# CHAPTER 14

# STATE SPACE FORMULATION OF STRUCTURAL SYSTEMS

The aim of this chapter is to introduce the state space formulation of linear time-invariant systems with reference to discretized models of vibrating structures.

## 14.1 Preliminary considerations

We have extensively presented in the previous chapters the equations governing the motion of *time-invariant* flexible structural systems. In particular, we have assumed that the flexibility induces *small* motion about an equilibrium condition, and so we have shown how to write and solve the partial and ordinary differential equations of vibrating structures. We have also highlighted that, in most practical problems, one is forced to rely on approximate methods to compute the dynamic solution. The most common displacement-based approaches, like the Ritz method and the finite element method, are based on spatial discretization techniques. As a result, the corresponding *linear time-invariant* (LTI) dynamic models are described by ordinary differential equations with constant coefficients.

As any other discipline in applied science, structural dynamics tends to formulate models in a way that best suits the problems of interest and satisfies certain objectives such as ease of computation. For structural dynamics, the equations of motion are naturally presented in the form of *second-order* differential equations which involve inertial, damping and stiffness contribution. For the sake of convenience, the general form is here recalled:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \tag{14.1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}(t)$ is the vector containing the generalized displacement coordinates (it may contain physical nodal coordinates for finite element models), and $\mathbf{f}(t)$ is the vector of generalized forces (it may contain nodal forces for finite element models). As already outlined, the above matrices are considered to be constant (i.e., time-independent) and symmetric. Let us assume that Eq. (14.1) is a set of $N$ equations of motion for the $N$ degrees of freedom of the problem. Therefore, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are $N \times N$ constant symmetric matrices.

In some cases, it could be more convenient to convert the above set of second-order equations into a set of *first-order* equations. Even though the form of the differential equation is changed, obviously the solution has to be the same. The first-order representation is typically referred to as the *state space formulation*. In a state space formulation, the unknowns are those quantities that are necessary to completely describe the state of the system at any time. Those unknowns are called *state variables*. Since the state of a non-dissipative structural system can be represented through its kinetic and potential energy, the state variables in structural dynamics can be considered to be the displacements and velocities, or any combination thereof. Let us briefly discuss some typical cases.

### 14.1.1 Physical (nodal) formulations

A common state space representation of Eq. (14.1) is to define a state vector $\mathbf{x}(t)$ as the collection of $\mathbf{u}$ and $\dot{\mathbf{u}}$ as follows

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} \tag{14.2}$$

By splitting the state vector into two subsets $\mathbf{x}_1$ and $\mathbf{x}_2$, the above can be also written as

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{Bmatrix} \tag{14.3}$$

where $\mathbf{x}_1(t) = \mathbf{u}(t)$ and $\mathbf{x}_2(t) = \dot{\mathbf{u}}(t)$. Provided that the mass matrix is non-singular [1], Equation (14.1) can be rewritten as

$$\ddot{\mathbf{u}}(t) = -\mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{u}}(t) - \mathbf{M}^{-1}\mathbf{K}\mathbf{u}(t) + \mathbf{M}^{-1}\mathbf{f}(t) \tag{14.4}$$

The corresponding first-order formulation is given by

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) &= -\mathbf{M}^{-1}\mathbf{C}\mathbf{x}_2(t) - \mathbf{M}^{-1}\mathbf{K}\mathbf{x}_1(t) + \mathbf{M}^{-1}\mathbf{f}(t) \end{aligned} \tag{14.5}$$

In matrix form, we can write

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{f}(t) \tag{14.6}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \tag{14.7}$$

is the state system (or dynamic) matrix and

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \tag{14.8}$$

is the state input matrix relating the external force vector $\mathbf{f}(t)$ to the state vector $\mathbf{x}(t)$.

Note that the problem is now described by the first-order set (14.6), which involves a total of $2N$ variables. Thus, the size of the problem has been doubled by converting from second-order to first-order formulation. Indeed, the system matrix $\mathbf{A}$ has dimension $2N \times 2N$. Since the number $N$ of physical degrees of freedom can be large for structural systems and the computational efficiency of the solution can become an issue, one has to consider such an aspect when switching to the space state formulation. Note also that the dynamic properties of the system are embedded in the system matrix $\mathbf{A}$, which is not symmetric.

The physical state space formulation presented so far can be written slightly differently. This is the typical case of control applications. If our structural system is supposed to be forced by $N_l$ external loads $f_j(t)$ (they may be disturbance forces or control inputs), the corresponding second-order model is typically written as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \sum_{j=1}^{N_l} \mathbf{L}_j f_j(t) \tag{14.9}$$

where $\mathbf{L}_j$ is the $N \times 1$ input force influence vector, indicating how the $j$th load is distributed over the structural degrees of freedom. Using the same definition of the state vector in Eq. (14.2), the first-order set of equations of motion are given by

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) &= -\mathbf{M}^{-1}\mathbf{C}\mathbf{x}_2(t) - \mathbf{M}^{-1}\mathbf{K}\mathbf{x}_1(t) + \sum_{j=1}^{N_l} \mathbf{M}^{-1}\mathbf{L}_j f_j(t) \end{aligned} \tag{14.10}$$

[1] In most cases, the mass matrix $\mathbf{M}$ is a positive definite matrix since the kinetic energy is a positive quantity. However, we can eventually come up with models for which there are degrees of freedom without inertia. We have already shown that, using a static condensation technique, the problem can be recast into an equivalent form with a non-singular matrix. Therefore, without any loss of generality, we can assume that the mass matrix $\mathbf{M}$ is invertible.

In matrix form, we can write

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{j=1}^{N_l} \mathbf{b}_j f_j(t) \tag{14.11}$$

where

$$\mathbf{b}_j = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{L}_j \end{bmatrix} \tag{14.12}$$

is the state input vector. In this way, the external physical force appears explicitly in the state formulation so that it can be directly manipulated, as it will be shown later.

### 14.1.2 Modal formulations

We have seen that a powerful and efficient way of computing the dynamic solution of time-invariant structural systems is the modal analysis approach. Owing to the frequency information associated to each modal coordinate, the modal representation is also a valuable method to obtain reduced-order dynamic models.

For the sake of simplicity, let consider the common case of lightly damped structures. After solving the eigenvalue problem related to the undamped model in Eq. (14.1), if $n$ low-frequency modes (typically $n \ll N$) are used to compute an accurate solution for the problem under investigation, we can use the following transformation

$$\mathbf{u}(t) = \mathbf{U}\mathbf{q}(t) \tag{14.13}$$

where $\mathbf{U}$ is the $N \times n$ eigenvector matrix

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \tag{14.14}$$

and $\mathbf{q}(t)$ is the corresponding $n \times 1$ vector of modal coordinates retained in the dynamic model, such that the following set of second-order equations is obtained

$$\text{Diag}\left\{m_i\right\}\ddot{\mathbf{q}}(t) + \text{Diag}\left\{2m_i\xi_i\omega_i\right\}\dot{\mathbf{q}}(t) + \text{Diag}\left\{m_i\omega_i^2\right\}\mathbf{q}(t) = \sum_{j=1}^{N_l} \mathbf{U}^T\mathbf{L}_j f_j(t) \tag{14.15}$$

where $i = 1, \dots, n$. The corresponding state space representation can be done in different equivalent ways. Four examples are presented in the following.

In the first state model, the state vector is defined as

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} \tag{14.16}$$

This is the most straightforward and common approach, and corresponds to the procedure used before for nodal models. Since

$$\ddot{\mathbf{q}}(t) = -\text{Diag}\left\{2\xi_i\omega_i\right\}\dot{\mathbf{q}}(t) - \text{Diag}\left\{\omega_i^2\right\}\mathbf{q}(t) + \sum_{j=1}^{N_l} \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}^T\mathbf{L}_j f_j(t) \tag{14.17}$$

the first-order dynamics of the system is again described by Eq. (14.11) with state matrix and input vector given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\text{Diag}\left\{\omega_i^2\right\} & -\text{Diag}\left\{2\xi_i\omega_i\right\} \end{bmatrix} \tag{14.18}$$

and

$$\mathbf{b}_j = \begin{bmatrix} \mathbf{0} \\ \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}^T\mathbf{L}_j \end{bmatrix} \tag{14.19}$$

The previous model can be slightly modified by defining the state vector as follows

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_n(t) \end{Bmatrix} \tag{14.20}$$

where the generic $\mathbf{x}_i(t)$ variable corresponding to the mode $i$ is given by

$$\mathbf{x}_i(t) = \begin{Bmatrix} q_i(t) \\ \dot{q}_i(t) \end{Bmatrix} \tag{14.21}$$

As a result, the state matrix $\mathbf{A}$ is a block diagonal matrix obtained as

$$\mathbf{A} = \text{Diag}\,\{\mathbf{A}_i\} \qquad\qquad (i = 1, \ldots, n) \tag{14.22}$$

where each block is

$$\mathbf{A}_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\xi_i\omega_i \end{bmatrix} \tag{14.23}$$

Accordingly, the input vector is given by

$$\mathbf{b}_j = \begin{Bmatrix} \mathbf{b}_{j1} \\ \mathbf{b}_{j2} \\ \vdots \\ \mathbf{b}_{jn} \end{Bmatrix} \tag{14.24}$$

where each element $\mathbf{b}_i$ is

$$\mathbf{b}_{ji} = \begin{Bmatrix} 0 \\ \frac{1}{m_i}\mathbf{u}_i^T\mathbf{L}_j \end{Bmatrix} \tag{14.25}$$

Another state-space modal model can be obtained by defining the state vector as follows

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{Bmatrix} = \begin{Bmatrix} \text{Diag}\,\{\omega_i\}\,\mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} \tag{14.26}$$

Since

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \text{Diag}\,\{\omega_i\}\,\mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) &= -\text{Diag}\,\{2\xi_i\omega_i\}\,\mathbf{x}_2(t) - \text{Diag}\,\{\omega_i\}\,\mathbf{x}_1(t) + \sum_{j=1}^{N_l} \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}^T\mathbf{L}_j f_j(t) \end{aligned} \tag{14.27}$$

the corresponding state matrix and input vectors are given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \text{Diag}\,\{\omega_i\} \\ -\text{Diag}\,\{\omega_i\} & -\text{Diag}\,\{2\xi_i\omega_i\} \end{bmatrix} \tag{14.28}$$

and

$$\mathbf{b}_j = \begin{bmatrix} \mathbf{0} \\ \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}^T\mathbf{L}_j \end{bmatrix} \tag{14.29}$$

Similarly to what done for the previous state model, the state vector can be recast as in Eq. (14.20), where now each element $\mathbf{x}_i(t)$ is expressed as

$$\mathbf{x}_i(t) = \begin{Bmatrix} \omega_i q_i(t) \\ \dot{q}_i(t) \end{Bmatrix} \tag{14.30}$$

Therefore, each block of the block diagonal state matrix in Eq. (14.22) is given by

$$\mathbf{A}_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\xi_i\omega_i \end{bmatrix} \tag{14.31}$$

and the input vectors are the same as Eqs. (14.24) and (14.25).

### 14.1.3 Output variables

Although the state of the system is fundamental, there are many situations in which one is not interested directly in the state vector, but on different quantities which can be derived from the state. Such quantities are usually called *output variables*, since they express the response of the structural system to the prescribed external loads. In such a context, the state is considered to be an internal variable since it describes the internal dynamics of the system, which is then reflected on the output variables. Two different types of output variables can be introduced:

- measured outputs
- performance outputs

Measured outputs, as the name implies, are representative of dynamic structural quantities which are measured using suitable sensors. The most common sensors used in structural dynamics problems are displacement sensors, velocity sensors, acceleration sensors and deformation sensors. Accordingly, measured outputs typically include displacements, velocities, accelerations and deformations.

Performance outputs are variables, not directly measured by sensors attached to the structure, which are of interest in order to evaluate in some way the performance of the system. A typical example is the kinetic energy of the structure, which can be considered as a overall measure of the vibration response and used to evaluate different structural designs. Performance outputs are usually introduced in control problems. In this case, they are typically referred to as regulated or error outputs since they represent signals important in the design of the control system for performance, contrary to measured outputs which are signals required for implementation of the control system.

### 14.1.4 Output equation

The *output equation* is the equation which describes how the measured outputs of the system are expressed in terms of the state variables adopted in the first-order representation. As shown in the following, it is possible to express the output equation in a general form, similarly to what done for the state equation (14.11). Let consider two cases which are useful to present the overall procedure.

The first case refers to a flexible vibrating slender beam modelled using the Ritz method. The transverse displacement at any point $x$ along the beam is approximated by the following $N$-terms expansion

$$w(x,t) = \mathbf{N}(x)\mathbf{u}(t) \tag{14.32}$$

where $\mathbf{u}(t)$ is the vector of Ritz coordinates. If the state vector is taken as in Eq. (14.2) and an ideal displacement sensor measuring the transverse deflection of the beam is placed at location $\bar{x}$, the output variable is written as

$$y(t) = w(\bar{x},t) = \mathbf{N}(\bar{x})\mathbf{u}(t) = \begin{bmatrix}\mathbf{N}(\bar{x}) & \mathbf{0}\end{bmatrix} \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} = \mathbf{C}\mathbf{x}(t) \tag{14.33}$$

where

$$\mathbf{C} = \begin{bmatrix}\mathbf{N}(\bar{x}) & \mathbf{0}\end{bmatrix} \tag{14.34}$$

is the row matrix relating the state vector to the displacement output. If $n_s$ displacement sensors are located at $\bar{x}_i$ $(i = 1, \dots, n_s)$, the output variables can be collected into an output vector which is expressed as

$$\mathbf{y}(t) = \begin{Bmatrix} w(\bar{x}_1,t) \\ w(\bar{x}_2,t) \\ \vdots \\ w(\bar{x}_{n_s},t) \end{Bmatrix} = \begin{bmatrix} \mathbf{N}(\bar{x}_1) & \mathbf{0} \\ \mathbf{N}(\bar{x}_2) & \mathbf{0} \\ \vdots & \\ \mathbf{N}(\bar{x}_{n_s}) & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} = \mathbf{C}\mathbf{x}(t) \tag{14.35}$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{N}(\bar{x}_1) & \mathbf{0} \\ \mathbf{N}(\bar{x}_2) & \mathbf{0} \\ \vdots & \\ \mathbf{N}(\bar{x}_{n_s}) & \mathbf{0} \end{bmatrix} \tag{14.36}$$

is the matrix of $n_s$ rows and $2N$ columns relating the state vector to the output vector. The $\mathbf{C}$ matrix is usually called *output matrix*.

A similar procedure can be applied if velocity sensors are used instead of displacement sensors. In this case, we can write the output vector as

$$\mathbf{y}(t) = \begin{Bmatrix} \dot{w}(\bar{x}_1, t) \\ \dot{w}(\bar{x}_2, t) \\ \vdots \\ \dot{w}(\bar{x}_{n_s}, t) \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{N}(\bar{x}_1) \\ \mathbf{0} & \mathbf{N}(\bar{x}_2) \\ \vdots & \\ \mathbf{0} & \mathbf{N}(\bar{x}_{n_s}) \end{bmatrix} \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} \tag{14.37}$$

which can be again written formally as before

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{14.38}$$

where now the matrix relating the state vector to the velocity output vector is given by

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{N}(\bar{x}_1) \\ \mathbf{0} & \mathbf{N}(\bar{x}_2) \\ \vdots & \\ \mathbf{0} & \mathbf{N}(\bar{x}_{n_s}) \end{bmatrix} \tag{14.39}$$

A slightly different result is obtained if we assume to have $n_s$ ideal accelerometers. In this case, the output vector is given by

$$\mathbf{y}(t) = \begin{Bmatrix} \ddot{w}(\bar{x}_1, t) \\ \ddot{w}(\bar{x}_2, t) \\ \vdots \\ \ddot{w}(\bar{x}_{n_s}, t) \end{Bmatrix} \tag{14.40}$$

Each component can be expressed as

$$\ddot{w}(\bar{x}_i, t) = \mathbf{N}(\bar{x}_i)\ddot{\mathbf{u}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{N}(\bar{x}_i) \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}}(t) \\ \ddot{\mathbf{u}}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{N}(\bar{x}_i) \end{bmatrix} \dot{\mathbf{x}}(t) \tag{14.41}$$

Since, from Eq. (14.11), the state derivative is expressed through the state vector and the external loads, we can write

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{N}(\bar{x}_1) \\ \mathbf{0} & \mathbf{N}(\bar{x}_2) \\ \vdots & \\ \mathbf{0} & \mathbf{N}(\bar{x}_{n_s}) \end{bmatrix} \left( \mathbf{A}\mathbf{x}(t) + \sum_{j=1}^{N_l} \mathbf{b}_j f_j(t) \right) \tag{14.42}$$

or, in more general form,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \sum_{j=1}^{N_l} \mathbf{d}_j f_j(t) \tag{14.43}$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{N}(\bar{x}_1) \\ \mathbf{0} & \mathbf{N}(\bar{x}_2) \\ \vdots & \\ \mathbf{0} & \mathbf{N}(\bar{x}_{n_s}) \end{bmatrix} \mathbf{A} \qquad \mathbf{d}_j = \begin{bmatrix} \mathbf{0} & \mathbf{N}(\bar{x}_1) \\ \mathbf{0} & \mathbf{N}(\bar{x}_2) \\ \vdots & \\ \mathbf{0} & \mathbf{N}(\bar{x}_{n_s}) \end{bmatrix} \mathbf{b}_j \tag{14.44}$$

It is clear that the output equation contains a term relating the state vector to the output vector, as the previous cases, and additional terms relating directly the external loads to the output variables, without the intervention of the state $\mathbf{x}(t)$. Such terms are usually called *direct feedthrough* terms.

