CHAPTER 17

STATE SPACE APPROACH TO CONTROL SYSTEM DESIGN

draft draft

17.1 Introduction

[...]

17.2 Pole-Placement technique

A simple method of designing an active control for a LTI system in which all the state variables are accessible (i.e., they can be measured in some way) is the so-called *pole-placement technique*. It is based on enforcing a desired closed-loop response by specifying the closed-loop poles of the system. *If the system is fully controllable*, the closed-loop poles can be located *anywhere we wish* in the complex plane.

Let consider a structural system with a *single control input* $u(t)$. The structure is assumed to be excited by a generic disturbance input $d(t)$. Therefore, the dynamics of the system is represented by the following state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}_u u(t) + \mathbf{b}_d d(t) \tag{17.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b}_u \in \mathbb{R}^{n\times 1}$, $\mathbf{b}_d \in \mathbb{R}^{n\times 1}$, and n is the number of states. The pole-placement approach is implemented by a full state feedback law as follows

$$u(t) = -\mathbf{g}\,\mathbf{x}(t) \tag{17.2}$$

where $\mathbf{g} \in \mathbb{R}^{1\times n}$ is the row vector of control gains, which should be selected to achieve desirable properties of the closed-loop dynamics. For a LTI system with n states, there are n feedback gains g_i that can be adjusted independently. Substituting Eq. (17.2) into Eq. (17.1) yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{b}_d d(t) \tag{17.3}$$

where

$$\mathbf{A}_c = \mathbf{A} - \mathbf{b}_u\mathbf{g} \tag{17.4}$$

is the closed-loop system matrix. The eigenvalues of $\mathbf{A}_c$ are the closed-loop poles of the system. The objective of the design is to find the vector of control gains $\mathbf{g}$ which places the poles of $\mathbf{A}_c$ at desired locations. Therefore, the first step is to select the desired locations of the closed-loop poles. Then, a technique is sought to compute $\mathbf{g}$ such that $\mathbf{A}_c$ will have the prescribed eigenvalues.

For illustrative purposes, let consider the following example. A lightly-damped homogeneous simply-supported beam of length ℓ, mass per unit length m and bending stiffness EJ is equipped with an ideal force actuator located at $\ell/4$ aimed at reducing the transverse vibration. We would like to design a pole-placement control system based on a reduced-order model of the beam involving only the fundamental mode of the system.

The state-space equation describing the modal response $q(t)$ of the lowest mode of the beam is given by

$$\begin{Bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\xi_1\omega_1 \end{bmatrix} \begin{Bmatrix} q(t) \\ \dot{q}(t) \end{Bmatrix} + \begin{bmatrix} 0 \\ \sqrt{2}/m\ell \end{bmatrix} u(t) \tag{17.5}$$

where $u(t)$ is the control force, and ω_1 and ξ_1 are the natural frequency and damping factor, respectively, of the open-loop system. Therefore, the open-loop poles are located at

$$s_{1,2}^{\mathrm{O}} = -\xi_1\omega_1 \pm j\omega_1\sqrt{1-\xi_1^2} \tag{17.6}$$

Let assume that the desired closed-loop dynamics will correspond to the following closed-loop poles

$$s_{1,2}^{\mathrm{C}} = -\xi_c\omega_c \pm j\omega_c\sqrt{1-\xi_c^2} \tag{17.7}$$

where ω_c and ξ_c are the prescribed closed-loop natural frequency and damping factor. The state feedback in Eq. (17.2) is given in this case by

$$u(t) = -g_d q(t) - g_v\dot{q}(t) = -\begin{bmatrix} g_d & g_v \end{bmatrix} \begin{Bmatrix} q(t) \\ \dot{q}(t) \end{Bmatrix} \tag{17.8}$$

where g_d is the control gain related to the modal displacement and g_v is the control gain associated with the modal velocity. It is seen that the selected feedback state control corresponds to the classical PD control. The closed-loop system matrix is

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\xi_1\omega_1 \end{bmatrix} - \begin{bmatrix} 0 \\ \sqrt{2}/m\ell \end{bmatrix} \begin{bmatrix} g_d & g_v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_1^2 - \frac{\sqrt{2}}{m\ell}g_d & -2\xi_1\omega_1 - \frac{\sqrt{2}}{m\ell}g_v \end{bmatrix} \tag{17.9}$$

which indicates that the control gain g_d is associated with the stiffness term and the control gain g_v is related to the damping term. By varying g_d, the open-loop natural frequency associated with the fundamental mode of the beam can be modified. On the other hand, the open-loop modal damping of the first mode can be improved, if needed, by increasing g_v. The eigenvalues of the matrix $\mathbf{A}_c$ are determined by solving the characteristic equation

$$\det\left[s\mathbf{I} - \mathbf{A}_c\right] = 0 \tag{17.10}$$

which gives

$$s^2 + \left(2\xi_1\omega_1 + \frac{\sqrt{2}}{m\ell}g_v\right)s + \left(\omega_1^2 + \frac{\sqrt{2}}{m\ell}g_d\right) = 0 \tag{17.11}$$

According to the desired closed-loop poles, the desired characteristic equation is

$$\left(s - s_1^{\mathrm{C}}\right)\left(s - s_2^{\mathrm{C}}\right) = s^2 + 2\xi_c\omega_c s + \omega_c^2 = 0 \tag{17.12}$$

Therefore, by equating the coefficients of the actual characteristic equation with those of the desired characteristic equation we obtain

$$\begin{aligned} 2\xi_1\omega_1 + \frac{\sqrt{2}}{m\ell}g_v &= 2\xi_c\omega_c \\ \omega_1^2 + \frac{\sqrt{2}}{m\ell}g_d &= \omega_c^2 \end{aligned} \tag{17.13}$$

It is noted that each equation contains one control gain. The control gains are computed as

$$g_v = \frac{2m\ell}{\sqrt{2}}\left(\xi_c\omega_c - \xi_1\omega_1\right)$$
$$g_d = \frac{m\ell}{\sqrt{2}}\left(\omega_c^2 - \omega_1^2\right) \tag{17.14}$$

Note also that, if the design of the control system was carried out with the aim of increasing the damping of the fundamental mode without changing the corresponding frequency value, we have $\omega_c = \omega_1$ and so

$$g_v = \frac{2m\ell}{\sqrt{2}}\left(\xi_c - \xi_1\right)\omega_1$$
$$g_d = 0 \tag{17.15}$$

In this case, the proportional gain g_d is null since the desired performance can be achieved by the single derivative gain g_v.

What presented for the previous second-order system can be extended to any order. The characteristic equation will be expressed through a polynomial with n coefficients that are functions of the n control gains. Equating these functions to the numerical values desired for the polynomial coefficients will result in n simultaneous equations the solution of which will yield the desired gains $g_1, \ldots, g_n$. For example, referring to the previous beam problem, if we want to design a pole-placement control targeted on the first two modes instead on the single fundamental mode, the state and input matrices can be expressed as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\xi_1\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & -2\xi_2\omega_2 \end{bmatrix} \qquad \mathbf{b}_u = \begin{bmatrix} 0 \\ \sqrt{2}/m\ell \\ 0 \\ 2/m\ell \end{bmatrix} \tag{17.16}$$

The feedback control law is written as

$$u(t) = -\begin{bmatrix} g_1 & g_2 & g_3 & g_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{Bmatrix} \tag{17.17}$$

Therefore, the characteristic equation associated with the closed-loop matrix

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\xi_1\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & -2\xi_2\omega_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sqrt{2}/m\ell \\ 0 \\ 2/m\ell \end{bmatrix} \begin{bmatrix} g_1 & g_2 & g_3 & g_4 \end{bmatrix} \tag{17.18}$$

is given by

$$\begin{aligned} s^4 &+ \left(2\xi_1\omega_1 + 2\xi_2\omega_2 + \frac{2g_4 + \sqrt{2}g_2}{m\ell}\right)s^3 + \left(\omega_1^2 + \omega_2^2 + 4\xi_1\xi_2\omega_1\omega_2 + \frac{2g_3 + \sqrt{2}g_1 + 4g_4\xi_1\omega_1 + 2\sqrt{2}g_2\xi_2\omega_2}{m\ell}\right)s^2 \\ &+ \left(2\xi_1\omega_1\omega_2^2 + 2\xi_2\omega_1^2\omega_2 + \frac{2g_4\omega_1^2 + 4g_3\xi_1\omega_1 + \sqrt{2}g_2\omega_2^2 + 2\sqrt{2}g_1\xi_2\omega_2}{m\ell}\right)s \\ &+ \left(\omega_1^2\omega_2^2 + \frac{2g_3\omega_1^2 + \sqrt{2}g_1\omega_2^2}{m\ell}\right) = 0 \end{aligned}$$

The desired characteristic equation can be expressed in the general form as

$$s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 = 0 \tag{17.19}$$

where α_i are prescribed coefficients. The control gains g_i $(i = 1, \dots, 4)$ are computed by solving the following set of equations

$$\begin{cases} 2\xi_1\omega_1 + 2\xi_2\omega_2 + \dfrac{2g_4 + \sqrt{2}g_2}{m\ell} = \alpha_1 \\ \omega_1^2 + \omega_2^2 + 4\xi_1\xi_2\omega_1\omega_2 + \dfrac{2g_3 + \sqrt{2}g_1 + 4g_4\xi_1\omega_1 + 2\sqrt{2}g_2\xi_2\omega_2}{m\ell} = \alpha_2 \\ 2\xi_1\omega_1\omega_2^2 + 2\xi_2\omega_1^2\omega_2 + \dfrac{2g_4\omega_1^2 + 4g_3\xi_1\omega_1 + \sqrt{2}g_2\omega_2^2 + 2\sqrt{2}g_1\xi_2\omega_2}{m\ell} = \alpha_3 \\ \omega_1^2\omega_2^2 + \dfrac{2g_3\omega_1^2 + \sqrt{2}g_1\omega_2^2}{m\ell} = \alpha_4 \end{cases} \tag{17.20}$$

The previous procedure is not efficient if the order of the system is high. It becomes straightforward only if the state equation is written in a canonical form. By referring to the first companion form we introduced in a previous chapter, the system to be controlled should be expressed by the following state matrix and input vector

$$\mathbf{A}_{\mathrm{CF}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \qquad \mathbf{b}_{\mathrm{CF}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{17.21}$$

The related characteristic polynomial is given by

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1}s + a_n \tag{17.22}$$

where the coefficients correspond to the elements of the last row of the state matrix. By considering the full state feedback law

$$u(t) = -\begin{bmatrix} g_1 & g_2 & \dots & g_{n-1} & g_n \end{bmatrix} \mathbf{x}(t) \tag{17.23}$$

the closed-loop system matrix is given in this case by

$$\mathbf{A}_c = \mathbf{A}_{\mathrm{CF}} - \mathbf{b}_{\mathrm{CF}}\mathbf{g} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ -(a_n + g_1) & -(a_{n-1} + g_2) & -(a_{n-2} + g_3) & \dots & -(a_1 + g_n) \end{bmatrix} \tag{17.24}$$

The characteristic equation is then expressed as

$$s^n + (a_1 + g_n)s^{n-1} + (a_2 + g_{n-1})s^{n-2} + \dots + (a_{n-1} + g_2)s + (a_n + g_1) = 0 \tag{17.25}$$

After selecting the locations of the closed-loop poles s_i^{C}, the desired characteristic equation takes the following form

$$\prod_{i=1}^{n} \left(s - s_i^{\mathrm{C}}\right) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \dots + \alpha_{n-1}s + \alpha_n = 0 \tag{17.26}$$

Therefore, by equating the polynomial coefficients, the control gains are determined as follows

$$\begin{cases} g_1 = \alpha_n - a_n \\ g_2 = \alpha_{n-1} - a_{n-1} \\ \vdots \\ g_{n-1} = \alpha_2 - a_2 \\ g_n = \alpha_1 - a_1 \end{cases} \tag{17.27}$$

The previous design procedure requires that the state equation is expressed through the first companion form. However, in practical applications, this condition is rarely met. What can be done is to transform the original state-space formulation of the problem under investigation to the first companion form. Using the linear transformation with the nonsingular matrix $\mathbf{T}$

$$\mathbf{x}_{\mathrm{CF}}(t) = \mathbf{T}\mathbf{x}(t) \tag{17.28}$$

the original state-space representation $(\mathbf{A}, \mathbf{b}_u)$ is expressed by the matrices $(\mathbf{A}_{\mathrm{CF}}, \mathbf{b}_{\mathrm{CF}})$ in Eq. (17.21) such that

$$\mathbf{A}_{\mathrm{CF}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \qquad \mathbf{b}_{\mathrm{CF}} = \mathbf{T}\mathbf{b}_u \tag{17.29}$$

