CHAPTER 10

DYNAMIC RESPONSE BY MODAL ANALYSIS

This section deals with the computation of the dynamic response of vibrating structures modelled according to the discretization techniques presented in the previous chapters.

10.1 Introduction

Previous chapters have presented some techniques commonly used in structural dynamics in order to derive approximate models of vibrating structures. We have seen that all the methods yield the following set of ordinary differential equations with constant coefficients governing the vibration of the system

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \tag{10.1}$$

with mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$. The vectors $\mathbf{u}(t)$ and $\mathbf{f}(t)$ contain N displacement and load variables, respectively, arising from the spatial discretization of the problem. Therefore, continuous structures with infinite number of degrees of freedom have been reduced to multi-degrees-of-freedom dynamic systems having a finite number N of degrees of freedom. When $N = 1$, the structure is represented by an equivalent single-degree-of-freedom system. Attention is focused in this chapter on the computation of the dynamic response of discretized structures described by Eq. (10.1).

10.2 Dynamic response of single-degree-of-freedom systems

The simplest discretized model of a continuous structure involves only a single degree of freedom. According to the approximate modelling methods previously presented, equivalent single-degree-of-freedom (SDOF) systems of continuous structures may be derived from a lumped-parameter approach with a single mass quantity or a Ritz-Galerkin discretization using a single-term in the series approximation of the structural displacement field. In this section, computation of the dynamic response of SDOF systems is illustrated by considering models arising both from lumping and Ritz discretization.

It is noticed that the basic principles of dynamics of SDOF mechanical systems are assumed to be well known from a first course in vibration. Moreover, when discussing in Chapter 3 the dynamic response of continuous structures through the modal analysis approach, we have found that the modal equations represent a set of uncoupled second-order SDOF systems in modal coordinates. Some general solutions and particular cases related to specific examples have been already roughly presented. Therefore, the examples provided in the following will be mainly used as a refresher of some important concepts and methods.

10.2.1 SDOF lumped-parameter models

10.2.1.1 Free undamped response. As an illustrative example of a SDOF lumped-parameter model, let consider the first case introduced in Section 5.3.2. The system consists of a flexible cantilever beam of length ℓ and bending rigidity EJ carrying a tip mass M. Since the beam is assumed to be massless, the structure is *equivalent to a* SDOF *undamped spring-mass system* of mass M and stiffness $K = 3EJ/\ell^3$. We have previously shown that the equation governing the free motion of the equivalent system is written as

$$M\ddot{z}(t) + Kz(t) = 0 \tag{10.2}$$

where $z(t)$ is the displacement of the tip mass. Dividing through M, the equation of motion can be also written in the form

$$\ddot{z}(t) + \omega_0^2 z(t) = 0 \tag{10.3}$$

where $\omega_0 = \sqrt{K/M}$ is the natural frequency of the system. The time history of $z(t)$ can be determined once the initial displacement $z(t_0)$ and velocity $\dot{z}(t_0)$ are known. Without any loss of generality, let $t_0 = 0$. Therefore, the initial conditions are specified as

$$z(0) = z_0 \qquad\qquad \dot{z}(0) = \dot{z}_0 \tag{10.4}$$

The second-order homogeneous differential equation (10.3) has the following general solution

$$z(t) = Ce^{\lambda t} \tag{10.5}$$

where the arbitrary constant C and λ are to be determined. Inserting Eq. (10.5) into Eq. (10.3) yields

$$\lambda^2 + \omega_0^2 = 0 \tag{10.6}$$

which has the following *two purely imaginary conjugate roots*

$$\lambda_1 = j\omega_0 \qquad\qquad \lambda_2 = -j\omega_0 \tag{10.7}$$

Therefore, the displacement solution is given by

$$z(t) = C_1 e^{j\omega_0 t} + C_2 e^{-j\omega_0 t} \tag{10.8}$$

where C_1 and C_2 are arbitrary constants of integration to be determined from initial conditions. Equation (10.8) represents *harmonic oscillations*. Using Euler's formula, we can also write

$$z(t) = (C_1 + C_2)\cos\omega_0 t + j\,(C_1 - C_2)\sin\omega_0 t \tag{10.9}$$

Note that, in order for the motion to be real, $C_1 + C_2$ must be real and $C_1 - C_2$ must be imaginary. As a result, one constant must be the complex conjugate of the other. The value of constants C_1 and C_2 are given by

$$C_1 = \frac{1}{2}\left(z_0 + \frac{\dot{z}_0}{j\omega_0}\right) \qquad\qquad C_2 = \frac{1}{2}\left(z_0 - \frac{\dot{z}_0}{j\omega_0}\right) \tag{10.10}$$

Hence, the harmonic solution can be expressed in the form

$$\boxed{z(t) = z_0\cos\omega_0 t + \frac{\dot{z}_0}{\omega_0}\sin\omega_0 t} \tag{10.11}$$

10.2.1.2 Free damped response. If we assume that the tip mass of the previous example is connected to a grounded viscous damper C, the lumped-paramater model of the structure is equivalent to a SDOF spring-mass-damper system. The equation of motion in this case is expressed as

$$M\ddot{z}(t) + C\dot{z}(t) + Kz(t) = 0 \tag{10.12}$$

Dividing as before by M, Equation (10.12) can be written as

$$\ddot{z}(t) + 2\xi_0\omega_0\dot{z}(t) + \omega_0^2 z(t) = 0 \tag{10.13}$$

where the dimensionless *damping factor* $\xi_0 = C/2M\omega_0 = C/2\sqrt{MK}$. The solution is again in the exponential form Eq. (10.5). Inserting such solution into the second-order differential equation (10.13) yields

$$\lambda^2 + 2\xi_0\omega_0\lambda + \omega_0^2 = 0 \tag{10.14}$$

which has the following two roots

$$\lambda_1 = -\xi_0\omega_0 + \omega_0\sqrt{\xi_0^2 - 1} \qquad \lambda_2 = -\xi_0\omega_0 - \omega_0\sqrt{\xi_0^2 - 1} \tag{10.15}$$

It is clear that, when $\xi_0 = 0$, the undamped solution is obtained. From Eq. (10.15), three different kinds of stable solutions can be distinguished:

- $0 < \xi_0 < 1$ (underdamped case)
- $\xi_0 = 1$ (critically damped case)
- $\xi_0 > 1$ (overdamped case)

For underdamped systems, which represent the most common situation in real structures, the roots are complex conjugates and are conveniently written as

$$\lambda_1 = -\xi_0\omega_0 + j\omega_{0d} \qquad \lambda_2 = -\xi_0\omega_0 - j\omega_{0d} \tag{10.16}$$

where

$$\omega_{0d} = \omega_0\sqrt{1 - \xi_0^2} \tag{10.17}$$

is the so-called *damped natural frequency*. The displacement solution is given by

$$z(t) = C_1 e^{-\xi_0\omega_0 t} e^{j\omega_{0d}t} + C_2 e^{-\xi_0\omega_0 t} e^{-j\omega_{0d}t} \tag{10.18}$$

As previously done for the undamped case, the above solution can be put in the trigonometric form

$$z(t) = e^{-\xi_0\omega_0 t}\left[(C_1 + C_2)\cos\omega_{0d}t + j\,(C_1 - C_2)\sin\omega_{0d}t\right] \tag{10.19}$$

The integration constants are determined from initial conditions. We have

$$C_1 = \frac{1}{2}\left(\frac{j\omega_{0d} + \xi_0\omega_0}{j\omega_{0d}} z_0 + \frac{\dot{z}_0}{j\omega_{0d}}\right) \qquad C_2 = \frac{1}{2}\left(\frac{j\omega_{0d} - \xi_0\omega_0}{j\omega_{0d}} z_0 - \frac{\dot{z}_0}{j\omega_{0d}}\right) \tag{10.20}$$

Therefore, the damped solution may be written as

$$\boxed{z(t) = e^{-\xi_0\omega_0 t}\left[z_0\cos\omega_{0d}t + \frac{\xi_0\omega_0 z_0 + \dot{z}_0}{\omega_{0d}}\sin\omega_{0d}t\right]} \tag{10.21}$$

The preceding analysis considers the motion of equivalent SDOF structural systems as a result of non-zero initial conditions. In the following, we want to introduce external loads in order to study the forced dynamic response.

10.2.1.3 Impulse response (undamped case). Referring again to the massless cantilever beam carrying a tip mass, it is first assumed that the mass is excited by an impulse force $F(t) = \delta(t)$. The inhomogeneous problem to be solved is

$$M\ddot{z}(t) + Kz(t) = \delta(t) \tag{10.22}$$

with *null initial conditions*, i.e., the system is at rest prior to the application of the impulse.

We need to recall that integration of ordinary differential equations requires initial conditions at time $t = 0_+$, whereas physical initial conditions are those corresponding to time $t = 0_-$. Such a distinction is actually superfluous if the forcing function is continuous in $t = 0$ or with a discontinuity of the first kind at most. However, the application of an impulsive force $\delta(t)$ has the effect of changing the state of the system at time $t = 0$. In other words, the conditions of the system at time $t = 0_+$ are different compared with the conditions at time $t = 0_-$ and the former should be derived from the latter. Therefore, the initial conditions of the problem are specified at $t = 0_-$ as

$$z(0_-) = 0 \qquad \dot{z}(0_-) = 0 \tag{10.23}$$

Looking at Eq. (10.22), it is noted that the impulse on the right-hand side must be balanced with an impulse on the left-hand side. This impulse cannot lie in the z term, otherwise the $\ddot{z}$ term should contain the double time derivative of the unit impulse, which does not appear in the right-hand side. This means that $z(t)$ is not impulsive, and so it is bounded at the origin.

The effect of the impulse on the system under study can be determined as follows. Integrating each term of Eq. (10.22) over a symmetric interval $[-\epsilon, +\epsilon]$ about $t = 0$ yields

$$M\left[\dot{z}(+\epsilon) - \dot{z}(-\epsilon)\right] + K\int_{-\epsilon}^{+\epsilon} z(t)dt = \int_{-\epsilon}^{+\epsilon} \delta(t)dt \tag{10.24}$$

By definition of the unit impulse, the integral in the right-hand side is equal to one for any ϵ, i.e.,

$$\int_{-\epsilon}^{+\epsilon} \delta(t)dt = 1$$

By letting $\epsilon \to 0$, since $z(t)$ is finite at the origin, the integral of z vanishes, i.e.,

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} z(t)dt = 0 \tag{10.25}$$

Therefore, we obtain

$$M\left[\dot{z}(0_+) - \dot{z}(0_-)\right] = 1 \tag{10.26}$$

Using the fact the system is at rest before the application of the impulsive force ($\dot{z}(0_-) = 0$), we conclude that the effect of the impulse excitation is to produce an equivalent initial velocity

$$\dot{z}(0_+) = \frac{1}{M} \tag{10.27}$$

At this point, instead of solving the inhomogeneous equation (10.22) with null initial conditions, we solve the following homogeneous equation

$$M\ddot{z}(t) + Kz(t) = 0 \tag{10.28}$$

with inhomogeneous initial conditions (at time $t = 0_+$)

$$z(0) = 0 \qquad \dot{z}(0) = \frac{1}{M} \tag{10.29}$$

Recalling Eq. (10.11), the solution is given for $t > 0$ by

$$\boxed{z(t) = \frac{1}{\omega_0 M}\sin\omega_0 t} \tag{10.30}$$

where $\omega_0^2 = 3EJ/M\ell^3$ is the square value of the approximate natural frequency.

10.2.1.4 Doublet response (undamped case). Let now consider the case of $F(t) = \dot{\delta}(t)$, where the time derivative of the unit impulse is denoted as the *doublet* function. The governing equation is

$$M\ddot{z}(t) + Kz(t) = \dot{\delta}(t) \tag{10.31}$$

with null initial conditions

$$z(0_-) = 0 \qquad \dot{z}(0_-) = 0 \tag{10.32}$$

From Eq. (10.31), the doublet appearing in the right-hand side must be balanced with a doublet on the left-hand side. This doublet cannot appear in z, otherwise we should have a double derivative of a doublet in the right-hand side. Therefore, the doublet appears in $\ddot{z}$. This implies that z comprises a step function and is thus finite about the origin.

By following the same procedure of the previous case, we integrate Eq. (10.31) first between 0_- and t, thus obtaining

$$M\dot{z}(t) + \int_{0_-}^{t} z(t)d\tau = \delta(t) \tag{10.33}$$

Next, we integrate again between 0_- and 0_+

$$M\left[z(0_+) - z(0_-)\right] + \int_{0_-}^{0_+}\int_{0_-}^{t} z(t)d\tau dt = 1 \tag{10.34}$$

The integral in the above equation is zero since z is finite in the neighborhood of the origin. Since $z(0_-) = 0$, we obtain

$$z(0_+) = \frac{1}{M} \tag{10.35}$$

The initial force problem with null initial conditions is then transformed to the following unforced system with nonzero initial conditions (at time $t = 0_+$)

$$\begin{aligned} &\ddot{z}(t) + \omega_0^2 z(t) = 0 \\ &z(0) = \frac{1}{M} \\ &\dot{z}(0) = 0 \end{aligned}$$

whose solution is expressed for $t > 0$ as

$$\boxed{z(t) = \frac{1}{M}\cos(\omega_0 t)} \tag{10.36}$$

10.2.1.5 Impulse response (damped case). Referring now to the case where the tip mass is connected to a viscous damper, the impulse response is governed by the equation

$$\ddot{z}(t) + 2\xi_0\omega_0\dot{z}(t) + \omega_0^2 z(t) = \frac{1}{M}\delta(t) \tag{10.37}$$

with null initial displacement and velocity at time $t = 0_-$. Similarly to what discussed for the undamped case, balance of impulses between left- and right-hand sides of Eq. (10.37) implies that both $\dot{z}$ and z do not contain impulsive functions, and thus are finite quantities at the origin. Integrating between 0_- and 0_+ yield

$$\dot{z}(0_+) - \dot{z}(0_-) = \frac{1}{M} \tag{10.38}$$

Therefore, using the initial condition on the velocity we have

$$\dot{z}(0_+) = \frac{1}{M} \tag{10.39}$$

The problem is then transformed into the following

$$\ddot{z}(t) + 2\xi_0\omega_0 z(t) + \omega_0^2 z(t) = 0 \tag{10.40}$$

with initial conditions

$$z(0_+) = 0 \qquad\qquad \dot{z}(0_+) = \frac{1}{M} \tag{10.41}$$

Referring to the solution in Eq. (10.21), we can write for $t > 0$

$$\boxed{z(t) = \frac{1}{M\omega_{0d}}e^{-\xi_0\omega_0 t}\sin\omega_{0d}t} \tag{10.42}$$

10.2.1.6 Doublet response (damped case). The governing equation in this case is the following

$$\ddot{z}(t) + 2\xi_0\omega_0\dot{z}(t) + \omega_0^2 z(t) = \frac{1}{M}\dot{\delta}(t) \tag{10.43}$$

Using the same procedure adopted in the undamped case, it is left to the reader to verify that the solution in this case for $t > 0$ is given by

$$\boxed{z(t) = \frac{1}{M}e^{-\xi_0\omega_0 t}\left[\cos\omega_{0d}t + \frac{\xi_0\omega_0}{\omega_{0d}}\sin\omega_{0d}t\right]} \tag{10.44}$$

10.2.1.7 Step response (undamped case). Another response of great importance in vibration is the response to step loads. Referring back to the undamped massless cantilever beam with tip mass, the problem is governed by the following equation

$$\ddot{z}(t) + \omega_0^2 z(t) = \frac{1}{M}H(t) \tag{10.45}$$

where $H(t)$ is the unit Heaviside function. The general solution for $t > 0$ takes the form

$$z(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{1}{M\omega_0^2} \tag{10.46}$$

where C_1 and C_2 must be determined according to the initial conditions. It follows that

$$\boxed{z(t) = z_0\cos\omega_0 t + \frac{\dot{z}_0}{\omega_0}\sin\omega_0 t + \frac{1}{K}\left(1 - \cos\omega_0 t\right)} \tag{10.47}$$