The second case selected to show the procedure to obtain the output equation of a state space formulation refers again to the previous beam modelled according to the Ritz method. However, now the first-order representation is expressed using a reduced set of modal coordinates. Accordingly, we assume that the state vector has the form in Eq. (14.16). The output equation when $n_s$ displacement sensors are placed along the beam is written as

$$\mathbf{y}(t) = \begin{Bmatrix} w(\bar{x}_1, t) \\ w(\bar{x}_2, t) \\ \vdots \\ w(\bar{x}_{n_s}, t) \end{Bmatrix} = \begin{bmatrix} \mathbf{N}(\bar{x}_1)\mathbf{U} & \mathbf{0} \\ \mathbf{N}(\bar{x}_2)\mathbf{U} & \mathbf{0} \\ \vdots & \\ \mathbf{N}(\bar{x}_{n_s})\mathbf{U} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} = \mathbf{C}\mathbf{x}(t) \tag{14.45}$$

Note that the above output relation expressed by the matrix $\mathbf{C}$ takes the same form we have derived so far using Ritz coordinates. The difference lies in the contents of the output matrix $\mathbf{C}$.

Similarly, the output of $n_s$ velocity sensors is given by

$$\mathbf{y}(t) = \begin{Bmatrix} \dot{w}(\bar{x}_1, t) \\ \dot{w}(\bar{x}_2, t) \\ \vdots \\ \dot{w}(\bar{x}_{n_s}, t) \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{N}(\bar{x}_1)\mathbf{U} \\ \mathbf{0} & \mathbf{N}(\bar{x}_2)\mathbf{U} \\ \vdots & \\ \mathbf{0} & \mathbf{N}(\bar{x}_{n_s})\mathbf{U} \end{bmatrix} \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} = \mathbf{C}\mathbf{x}(t) \tag{14.46}$$

From what presented so far, it should be clear that the general form of the output equation related to a state-space formulation can be written as in Eq. (14.43). Even though only two simple examples have been discussed, they have shown the procedure to be followed to derive the output equation. It mainly consists of three steps:

1. Express the output vector in terms of the physical variables which are actually measured by the sensors located on the structure.

2. Express each measured physical variable using the spatial discretization technique adopted to model the flexible structure.

3. Rearrange the previous relations in order to relate the output vector to the assumed state vector and the external (disturbance and control) loads acting on the structure.

More examples will be presented in the following, which can be useful to better illustrate the above procedure.

### 14.1.5 Performance equation

As outlined in the previous section, in many cases one is interested in dynamic quantities which are not directly measured by sensors. They are usually called performance output variables since they are related to the design of the structure. The equation describing how such outputs are expressed in terms of the state vector is called *performance equation*. The procedure to obtain the performance equation is completely similar to what presented before for the output equation.

As an illustrative example, let consider the slender beam of the previous section. We assume that the overall kinetic energy of the beam is taken as a performance measure of the dynamic behaviour of the system. In our flexural case, after neglecting the contribution due to rotary inertia, the kinetic energy is given by

$$T = \frac{1}{2}\int_{\ell} m\dot{w}^2 dx = \frac{1}{2}\dot{\mathbf{u}}^T(t)\mathbf{M}\dot{\mathbf{u}}(t) \tag{14.47}$$

where $\mathbf{M} = \int_{\ell} m\mathbf{N}^T\mathbf{N}dx$ is the symmetric positive definite mass matrix arising from the Ritz discretization. Using the Cholesky decomposition of the mass matrix

$$\mathbf{M} = \mathbf{L}\mathbf{L}^T \tag{14.48}$$

where $\mathbf{L}$ is a lower triangular matrix, the quadratic form in Eq. (14.47) can be rewritten as

$$T = \frac{1}{2}\dot{\mathbf{u}}^T(t)\mathbf{L}\mathbf{L}^T\dot{\mathbf{u}}(t) \tag{14.49}$$

or, alternatively,

$$T = \frac{1}{2}\mathbf{z}^T(t)\mathbf{z}(t) \tag{14.50}$$

where

$$\mathbf{z}(t) = \mathbf{L}^T\dot{\mathbf{u}}(t) \tag{14.51}$$

Therefore, the performance measure in terms of kinetic energy of the beam can be expressed as a quadratic form involving the new variable $\mathbf{z}$, which can be called performance output variable. If the state vector is taken as in Eq. (14.2), the performance output can be expressed as

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{L}^T \end{bmatrix} \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} = \mathbf{C}_z\mathbf{x}(t) \tag{14.52}$$

where

$$\mathbf{C}_z = \begin{bmatrix} \mathbf{0} & \mathbf{L}^T \end{bmatrix} \tag{14.53}$$

is the *performance output matrix*.

If the same beam is represented by a reduced-order modal model computed from the discretized Ritz model, the state vector is taken as in Eq. (14.16) and the kinetic energy is written as

$$T = \frac{1}{2}\dot{\mathbf{q}}^T(t)\text{Diag}\left\{m_i\right\}\dot{\mathbf{q}}(t) \tag{14.54}$$

In this case, the performance output variable can be defined as

$$\mathbf{z}(t) = \text{Diag}\left\{\sqrt{m_i}\right\}\dot{\mathbf{q}}(t) \tag{14.55}$$

or, in terms of the state vector, as

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{0} & \text{Diag}\left\{\sqrt{m_i}\right\} \end{bmatrix} \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} = \mathbf{C}_z\mathbf{x}(t) \tag{14.56}$$

where

$$\mathbf{C}_z = \begin{bmatrix} \mathbf{0} & \text{Diag}\left\{\sqrt{m_i}\right\} \end{bmatrix} \tag{14.57}$$

### 14.1.6 Generalized state-space models

From the previous discussion and examples, it should be clear that we can represent a LTI structural system in state-space form by three equations:

- the state equation
- the output equation

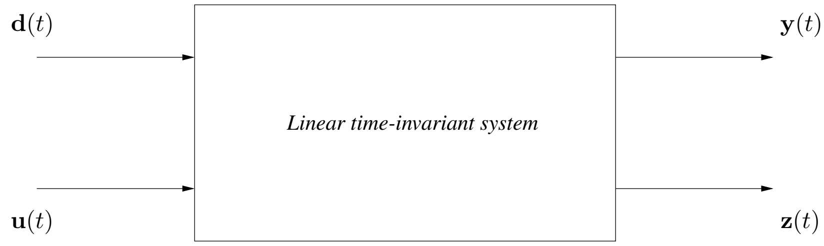


**Figure 14.1** Generalized state-space model.

- the performance equation

We have seen that these equations mainly involve five vector variables, which can be grouped as two input variables, two output variables and one internal variable. Using the classical notation adopted in the control literature, the input variables are:

- disturbance inputs, denoted by $\mathbf{d}(t) \in \mathbb{R}^{n_d \times 1}$, which are typically external prescribed loads, also called exogenous or environmental inputs, that are responsible of exciting the system in an undesired manner;
- control inputs, denoted by $\mathbf{u}(t) \in \mathbb{R}^{n_u \times 1}$, which are controllable variables used to regulate in some way the dynamic response of the system such that the system subjected to $\mathbf{d}$ and $\mathbf{u}$ exhibits a desired dynamic behaviour. The input vector $\mathbf{u}$ is usually associated with the design of a control system.

As already discussed in the previous section, the output variables can be distinguished between:

- measured output variables, denoted by $\mathbf{y}(t) \in \mathbb{R}^{n_y \times 1}$, which are physical quantities measured by sensors and representing the system response as perceived by an observer;
- performance variables, denoted by $\mathbf{z}(t) \in \mathbb{R}^{n_z \times 1}$, which are variables related to a performance measure of the structural response and typically introduced as error outputs when a control system is to be designed.

The internal variable of the state-space representation is given by the state vector $\mathbf{x}(t) \in \mathbb{R}^{n_x \times 1}$ of the system, which involves the state variables expressing the structural dynamics and any other internal dynamics of interest, as shown later. Accordingly, the block diagram representation of a LTI system is depicted in Figure 14.1, which is usually called *generalized state space model.*

Since measured outputs are signals coming from sensor devices, they are affected by measurement noise, which should be kept as small as possible, but it is never exactly equal to zero. Measurement noise is typically modelled as an additive noise $\mathbf{r}(t) \in \mathbb{R}^{n_r \times 1}$, which is considered as an exogenous variable since it is prescribed (i.e., not controllable).

According to the above notation and the presence of measurement noise, the most general form of the state-space representation of a linear time-invariant system can be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_d\mathbf{d}(t) \\ \mathbf{y}(t) &= \mathbf{C}_y\mathbf{x}(t) + \mathbf{D}_{yu}\mathbf{u}(t) + \mathbf{D}_{yd}\mathbf{d}(t) + \mathbf{D}_{yr}\mathbf{r}(t) \\ \mathbf{z}(t) &= \mathbf{C}_z\mathbf{x}(t) + \mathbf{D}_{zu}\mathbf{u}(t) + \mathbf{D}_{zd}\mathbf{d}(t) \end{aligned} \tag{14.58}$$

where

- $\mathbf{A}$ is the $n_x \times n_x$ state matrix
- $\mathbf{B}_u$ is the $n_x \times n_u$ input matrix (the $i$th column is referred to the influence vector of the $i$th control input)
- $\mathbf{B}_d$ is the $n_x \times n_d$ disturbance matrix (the $i$th column is referred to the influence vector of the $i$th disturbance input)
- $\mathbf{C}_y$ is the $n_y \times n_x$ output matrix (the $i$th row is referred to the $i$th output variable)

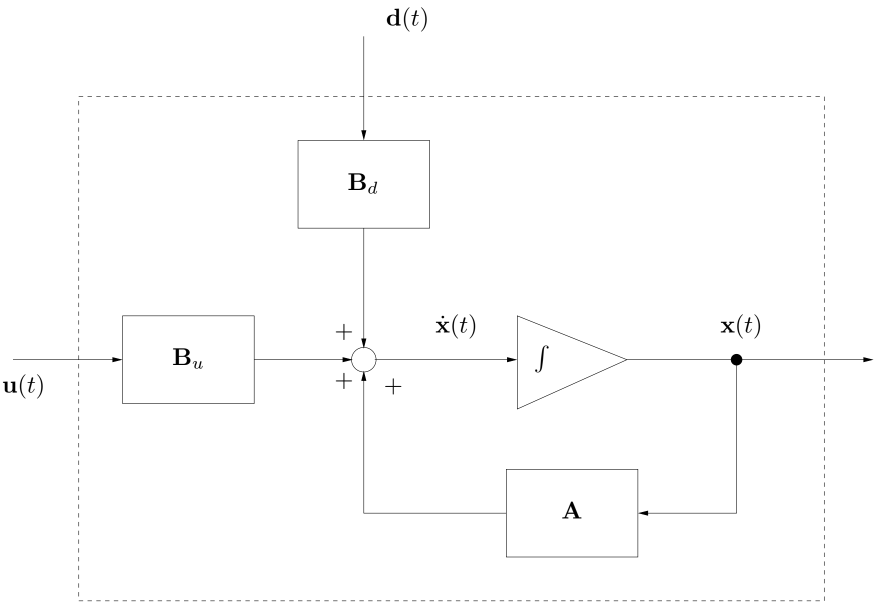


**Figure 14.2** Block diagram representation of the state equation.

- $\mathbf{D}_{yu}$ is the $n_y \times n_u$ direct feedthrough matrix (the element $(i, j)$ is the direct feedthrough term of the $j$th control input on the $i$th output)

- $\mathbf{D}_{yd}$ is the $n_y \times n_d$ disturbance direct feedthrough matrix (the element $(i, j)$ is the direct feedthrough term of the $j$th disturbance input on the $i$th output)

- $\mathbf{D}_{yr}$ is the $n_y \times n_r$ noise matrix (the element $(i, j)$ relates the $j$th noise variable on the $i$th output variable)

- $\mathbf{C}_z$ is the $n_z \times n_x$ performance matrix (the $i$th row is referred to the $i$th performance variable)

- $\mathbf{D}_{zu}$ is the $n_z \times n_u$ direct feedthrough matrix of the performance (the element $(i, j)$ is the direct feedthrough term of the $j$th control input on the $i$th performance variable)

- $\mathbf{D}_{zd}$ is the $n_z \times n_d$ disturbance direct feedthrough matrix of the performance (the element $(i, j)$ is the direct feedthrough term of the $j$th disturbance input on the $i$th performance variable)

The block diagram representations of the state equation, the output equation and the performance equation are reported in Figures 14.2, 14.3, and 14.4, respectively. Note that, in most cases, the noise matrix $\mathbf{D}_{yr}$ is given by the identity matrix. Moreover, the presence of the $\mathbf{D}_{yu}$ and $\mathbf{D}_{yd}$ matrices in Eqs. (14.58) is questionable in practical applications. Indeed, a $\mathbf{D}$ matrix means a direct connection between the input $\mathbf{u}$ or the disturbance $\mathbf{d}$ on the system output $\mathbf{y}$, without any dynamics involved in the state vector $\mathbf{x}$. In other words, the external input variables are directly reflected on the output variables through a static relation. Since both the structure and the sensor and actuator devices are dynamic systems, a static relation should arise from a modelling approximation, which typically implies some sort of static residualization of the dynamics.

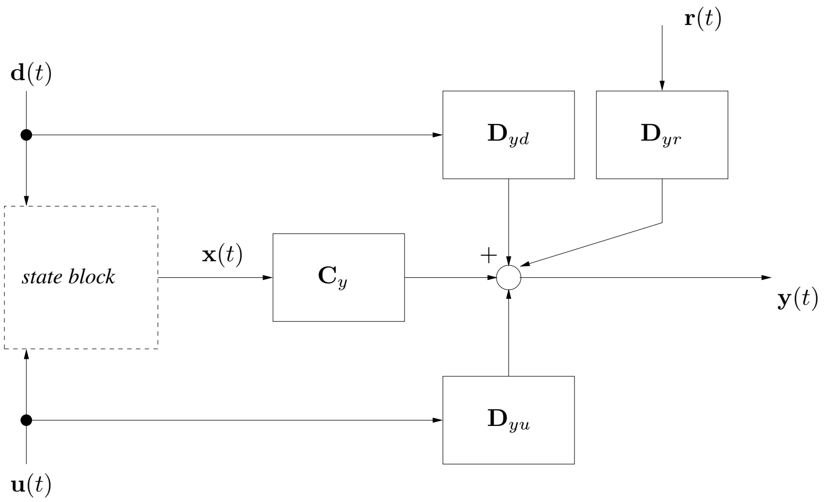


**Figure 14.3** Block diagram representation of the output equation.

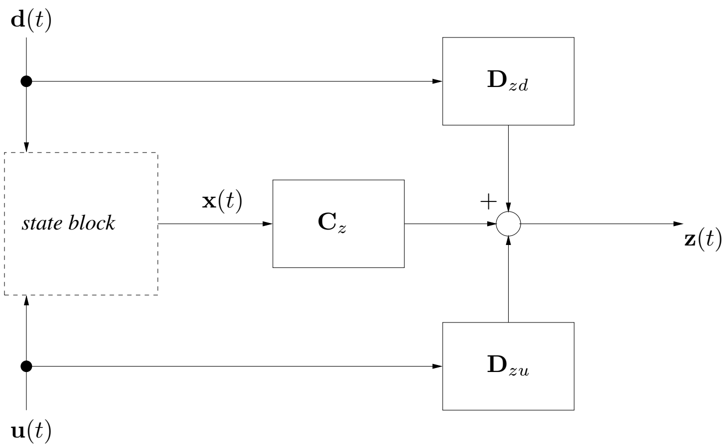


**Figure 14.4** Block diagram representation of the performance equation.

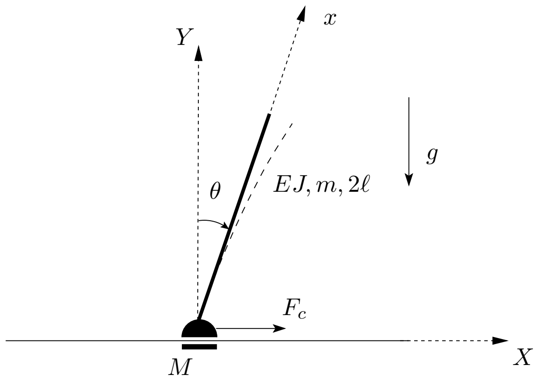


**Figure 14.5** The inverted pendulum.

## 14.2 Examples of state space models

### 14.2.1 Rigid pendulum

Let consider the frictionless inverted pendulum described in Section 8.5.2 and depicted in Figure 14.5. Assuming the pendulum beam as rigid, the *linearized* set of equations of motion for *small* rotation $\theta(t)$ were derived as follows

$$\begin{cases} (M+\mu)\ddot{x} + \mu\ell\ddot{\theta} = F_c \\ \\ \dfrac{4}{3}\mu\ell^2\ddot{\theta} + \mu\ell\ddot{x} - \mu g\ell\theta = 0 \end{cases} \tag{14.59}$$

where $x(t)$ is the cart position, $M$ is the cart mass, $\mu$ is the pendulum mass, $2\ell$ is the pendulum length and $F_c$ is the control force applied on the cart. Note that the system dynamics is described by two second-order differential equations in the unknowns $x$ and $\theta$.