We have already shown that $\mathbf{A}$ and the transformed state matrix are similar, i.e., they have the same eigenvalues. Therefore, we can assign the poles of the system in the first companion form since they are the same poles of the original system. The state feedback law is then written as

$$u(t) = -\mathbf{g}_{\mathrm{CF}}\mathbf{x}_{\mathrm{CF}}(t) \tag{17.30}$$

where $\mathbf{g}_{\mathrm{CF}}$ can be determined as described above in Eq. (17.27). Now, using the transformation (17.28), we can write

$$u(t) = -\mathbf{g}_{\mathrm{CF}}\mathbf{T}\mathbf{x}(t) = -\mathbf{g}\,\mathbf{x}(t) \tag{17.31}$$

Therefore, the control gains associated with the original system are given by

$$\mathbf{g} = \mathbf{g}_{\mathrm{CF}}\mathbf{T} \tag{17.32}$$

The transformation matrix $\mathbf{T}$ can be determined by using the relation between the controllability matrix $\boldsymbol{\mathcal{C}}$ of the original system and the controllability matrix of the transformed system, which is given by

$$\boldsymbol{\mathcal{C}}_{\mathrm{CF}} = \mathbf{T}\boldsymbol{\mathcal{C}} \tag{17.33}$$

Therefore, we have

$$\mathbf{T} = \boldsymbol{\mathcal{C}}_{\mathrm{CF}}\boldsymbol{\mathcal{C}}^{-1} \tag{17.34}$$

where

$$\boldsymbol{\mathcal{C}}_{\mathrm{CF}} = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \dots & 1 & 0 \\ \vdots & & & & \\ a_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}^{-1} \qquad \text{(note the inverse!)} \tag{17.35}$$

The row vector of control gains is then expressed as

$$\mathbf{g} = \mathbf{g}_{\mathrm{CF}}\boldsymbol{\mathcal{C}}_{\mathrm{CF}}\boldsymbol{\mathcal{C}}^{-1} \tag{17.36}$$

which is known as *Bass-Gura formula*. A simple MATLAB code of the algorithm is given below.

MATLAB code 17.1

```
function g = bassgura(A,b,p)

n = length(b);
a1 = poly(p);

alpha = [a1(n:-1:2),1];
C = ctrb(A,b);
a = poly(A);
aa = [a(n:-1:2),1];
L = hankel(aa);
g = (a1(n+1:-1:2)-a(n+1:-1:2))*inv(L)*inv(C);
```

Note that, if the system is not controllable, then $\mathcal{C}^{-1}$ does not exist and there is no general method of transforming the original system. Indeed, it is not possible to place the closed-loop poles anywhere one desires. Thus, controllability is an essential requirement of system design by pole-placement.

Until now, we have considered the design of single-input systems. If the system under consideration has more than one control input, it means that the state feedback control law is expressed as

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t) \tag{17.37}$$

where $\mathbf{G}$ is now a matrix with n_u rows (as the number of inputs) and n columns. The closed-loop dynamics is expressed as

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{b}_d d(t) \tag{17.38}$$

where

$$\mathbf{A}_c = \mathbf{A} - \mathbf{B}_u\mathbf{G} \tag{17.39}$$

is the closed-loop system matrix. Therefore, the design process involves the computation of the control matrix $\mathbf{G}$. Since each row of $\mathbf{G}$ furnishes n gains that can be adjusted, it is clear that in a controllable system there will be more gains available than are needed to place all of the closed-loop poles. This can be considered as an increased flexibility for the designer since he/she can place the poles and, at the same time, satisfy some other requirements. For example, the control system structure can be simplified by setting some of the gains to zero. Another possibility is to use the extra degrees of freedom to find a solution which minimizes the sensitivity of the closed-loop poles to perturbations in the $\mathbf{A}$ and $\mathbf{B}$ matrices. This method is adopted by the MATLAB function

```
G = place(A,B,p)
```

where $(\mathbf{A}, \mathbf{B})$ is the state-space model, and p is a vector containing the expected pole locations. The returned variable is the state feedback gain matrix. It should be pointed out that the above function can be used with *distinct* pole positions, i.e., the multiplicity of any desired pole cannot be greater than one.

17.3 Steady-state linear-quadratic control (LQR)

17.3.1 Preliminaries

Through the assignment of the closed-loop poles, the pole-placement technique is capable of specifying the speed (bandwidth) and damping of the dynamic response of LTI systems. However, it suffers from some shortcomings. They are briefly discussed in the following.

The first shortcoming is related to the multiple input case. We have seen that, in this case, there are more gains than those needed. The resulting flexibility provides an undetermined control solution, since there are infinitely many ways by which the same closed-loop poles can be attained. The lack of a definitive algorithm gives rise to a natural question: which way of assigning poles is best?

Another shortcoming of the pole-placement technique is that the designer may not really know the desirable closed-loop pole locations. Furthermore, he/she has no quantitative information on the control effort associated with a particular solution. Choosing pole locations far from the open-loop poles (and from the origin) will typically require large control signals, which can exceed the available power source. However, it is very difficult to predict the amount of control effort and to identify the poles that dominate the response.

Finally, it is well known that the transient response of LTI systems can be also strongly affected by the zeros. The pole-placement technique is focused only on the pole locations.

17.3.2 Problem definition

The above limitations can be overcome by using another state-space design technique, which is known as *steady-state optimal linear quadratic* control or *linear quadratic regulator* (LQR). The LQR problem is formulated as follows. As done before with the pole-placement approach, again we seek a *stabilizing* linear state feedback with constant gain matrix

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t) \tag{17.40}$$

Here, however, the gain matrix $\mathbf{G}$ is selected such that the following quadratic cost functional (or performance index) is minimised

$$J = \frac{1}{2}\int_0^{\infty} \left(\mathbf{z}^T \mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T \mathbf{W}_{uu}\mathbf{u}\right) dt \tag{17.41}$$

where

- $\mathbf{z}(t)$ is a selected performance vector
- $\mathbf{W}_{zz}$ is a nonnegative symmetric weighting matrix ($\mathbf{W}_{zz} \geq 0$) associated with the performance
- $\mathbf{W}_{uu}$ is a symmetric weighting matrix related to the control effort

Some remarks on the cost functional J are in order.

First, minimization of J also minimizes αJ, where α is any positive constant. So the problem is not altered if we multiply the cost functional by any positive value.

Second, minimization of J will depend on the selection of the weighting matrices $\mathbf{W}_{zz}$ and $\mathbf{W}_{uu}$. The corresponding solution will then be strongly affected by the weighting matrices, which therefore play a fundamental role in the design process.

Finally, it is noted that the cost functional contains two contributions. The first is the quadratic form $\mathbf{z}^T\mathbf{W}_{zz}\mathbf{z}$, which represents the penalty on the deviation of the performance vector $\mathbf{z}$ from the origin. The second quadratic form $\mathbf{u}^T\mathbf{W}_{uu}\mathbf{u}$ represents the *cost of control* and is included in order to limit the magnitude of the control variables. In most practical cases, the control weighting matrix is selected large enough to avoid saturation of the control actuators under nominal conditions of operation.

The weighting matrices can be used to specify the relative importance of the various components of the performance vector and the control input vector. For example, for a system with two components of the performance vector z_1 and z_2, the selection of the following weighting matrix

$$\mathbf{W}_{zz} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{17.42}$$

will introduce a penalty for the first performance variable z_1 without imposing any restriction for z_2. Since the previous choice might lead to values of z_2 larger than desired, a limitation can be imposed by selecting the following diagonal weighting matrix

$$\mathbf{W}_{zz} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \tag{17.43}$$

where the weight c can be regulated to achieve a desired behavior of z_2 compared to the behavior of z_1. The same procedure can be applied to the control inputs. Referring to a LTI system with three control inputs u_1, u_2 and u_3, a diagonal control weighting matrix $\mathbf{W}_{uu}$ as follows

$$\mathbf{W}_{uu} = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix} \tag{17.44}$$

can be used to regulate the relative control effort of the three components by tuning the ratios ρ_2/ρ_1 and ρ_3/ρ_1.

It is also observed that it is very difficult and in most cases impractical to accurately predict the effect of the selection of a given pair of weighting matrices on the closed-loop dynamic response of the system. This is one of the most restrictive aspect of the LQR design. It means that there is a gap between what the LQR controller achieves and the desired control system performance. The optimization problem expressed by the minimization of the cost functional J may have very little to do with more meaningful control system specifications like levels of disturbance rejection, overshoot in tracking, stability margins, and so on. This aspect should always be kept in mind when using the LQR technique. A LQR design which is optimal does not imply that it meets the performance goals, since performance requirements are not given in terms of minimizing quadratic costs. It is the job of the designer to use the LQR tool wisely. According to what discussed, the design process is typically carried out iteratively, by starting from a guess pair of $\mathbf{W}_{zz}$ and $\mathbf{W}_{uu}$, then simulating the closed-loop response, and changing the weighting matrices so that the corresponding gain matrix $\mathbf{G}$ will produce the response closest to the design objectives.

17.3.3 Optimal solution

Let now present the result of the minimization of J. By referring to the dynamics of the system which can be managed by the control, the LTI model for the LQR design is the following

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)\\ \mathbf{z}(t) &= \mathbf{C}_z\mathbf{x}(t) + \mathbf{D}_{zu}\mathbf{u}(t)\end{aligned} \tag{17.45}$$

Based on the state feedback in Eq. (17.40), the corresponding closed-loop response is governed by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}_c\mathbf{x}(t)\\ \mathbf{z}(t) &= \mathbf{C}_{zc}\mathbf{x}(t)\end{aligned} \tag{17.46}$$

where

$$\mathbf{A}_c = \mathbf{A} - \mathbf{B}_u\mathbf{G} \qquad\qquad \mathbf{C}_{zc} = \mathbf{C}_z - \mathbf{D}_{zu}\mathbf{G} \tag{17.47}$$

The cost function (17.41) associated with the closed-loop system is then written as

$$\begin{aligned}J &= \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{C}_{zc}^T\mathbf{W}_{zz}\mathbf{C}_{zc}\mathbf{x} + \mathbf{x}^T\mathbf{G}^T\mathbf{W}_{uu}\mathbf{G}\mathbf{x}\right) dt\\ &= \frac{1}{2}\int_0^\infty \mathbf{x}^T\mathbf{W}(\mathbf{G})\mathbf{x}\, dt\end{aligned} \tag{17.48}$$

where

$$\begin{aligned}\mathbf{W}(\mathbf{G}) &= \mathbf{C}_{zc}^T\mathbf{W}_{zz}\mathbf{C}_{zc} + \mathbf{G}^T\mathbf{W}_{uu}\mathbf{G}\\ &= \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z - \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{D}_{zu}\mathbf{G} - \mathbf{G}^T\mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z + \mathbf{G}^T\left(\mathbf{W}_{uu} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}\right)\mathbf{G}\end{aligned} \tag{17.49}$$

If the initial state vector at time $t = 0$ is denoted as $\mathbf{x}_0$, the closed-loop response of the system is expressed as

$$\mathbf{x}(t) = e^{\mathbf{A}_c t}\mathbf{x}_0 \tag{17.50}$$

Note that the initial vector can be considered as the disturbance source of the motion of the system. Accordingly, the cost functional can be written as

$$\begin{aligned}J &= \frac{1}{2}\mathbf{x}_0^T\int_0^\infty e^{\mathbf{A}_c^T t}\mathbf{W}(\mathbf{G})e^{\mathbf{A}_c t}dt\,\mathbf{x}_0\\ &= \frac{1}{2}\mathbf{x}_0^T\mathbf{P}\mathbf{x}_0\end{aligned} \tag{17.51}$$

where the following *nonnegative symmetric* matrix $\mathbf{P}$ is introduced

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}_c^T t}\mathbf{W}(\mathbf{G})e^{\mathbf{A}_c t}dt \tag{17.52}$$

Since J is a scalar quantity, we can take the trace of J without changing the result. Therefore, the cost functional can be equivalently expressed by

$$J = \mathrm{Tr}\left[\frac{1}{2}\mathbf{x}_0^T\mathbf{P}\mathbf{x}_0\right] \tag{17.53}$$