10.2.1.8 General response (undamped case). Previous cases are referred to the dynamic response of SDOF systems due to specific loading conditions (impulse, doublet, step). Let consider now the computation of the general response of an undamped spring-mass system to *arbitrary excitation* $f(t)$ and non-null initial conditions $z(0) = z_0$ and $\dot{z}(0) = \dot{z}_0$. The governing equation is the following

$$\ddot{z}(t) + \omega_0^2 z(t) = \frac{1}{M}f(t) \tag{10.48}$$

Let's derive the general solution using the Laplace transformation method, which is widely adopted in the study of linear time-invariant systems. It can be considered as an efficient method for solving linear ordinary differential equations with constant coefficients since it transforms the differential problem into a simple algebraic method and it can treat discontinuous functions without any particular difficulty. If we denote by $Z(s)$ the Laplace transform of $z(t)$, we know that the first and second time derivative of $z(t)$ will have the following Laplace transforms, respectively,

$$\mathcal{L}\left[\dot{z}(t)\right] = sZ(s) - z(0) = sZ(s) - z_0 \tag{10.49}$$

and

$$\mathcal{L}\left[\ddot{z}(t)\right] = s^2 Z(s) - sz(0) - \dot{z}(0) = s^2 Z(s) - sz_0 - \dot{z}_0 \tag{10.50}$$

Therefore, the equation of motion (10.48) can be written as

$$s^2 Z(s) + \omega_0^2 Z(s) = \frac{1}{M}F(s) + sz_0 + \dot{z}_0 \tag{10.51}$$

where $F(s)$ is the Laplace transform of $f(t)$. Rearranging terms, we obtain the response in this form

$$Z(s) = \frac{1}{M}\frac{1}{s^2 + \omega_0^2}F(s) + \frac{s}{s^2 + \omega_0^2}z_0 + \frac{1}{s^2 + \omega_0^2}\dot{z}_0 \tag{10.52}$$

The corresponding time-domain solution is computed by taking the inverse Laplace transform of each quantity in the previous relation. We can write

$$z(t) = \frac{1}{M}\mathcal{L}^{-1}\left[\frac{1}{s^2 + \omega_0^2}F(s)\right] + \mathcal{L}^{-1}\left[\frac{s}{s^2 + \omega_0^2}\right]z_0 + \mathcal{L}^{-1}\left[\frac{1}{s^2 + \omega_0^2}\right]\dot{z}_0 \tag{10.53}$$

Since we have

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega_0^2}F(s)\right] = \frac{1}{\omega_0}\int_0^t \sin\left[\omega_0(t-\tau)\right] f(\tau)d\tau \tag{10.54}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega_0^2}\right] = \cos\omega_0 t \tag{10.55}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega_0^2}\right] = \frac{1}{\omega_0}\sin\omega_0 t \tag{10.56}$$

the general response can be written as

$$\boxed{z(t) = \frac{1}{M\omega_0}\int_0^t \sin\left[\omega_0(t-\tau)\right] f(\tau)d\tau + z_0\cos\omega_0 t + \frac{\dot{z}_0}{\omega_0}\sin\omega_0 t} \tag{10.57}$$

Note that the Laplace transformation method has obtained both the response to external excitation and the response due to non-null initial conditions simultaneously. Note also that, since Eq. (10.57) is the most general form of the solution, all the cases presented so far can be considered as particular cases. Indeed, as illustrative examples, let refer to the impulse and step responses. Let us assume that the initial conditions of the problem are null in both cases. The unit impulse response can be obtained as

$$z(t) = \frac{1}{M\omega_0}\int_0^t \sin\left[\omega_0(t-\tau)\right]\delta(\tau)d\tau = \frac{\sin\omega_0 t}{M\omega_0} \tag{10.58}$$

The unit step response is given by

$$\begin{aligned} z(t) &= \frac{1}{M\omega_0}\int_0^t \sin\left[\omega_0(t-\tau)\right] H(\tau)d\tau = \frac{1}{M\omega_0}\int_0^t \sin\left[\omega_0(t-\tau)\right] d\tau \\ &= \frac{1}{M\omega_0^2}\cos\omega_0(t-\tau)\Big|_0^t = \frac{1}{K}\left(1-\cos\omega_0 t\right) \end{aligned} \tag{10.59}$$

10.2.1.9 General response (damped case). Let consider now the computation of the general response of a viscously damped second-order system to arbitrary excitation $f(t)$ and non-null initial conditions $z(0) = z_0$ and $\dot{z}(0) = \dot{z}_0$. The governing equation is the following

$$\ddot{z}(t) + 2\xi_0\omega_0\dot{z}(t) + \omega_0^2 z(t) = \frac{1}{M}f(t) \tag{10.60}$$

Let's derive the general solution using again the Laplace transformation method. The equation of motion is written as

$$\left(s^2 + 2\xi_0\omega_0 s + \omega_0^2\right)Z(s) = \frac{1}{M}F(s) + \left(s + 2\xi_0\omega_0\right)z_0 + \dot{z}_0 \tag{10.61}$$

Therefore, the Laplace transform of the displacement variable z is given by

$$Z(s) = \frac{1}{M}\frac{1}{s^2+2\xi_0\omega_0 s+\omega_0^2}F(s) + \frac{s+2\xi_0\omega_0}{s^2+2\xi_0\omega_0 s+\omega_0^2}z_0 + \frac{1}{s^2+2\xi_0\omega_0 s+\omega_0^2}\dot{z}_0 \tag{10.62}$$

The corresponding time-domain solution is computed by taking the inverse Laplace transform of each quantity in the previous relation as follows

$$z(t) = \frac{1}{M}\mathcal{L}^{-1}\left[\frac{1}{s^2+2\xi_0\omega_0 s+\omega_0^2}F(s)\right] + \mathcal{L}^{-1}\left[\frac{s+2\xi_0\omega_0}{s^2+2\xi_0\omega_0 s+\omega_0^2}\right]z_0 + \mathcal{L}^{-1}\left[\frac{1}{s^2+2\xi_0\omega_0 s+\omega_0^2}\right]\dot{z}_0 \tag{10.63}$$

Using the partial fractions method, we have

$$\frac{1}{s^2+2\xi_0\omega_0 s+\omega_0^2} = \frac{1}{\lambda_1-\lambda_2}\left(\frac{1}{s-\lambda_1} - \frac{1}{s-\lambda_2}\right) \tag{10.64}$$

and

$$\frac{s+2\xi_0\omega_0}{s^2+2\xi_0\omega_0 s+\omega_0^2} = \frac{2\xi_0\omega_0+\lambda_1}{\lambda_1-\lambda_2}\frac{1}{s-\lambda_1} - \frac{2\xi_0\omega_0+\lambda_2}{\lambda_1-\lambda_2}\frac{1}{s-\lambda_2} \tag{10.65}$$

where

$$\lambda_1 = -\xi_0\omega_0 + j\omega_{0d} \qquad \lambda_2 = -\xi_0\omega_0 - j\omega_{0d} \tag{10.66}$$

Therefore,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^2+2\xi_0\omega_0 s+\omega_0^2}\right] &= \frac{1}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)\\ &= \frac{1}{2j\omega_{0d}}e^{-\xi_0\omega_0 t}\left(e^{j\omega_{0d}t}-e^{-j\omega_{0d}t}\right)\\ &= \frac{1}{\omega_{0d}}e^{-\xi_0\omega_0 t}\sin\omega_{0d}t\end{aligned} \tag{10.67}$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s+2\xi_0\omega_0}{s^2+2\xi_0\omega_0 s+\omega_0^2}\right] &= \frac{\xi_0\omega_0+j\omega_{0d}}{2j\omega_{0d}}e^{-\xi_0\omega_0 t}e^{j\omega_{0d}t} - \frac{\xi_0\omega_0-j\omega_{0d}}{2j\omega_{0d}}e^{-\xi_0\omega_0 t}e^{-j\omega_{0d}t}\\ &= e^{-\xi_0\omega_0 t}\left[\left(\frac{\xi_0\omega_0}{2j\omega_{0d}}+\frac{1}{2}\right)e^{j\omega_{0d}t} - \left(\frac{\xi_0\omega_0}{2j\omega_{0d}}-\frac{1}{2}\right)e^{-j\omega_{0d}t}\right]\\ &= e^{-\xi_0\omega_0 t}\left[\cos\omega_{0d}t+\frac{\xi_0\omega_0}{\omega_{0d}}\sin\omega_{0d}t\right]\end{aligned} \tag{10.68}$$

The response of the system takes the following form

$$\boxed{z(t) = \frac{1}{M\omega_{0d}}\int_0^t e^{-\xi_0\omega_0(t-\tau)}\sin\left[\omega_{0d}(t-\tau)\right]f(\tau)d\tau + z_0 e^{-\xi_0\omega_0 t}\cos\omega_{0d}t + \frac{\dot{z}_0+z_0\xi_0\omega_0}{\omega_{0d}}\sin\omega_{0d}t} \tag{10.69}$$

It is worth noting that when $\xi_0 = 0$, the above solution will yield the undamped system response derived in the last paragraph.

10.2.2 SDOF Ritz-Galerkin models

As already outlined, equivalent SDOF vibrating systems can be also obtained using a single admissible function in the Ritz-Galerkin discretization procedure. Let consider, for example, the simply supported uniform beam discussed in Chapter 7. The beam has length ℓ, mass per unit length m and flexural rigidity EJ. Since in this case we are interested in the computation of the dynamic response, the beam is assumed to be subjected to a transverse distributed load per unit length $p(x,t) = p_0 H(t)$, where $H(t)$ is the unit step function. According to the Euler-Bernoulli beam theory and neglecting rotary inertia, the principle of virtual work for the problem under study may be written as follows

$$\int_0^\ell \delta w_{/xx}(x,t)EJw_{/xx}(x,t)\,dx = -\int_0^\ell \delta w(x,t)m\ddot{w}(x,t)\,dx + \int_0^\ell \delta w(x,t)p_0\,dx\,H(t) \tag{10.70}$$

where, as usual, $w(x,t)$ is the transverse displacement of a generic point of the beam at coordinate x and time instant t. The origin of the coordinate system is taken at the left boundary. The problem has an exact solution, which was previously derived using an infinite summation of exact eigenfunctions $\sin(n\pi x/\ell)$. The mode displacement solution is reported here for the sake of completeness in terms of transverse motion, bending moment and shear force as

$$w(x,t) = \frac{4p_0\ell^4}{EJ\pi^5}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^5}\sin\left(\frac{n\pi x}{\ell}\right)\left[1-\cos\left(\omega_n t\right)\right] \tag{10.71}$$

$$M(x,t) = -\frac{4p_0\ell^2}{\pi^3}\sum_{n=1,3,5,\dots}^{N}\frac{1}{n^3}\sin\left(\frac{n\pi x}{\ell}\right)\left[1-\cos(\omega_n t)\right] \tag{10.72}$$

$$V(x,t) = -\frac{4p_0\ell}{\pi^2}\sum_{n=1,3,5,\dots}^{N}\frac{1}{n^2}\cos\left(\frac{n\pi x}{\ell}\right)\left[1-\cos(\omega_n t)\right] \tag{10.73}$$

The exact solution is used here again as a comparison with regard to the approximate one-term solutions adopted in Chapter 7.

The first approximate single-term solution is sought by taking

$$w(x,t) = W(x)u(t) \tag{10.74}$$

where the admissible function is

$$W(x) = x\,(\ell - x)$$

After putting the above approximate function into the principle of virtual work, the following ordinary differential equation is obtained

$$\tilde{m}\,\ddot{u}(t) + \tilde{k}\,u(t) = \tilde{f}(t) \tag{10.75}$$

where the generalized mass, stiffness and load terms are given, respectively, by

$$\tilde{m} = \int_0^\ell mW^2(x)\,dx = \frac{m\ell^5}{30}$$

$$\tilde{k} = \int_0^\ell EJW^2_{/xx}(x)\,dx = 4EJ\ell$$

and

$$\tilde{f}(t) = \int_0^\ell p(x,t)W(x)\,dx = \frac{p_0\ell^3}{6}H(t)$$

The equation of motion (10.75) can be also written as

$$\ddot{u}(t) + \tilde{\omega}^2 u(t) = \frac{5p_0}{m\ell^2}H(t) \tag{10.76}$$

which describes the forced motion of an undamped single-degree-of-freedom oscillator having natural frequency

$$\tilde{\omega} = \sqrt{120}\sqrt{\frac{EJ}{m\ell^4}} \approx 10.9545\sqrt{\frac{EJ}{m\ell^4}} \tag{10.77}$$

Assuming the beam at rest before the application of the step loading, the solution of Eq. (10.76) in terms of the generalized coordinate $u(t)$ may be written as

$$u(t) = \frac{5p_0}{m\ell^2\tilde{\omega}^2}\left[1 - \cos(\tilde{\omega}t)\right] = \frac{p_0\ell^2}{24EJ}\left[1 - \cos(\tilde{\omega}t)\right] \qquad (t > 0)$$

The approximate transverse displacement is thus given by

$$w(x,t) = x\,(\ell - x)\,\frac{p_0\ell^2}{24EJ}\left[1 - \cos(\tilde{\omega}t)\right] \qquad (t > 0) \tag{10.78}$$

Direct recovery of bending moment and shear force along the beam is obtained by spatial derivatives of the above displacement solution as follows

$$M(x,t) = EJw_{/xx}(x,t) = -\frac{p_0\ell^2}{12}\left[1 - \cos(\tilde{\omega}t)\right] \qquad (t > 0)$$

$$V(x,t) = EJw_{/xxx}(x,t) = 0$$

We can notice that the approximate bending moment solution is composed of a quasi-static contribution $-p_0\ell^2/12$ and a dynamic contribution. Instead, approximate shearing force is exactly null. The resulting estimates are clearly very far from the exact solutions, as shown in Figure 10.1 where exact bending moment at $x = \ell/2$ and shearing force at $x = 0$ obtained with a superposition of $N = 5$ eigenfunctions are compared with the above approximate solutions.

Recovery of bending moment and shear force by the mode acceleration method yields better estimates. Following a displacement-based approach, the equation of motion of the vibrating beam under study for $t > 0$ can be written as follows

$$EJw_{/xxxx}(x,t) = p_0 - m\ddot{w}(x,t)$$

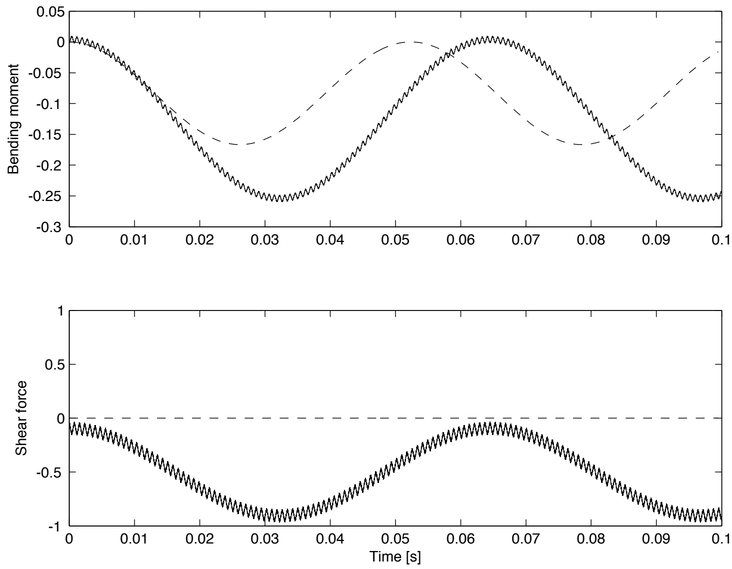

Figure 10.1 Direct recovery of bending moment and shearing force through the mode displacement method. Comparison of exact (solid lines) and approximate (dash lines) solutions based on single trial function $W(x) = x(\ell - x)$.