The above set can be also written in matrix form as

$$\begin{bmatrix} M+\mu & \mu\ell \\ \mu\ell & \frac{4}{3}\mu\ell^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\mu g\ell \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_c \tag{14.60}$$

or, alternatively,

$$\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} = -\begin{bmatrix} M+\mu & \mu\ell \\ \mu\ell & \frac{4}{3}\mu\ell^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -\mu g\ell \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} + \begin{bmatrix} M+\mu & \mu\ell \\ \mu\ell & \frac{4}{3}\mu\ell^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_c \tag{14.61}$$

More explicitly,

$$\begin{aligned} \ddot{x} &= \frac{3\mu g}{4M+\mu}\theta + \frac{4}{4M+\mu}F_c \\ \ddot{\theta} &= -\frac{3(M+\mu)g}{\ell(4M+\mu)}x - \frac{3}{\ell(4M+\mu)}F_c \end{aligned} \tag{14.62}$$

Therefore, a state-space representation of the system can be obtained for example[2] by defining the state vector as

$$\mathbf{x}(t) = \begin{Bmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{Bmatrix} \tag{14.63}$$

[2]As seen before, we can derive many different state space formulation of the same physical system. They are all equivalent as shown in the following.

It follows that

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \dot{x}(t) \\ \dot{\theta}(t) \\ \ddot{x}(t) \\ \ddot{\theta}(t) \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3\mu g}{4M+\mu} & 0 & 0 \\ -\frac{3(M+\mu)g}{\ell(4M+\mu)} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{4}{4M+\mu} \\ -\frac{3}{\ell(4M+\mu)} \end{bmatrix} F_c \tag{14.64}$$

If we assume that the system is equipped with two noiseless sensors, one measuring the cart position $x(t)$ and one measuring the pendulum rotation $\theta(t)$, the output equation is given by

$$\mathbf{y}(t) = \begin{Bmatrix} x(t) \\ \theta(t) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{Bmatrix} \tag{14.65}$$

Using the notation introduced in the previous paragraph, the matrices of the present state-space formulation are

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{3\mu g}{4M+\mu} & 0 & 0 \\ -\frac{3(M+\mu)g}{\ell(4M+\mu)} & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{b}_u = \begin{bmatrix} 0 \\ 0 \\ \frac{4}{4M+\mu} \\ -\frac{3}{\ell(4M+\mu)} \end{bmatrix} \tag{14.66}$$

$$\mathbf{C}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{d}_{yu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{14.67}$$

### 14.2.2 Flexible pendulum

The linearized (small rotation) equations of motion of the previous pendulum when the pendulum beam is flexible may be written as

$$\begin{cases} (M + 2m\ell)\ddot{u}_0 + 2m\ell^2\ddot{u}_1 + m\displaystyle\int_0^{2\ell} \mathbf{N}dx\,\ddot{\mathbf{u}} = F_c \\[2ex] \dfrac{8}{3}m\ell^3\ddot{u}_1 + 2m\ell^2\ddot{u}_0 + m\displaystyle\int_0^{2\ell} \mathbf{N}xdx\,\ddot{\mathbf{u}}(t) - 2mg\ell^2 u_1 - mg\int_0^{2\ell}(2\ell - x)\mathbf{N}_{/x}dx\,\mathbf{u}(t) = 0 \\[2ex] m\displaystyle\int_0^{2\ell} \mathbf{N}^T dx\,\ddot{u}_0(t) + m\int_0^{2\ell} \mathbf{N}^T xdx\,\ddot{u}_1(t) + m\int_0^{2\ell} \mathbf{N}^T\mathbf{N}dx\,\ddot{\mathbf{u}}(t) + EJ\int_0^{2\ell} \mathbf{N}^T_{/xx}\mathbf{N}_{/xx}dx\,\mathbf{u}(t) \\ \qquad -mg\displaystyle\int_0^{2\ell}(2\ell - x)\mathbf{N}^T_{/x}dx\,u_1(t) - mg\int_0^{2\ell}(2\ell - x)\mathbf{N}^T_{/x}\mathbf{N}_{/x}dx\,\mathbf{u}(t) = \mathbf{0} \end{cases} \tag{14.68}$$

where $u_0(t)$ is the rigid-body translation, $u_1(t)$ is the small rigid-body rotation, $\mathbf{u}(t)$ contains the coordinates describing the flexible motion, $\mathbf{N}(x)$ is the $1 \times N$ row matrix of Ritz admissible functions, $m$ is the mass per unit length of the pendulum beam and $EJ$ is the flexural stiffness of the pendulum beam. Note that the above set is composed by $2 + N$ equations, where $N$ is the number of Ritz functions used to take into account the flexibility of the beam. We can write the system dynamics in the following matrix form

$$\begin{bmatrix} M + 2m\ell & 2m\ell^2 & \mathbf{m}_{0u} \\ 2m\ell^2 & \frac{8}{3}m\ell^3 & \mathbf{m}_{1u} \\ \mathbf{m}^T_{0u} & \mathbf{m}^T_{1u} & \mathbf{M}_{uu} \end{bmatrix} \begin{Bmatrix} \ddot{u}_0 \\ \ddot{u}_1 \\ \ddot{\mathbf{u}} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & \mathbf{0}_{1\times N} \\ 0 & -2mg\ell^2 & -\mathbf{k}_{1u} \\ \mathbf{0}_{N\times 1} & -\mathbf{k}^T_{1u} & \mathbf{K}_{uu} \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ \mathbf{u} \end{Bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \mathbf{0}_{N\times 1} \end{bmatrix} F_c \tag{14.69}$$

where

$$
\begin{aligned}
\mathbf{m}_{0u} &= m\int_0^{2\ell}\mathbf{N}dx \\
\mathbf{m}_{1u} &= m\int_0^{2\ell}\mathbf{N}xdx \\
\mathbf{M}_{uu} &= m\int_0^{2\ell}\mathbf{N}^T\mathbf{N}dx \\
\mathbf{k}_{1u} &= mg\int_0^{2\ell}(2\ell-x)\mathbf{N}_{/x}dx \\
\mathbf{K}_{uu} &= EJ\int_0^{2\ell}\mathbf{N}_{/xx}^T\mathbf{N}_{/xx}dx - mg\int_0^{2\ell}(2\ell-x)\mathbf{N}_{/x}^T\mathbf{N}_{/x}dx
\end{aligned}
\tag{14.70}
$$

By defining the mass and stiffness matrices as

$$
\mathbf{M} = \begin{bmatrix} M+2m\ell & 2m\ell^2 & \mathbf{m}_{0u} \\ 2m\ell^2 & \frac{8}{3}m\ell^3 & \mathbf{m}_{1u} \\ \mathbf{m}_{0u}^T & \mathbf{m}_{1u}^T & \mathbf{M}_{uu} \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 0 & 0 & \mathbf{0}_{1\times N} \\ 0 & -2mg\ell^2 & -\mathbf{k}_{1u} \\ \mathbf{0}_{N\times 1} & -\mathbf{k}_{1u}^T & \mathbf{K}_{uu} \end{bmatrix} \tag{14.71}
$$

and introducing the following state vector

$$
\mathbf{x}(t) = \begin{Bmatrix} u_0(t) \\ u_1(t) \\ \mathbf{u}(t) \\ \dot{u}_0(t) \\ \dot{u}_1(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} \tag{14.72}
$$

a state-space formulation of the flexible pendulum takes the form

$$
\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}F_c(t) \tag{14.73}
$$

where

$$
\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \tag{14.74}
$$

and

$$
\mathbf{b}_u = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\begin{Bmatrix} 1 \\ 0 \\ \mathbf{0}_{N\times 1} \end{Bmatrix} \end{bmatrix} \tag{14.75}
$$

If we assume that the system is equipped with one sensor measuring the rigid-body translation $u_0(t)$ placed on the cart and one sensor measuring the rigid-body rotation $u_1(t)$ placed on the pendulum hinge, the output equation is given by

$$
\mathbf{y}(t) = \begin{Bmatrix} u_0(t) \\ u_1(t) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & \mathbf{0}_{1\times N} & 0 & 0 & \mathbf{0}_{1\times N} \\ 0 & 1 & \mathbf{0}_{1\times N} & 0 & 0 & \mathbf{0}_{1\times N} \end{bmatrix} \begin{Bmatrix} u_0(t) \\ u_1(t) \\ \mathbf{u}(t) \\ \dot{u}_0(t) \\ \dot{u}_1(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} = \mathbf{C}_y\mathbf{x}(t) \tag{14.76}
$$

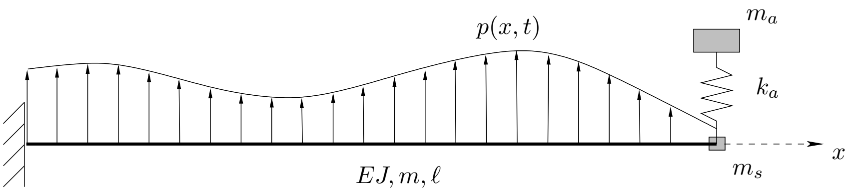


**Figure 14.6** A cantilever beam with a tip inertial actuator.

### 14.2.3 Cantilever beam with a tip inertial actuator

Let consider a cantilever beam of length $\ell$, mass per unit length $m(x)$ and flexural rigidity $EJ(x)$ (see Figure 14.6). The beam is induced to vibrate by a distributed transverse load $p(x,t)$ with prescribed spatial distribution $p_0(x)$ and prescribed temporal variation $d(t)$, such that $p(x,t) = p_0(x)d(t)$. The beam is equipped at the tip with an accelerometer of mass $m_s$ and an inertial actuator, which is modelled as a SDOF system of mass $m_a$ and spring $k_a$. Starting from a finite element model of the beam with the tip sensor, we would like to write the state-space model of the system by coupling a reduced-order modal model of the beam dynamics with the equations of the inertial actuator.

Since the problem requires to couple a modal representation of the beam dynamics with the dynamics of the tip inertial actuator, the first step is to write the equations of motion of the beam without the actuator. The effect of the inertial actuator is considered by introducing the force $F_{\text{tip}}(t)$ which is transmitted to the beam by the actuator. As a result, the beam with the tip sensor is forced by the disturbance distributed load $p(x,t)$ and the tip force $F_{\text{tip}}(t)$ as shown in Figure 14.7.

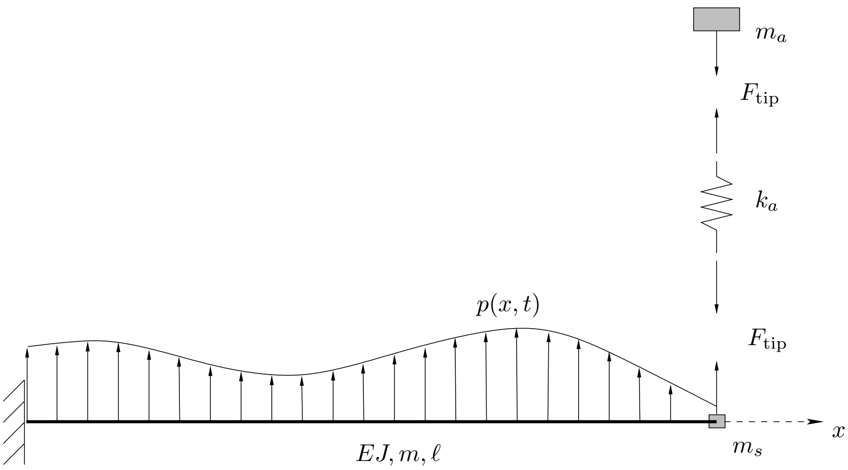


**Figure 14.7** A cantilever beam with a tip inertial actuator.

The dynamic equilibrium of the beam can be written through the principle of virtual work as follows

$$\int_0^\ell \delta w_{/xx}^T EJ(x) w_{/xx} dx + \int_0^\ell \delta w^T m(x) \ddot{w} dx + \delta w^T(\ell,t) m_s \ddot{w}(\ell,t) = \int_0^\ell \delta w^T p_0(x) dx\, d(t) + \delta w^T(\ell,t) F_{\text{tip}}(t) \tag{14.77}$$

where $w = w(x,t)$ is the transverse displacement of a generic point of the beam. According to a finite element formulation (see chapter 9), the beam length is divided into elements so that each term in the above equation is expressed in the form of a sum over the individual elements to yield

$$\begin{aligned}\sum_{j=1}^{n}\left[\int_{(j-1)h}^{jh}\delta w_{/xx}^{T}EJ(x)w_{/xx}dx+\int_{(j-1)h}^{jh}\delta w^{T}m(x)\ddot{w}dx+\delta_{jn}\delta w^{T}(\ell,t)m_{s}\ddot{w}(\ell,t)\right]=\\ \sum_{j=1}^{n}\left[\int_{(j-1)h}^{jh}\delta w^{T}p_{0}dx\,d(t)+\delta_{jn}\delta w^{T}(\ell,t)F_{\mathrm{tip}}(t)\right]\end{aligned}\tag{14.78}$$

where $n$ is the total number of elements, $h$ is the length of each element, and $\delta_{jn}$ is the Kronecker symbol (i.e., $\delta_{jn}=1$ if $j=n$, otherwise it is equal to zero). Note that the beam has been divided by elements of the same length $h$. After replacing the beam coordinate $x$ with the local dimensionless coordinates $\xi$, inside each element $j$, the transverse displacement is approximated by the following expression

$$w(\xi,t)=N_{1}(\xi)w_{j-1}(t)+N_{2}(\xi)\theta_{j-1}(t)+N_{3}(\xi)w_{j}(t)+N_{4}(\xi)\theta_{j}(t)=\mathbf{N}(\xi)\mathbf{u}_{j}(t)\qquad 0<\xi<1\tag{14.79}$$

where

$$\mathbf{u}_{j}(t)=\begin{Bmatrix}w_{j-1}(t)\\ \theta_{j-1}(t)\\ w_{j}(t)\\ \theta_{j}(t)\end{Bmatrix}\tag{14.80}$$

is the vector of nodal degrees of freedom of the $j$th element, and

$$\begin{aligned}N_{1}(\xi)&=\left(1-3\xi^{2}+2\xi^{3}\right)\\ N_{2}(\xi)&=h\left(\xi-2\xi^{2}+\xi^{3}\right)\\ N_{3}(\xi)&=\left(3\xi^{2}-2\xi^{3}\right)\\ N_{4}(\xi)&=h\left(\xi^{3}-\xi^{2}\right)\end{aligned}\tag{14.81}$$

is the set of Hermitian interpolating functions. Accordingly, the stiffness term of any element $j$ is given by

$$\int_{(j-1)h}^{jh}\delta w_{/xx}^{T}EJ(x)w_{/xx}dx=\delta\mathbf{u}_{j}^{T}(t)\mathbf{K}_{j}\mathbf{u}_{j}(t)\tag{14.82}$$

where

$$\mathbf{K}_{j}=\frac{1}{h^{3}}\int_{0}^{1}EJ\left[h\xi+(j-1)h\right]\mathbf{N}_{/\xi\xi}^{T}\mathbf{N}_{/\xi\xi}d\xi\tag{14.83}$$

The mass term of any element $j$ is

$$\int_{(j-1)h}^{jh}\delta w^{T}m(x)\ddot{w}dx+\delta_{jn}\delta w^{T}(\ell,t)m_{s}\ddot{w}(\ell,t)=\delta\mathbf{u}_{j}^{T}(t)\left[\mathbf{M}_{j}+\delta_{jn}\mathbf{M}_{s}\right]\ddot{\mathbf{u}}_{j}(t)\tag{14.84}$$

where

$$\mathbf{M}_{j}=h\int_{0}^{1}m\left[h\xi+(j-1)h\right]\mathbf{N}^{T}\mathbf{N}d\xi\tag{14.85}$$

and

$$\mathbf{M}_{s}=m_{s}\mathbf{N}^{T}(1)\mathbf{N}(1)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&m_{s}&0\\0&0&0&0\end{bmatrix}\tag{14.86}$$

The load term of any element $j$ is given by

$$\int_{(j-1)h}^{jh}\delta w^{T}p_{0}(x)dx\,d(t)+\delta_{jn}\delta w^{T}(\ell,t)F_{\mathrm{tip}}(t)=\delta\mathbf{u}_{j}^{T}(t)\mathbf{L}_{j}^{d}d(t)+\delta\mathbf{u}_{j}^{T}(t)\mathbf{L}_{j}^{c}F_{\mathrm{tip}}(t)\tag{14.87}$$

where

$$\mathbf{L}_j^d = h \int_0^1 p_0 \left[h\xi + (j-1)h\right] \mathbf{N}^T d\xi \tag{14.88}$$

and

$$\mathbf{L}_j^c = \delta_{jn} \mathbf{N}^T(1) = \delta_{jn} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{14.89}$$

Referring back to the principle of virtual work and using what derived so far, the dynamic equilibrium of the beam with the tip sensor can be written as

$$\begin{aligned} &\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t) \left[\mathbf{M}_j + \delta_{jn}\mathbf{M}_s\right] \ddot{\mathbf{u}}_j(t) + \sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{K}_j\mathbf{u}_j(t) \\ &\quad = \sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{L}_j^d d(t) + \sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{L}_j^c F_{\text{tip}}(t) \end{aligned} \tag{14.90}$$