Using the properties of the trace operator (see Appendix A), we can write

$$\begin{aligned}J &= \frac{1}{2}\mathrm{Tr}\left[\mathbf{x}_0^T\mathbf{P}\mathbf{x}_0\right]\\ &= \frac{1}{2}\mathrm{Tr}\left[\mathbf{x}_0\mathbf{x}_0^T\mathbf{P}\right]\\ &= \frac{1}{2}\mathrm{Tr}\left[\mathbf{P}\mathbf{X}_0\right]\end{aligned} \tag{17.54}$$

where $\mathbf{X}_0 = \mathbf{x}_0\mathbf{x}_0^T$ is the matrix containing all the combinations of initial conditions. The matrix $\mathbf{P}$ satisfies the following Lyapunov equation

$$\mathbf{A}_c^T\mathbf{P} + \mathbf{P}\mathbf{A}_c + \mathbf{W}(\mathbf{G}) = \mathbf{0} \tag{17.55}$$

This can be verified by inspection as follows

$$\begin{aligned}
&\int_0^\infty \mathbf{A}_c^T e^{\mathbf{A}_c^T t}\mathbf{W}(\mathbf{G})e^{\mathbf{A}_c t}dt + \int_0^\infty e^{\mathbf{A}_c^T t}\mathbf{W}(\mathbf{G})e^{\mathbf{A}_c t}\mathbf{A}_c dt + \mathbf{W}(\mathbf{G}) = \\
&\int_0^\infty \frac{d}{dt}\left[e^{\mathbf{A}_c^T t}\mathbf{W}(\mathbf{G})e^{\mathbf{A}_c t}\right]dt + \mathbf{W}(\mathbf{G}) = \\
&e^{\mathbf{A}_c^T t}\mathbf{W}(\mathbf{G})e^{\mathbf{A}_c t}\Big|_0^\infty + \mathbf{W}(\mathbf{G}) = \\
&-\mathbf{W}(\mathbf{G}) + \mathbf{W}(\mathbf{G}) = \mathbf{0}
\end{aligned}$$

Therefore, the minimization of J, which involves the matrix $\mathbf{P}$, must be performed by considering the Lyapunov equation (17.55) as a constraint equation. The minimization of J can be solved as well as a free minimization problem by using the Lagrange multipliers. Therefore, the new cost functional to be minimized is

$$J = \frac{1}{2}\mathrm{Tr}\left[\mathbf{P}\mathbf{X}_0 + \mathbf{\Lambda}\left(\mathbf{A}_c^T\mathbf{P} + \mathbf{P}\mathbf{A}_c + \mathbf{W}(\mathbf{G})\right)\right] \tag{17.56}$$

where $\mathbf{\Lambda}$ is the symmetric matrix of Lagrange multipliers. Since we have three unknown matrices, $\mathbf{\Lambda}$, $\mathbf{P}$ and $\mathbf{G}$, the minimization of J implies that

$$\begin{aligned}
\frac{\partial J}{\partial \mathbf{\Lambda}} &= \mathbf{0} \\
\frac{\partial J}{\partial \mathbf{P}} &= \mathbf{0} \\
\frac{\partial J}{\partial \mathbf{G}} &= \mathbf{0}
\end{aligned} \tag{17.57}$$

The first condition yields the constraint equation

$$\mathbf{A}_c^T\mathbf{P} + \mathbf{P}\mathbf{A}_c + \mathbf{W}(\mathbf{G}) = \mathbf{0} \tag{17.58}$$

Using the properties of the trace and its derivatives, we can write

$$\frac{\partial J}{\partial \mathbf{P}} = \frac{1}{2}\frac{\partial}{\partial \mathbf{P}}\mathrm{Tr}\left[\mathbf{X}_0\mathbf{P} + \mathbf{\Lambda}\mathbf{A}_c^T\mathbf{P} + \mathbf{A}_c\mathbf{\Lambda}\mathbf{P} + \mathbf{\Lambda}\mathbf{W}(\mathbf{G})\right]$$

Therefore, the second condition yields the following Lyapunov equation

$$\mathbf{A}_c\mathbf{\Lambda} + \mathbf{\Lambda}\mathbf{A}_c^T + \mathbf{X}_0 = \mathbf{0} \tag{17.59}$$

The partial derivative of the cost functional with respect to the gain matrix $\mathbf{G}$ can be written as

$$\begin{aligned}
\frac{\partial J}{\partial \mathbf{G}} &= \frac{1}{2}\frac{\partial}{\partial \mathbf{G}}\mathrm{Tr}\left[\mathbf{X}_0\mathbf{P} + \mathbf{\Lambda}\mathbf{A}_c^T\mathbf{P} + \mathbf{A}_c\mathbf{\Lambda}\mathbf{P} + \mathbf{W}(\mathbf{G})\right] \\
&= \frac{1}{2}\frac{\partial}{\partial \mathbf{G}}\mathrm{Tr}\left[\mathbf{X}_0\mathbf{P} + \mathbf{\Lambda}\mathbf{A}^T\mathbf{P} - \mathbf{\Lambda}\mathbf{G}^T\mathbf{B}_u^T\mathbf{P} + \mathbf{\Lambda}\mathbf{P}\mathbf{A} - \mathbf{\Lambda}\mathbf{P}\mathbf{B}_u\mathbf{G}\right. \\
&\qquad + \mathbf{\Lambda}\mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z - \mathbf{\Lambda}\mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{D}_{zu}\mathbf{G} - \mathbf{\Lambda}\mathbf{G}^T\mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z \\
&\qquad \left. + \mathbf{\Lambda}\mathbf{G}^T\left(\mathbf{W}_{uu} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}\right)\mathbf{G}\right] \\
&= \frac{1}{2}\frac{\partial}{\partial \mathbf{G}}\mathrm{Tr}\left[\mathbf{X}_0\mathbf{P} + \mathbf{\Lambda}\mathbf{A}^T\mathbf{P} - \mathbf{B}_u^T\mathbf{P}\mathbf{\Lambda}\mathbf{G}^T + \mathbf{\Lambda}\mathbf{P}\mathbf{A} - \mathbf{G}^T\mathbf{B}_u^T\mathbf{P}\mathbf{\Lambda}\right. \\
&\qquad + \mathbf{\Lambda}\mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z - \mathbf{G}^T\mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z\mathbf{\Lambda} - \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z\mathbf{\Lambda}\mathbf{G}^T \\
&\qquad \left. + \mathbf{\Lambda}\mathbf{G}^T\left(\mathbf{W}_{uu} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}\right)\mathbf{G}\right] \\
&= \frac{1}{2}\frac{\partial}{\partial \mathbf{G}}\mathrm{Tr}\left[\mathbf{X}_0\mathbf{P} + \mathbf{\Lambda}\mathbf{A}^T\mathbf{P} - \mathbf{B}_u^T\mathbf{P}\mathbf{\Lambda}\mathbf{G}^T + \mathbf{\Lambda}\mathbf{P}\mathbf{A} - \mathbf{B}_u^T\mathbf{P}\mathbf{\Lambda}\mathbf{G}^T\right. \\
&\qquad + \mathbf{\Lambda}\mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z - \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z\mathbf{\Lambda}\mathbf{G}^T - \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z\mathbf{\Lambda}\mathbf{G}^T \\
&\qquad \left. + \mathbf{G}^T\left(\mathbf{W}_{uu} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}\right)\mathbf{G}\mathbf{\Lambda}\right]
\end{aligned}$$

The minimum condition is then expressed by

$$-\mathbf{B}_u^T\mathbf{P}\mathbf{\Lambda} - \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z\mathbf{\Lambda} + \left(\mathbf{W}_{uu} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}\right)\mathbf{G}\mathbf{\Lambda} = \mathbf{0} \tag{17.60}$$

From this last equation, we can derive the expression for the feedback gain matrix

$$\mathbf{G} = \left(\mathbf{W}_{uu} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}\right)^{-1}\left(\mathbf{B}_u^T\mathbf{P} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z\right) \tag{17.61}$$

The above equation can be written in a more compact form as follows. It is noted that the initial form of the cost functional (17.41) is expressed in terms of the performance vector $\mathbf{z}$ and the control input vector $\mathbf{u}$. Using the performance equation, J can be written in terms of the state vector $\mathbf{x}$ and the control input $\mathbf{u}$. We have

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z\mathbf{x} + \mathbf{x}^T\mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{D}_{zu}\mathbf{u} + \mathbf{u}^T\mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{C}_z\mathbf{x} + \mathbf{u}^T\mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}\mathbf{u} + \mathbf{u}^T\mathbf{W}_{uu}\mathbf{u}\right)dt$$

or, in more compact notation,

$$\begin{aligned} J &= \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{S}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt \\ &= \frac{1}{2}\int_0^\infty \begin{Bmatrix}\mathbf{x}\\ \mathbf{u}\end{Bmatrix}^T \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^T & \mathbf{R}\end{bmatrix}\begin{Bmatrix}\mathbf{x}\\ \mathbf{u}\end{Bmatrix} dt \end{aligned} \tag{17.62}$$

where

- $\mathbf{Q} = \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z$ is the nonnegative symmetric weighting matrix related to the states
- $\mathbf{R} = \mathbf{W}_{uu} + \mathbf{D}_{zu}^T\mathbf{W}_{zz}\mathbf{D}_{zu}$ is the positive definite symmetric weighting matrix related to the control effort
- $\mathbf{S} = \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{D}_{zu}$ is the symmetric coupled weighting matrix

Accordingly, the gain matrix $\mathbf{G}$ can be written as

$$\mathbf{G} = \mathbf{R}^{-1}\left(\mathbf{B}_u^T\mathbf{P} + \mathbf{S}^T\right) \tag{17.63}$$

where it can be observed that the matrix $\mathbf{R}$ must be positive definite in order the compute the solution. If the coupled weighting matrix $\mathbf{S}$ is equal to zero (this occurs if the feedthrough matrix $\mathbf{D}_{zu}$ in the performance equation is null), the solution for the feedback gains is reduced to

$$\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} \tag{17.64}$$

The gain matrix can be determined using Eq. (17.63) or Eq. (17.64) if the matrix $\mathbf{P}$ is known. The $\mathbf{P}$ matrix can be computed by putting the gain solution into the constraint Lyapunov equation. It follows that

$$\begin{aligned} &\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} - \mathbf{S}\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{S}^T \\ &+ \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{S}^T - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T \\ &+ \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u\mathbf{P} + \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{S}^T + \mathbf{S}\mathbf{R}^{-1}\mathbf{B}_u\mathbf{P} + \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T = \mathbf{0} \end{aligned}$$

and, deleting the opposite terms, we obtain

$$\mathbf{P}\left(\mathbf{A} - \mathbf{B}_u\mathbf{R}^{-1}\mathbf{S}^T\right) + \left(\mathbf{A}^T - \mathbf{S}\mathbf{R}^{-1}\mathbf{B}_u^T\right)\mathbf{P} - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} + \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T = \mathbf{0} \tag{17.65}$$

which is known as the algebraic *Riccati equation*. When the coupled weighting matrix $\mathbf{S}$ is null, the Riccati equation for the computation of the $\mathbf{P}$ matrix is expressed as

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{17.66}$$

As a summary, the design of the LQR control involves the following steps:

1. Select the performance $\mathbf{z}$ and write the corresponding performance equation

2. Select the weighting matrices $\mathbf{W}_{zz}$ and $\mathbf{W}_{uu}$ associated with the penalty of the performance vector and the cost of control, respectively
3. Compute the corresponding weighting matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$
4. Solve the Riccati equation (17.65) to compute the matrix $\mathbf{P}$
5. Compute the gain matrix $\mathbf{G}$ using Eq. (17.63)

Note that the cost functional expressed in Eq. (17.62) can be always written in a form which does not involve the coupled quadratic forms. Indeed, if we perform the following transformation

$$\begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{T} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix} \tag{17.67}$$

the cost functional in Eq. (17.62) is written as

$$\begin{aligned} J &= \frac{1}{2} \int_0^\infty \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix}^T \begin{bmatrix} \mathbf{I} & \mathbf{T}^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{T} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix} dt \\ &= \frac{1}{2} \int_0^\infty \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix}^T \begin{bmatrix} \mathbf{Q} + \mathbf{S}\mathbf{T} + \mathbf{T}^T\mathbf{S}^T + \mathbf{T}^T\mathbf{R}\mathbf{T} & \mathbf{S} + \mathbf{T}^T\mathbf{R} \\ \mathbf{S}^T + \mathbf{R}\mathbf{T} & \mathbf{R} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix} dt \end{aligned} \tag{17.68}$$

The coupling terms are set to zero by imposing

$$\mathbf{S}^T\mathbf{T}\mathbf{R} = \mathbf{0} \tag{17.69}$$

Therefore, the transformation matrix $\mathbf{T}$ in order to avoid coupling between state and control inputs in the LQR cost functional is given by

$$\mathbf{T} = -\mathbf{R}^{-1}\mathbf{S}^T \tag{17.70}$$

According, the resulting J is expressed as

$$\begin{aligned} J &= \frac{1}{2} \int_0^\infty \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix}^T \begin{bmatrix} \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix} dt \\ &= \frac{1}{2} \int_0^\infty \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix}^T \begin{bmatrix} \hat{\mathbf{Q}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{u}} \end{Bmatrix} dt \end{aligned} \tag{17.71}$$

17.3.4 Remarks

It can be shown, provided that $(\mathbf{A}, \mathbf{B}_u)$ is stabilizable and $(\sqrt{\mathbf{Q}}, \mathbf{A})$ is detectable, that:

1. There is a unique symmetric positive-definite solution $\mathbf{P}$ of the algebraic Riccati equation (17.66).
2. The closed-loop system $\mathbf{A}_c$ is asymptotically stable.