The inertial force $-m\ddot{w}$ is approximated by taking the second derivative of the displacement solution in Eq. (10.78). It follows that

$$EJw_{/xxxx}(x,t) = p_0 - \frac{5p_0}{\ell^2} x(\ell - x)\cos(\tilde{\omega}t)$$

Integrating two times yields

$$\begin{aligned} EJw_{/xxx} &= p_0 x - \frac{5p_0}{\ell^2}\int x(\ell - x)\,dx\ \cos(\tilde{\omega}t) + c_1 \\ &= p_0 x + \frac{5p_0}{6\ell^2}\left(2x^3 - 3\ell x^2\right)\cos(\tilde{\omega}t) + c_1 \end{aligned}$$

$$\begin{aligned} EJw_{/xx} &= \frac{p_0 x^2}{2} + \frac{5p_0}{6\ell^2}\int \left(2x^3 - 3\ell x^2\right)dx\ \cos(\tilde{\omega}t) + c_1 x + c_2 \\ &= \frac{p_0 x^2}{2} + \frac{5p_0}{12\ell^2}\left(x^4 - 2\ell x^3\right)\cos(\tilde{\omega}t) + c_1 x + c_2 \end{aligned}$$

Constants of integration c_1 and c_2 are obtained by imposing the natural boundary conditions of null bending moment at boundaries $EJw_{/xx}(0) = 0$ and $EJw_{/xx}(\ell) = 0$. We obtain

$$c_2 = 0$$

and

$$c_1 = -\frac{p_0\ell}{2} + \frac{5p_0\ell}{12}\cos(\tilde{\omega}t)$$

Therefore, approximate bending moment and shearing force are given in this case by

$$M(x,t) = \frac{p_0 x}{2}(x - \ell) + \frac{5p_0\ell x}{12}\left[1 + \left(\frac{x}{\ell}\right)^3 - 2\left(\frac{x}{\ell}\right)^2\right]\cos(\tilde{\omega}t) \tag{10.79}$$

$$V(x,t) = p_0\left(x - \frac{\ell}{2}\right) + \frac{5p_0\ell}{12}\left[1 + 4\left(\frac{x}{\ell}\right)^3 - 6\left(\frac{x}{\ell}\right)^2\right]\cos(\tilde{\omega}t) \tag{10.80}$$

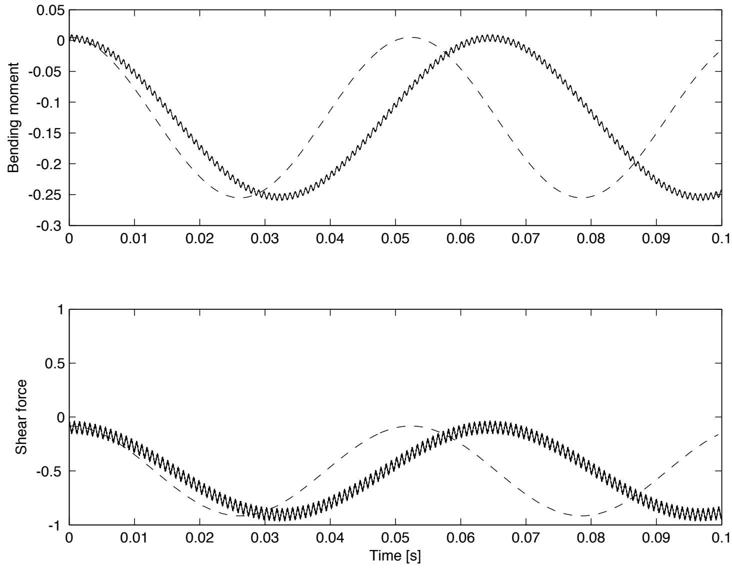

Figure 10.2 Recovery of bending moment and shearing force through the mode acceleration method. Comparison of exact (solid lines) and approximate (dash lines) solutions based on single trial function $W(x) = x(\ell - x)$.

Now, the quasi-static term is recovered exactly. This helps in improving the estimates as shown in Figure 10.2. Note that the maximum and minimum values of the stress resultants are correctly estimated compared to the exact solution obtained using five eigenfunctions. However, the time response is badly approximated since the adopted trial function yields a large error in estimating the oscillation period.

The above dynamic solution has been derived through the mode acceleration method based on a displacement approach. It is worth noting that, as done for the exact response of continuous systems, we can rely on a direct-summation-of-forces approach to recover stress resultants through the mode acceleration method. Referring back to Figure 3.19, equilibrium of moments about $x = \ell$ is given in this case by

$$X_0\ell + \int_0^\ell \left[p_0 - \frac{5p_0}{\ell^2}x(\ell - x)\cos(\tilde{\omega}t)\right](\ell - x)dx = 0$$

From this equation it follows that

$$X_0(t) = \frac{5p_0\ell}{12}\cos(\tilde{\omega}t) - \frac{p_0\ell}{2}$$

Considerations on symmetry yields

$$X_\ell(t) = X_0(t)$$

Therefore, referring back to Figure 3.20, we have

$$X_0(t) + \int_0^x \left[p_0 - \frac{5p_0}{\ell^2}\xi(\ell - \xi)\cos(\tilde{\omega}t)\right]d\xi = V(x,t)$$

and

$$X_0(t)x + \int_0^x \left[p_0 - \frac{5p_0}{\ell^2}\xi(\ell - \xi)\cos(\tilde{\omega}t)\right](\ell - \xi)d\xi = M(x,t)$$

It is left to the reader to verify that the above equations lead to results in Eqs. (10.79) and (10.80).

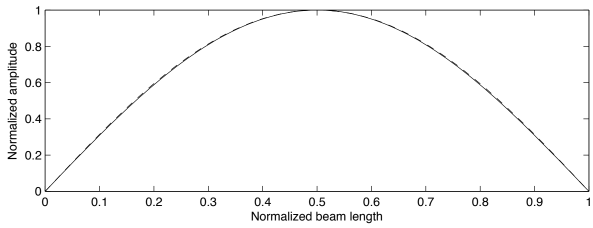

Figure 10.3 Comparison between exact (solid line) and approximate (dash line) shape of the fundamental natural mode. Approximation is based on the trial function $W(x) = x^4 - 2\ell x^3 + \ell^3 x$.

A second way of computing the dynamic response of the equivalent SDOF model of the simply supported beam under investigation is to use the following fourh-order approximation for the transverse displacement

$$w(x,t) = \left(x^4 - 2\ell x^3 + \ell^3 x\right) u(t) \tag{10.81}$$

The corresponding equation of motion is

$$\tilde{m}\,\ddot{u}(t) + \tilde{k}\,u(t) = \tilde{f}(t) \tag{10.82}$$

where, in this case, the generalized mass, stiffness and load terms are given, respectively, by

$$\tilde{m} = \int_0^\ell m\left(x^4 - 2\ell x^3 + \ell^3 x\right)^2\,dx = \frac{31}{630} m\ell^9$$

$$\tilde{k} = \int_0^\ell EJ\left[12x(x-\ell)\right]^2\,dx = \frac{24}{5} EJ\ell^5$$

and

$$\tilde{f}(t) = \int_0^\ell p(x,t)\left(x^4 - 2\ell x^3 + \ell^3 x\right)\,dx = \frac{p_0 \ell^5}{5} H(t)$$

Equation of motion (10.82) can be also written as

$$\ddot{u}(t) + \tilde{\omega}^2 u(t) = \frac{126}{31}\frac{p_0}{m\ell^4} H(t) \tag{10.83}$$

where the approximate natural frequency is given by

$$\tilde{\omega} = \sqrt{\frac{3024}{31}}\sqrt{\frac{EJ}{m\ell^4}} \approx 9.8767\sqrt{\frac{EJ}{m\ell^4}} \tag{10.84}$$

The solution of Eq. (10.83) in terms of generalized coordinate $u(t)$ may be written as

$$u(t) = \frac{1}{24}\frac{p_0}{EJ}\left[1 - \cos(\tilde{\omega}t)\right] \qquad (t > 0)$$

The approximate transverse displacement is thus given by

$$w(x,t) = \left(x^4 - 2\ell x^3 + \ell^3 x\right)\frac{1}{24}\frac{p_0}{EJ}\left[1 - \cos(\tilde{\omega}t)\right] \qquad (t > 0) \tag{10.85}$$

Direct recovery of bending moment $M(x,t)$ and shearing force $V(x,t)$ yields, respectively,

$$M(x,t) = \frac{p_0 x}{2}(x - \ell)\left[1 - \cos(\tilde{\omega}t)\right] \qquad (t > 0)$$

$$V(x,t) = p_0\left(x - \frac{\ell}{2}\right)\left[1 - \cos(\tilde{\omega}t)\right] \qquad (t > 0)$$

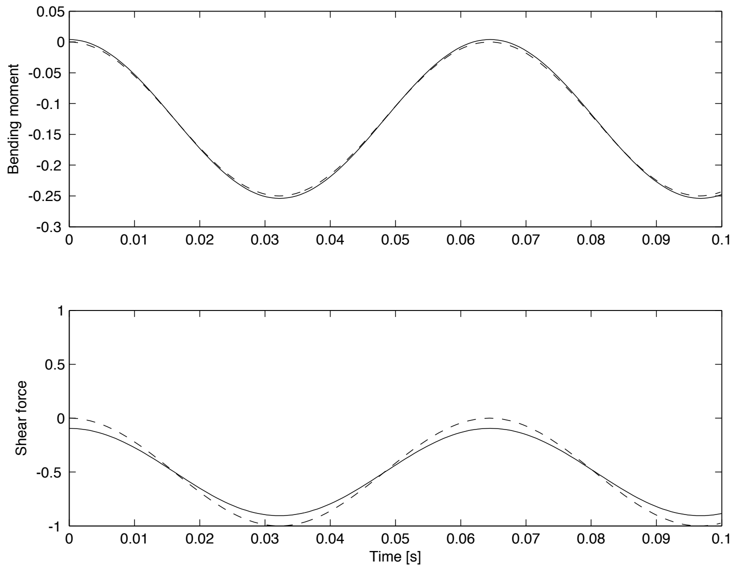

Figure 10.4 Recovery of bending moment and shearing force through the mode displacement method (direct recovery). Comparison of exact (solid lines) and approximate (dash lines) solutions based on single trial function $W(x) = \left(x^4 - 2\ell x^3 + \ell^3 x\right)$.

Following the procedure previously outlined, recovery of bending moment and shearing force through the mode acceleration method gives

$$M(x,t) = \frac{p_0 x}{2}(x-\ell) + \frac{21}{155}p_0 \ell x \left[3 - \left(\frac{x}{\ell}\right)^5 + 3\left(\frac{x}{\ell}\right)^4 - 5\left(\frac{x}{\ell}\right)^2\right]\cos(\tilde{\omega}t)$$

$$V(x,t) = p_0\left(x - \frac{\ell}{2}\right) + \frac{63}{155}p_0 \ell \left[1 - 2\left(\frac{x}{\ell}\right)^5 + 5\left(\frac{x}{\ell}\right)^4 - 5\left(\frac{x}{\ell}\right)^2\right]\cos(\tilde{\omega}t)$$

Comparisons of above approximate results with exact one-mode solutions are shown in Figure 10.4 and 10.5. It may be noticed that, in this case, very good estimate of bending moment at $x = \ell/2$ is obtained even with direct recovery from displacement solution. Accurate approximation of shearing force at $x = 0$ requires the adoption of the mode acceleration method.

EXAMPLE 10.1 Fixed-simply-supported beam

Consider a uniform fixed-simply-supported beam (see Figure 10.6) of length ℓ, mass per unit length m and flexural rigidity EJ, subjected to a distributed step load $p(x,t) = p_0 H(t)$. Determine the one-term solution for the dynamic response of the system in terms of bending moment through the mode acceleration method using a direct-summation-of-forces approach. Assume a trial function as an appropriate fourth-order polynomial.

As required, the single trial function has the general form

$$W(x) = ax^4 + bx^3 + cx^2 + dx + e \tag{10.86}$$

The actual polynomial can be selected by searching for the coefficients of $W(x)$ which satisfy the essential and natural boundary conditions of the problem. As seen before, this corresponds to the static displacement curve of the beam under its own weight.

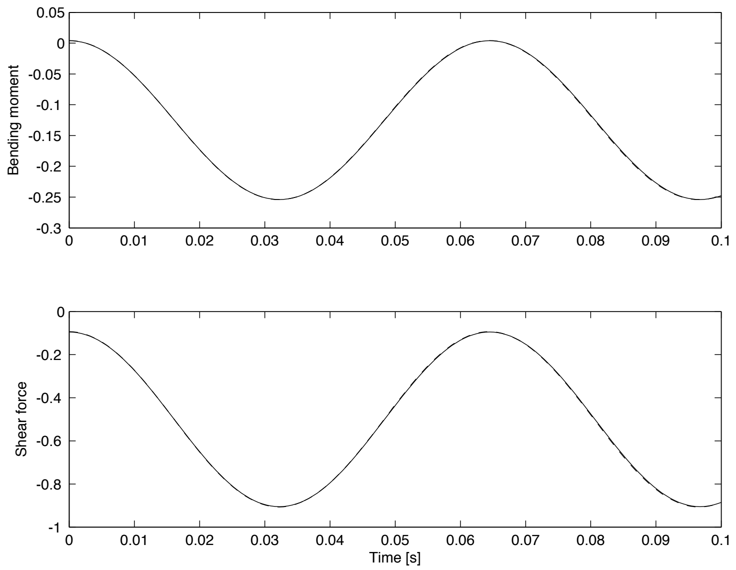

Figure 10.5 Recovery of bending moment and shearing force through the mode acceleration method. Comparison of exact (solid lines) and approximate (dash lines) solutions based on single trial function $W(x) = \left(x^4 - 2\ell x^3 + \ell^3 x\right)$.

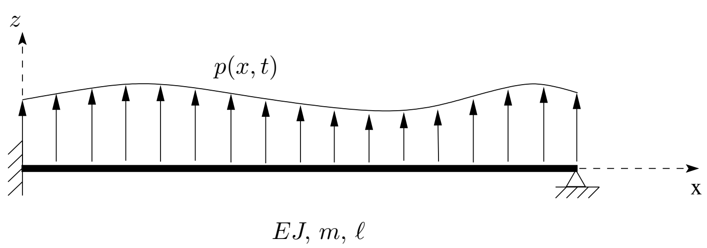

Figure 10.6 A uniform fixed-simply-supported beam subjected to a transverse distributed load $p(x,t)$.