The assembly procedure described in chapter 9 yields

$$\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t) \left[\mathbf{M}_j + \delta_{jn}\mathbf{M}_s\right] \ddot{\mathbf{u}}_j(t) = \delta\mathbf{u}^T(t)\mathbf{M}_{beam}\ddot{\mathbf{u}}(t) \tag{14.91}$$

$$\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{K}_j\mathbf{u}_j(t) = \delta\mathbf{u}^T(t)\mathbf{K}_{beam}\mathbf{u}(t) \tag{14.92}$$

and

$$\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{L}_j^d d(t) = \delta\mathbf{u}^T(t)\mathbf{L}_d d(t) \tag{14.93}$$

$$\sum_{j=1}^{n} \delta\mathbf{u}_j^T(t)\mathbf{L}_j^c F_{\text{tip}}(t) = \delta\mathbf{u}^T(t)\mathbf{L}_c F_{\text{tip}}(t) \tag{14.94}$$

where $\mathbf{u}(t)$ contains the nodal degrees of freedom

$$\mathbf{u}(t) = \begin{Bmatrix} w_0(t) \\ \theta_0(t) \\ w_1(t) \\ \theta_1(t) \\ \vdots \\ w_n(t) \\ \theta_n(t) \end{Bmatrix} \tag{14.95}$$

As a final step, since the beam is clamped at $x = 0$, the corresponding nodal variables are zero, i.e., $w_0 = \theta_0 = 0$. Therefore, the vector $\mathbf{u}(t)$ is actually given by

$$\mathbf{u}(t) = \begin{Bmatrix} w_1(t) \\ \theta_1(t) \\ \vdots \\ w_n(t) \\ \theta_n(t) \end{Bmatrix} \tag{14.96}$$

and the first two rows and columns of the mass and stiffness matrices and the first two components of $\mathbf{L}_d$ and $\mathbf{L}_c$ are omitted. Consequently, $\mathbf{M}_{beam}$ and $\mathbf{K}_{beam}$ have dimensions $2n \times 2n$, and $\mathbf{L}_d$ and $\mathbf{L}_c$ have dimensions $2n \times 1$. By exploiting the arbitrariness of $\delta\mathbf{u}$, the following governing equations of the beam are obtained:

$$\mathbf{M}_{beam}\ddot{\mathbf{u}}(t) + \mathbf{K}_{beam}\mathbf{u}(t) = \mathbf{L}_d d(t) + \mathbf{L}_c F_{\text{tip}}(t) \tag{14.97}$$

Accuracy considerations typically dictate the use of a large number $n$ of elements. A reduced-order model can be extracted from the previous finite element model by using the modal approach. Once a reduced set of low-frequency eigenpairs is computed from the solution of the eigenvalue problem associated with Eq. (14.97), i.e.,

$$\left(\mathbf{K}_{beam} - \omega^2\mathbf{M}_{beam}\right)\hat{\mathbf{u}} = \mathbf{0} \tag{14.98}$$

the nodal vector $\mathbf{u}(t)$ is transformed into the modal vector $\mathbf{q}_{\text{L}}(t)$ as follows

$$\mathbf{u}(t) = \mathbf{U}_{\text{L}}\mathbf{q}_{\text{L}}(t) \tag{14.99}$$

where $\mathbf{U}_{\text{L}}$ is the rectangular matrix containing the first $n_m \ll n$ eigenvectors. Equation (14.97) becomes

$$\text{Diag}\left\{m_i\right\}\ddot{\mathbf{q}}_{\text{L}}(t) + \text{Diag}\left\{m_i\omega_i^2\right\}\mathbf{q}_{\text{L}}(t) = \mathbf{U}_{\text{L}}^T\mathbf{L}_d d(t) + \mathbf{U}_{\text{L}}^T\mathbf{L}_c F_{\text{tip}}(t) \tag{14.100}$$

Let consider now the inertial actuator. The dynamic equilibrium of the sismic mass $m_a$ of the actuator is expressed as

$$m_a\ddot{z}(t) + F_{\text{tip}}(t) = 0 \tag{14.101}$$

where $z = z(t)$ is the absolute displacement of the mass $m_a$. The elongation $\delta(t)$ of the spring $k_a$ is

$$\delta(t) = \frac{F_{\text{tip}}(t)}{k_a} + \delta_c(t) \tag{14.102}$$

where the first quantity in the right-end side is the elastic contribution and $\delta_c(t)$ is the contribution due to the actuating device, i.e., the additional elongation arising when the actuator is driven by an external control source. Since we can also write the total elongation as

$$\delta(t) = z(t) - w(\ell, t) \tag{14.103}$$

where $w(\ell, t)$ is the transverse tip displacement of the beam, the tip force is given by

$$F_{\text{tip}}(t) = k_a\left[\delta(t) - \delta_c(t)\right] = k_a\left[z(t) - w(\ell, t) - \delta_c(t)\right] \tag{14.104}$$

The above expression can be also written in an alternative form as

$$F_{\text{tip}}(t) = k_a\left[z(t) - w(\ell, t)\right] - F_c(t) \tag{14.105}$$

where

$$F_c(t) = k_a\delta_c(t) \tag{14.106}$$

is the actuator (control) force, i.e., the equivalent force of the actuator when it is driven by an external control source. Note that, according to the modal model of the beam derived above, the tip transverse displacement is

$$\begin{aligned} w(\ell, t) &= \mathbf{L}_c^T\mathbf{u}(t) \\ &= \mathbf{L}_c^T\mathbf{U}_{\text{L}}\mathbf{q}_{\text{L}}(t) \end{aligned} \tag{14.107}$$

Therefore, the tip force is given by

$$F_{\text{tip}}(t) = k_a z(t) - k_a\mathbf{L}_c^T\mathbf{U}_{\text{L}}\mathbf{q}_{\text{L}}(t) - F_c(t) \tag{14.108}$$

Substituting Eq. (14.108) into Eq. (14.100), the dynamics of the beam may be written as

$$\text{Diag}\left\{m_i\right\}\ddot{\mathbf{q}}_{\text{L}}(t) + \text{Diag}\left\{m_i\omega_i^2\right\}\mathbf{q}_{\text{L}}(t) = \mathbf{U}_{\text{L}}^T\mathbf{L}_d d(t) + \mathbf{U}_{\text{L}}^T\mathbf{L}_c\left[k_a z(t) - k_a\mathbf{L}_c^T\mathbf{U}_{\text{L}}\mathbf{q}_{\text{L}}(t) - F_c(t)\right] \tag{14.109}$$

Substituting Eq. (14.108) into Eq. (14.101), the dynamics of the actuator may be written as

$$m_a \ddot{z}(t) + k_a z(t) - k_a \mathbf{L}_c^T \mathbf{U}_{\mathrm{L}} \mathbf{q}_{\mathrm{L}}(t) - F_c(t) = 0 \tag{14.110}$$

It is clear that the above governing equations are coupled through the spring $k_a$. Note also that the beam equipped with the inertial actuator is equivalent to a beam with a spring-mass system and a pair of equal and opposite internal control forces $F_c$. The above equations are rearranged to yield

$$\begin{cases} \text{Diag}\{m_i\}\ddot{\mathbf{q}}_{\mathrm{L}}(t) + \left[\text{Diag}\{m_i\omega_i^2\} + \mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c k_a \mathbf{L}_c^T \mathbf{U}_{\mathrm{L}}\right]\mathbf{q}_{\mathrm{L}}(t) - \mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c k_a z(t) = -\mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c F_c(t) + \mathbf{U}_{\mathrm{L}}^T\mathbf{L}_d d(t) \\ m_a\ddot{z}(t) + k_a z(t) - k_a\mathbf{L}_c^T\mathbf{U}_{\mathrm{L}}\mathbf{q}_{\mathrm{L}}(t) = F_c(t) \end{cases} \tag{14.111}$$

or, in compact matrix form,

$$\begin{aligned} &\begin{bmatrix} \text{Diag}\{m_i\} & \mathbf{0}_{n_m\times 1} \\ \mathbf{0}_{1\times n_m} & m_a \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_{\mathrm{L}} \\ \ddot{z} \end{Bmatrix} + \begin{bmatrix} \text{Diag}\{m_i\omega_i^2\} + \mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c k_a \mathbf{L}_c^T\mathbf{U}_{\mathrm{L}} & -\mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c k_a \\ -k_a\mathbf{L}_c^T\mathbf{U}_{\mathrm{L}} & k_a \end{bmatrix} \begin{Bmatrix} \mathbf{q}_{\mathrm{L}} \\ z \end{Bmatrix} = \\ &= \begin{bmatrix} -\mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c \\ 1 \end{bmatrix} F_c(t) + \begin{bmatrix} \mathbf{U}_{\mathrm{L}}^T\mathbf{L}_d \\ 0 \end{bmatrix} d(t) \end{aligned} \tag{14.112}$$

A state-space representation can be obtained by introducing the state vector

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{q}_{\mathrm{L}}(t) \\ z(t) \\ \dot{\mathbf{q}}_{\mathrm{L}}(t) \\ \dot{z}(t) \end{Bmatrix} \tag{14.113}$$

The corresponding state equation is written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_u F_c(t) + \mathbf{b}_d d(t) \tag{14.114}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{n_m\times n_m} & \mathbf{0}_{n_m\times 1} & \mathbf{I} & \mathbf{0}_{n_m\times 1} \\ \mathbf{0}_{1\times n_m} & 0 & \mathbf{0}_{1\times n_m} & 1 \\ -\text{Diag}\{\omega_i^2\} - \text{Diag}\{1/m_i\}\mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c k_a\mathbf{L}_c^T\mathbf{U}_{\mathrm{L}} & \text{Diag}\{1/m_i\}\mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c k_a & \mathbf{0}_{n_m\times n_m} & \mathbf{0}_{n_m\times 1} \\ (k_a/m_a)\mathbf{L}_c^T\mathbf{U}_{\mathrm{L}} & -k_a/m_a & \mathbf{0}_{1\times n_m} & 0 \end{bmatrix} \tag{14.115}$$

$$\mathbf{b}_u = \begin{bmatrix} \mathbf{0}_{n_m\times 1} \\ 0 \\ -\text{Diag}\{1/m_i\}\mathbf{U}_{\mathrm{L}}^T\mathbf{L}_c \\ (1/m_a) \end{bmatrix} \tag{14.116}$$

and

$$\mathbf{b}_d = \begin{bmatrix} \mathbf{0}_{n_m\times 1} \\ 0 \\ \text{Diag}\{1/m_i\}\mathbf{U}_{\mathrm{L}}^T\mathbf{L}_d \\ 0 \end{bmatrix} \tag{14.117}$$

The output equation is given by

$$\begin{aligned} y(t) = \ddot{w}_{\mathrm{tip}}(t) = \mathbf{L}_c^T\ddot{\mathbf{u}}(t) &= \begin{bmatrix} \mathbf{0}_{1\times n_m} & 0 & \mathbf{L}_c^T\mathbf{U}_{\mathrm{L}} & 0 \end{bmatrix}\dot{\mathbf{x}}(t) \\ &= \mathbf{c}_y\mathbf{x}(t) + d_{yu}F_c(t) + d_{yd}d(t) \end{aligned} \tag{14.118}$$

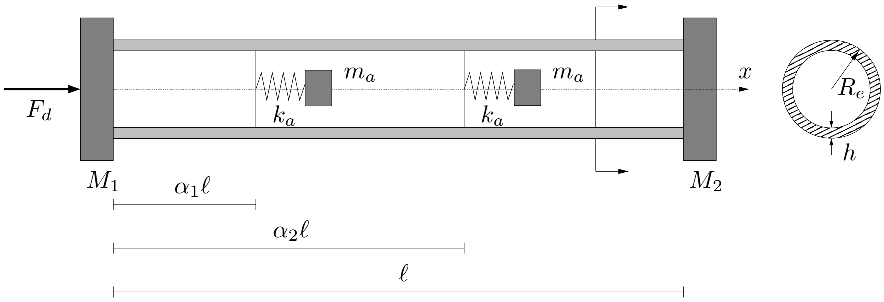


**Figure 14.8** Rod with end masses and equipped with two inertial actuators.

where

$$\mathbf{c}_y = \begin{bmatrix}\mathbf{0}_{1\times n_m} & 0 & \mathbf{L}_c^T\mathbf{U}_\mathrm{L} & 0\end{bmatrix}\mathbf{A} \tag{14.119}$$

$$d_{yu} = \begin{bmatrix}\mathbf{0}_{1\times n_m} & 0 & \mathbf{L}_c^T\mathbf{U}_\mathrm{L} & 0\end{bmatrix}\mathbf{b}_u \tag{14.120}$$

and

$$d_{yd} = \begin{bmatrix}\mathbf{0}_{1\times n_m} & 0 & \mathbf{L}_c^T\mathbf{U}_\mathrm{L} & 0\end{bmatrix}\mathbf{b}_d \tag{14.121}$$

### 14.2.4 Rod with band-limited inertial actuators

Consider the homogeneous hollow cylindrical rod in Figure 14.8 of length $\ell$, external radius $R_e$ and thickness $h$. The rod is made of aluminium and is connected to two rigid masses $M_1$ and $M_2$. The system is subjected to a disturbance force $F_d(t)$ on the first mass. The rod accommodates internally two identical inertial actuators at locations $x = \alpha_1\ell$ $(0 < \alpha_1 < 1)$ and $x = \alpha_2\ell$ $(0 < \alpha_2 < 1)$, respectively, with $\alpha_1 \neq \alpha_2$. A point sensor is also placed on the mass $M_2$ measuring the longitudinal displacement.

Each actuator is modelled as a SDOF system of mass $m_a$ and stiffness $k_a$ and is supposed to have an internal second-order dynamics expressed in the Laplace domain as

$$\delta_i(s) = \frac{\omega_a^2}{s^2 + 2\xi_a\omega_a s + \omega_a^2}\delta_{ci}(s) + \frac{F_i(s)}{k_a} \qquad (i = 1, 2) \tag{14.122}$$

where $\delta_i$ is the elongation of actuator $i$, $\delta_{ci}$ is the elongation due to the control action, and $F_i$ is the force transmitted by the actuator to the rod. $\omega_a$ and $\xi_a$ are characteristic parameters of the actuators, which are properly tuned to represent the actual dynamics of the control devices according to the assumed second-order model. Starting from a finite element model of the rod with the end rigid masses, we would like to write the state-space model of the system by coupling a reduced-order modal model of the rod dynamics with the equations of the inertial actuators.

Similarly to what done in the previous example, the problem requires to couple a modal representation of the rod dynamics with the dynamics of the two inertial actuators. The first step is to write the equations of motion without the actuators. The effect of the inertial actuators is considered by introducing the forces $F_1(t)$ and $F_2(t)$ which are transmitted by the first and the second actuator, respectively, to the rod. The rod with the end masses is then forced by the disturbance load $F_d(t)$ on the mass $M_1$ and by the forces $F_1(t)$ and $F_2(t)$ in correspondence of the actuator locations as shown in Figure 14.9.