This result means that, as long as the system and the performance index satisfy certain basic controllability and observability requirements, the steady-state LQR control will yield gains that stabilize the system. Considering the difficulty encountered by classical control techniques in stabilizing multi-input systems, this is a remarkable property. As already outlined, the closed-loop poles will depend on the selection of the design matrices $\mathbf{Q}$ and $\mathbf{R}$; however, the poles will always be stable as long as the designer selects $\mathbf{R} > 0$ and $\mathbf{Q} \geq 0$ with $(\sqrt{\mathbf{Q}}, \mathbf{A})$ observable. Thus, the elements of $\mathbf{Q}$ and $\mathbf{R}$ may be varied during an interactive computer-aided design procedure to obtain suitable closed-loop performance.

In addition to guaranteed closed-loop stability and ease of design by solving matrix design equations, the steady-state LQR approach has also certain guaranteed robustness properties. In particular, it can be shown that:

1. The LQR control with full state feedback has an *infinite gain margin*.
2. The LQR control with full state feedback has a *guaranteed phase margin of at least 60^o*.

17.3.5 Example

The LQR solution procedure is illustrated by the following simple example. It consists of an undamped SDOF oscillator with mass m and unit stiffness $k = 1$. Without any control, initial conditions on the displacement or the velocity of the mass will produce a persistent oscillatory motion at a frequency $\omega = 1/\sqrt{m}$. Let assume that we want to regulate the position of the mass to its equilibrium value $x(t) = 0$ using a control force $u(t)$ applied onto the mass. The equations of motion can be written as

$$\dot{\mathbf{x}}(t) = \begin{Bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} + \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix} u(t)$$

The performance variable is the displacement $x(t)$ of the mass, i.e.,

$$z(t) = x(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

The cost functional in this case can be expressed as

$$J = \frac{1}{2} \int_0^\infty \left(z^2 + \rho u^2 \right) dt$$

where $W_{zz} = 1$ and $W_{uu} = \rho$. The value of ρ can be tuned to achieve a balance between the performance and the control effort. Using the performance equation, the cost functional can be put into the following form

$$J = \frac{1}{2} \int_0^\infty \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \rho u^2 \right) dt$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{R} = \rho$$

It is noted that the coupling weighting term is lacking. The LQR controller is implemented as

$$u(t) = -\mathbf{g}\mathbf{x}(t) = -\begin{bmatrix} g_1 & g_2 \end{bmatrix} \mathbf{x}(t)$$

where

$$\mathbf{g} = \frac{1}{\rho} \begin{bmatrix} 0 & \omega^2 \end{bmatrix} \mathbf{P} = \frac{1}{\rho} \begin{bmatrix} 0 & \omega^2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

and the matrix $\mathbf{P}$ is the solution of the following Riccati equation

$$\begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{\rho} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix} \begin{bmatrix} 0 & \omega^2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Carrying out the matrix multiplication leads to the following three equations

$$\begin{cases} 2\omega^2 p_{12} - 1 + \dfrac{1}{\rho} \omega^4 p_{12}^2 = 0 \\ p_{11} - \omega^2 p_{22} - \dfrac{1}{\rho} \omega^4 p_{12} p_{22} = 0 \\ 2p_{12} - \dfrac{1}{\rho} \omega^4 p_{22}^2 = 0 \end{cases}$$

Since the matrix $\mathbf{P}$ must be nonnegative, we must have

$$p_{11} \geq 0 \qquad p_{11} p_{22} - p_{12}^2 \geq 0$$

From the second inequality, we find that $p_{22} \geq 0$ and, from the last of the three previous equations, also $p_{12} \geq 0$. Therefore, the solution is given by

$$p_{12} = \frac{-\rho + \rho\sqrt{1 + \frac{1}{\rho}}}{\omega^2}$$
$$p_{22} = \sqrt{\frac{2\rho}{\omega^4} p_{12}} = \frac{\rho}{\omega^3}\sqrt{2\left(\sqrt{1 + \frac{1}{\rho}} - 1\right)}$$
$$p_{11} = \omega^2 p_{22} + \frac{1}{\rho}\omega^4 p_{12} p_{22} = \frac{\rho}{\omega}\sqrt{2\left(1 + \frac{1}{\rho}\right)\left(\sqrt{1 + \frac{1}{\rho}} - 1\right)}$$

and the control gains are computed in terms of the control weighting ρ as

$$g_1 = \sqrt{1 + \frac{1}{\rho}} - 1$$
$$g_2 = \frac{1}{\omega}\sqrt{2\left(\sqrt{1 + \frac{1}{\rho}} - 1\right)}$$

According to the designed LQR control, the closed-loop dynamics is governed by the system matrix

$$\mathbf{A}_c = \mathbf{A} - \mathbf{b}_u \mathbf{g} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2(1 + g_1) & -\omega^2 g_2 \end{bmatrix}$$

The corresponding closed-loop poles are given by

$$s_{1,2} = -\frac{\omega}{2}\sqrt{2\left(\sqrt{1 + \frac{1}{\rho}} - 1\right)} \pm j\frac{\omega}{2}\sqrt{2\left(\sqrt{1 + \frac{1}{\rho}} + 1\right)}$$

As the value of ρ decreases from ∞ to 0, the closed-loop poles will move in the complex plane by following the path represented in Figure xxx. Note that, when $\rho = \infty$, the poles are purely imaginary, since the penalty on the control effort is so large that the closed-loop response is equal to the open-loop response. It can be observed that the poles move to the left half-plane, so that the open-loop oscillating behavior is replaced by a damped response. As ρ decreases, the system becomes more and more fast. However, the power demanded to the control action is larger and larger. This is clear by looking at the expression of the gains g_1 and g_2.

17.4 Steady-state tracking

The state feedback law $\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t)$, by proper tuning of the gain matrix $\mathbf{G}$ through pole placement or LQR design, can conveniently modify the transient response of a dynamic system according to some specified requirements. However, there is no direct control on the steady-state value of the system. Steady-state tracking can be of utmost importance in some applications, i.e., servomechanisms. The design of state feedback control systems involving steady-state tracking is addressed in the following through two different approaches.

17.4.1 Feedforward input

This approach is based on writing the feedback law in this form

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{M}\mathbf{r}(t) \tag{17.72}$$

where $\mathbf{r}(t)$ is the reference signal and $\mathbf{M}$ is a new gain matrix to be determined so that the steady-state error for step reference input $\mathbf{r}(t) = \mathbf{r}$ is equal to zero. It is clear that the control action expressed in Eq. (17.72) contains two contributions. The first term is the usual feedback law proportional to the state vector. The corresponding gain matrix $\mathbf{G}$

is designed as before using a pole placement technique or a LQR approach. The second term in Eq. (17.72) represents a *feedforward control input* proportional to the reference input through the gain matrix $\mathbf{M}$. To obtain an expression for $\mathbf{M}$, we can proceed as follows.

Using Eq. (17.72), the closed-loop system dynamics is written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}_u\mathbf{G})\,\mathbf{x}(t) + \mathbf{B}_u\mathbf{M}\mathbf{r} \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{17.73}$$

For a constant reference input, the steady-state condition corresponds to an equilibrium condition for the closed-loop state equation involving an equilibrium state denoted as $\mathbf{x}_{\text{ss}}$ that satisfies

$$\mathbf{0} = (\mathbf{A} - \mathbf{B}_u\mathbf{G})\,\mathbf{x}_{\text{ss}} + \mathbf{B}_u\mathbf{M}\mathbf{r} \tag{17.74}$$

Therefore, the steady state solution $\mathbf{x}_{\text{ss}}$ is given by

$$\mathbf{x}_{\text{ss}} = -\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\right)^{-1}\mathbf{B}_u\mathbf{M}\mathbf{r} \tag{17.75}$$

It is required that the closed-loop system matrix $\mathbf{A} - \mathbf{B}_u\mathbf{G}$ is capable of providing an asymptotically stable system, i.e., the gain matrix $\mathbf{G}$ is selected to have strictly negative real-part eigenvalues. The corresponding steady-state output is

$$\mathbf{y}_{\text{ss}} = \mathbf{C}\mathbf{x}_{\text{ss}} = -\mathbf{C}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\right)^{-1}\mathbf{B}_u\mathbf{M}\mathbf{r} \tag{17.76}$$

By imposing that $\mathbf{y}_{\text{ss}} = \mathbf{r}$, we obtain

$$-\mathbf{C}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\right)^{-1}\mathbf{B}_u\mathbf{M} = \mathbf{I} \tag{17.77}$$

which is used to express the gain matrix $\mathbf{M}$ as

$$\mathbf{M} = -\left[\mathbf{C}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\right)^{-1}\mathbf{B}_u\right]^{-1} \tag{17.78}$$

Note that the previous result is valid for MIMO systems, provided that the open-loop state equation has at least as many inputs as the number of outputs. It is also noted that the matrix $\mathbf{C}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\right)^{-1}\mathbf{B}_u$ must be nonsingular.

17.4.2 Integral action

The method previously described requires accurate knowledge of the open-loop state equations coefficient matrices in order to obtain $\mathbf{M}$. In real situations, there are many aspects (model uncertainty, parameter variations, approximations,...) that result in deviations between the nominal coefficient matrices and the actual system. Thus, a significant difference between the actual and the estimated steady-state behaviour can arise. Hence, more robust methods to design state-space feedback control systems with steady-state tracking capabilities that can deal with system parameters uncertainties are needed.

One common approach is based on adding an integral term to the state feedback law. For the sake of simplicity, the procedure is presented in the following for SISO systems.

The integral action is provided by assuming a control law in this form

$$u(t) = -\mathbf{g}\mathbf{x}(t) - g_I\int e(t)dt \tag{17.79}$$

where

$$e(t) = r - y(t) \tag{17.80}$$

is the tracking error with respect to the step reference input r. After introducing a new state variable corresponding to the integral term as

$$x_I(t) = \int e(t)dt \tag{17.81}$$

the control law can be written as

$$u(t) = -\begin{bmatrix}\mathbf{g} & g_I\end{bmatrix}\begin{Bmatrix}\mathbf{x}(t) \\ x_I(t)\end{Bmatrix} \tag{17.82}$$

which can be interpreted as a state feedback law involving the augmented state vector consisting of the open-loop state vector $\mathbf{x}(t)$ together with the integrator state variable $x_I(t)$. According to the definition in Eq. (17.81), the integrator state is governed by the following dynamics

$$\dot{x}_I(t) = e(t) = r - y(t) = r - \mathbf{c}\mathbf{x}(t) \tag{17.83}$$

Therefore, the dynamics of the augmented state vector is expressed as

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{x}_I(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{c} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ x_I(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r + \begin{bmatrix} \mathbf{b}_u \\ 0 \end{bmatrix} u \tag{17.84}$$

Once the gains are selected, the resulting closed-loop dynamics is given by

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{x}_I(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{b}_u\mathbf{g} & -\mathbf{b}_u g_I \\ -\mathbf{c} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ x_I(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \tag{17.85}$$

17.5 State reconstruction: the linear observer

The pole-placement technique and the LQR control are based on the feedback of the *full* state vector. Unfortunately, in many practical situations, $\mathbf{x}(t)$ is not fully directly accessible to measurement.