Since the beam is fixed-simply-supported, we have to impose the following conditions

$$\begin{aligned} &W(0) = 0 \\ &W_{/x}(0) = 0 \\ &W(\ell) = 0 \\ &W_{/xx}(\ell) = 0 \end{aligned}$$

where the last equation implies the vanishing of bending moment at the simple support. It follows that

$$\begin{aligned} &e = 0 \\ &d = 0 \\ &b = -\frac{5}{2}a\ell \\ &c = \frac{3}{2}a\ell^2 \end{aligned}$$

The resulting shape function is defined up to a multiplicative coefficient a, which is taken as the generalized coordinate of the approximate problem. Then, we may write the assumed solution for the transverse displacement of the beam as

$$w(x,t) = \left(2x^4 - 5\ell x^3 + 3\ell^2 x^2\right) u(t)$$

Inserting the assumed solution into the principle of virtual work and exploiting the arbitrariness of the virtual variation $\delta u(t)$ we find the governing equation of motion

$$\tilde{m}\,\ddot{u}(t) + \tilde{k}\,u(t) = \tilde{f}(t) \tag{10.87}$$

where the generalized mass, stiffness and load terms are given, respectively, by

$$\tilde{m} = m \int_0^\ell \left(2x^4 - 5\ell x^3 + 3\ell^2 x^2\right)^2 dx = \frac{19}{630} m\ell^9$$

$$\tilde{k} = EJ \int_0^\ell \left(24x^2 - 30\ell x + 6\ell^2\right)^2 dx = \frac{36}{5} EJ\ell^5$$

and

$$\tilde{f}(t) = \int_0^\ell p(x,t)\left(2x^4 - 5\ell x^3 + 3\ell^2 x^2\right) dx = \frac{3}{20} p_0 \ell^5 H(t)$$

Equation of motion (10.87) can be also written as

$$\ddot{u}(t) + \tilde{\omega}^2 u(t) = \frac{189}{38}\frac{p_0}{m\ell^4} H(t) \tag{10.88}$$

where the approximate natural frequency is given by

$$\tilde{\omega} = \sqrt{\frac{4536}{19}}\sqrt{\frac{EJ}{m\ell^4}} \approx 15.4511\sqrt{\frac{EJ}{m\ell^4}} \tag{10.89}$$

Comparing this result with the exact value of the fundamental mode given by[1]

$$\omega_1 = (3.9266)^2 \sqrt{\frac{EJ}{m\ell^4}} \approx 15.4182\sqrt{\frac{EJ}{m\ell^4}}$$

we note that the error in the estimation is

$$e = \frac{\tilde{\omega} - \omega_1}{\omega_1} = 0.21\% \tag{10.90}$$

A good estimate has been obtained since the trial function is close to the actual fundamental mode shape. The solution of Eq. (10.88) in terms of the generalized coordinate $u(t)$ may be written as

$$u(t) = \frac{1}{48}\frac{p_0}{EJ}\left[1 - \cos(\tilde{\omega} t)\right] \qquad (t > 0)$$

The approximate transverse displacement is thus given by

$$w(x,t) = \frac{p_0}{48EJ} W(x)\left[1 - \cos(\tilde{\omega}t)\right] \qquad (t > 0) \tag{10.91}$$

The second temporal derivative is

$$\ddot{w}(x,t) = \frac{189}{38}\frac{p_0}{m\ell^4} W(x)\cos(\tilde{\omega}t) \qquad (t > 0) \tag{10.92}$$

Since in this case the beam is statically indeterminate, we must first solve for the indeterminate. Taking as indeterminate the reaction force X_ℓ at the simple support, the equilibrium in the vertical direction is written as follows

$$X_0 + X_\ell + p_0\ell - \frac{189}{38}\frac{p_0}{\ell^4}\int_0^\ell W(x)dx\cos(\tilde{\omega}t) = 0$$

whereas, equilibrium of moments about $x = 0$ gives

$$X_\ell \ell + \frac{p_0\ell^2}{2} - \frac{189}{38}\frac{p_0}{\ell^4}\int_0^\ell W(x)xdx\cos(\tilde{\omega}t) - M_0 = 0$$

Therefore, reaction force and moment at clamped end are given by

$$\begin{aligned} X_0 &= \frac{567}{760} p_0\ell\cos(\tilde{\omega}t) - X_\ell - p_0\ell \\ M_0 &= X_\ell\ell + \frac{p_0\ell^2}{2} - \frac{63}{152}p_0\ell^2\cos(\tilde{\omega}t) \end{aligned}$$

The internal moment is expressed as

$$\begin{aligned} M(x,t) &= M_0 + X_0 x + \int_0^x p_0(x-\xi)d\xi - \int_0^x m\ddot{w}(x-\xi)d\xi \\ &= X_\ell(\ell - x) + \frac{p_0}{2}(\ell - x)^2 - \frac{63}{152}p_0\ell^2\cos(\tilde{\omega}t) \\ &+ \frac{567}{760}p_0\ell x\cos(\tilde{\omega}t) - \frac{189}{38}\frac{p_0}{\ell^4}\int_0^x W(x)(x-\xi)d\xi \end{aligned} \tag{10.93}$$

The dummy internal moment is given by

$$M^\star(x,t) = \ell - x$$

The indeterminate can be obtained by solving the following equation

$$\int_0^\ell \frac{MM^\star}{EJ}dx = 0$$

Once X_ℓ has been determined, its value is substituted into Eq. (10.93) to yield the variation of bending moment along the beam.

10.3 Multi-degrees-of-freedom systems: reduced-order models

The SDOF models discussed in the previous section with equivalent *scalar* mass, damping and stiffness properties are in many practical cases a very rough approximation of the real dynamic behaviour. More degrees of freedom are typically required for an accurate estimation of the dynamic response. Therefore, the corresponding multi-degrees-of-freedom (MDOF) models are represented by mass, damping and stiffness *matrices* as in Eq. (10.1).

Many numerical methods, known as implicit or explicit direct integration methods, are available for the time-domain solution of the set of ordinary differential equations expressed in Eq. (10.1). Some of them will be presented in a later section.

[1] The exact solution can be derived by the procedure discussed in chapter **??**. In this case, the frequency equation is $\tan(\beta\ell) = \tanh(\beta\ell)$, which must be solved numerically to give the denumerable set of discrete values $\beta_n\ell$ and the corresponding natural frequencies $\omega_n = (\beta_n\ell)\sqrt{EJ/m\ell^4}$.

Alternatively, one might rely on the direct solution of the linear set of equations in the frequency-domain after a Fourier transform. This can be done if the system is asymptotically stable[2] and the forcing functions are band-limited. Introducing the Fourier transform of the generalized coordinates and loads as

$$\mathbf{u}(\omega) = \int_{-\infty}^{+\infty} \mathbf{u}(t) e^{-j\omega t} dt \qquad \mathbf{f}(\omega) = \int_{-\infty}^{+\infty} \mathbf{f}(t) e^{-j\omega t} dt$$

the model in the frequency-domain is described by

$$\left(\mathbf{K} + j\omega\mathbf{C} - \omega^2\mathbf{M}\right)\mathbf{u}(\omega) = \mathbf{f}(\omega) \tag{10.94}$$

which is a set of algebraic linear equations with complex coefficients in the unknowns $\mathbf{u}(\omega)$ for each discrete value of the circular frequency ω. Once solved with the aim of an appropriate numerical method, the time-domain solution is recovered by an inverse Fourier transform

$$\mathbf{u}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{u}(\omega) e^{j\omega t} d\omega$$

Note that the above inverse operation can be performed analytically only in few simple cases. Typically, one must resort to some numerical technique.

As outlined above, accurate analysis of complex vibrating systems typically requires a very large number of approximating functions. This is particularly true when functions with compact support are used such as in the finite element method. The resulting discretized MDOF models in Eq. (10.1) or their equivalent frequency-domain models in Eq. (10.94) can be of very high dimensions and the computation of the related time or frequency response can therefore be highly demanding.

At this point a question naturally arises: is it possible to reduce the size of the approximate model without giving up much accuracy? In other words, an efficient strategy is sought in order to construct a proper reduced-order model including the most significant dynamics of the original large model.

There are many ways of going about constructing discretized reduced-order models. The approach adopted here, which is the most common in the field of structural dynamics and takes the cue from the modal analysis approach for continuous systems, relies on *expressing the dynamics of the system in terms of global undamped modes*. As we will see, this can be accomplished for particular models of damping. It is the hope and expectation that a relatively small number of such modes will prove to be adequate to describe with good accuracy the vibration field. Such modes represent an approximation of the true eigenmodes of the elastic continuum since they are obtained by solving the discretized problem corresponding to the homogeneous model of Eq. (10.1) without damping. However, they form a complete set and any displacement field distribution can be expressed in terms of such modes. Within this approach, we can say that a reduced-order model is a *modal model including a small number of modes of interest for the problem under investigation*. Typically, we are interested in the dynamic response in the low-frequency range because this range corresponds to the frequency content of the most common excitations. Therefore, a reduced-order modal model contains the modes corresponding to the low frequencies of the system. All we need to do is to build a sufficiently accurate high-dimensional model using for example a Ritz-Galerkin approximation and then calculate the eigenvalues and eigenvectors of the corresponding homogeneous undamped problem. A relatively small number of such eigenpairs is then used to predict

[2]Note that the model in Eq. (10.1) is asymptotically stable if the damping matrix $\mathbf{C}$ is positive definite. This can be seen through the Lyapunov method. The total energy of the system given by the sum of the kinetic and potential energy is

$$E = \frac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}$$

Its time variation is given by

$$\frac{dE}{dt} = \dot{\mathbf{u}}^T\left[\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u}\right]$$

which can be written as

$$\frac{dE}{dt} = -\dot{\mathbf{u}}^T\mathbf{C}\dot{\mathbf{u}}$$

Therefore, the time variation of the energy of the system is always negative if and only if $\mathbf{C} > 0$.

the dynamic response. It is clear that the method is very similar to the analytical modal analysis used for continuous systems. The starting point for the calculation of the exact eigenmodes was a (system of) partial differential equation(s) and, as a result, a denumerably infinite number of eigenfrequencies and eigenfunctions was obtained. On the other hand, the calculation of the approximate modes to be used in a reduced-order discrete model is performed on a large- but finite-order set of ordinary differential equations and the result is a *finite number of eigenvalues and eigenvectors*. Note that, as shown later, such eigenvectors can be associated to a Ritz approximation involving continuous functions. Hence, the resulting modes have the same nature, although approximate, of the eigenfunctions of the corresponding continuous system.

From what discussed above, it is apparent that a careful analysis of the convergence properties of the solution will be required to determine the minimum number of modes to be retained to achieve a desired accuracy. This is discussed in the following.

Furthermore, since we are interested in only a small subset of all eigenvalues and eigenvectors of the system, it would be highly desirable to rely on computationally efficient and accurate numerical techniques for getting only a few eigenpairs from a large eigenvalue problem. This topic will be introduced in a later section.

10.4 Undamped modes

Irrespective of the type of Ritz approximation (global or local), let us consider a Ritz-Galerkin model of a linearly elastic continuum yielding a discretized N-degrees-of-freedom system. Since the system is linear and time-invariant, we can perform a modal analysis in order to find the approximate undamped natural frequencies and mode shapes.

Similarly to what outlined for continuous systems, the solution of discrete systems based on classical modal analysis is expressed as a superposition of freely vibrating undamped modes. Free undamped vibration is described by the following equations

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{0} \tag{10.95}$$

where, as usual, $\mathbf{M}$ and $\mathbf{K}$ are the discretized symmetric mass and stiffness matrices. After substituting into Eq. (10.95) an harmonic solution of the type

$$\mathbf{u}(t) = \mathbf{u}e^{j\omega t}$$

where $\mathbf{u}$ is a column matrix of the amplitudes of the generalized coordinates $\mathbf{u}(t)$, we obtain the well-known generalized eigenvalue problem

$$\boxed{\mathbf{K}\mathbf{u} = \omega^2\mathbf{M}\mathbf{u}} \tag{10.96}$$

which consists of a set of N homogeneous algebraic equations in the unknown vector $\mathbf{u}$. This set has always a trivial solution $\mathbf{u} = \mathbf{0}$ which implies no motion, i.e., $\mathbf{u}(t) = \mathbf{0}$. It has non-trivial solutions if

$$\det\left[\mathbf{K} - \omega^2\mathbf{M}\right] = 0 \tag{10.97}$$

which yields the characteristic or frequency equation of the system.

Note that, differently from the continuous case, the determinant of the symmetric algebraic eigenvalue problem (10.97), when expanded, leads always to a polynomial of order N which has N roots or eigenvalues $\lambda_i = \omega_i^2$. These roots are real and positive if the mass and stiffness matrices are symmetric and positive definite, and can be arranged from the smallest to the largest $\lambda_1 < \lambda_2 < \cdots < \lambda_N$.

In the case of a structure with rigid-body motions, the number of eigenvalues include also zero frequencies (they are six for a completely unconstrained 3-D body). For $\omega = 0$, $\mathbf{u}(t) = \mathbf{u} = \mathbf{u}_R$ and $\ddot{\mathbf{u}}(t) = \mathbf{0}$. Therefore, we have

$$\mathbf{K}\mathbf{u}_R = \mathbf{0}$$

which is obviously satisfied since the stiffness matrix is singular, i.e., any rigid-body motion does not produce elastic restoring forces.

It is also worth noting that the approximate natural frequencies of the system can be found directly by Eq. (10.97) if the size of the determinant is not too large. For most practical problems involving a large number N of degrees-of-freedom, various numerical methods are available to solve efficiently a symmetric algebraic eigenvalue problem.

For each root ω_i^2, the corresponding system

$$\mathbf{K}\mathbf{u}_i = \mathbf{M}\mathbf{u}_i\omega_i^2 \qquad (i = 1, 2, \ldots, N) \tag{10.98}$$

can be solved for the eigenvector $\mathbf{u}_i$ to within a multiplicative constant. The N eigenvalues and eigenvectors can be conveniently grouped into matrices. The eigenvalues can be assembled into the following diagonal matrix

$$\mathbf{\Omega}^2 = \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & \omega_N^2 \end{bmatrix} = \text{Diag}\left\{\omega_i^2\right\} \tag{10.99}$$

Eigenvectors can be assembled into a modal matrix $\in \mathbb{R}^{N\times N}$ as follows

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_N \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1N} \\ u_{21} & u_{22} & \dots & u_{2N} \\ \vdots & \vdots & & \\ u_{N1} & u_{N2} & \dots & u_{NN} \end{bmatrix} \tag{10.100}$$

where u_{ji} is the j-th component of the i-th eigenvector. By using above matrices, it is possible to convert all the relations in Eq. (10.98)

$$\begin{aligned} \mathbf{K}\mathbf{u}_1 &= \mathbf{M}\mathbf{u}_1\omega_1^2 \\ \mathbf{K}\mathbf{u}_2 &= \mathbf{M}\mathbf{u}_2\omega_2^2 \\ &\dots \\ \mathbf{K}\mathbf{u}_N &= \mathbf{M}\mathbf{u}_N\omega_N^2 \end{aligned}$$

into the following compact form

$$\boxed{\mathbf{K}\mathbf{U} = \mathbf{M}\mathbf{U}\text{Diag}\left\{\omega_i^2\right\}} \tag{10.101}$$

10.5 Modal superposition

Since any set of N independent vectors can be used as a basis for representing any other vector of order N, we may write

$$\boxed{\mathbf{u}(t) = \sum_{i=1}^{N} \mathbf{u}_i q_i(t) = \mathbf{U}\mathbf{q}(t)} \tag{10.102}$$

where we have used the approximate natural modes as such a basis. As in the continuous case, the coordinates $q_i(t)$ are called normal or modal coordinates. The physical displacement function will then be expressed as

$$\boxed{\mathbf{s}(\mathbf{x},t) = \mathbf{N}(\mathbf{x})\mathbf{u}(t) = \mathbf{N}(\mathbf{x})\mathbf{U}\mathbf{q}(t)} \tag{10.103}$$

where $\mathbf{N}(\mathbf{x})\mathbf{U}$ are the approximate eigenfunctions of the problem. The above relation can be viewed as a coordinate transformation from the infinite-dimensional physical space to the finite-dimensional space of modal coordinates. Note the similarity between Eq. (10.103) and the mode superposition expression in Eq. (3.64) for continuous systems. By substituting the above solution Eq. (10.103) into the principle of virtual work we obtain

$$\begin{aligned} &\delta\mathbf{q}^T\,\mathbf{U}^T\left[\int_{V_0} \mathbf{N}^T(\mathbf{x})\rho_0\mathbf{N}(\mathbf{x})\,dV_0\right]\mathbf{U}\,\ddot{\mathbf{q}} \\ &\quad + \delta\mathbf{q}^T\,\mathbf{U}^T\left[\int_{V_0} (\mathcal{D}\mathbf{N}(\mathbf{x}))^T\mathbf{C}\,(\mathcal{D}\mathbf{N}(\mathbf{x}))\,dV_0\right]\mathbf{U}\,\mathbf{q} = \\ &\delta\mathbf{q}^T\,\mathbf{U}^T\left[\int_{V_0} \mathbf{N}^T(\mathbf{x})\mathbf{f}(\mathbf{x},t)\,dV_0 + \int_{A_0^N} \mathbf{N}^T(\mathbf{x})\mathbf{t}(\mathbf{x},t)\,dA_0\right] \end{aligned}$$

The resulting equations of motion in the modal coordinates, after introducing the damping term, are

$$\boxed{\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\mathbf{q}}(t) + \mathbf{U}^T\mathbf{C}\mathbf{U}\dot{\mathbf{q}}(t) + \mathbf{U}^T\mathbf{K}\mathbf{U}\mathbf{q}(t) = \mathbf{U}^T\mathbf{f}(t)} \tag{10.104}$$

or, alternatively,

$$\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\mathbf{q}}(t) + \mathbf{U}^T\mathbf{C}\mathbf{U}\dot{\mathbf{q}}(t) + \mathbf{U}^T\mathbf{K}\mathbf{U}\mathbf{q}(t) = \mathbf{Q}(t) \tag{10.105}$$

where $\mathbf{Q}(t) = \mathbf{U}^T\mathbf{f}(t)$.