After assuming a rod axis $x$ with origin coincident with mass $M_1$, the dynamic equilibrium of the system in Figure 14.9 can be written through the principle of virtual work as follows

$$\begin{aligned}&\int_0^\ell \delta u_{/x}^T EA u_{/x} dx + \int_0^\ell \delta u^T m\ddot{u} dx + \delta u^T(0,t) M_1 \ddot{u}(0,t) + \delta u^T(\ell,t) M_2 \ddot{u}(\ell,t) \\ &\quad = \delta u^T(0,t) F_d(t) + \delta u^T(\alpha_1\ell,t) F_1(t) + \delta u^T(\alpha_2\ell,t) F_2(t)\end{aligned} \tag{14.123}$$

where $u = u(x,t)$ is the longitudinal displacement of a generic point of the rod, $EA = E\pi h\,(2R_e - h)$ is the axial stiffness and $m = \rho\pi h\,(2R_e - h)$ is the mass per unit length. According to a finite element formulation, the rod length

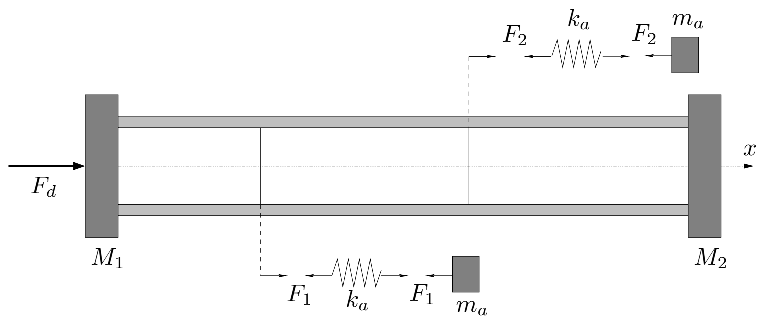


**Figure 14.9** Forces transmitted by each inertial actuator on the rod.

is divided into elements so that each term in the above equation is expressed in the form of a sum over the individual elements to yield

$$\sum_{j=1}^{n}\left[\int_{(j-1)h}^{jh}\delta u_{/x}^{T}EA(x)u_{/x}dx+\int_{(j-1)h}^{jh}\delta u^{T}m(x)\ddot{u}dx+\delta_{j1}\delta u^{T}(0,t)M_{1}\ddot{u}(0,t)+\delta_{jn}\delta u^{T}(\ell,t)M_{2}\ddot{u}(\ell,t)\right]$$
$$=\sum_{j=1}^{n}\left[\delta_{j1}\delta u^{T}(0,t)F_{d}(t)+\delta_{jn_{a1}}\delta u^{T}(\alpha_{1}\ell,t)F_{1}(t)+\delta_{jn_{a2}}\delta u^{T}(\alpha_{2}\ell,t)F_{2}(t)\right] \tag{14.124}$$

where $n$ is the total number of elements, $h$ is the length of each element, $\delta_{xy}$ is the Kronecker symbol, and $n_{a1}$ and $n_{a2}$ is the element number corresponding to the location of actuator 1 and 2, respectively. Note that the rod has been divided by elements of the same length $h$. Inside each element $j$, the longitudinal displacement can be approximated by the following expression

$$u(\xi,t)=N_{1}(\xi)u_{j-1}(t)+N_{2}(\xi)u_{j}(t)=\mathbf{N}(\xi)\mathbf{u}_{j}(t)\qquad 0<\xi<1 \tag{14.125}$$

where

$$\mathbf{u}_{j}(t)=\begin{Bmatrix}u_{j-1}(t)\\ u_{j}(t)\end{Bmatrix} \tag{14.126}$$

is the vector of nodal degrees of freedom of the $j$th element, and the interpolating hat functions are

$$\begin{aligned}N_{1}(\xi)&=1-\xi\\ N_{2}(\xi)&=\xi\end{aligned} \tag{14.127}$$

Accordingly, the stiffness term of any element $j$ is given by

$$\int_{(j-1)h}^{jh}\delta u_{/x}^{T}EAu_{/x}dx=\delta\mathbf{u}_{j}^{T}(t)\mathbf{K}_{j}\mathbf{u}_{j}(t) \tag{14.128}$$

where

$$\mathbf{K}_{j}=\frac{EA}{h}\begin{bmatrix}1&-1\\-1&1\end{bmatrix} \tag{14.129}$$

The mass term of the $j$th element is given by

$$\begin{aligned}&\int_{(j-1)h}^{jh}\delta u^{T}m(x)\ddot{u}dx+\delta_{j1}\delta u^{T}(0,t)M_{1}\ddot{u}(0,t)+\delta_{jn}\delta u^{T}(\ell,t)M_{2}\ddot{u}(\ell,t)\\ &\qquad=\delta\mathbf{u}_{j}^{T}(t)\left[\mathbf{M}_{j}+\delta_{j1}\mathbf{M}_{1}+\delta_{jn}\mathbf{M}_{2}\right]\ddot{\mathbf{u}}_{j}(t)\end{aligned} \tag{14.130}$$

where

$$\mathbf{M}_j = \frac{mh}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{14.131}$$

and

$$\mathbf{M}_1 = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} 0 & 0 \\ 0 & M_2 \end{bmatrix} \tag{14.132}$$

If we assume that the actuator forces $F_1$ and $F_2$ are applied on the left node of the element $n_{a1}$ and $n_{a2}$, respectively, the load term of any element $j$ is

$$\begin{aligned} &\delta_{j1}\delta u^T(0,t)F_d(t) + \delta_{jn_{a1}}\delta u^T(\alpha_1\ell,t)F_1(t) + \delta_{jn_{a2}}\delta u^T(\alpha_2\ell,t)F_2(t) \\ &\quad = \delta\mathbf{u}_j^T(t)\delta_{j1}\mathbf{N}^T(0)F_d(t) + \delta\mathbf{u}_j^T(t)\delta_{jn_{a1}}\mathbf{N}^T(0)F_1(t) + \delta\mathbf{u}_j^T(t)\delta_{jn_{a2}}\mathbf{N}^T(0)F_2(t) \\ &\quad = \delta\mathbf{u}_j^T(t)\mathbf{L}_j^dF_d(t) + \delta\mathbf{u}_j^T(t)\mathbf{L}_j^1F_1(t) + \delta\mathbf{u}_j^T(t)\mathbf{L}_j^2F_2(t) \end{aligned} \tag{14.133}$$

where

$$\mathbf{L}_j^d = \delta_{j1}\mathbf{N}^T(0) = \delta_{j1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{14.134}$$

$$\mathbf{L}_j^1 = \delta_{jn_{a1}}\mathbf{N}^T(0) = \delta_{jn_{a1}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{14.135}$$

$$\mathbf{L}_j^2 = \delta_{jn_{a2}}\mathbf{N}^T(0) = \delta_{jn_{a2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{14.136}$$

Using what derived thus far, the dynamic equilibrium of the rod can be expressed as follows

$$\begin{aligned} &\sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\left[\mathbf{M}_j + \delta_{j1}\mathbf{M}_1 + \delta_{jn}\mathbf{M}_2\right]\ddot{\mathbf{u}}_j(t) + \sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{K}_j\mathbf{u}_j(t) \\ &\quad = \sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{L}_j^dF_d(t) + \sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{L}_j^1F_1(t) + \sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{L}_j^2F_2(t) \end{aligned} \tag{14.137}$$

Since the nodal variable $u_j$ appears twice in the above summation, the assembly procedure yields

$$\sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\left[\mathbf{M}_j + \delta_{j1}\mathbf{M}_1 + \delta_{jn}\mathbf{M}_2\right]\ddot{\mathbf{u}}_j(t) = \delta\mathbf{u}^T(t)\mathbf{M}_{rod}\ddot{\mathbf{u}}(t) \tag{14.138}$$

and

$$\sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{K}_j\mathbf{u}_j(t) = \delta\mathbf{u}^T(t)\mathbf{K}_{rod}\mathbf{u}(t) \tag{14.139}$$

where $\mathbf{u}(t)$ is the vector of nodal displacements

$$\mathbf{u}(t) = \begin{Bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_n(t) \end{Bmatrix} \tag{14.140}$$

The previous assembly procedure applied to loading terms yields

$$\sum_{j=1}^{n}\delta\mathbf{u}_j^T(t)\mathbf{L}_j^dF_d(t) = \delta\mathbf{u}^T(t)\mathbf{L}_dF_d(t) \tag{14.141}$$

$$\sum_{j=1}^{n} \delta \mathbf{u}_j^T(t)\mathbf{L}_j^1 F_1(t) = \delta \mathbf{u}^T(t)\mathbf{L}_{c1}F_1(t) \tag{14.142}$$

$$\sum_{j=1}^{n} \delta \mathbf{u}_j^T(t)\mathbf{L}_j^2 F_2(t) = \delta \mathbf{u}^T(t)\mathbf{L}_{c2}F_2(t) \tag{14.143}$$

where the resulting $\mathbf{L}$ vectors are obtained by adding the bottom component of the $j$th element vector to the top component of the $(j+1)$th element vector. Exploiting the arbitrariness of virtual variation $\delta \mathbf{u}^T$, the finite element model of the present rod is written as

$$\mathbf{M}_{rod}\ddot{\mathbf{u}}(t) + \mathbf{K}_{rod}\mathbf{u}(t) = \mathbf{L}_d F_d(t) + \mathbf{L}_c\mathbf{F}(t) \tag{14.144}$$

where

$$\mathbf{L}_c = \begin{bmatrix} \mathbf{L}_{c1} & \mathbf{L}_{c2} \end{bmatrix} \tag{14.145}$$

and

$$\mathbf{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \tag{14.146}$$

The corresponding reduced-order modal model is given by

$$\text{Diag}\left\{m_i\right\}\ddot{\mathbf{q}}_{\text{L}}(t) + \text{Diag}\left\{m_i\omega_i^2\right\}\mathbf{q}_{\text{L}}(t) = \mathbf{U}_{\text{L}}^T\mathbf{L}_d F_d(t) + \mathbf{U}_{\text{L}}^T\mathbf{L}_c\mathbf{F}(t) \tag{14.147}$$

where $\omega_i^2$ are the natural frequencies associated with the eigenvalue problem $\left(\mathbf{K}_{rod} - \omega^2\mathbf{M}_{rod}\right)\hat{\mathbf{u}} = \mathbf{0}$ and $\mathbf{U}_{\text{L}}$ is the truncated eigenvector matrix including the first $n_m \ll n$ modes.

Let consider now the two inertial actuators. The dynamic equilibrium of each actuator mass is given by

$$m_a\ddot{z}_i(t) + F_i(t) = 0 \qquad (i = 1, 2) \tag{14.148}$$

where $z_i$ is the absolute displacement related to the $i$th actuator. In matrix form, we can write

$$\text{Diag}\left\{m_a\right\}\ddot{\mathbf{z}}(t) + \mathbf{F}(t) = \mathbf{0} \tag{14.149}$$

where

$$\text{Diag}\left\{m_a\right\} = \begin{bmatrix} m_a & 0 \\ 0 & m_a \end{bmatrix} \tag{14.150}$$

and

$$\mathbf{z}(t) = \begin{Bmatrix} z_1(t) \\ z_2(t) \end{Bmatrix} \qquad \mathbf{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \tag{14.151}$$

The second-order internal dynamics of each actuator can be included as follows. Equation (14.122) is rewritten as

$$\delta_i(s) = v_i(s) + \frac{F_i(s)}{k_a} \qquad (i = 1, 2) \tag{14.152}$$

where

$$\delta_i = z_i - u(\alpha_i\ell) \qquad (i = 1, 2) \tag{14.153}$$

is the total elongation of the $i$th actuator and the new variables $v_i$ are defined as

$$v_i(s) = \frac{\omega_a^2}{s^2 + 2\xi_a\omega_a + \omega_a^2}\delta_{ci}(s) \qquad (i = 1, 2) \tag{14.154}$$

The previous relation can be also rearranged as

$$\left(s^2 + 2\xi_a\omega_a + \omega_a^2\right)v_i(s) = \omega_a^2\delta_{ci}(s) \qquad (i = 1, 2) \tag{14.155}$$

Transforming it back to the time domain, the previous equation becomes

$$\ddot{v}_i(t) + 2\xi_a\omega_a\dot{v}_i(t) + \omega_a^2 v_i(t) = \omega_a^2\delta_{ci}(t) \qquad (i = 1, 2) \tag{14.156}$$

Since the vector of actuator elongations can be expressed as

$$\boldsymbol{\delta}(t) = \mathbf{z}(t) - \mathbf{L}_c^T \mathbf{u}(t) \tag{14.157}$$

the equations (14.152) and (14.156) describing the internal dynamics of the inertial actuators can be put into matrix form to yield

$$\mathbf{F}(t) = \text{Diag}\left\{k_a\right\} \mathbf{z}(t) - \text{Diag}\left\{k_a\right\} \mathbf{L}_c^T \mathbf{U}_{\mathrm{L}} \mathbf{q}_{\mathrm{L}}(t) - \text{Diag}\left\{k_a\right\} \mathbf{v}(t) \tag{14.158}$$

and

$$\ddot{\mathbf{v}}(t) + \text{Diag}\left\{2\xi_a\omega_a\right\} \dot{\mathbf{v}}(t) + \text{Diag}\left\{\omega_a^2\right\} \mathbf{v}(t) = \text{Diag}\left\{\omega_a^2\right\} \boldsymbol{\delta}_c(t) \tag{14.159}$$

where

$$\text{Diag}\left\{k_a\right\} = \begin{bmatrix} k_a & 0 \\ 0 & k_a \end{bmatrix} \qquad \text{Diag}\left\{2\xi_a\omega_a\right\} = \begin{bmatrix} 2\xi\omega_a & 0 \\ 0 & 2\xi_a\omega_a \end{bmatrix} \qquad \text{Diag}\left\{\omega_a^2\right\} = \begin{bmatrix} \omega_a^2 & 0 \\ 0 & \omega_a^2 \end{bmatrix} \tag{14.160}$$

and

$$\boldsymbol{\delta}_c = \begin{Bmatrix} \delta_{c1} \\ \delta_{c2} \end{Bmatrix} \tag{14.161}$$

is the vector containing the prescribed elongations arising from the control source. Inserting Eq. (14.158) into Eq. (14.147) and Eq. (14.149), the equations governing the motion of the system are given by

$$\begin{aligned}
&\text{Diag}\left\{m_i\right\} \ddot{\mathbf{q}}_{\mathrm{L}}(t) + \left[\text{Diag}\left\{m_i\omega_i^2\right\} + \mathbf{U}_{\mathrm{L}}^T \mathbf{L}_c \text{Diag}\left\{k_a\right\} \mathbf{L}_c^T \mathbf{U}_{\mathrm{L}}\right] \mathbf{q}_{\mathrm{L}}(t) - \mathbf{U}_{\mathrm{L}}^T \mathbf{L}_c \text{Diag}\left\{k_a\right\} \mathbf{z}(t) \\
&\quad + \mathbf{U}_{\mathrm{L}}^T \mathbf{L}_c \text{Diag}\left\{k_a\right\} \mathbf{v}(t) = \mathbf{U}_{\mathrm{L}}^T \mathbf{L}_d F_d(t) \\
&-\text{Diag}\left\{k_a\right\} \mathbf{L}_c^T \mathbf{U}_{\mathrm{L}} \mathbf{q}_{\mathrm{L}}(t) + \text{Diag}\left\{m_a\right\} \ddot{\mathbf{z}}(t) + \text{Diag}\left\{k_a\right\} \mathbf{z}(t) - \text{Diag}\left\{k_a\right\} \mathbf{v}(t) = \mathbf{0} \\
&\ddot{\mathbf{v}}(t) + \text{Diag}\left\{2\xi_a\omega_a\right\} \dot{\mathbf{v}}(t) + \text{Diag}\left\{\omega_a^2\right\} \mathbf{v}(t) = \text{Diag}\left\{\omega_a^2\right\} \boldsymbol{\delta}_c(t)
\end{aligned} \tag{14.162}$$

or, alternatively,

$$\begin{aligned}
&\begin{bmatrix} \text{Diag}\left\{m_i\right\} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Diag}\left\{m_a\right\} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_{\mathrm{L}} \\ \ddot{\mathbf{z}} \\ \ddot{\mathbf{v}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \text{Diag}\left\{2\xi\omega_a\right\} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_{\mathrm{L}} \\ \dot{\mathbf{z}} \\ \dot{\mathbf{v}} \end{Bmatrix} \\
&+ \begin{bmatrix} \text{Diag}\left\{m_i\omega_i^2\right\} + \mathbf{U}_{\mathrm{L}}^T \mathbf{L}_c \text{Diag}\left\{k_a\right\} \mathbf{L}_c^T \mathbf{U}_{\mathrm{L}} & -\mathbf{U}_{\mathrm{L}}^T \mathbf{L}_c \text{Diag}\left\{k_a\right\} & \mathbf{U}_{\mathrm{L}}^T \mathbf{L}_c \text{Diag}\left\{k_a\right\} \\ -\text{Diag}\left\{k_a\right\} \mathbf{L}_c^T \mathbf{U}_{\mathrm{L}} & \text{Diag}\left\{k_a\right\} & -\text{Diag}\left\{k_a\right\} \\ \mathbf{0} & \mathbf{0} & \text{Diag}\left\{\omega_a^2\right\} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_{\mathrm{L}} \\ \mathbf{z} \\ \mathbf{v} \end{Bmatrix} \\
&= \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \text{Diag}\left\{\omega_a^2\right\} \end{bmatrix} \boldsymbol{\delta}_c(t) + \begin{bmatrix} \mathbf{U}_{\mathrm{L}}^T \mathbf{L}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} F_d(t)
\end{aligned} \tag{14.163}$$

A state-space representation can be obtained by introducing the state vector

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{q}_{\mathrm{L}}(t) \\ \mathbf{z}(t) \\ \mathbf{v}(t) \\ \dot{\mathbf{q}}_{\mathrm{L}}(t) \\ \dot{\mathbf{z}}(t) \\ \dot{\mathbf{v}}(t) \end{Bmatrix} \tag{14.164}$$

The corresponding state equation is written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u \boldsymbol{\delta}_c(t) + \mathbf{b}_d F_d(t) \tag{14.165}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \mathbf{A}_{12} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \qquad \mathbf{A}_{22} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\text{Diag}\{2\xi_a\omega_a\} \end{bmatrix}$$

$$\mathbf{A}_{21} = \begin{bmatrix} -\text{Diag}\{\omega_i^2\} - \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}_{\text{L}}^T\mathbf{L}_c\text{Diag}\{k_a\}\mathbf{L}_c^T\mathbf{U}_{\text{L}} & \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}_{\text{L}}^T\mathbf{L}_c\text{Diag}\{k_a\} & -\text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}_{\text{L}}^T\mathbf{L}_c\text{Diag}\{k_a\} \\ \text{Diag}\left\{\frac{k_a}{m_a}\right\}\mathbf{L}_c^T\mathbf{U}_{\text{L}} & -\text{Diag}\left\{\frac{k_a}{m_a}\right\} & \text{Diag}\left\{\frac{k_a}{m_a}\right\} \\ \mathbf{0} & \mathbf{0} & -\text{Diag}\{\omega_a^2\} \end{bmatrix}$$

$$\mathbf{B}_u = \begin{bmatrix} \text{Diag}\left\{\frac{1}{m_i}\right\} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Diag}\left\{\frac{1}{m_a}\right\} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \text{Diag}\{\omega_a^2\} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \text{Diag}\{\omega_a^2\} \end{bmatrix}$$

$$\mathbf{b}_d = \begin{bmatrix} \text{Diag}\left\{\frac{1}{m_i}\right\} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Diag}\left\{\frac{1}{m_a}\right\} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\text{L}}^T\mathbf{L}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \text{Diag}\left\{\frac{1}{m_i}\right\}\mathbf{U}_{\text{L}}^T\mathbf{L}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

The output equation is given by

$$y(t) = u(\ell, t) = \mathbf{L}_s^T\mathbf{u}(t) = \mathbf{L}_s^T\mathbf{U}_{\text{L}}\mathbf{q}_{\text{L}}(t) = \mathbf{c}_y\mathbf{x}(t) \tag{14.166}$$

where

$$\mathbf{c}_y = \begin{bmatrix} \mathbf{L}_s^T\mathbf{U}_{\text{L}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and $\mathbf{L}_s$ is obtained by assembling the element vectors

$$\mathbf{L}_j^s = \delta_{jn}\mathbf{N}^T(1) = \delta_{jn}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

### 14.2.5 Cantilever plate with derivator circuits and subjected to band-limited disturbance load

Let consider the undamped cantilever thin isotropic plate in Figure xxx of length $\ell_x$, width $\ell_y$ and thickness $h$, equipped with four ideal control actuators capable of applying concentrated transverse forces at locations $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, and $(x_4, y_4)$. Four collocated sensors of mass $m_s$ measuring the out-of-plane displacement are also attached onto the plate. Each output signal $y_{disp}$ is filtered by a real derivator circuit in order to obtain the corresponding velocity signal. The derivator has the following transfer function

$$y_{vel}(s) = \frac{s}{s+a}y_{disp}(s) \tag{14.167}$$

where $a > 0$ identifies the derivator bandwidth. The plate is subjected to a band-limited transverse distributed load having uniform spatial distribution $p_0$ and frequency content expressed by a first-order transfer function of the form $b/(s+b)$, where $b > 0$ is a characteristic parameter. It is required to write a state space model of the system using a suitable Ritz-Galerkin discretization of the plate dynamics.