In some cases, it is possible to have access to a subset of state variables and to estimate those state variables that are not accessible to measurement using the measurement data from those state variables that are accessible. Let consider for example the rigid inverted pendulum previously discussed. The dynamics of the system can be written in terms of the cart displacement $x(t)$ and the small rotation angle $\theta(t)$ of the pendulum beam. The corresponding state-space formulation will also involve the related velocity variables (i.e., velocity of the cart and angular velocity of the beam). If the pendulum is equipped with a sensor measuring the displacement of the cart and another sensor measuring the rotation of the beam, the state variables $\dot{x}(t)$ and $\dot{\theta}(t)$ can be estimated by taking the time derivative of the displacement and rotation signals. Note that, in some circumstances, estimates of some state variables may even be preferable to direct measurements, since the noise associated with the real measurement device can be larger than the error introduced in the above estimation.

The previous condition is rarely met when flexible structures are considered. Indeed, as extensively discussed in previous chapters, the flexible dynamics is typically described by a modal representation. As such, the state vector will contain modal coordinates $q(t)$ and modal velocities $\dot{q}(t)$. Since modal sensors are typically not available or are very difficult to be designed, no state variable is directly accessible to measurement. This implies that there is the need of estimating in some way the entire state vector. This job is undertaken by the so-called *linear observer*.

A linear observer is a dynamic system whose state variables are the estimates of the state variables of another system. Thus, the aim of the observer is to reconstruct the state vector from a model of the system. The approach we will show in the following is based on the reconstruction of $\mathbf{x}(t)$ from the input vector $\mathbf{u}(t)$ and the output measurement $\mathbf{y}(t)$.

Let consider that an accurate model of the system is available as follows

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}_y\mathbf{x}(t) \end{aligned} \tag{17.86}$$

Note that the direct feedthrough term $\mathbf{D}_{yu}\mathbf{u}(t)$ is lacking. Indeed, the more generic output

$$\mathbf{y}(t) = \mathbf{C}_y\mathbf{x}(t) + \mathbf{D}_{yu}\mathbf{u}(t) \tag{17.87}$$

can be treated by defining a modified output

$$\hat{\mathbf{y}}(t) = \mathbf{y}(t) - \mathbf{D}_{yu}\mathbf{u}(t) \tag{17.88}$$

and working with $\hat{\mathbf{y}}(t)$ instead of $\mathbf{y}(t)$, since the feedthrough term depends on the input variable $\mathbf{u}(t)$, which is known and controllable. Note also that the direct coupling $\mathbf{D}_{yu}\mathbf{u}$ from the input to the output is absent in most physical plants.

The equation of the full-state observer is assumed to be represented by

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t) \tag{17.89}$$

where $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}_u$ and $\mathbf{L}$ are undetermined matrices. The output of the above dynamic system is the estimate of the state vector $\hat{\mathbf{x}}(t)$ at any time instant t. Note that the state estimation is achieved by using as inputs the control input vector $\mathbf{u}(t)$ and the output vector $\mathbf{y}(t)$ arising from measurement.

Since a perfect estimate of the state is impossible, there will be a difference between the actual state $\mathbf{x}$ and the reconstructed state $\hat{\mathbf{x}}$. This can be considered as an error associated with the observer and is called *observation error*. It is defined as

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) \tag{17.90}$$

After taking the time derivative of Eq. (17.90), the dynamics of the error is represented by

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) - \hat{\mathbf{A}}\hat{\mathbf{x}}(t) - \hat{\mathbf{B}}_u\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) - \hat{\mathbf{A}}\left[\mathbf{x}(t) - \mathbf{e}(t)\right] - \hat{\mathbf{B}}_u\mathbf{u}(t) - \mathbf{L}\mathbf{C}_y\mathbf{x}(t) \\ &= \hat{\mathbf{A}}\mathbf{e}(t) + \left(\mathbf{A} - \hat{\mathbf{A}} - \mathbf{L}\mathbf{C}_y\right)\mathbf{x}(t) + \left(\mathbf{B}_u - \hat{\mathbf{B}}_u\right)\mathbf{u}(t)\end{aligned} \tag{17.91}$$

If we demand that the error go to zero asymptotically, independent to $\mathbf{x}$ and $\mathbf{u}$, then the coefficients of $\mathbf{x}$ and $\mathbf{u}$ in Eq. (17.91) must be zero and the state matrix $\hat{\mathbf{A}}$ must correspond to an asymptotically stable system. This implies that

$$\begin{aligned}\hat{\mathbf{A}} &= \mathbf{A} - \mathbf{L}\mathbf{C}_y \\ \hat{\mathbf{B}}_u &= \mathbf{B}_u\end{aligned} \tag{17.92}$$

Therefore, we started from an assumed model of the observer with arbitrary matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}_u$ and $\mathbf{L}$, and we have found that we cannot pick them arbitrarily. Indeed, there is no choice at all in the selection of $\hat{\mathbf{B}}_u$: it must be the input matrix $\mathbf{B}_u$. Once the matrix $\mathbf{L}$ is selected, the matrix $\hat{\mathbf{A}}$ is determined. The only freedom is the selection of the matrix $\mathbf{L}$, which is the design process of the observer.

According to Eq. (17.92), we can write the dynamics of the observation error as

$$\dot{\mathbf{e}}(t) = \left(\mathbf{A} - \mathbf{L}\mathbf{C}_y\right)\mathbf{e}(t) \tag{17.93}$$

and the equation of the observer as follows

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \left(\mathbf{A} - \mathbf{L}\mathbf{C}_y\right)\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t) \\ &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{L}\left[\mathbf{y}(t) - \mathbf{C}_y\hat{\mathbf{x}}(t)\right]\end{aligned} \tag{17.94}$$

The last equation shows that the observer has the same form of the process, with an additional input represented by $\mathbf{L}\left[\mathbf{y}(t) - \mathbf{C}_y\hat{\mathbf{x}}(t)\right]$, where $\mathbf{y}(t) - \mathbf{C}_y\hat{\mathbf{x}}(t)$ is the difference between the actual output and the estimated output. The additional input can be considered as a feedback term proportional to the observation error

$$\mathbf{L}\mathbf{C}_y\left[\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right] = \mathbf{L}\mathbf{C}_y\mathbf{e}(t) \tag{17.95}$$

and the matrix $\mathbf{L}$ can be considered as the *observer gain matrix*.

17.6 Observer design by pole-placement

As shown in the last section, the equation governing the dynamics of the observation error shows that the error goes to zero if the matrix $\hat{\mathbf{A}} = \mathbf{A} - \mathbf{L}\mathbf{C}_y$ corresponds to an asymptotically stable system, i.e., all eigenvalues of $\hat{\mathbf{A}}$ have negative real parts. The eigenvalues of $\hat{\mathbf{A}}$ are typically called *observer poles*.

Therefore, the design of the observer can be carried out by selecting the matrix $\mathbf{L}$ such that the matrix

$$\mathbf{A} - \mathbf{L}\mathbf{C}_y \tag{17.96}$$

has poles at desired locations. By comparing this approach with the pole-placement technique used in the design of the controller, we have

(controller)	(observer)
poles of $\mathbf{A} - \mathbf{B}_u\mathbf{G}$	poles of $\mathbf{A} - \mathbf{L}\mathbf{C}_y$

Since the eigenvalues of a matrix are equal to the eigenvalues of the matrix transposed, the observer poles can be determined on

$$\mathbf{A}^T - \mathbf{C}_y^T \mathbf{L}^T \tag{17.97}$$

Therefore, the comparison is

(controller)	(observer)
poles of $\mathbf{A} - \mathbf{B}_u \mathbf{G}$	poles of $\mathbf{A}^T - \mathbf{C}_y^T \mathbf{L}^T$

Now it is evident that the design of the observer involves the same steps as the design of the controller with $\mathbf{A}^T$ instead of $\mathbf{A}$ and $\mathbf{C}_y^T$ instead of $\mathbf{B}_u$. By using the MATLAB command for pole-placement, we have

```
Lt = place(A',C',po)
```

where *po* is the vector of desired observer poles. Note that the output of the previous command is $\mathbf{L}^T$.

We have seen that the system must be controllable in order to place the closed-loop poles anywhere we wish in the complex plane. Likewise, *the system must be observable* in order to locate arbitrarily the observer poles.

If one wants the controller poles to dominate the closed-loop response, the observer poles should be faster than the controller poles. This means that the observer poles must be to the left of the controller poles. By doing so, the observation error will decay faster than the desired closed-loop dynamics, so that when the state feedback control law is applied as

$$\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}(t) \tag{17.98}$$

the estimated state follows closely the actual state. As a rule of thumb, an ideal design would require that the observer poles should be 2 to 6 times faster than the controller poles.

The above lower limitation is related to the requirement of having the reconstructed state close to the actual state vector when the loop is closed. On the other hand, the upper limit is due to noisy measurements. Indeed, if the observer poles are too fast, the bandwidth of the observer is so large that the measurement noise will strongly affect the state reconstruction. As a result, the reconstructed state will be very noisy since we are amplifying the effect of noise using large observer gains $\mathbf{L}$. Therefore, the bandwidth of the observer must be decreased in order to produce some filtering of the measurement noise. The disadvantage of this limitation is that the observer poles will have a more significant influence on the closed-loop response.

It is clear that the observer design arises from a compromise between fast convergence of the reconstruction error and noisy measurements. A later section will present an *optimal* compromise.

17.7 Compensator design: the separation principle

In the previous sections, it has been shown that a LTI system of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}_y \mathbf{x}(t) \end{aligned} \tag{17.99}$$

can have a desired closed-loop response when the control input $\mathbf{u}(t)$ is taken as being proportional to the state vector $\mathbf{x}(t)$ through a properly designed constant gain matrix $\mathbf{G}$. Since the actual state is not directly available from measurements, it can be reconstructed using a linear observer. Therefore, the control action is performed by the following equations

$$\begin{aligned} \mathbf{u}(t) &= -\mathbf{G}\hat{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) &= (\mathbf{A} - \mathbf{L}\mathbf{C}_y)\,\hat{\mathbf{x}}(t) + \mathbf{B}_u \mathbf{u}(t) \mathbf{L}\mathbf{y}(t) \end{aligned} \tag{17.100}$$

where the observer gain matrix $\mathbf{L}$ can be designed by placing the observer poles at desired locations.

According to the procedures outlined before, the full-state feedback control has been designed by assuming that the full state vector is available and the linear observer has been designed without considering the feedback controller. We have implicitly assumed that the controller design and the observer design can be carried out *separately* or independently, in such a way that one design does not affect the other. This actually *does* work as shown in the following.

Since the reconstructed state $\hat{\mathbf{x}}(t)$ can be expressed in terms of the observation error as

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{e}(t) \tag{17.101}$$

the feedback law is given by

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{G}\mathbf{e}(t) \tag{17.102}$$

Therefore, the closed-loop dynamics is written as

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}_u\mathbf{G})\,\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{e}(t) \tag{17.103}$$

By coupling the above equation with the equation governing the dynamics of the observation error, one can write

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}_u\mathbf{G} & \mathbf{B}_u\mathbf{G} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C}_y \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{Bmatrix} \tag{17.104}$$

Since the above system matrix is block triangular, the eigenvalues of the closed-loop system are the eigenvalues of the diagonal blocks $\mathbf{A} - \mathbf{B}_u\mathbf{G}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}_y$, i.e.,

$$\det(s\mathbf{I} - \mathbf{A}) = \det(s\mathbf{I} - \mathbf{A} + \mathbf{B}_u\mathbf{G})\det(s\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_y) \tag{17.105}$$

As a result, the poles of the controller are not affected by the poles of the observer (also the opposite is true) when the two subsystems are put together and the controller design and the observer design can be carried out independently. This is known as *separation principle*, which assures that the poles of the closed-loop dynamic system will be the poles of the full-state feedback system and those selected for the state reconstruction.