10.6 Orthogonality properties

As in the continuous case, the undamped eigenmodes $\mathbf{u}_i$ fulfill orthogonality conditions. Let us consider the i-th and j-th eigenmode. They satisfy the following relations

$$\begin{aligned}\mathbf{K}\mathbf{u}_i &= \omega_i^2\mathbf{M}\mathbf{u}_i\\ \mathbf{K}\mathbf{u}_j &= \omega_j^2\mathbf{M}\mathbf{u}_j\end{aligned}$$

After premultiplying by $\mathbf{u}_j^T$ the first equation and by $\mathbf{u}_i^T$ the second equation, we may write

$$\begin{aligned}\mathbf{u}_j^T\mathbf{K}\mathbf{u}_i &= \omega_i^2\mathbf{u}_j^T\mathbf{M}\mathbf{u}_i\\ \mathbf{u}_i^T\mathbf{K}\mathbf{u}_j &= \omega_j^2\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j\end{aligned}$$

The first can be transposed and, since $\mathbf{M}$ and $\mathbf{K}$ are symmetric, we have

$$\begin{aligned}\mathbf{u}_i^T\mathbf{K}\mathbf{u}_j &= \omega_i^2\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j\\ \mathbf{u}_i^T\mathbf{K}\mathbf{u}_j &= \omega_j^2\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j\end{aligned}$$

Subtraction of the second equation from the first yields

$$\left(\omega_i^2 - \omega_j^2\right)\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j = 0$$

Then, if the eigenfrequencies are distinct, i.e., $\omega_i^2 \neq \omega_j^2$, we have

$$\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j = 0 \qquad \text{for} \quad i \neq j \tag{10.106}$$

which also implies that

$$\mathbf{u}_i^T\mathbf{K}\mathbf{u}_j = 0 \qquad \text{for} \quad i \neq j \tag{10.107}$$

As well known, orthogonality properties shown above mean that the inertial (stiffness) forces developed in a given mode do not affect the motion of the other modes. In other words, the modes are mechanically independent.

Note that previous conditions (10.106) and (10.107) are not necessarily satisfied by rigid modes for which $\omega_i = 0$ and by multiple eigenmodes associated to the same eigenfrequency for which $\omega_i^2 = \omega_j^2$ $(i \neq j)$. Indeed, for generalized eigenvalue problems such as Eq. (10.96) with symmetric matrices, the multiplicity of a multiple eigenvalue is equal to the number of corresponding eigenvectors, i.e. algebraic multiplicity is always equal to geometric multiplicity. Nothing can be said about orthogonality properties of such eigenmodes with respect to the mass and stiffness matrix, since an arbitrary linear combination of those eigenmodes is again an eigenmode. However, orthogonality conditions for such eigenmodes can be always imposed *a posteriori* through a Gram-Schmidt orthogonalization procedure. Therefore, we can assume orthogonality properties for all approximate eigenmodes calculated by solving the eigenvalue problem (10.96).

The above orthogonality results can be written in matrix form as follows

$$\boxed{\mathbf{U}^T\mathbf{M}\mathbf{U} = \text{Diag}\left\{m_i\right\}} \tag{10.108}$$

$$\boxed{\mathbf{U}^T\mathbf{K}\mathbf{U} = \text{Diag}\left\{m_i\omega_i^2\right\}} \tag{10.109}$$

where $m_i = \mathbf{u}_i^T\mathbf{M}\mathbf{u}_i$ is the modal mass of mode i. Since the mode shapes can be scaled arbitrarily, it is a common practice to normalize them in such a way that $m_i = 1$.

10.7 Rayleigh's quotient

From what derived above, the i-th approximate eigenfrequency of the system can be obtained as follows

$$\omega_i^2 = \frac{\mathbf{u}_i^T \mathbf{K} \mathbf{u}_i}{\mathbf{u}_i^T \mathbf{M} \mathbf{u}_i} \qquad (i = 1, 2, \ldots, N) \tag{10.110}$$

where we can observe that the numerator is proportional to the potential energy of the system vibrating in the i-th mode

$$V = \frac{1}{2} \mathbf{u}_i^T \mathbf{K} \mathbf{u}_i \tag{10.111}$$

and the denominator is a measure of the kinetic energy. Equation (10.110) can be generalized by replacing ω_i^2 with ω^2 and $\mathbf{u}_i$ with an arbitrary N-dimensional vector $\mathbf{v}$

$$\mathcal{R}(\mathbf{v}) = \omega^2 = \frac{\mathbf{v}^T \mathbf{K} \mathbf{v}}{\mathbf{v}^T \mathbf{M} \mathbf{v}} \tag{10.112}$$

which is the expression of *Rayleigh's quotient* for an N-degrees-of-freedom system. Since any N-dimensional vector can be expressed as a linear combination of the mass-normalized eigenvectors $\mathbf{u}_i$ $(i = 1, 2, \ldots, N)$ we may write

$$\mathbf{v} = \sum_{i=1}^{N} c_i \mathbf{u}_i = \mathbf{U} \mathbf{c} \tag{10.113}$$

where vectors $\mathbf{u}_i$ can be considered the axes of a N-dimensional vector space. Then, using orthogonality properties (10.108) and (10.109) with $m_i = 1$, Rayleigh's quotient (10.112) becomes

$$\mathcal{R}(\mathbf{c}) = \frac{\mathbf{c}^T \mathbf{U}^T \mathbf{K} \mathbf{U} \mathbf{c}}{\mathbf{c}^T \mathbf{U}^T \mathbf{M} \mathbf{U} \mathbf{c}} = \frac{\mathbf{c}^T \boldsymbol{\Omega}^2 \mathbf{c}}{\mathbf{c}^T \mathbf{c}} = \frac{\displaystyle\sum_{i=1}^{N} \omega_i^2 c_i^2}{\displaystyle\sum_{i=1}^{N} c_i^2} \tag{10.114}$$

Similarly to what done directly on the single-term Ritz approximation, let us suppose that the trial vector $\mathbf{v}$ is in the neighborhood of the r-th modal vector $\mathbf{u}_r$. As such, all the projections of $\mathbf{v}$ on $\mathbf{u}_i$ with $i \neq r$ are small. We can state that

$$\frac{c_i}{c_r} = \epsilon_i \qquad (i = 1, \ldots, N;\, i \neq r) \tag{10.115}$$

where ϵ_i is a small quantity. Dividing top and bottom of Eq. (10.114) by c_r^2 and ignoring higher-order terms in ϵ_i^2 yields

$$\begin{aligned} \mathcal{R} = \frac{\omega_r^2 + \displaystyle\sum_{\substack{i=1 \\ i\neq r}}^{N} \omega_i^2 \epsilon_i^2}{1 + \displaystyle\sum_{\substack{i=1 \\ i\neq r}}^{N} \epsilon_i^2} &\approx \left(\omega_r^2 + \sum_{\substack{i=1 \\ i\neq r}}^{N} \omega_i^2 \epsilon_i^2 \right) \left(1 - \sum_{\substack{i=1 \\ i\neq r}}^{N} \epsilon_i^2 \right) \\ &\approx \omega_r^2 + \sum_{i=1}^{N} \left(\omega_i^2 - \omega_r^2 \right) \epsilon_i^2 \end{aligned}$$

We have found that Rayleigh's quotient is stationary in correspondence of an eigenvector. When $r = 1$, we have

$$\mathcal{R} = \omega_1^2 + \sum_{i=2}^{N} \left(\omega_i^2 - \omega_1^2 \right) \epsilon_i^2$$

Since $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_N$, we can conclude that

$$\mathcal{R} \geq \omega_1^2$$

10.8 Damped systems

10.8.1 Proportional damping

The discussion above shows that Equation (10.105) becomes

$$\boxed{\text{Diag}\,\{m_i\}\,\ddot{\mathbf{q}}(t) + \mathbf{U}^T\mathbf{C}\mathbf{U}\dot{\mathbf{q}}(t) + \text{Diag}\,\{m_i\omega_i^2\}\,\mathbf{q}(t) = \mathbf{Q}(t)} \tag{10.116}$$

or, in indicial notation $(i = 1, 2, \ldots, N)$,

$$m_i\ddot{q}_i(t) + \sum_{j=1}^{N} \mathbf{u}_i^T\mathbf{C}\mathbf{u}_j\,\dot{q}_j(t) + m_i\omega_i^2 q_i(t) = Q_i(t) \tag{10.117}$$

where

$$Q_i(t) = \mathbf{u}_i^T\mathbf{f}(t) \tag{10.118}$$

In general, $\mathbf{U}^T\mathbf{C}\mathbf{U}$ is a *full* modal damping matrix, i.e., the modal matrix is not able to diagonalize the damping matrix $\mathbf{C}$. This situation can occur in many applications, such as modeling of structures having concentrated devices including some damping mechanism.

There is a case in which the modal matrix diagonalizes the damping matrix, namely, when the damping matrix can be expressed as a linear combination of the mass and stiffness matrices as follows

$$\boxed{\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}} \tag{10.119}$$

where α and β are given constant scalars. Indeed, we have

$$\begin{aligned}\mathbf{U}^T\mathbf{C}\mathbf{U} &= \mathbf{U}^T\left(\alpha\mathbf{M} + \beta\mathbf{K}\right)\mathbf{U}\\ &= \alpha\,\text{Diag}\,\{m_i\} + \beta\,\text{Diag}\,\{m_i\omega_i^2\} = \text{Diag}\,\{m_i\left(\alpha + \beta\omega_i^2\right)\}\end{aligned}$$

This special case of damping is called *proportional damping* or *Rayleigh damping*. Assuming a damping of this form, the modal equations may be written as follows

$$\boxed{\text{Diag}\,\{m_i\}\,\ddot{\mathbf{q}}(t) + \text{Diag}\,\{2m_i\xi_i\omega_i\}\,\dot{\mathbf{q}}(t) + \text{Diag}\,\{m_i\omega_i^2\}\,\mathbf{q}(t) = \mathbf{Q}(t)} \tag{10.120}$$

where we have introduced the following notation

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right) \qquad (i = 1, 2, \ldots, N)$$

in which ξ_i are so-called *modal damping factors*. As a result, the present damping model generates a higher damping in the lower and higher frequency ranges, and a lower one in the intermediate range, as shown in Figure 10.7.

Under assumption (10.119), we may write the homogeneous equations of motion as follows

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \alpha\mathbf{M}\dot{\mathbf{u}}(t) + \beta\mathbf{K}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{0} \tag{10.121}$$

Transforming the above equation into the Laplace domain, we have

$$\left(\mathbf{M}s^2 + \alpha\mathbf{M}s + \beta\mathbf{K}s + \mathbf{K}\right)\mathbf{u}(s) = \mathbf{0} \tag{10.122}$$

Rearranging, we obtain

$$\left[\mathbf{K}\left(\beta s + 1\right) + \mathbf{M}\left(s^2 + \alpha s\right)\right]\mathbf{u}(s) = \mathbf{0} \tag{10.123}$$

and, introducing the parameter

$$\omega_d^2 = -\frac{s^2 + \alpha s}{\beta s + 1} \tag{10.124}$$

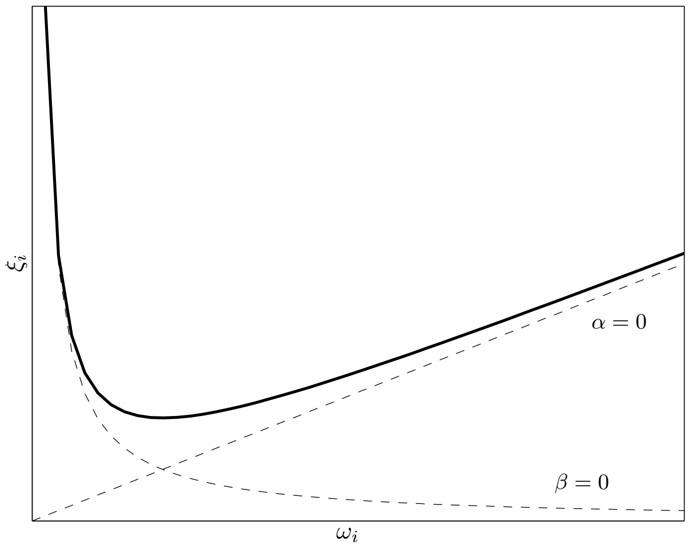

Figure 10.7 Proportional damping.

we can write the corresponding eigenvalue problem as

$$\left[\mathbf{K} - \omega_d^2 \mathbf{M}\right] \mathbf{u} = \mathbf{0} \tag{10.125}$$

This shows that, in the case of proportional damping, the solution may be expressed in terms of undamped modes. In other words, the analysis of damped systems with assumed proportional damping can be carried out using the theory of undamped modes. The difference is that modes are now associated to real roots ω_d^2, which corresponds to damped eigenvalues s obtained by solving Eq. (10.124).

Note also that Eq. (10.124) holds for each mode while keeping fixed the parameters α and β. From experimental vibration testing, it is observed that it is highly unlikely that the same α and β are associated to all actual frequencies. Indeed, it is not realistic that a complex phenomenon such as damping of a multiple degrees-of-freedom system can be fully described by only two parameters. It is argued that the proportional damping model has no real physical foundation.

10.8.2 Lightly damped structures

Even though the proportional damping can be considered as an unphysical damping model for real structures, it can be shown that a diagonal modal damping matrix is a good approximation for *lightly damped structures*.

