \frac{1}{h} N_{0j}(\xi) w_{/\xi} \frac{1}{h} h d\xi \\ &= \delta \mathbf{u}_j^T(t) \frac{1}{h} \int_0^1 N_{0j}(\xi) \mathbf{N}_{/\xi}^T \mathbf{N}_{/\xi} d\xi \, \mathbf{u}_j(t) = \delta \mathbf{u}_j^T(t) \mathbf{K}_j^G \mathbf{u}_j(t)\end{aligned} \tag{9.83}$$

where

$$\mathbf{K}_j^G = \frac{1}{h} \int_0^1 N_{0j}(\xi) \mathbf{N}_{/\xi}^T \mathbf{N}_{/\xi} d\xi \tag{9.84}$$

is the geometric stiffness matrix of the beam element and

$$N_{0j}(\xi) = N_0 \left[h\xi + (j-1)h \right] \tag{9.85}$$

When N_{0j} can be assumed to be constant in each element, the element geometric stiffness matrix is written explicitly as

$$\mathbf{K}_j^G = \frac{N_{0j}}{h} \int_0^1 \mathbf{N}_{/\xi}^T \mathbf{N}_{/\xi} d\xi = \frac{N_{0j}}{30h} \begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} \tag{9.86}$$

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