The combination of the state feedback controller and the observer gives rise to the so-called *compensator*. Using the feedback law in the observer equation, one can write

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C}_y - \mathbf{B}_u\mathbf{G})\,\hat{\mathbf{x}}(t) + \mathbf{L}\mathbf{y}(t) \tag{17.106}$$

In the Laplace domain we have the following transfer function

$$\hat{\mathbf{x}}(s) = (s\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_y\mathbf{B}_u\mathbf{G})^{-1}\,\mathbf{L}\mathbf{y}(s) \tag{17.107}$$

Since $\mathbf{u}(s) = -\mathbf{G}\hat{\mathbf{x}}(s)$, the transfer function of the compensator is given by

$$\mathbf{u}(s) = -\mathbf{G}\,(s\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{C}_y\mathbf{B}_u\mathbf{G})^{-1}\,\mathbf{L}\mathbf{y}(s) \tag{17.108}$$

In this way, we can represent the control scheme as done in classical methods (see Figure xxx). Note that, for multiple-input multiple-output system, the compensator is given by a matrix of transfer functions. Note also that, differently to the design procedure adopted in classical methods where the structure and shaping of the compensator is imposed directly to achieve desired closed-loop properties, the compensator in Eq. (17.108) is never directly addressed and will result from the design of the controller and the observer as previously discussed. Indeed, the compensator is always of the same order as the system.

17.8 Spillover effects

It is crucial to highlight that the separation principle applies only when the model of the process used in the observer agrees exactly with the actual dynamics of the physical process. It is not possible to meet this requirement in practice and, hence, the separation principle is an approximation at best. This condition can be particularly critical when dealing with control of flexible structures.

We have extensively seen in previous chapters that flexible structures are distributed parameter systems having an infinite number of degrees of freedom. Since the exact treatment of continuous systems is unpractical in many real situations, the structure is spatially discretized by a finite number of generalized coordinates using appropriate modeling tools. Furthermore, since the number of degrees-of-freedom of the related discretized model typically is too high, a reduced-order model is developed, which includes the few dominant low-frequency modes. This reduced-order modal model is used in the control design. The flexible modes not included in the design model are called residual modes. Therefore, even if the sensors and actuators adopted to implement the control system can be considered to be ideal since their own dynamics is far beyond the control bandwidth, the model of the flexible structure does not agree exactly with the actual dynamics of the physical system due to the presence of the residual modes. Since in many practical applications those modes are inherently lightly damped, there is a danger that the state feedback will affect the residual modes such that an instability occurs. The effect of the control on the residual dynamics is known as *spillover*.

In order to show the spillover effect, the state vector is partitioned into controlled modes, which are included in the control design model, and residual modes, such that the dynamics of the system is written as

$$\begin{aligned}\dot{\mathbf{x}}_c(t) &= \mathbf{A}_c\mathbf{x}_c(t) + \mathbf{B}_{uc}\mathbf{u}(t)\\ \dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_{ur}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}_{yc}\mathbf{x}_c(t) + \mathbf{C}_{yr}\mathbf{x}_r(t)\end{aligned} \tag{17.109}$$

where $\mathbf{x}_c$ and $\mathbf{x}_r$ are the states referred to controlled and residual modes, and

$$\mathbf{A}_c = \text{Diag}\,\{\mathbf{A}_i\} \qquad (i = 1, \dots, n_c) \qquad \mathbf{A}_r = \text{Diag}\,\{\mathbf{A}_i\} \qquad (i = n_c + 1, \dots, n_c + n_r) \tag{17.110}$$

where each block is

$$\mathbf{A}_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\xi_i\omega_i \end{bmatrix} \tag{17.111}$$

Accordingly, the state feedback is expressed as

$$\mathbf{u}(t) = -\mathbf{G}\hat{\mathbf{x}}_c(t) = -\mathbf{G}\mathbf{x}_c(t) + \mathbf{G}\mathbf{e}(t) \tag{17.112}$$

and the closed-loop controlled and residual dynamics are given, respectively, by

$$\dot{\mathbf{x}}_c(t) = (\mathbf{A}_c - \mathbf{B}_{uc}\mathbf{G})\,\mathbf{x}_c(t) + \mathbf{B}_{uc}\mathbf{G}\mathbf{e}(t) \tag{17.113}$$

and

$$\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t) - \mathbf{B}_{ur}\mathbf{G}\mathbf{x}_c(t) + \mathbf{B}_{ur}\mathbf{G}\mathbf{e}(t) \tag{17.114}$$

The error dynamics is written as

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}_c(t) - \dot{\hat{\mathbf{x}}}_c(t)\\ &= \mathbf{A}_c\mathbf{x}_c(t) + \mathbf{B}_{uc}\mathbf{u}(t) - \hat{\mathbf{A}}_c\hat{\mathbf{x}}_c(t) - \hat{\mathbf{B}}_{uc}\mathbf{u}(t) - \mathbf{L}\mathbf{u}(t)\\ &= \mathbf{A}_c\mathbf{x}_c(t) + \mathbf{B}_{uc}\mathbf{u}(t) - \hat{\mathbf{A}}_c\mathbf{x}_c(t) + \hat{\mathbf{A}}_c\mathbf{e}(t) - -\hat{\mathbf{B}}_{uc}\mathbf{u}(t) - \mathbf{L}\mathbf{C}_{yc}\mathbf{x}_c(t) - \mathbf{L}\mathbf{C}_{yr}\mathbf{x}_r(t)\\ &= \hat{\mathbf{A}}_c\mathbf{e}(t) + \left(\mathbf{A}_c - \hat{\mathbf{A}}_c - \mathbf{L}\mathbf{C}_{yc}\right)\mathbf{x}_c(t) + \left(\mathbf{B}_{uc} - \hat{\mathbf{B}}_{uc}\right)\mathbf{u}(t) - \mathbf{L}\mathbf{C}_{yr}\mathbf{x}_r(t)\\ &= (\mathbf{A}_c - \mathbf{L}\mathbf{C}_{yc})\,\mathbf{e}(t) - \mathbf{L}\mathbf{C}_{yr}\mathbf{x}_r(t)\end{aligned} \tag{17.115}$$

Therefore, the closed-loop dynamics of the system can be written in matrix form as follows

$$\begin{Bmatrix} \dot{\mathbf{x}}_c(t) \\ \dot{\mathbf{e}}(t) \\ \dot{\mathbf{x}}_r(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A}_c - \mathbf{B}_{uc}\mathbf{G} & \mathbf{B}_{uc}\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_c - \mathbf{L}\mathbf{C}_{yc} & -\mathbf{L}\mathbf{C}_{yr} \\ -\mathbf{B}_{ur}\mathbf{G} & \mathbf{B}_{ur}\mathbf{G} & \mathbf{A}_r \end{bmatrix} \begin{Bmatrix} \mathbf{x}_c(t) \\ \mathbf{e}(t) \\ \mathbf{x}_r(t) \end{Bmatrix} \tag{17.116}$$

It is clearly observed that the above system matrix is neither block diagonal nor block triangular. This implies that the poles of the closed-loop system cannot be considered as the union of the poles of the controller, the poles of the observer and the poles associated with the residual modes. The poles of the controller and the observer are affected by the residual dynamics through the terms $\mathbf{L}\mathbf{C}_{yr}$ and $\mathbf{B}_{ur}\mathbf{G}$. The first quantity arises from the sensor output being contaminated by the residual modes due to the term $\mathbf{C}_{yr}\mathbf{x}_r(t)$ in the output equation. This is called *observation spillover*. The second quantity is related to the excitation of the residual modes by the feedback control via the term $\mathbf{B}_{ur}\mathbf{u}(t)$ and is known as *control spillover*.

17.9 Stochastic LQR

In this section we want to extend the linear quadratic control (LQR) to LTI systems subjected to stochastic disturbances $\mathbf{d}(t)$. The disturbances are assumed to be random white noise processes with zero mean value. It is known that we can always transform our system as subjected to random loads even if the disturbance is not a white noise by including suitable shape filters.

Let consider the following state-space formulation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_d\mathbf{d}(t) \\ \mathbf{z}(t) &= \mathbf{C}_z\mathbf{x}(t) + \mathbf{D}_{zu}\mathbf{u}(t) \end{aligned} \tag{17.117}$$

Using the full-state feedback law, the closed-loop dynamics is described by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_d\mathbf{d}(t) \\ \mathbf{z}(t) &= \mathbf{C}_{zc}\mathbf{x}(t) \end{aligned} \tag{17.118}$$

Since we are dealing with a stochastic response, the cost function to minimize is represented in this case by the expected value of the quadratic forms associated with the performance and control. We can write

$$J = \mathrm{E}\left[\mathbf{z}^T\mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T\mathbf{W}_{uu}\mathbf{u}\right] \tag{17.119}$$

and, using the closed-loop dynamics derived above,

$$J = \mathrm{E}\left[\mathbf{x}^T\mathbf{W}(\mathbf{G})\mathbf{x}\right] \tag{17.120}$$

where

$$\begin{aligned} \mathbf{W}(\mathbf{G}) &= \mathbf{C}_{zc}^T\mathbf{W}_{zz}\mathbf{C}_{zc} + \mathbf{G}^T\mathbf{W}_{uu}\mathbf{G} \\ &= \mathbf{Q} - \mathbf{S}\mathbf{G} - \mathbf{G}^T\mathbf{S}^T + \mathbf{G}^T\mathbf{R}\mathbf{G} \end{aligned} \tag{17.121}$$

The cost functional J can be also expressed by taking the trace of Eq. (17.120) without changing the result, i.e.,

$$J = \mathrm{Tr}\left\{\mathrm{E}\left[\mathbf{x}^T\mathbf{W}(\mathbf{G})\mathbf{x}\right]\right\} \tag{17.122}$$

Since the trace and the expected value are linear operators, one can write using the properties of the trace

$$\begin{aligned} J &= \mathrm{E}\left\{\mathrm{Tr}\left[\mathbf{x}^T\mathbf{W}(\mathbf{G})\mathbf{x}\right]\right\} \\ &= \mathrm{E}\left\{\mathrm{Tr}\left[\mathbf{W}(\mathbf{G})\mathbf{x}\mathbf{x}^T\right]\right\} \\ &= \mathrm{Tr}\left\{\mathrm{E}\left[\mathbf{W}(\mathbf{G})\mathbf{x}\mathbf{x}^T\right]\right\} \\ &= \mathrm{Tr}\left\{\mathbf{W}(\mathbf{G})\mathrm{E}\left[\mathbf{x}\mathbf{x}^T\right]\right\} \\ &= \mathrm{Tr}\left[\mathbf{W}(\mathbf{G})\boldsymbol{\sigma}^2_{\mathbf{xx}}\right] \end{aligned} \tag{17.123}$$

Therefore, the minimization of J implies the minimization of the variance matrix of the state vector. We know that $\boldsymbol{\sigma}^2_{\mathbf{xx}}$ must satisfy the following Lyapunov equation corresponding to the closed-loop dynamics

$$\mathbf{A}_c\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\sigma}^2_{\mathbf{xx}}\mathbf{A}_c^T + \mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T = \mathbf{0} \tag{17.124}$$

where $\mathbf{W}_{dd}$ is the matrix of intensities of the white noise disturbances. The above equation represents the constraint equation of the minimization process. An unconstrained minimization can be carried out if we introduce into J the constraint equation using the Lagrange multipliers. Therefore, the cost functional to be minimized is rewritten as

$$J = \mathrm{Tr}\left[\mathbf{W}(\mathbf{G})\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\Lambda}\left(\mathbf{A}_c\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\sigma}^2_{\mathbf{xx}}\mathbf{A}_c^T + \mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T\right)\right] \tag{17.125}$$

where $\boldsymbol{\Lambda}$ is the symmetric matrix of Lagrange multipliers. Since we have three unknown matrices, $\boldsymbol{\Lambda}$, $\boldsymbol{\sigma}^2_{\mathbf{xx}}$ and $\mathbf{G}$, the minimization of J implies that

$$\begin{aligned} \frac{\partial J}{\partial \boldsymbol{\Lambda}} &= \mathbf{0} \\ \frac{\partial J}{\partial \boldsymbol{\sigma}^2_{\mathbf{xx}}} &= \mathbf{0} \\ \frac{\partial J}{\partial \mathbf{G}} &= \mathbf{0} \end{aligned} \tag{17.126}$$

The first condition yields the constraint equation

$$\mathbf{A}_c\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\sigma}^2_{\mathbf{xx}}\mathbf{A}_c^T + \mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T = \mathbf{0} \tag{17.127}$$

Using the properties of the trace and its derivatives, we can write

$$\frac{\partial J}{\partial \boldsymbol{\sigma}^2_{\mathbf{xx}}} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\sigma}^2_{\mathbf{xx}}}\mathrm{Tr}\left[\mathbf{W}(\mathbf{G})\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\Lambda}\mathbf{A}_c\boldsymbol{\sigma}^2_{\mathbf{xx}} + \mathbf{A}_c^T\boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\Lambda}\mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T\right]$$

and, hence,

$$\boldsymbol{\Lambda}\mathbf{A}_c + \mathbf{A}_c^T\boldsymbol{\Lambda} + \mathbf{W}(\mathbf{G}) = \mathbf{0} \tag{17.128}$$