Let us consider the free vibration problem associated to a discretized model with viscous damping matrix $\mathbf{C}$. For the sake of convenience, the equations of motions are rewritten here

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{0}$$

Solutions of the above set of ordinary differential equations with constant coefficients are of the form

$$\mathbf{u}(t) = \mathbf{v}e^{st} \tag{10.126}$$

Substituting Eq. (10.126) into the equations of motion yields for the i-th damped eigenvector $\mathbf{v}_i$

$$\left(s_i^2 \mathbf{M} + s_i \mathbf{C} + \mathbf{K}\right) \mathbf{v}_i = \mathbf{0} \tag{10.127}$$

where s_i is the damped eigenvalue. We know that, in the undamped case, $s_i = \lambda_i = j\omega_i$ and the i-th eigenvector $\mathbf{u}_i$ is solution of

$$\left(\mathbf{K} - \omega_i^2\mathbf{M}\right)\mathbf{u}_i = \left(\mathbf{K} + \lambda_i^2\mathbf{M}\right)\mathbf{u}_i = \mathbf{0} \tag{10.128}$$

If we assume that the structure under study is lightly damped (i.e., $\mathbf{C}$ is small), we can perform a linearization by supposing that the damped eigenmodes differ only slightly from the undamped ones. Therefore, we may write

$$s_i = \lambda_i + \Delta\lambda_i$$

and

$$\mathbf{v}_i = \mathbf{u}_i + \Delta\mathbf{u}_i$$

where $\Delta\lambda_i$ and $\Delta\mathbf{u}_i$ are small deviations from the corresponding undamped quantities. Substituting such approximation into the damped eigenvalue problem (10.127) yields

$$\left[\left(\lambda_i^2 + \Delta\lambda_i^2 + 2\lambda_i\Delta\lambda_i\right)\mathbf{M} + \lambda_i\mathbf{C} + \Delta\lambda_i\mathbf{C} + \mathbf{K}\right](\mathbf{u}_i + \Delta\mathbf{u}_i) = \mathbf{0}$$

and, neglecting higher-order terms, we have

$$\left[\left(\lambda_i^2 + 2\lambda_i\Delta\lambda_i\right)\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K}\right]\mathbf{u}_i + \left(\lambda_i^2\mathbf{M} + \mathbf{K}\right)\Delta\mathbf{u}_i \approx \mathbf{0}$$

Taking into account Eq. (10.128), we obtain

$$\lambda_i\left(\mathbf{C} + 2\Delta\lambda_i\mathbf{M}\right)\mathbf{u}_i + \left(\mathbf{K} + \lambda_i^2\mathbf{M}\right)\Delta\mathbf{u}_i \approx \mathbf{0} \tag{10.129}$$

Finally, after pre-multiplying by $\mathbf{u}_i^T$, we get

$$\mathbf{u}_i^T\mathbf{C}\mathbf{u}_i + 2\Delta\lambda_i\,\mathbf{u}_i^T\mathbf{M}\mathbf{u}_i \approx 0$$

Therefore, we can conclude that

$$\Delta\lambda_i \approx -\frac{1}{2m_i}\mathbf{u}_i^T\mathbf{C}\mathbf{u}_i \tag{10.130}$$

which shows that *the first-order correction of the undamped solutions involves only diagonal terms of the modal damping matrix*. After calling the modal damping coefficients

$$\beta_{ij} = \mathbf{u}_i^T\mathbf{C}\mathbf{u}_j \tag{10.131}$$

the i-th damped eigenvalue is related to the undamped eigenvalue by the following equation

$$s_i \approx -\frac{\beta_{ii}}{2m_i} + j\omega_i \tag{10.132}$$

Since the undamped eigenmodes form a complete set (a vector basis spanning the N-dimensional space), the eigenvector first-order correction $\Delta\mathbf{u}_i$ can be expressed as a superposition of the undamped eigenvectors as follows

$$\Delta\mathbf{u}_i = \sum_{\substack{s=1\\ s\neq i}}^{N}\alpha_s\mathbf{u}_s$$

Substituting such expression into Eq. (10.129) and pre-multiplying by $\mathbf{u}_k^T$ where $k \neq i$ yields

$$\lambda_i\mathbf{u}_k^T\left(\mathbf{C} + 2\Delta\lambda_i\mathbf{M}\right)\mathbf{u}_i + \mathbf{u}_k^T\left(\mathbf{K} + \lambda_i^2\mathbf{M}\right)\sum_{\substack{s=1\\ s\neq i}}^{N}\alpha_s\mathbf{u}_s \approx 0 \tag{10.133}$$

and, taking into account eigenmodes orthogonality,

$$\lambda_i\mathbf{u}_k^T\mathbf{C}\mathbf{u}_i + \alpha_k\mathbf{u}_k^T\mathbf{K}\mathbf{u}_k + \alpha_k\lambda_i^2\mathbf{u}_k^T\mathbf{M}\mathbf{u}_k \approx 0$$

or

$$j\omega_i\beta_{ki} + \alpha_k m_k \omega_k^2 - \alpha_k m_k \omega_i^2 \approx 0$$

Therefore, we can write

$$\alpha_k \approx \frac{j\omega_i\beta_{ki}}{m_k\left(\omega_i^2 - \omega_k^2\right)}$$

which is valid if $\left(\omega_i^2 - \omega_k^2\right) \neq 0$. The i-th damped eigenvector is related to the corresponding undamped eigenvector by the following expression

$$\mathbf{v}_i \approx \mathbf{u}_i + \sum_{\substack{s=1 \\ s\neq i}}^{N} \frac{j\omega_i\beta_{si}}{m_s\left(\omega_i^2 - \omega_s^2\right)}\mathbf{u}_s \tag{10.134}$$

which shows that, provided the term $\left(\omega_i^2 - \omega_k^2\right)$ remains finite, the correction of the eigenvector is of the same order of magnitude of the coupled modal damping coefficients β_{si}. In other words, *the damped eigenvectors can be considered approximately equal to the undamped eigenvectors as long as the structure is lightly damped and the undamped modes are well separated*.

10.8.3 Time response in the case of diagonal damping matrix

According to the modal analysis approach, the time response of systems with diagonal damping matrix to non-zero initial conditions is obtained by solving the following N decoupled equations of motion

$$m_i\ddot{q}_i(t) + 2m_i\xi_i\omega_i\dot{q}_i(t) + m_i\omega_i^2 q_i(t) = 0 \qquad i = 1,2,\ldots,N$$

or, after dividing by m_i,

$$\ddot{q}_i(t) + 2\xi_i\omega_i\dot{q}_i(t) + \omega_i^2 q_i(t) = 0 \qquad i = 1,2,\ldots,N \tag{10.135}$$

with

$$q_i(0) = q_{i0}$$
$$\dot{q}_i(0) = \dot{q}_{i0}$$

According to Eq. (10.21), the solution in terms of modal coordinate q_i is given by

$$\boxed{q_i(t) = e^{-\xi_i\omega_i t}\left[q_{i0}\cos\omega_{id}t + \frac{\xi_i\omega_i q_{i0} + \dot{q}_{i0}}{\omega_{id}}\sin\omega_{id}t\right]} \tag{10.136}$$

where

$$\omega_{id} = \omega_i\sqrt{1-\xi_i^2} \tag{10.137}$$

is the damped natural frequency of the ith mode. The response in terms of physical displacements can be directly recovered from Eq. (10.102) and Eq. (10.103) as follows

$$\boxed{\mathbf{s}(\mathbf{x},t) = \mathbf{N}(\mathbf{x})\sum_{i=1}^{N}\mathbf{u}_i q_i(t) = \mathbf{N}(\mathbf{x})\sum_{i=1}^{N}\mathbf{u}_i e^{-\xi_i\omega_i t}\left[q_{i0}\cos\omega_{id}t + \frac{\xi_i\omega_i q_{i0} + \dot{q}_{i0}}{\omega_{id}}\sin\omega_{id}t\right]} \tag{10.138}$$

Response to arbitrary excitation may be determined through the Laplace transform method. The set of N equations of motion is

$$m_i\ddot{q}_i(t) + 2m_i\xi_i\omega_i\dot{q}_i(t) + m_i\omega_i^2 q_i(t) = Q_i(t) \qquad i = 1,2,\ldots,N$$

Recalling Eq. (10.69), we can write

$$\boxed{q_i(t) = \frac{1}{m_i\omega_{id}}\int_0^t e^{-\xi_i\omega_i(t-\tau)}\sin\left[\omega_{id}(t-\tau)\right]Q_i(\tau)d\tau + q_{i0}e^{-\xi_i\omega_i t}\cos\omega_{id}t + \frac{\dot{q}_{i0} + q_{i0}\xi_i\omega_i}{\omega_{id}}\sin\omega_{id}t} \tag{10.139}$$

The corresponding physical displacement is given by

$$\boxed{\begin{aligned}\mathbf{s}(\mathbf{x},t) = \mathbf{N}(\mathbf{x})\sum_{i=1}^{N}\mathbf{u}_i q_i(t) = \mathbf{N}(\mathbf{x})\sum_{i=1}^{N}\mathbf{u}_i \left\{\frac{1}{m_i\omega_{id}}\int_0^t e^{-\xi_i\omega_i(t-\tau)}\sin\left[\omega_{id}(t-\tau)\right]Q_i(\tau)d\tau\right.\\ \left. +q_{i0}e^{-\xi_i\omega_i t}\cos\omega_{id}t + \frac{\dot{q}_{i0}+q_{i0}\xi_i\omega_i}{\omega_{id}}\sin\omega_{id}t\right\}\end{aligned}} \tag{10.140}$$

Issues related to truncation and convergence of the above solutions are discussed later.

As discussed previously for exact modal analysis of continuous systems, the initial conditions on the modal coordinates q_{i0} and $\dot{q}_{i0}$ must be derived from the initial conditions on the physical variables. Let the discretized problem be subjected to the following conditions at initial time $t = 0$:

$$\begin{aligned}\mathbf{u}(0) &= \mathbf{u}_0\\ \dot{\mathbf{u}}(0) &= \mathbf{v}_0\end{aligned} \tag{10.141}$$

Using the modal representation in Eq. (10.102), we can write

$$\begin{aligned}\mathbf{u}(0) &= \mathbf{U}\mathbf{q}(0) = \mathbf{u}_0\\ \dot{\mathbf{u}}(0) &= \mathbf{U}\dot{\mathbf{q}}(0) = \mathbf{v}_0\end{aligned} \tag{10.142}$$

After multiplying both sides of the above relations by $\mathbf{U}^T\mathbf{M}$ and using the orthogonality property of the modes with respect to the mass matrix, we obtain

$$\begin{aligned}\text{Diag}\left\{m_i\right\}\mathbf{q}(0) &= \mathbf{U}^T\mathbf{M}\mathbf{u}_0\\ \text{Diag}\left\{m_i\right\}\dot{\mathbf{q}}(0) &= \mathbf{U}^T\mathbf{M}\mathbf{v}_0\end{aligned} \tag{10.143}$$

Therefore, the vectors of modal initial conditions can be computed as

$$\begin{aligned}\mathbf{q}(0) &= \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}^T\mathbf{M}\mathbf{u}_0\\ \dot{\mathbf{q}}(0) &= \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}^T\mathbf{M}\mathbf{v}_0\end{aligned} \tag{10.144}$$

10.8.4 Frequency response in the case of diagonal damping matrix

The frequency response of lightly damped systems can be obtained by assuming

$$\begin{aligned}\mathbf{f}(t) &= \mathbf{f}e^{j\omega t}\\ \mathbf{q}(t) &= \mathbf{q}e^{j\omega t}\end{aligned}$$

Introducing the above into Eq. (10.116) we have

$$\left(-\omega^2\text{Diag}\left\{m_i\right\} + j\omega\mathbf{U}^T\mathbf{C}\mathbf{U} + \text{Diag}\left\{m_i\omega_i^2\right\}\right)\mathbf{q} = \mathbf{U}^T\mathbf{f}$$

Modal damping assumption allows us to write

$$\left(-\omega^2\text{Diag}\left\{m_i\right\} + j\omega\text{Diag}\left\{2m_i\xi_i\omega_i\right\} + \text{Diag}\left\{m_i\omega_i^2\right\}\right)\mathbf{q} = \mathbf{U}^T\mathbf{f}$$

Hence, the vector of modal amplitudes is given by

$$\mathbf{q} = \text{Diag}\left\{\frac{1}{m_i\left(\omega_i^2-\omega^2+2j\xi_i\omega_i\omega\right)}\right\}\mathbf{U}^T\mathbf{f} \tag{10.145}$$

The response in terms of physical displacements can be written as

$$\begin{aligned}\mathbf{s}(\mathbf{x}) &= \mathbf{N}(\mathbf{x})\mathbf{U}\mathbf{q}\\ &= \mathbf{N}(\mathbf{x})\mathbf{U}\text{Diag}\left\{\frac{1}{m_i\left(\omega_i^2-\omega^2+2j\xi_i\omega_i\omega\right)}\right\}\mathbf{U}^T\mathbf{f}\end{aligned} \tag{10.146}$$

or, alternatively,

$$\boxed{\mathbf{s}(\mathbf{x}) = \mathbf{N}(\mathbf{x}) \sum_{i=1}^{N} \frac{\mathbf{u}_i \mathbf{u}_i^T}{m_i \left(\omega_i^2 - \omega^2 + 2j\xi_i\omega_i\omega\right)} \mathbf{f}} \tag{10.147}$$

10.8.5 Arbitrary viscous damping

[...] TODO ?

10.9 Modal truncation and convergence

10.9.1 Truncated models

The difference between models expressed through Eq. (10.1) and Eq. (10.116) lies in the change of coordinate in Eq. (10.102) from generalized (physical) variables to modal variables. If a square modal matrix $\mathbf{U} \in \mathbb{R}^{N \times N}$ is used, the two representations are totally equivalent and the number of degrees of freedom of the modal model is equal to the number of generalized or physical coordinates.

However, as discussed above, a modal analysis is really useful if a subset of global modes can be used as a basis for a reduced-order (small size) model capable of representing the dynamic response in a limited bandwidth with sufficient accuracy. By limiting our analysis to a bandwidth going from zero up to a certain maximum frequency, corresponding to the typical frequency content of external excitations, the modal matrix can be partitioned as follows

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_{\mathrm{L}} & \mathbf{U}_{\mathrm{H}} \end{bmatrix} \tag{10.148}$$

where $\mathbf{U}_{\mathrm{L}}$ are the eigenmodes corresponding to natural frequencies which fall inside in the bandwidth of interest (*low-frequency* or slow modes) and $\mathbf{U}_{\mathrm{H}}$ are all the other eigenmodes corresponding to natural frequencies outside the bandwidth (*high-frequency* or fast modes). If $n < N$ is the number of low-frequency modes, the matrix $\mathbf{U}_{\mathrm{L}}$ is $\in \mathbb{R}^{N \times n}$

$$\mathbf{U}_{\mathrm{L}} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \tag{10.149}$$

whereas the matrix $\mathbf{U}_{\mathrm{H}}$ is $\in \mathbb{R}^{N \times (N-n)}$

$$\mathbf{U}_{\mathrm{H}} = \begin{bmatrix} \mathbf{u}_{n+1} & \mathbf{u}_{n+2} & \dots & \mathbf{u}_N \end{bmatrix} \tag{10.150}$$

A reduced-order modal can be obtained by using the following change of coordinate

$$\boxed{\mathbf{u}(t) = \mathbf{U}_{\mathrm{L}}\mathbf{q}_{\mathrm{L}}(t)} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \dots \\ q_n(t) \end{Bmatrix} \tag{10.151}$$

where $\mathbf{q}_{\mathrm{L}}$ is the vector of modal coordinates corresponding to the low-frequency modes. This strategy is known as *modal truncation*. In other words, the global modes belonging to the selected frequency range of interest are used to represent the dynamic response, whereas all the others are completely discarded. Typically, we have $n \ll N$, therefore the number of degrees of freedom contributing effectively to the response is drastically reduced when modal coordinates are used. It has to be noted that, as will be shown later, the actual number of modes to be retained in the reduced-order model in order to achieve a desired accuracy depends on the convergence properties of the truncated approximation.

The equations of motion of a reduced-order model obtained by modal truncation are given by

$$\mathbf{U}_{\mathrm{L}}^T\mathbf{M}\mathbf{U}_{\mathrm{L}}\ddot{\mathbf{q}}_{\mathrm{L}}(t) + \mathbf{U}_{\mathrm{L}}^T\mathbf{C}\mathbf{U}_{\mathrm{L}}\dot{\mathbf{q}}_{\mathrm{L}}(t) + \mathbf{U}_{\mathrm{L}}^T\mathbf{K}\mathbf{U}_{\mathrm{L}}\mathbf{q}_{\mathrm{L}}(t) = \mathbf{U}_{\mathrm{L}}^T\mathbf{f}(t) \tag{10.152}$$

and, using the orthogonality properties of the low-frequency eigenmodes,

$$\text{Diag}\left\{m_i\right\}\ddot{\mathbf{q}}_{\mathrm{L}}(t) + \mathbf{U}_{\mathrm{L}}^T\mathbf{C}\mathbf{U}_{\mathrm{L}}\dot{\mathbf{q}}_{\mathrm{L}}(t) + \text{Diag}\left\{m_i\omega_i^2\right\}\mathbf{q}_{\mathrm{L}}(t) = \mathbf{U}_{\mathrm{L}}^T\mathbf{f}(t) \tag{10.153}$$

or, alternatively,

$$m_i \ddot{q}_{\mathrm{L}i}(t) + \sum_{j=1}^{n} \mathbf{u}_i^T \mathbf{C} \mathbf{u}_j \, \dot{q}_{\mathrm{L}j}(t) + m_i \omega_i^2 q_{\mathrm{L}i}(t) = Q_{\mathrm{L}i}(t) \qquad (i = 1, 2, \ldots, n) \tag{10.154}$$

where

$$Q_{\mathrm{L}i}(t) = \mathbf{u}_i^T \mathbf{f}(t) \qquad (i = 1, 2, \ldots, n) \tag{10.155}$$

Note that, now, index i lies between 1 and n. If a diagonal modal damping model is assumed, the reduced set of equations of motion is decoupled and we may write

$$\mathrm{Diag}\,\{m_i\}\, \ddot{\mathbf{q}}_{\mathrm{L}}(t) + \mathrm{Diag}\,\{2 m_i \xi_i \omega_i\}\, \dot{\mathbf{q}}_{\mathrm{L}}(t) + \mathrm{Diag}\,\{m_i \omega_i^2\}\, \mathbf{q}_{\mathrm{L}}(t) = \mathbf{U}_{\mathrm{L}}^T \mathbf{f}(t) \tag{10.156}$$

It should be remarked again that what is important in a modal reduction technique is not the availability of a diagonal model but the reduced size n of the model.