Since the plate is isotropic and subjected only to transverse loads, the dynamics of interest involves only the flexural motion of the plate. The model of the plate without the dynamics of the disturbance load and the dynamics of the derivator circuits (they will be introduced later) can be derived by the principle of virtual work, which can be written in this case as

$$\begin{aligned}\int_0^{\ell_x}\int_{-\ell_y/2}^{\ell_y/2}\delta\boldsymbol{\epsilon}_1^T\mathbf{D}\boldsymbol{\epsilon}_1 dxdy = &-\int_0^{\ell_x}\int_{-\ell_y/2}^{\ell_y/2}\delta w^T\rho h\ddot{w}dxdy - \sum_{i=1}^{4}\delta w^T(x_i,y_i)m_s\ddot{w}(x_i,y_i)\\ &+\sum_{i=1}^{4}\delta w^T(x_i,y_i)F_{ci}(t)+\int_0^{\ell_x}\int_{-\ell_y/2}^{\ell_y/2}\delta w^T p_0 dxdyp(t)\end{aligned} \tag{14.168}$$

where

$$\boldsymbol{\epsilon}_1 = \begin{Bmatrix} w_{/xx} \\ w_{/yy} \\ 2w_{/xy}\end{Bmatrix} \qquad \mathbf{D} = D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}$$

and $w = w(x,y,t)$ is the transverse displacement of the middle surface of the plate, $D = Eh^3/12(1-\nu^2)$ is the plate bending stiffness, $\rho$ is the mass density per unit area, $F_{ci}(t)$ is the control force of the $i$th actuator and $p(t)$ expresses the temporal variation of the disturbance load acting on the plate after being filtered by the disturbance transfer function.

As required, the plate dynamics is approximated using a global set of Ritz admissible functions such that

$$w(x,y,t) = \mathbf{N}(x,y)\mathbf{u}(t) \tag{14.169}$$

The same discretization is also applied to virtual variations $\delta w$. Since the plate has rectangular shape, the generic element $(r,s)$ of the row matrix $\mathbf{N} = [\ldots N_{rs}\ldots]$ can be selected as

$$N_{rs}(x,y) = X_r(x)Y_s(y) \tag{14.170}$$

One possible choice for $X_r$ and $Y_s$ is the following

$$\begin{aligned} X_r(x) &= \left(\frac{x}{\ell_x}\right)^{r+1} \qquad & r = 1,2,\ldots,R\\ Y_s(y) &= \left(\frac{y}{\ell_y}\right)^{s-1} \qquad & s = 1,2,\ldots,S\end{aligned} \tag{14.171}$$

The order $R\times S$ of the previous Ritz-Galerkin approximation should be chosen according to the frequency range of interest for the problem under investigation. In this case, the external disturbance is band-limited and is characterized by a first-order low-pass dynamics governed by the value of $b$. As a result, the plate will be mostly excited within the range $[0,b]$ since higher frequencies are less and less excited due to the low-pass filtering action of the disturbance. Therefore, the discretized model of the plate must provide accurate modal representation at least up to $b$. This is practically checked by a convergence analysis on the natural modes falling in that frequency range as the number of Ritz functions increases. Since the filtering capabilities of the disturbance in this problem are only of first-order (which correspond to an attenuation of 20 dB/decade) the bandwidth of interest should not be strictly limited to $[0,b]$, but properly enlarged to include a suitable number of modes above $b$.

After inserting the approximation directly into the variational statement, and exploiting the arbitrariness of the virtual variations $\delta\mathbf{u}(t)$, we obtain the following set of equations of motion

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{L}_c\mathbf{F}_c(t) + \mathbf{L}_p p(t) \tag{14.172}$$

where

$$\mathbf{M} = \rho h\int_0^{\ell_x}\int_{-\ell_y/2}^{\ell_y/2}\mathbf{N}^T\mathbf{N}dxdy + \sum_{i=1}^{4}m_s\mathbf{N}^T(x_i,y_i)\mathbf{N}(x_i,y_i)$$

$$\mathbf{K} = D\int_0^{\ell_x}\int_{-\ell_y/2}^{\ell_y/2}\left[\mathbf{N}_{/xx}^T\left(\mathbf{N}_{/xx}+\nu\mathbf{N}_{/yy}\right)+\mathbf{N}_{/yy}^T\left(\mathbf{N}_{/yy}+\nu\mathbf{N}_{/xx}\right)+2(1-\nu)\mathbf{N}_{/xy}^T\mathbf{N}_{/xy}\right]dxdy$$

$$\mathbf{L}_c = \begin{bmatrix}\mathbf{N}^T(x_1,y_1) & \mathbf{N}^T(x_2,y_2) & \mathbf{N}^T(x_3,y_3) & \mathbf{N}^T(x_4,y_4)\end{bmatrix}$$

and

$$\mathbf{L}_p = p_0 \int_0^{\ell_x} \int_{-\ell_y/2}^{\ell_y/2} \mathbf{N}^T dxdy \qquad \mathbf{F}_c = \begin{Bmatrix} F_{c1} \\ F_{c2} \\ F_{c3} \\ F_{c4} \end{Bmatrix}$$

It is noted that the mass and stiffness matrices have dimensions $RS \times RS$ according to the Ritz expansion of the transverse displacement. The matrix $\mathbf{L}_c$ relating the four control forces with the vector of Ritz generalized coordinates has dimensions $RS \times 4$. $\mathbf{L}_p$ is a column vector with $RS$ components.

The second-order representation in Eq. (14.172) can be put into a state space form by defining a state vector collecting the Ritz generalized coordinates and the corresponding derivatives as follows

$$\mathbf{x}_{plate}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} \tag{14.173}$$

Therefore, the plate dynamics can be described by the state equation

$$\dot{\mathbf{x}}_{plate}(t) = \mathbf{A}_{plate}\mathbf{x}_{plate}(t) + \mathbf{B}_u\mathbf{F}_c(t) + \mathbf{b}_p p(t) \tag{14.174}$$

where

$$\mathbf{A}_{plate} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \qquad \mathbf{B}_u = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{L}_c \end{bmatrix} \qquad \mathbf{b}_p = \begin{bmatrix} 0 \\ \mathbf{M}^{-1}\mathbf{L}_p \end{bmatrix} \tag{14.175}$$

The state matrix $\mathbf{A}_{plate}$ has dimensions $2RS \times 2RS$. The control input matrix $\mathbf{B}_u$ and disturbance matrix $\mathbf{b}_p$ have dimensions $2RS \times 4$ and $2RS \times 1$, respectively.

The output variables of the plate model are the transverse displacements at the sensor locations

$$\mathbf{y}_{plate}(t) = \begin{Bmatrix} w(x_1, y_1, t) \\ w(x_2, y_2, t) \\ w(x_3, y_3, t) \\ w(x_4, y_4, t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{N}(x_1, y_1)\mathbf{u}(t) \\ \mathbf{N}(x_2, y_2)\mathbf{u}(t) \\ \mathbf{N}(x_3, y_3)\mathbf{u}(t) \\ \mathbf{N}(x_4, y_4)\mathbf{u}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{N}(x_1, y_1) \\ \mathbf{N}(x_2, y_2) \\ \mathbf{N}(x_3, y_3) \\ \mathbf{N}(x_4, y_4) \end{bmatrix} \mathbf{u}(t) = \mathbf{L}_s\mathbf{u}(t) \tag{14.176}$$

where, since the sensors are collocated with the actuators,

$$\mathbf{L}_s = \mathbf{L}_c^T \tag{14.177}$$

The output equation is obtained by writing Eq. (14.176) in terms of the state vector $\mathbf{x}_{plate}$ as follows

$$\mathbf{y}_{plate}(t) = \mathbf{C}_{plate}\mathbf{x}_{plate}(t) \tag{14.178}$$

where

$$\mathbf{C}_{plate} = \begin{bmatrix} \mathbf{L}_s & \mathbf{0} \end{bmatrix} \tag{14.179}$$

The model in Eq. (14.172) or, alternatively, in Eq. (14.174) and (14.178), is referred to a plate with displacement outputs and subjected to a distributed load having spatial distribution $p_0$ and generic temporal variation $p(t)$. Therefore, we need to include additional dynamics to take into account the derivator circuits and the band-limited disturbance.

Since in this case the disturbance load has a prescribed band-limited frequency content, the quantity $p(t)$ must express the variation corresponding to that frequency behaviour. This requirement can be satisfied by representing $p$ as the output of the filter which expresses the disturbance dynamics. The filter input is denoted by $d$ and corresponds to a signal with infinite bandwidth of unit amplitude, i.e., a Dirac delta function. Accordingly, the load $p(s)$ acting on the plate is given by

$$p(s) = \frac{b}{s+b}d(s) \tag{14.180}$$

where $b/(s+b)$ can be considered as a *shaping filter* of the disturbance $d$. The previous relation can be written in the time domain as

$$\dot{p}(t) = -bp(t) + bd(t) \tag{14.181}$$

which is the equation governing the load dynamics. It is already in state space form, where the variable $p$ is both the state variable and the output variable, and the quantity $d$ is the disturbance input variable. According to the usual notation for the state variable, the state space formulation of the load dynamics is rewritten as follows

$$\begin{aligned} \dot{\mathrm{x}}_{load}(t) &= -b\,\mathrm{x}_{load}(t) + bd(t) \\ p(t) &= \mathrm{x}_{load}(t) \end{aligned} \tag{14.182}$$

As previously stated, each derivator circuit is characterized by the following input-output relation in the Laplace domain

$$\frac{v_i}{w_i} = \frac{s}{s+a} \qquad (i = 1, \ldots, 4) \tag{14.183}$$

where $w_i(s) = w(x_i, y_i, s)$ is the plate displacement of the $i$th sensor and $v_i(s)$ is the corresponding velocity signal. The above relation can be equivalently written as

$$\frac{v_i}{z_i}\frac{z_i}{w_i} = \frac{s}{s+a} \qquad (i = 1, \ldots, 4) \tag{14.184}$$

where $z_i = z_i(s)$ is the state variable of the derivator. Therefore, we can set

$$\begin{cases} \dfrac{v_i}{z_i} = s \\[2ex] \dfrac{z_i}{w_i} = \dfrac{1}{s+a} \end{cases} \tag{14.185}$$

which correspond to the time-domain relations

$$\begin{aligned} v_i(t) &= \dot{z}_i(t) \\ \dot{z}_i(t) &= -az_i(t) + w_i(t) \end{aligned} \tag{14.186}$$

Using the second equation into the first equation, we can write for $i = 1, \ldots, 4$

$$\begin{aligned} \dot{z}_i(t) &= -az_i(t) + w_i(t) \\ v_i(t) &= -az_i(t) + w_i(t) \end{aligned} \tag{14.187}$$

The above representation is the state space form of the dynamics governing the $i$th derivator, where $v_i(t)$ is the output variable and $w_i(t) = w(x_i, y_i, t)$ is the input variable. Therefore, the first equation represents the state equation and the second equation is the output equation of the derivator system. After defining the state vector

$$\mathbf{x}_{deriv}(t) = \begin{Bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{Bmatrix} \tag{14.188}$$

and recalling the definition of the plate output vector, the state equation of *all* the derivator circuits is given by

$$\dot{\mathbf{x}}_{deriv}(t) = -\mathrm{Diag}\,\{a\}\,\mathbf{x}_{deriv}(t) + \mathbf{y}_{plate}(t) \tag{14.189}$$

Accordingly, the output equation of the dynamic system representing the derivator circuits is

$$\mathbf{y}_{deriv}(t) = -\mathrm{Diag}\,\{a\}\,\mathbf{x}_{deriv}(t) + \mathbf{y}_{plate}(t) \tag{14.190}$$

where the vector $\mathbf{y}_{deriv}$ collects the four velocity signals.

Using what previously derived, the dynamics of the overall system is represented by the series of three dynamic blocks (see Figure xxx). The first block is the system expressing the dynamics of the disturbance load acting on the plate (the load shaping filter). It has one input variable $d(t)$, one output variable $p(t)$ and the scalar internal variable corresponding to the state $\mathrm{x}_{load}(t)$. The second block represents the discretized dynamics of the cantilever plate. The state vector $\mathbf{x}_{plate}(t)$ contains the generalized Ritz coordinates and their derivatives. This block has two input variables, the control input vector $\mathbf{F}_c(t)$ and the load variable $p(t)$, which is also the output (and state) variable of the first block. This establishes the series connection between the block of load dynamics and the block of plate dynamics. The output of the second block is the vector of displacements at sensor locations. This vector is also the input vector for the third block representing the dynamics of the derivators, since each derivator is driven by the displacement signal and converts it into a velocity signal. This series relationship links the second block with the dynamic block of derivators, represented by the state vector $\mathbf{x}_{deriv}(t)$.

Therefore, we can define a state vector $\mathbf{x}(t)$ of the overall system by collecting the state vectors of the three dynamic blocks. By taking the dynamics of the structural system as a reference, the state $\mathbf{x}(t)$ can be considered as an *augmented state vector*, since the plate dynamics is augmented by the dynamics of the disturbance load and the derivator circuits. The equation expressing the dynamics of the state $\mathbf{x}(t)$ is obtained after substituting the series relations in the state equations of each block as follows

$$\begin{aligned}
\dot{\mathrm{x}}_{load}(t) &= -b\,\mathrm{x}_{load}(t) + bd(t)\\
\dot{\mathbf{x}}_{plate}(t) &= \mathbf{A}_{plate}\mathbf{x}_{plate}(t) + \mathbf{B}_u\mathbf{F}_c(t) + \mathbf{b}_p\mathrm{x}_{load}(t)\\
\dot{\mathbf{x}}_{deriv}(t) &= -\mathrm{Diag}\,\{a\}\,\mathbf{x}_{deriv}(t) + \mathbf{C}_{plate}\mathbf{x}_{plate}(t)
\end{aligned} \tag{14.191}$$

and writing the above set in matrix form

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \dot{\mathrm{x}}_{load}(t) \\ \dot{\mathbf{x}}_{plate}(t) \\ \dot{\mathbf{x}}_{deriv}(t) \end{Bmatrix} = \begin{bmatrix} -b & \mathbf{0} & \mathbf{0} \\ \mathbf{b}_p & \mathbf{A}_{plate} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{plate} & -\mathrm{Diag}\,\{a\} \end{bmatrix} \begin{Bmatrix} \mathrm{x}_{load}(t) \\ \mathbf{x}_{plate}(t) \\ \mathbf{x}_{deriv}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_u \\ \mathbf{0} \end{bmatrix} \mathbf{F}_c(t) + \begin{bmatrix} b \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} d(t) \tag{14.192}$$

The output vector of the overall system contains the velocity signals. Therefore, the output equation is expressed as

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{C}_{plate} & -\mathrm{Diag}\,\{a\} \end{bmatrix} \mathbf{x}(t) \tag{14.193}$$

## 14.3 State space fundamentals

For the sake of convenience, the generalized form of the state and output equations reported in Eq. (14.58) will be simplified in the following by grouping the control, disturbance and noise inputs into a single input vector denoted as $\mathbf{u}(t)$. Accordingly, the state equation will have a single $\mathbf{B}$ input matrix, without any distinction between control and disturbance related terms, and the output equation will be written in terms of the output matrix $\mathbf{C}$ and a single direct feedthrough matrix $\mathbf{D}$.