The partial derivative of the cost functional with respect to the gain matrix $\mathbf{G}$ can be written as

$$\begin{aligned}
\frac{\partial J}{\partial \mathbf{G}} &= \frac{1}{2}\frac{\partial}{\partial \mathbf{G}}\mathrm{Tr}\left[\mathbf{W}(\mathbf{G})\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\Lambda}\mathbf{A}_c\boldsymbol{\sigma}^2_{\mathbf{xx}} + \mathbf{A}_c^T\boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\Lambda}\mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T\right] \\
&= \frac{1}{2}\frac{\partial}{\partial \mathbf{G}}\mathrm{Tr}\left[\left(\mathbf{Q} - \mathbf{S}\mathbf{G} - \mathbf{G}^T\mathbf{S}^T + \mathbf{G}^T\mathbf{R}\mathbf{G}\right)\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\Lambda}\mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{xx}}\right. \\
&\qquad \left. - \boldsymbol{\Lambda}\mathbf{B}_u\mathbf{G}\boldsymbol{\sigma}^2_{\mathbf{xx}} + \mathbf{A}^T\boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{xx}} - \mathbf{G}^T\mathbf{B}_u^T\boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\Lambda}\mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T\right] \\
&= \frac{1}{2}\frac{\partial}{\partial \mathbf{G}}\mathrm{Tr}\left[\mathbf{Q}\boldsymbol{\sigma}^2_{\mathbf{xx}} - 2\mathbf{S}^T\boldsymbol{\sigma}^2_{\mathbf{xx}}\mathbf{G}^T + \mathbf{G}^T\mathbf{R}\mathbf{G}\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\Lambda}\mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{xx}} + \mathbf{A}^T\boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{xx}}\right. \\
&\qquad \left. - 2\mathbf{B}_u^T\boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{xx}}\mathbf{G}^T + \boldsymbol{\Lambda}\mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T\right]
\end{aligned}$$

The minimum condition is then expressed by

$$-\mathbf{B}_u^T\boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{xx}} - \mathbf{S}^T\boldsymbol{\sigma}^2_{\mathbf{xx}} + \mathbf{R}\mathbf{G}\boldsymbol{\sigma}^2_{\mathbf{xx}} = \mathbf{0} \tag{17.129}$$

From this last equation, we can derive the expression for the feedback gain matrix

$$\mathbf{G} = \mathbf{R}^{-1}\left(\mathbf{B}_u^T\boldsymbol{\Lambda} + \mathbf{S}^T\right) \tag{17.130}$$

The matrix $\boldsymbol{\Lambda}$ in the expression of the gain matrix can be determined from the solution of the Lyapunov equation (17.128). Using the value of $\mathbf{G}$ derived above, equation (17.128) becomes the following Riccati equation

$$\boldsymbol{\Lambda}\left(\mathbf{A} - \mathbf{B}_u\mathbf{R}^{-1}\mathbf{S}^T\right) + \left(\mathbf{A}^T - \mathbf{S}\mathbf{R}^{-1}\mathbf{B}_u^T\right)\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\boldsymbol{\Lambda} + \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T = \mathbf{0} \tag{17.131}$$

It is clearly observed that the design of the stochastic LQR is identical to the design of the deterministic LQR.

17.10 Kalman filter: the optimal observer

The linear observer presented in Section 17.5 did not directly take into account the random disturbances acting on the physical system and the random noise associated with measurements. Indeed, it has been said that in the selection of an observer for a given system a certain arbitrariness remains in the choice of the gain matrix $\mathbf{L}$, since the design should be a compromise between speed of state reconstruction and immunity to observation noise. In this section, we present a method for finding the *optimal* observer gain matrix. To this end, the effects of disturbances and noise are properly considered in the following.

Let have the following state-space formulation

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_d\mathbf{d}(t) \\
\mathbf{y}(t) &= \mathbf{C}_y\mathbf{x}(t) + \mathbf{r}(t)
\end{aligned} \tag{17.132}$$

where $\mathbf{d}(t)$ and $\mathbf{r}(t)$ are assumed to be white noise processes. The dynamics of the observer is again described by

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}_u\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t) \tag{17.133}$$

Accordingly, the observation error is expressed as

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_d\mathbf{d}(t) - \hat{\mathbf{A}}\hat{\mathbf{x}}(t) - \hat{\mathbf{B}}_u\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_d\mathbf{d}(t) - \hat{\mathbf{A}}\left[\mathbf{x}(t) - \mathbf{e}(t)\right] - \hat{\mathbf{B}}_u\mathbf{u}(t) - \mathbf{L}\mathbf{C}_y\mathbf{x}(t) - \mathbf{L}\mathbf{r}(t) \\ &= \hat{\mathbf{A}}\mathbf{e}(t) + \left(\mathbf{A} - \hat{\mathbf{A}} - \mathbf{L}\mathbf{C}_y\right)\mathbf{x}(t) + \left(\mathbf{B}_u - \hat{\mathbf{B}}_u\right)\mathbf{u}(t) + \mathbf{B}_d\mathbf{d}(t) - \mathbf{L}\mathbf{r}(t)\end{aligned} \tag{17.134}$$

Imposing the dynamics of the error independent of the actual state and input vectors yields

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \left(\mathbf{A} - \mathbf{L}\mathbf{C}_y\right)\mathbf{e}(t) + \mathbf{B}_d\mathbf{d}(t) - \mathbf{L}\mathbf{r}(t) \\ &= \left(\mathbf{A} - \mathbf{L}\mathbf{C}_y\right)\mathbf{e}(t) + \begin{bmatrix}\mathbf{B}_d & -\mathbf{L}\end{bmatrix}\begin{Bmatrix}\mathbf{d}(t) \\ \mathbf{r}(t)\end{Bmatrix}\end{aligned} \tag{17.135}$$

The variance matrix of the observation error must satisfy the following Lyapunov equation

$$\left(\mathbf{A} - \mathbf{L}\mathbf{C}_y\right)\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\sigma}^2_{\mathbf{ee}}\left(\mathbf{A}^T - \mathbf{C}_y^T\mathbf{L}^T\right) + \begin{bmatrix}\mathbf{B}_d & -\mathbf{L}\end{bmatrix}\begin{bmatrix}\mathbf{W}_{dd} & \mathbf{W}_{dr} \\ \mathbf{W}_{rd} & \mathbf{W}_{rr}\end{bmatrix}\begin{bmatrix}\mathbf{B}_d^T \\ -\mathbf{L}^T\end{bmatrix} = \mathbf{0} \tag{17.136}$$

Assuming the disturbance and measurement noise being uncorrelated as in most practical situations, i.e., $\mathbf{W}_{dr} = \mathbf{0}$ and $\mathbf{W}_{rd} = \mathbf{0}$, Equation (17.136) is written as

$$\left(\mathbf{A} - \mathbf{L}\mathbf{C}_y\right)\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\sigma}^2_{\mathbf{ee}}\left(\mathbf{A}^T - \mathbf{C}_y^T\mathbf{L}^T\right) + \mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T + \mathbf{L}\mathbf{W}_{rr}\mathbf{L}^T = \mathbf{0} \tag{17.137}$$

The optimal observer problem is defined as the problem related to the design of a linear observer which minimizes the following quantity

$$J = \mathrm{Tr}\left[\boldsymbol{\sigma}^2_{\mathbf{ee}}\right] \tag{17.138}$$

Since the variance of the observation error must satisfy Eq. (17.137), the unconstrained minimization can be written as

$$J = \mathrm{Tr}\left\{\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\Lambda}\left[\left(\mathbf{A} - \mathbf{L}\mathbf{C}_y\right)\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\sigma}^2_{\mathbf{ee}}\left(\mathbf{A}^T - \mathbf{C}_y^T\mathbf{L}^T\right) + \mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T + \mathbf{L}\mathbf{W}_{rr}\mathbf{L}^T\right]\right\} \tag{17.139}$$

where $\boldsymbol{\Lambda}$ is the symmetric matrix of Lagrange multipliers. The minimization process implies that

$$\begin{aligned}\frac{\partial J}{\partial \boldsymbol{\Lambda}} &= \mathbf{0} \\ \frac{\partial J}{\partial \boldsymbol{\sigma}^2_{\mathbf{ee}}} &= \mathbf{0} \\ \frac{\partial J}{\partial \mathbf{L}} &= \mathbf{0}\end{aligned} \tag{17.140}$$

From the first condition, we have

$$\mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{ee}} - \mathbf{L}\mathbf{C}_y\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{A}^T - \boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{C}_y^T\mathbf{L}^T + \mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T + \mathbf{L}\mathbf{W}_{rr}\mathbf{L}^T = \mathbf{0} \tag{17.141}$$

From the second conditions, we have

$$\mathbf{I} + \boldsymbol{\Lambda}\mathbf{A} - \boldsymbol{\Lambda}\mathbf{L}\mathbf{C}_y + \mathbf{A}^T\boldsymbol{\Lambda} - \mathbf{C}_y^T\mathbf{L}^T\boldsymbol{\Lambda} = \mathbf{0} \tag{17.142}$$

The third condition can be written as

$$\begin{aligned}\frac{\partial J}{\partial \mathbf{L}} &= \frac{\partial}{\partial \mathbf{L}}\mathrm{Tr}\left\{\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\Lambda}\mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{ee}} - \boldsymbol{\Lambda}\mathbf{L}\mathbf{C}_y\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{A}^T - \boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{C}_y^T\mathbf{L}^T + \boldsymbol{\Lambda}\mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T + \boldsymbol{\Lambda}\mathbf{L}\mathbf{W}_{rr}\mathbf{L}^T\right\} \\ &= \frac{\partial}{\partial \mathbf{L}}\mathrm{Tr}\left\{\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\Lambda}\mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{ee}} - \boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{C}_y^T\mathbf{L}^T + \boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{A}^T - \boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{C}_y^T\mathbf{L}^T + \boldsymbol{\Lambda}\mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T + \boldsymbol{\Lambda}\mathbf{L}\mathbf{W}_{rr}\mathbf{L}^T\right\}\end{aligned}$$

which yields

$$-2\boldsymbol{\Lambda}\boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{C}_y^T + 2\boldsymbol{\Lambda}\mathbf{L}\mathbf{W}_{rr} = \mathbf{0} \tag{17.143}$$

The optimal observer gain is then found as

$$\mathbf{L} = \boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{C}_y^T\mathbf{W}_{rr}^{-1} \tag{17.144}$$

By putting the above solution into Eq. (17.141) we have

$$\mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{ee}} + \boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{A}^T - \boldsymbol{\sigma}^2_{\mathbf{ee}}\mathbf{C}_y^T\mathbf{W}_{rr}^{-1}\mathbf{C}_y\boldsymbol{\sigma}^2_{\mathbf{ee}} + \mathbf{B}_d\mathbf{W}_{dd}\mathbf{B}_d^T = \mathbf{0} \tag{17.145}$$

which is the algebraic Riccati equation to solve in order to determine the variance matrix of the observation error.

17.11 Direct output optimal control

We have shown that the practical implementation of a deterministic or stochastic LQR control system requires the availability of the state vector $\mathbf{x}(t)$ or its estimation $\hat{\mathbf{x}}(t)$ as provided by a linear observer. We may wonder if there exists an optimal solution for a control law which is based on the direct feedback of the output vector $\mathbf{y}(t)$ instead of the full state vector. The related approach can be highly appealing since it avoids completely the need of reconstructing the state using an appropriate dynamic system. The solution is presented in the following.