10.9.2 Convergence of truncated modal models

At this point, it should be clear that the resulting model is efficient if the number n of modal coordinates to be included in the condensed model is actually much lower than the number N of original generalized (physical) coordinates. Similarly to what discussed for continuous systems, the required number of modes which yields a desired accuracy is related to the convergence of modal series, i.e., how rapidly the terms of a modal series decrease. General convergence properties of discretized systems can be presented more easily in the frequency domain. For simplicity, we limit the presentation to the common case of lightly damped structures, for which a diagonal modal damping can be assumed.

Taking the Laplace transform of both sides of the truncated model in Eq. (10.156) yields the following equation

$$\mathrm{Diag}\,\{m_i s^2 + 2 m_i \xi_i \omega_i s + m_i \omega_i^2\}\, \mathbf{q}_{\mathrm{L}}(s) = \mathbf{U}_{\mathrm{L}}^T \mathbf{f}(s) \tag{10.157}$$

As a result, the modal coordinates will be given by

$$\mathbf{q}_{\mathrm{L}}(s) = \mathrm{Diag}\left\{ \frac{1}{m_i \left(s^2 + 2\xi_i \omega_i s + \omega_i^2\right)} \right\} \mathbf{U}_{\mathrm{L}}^T \mathbf{f}(s) \tag{10.158}$$

From Eq. (10.151) we obtain the Laplace transform of the generalized coordinates as follows

$$\begin{aligned} \mathbf{u}(s) = \mathbf{U}_{\mathrm{L}} \mathbf{q}_{\mathrm{L}}(s) &= \mathbf{U}_{\mathrm{L}} \mathrm{Diag}\left\{ \frac{1}{m_i \left(s^2 + 2\xi_i \omega_i s + \omega_i^2\right)} \right\} \mathbf{U}_{\mathrm{L}}^T \mathbf{f}(s) \\ &= \sum_{i=1}^{n} \frac{\mathbf{u}_i \mathbf{u}_i^T}{m_i \left(s^2 + 2\xi_i \omega_i s + \omega_i^2\right)} \mathbf{f}(s) \end{aligned} \tag{10.159}$$

Last equation shows that the frequency response in terms of displacements can be seen as a modal series of n terms which has a relatively fast rate of convergence as $1/\omega_i^2$. Note also that, if the distribution of forces is spatially smooth, then the generalized forces $\mathbf{U}_{\mathrm{L}}^T \mathbf{f}(s)$ will themselves diminish with increasing mode number.

Results on rate of convergence obtained thus far for displacements do not apply to elastic forces, which can be viewed as an indicator for the evaluation of the rate of convergence of strains and stresses. Elastic forces in the Laplace domain may be recovered directly from a truncated modal model as follows

$$\mathbf{K}\mathbf{u}(s) = \mathbf{K}\mathbf{U}_{\mathrm{L}} \mathrm{Diag}\left\{ \frac{1}{m_i \left(s^2 + 2\xi_i \omega_i s + \omega_i^2\right)} \right\} \mathbf{U}_{\mathrm{L}}^T \mathbf{f}(s) \tag{10.160}$$

This is the discretized version of direct recovery of stress resultants through mode displacement method discussed previously for continuous systems. Since

$$\mathbf{K}\mathbf{U}_{\mathrm{L}} = \mathbf{M}\mathbf{U}_{\mathrm{L}} \mathrm{Diag}\,\{\omega_i^2\}$$

with $1 \leq i \leq n$, we may write

$$\mathbf{K}\mathbf{u}(s) = \mathbf{M}\mathbf{U}_{\mathrm{L}} \mathrm{Diag}\left\{ \frac{\omega_i^2}{m_i \left(s^2 + 2\xi_i \omega_i s + \omega_i^2\right)} \right\} \mathbf{U}_{\mathrm{L}}^T \mathbf{f}(s) \tag{10.161}$$

or, alternatively,

$$\boxed{\mathbf{Ku}(s) = \mathbf{M}\sum_{i=1}^{n}\frac{\omega_i^2\mathbf{u}_i\mathbf{u}_i^T}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\mathbf{f}(s)} \tag{10.162}$$

which shows that, differently from the case of displacements, the modal series expressing the elastic forces converges only if the distribution of forces $\mathbf{U}_L^T\mathbf{f}(s)$ has some decreasing behavior with frequency. As mentioned, this occurs if applied forces are spatially smooth. Note also that the expression in Eq. (10.161) contains also the mass matrix $\mathbf{M}$. Hence, if mass distribution lacks of regularity, such as the case of big concentrated masses, further issues related to convergence arise.

We can conclude that mode displacement method exhibits bad convergence and thus direct recovery of elastic forces from displacement solutions obtained from truncated modal models is not effective nor efficient. This shortcoming is compensated by the mode acceleration method.

10.10 Mode acceleration method

10.10.1 Useful properties

Before presenting the mode acceleration method for discretized models, some useful properties are first introduced, which will be extensively used in the following.

In the case of diagonal modal damping, the free vibration solution is given as the superposition of terms of the form

$$\mathbf{u}_i q_i(t) = \mathbf{u}_i e^{s_i t}$$

where, as previously shown, the i-th damped eigenvalue is given by

$$s_i = -\xi_i\omega_i \pm j\omega_{id} = -\xi_i\omega_i \pm j\omega_i\sqrt{1-\xi_i^2}$$

Therefore, a system vibrating according to the i-th eigenmode satisfies the following equations

$$\begin{aligned}
&\mathbf{M}s_i^2\mathbf{u}_i e^{s_i t} + \mathbf{C}s_i\mathbf{u}_i e^{s_i t} + \mathbf{Ku}_i e^{s_i t} = \mathbf{0}\\
&\mathbf{M}s_i^2\mathbf{u}_i + \mathbf{C}s_i\mathbf{u}_i + \mathbf{Ku}_i = \mathbf{0}\\
&\mathbf{M}\left(\xi_i^2\omega_i^2 - \omega_{id}^2 \mp j2\xi_i\omega_i\omega_{id}\right)\mathbf{u}_i + \mathbf{C}\left(-\xi_i\omega_i \pm j\omega_{id}\right)\mathbf{u}_i + \mathbf{Ku}_i = \mathbf{0}
\end{aligned}$$

The last equation can be rewritten by separating the real and imaginary part, respectively, as follows

$$\mathbf{Mu}_i\left(\xi_i^2\omega_i^2 - \omega_{id}^2\right) - \mathbf{Cu}_i\xi_i\omega_i + \mathbf{Ku}_i = \mathbf{0} \tag{10.163}$$

$$\mathbf{Mu}_i 2\xi_i\omega_i\omega_{id} - \mathbf{Cu}_i\omega_{id} = \mathbf{0} \tag{10.164}$$

Since the above must hold for each mode i, we can write Eqs. (10.163) and (10.164) in the following compact matrix form

$$\mathbf{MU}\text{Diag}\left\{\left(2\xi_i^2-1\right)\omega_i^2\right\} - \mathbf{CU}\text{Diag}\left\{\xi_i\omega_i\right\} + \mathbf{KU} = \mathbf{0} \tag{10.165}$$

$$\boxed{\mathbf{MU}\text{Diag}\left\{2\xi_i\omega_i\right\} - \mathbf{CU} = \mathbf{0}} \tag{10.166}$$

Note that Eq. (10.166) can be also written as

$$\mathbf{MU}\text{Diag}\left\{\omega_i^2\right\}\text{Diag}\left\{\frac{2\xi_i}{\omega_i}\right\} - \mathbf{CU} = \mathbf{0}$$

Then, recalling that $\mathbf{MU}\text{Diag}\left\{\omega_i^2\right\} = \mathbf{KU}$, we obtain

$$\boxed{\mathbf{KU}\text{Diag}\left\{\frac{2\xi_i}{\omega_i}\right\} - \mathbf{CU} = \mathbf{0}} \tag{10.167}$$

10.10.2 Description of the method

The equations of motion of a discretized system can be put in the following form

$$\mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) - \mathbf{C}\dot{\mathbf{u}}(t) - \mathbf{M}\ddot{\mathbf{u}}(t) \tag{10.168}$$

Using a modal superposition involving the subset $\mathbf{U}_\mathrm{L}$ of system eigenvectors, the displacement solution is given by

$$\mathbf{u}(t) = \mathbf{K}^{-1}\mathbf{f}(t) - \mathbf{K}^{-1}\mathbf{C}\mathbf{U}_\mathrm{L}\dot{\mathbf{q}}_\mathrm{L}(t) - \mathbf{K}^{-1}\mathbf{M}\mathbf{U}_\mathrm{L}\ddot{\mathbf{q}}_\mathrm{L}(t) \tag{10.169}$$

Therefore, using what derived above, Equation (10.169) becomes

$$\mathbf{u}(t) = \mathbf{K}^{-1}\mathbf{f}(t) - \mathbf{K}^{-1}\mathbf{K}\mathbf{U}_\mathrm{L}\mathrm{Diag}\left\{\frac{2\xi_i}{\omega_i}\right\}\dot{\mathbf{q}}_\mathrm{L}(t) - \mathbf{K}^{-1}\mathbf{K}\mathbf{U}_\mathrm{L}\mathrm{Diag}\left\{\frac{1}{\omega_i^2}\right\}\ddot{\mathbf{q}}_\mathrm{L}(t) \tag{10.170}$$

or, alternatively,

$$\boxed{\mathbf{u}(t) = \mathbf{K}^{-1}\mathbf{f}(t) - \sum_{i=1}^{n}\frac{2\xi_i}{\omega_i}\mathbf{u}_i\dot{q}_i(t) - \sum_{i=1}^{n}\frac{1}{\omega_i^2}\mathbf{u}_i\ddot{q}_i(t)} \tag{10.171}$$

which is the approximation of the generalized (or physical) coordinates through the mode acceleration method. The first term on the right side of Eq. (10.171) is the *exact quasi-static response* since the full stiffness matrix $\mathbf{K}$ is used. Therefore, we can also write

$$\mathbf{u}(t) = \mathbf{u}_{qs}(t) - \sum_{i=1}^{n}\frac{2\xi_i}{\omega_i}\mathbf{u}_i\dot{q}_i(t) - \sum_{i=1}^{n}\frac{1}{\omega_i^2}\mathbf{u}_i\ddot{q}_i(t) \tag{10.172}$$

Note that the computation of the quasi-static response implies the solution of the following linear system at each time instant

$$\mathbf{K}\mathbf{u}_{qs}(t) = \mathbf{f}(t) \tag{10.173}$$

Note also that some care must be taken in the case of a system with rigid-body modes because the stiffness matrix is singular. This will be discussed later. Time derivatives of modal coordinates q_i involved in the second and third term of Eq. (10.171) are computed from the response given by Eq. (10.139).

10.10.3 Convergence analysis

By taking the Laplace transform of Eq. (10.168), we can write

$$\mathbf{K}\mathbf{u}(s) = \mathbf{f}(s) - s^2\mathbf{M}\mathbf{u}(s) - s\mathbf{C}\mathbf{u}(s)$$

According to a modal reduction technique including the low-frequency modes, the above expression becomes

$$\begin{aligned}\mathbf{K}\mathbf{u}(s) &= \mathbf{f}(s) - \left(s^2\mathbf{M} + s\mathbf{C}\right)\mathbf{U}_\mathrm{L}\mathbf{q}_\mathrm{L}(s) \\ &= \mathbf{f}(s) - \left(s^2\mathbf{M} + s\mathbf{C}\right)\mathbf{U}_\mathrm{L}\mathrm{Diag}\left\{\frac{1}{m_i\left(s^2 + 2\xi_i\omega_i s + \omega_i^2\right)}\right\}\mathbf{U}_\mathrm{L}^T\mathbf{f}(s)\end{aligned} \tag{10.174}$$

Using Eq. (10.166), the Laplace transform of the elastic forces recovered through the mode acceleration method can be written as follows

$$\mathbf{K}\mathbf{u}(s) = \mathbf{f}(s) - \mathbf{M}\mathbf{U}_\mathrm{L}\mathrm{Diag}\left\{\frac{s\left(s + 2\xi_i\omega_i\right)}{m_i\left(s^2 + 2\xi_i\omega_i s + \omega_i^2\right)}\right\}\mathbf{U}_\mathrm{L}^T\mathbf{f}(s) \tag{10.175}$$

or, alternatively,

$$\boxed{\mathbf{K}\mathbf{u}(s) = \mathbf{f}(s) - \mathbf{M}\sum_{i=1}^{n}\frac{s\left(s + 2\xi_i\omega_i\right)\mathbf{u}_i\mathbf{u}_i^T}{m_i\left(s^2 + 2\xi_i\omega_i s + \omega_i^2\right)}\mathbf{f}(s)} \tag{10.176}$$

which is composed by a quasi-static response (first term in the right-hand side of Eq. (10.176)) and a dynamic contribution in the form of a modal series (second term in the right-hand side of Eq. (10.176)). The quasi-static term is computed exactly and allows us to recover the natural boundary conditions of the problem, which are not included into the mode superposition since eigenmodes are computed with homogeneous boundary conditions. The dynamic contribution now exhibits better convergence than the case of direct recovery from displacement solution. Note that, if we assume that modal damping factors ξ_i are small[3], the dynamic part of the response presents the generic term with an approximate decrease $O(1/\omega_i^2)$. The series might converge even faster if applied loads are spatially smooth.