### 14.3.1 Homogeneous solution

The homogeneous form of the state equation is written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{14.194}$$

where $\mathbf{A}$ is, by assumption, a constant matrix, i.e., not function of time. We want to find the solution of the above dynamics with the following initial conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{14.195}$$

The solution of Eq. (14.194) can be expressed as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c} \tag{14.196}$$

where $\mathbf{c}$ is a vector of constant coefficients to be determined from initial conditions, and $e^{\mathbf{A}t}$ is the exponential matrix defined as

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \dots \tag{14.197}$$

The result in Eq. (14.196) can be verified by writing

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \frac{d}{dt}\left(e^{\mathbf{A}t}\mathbf{c}\right) = \frac{d}{dt}\left(e^{\mathbf{A}t}\right)\mathbf{c} \\ &= \left(\mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3\frac{t^2}{2!} + \dots\right)\mathbf{c} \\ &= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \dots\right)\mathbf{c} = \mathbf{A}e^{\mathbf{A}t}\mathbf{c} = \mathbf{A}\mathbf{x}(t)\end{aligned} \tag{14.198}$$

Putting Eq. (14.196) into the initial condition yields

$$\mathbf{x}_0 = e^{\mathbf{A}t_0}\mathbf{c} \tag{14.199}$$

The vector $\mathbf{c}$ is then given by

$$\mathbf{c} = \left(e^{\mathbf{A}t_0}\right)^{-1}\mathbf{x}_0 = e^{-\mathbf{A}t_0}\mathbf{x}_0 \tag{14.200}$$

Therefore, the homogenous solution of Eq. (14.194) is expressed as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 \tag{14.201}$$

where the matrix $e^{\mathbf{A}(t-t_0)}$ is called the *state transition matrix* of a linear time-invariant system, i.e.,

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)} \tag{14.202}$$

As a result, the homogeneous solution can be rewritten as

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 \tag{14.203}$$

Note that $\mathbf{\Phi}(t, t) = \mathbf{I}$ for any $t$. It should be also noted that the above state transition matrix is a function of the difference $t - t_0$ between the initial time $t_0$ and the current time $t$. This occurs since the system is time-invariant.

The corresponding homogenous output solution will be given by

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 \tag{14.204}$$

or, alternatively,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{\Phi}(t, t_0)\mathbf{x}_0 \tag{14.205}$$

### 14.3.2 Complete solution

Let us now refer to the inhomogeneous case. We seek for the complete solution of the following state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{14.206}$$

with the initial condition in Eq. (14.195). Note that we have considered a state equation having one input term only, which has been considered to be coincident with the vector of control inputs $\mathbf{u}(t)$. This is not a restriction of the following analysis, since the same procedure can be applied to the system forced by the disturbance vector $\mathbf{d}(t)$. Moreover, since the system is linear, the most general solution can be expressed as the superposition of the solutions due to $\mathbf{u}$ and $\mathbf{d}$.

First, let us assume a particular solution of the form

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c}(t) \tag{14.207}$$

where now $\mathbf{c}(t)$ is a vector function of time. Substituting such solution into the state equation (14.206) yields

$$\mathbf{A}e^{\mathbf{A}t}\mathbf{c}(t) + e^{\mathbf{A}t}\dot{\mathbf{c}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{c}(t) + \mathbf{B}\mathbf{u}(t) \tag{14.208}$$

or, upon cancelling the $\mathbf{A}e^{\mathbf{A}t}\mathbf{c}(t)$ terms,

$$\dot{\mathbf{c}}(t) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{14.209}$$

The vector $\mathbf{c}(t)$ is then obtained by simple integration as

$$\mathbf{c}(t) = \int_{\tilde{t}}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{14.210}$$

where the lower limit $\tilde{t}$ will be specified later. Therefore, the particular solution is given by

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\int_{\tilde{t}}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \\ &= \int_{\tilde{t}}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\end{aligned} \tag{14.211}$$

The complete solution is obtained by superimposing the homogeneous solution derived in the previous paragraph and the particular solution in Eq. (14.211). The result is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{\tilde{t}}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{14.212}$$

The above expression, for $t = t_0$, becomes

$$\mathbf{x}(t_0) = \mathbf{x}_0 + \int_{\tilde{t}}^{t_0} e^{\mathbf{A}(t_0-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{14.213}$$

which implies that the integral must be zero for any $\mathbf{u}(t)$. This is possible only if $\tilde{t} = t_0$. As such, the complete solution of the state equation (14.206) can be expressed as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{14.214}$$

or, alternatively,

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \tag{14.215}$$

Note that there is no requirement for the above equation to be valid that $t \geq t_0$. Furthermore, it is observed that the integral term due to the input is a *convolution integral*, where the function $e^{\mathbf{A}t}\mathbf{B}$ has the role of the impulse response of the system whose input is $\mathbf{u}(t)$ and whose output is the state $\mathbf{x}(t)$.

Assuming that the output equation is given by

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{14.216}$$

the system output will take the following form

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \tag{14.217}$$

or, alternatively,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{C}\mathbf{\Phi}(t, \tau)\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \tag{14.218}$$

### 14.3.3 Impulse response

Without any loss of generality, we assume that $t_0 = 0$ and $\mathbf{x}_0 = \mathbf{0}$. The output response becomes

$$\mathbf{y}(t) = \int_{0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \tag{14.219}$$

If $\mathbf{u}(t) = \boldsymbol{\delta}(t)$, we obtain the impulse response of the system as

$$\mathbf{h}(t) = \int_{0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\boldsymbol{\delta}(\tau)d\tau + \mathbf{D}\boldsymbol{\delta}(t) \tag{14.220}$$

Note that, if the system has $n_y$ outputs and $n_u$ inputs, the impulse response is a matrix of dimensions $n_y \times n_u$. Using the sampling property of the Dirac delta function, we can write

$$\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\boldsymbol{\delta}(t) \tag{14.221}$$

The delta function appears in the previous equation due to the direct feedthrough term $\mathbf{D}$ from the input to the output. For systems without $\mathbf{D}$, the impulse response matrix will not contain an impulse term.

### 14.3.4 Transfer function matrix

The Laplace transform of the state equation is given by

$$s\mathbf{x}(s) = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s) \tag{14.222}$$

Therefore, the state can be written as a function of the input vector as

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\mathbf{u}(s) \tag{14.223}$$

Since the Laplace transform of the output equation is given by

$$\mathbf{y}(s) = \mathbf{C}\mathbf{x}(s) + \mathbf{D}\mathbf{u}(s) \tag{14.224}$$

we have the following input-output relation

$$\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s) \tag{14.225}$$

where

$$\mathbf{H}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} \tag{14.226}$$

is the transfer function matrix. It has dimensions $n_y \times n_u$. Note that the inverse Laplace transform of $\mathbf{H}(s)$ is the impulse response matrix defined in Eq. (14.221), i.e.,

$$\mathcal{L}^{-1}\left[\mathbf{H}(s)\right] = \mathbf{h}(t)$$

For dynamic systems without the direct feedthrough matrix $\mathbf{D}$, the transfer function is reduced to

$$\mathbf{H}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} \tag{14.227}$$

In this case, since the determinant of $s\mathbf{I} - \mathbf{A}$ is of degree $n_x$, where $n_x$ is the number of state variables, and the adjoint matrix of $s\mathbf{I} - \mathbf{A}$ is of degree $n_x - 1$, it follows that each element of the transfer function matrix is a strictly proper transfer function, i.e., a rational function of $s$ with the numerator of degree $n_x - 1$ (or less) and the denominator of degree $n_x$.

### 14.3.5 Transformation of state variables

We have already outlined that the same structural system can have many different state space formulations. Here, we will briefly present the theory related to the transformation of state variables and explain why all the state space formulations are equivalent, in the sense that they share the same input-output relations and, hence, they represent the same physical system.

Each state space formulation is based on a different set of state variables. If one formulation is assumed to be expressed by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{14.228}$$

a change of state variables can be represented by a linear transformation

$$\mathbf{z}(t) = \mathbf{T}\mathbf{x}(t) \tag{14.229}$$

where $\mathbf{z}$ is the new set of states and $\mathbf{T}$ is a constant nonsingular matrix, such that we can recover the original state vector $\mathbf{x}$ from the new vector $\mathbf{z}$ by

$$\mathbf{x}(t) = \mathbf{T}^{-1}\mathbf{z}(t) \tag{14.230}$$

Substitution of $\mathbf{x}$ into Eq. (14.228) yields

$$\begin{aligned}\mathbf{T}^{-1}\dot{\mathbf{z}}(t) &= \mathbf{A}\mathbf{T}^{-1}\mathbf{z}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{T}^{-1}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned} \tag{14.231}$$

or, rearranging,

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \hat{\mathbf{A}}\mathbf{z}(t) + \hat{\mathbf{B}}\mathbf{u}(t)\\ \mathbf{y}(t) &= \hat{\mathbf{C}}\mathbf{z}(t) + \hat{\mathbf{D}}\mathbf{u}(t)\end{aligned} \tag{14.232}$$

where

$$\hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \qquad \hat{\mathbf{B}} = \mathbf{T}\mathbf{B} \qquad \hat{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} \qquad \hat{\mathbf{D}} = \mathbf{D} \tag{14.233}$$

are the new matrices of the transformed state space representation. What presented above is the general procedure behind any state transformation.

The input-output relation of the new state formulation is represented by the corresponding transfer function, which is obtained by taking the Laplace transform of Eqs. (14.232). It is given by

$$\begin{aligned}\hat{\mathbf{H}}(s) &= \hat{\mathbf{C}}\left(s\mathbf{I} - \hat{\mathbf{A}}\right)^{-1}\hat{\mathbf{B}} + \hat{\mathbf{D}}\\ &= \mathbf{C}\mathbf{T}^{-1}\left(s\mathbf{I} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)^{-1}\mathbf{T}\mathbf{B} + \mathbf{D}\\ &= \mathbf{C}\mathbf{T}^{-1}\left(s\mathbf{T}\mathbf{T}^{-1} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)^{-1}\mathbf{T}\mathbf{B} + \mathbf{D}\\ &= \mathbf{C}\mathbf{T}^{-1}\left[\mathbf{T}\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{T}^{-1}\right]^{-1}\mathbf{T}\mathbf{B} + \mathbf{D}\\ &= \mathbf{C}\mathbf{T}^{-1}\mathbf{T}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{T}^{-1}\mathbf{T}\mathbf{B} + \mathbf{D}\\ &= \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\end{aligned} \tag{14.234}$$

Therefore, the transfer function matrix of the new state formulation is equal to the transfer function matrix of the original formulation. This implies that the input-output relations of a LTI system do not depend on how the state variables are defined.

Note also that the original matrix $\mathbf{A}$ and the transformed matrix $\hat{\mathbf{A}}$ are similar. Indeed, they have the same eigenvalues as shown in the following

$$\begin{aligned}\det\left(s\mathbf{I} - \hat{\mathbf{A}}\right) &= \det\left(s\mathbf{I} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)\\ &= \det\left(s\mathbf{T}\mathbf{T}^{-1} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)\\ &= \det\left[\mathbf{T}\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{T}^{-1}\right]\\ &= \det(\mathbf{T})\det\left(s\mathbf{I} - \mathbf{A}\right)\det(\mathbf{T}^{-1})\\ &= \det(\mathbf{T})\det\left(s\mathbf{I} - \mathbf{A}\right)\frac{1}{\det(\mathbf{T})}\\ &= \det\left(s\mathbf{I} - \mathbf{A}\right)\end{aligned} \tag{14.235}$$

Therefore, the nonsingular matrix $\mathbf{T}$ describes a similarity transformation.

### 14.3.6 Canonical forms

We have shown how to determine the transfer function of a LTI system, given its state space representation. In many cases, it is necessary to go in the opposite direction, i.e., derive the state space form from the input-output relation. This need arises if one wants to use state space methods but one or more subsystems within larger systems are described by a transfer function model. We have already seen some examples in section 14.2 (band-limited actuators and disturbance, derivator circuits).

At this stage, we know that there are innumerable equivalent state space models that have the same transfer function. Therefore, the state space representation of a transfer function is not unique. In the following, two possible canonical representations, called *first companion form* and *Jordan form*, are developed.

***14.3.6.1 First companion form.*** Let first consider a general transfer function of a single-input, single-output system

$$H(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} \tag{14.236}$$

After the introduction of the intermediate variable $z(s)$, the previous function can be alternatively written as

$$H(s) = \frac{y(s)}{u(s)} = \frac{y(s)}{z(s)}\frac{z(s)}{u(s)} \tag{14.237}$$

where the first factor is identified with the numerator of $H(s)$

$$\frac{y(s)}{z(s)} = b_0 s^n + b_1 s^{n-1} + \cdots + b_n \tag{14.238}$$

and the second factor is identified with the denominator

$$\frac{z(s)}{u(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n} \tag{14.239}$$

They can be also written as

$$\begin{gathered} \left(s^n + a_1 s^{n-1} + \cdots + a_n\right) z(s) = u(s) \\ y(s) = \left(b_0 s^n + b_1 s^{n-1} + \cdots + b_n\right) z(s) \end{gathered} \tag{14.240}$$

The differential equations corresponding to Eqs. (14.240) are

$$\frac{d^n}{dt^n} z(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} z(t) + \cdots + a_n z(t) = u(t) \tag{14.241}$$

and

$$y(t) = b_0 \frac{d^n}{dt^n} z(t) + b_1 \frac{d^{n-1}}{dt^{n-1}} z(t) + \cdots + b_n z(t) \tag{14.242}$$

The $n$th-order equation (14.241) can be put into a first-order representation by defining the following state variables

$$\begin{aligned} x_1(t) &= z(t) \\ x_2(t) &= \frac{d}{dt} z(t) = \dot{x}_1(t) \\ x_3(t) &= \frac{d^2}{dt^2} z(t) = \dot{x}_2(t) \\ &\vdots \\ x_n(t) &= \frac{d^{n-1}}{dt^{n-1}} z(t) = \dot{x}_{n-1}(t) \end{aligned} \tag{14.243}$$

After collecting the state variables into the state vector $\mathbf{x}(t)$ as follows

$$\mathbf{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{Bmatrix} \tag{14.244}$$

Equation (14.241) can be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{Bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \end{aligned} \tag{14.245}$$

By substituting the $n$th-order derivative of Eq. (14.241) into Eq. (14.242) we obtain

$$\begin{aligned} y(t) &= b_0 \left[ -a_1 \frac{d^{n-1}}{dt^{n-1}} z(t) - \cdots - a_n z(t) + u(t) \right] + b_1 \frac{d^{n-1}}{dt^{n-1}} z(t) + \cdots + b_n z(t) \\ &= (b_1 - a_1 b_0) \frac{d^{n-1}}{dt^{n-1}} z(t) + (b_2 - a_2 b_0) \frac{d^{n-2}}{dt^{n-2}} z(t) + \cdots + (b_n - a_n b_0) z(t) + b_0 u(t) \end{aligned} \tag{14.246}$$

According to the definition of the state vector, it follows that

$$\begin{aligned} y(t) &= \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \ldots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{Bmatrix} + b_0 u(t) \\ &= \mathbf{c}\mathbf{x}(t) + d u(t) \end{aligned} \tag{14.247}$$

In the case of multiple outputs, we can write the transfer function from the input $u(t)$ to the $i$th output $y_i(t)$ as

$$H_i(s) = \frac{b_{0i} s^n + b_{1i} s^{n-1} + \cdots + b_{ni}}{s^n + a_1 s^{n-1} + \cdots + a_n} \tag{14.248}$$

The same set of state variables serves for each transfer function. Therefore, the dynamic matrix $\mathbf{A}$ and input vector $\mathbf{b}$ are exactly as given earlier. After collecting all the output variables in the output vector $\mathbf{y}(t)$, the output matrix $\mathbf{C}$ will be expressed by

$$\mathbf{C} = \begin{bmatrix} b_{n1} - a_{n1} b_{01} & b_{n-1,1} - a_{n-1,1} b_{01} & \ldots & b_{21} - a_{21} b_{01} & b_{11} - a_{11} b_{01} \\ \cdots & & & & \\ b_{ni} - a_{ni} b_{0i} & b_{n-1,i} - a_{n-1,i} b_{0i} & \ldots & b_{2i} - a_{2i} b_{0i} & b_{1i} - a_{1i} b_{0i} \\ \cdots & & & & \end{bmatrix} \tag{14.249}$$

The direct feedthrough matrix is given by

$$\mathbf{d} = \begin{bmatrix} b_{01} \\ \vdots \\ b_{0i} \\ \vdots \end{bmatrix} \tag{14.250}$$

***14.3.6.2 Jordan form.*** The Jordan form follows directly from the partial fraction expansion of the transfer function. However, we need to distinguish between two cases.

The first case refers to systems with *all distinct poles*. The transfer function in Eq. (14.236) can be put into the following partial fraction form

$$H(s) = b_0 + \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n} \tag{14.251}$$

where $r_i$ are the residues and $p_i$ are the system poles. The previous form can be alternatively written as

$$y(s) = b_0 u(s) + r_1 x_1(s) + r_2 x_2(s) + \cdots + r_n x_n(s) \tag{14.252}$$

where

$$x_i(s) = \frac{1}{s - p_i} u(s) \qquad (i = 1, \ldots, n) \tag{14.253}$$

In the time-domain we have

$$\dot{x}_i(t) = p_i x_i(t) + u(t) \qquad (i = 1, \ldots, n) \tag{14.254}$$

which are the state variables of the model. The output equation becomes

$$y(t) = \sum_{i=1}^{n} r_i x_i(t) + b_0 u(t) \tag{14.255}$$

After collecting the state variables into the state vector

$$\mathbf{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} \tag{14.256}$$

the state representation is given by

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{Bmatrix} = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & p_n \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \tag{14.257}$$

Note that the state matrix $\mathbf{A}$ is a diagonal matrix. The output equation is expressed by

$$y(t) = \begin{bmatrix} r_1 & r_2 & \dots & r_n \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} + b_0 u(t) = \mathbf{c}\mathbf{x}(t) + du(t) \tag{14.258}$$

in which the output matrix $\mathbf{c}$ contains the residues.