Expression in Eq. (10.175) can be derived in an alternative way. Let us first assume that the generalized (physical) displacement solution can be seen as the superposition of a quasi-static term $\mathbf{u}_{qs}(t)$ and a dynamic term $\mathbf{u}_d(t)$ as follows

$$\mathbf{u}(t) = \mathbf{u}_{qs}(t) + \mathbf{u}_d(t)$$

where the quasi-static term is solution of

$$\mathbf{K}\mathbf{u}_{qs}(t) = \mathbf{f}(t)$$

Inserting the above into the discretized equations of motion yields

$$\mathbf{M}\ddot{\mathbf{u}}_d(t) + \mathbf{M}\mathbf{K}^{-1}\ddot{\mathbf{f}}(t) + \mathbf{C}\dot{\mathbf{u}}_d(t) + \mathbf{C}\mathbf{K}^{-1}\dot{\mathbf{f}}(t) + \mathbf{K}\mathbf{u}_d(t) = \mathbf{0}$$

After taking the Laplace transform and rearranging, we obtain

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{u}_d(s) = -\left(s^2\mathbf{M} + s\mathbf{C}\right)\mathbf{K}^{-1}\mathbf{f}(s)$$

The dynamic part can be expanded using a subset of modes $\mathbf{u}_d(s) = \mathbf{U}_{\mathrm{L}}\mathbf{q}_{\mathrm{L}}(s)$. Then, pre-multiplying by $\mathbf{U}_{\mathrm{L}}^T$ and taking into account the orthogonality properties of eigenmodes, we can write

$$\mathrm{Diag}\left\{s^2 m_i + s2m_i\xi_i\omega_i + m_i\omega_i^2\right\}\mathbf{q}_{\mathrm{L}}(s) = -s\left(s\mathbf{U}_{\mathrm{L}}^T\mathbf{M} + \mathbf{U}_{\mathrm{L}}^T\mathbf{C}\right)\mathbf{K}^{-1}\mathbf{f}(s)$$

Since the transpose of Eq. (10.166) involving the subset $\mathbf{U}_{\mathrm{L}}$ leads to

$$\mathbf{U}_{\mathrm{L}}^T\mathbf{C} = \mathrm{Diag}\left\{2\xi_i\omega_i\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{M}$$

the vector of modal coordinates can be written as follows

$$\mathbf{q}_{\mathrm{L}}(s) = -\mathrm{Diag}\left\{\frac{s\,(s+2\xi_i\omega_i)}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{M}\mathbf{K}^{-1}\mathbf{f}(s) \tag{10.177}$$

Therefore, elastic forces can be expressed as

$$\begin{aligned}
\mathbf{K}\mathbf{u}(s) &= \mathbf{K}\left[\mathbf{u}_{qs}(s) + \mathbf{u}_d(s)\right]\\
&= \mathbf{K}\mathbf{u}_{qs}(s) + \mathbf{K}\mathbf{U}_{\mathrm{L}}\mathbf{q}_{\mathrm{L}}(s)\\
&= \mathbf{f}(s) - \mathbf{K}\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{s\,(s+2\xi_i\omega_i)}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{M}\mathbf{K}^{-1}\mathbf{f}(s)\\
&= \mathbf{f}(s) - \mathbf{M}\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\omega_i^2\right\}\mathrm{Diag}\left\{\frac{s\,(s+2\xi_i\omega_i)}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{M}\mathbf{K}^{-1}\mathbf{f}(s)\\
&= \mathbf{f}(s) - \mathbf{M}\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\omega_i^2\right\}\mathrm{Diag}\left\{\frac{s\,(s+2\xi_i\omega_i)}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{M}\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{1}{m_i\omega_i^2}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)\\
&= \mathbf{f}(s) - \mathbf{M}\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\omega_i^2\right\}\mathrm{Diag}\left\{\frac{s\,(s+2\xi_i\omega_i)}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathrm{Diag}\left\{\frac{m_i}{m_i\omega_i^2}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)\\
&= \mathbf{f}(s) - \mathbf{M}\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{s\,(s+2\xi_i\omega_i)}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)
\end{aligned}$$

which is the same as Eq. (10.175)[4]. What presented thus far allows us to write the displacement solution as

$$\begin{aligned}
\mathbf{u}(s) &= \mathbf{u}_{qs}(s) - \mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{s\,(s+2\xi_i\omega_i)}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{M}\mathbf{K}^{-1}\mathbf{f}(s)\\
&= \mathbf{K}^{-1}\mathbf{f}(s) - \mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{s\,(s+2\xi_i\omega_i)}{m_i\omega_i^2\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)
\end{aligned} \tag{10.178}$$

[3]Recall that we have used the assumption of weak damping in deriving the result in Eq. (10.176).
[4]Note that, in the above derivation, we have used the following result: since $\mathbf{U}^T\mathbf{K}\mathbf{U} = \mathrm{Diag}\left\{m_i\omega_i^2\right\}$, the inverse is given by $\mathbf{U}^{-1}\mathbf{K}^{-1}\mathbf{U}^{-T} = \mathrm{Diag}\left\{1/m_i\omega_i^2\right\}$; then, after pre-multiplying by $\mathbf{U}$ and post-multiplying by $\mathbf{U}^T$, we get $\mathbf{K}^{-1} = \mathbf{U}\mathrm{Diag}\left\{1/m_i\omega_i^2\right\}\mathbf{U}^T$.

which shows that the displacement solution exhibits better convergence if calculated with the mode acceleration method (approximate decrease $O(1/\omega_i^3)$) rather than the mode displacement method (approximate decrease $O(1/\omega_i^2)$). Similar considerations discussed before about beneficial effects of regular spatial distribution of applied loads on the rate of convergence of modal series still hold.

10.10.4 Residualization

We have seen that the modal components of a discrete system of order N can be partitioned into a low-frequency $\mathbf{q}_{\mathrm{L}}$ subset of order n ($\ll N$) and a high-frequency $\mathbf{q}_{\mathrm{H}}$ subset of order $N-n$. Since the modes are mechanically independent, after assuming a diagonal modal damping, each modal subset satisfies the following decoupled equations

$$\mathrm{Diag}\left\{m_i\right\}\ddot{\mathbf{q}}_{\mathrm{L}}(t)+\mathrm{Diag}\left\{2m_i\xi_i\omega_i\right\}\dot{\mathbf{q}}_{\mathrm{L}}(t)+\mathrm{Diag}\left\{m_i\omega_i^2\right\}\mathbf{q}_{\mathrm{L}}(t)=\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(t)\qquad(i=1,\ldots,n)$$

and

$$\mathrm{Diag}\left\{m_i\right\}\ddot{\mathbf{q}}_{\mathrm{H}}(t)+\mathrm{Diag}\left\{2m_i\xi_i\omega_i\right\}\dot{\mathbf{q}}_{\mathrm{H}}(t)+\mathrm{Diag}\left\{m_i\omega_i^2\right\}\mathbf{q}_{\mathrm{H}}(t)=\mathbf{U}_{\mathrm{H}}^T\mathbf{f}(t)\qquad(i=n+1,\ldots,N)$$

The Laplace solutions of modal coordinates are thus given by

$$\mathbf{q}_{\mathrm{L}}(s)=\mathrm{Diag}\left\{\frac{1}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)$$

and

$$\mathbf{q}_{\mathrm{H}}(s)=\mathrm{Diag}\left\{\frac{1}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{H}}^T\mathbf{f}(s)$$

Using the whole set of modal coordinates, the displacement solution can be exactly recovered[5] and its Laplace transform is given by the following expression

$$\begin{aligned}\mathbf{u}(s)&=\mathbf{U}\mathbf{q}(s)=\mathbf{U}_{\mathrm{L}}\mathbf{q}_{\mathrm{L}}(s)+\mathbf{U}_{\mathrm{H}}\mathbf{q}_{\mathrm{H}}(s)\\&=\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{1}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)+\mathbf{U}_{\mathrm{H}}\mathrm{Diag}\left\{\frac{1}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{H}}^T\mathbf{f}(s)\end{aligned}\tag{10.179}$$

If we perform a modal truncation, the high-frequency modes are supposed to not contribute to the system response and are completely discarded. Therefore, expression in Eq. (10.159) is obtained.

Alternatively, we can assume that the system response is due to the contribution of modes from 1 to n responding dynamically and to contribution of modes from $n+1$ to N responding in a quasi-static manner. This is due to the fact that, if a structure is excited by a band-limited excitation, its response is largely dominated by those modes whose natural frequency lies in the bandwidth of excitation. The above assumption implies that Eq. (10.179) becomes

$$\boxed{\mathbf{u}(s)=\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{1}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)+\mathbf{U}_{\mathrm{H}}\mathrm{Diag}\left\{\frac{1}{m_i\omega_i^2}\right\}\mathbf{U}_{\mathrm{H}}^T\mathbf{f}(s)}\tag{10.180}$$

where the second term is a *quasi-static correction for the high-frequency modes*. The above equation can be also written as

$$\begin{aligned}\mathbf{u}(s)&=\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{1}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)\\&\quad-\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{1}{m_i\omega_i^2}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)+\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{1}{m_i\omega_i^2}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)\\&\quad+\mathbf{U}_{\mathrm{H}}\mathrm{Diag}\left\{\frac{1}{m_i\omega_i^2}\right\}\mathbf{U}_{\mathrm{H}}^T\mathbf{f}(s)\\&=\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{1}{m_i\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}-\frac{1}{m_i\omega_i^2}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)+\mathbf{U}\mathrm{Diag}\left\{\frac{1}{m_i\omega_i^2}\right\}\mathbf{U}^T\mathbf{f}(s)\\&=\mathbf{K}^{-1}\mathbf{f}(s)-\mathbf{U}_{\mathrm{L}}\mathrm{Diag}\left\{\frac{s\left(s+2\xi_i\omega_i\right)}{m_i\omega_i^2\left(s^2+2\xi_i\omega_i s+\omega_i^2\right)}\right\}\mathbf{U}_{\mathrm{L}}^T\mathbf{f}(s)\end{aligned}\tag{10.181}$$

[5]The word exact is used here to indicate that, if the modal model contains all the eigenmodes of the physical model, the modal solution coincides with the response obtained by solving the model in physical coordinates.

which is the recovery of displacement solution through the mode acceleration method derived in previous section. This means that the mode acceleration method can be considered as equivalent to a static residualization of high-frequency modes. However, note that the conceptual and physically meaningful equivalence between Eq. (10.180) and Eq. (10.181) does not imply an equivalence in terms of computational procedure. Indeed, use of Eq. (10.180) implies the computation of *all* eigenmodes of the discretized system, which is nowadays still impractical for huge finite element models of real structures. Instead, Eq. (10.181) implies the computation of only the low-frequency modes for the dynamic part, which are typically in small number, and the solution of algebraic linear systems $\mathbf{Ku} = \mathbf{f}$ for the quasi-static term, which can be efficiently performed through proper factorization of the stiffness matrix $\mathbf{K}$.

10.10.5 Systems with rigid-body modes

We have already mentioned that recovery of displacements by the method of mode acceleration, expressed in the Laplace domain by Eq. (10.178) or Eq. (10.181), needs some care if the system contains rigid-body modes. Indeed, the computation of the quasi-static displacements involves the inversion of the stiffness matrix $\mathbf{K}$, which is singular as many times as the number of rigid degrees of freedom. This problem can be solved in the following way.

Referring for simplicity to undamped systems, the displacements are partitioned into a rigid contribution and an elastic (flexible) contribution such that

$$\mathbf{u}(t) = \mathbf{u}_r(t) + \mathbf{u}_e(t) \tag{10.182}$$

Using the mode superposition we can write

$$\mathbf{u}(t) = \mathbf{U}_r\mathbf{q}_r(t) + \mathbf{U}_e\mathbf{q}_e(t) \tag{10.183}$$

where $\mathbf{U}_r$ is the matrix of eigenvectors whose columns are the rigid modes and $\mathbf{U}_e$ is the matrix of elastic modes. Note that the rigid modes are included in the analysis. According to the above partition, the equations of motion of the discretized system can be expressed as follows

$$\mathbf{MU}_r\ddot{\mathbf{q}}_r(t) + \mathbf{MU}_e\ddot{\mathbf{q}}_e(t) + \mathbf{KU}_e\mathbf{q}_e(t) = \mathbf{f}(t) \tag{10.184}$$

where we have used the fact that, by definition of rigid-body motion,

$$\mathbf{Ku}_r(t) = \mathbf{0}$$

The projection of Eq. (10.184) onto the set of rigid-body modes yields

$$\mathbf{U}_r^T\mathbf{MU}_r\ddot{\mathbf{q}}_r(t) + \mathbf{U}_r^T\mathbf{MU}_e\ddot{\mathbf{q}}_e(t) + \mathbf{U}_r^T\mathbf{KU}_e\mathbf{q}_e(t) = \mathbf{U}_r^t\mathbf{f}(t)$$

Using the orthogonality properties

$$\begin{aligned}
&\mathbf{U}_r^T\mathbf{MU}_r = \mathrm{Diag}\,\{m_i\} \qquad\qquad i = 1, \ldots, n_r \\
&\mathbf{U}_r^T\mathbf{MU}_e = \mathbf{0} \\
&\mathbf{U}_r^T\mathbf{KU}_e = \mathbf{0}
\end{aligned}$$

we can write

$$\ddot{\mathbf{q}}_r(t) = \mathrm{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}_r^T\mathbf{f}(t) \tag{10.185}$$

which is the dynamic equation governing the rigid-body motion. Once solved by double integration, the corresponding response is put into Eq. (10.183) to give the rigid contribution. The elastic contribution is obtained as follows. Modal solution in Eq. (10.185) is inserted into Eq. (10.184) and the following equations of motion of the elastic part are obtained

$$\begin{aligned}
\mathbf{MU}_e\ddot{\mathbf{q}}_e(t) + \mathbf{KU}_e\mathbf{q}_e(t) &= \mathbf{f}(t) - \mathbf{MU}_r\ddot{\mathbf{q}}_r(t) \\
&= \mathbf{f}(t) - \mathbf{MU}_r\mathrm{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}_r^T\mathbf{f}(t) \\
&= \left(\mathbf{I} - \mathbf{MU}_r\mathrm{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}_r^T\right)\mathbf{f}(t)
\end{aligned} \tag{10.186}$$

which can also put in the following form

$$\mathbf{M}\mathbf{U}_e\ddot{\mathbf{q}}_e(t) + \mathbf{K}\mathbf{U}_e\mathbf{q}_e(t) = \mathbf{f}_e(t) \tag{10.187}$$

where the vector of elastic forces is given by

$$\mathbf{f}_e(t) = \mathbf{R}^T\mathbf{f}(t)$$

and the matrix $\mathbf{R}$ is defined as

$$\mathbf{R} = \mathbf{I} - \mathbf{U}_r\text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}_r^T\mathbf{M}$$

The vector $\mathbf{f}_e(t)$ is the superposition of the external excitation and the inertia forces associated to the rigid motion. The $\mathbf{R}$ matrix, known as *inertia-relief matrix*, is such that

$$\begin{aligned}\mathbf{R}\mathbf{U}_r &= \mathbf{0}\\ \mathbf{R}\mathbf{U}_e &= \mathbf{U}_e\end{aligned}$$

Therefore, it has the important property of eliminating the rigid-body modes while leaving unchanged the elastic modes. The corresponding vector of loads $\mathbf{f}_e(t) = \mathbf{R}^T\mathbf{f}(t)$ is self-equilibrated since it is orthogonal to the rigid-body motions. In other words, the system is in equilibrium as a rigid body. As a result, we can apply arbitrary constraints such that to have a statically determinate structure, whose stiffness matrix can be inverted. Then, the general quasi-static solution is given by

$$\mathbf{u}_e(t) = \mathbf{G}_r\mathbf{f}_e(t) = \mathbf{G}_r\mathbf{R}^T\mathbf{f}(t)$$

where $\mathbf{G}_r$ is the flexibility matrix of the system with the applied constraints, which contains some component of rigid-body modes (the flexibility matrix is filled with zeros in the rows and columns corresponding to the constraints). This contribution is eliminated with the matrix $\mathbf{R}$, leading to

$$\mathbf{u}_e(t) = \mathbf{R}\mathbf{G}_r\mathbf{R}^T\mathbf{f}(t) \tag{10.188}$$

Therefore, the displacement vector can be computed as follows

$$\begin{aligned}\mathbf{u}(t) &= \mathbf{u}_r(t) + \mathbf{u}_e(t)\\ &= \mathbf{U}_r\mathbf{q}_r(t) + \mathbf{R}\mathbf{G}_r\mathbf{R}^T\mathbf{f}(t) - \sum_i^{n_e}\frac{1}{\omega_i^2}\mathbf{u}_i\ddot{q}_i(t)\end{aligned} \tag{10.189}$$

where all the rigid modes are included and the index i encompasses the first n_e elastic modes to be retained in the approximate dynamic model.

REFERENCES

1. M. Geradin, D. Rixen (1997). *Mechanical Vibrations: Theory and Application to Structural Dynamics*, Wiley.
2. R. R. Craig, A. J. Kurdila (2006). *Fundamentals of Structural Dynamics*, 2ed, Wiley.