The second case refers to systems with *repeated poles*. The partial fraction expansion is written as

$$H(s) = b_0 + H_1(s) + \cdots + H_{\bar{n}}(s) \tag{14.259}$$

where $\bar{n} < n$ is the number of distinct poles and

$$H_i(s) = \frac{r_{1i}}{s - p_i} + \frac{r_{2i}}{(s - p_i)^2} + \cdots + \frac{r_{m_i i}}{(s - p_i)^{m_i}} \tag{14.260}$$

where $m_i$ is the multiplicity of the $i$th pole.

Let focus on the subsystem $H_i(s)$. The corresponding input-output relation can be also written as

$$y_i(s) = r_{1i} x_{1i}(s) + r_{2i} x_{2i}(s) + \cdots + r_{m_i i} x_{m_i i}(s) \tag{14.261}$$

where

$$\begin{aligned} x_{1i}(s) &= \frac{1}{s - p_i} u(s) \\ x_{2i}(s) &= \frac{1}{(s - p_i)^2} u(s) = \frac{1}{s - p_i} x_{1i}(s) \\ x_{3i}(s) &= \frac{1}{(s - p_i)^3} u(s) = \frac{1}{s - p_i} x_{2i}(s) \\ &\dots \\ x_{m_i i}(s) &= \frac{1}{(s - p_i)^{m_i}} u(s) = \frac{1}{s - p_i} x_{(m_i - 1)i}(s) \end{aligned} \tag{14.262}$$

In the time domain, the state variables of this subsystem are governed by the following equations

$$\begin{aligned}
&\dot{x}_{1i}(t) = p_i x_{1i}(t) + u \\
&\dot{x}_{2i}(t) = p_i x_{2i}(t) + x_{1i}(t) \\
&\dots \\
&\dot{x}_{m_i i}(t) = p_i x_{m_i i}(t) + x_{(m_i-1)i}(t)
\end{aligned} \tag{14.263}$$

After collecting the above state variables into the state vector

$$\mathbf{x}_i(t) = \begin{Bmatrix} x_{m_i i}(t) \\ x_{(m_i-1)i}(t) \\ \vdots \\ x_{2i}(t) \\ x_{1i}(t) \end{Bmatrix} \tag{14.264}$$

the state representation for the subsystem is given by

$$\begin{aligned}
\dot{\mathbf{x}}_i(t) = \begin{Bmatrix} \dot{x}_{m_i i}(t) \\ \dot{x}_{(m_i-1)i}(t) \\ \vdots \\ \dot{x}_{2i}(t) \\ \dot{x}_{1i}(t) \end{Bmatrix} &= \begin{bmatrix} p_i & 1 & 0 & \dots & 0 \\ 0 & p_i & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & p_i \end{bmatrix} \begin{Bmatrix} x_{m_i i}(t) \\ x_{(m_i-1)i}(t) \\ \vdots \\ x_{2i}(t) \\ x_{1i}(t) \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \\
&= \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{b}_i u(t)
\end{aligned} \tag{14.265}$$

The above representation is also called Jordan block corresponding to the $i$th repeated pole of multiplicity $m_i$. Note that the $\mathbf{A}_i$ matrix has dimensions $m_i \times m_i$ and is not diagonal. It consists of two diagonals: the principal diagonal has the corresponding pole and the diagonal just above the main diagonal has all 1's. The corresponding output equation is

$$y_i(t) = \begin{bmatrix} r_{m_i i} & r_{(m_i-1)i} & \dots & r_{2i} & r_{1i} \end{bmatrix} \begin{Bmatrix} x_{m_i i}(t) \\ x_{(m_i-1)i}(t) \\ \vdots \\ x_{2i}(t) \\ x_{1i}(t) \end{Bmatrix} = \mathbf{c}_i \mathbf{x}_i(t) \tag{14.266}$$

The overall state space representation will be given by the concatenation of the state vectors of each Jordan block as follows

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_{\bar{n}}(t) \end{Bmatrix} \tag{14.267}$$

Accordingly, the $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ matrices of the overall system will be given by

$$\mathbf{A} = \text{blkdiag}\left(\mathbf{A}_i\right) \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_{\bar{n}} \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_{\bar{n}} \end{bmatrix} \tag{14.268}$$

## 14.4 Controllability and observability

### 14.4.1 Introduction

Controllability and observability are distinctive concepts strictly related to state-space methods. They can be considered as intrinsic properties of LTI systems, carrying useful information for structural testing and control.

We will see that the controllability property is connected to the capability of installed actuators of exciting *all* vibrating modes of a structure. Likewise, observability is related to the capability of detecting *all* structural modes by installed sensors. Therefore, they can be used as valuable guidelines for the placement of actuators and sensors in experimental modal testing or structural control applications.

However, the information on controllability and observability alone is often not enough in practical engineering problems. Indeed, we typically need to know *how much* a structure is controllable and/or observable. This quantitative information is provided by so-called controllability and observability grammians, which represent a degree of controllability and observability of a structural mode. Grammians will be also used as a valuable tool to reduce the order of a dynamic system. This tool is strictly related to a balanced representation of the state space model.

### 14.4.2 Definitions

The definition of controllability and observability are taken from Ref. [1] and are reported below.

*Definition of controllability.* A system is said to be controllable (or completely controllable) if and only if it is possible, by means of the input, to transfer the system from *any* initial state $\mathbf{x}(t_0)$ to *any* other state $\mathbf{x}(T)$ in a *finite* time interval $T - t_0 \geq 0$.

Note the use of words any and finite in the above definition. If it is only possible to transfer the system from some states to some other states, the system is not controllable. Likewise, if the transfer takes an infinite amount of time, the system is not controllable. Note also that the initial time $t_0$ is arbitrary and the terminal time $T$ is not fixed.

*Definition of observability.* An unforced system is said to be observable (or completely observable) if and only if it is possible to determine *any* state $\mathbf{x}(t)$ by using only a *finite* record $\mathbf{y}(\tau)$ for $t \leq \tau \leq T$, of the output.

It can be shown that every LTI system in state space form can be decomposed into four subsystems:

1. controllable and observable
2. uncontrollable but observable
3. controllable but unobservable
4. neither controllable nor observable

If a system contains an uncontrollable subsystems it is said to be uncontrollable. Likewise, if a system contains an unobservable subsystem it is said to be unobservable. An uncontrollable and/or unobservable system can be very dangerous if the corresponding uncontrollable or unobservable parts are unstable (imagine for example a disturbance capable of establishing a nonzero initial state; this condition will send the uncontrollable system off to infinity!). Therefore, a distinction is introduced between an uncontrollable system in which the uncontrollable part is stable and one in which the uncontrollable part is unstable.

*Definition of stabilizability.* A system is said to be stabilizable if the uncontrollable part is stable.

Similarly, there is a distinction between an unobservable system in which the unobservable part is stable and one in which the unobservable part is unstable.

*Definition of detectability.* A system is said to be detectable if the unobservable part is stable.

### 14.4.3 Controllability and observability matrices

Controllability and observability can be checked by evaluating the rank of special matrices called controllability and observability matrices. The following results are typically called algebraic test or criterion of controllability and observability. They are given without any proof.

A LTI system with $n$ states and $n_u$ inputs is completely controllable if and only if the following controllability matrix

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{14.269}$$

is such that

$$\text{rank}\,(\boldsymbol{\mathcal{C}}) = n \tag{14.270}$$

We say that the pair $(\mathbf{A}, \mathbf{B})$ is controllable. If $\boldsymbol{\mathcal{C}}$ is not full rank, the subspace spanned by its columns defines the controllable subsystem.

A LTI system with $n$ states and $n_y$ outputs is completely observable if and only if the following observability matrix

$$\boldsymbol{\mathcal{O}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{14.271}$$

is such that

$$\text{rank}\,(\boldsymbol{\mathcal{O}}) = n \tag{14.272}$$

In this case, we say that the pair $(\mathbf{A}, \mathbf{C})$ is observable.

Note that, since we have

$$\boldsymbol{\mathcal{O}}^T = \begin{bmatrix} \mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T & (\mathbf{A}^2)^T\mathbf{C}^T & \dots & (\mathbf{A}^{n-1})^T\mathbf{C}^T \end{bmatrix} \tag{14.273}$$

and the rank is not affected by the transpose operation, we can check the observability by checking the controllability of the pair $(\mathbf{A}^T, \mathbf{C}^T)$.

It is also noticed that, if $\boldsymbol{\mathcal{C}}$ is the controllability matrix of the pair $(\mathbf{A}, \mathbf{B})$, the controllability matrix of the transformed state space system with pair $(\hat{\mathbf{A}}, \hat{\mathbf{B}})$, where $\hat{\mathbf{A}} = \mathbf{TAT}^{-1}$ and $\hat{\mathbf{B}} = \mathbf{TB}$, is given by

$$\begin{aligned} \hat{\boldsymbol{\mathcal{C}}} &= \begin{bmatrix} \hat{\mathbf{B}} & \hat{\mathbf{A}}\hat{\mathbf{B}} & \hat{\mathbf{A}}^2\hat{\mathbf{B}} & \dots & \hat{\mathbf{A}}^{n-1}\hat{\mathbf{B}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{TB} & \mathbf{TAT}^{-1}\mathbf{TB} & (\mathbf{TAT}^{-1})^2\mathbf{TB} & \dots & (\mathbf{TAT}^{-1})^{n-1}\mathbf{TB} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{TB} & \mathbf{TAT}^{-1}\mathbf{TB} & (\mathbf{TAT}^{-1})(\mathbf{TAT}^{-1})\mathbf{TB} & \dots & (\mathbf{TAT}^{-1})(\mathbf{TAT}^{-1})\cdot\cdot(\mathbf{TAT}^{-1})\mathbf{TB} \end{bmatrix} \\ &= \mathbf{T}\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \\ &= \mathbf{T}\boldsymbol{\mathcal{C}} \end{aligned} \tag{14.274}$$

It can be shown that, for any nonsingular transformation $\mathbf{T}$, the rank of $\hat{\boldsymbol{\mathcal{C}}}$ is the same as that of $\boldsymbol{\mathcal{C}}$. Therefore, the property of controllability is preserved by any nonsingular transformation of the state variables.

In a similar manner, it can be shown that

$$\hat{\boldsymbol{\mathcal{O}}} = \boldsymbol{\mathcal{O}}\mathbf{T}^{-1} \tag{14.275}$$

### 14.4.4 Controllability and observability grammians

The above algebraic criteria are simple, but they have two serious drawbacks:

- The answer of each criterion is yes or no. This information can be poor in some situations. Consider for example a uniform simply supported beam of length $\ell$. We have seen that the exact flexural mode shapes are of the form

$\sin(n\pi x/\ell)$. Hence, if the structure is subject to a transverse point force (arising from a control actuator or an actuating device used for modal testing) acting at the midpoint of the beam, the modes having even number $n = 2, 4, \ldots$ are not controllable because they have a nodal point at the center. Similarly, if a sensor measuring the transverse displacement is located at the beam center, it is completely insensitive to even modes. From the rank tests, we can conclude that the system is uncontrollable and unobservable. Now, imagine to move the actuator and the sensor a small distance away from the nodal point. According to the above algebraic criteria, the rank deficiency disappears, and the system becomes both controllable and observable. However, such an actuator installed very close to the nodal point of a mode will have great difficulties in controlling or exciting the corresponding mode because the modal participation factor will be very small. Likewise, a sensor located very close to a nodal point of a mode will be very weakly sensitive to that mode. We say in this case that the mode is only *weakly* controllable or observable. The quantitative information on the degree of controllability and observability is not included in the rank tests presented above.

- The algebraic criteria are useful only for systems of small dimensions. This is clearly visible if we consider, for example, a structural system having 100 states. Such a number is not so high if we have to deal with flexible structures with many modes falling in the frequency range of interest. In order to use the rank tests, we have to build the corresponding controllability and observability matrices and, thus, compute powers of the state matrix $\mathbf{A}$ up to 99. Finding $\mathbf{A}^{99}$ is a numerical task that, for typical practical applications, easily results in numerical overflow.

An alternative to the algebraic tests on controllability and observability is to use grammians, which are *nonnegative* matrices free of the numerical difficulties mentioned above and capable of providing a quantitative measure of the degree of controllability and observability, as shown later.

For linear time-invariant asymptotically stable systems, the controllability and observability grammians are defined, respectively, as

$$\begin{aligned} \mathbf{W}_c &= \int_0^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau \\ \mathbf{W}_o &= \int_0^\infty e^{\mathbf{A}^T\tau}\mathbf{C}^T\mathbf{C} e^{\mathbf{A}\tau} d\tau \end{aligned} \tag{14.276}$$

A system is controllable (observable) if and only if the controllability (observability) grammian is strictly positive definite.

Instead of computing the grammians by evaluating the integrals from the above definitions, an alternative approach is to solve the following algebraic equations:

$$\begin{aligned} \mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}^T + \mathbf{B}\mathbf{B}^T &= \mathbf{0} \\ \mathbf{A}^T\mathbf{W}_o + \mathbf{W}_o\mathbf{A} + \mathbf{C}^T\mathbf{C} &= \mathbf{0} \end{aligned} \tag{14.277}$$

The above are Lyapunov equations in the unknowns $\mathbf{W}_c$ and $\mathbf{W}_o$.

The grammians related to a transformed state space system are related to those of the original system by the following relations

$$\begin{aligned} \hat{\mathbf{W}}_c &= \mathbf{T}^{-1}\mathbf{W}_c\mathbf{T}^{-T} \\ \hat{\mathbf{W}}_o &= \mathbf{T}^T\mathbf{W}_o\mathbf{T} \end{aligned} \tag{14.278}$$

## 14.5 Model order reduction

### 14.5.1 Modal representation (distinct eigenvalues)

We have previously shown that a single-input single-output transfer function with all distinct poles can be converted to a state space representation with diagonal dynamic matrix $\mathbf{A}$ with entries equal to the poles (i.e., eigenvalues) of the system. Now, we would like to present the general procedure to follow to transform a generic state space model to a state space model in diagonal form. This diagonal form is typically called *modal* representation since the eigenvalues of the system are explicit in it.

Let consider the generic single-input single-output state space formulation of order $n$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)\\ y(t) &= \mathbf{c}\mathbf{x}(t) + du(t)\end{aligned} \tag{14.279}$$

The eigenvalues of the system corresponds to the nontrivial solution of

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{14.280}$$

If the eigenvalues $\lambda_i$ $(i = 1, \dots, n)$ are all distinct, the associated eigenvectors $\mathbf{v}_i$ are linearly independent. This means that the eigenvector matrix

$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n\end{bmatrix} \tag{14.281}$$

is nonsingular. Then, we can write

$$\begin{aligned}\mathbf{A}\mathbf{V} &= \mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \dots & \mathbf{A}\mathbf{v}_n\end{bmatrix}\\ &= \begin{bmatrix}\lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \dots & \lambda_n\mathbf{v}_n\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n\end{bmatrix} = \mathbf{V}\hat{\mathbf{A}}\end{aligned} \tag{14.282}$$

where $\hat{\mathbf{A}}$ is the Jordan (diagonal) form of the matrix $\mathbf{A}$. Therefore, the relation between $\mathbf{A}$ and $\hat{\mathbf{A}}$ is given by

$$\hat{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} \tag{14.283}$$

This corresponds to the following transformation of the original state form

$$\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t) \tag{14.284}$$

We can state that a LTI system with distinct eigenvalues can be transformed to a diagonal representation by the similarity transformation (14.284), where the transformation matrix is represented by the matrix of eigenvectors. The above relation can be also inverted as follows

$$\mathbf{z}(t) = \mathbf{V}^{-1}\mathbf{x}(t) \tag{14.285}$$

Therefore, using what derived in section 14.3.5 with $\mathbf{T} = \mathbf{V}^{-1}$, the modal state space representation is given by

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{\Lambda}\mathbf{z}(t) + \mathbf{V}^{-1}\mathbf{b}u(t)\\ y(t) &= \mathbf{c}\mathbf{V}\mathbf{z}(t) + du(t)\end{aligned} \tag{14.286}$$

where $\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$. The above formulation, for multiple-input multiple-output systems, becomes

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{\Lambda}\mathbf{z}(t) + \mathbf{V}^{-1}\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{V}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned} \tag{14.287}$$

### 14.5.2 Modal representation (repeated eigenvalues)

If a matrix $\mathbf{A}$ has repeated eigenvalues it may or may not be possible to find an equivalent diagonal representation in the form of Eq. (14.287). The latter occurs when there are less independent eigenvectors than eigenvalues. In this case, the inverse of the eigenvector matrix does not exist and the transformation to diagonal form cannot be found. However, similarly to what presented in section 14.3.6, it is always possible to find an alternative set of linearly independent basis vectors that allow transformation to an *almost diagonal* form, which is called Jordan canonical form.
[...] TODO [...]

### 14.5.3 Modal representation (complex eigenvalues)

[...] TODO [...]

### 14.5.4 Modal reduction

[...] TODO [...]

### 14.5.5 Balanced representation

[...] TODO [...]

## REFERENCES

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2. A. Preumont (2011). *Vibration Control of Active Structure. An Introduction*, Springer, third edition.