Let consider the open-loop dynamics as described by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}_y\mathbf{x}(t)\\ \mathbf{z}(t) &= \mathbf{C}_z\mathbf{x}(t)\end{aligned} \tag{17.146}$$

We now consider the following feedback law

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{y}(t) \tag{17.147}$$

where $\mathbf{G}$ is the unknown gain matrix and $\mathbf{y}(t)$ is the set of output measurements. Using the output equation, we can also write

$$\mathbf{u}(t) = -\mathbf{G}\mathbf{C}_y\mathbf{x}(t) \tag{17.148}$$

which can be considered as a feedback control on a subset of state variables. If $\mathbf{C}_y = \mathbf{I}$, the full state feedback design is recovered. The closed-loop dynamics is expressed as usual by

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) \tag{17.149}$$

where the closed-loop state matrix is now given by

$$\mathbf{A}_c = \mathbf{A} - \mathbf{B}\mathbf{G}\mathbf{C}_y \tag{17.150}$$

The gain matrix $\mathbf{G}$ is designed such that the following cost functional is minimized

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{z}^T\mathbf{W}_{zz}\mathbf{z} + \mathbf{u}^T\mathbf{W}_{uu}\mathbf{u}\right)dt \tag{17.151}$$

It is remarked that it is assumed that a gain matrix $\mathbf{G}$ exists which stabilizes $\mathbf{A} - \mathbf{B}_u\mathbf{G}\mathbf{C}_y$. If no such $\mathbf{G}$ exists, then J is infinite and the foregoing discussion is meaningless. The above cost functional can be also put in the alternative form

$$J = \frac{1}{2}\int_0^\infty \mathbf{x}^T\mathbf{W}(\mathbf{G})\mathbf{x}dt \tag{17.152}$$

where

$$\mathbf{W}(\mathbf{G}) = \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z + \mathbf{C}_y^T\mathbf{G}^T\mathbf{W}_{uu}\mathbf{G}\mathbf{C}_y \tag{17.153}$$

Since the closed-loop response due to an initial state $\mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{\mathbf{A}_c t}\mathbf{x}_0 = e^{(\mathbf{A}-\mathbf{B}\mathbf{G}\mathbf{C}_y)t}\mathbf{x}_0 \tag{17.154}$$

the cost functional becomes

$$J = \frac{1}{2}\mathbf{x}_0^T\mathbf{P}\mathbf{x}_0 \tag{17.155}$$

where the matrix $\mathbf{P}$ is defined as

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}_c^T t}\mathbf{W}(\mathbf{G})e^{\mathbf{A}_c t}dt \tag{17.156}$$

and satisfies the Lyapunov equation

$$\mathbf{A}_c^T\mathbf{P} + \mathbf{P}\mathbf{A}_c + \mathbf{W}(\mathbf{G}) = \mathbf{0} \tag{17.157}$$

It is observed that all the above equations appear to be formally identical to what presented for the deterministic LQR design. However, it is noted that they differ in the definition of the closed-loop state matrix $\mathbf{A}_c$ and the matrix $\mathbf{W}(\mathbf{G})$.

By taking the trace of J, we can also write

$$J = \frac{1}{2}\mathrm{Tr}\left[\mathbf{P}\mathbf{X}_0\right] \tag{17.158}$$

Accordingly, the minimization of J with the constraint equation for $\mathbf{P}$ is equivalent to the free minimization problem of the following cost functional

$$\begin{aligned} J &= \frac{1}{2}\mathrm{Tr}\left[\mathbf{P}\mathbf{X}_0 + \mathbf{\Lambda}\left(\mathbf{A}_c^T\mathbf{P} + \mathbf{P}\mathbf{A}_c + \mathbf{W}(\mathbf{G})\right)\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\mathbf{P}\mathbf{X}_0 + \mathbf{\Lambda}\mathbf{A}^T\mathbf{P} - \mathbf{\Lambda}\mathbf{C}_y^T\mathbf{G}^T\mathbf{B}_u^T\mathbf{P} + \mathbf{\Lambda}\mathbf{P}\mathbf{A} - \mathbf{\Lambda}\mathbf{P}\mathbf{B}_u\mathbf{G}\mathbf{C}_y + \mathbf{\Lambda}\mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z + \mathbf{\Lambda}\mathbf{C}_y^T\mathbf{G}^T\mathbf{W}_{uu}\mathbf{G}\mathbf{C}_y\right] \end{aligned}$$

The minimum solution is then obtained by setting

$$\begin{aligned} \frac{\partial J}{\partial \mathbf{\Lambda}} &= \mathbf{0} \\ \frac{\partial J}{\partial \mathbf{P}} &= \mathbf{0} \\ \frac{\partial J}{\partial \mathbf{G}} &= \mathbf{0} \end{aligned} \tag{17.159}$$

The first two conditions yields the Lyapunov equations

$$\mathbf{A}_c^T\mathbf{P} + \mathbf{P}\mathbf{A}_c + \mathbf{W}(\mathbf{G}) = \mathbf{0} \tag{17.160}$$

and

$$\mathbf{A}_c\mathbf{\Lambda} + \mathbf{\Lambda}\mathbf{A}_c^T + \mathbf{X}_0 = \mathbf{0} \tag{17.161}$$

The last condition, using the properties of the trace, leads to

$$-2\mathbf{B}_u^T\mathbf{P}\mathbf{\Lambda}\mathbf{C}_y^T + 2\mathbf{W}_{uu}\mathbf{G}\mathbf{C}_y\mathbf{\Lambda}\mathbf{C}_y^T = \mathbf{0} \tag{17.162}$$

Therefore, the expression of the gain matrix is

$$\mathbf{G} = \mathbf{W}_{uu}^{-1}\mathbf{B}_u^T\mathbf{P}\mathbf{\Lambda}\mathbf{C}_y^T\left(\mathbf{C}_y\mathbf{\Lambda}\mathbf{C}_y^T\right)^{-1} \tag{17.163}$$

In the general case where $\mathbf{C}_y \neq \mathbf{I}$, we have obtained that the computation of $\mathbf{G}$ depends both on the matrix $\mathbf{P}$ and the matrix $\mathbf{\Lambda}$ simultaneously. Hence, the direct output optimal control has a completely different solution with respect to the LQR approach based on the full state feedback. In the LQR design, the gain matrix is dependent on a single unknown matrix $\mathbf{P}$ which can be determined from the solution of an algebraic Riccati equation. If a static output feedback law is assumed, the expression of the gain $\mathbf{G}$ contains two unknown matrices $\mathbf{P}$ and $\mathbf{\Lambda}$. Such matrices satisfy Lyapunov equations, which in turn contain the unknown gain matrix. Consequently, the overall solution involves the following fully coupled set of nonlinear matrix equations

$$\begin{aligned} &\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\mathbf{C}_y\right)^T\mathbf{P} + \mathbf{P}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\mathbf{C}_y\right) + \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z + \mathbf{C}_y^T\mathbf{G}^T\mathbf{W}_{uu}\mathbf{G}\mathbf{C}_y = \mathbf{0} \\ &\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\mathbf{C}_y\right)\mathbf{\Lambda} + \mathbf{\Lambda}\left(\mathbf{A} - \mathbf{B}_u\mathbf{G}\mathbf{C}_y\right)^T + \mathbf{X}_0 = \mathbf{0} \\ &\mathbf{G} = \mathbf{W}_{uu}^{-1}\mathbf{B}_u^T\mathbf{P}\mathbf{\Lambda}\mathbf{C}_y^T\left(\mathbf{C}_y\mathbf{\Lambda}\mathbf{C}_y^T\right)^{-1} \end{aligned} \tag{17.164}$$

There are different numerical techniques capable of providing a solution of the above problem. Two approaches are here briefly described.

The first approach is known as the Moerder-Calise algorithm. It involves the following steps:

1. Determine an initial guess matrix $\mathbf{G}_0$ such that the closed-loop system matrix $\mathbf{A} - \mathbf{B}_u\mathbf{G}_0\mathbf{C}_y$ is asymptotically stable. Note that if $\mathbf{A}$ corresponds to an open-loop asymptotically stable system, the initial gain matrix can be null.

2. At each iteration k

   (a) compute $\mathbf{P}_k$ and $\mathbf{\Lambda}_k$ by solving

$$
\begin{aligned}
&(\mathbf{A} - \mathbf{B}_u\mathbf{G}_k\mathbf{C}_y)^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}_u\mathbf{G}_k\mathbf{C}_y) + \mathbf{C}_z^T\mathbf{W}_{zz}\mathbf{C}_z + \mathbf{C}_y^T\mathbf{G}_k^T\mathbf{W}_{uu}\mathbf{G}_k\mathbf{C}_y = \mathbf{0} \\
&(\mathbf{A} - \mathbf{B}_u\mathbf{G}_k\mathbf{C}_y)\mathbf{\Lambda} + \mathbf{\Lambda}(\mathbf{A} - \mathbf{B}_u\mathbf{G}_k\mathbf{C}_y)^T + \mathbf{X}_0 = \mathbf{0}
\end{aligned}
$$

   (b) Evaluate $J = \mathrm{Tr}[\mathbf{P}_k\mathbf{X}_0]$

   (c) Evaluate gain update

$$
\Delta\mathbf{G} = \mathbf{W}_{uu}^{-1}\mathbf{B}_u^T\mathbf{P}_k\mathbf{\Lambda}_k\mathbf{C}_y^T\left(\mathbf{C}_y\mathbf{\Lambda}_k\mathbf{C}_y^T\right)^{-1} - \mathbf{G}_k
$$

   (d) Update the gain $\mathbf{G}_{k+1} = \mathbf{G}_k + \alpha\Delta\mathbf{G}$, where α is chosen such that the system is asymptotically stable and $J_{k+1} < J_k$. If $J_{k+1} - J_k < \mathrm{tol}$, go to 3. Otherwise, set $k = k + 1$, and go to 2.

3. Set $\mathbf{G} = \mathbf{G}_{k+1}$ and $J = J_{k+1}$. Terminate.

Another approach is based on directly using a gradient-based optimisation routine. The function to be minimized is the unconstrained cost function J expressed as

$$
J = \frac{1}{2}\mathrm{Tr}[\mathbf{P}\mathbf{X}_0] \tag{17.165}
$$

The gradient of J is given by

$$
\mathrm{grad}J = \frac{\partial J}{\partial \mathbf{G}} = \begin{bmatrix} \cdots & & \cdots \\ \cdots & \frac{\partial J}{\partial g_{ik}} & \cdots \\ & & \\ \cdots & & \cdots \end{bmatrix} \tag{17.166}
$$

where

$$
\frac{\partial J}{\partial g_{ik}} = \frac{\partial}{\partial g_{ik}}\mathrm{Tr}[\mathbf{P}\mathbf{X}_0] = \frac{1}{2}\mathrm{Tr}\left[\frac{\partial \mathbf{P}}{\partial g_{ik}}\mathbf{X}_0\right] \tag{17.167}
$$

The derivative of $\mathbf{P}$ with respect to the control gain g_{ik} can be obtained by taking the derivative of the Lyapunov equation (17.160) as follows

$$
\frac{\partial}{\partial g_{ik}}\left[\mathbf{A}_c^T\mathbf{P} + \mathbf{P}\mathbf{A}_c + \mathbf{W}(\mathbf{G})\right] = \mathbf{0} \tag{17.168}
$$

After performing the derivatives we can write

$$
\mathbf{A}_c^T\frac{\partial \mathbf{P}}{\partial g_{ik}} + \frac{\partial \mathbf{P}}{\partial g_{ik}}\mathbf{A}_c + \frac{\partial \mathbf{A}_c^T}{\partial g_{ik}}\mathbf{P} + \mathbf{P}\frac{\partial \mathbf{A}_c}{\partial g_{ik}} + \frac{\partial \mathbf{W}(\mathbf{G})}{\partial g_{ik}} = \mathbf{0} \tag{17.169}
$$

which is a set of Lyapunov equations. Note that this approach implies to solve each optimization iteration step as many Lyapunov equations as the number of control gains.

An alternative way of relying on a gradient-based optimization approach is to express the gradient of the cost function as follows

$$
\begin{aligned}
\frac{\partial J}{\partial g_{ik}} = \frac{1}{2}\mathrm{Tr}\left[\frac{\partial \mathbf{P}}{\partial g_{ik}}\mathbf{X}_0\right] &= -\frac{1}{2}\mathrm{Tr}\left[\frac{\partial \mathbf{P}}{\partial g_{ik}}\left(\mathbf{A}_c\mathbf{\Lambda} + \mathbf{\Lambda}\mathbf{A}_c^T\right)\right] \\
&= -\frac{1}{2}\mathrm{Tr}\left[\mathbf{\Lambda}\frac{\partial \mathbf{P}}{\partial g_{ik}}\mathbf{A}_c + \mathbf{\Lambda}\mathbf{A}_c^T\frac{\partial \mathbf{P}}{\partial g_{ik}}\right] \\
&= -\frac{1}{2}\mathrm{Tr}\left[\mathbf{\Lambda}\left(\mathbf{A}_c^T\frac{\partial \mathbf{P}}{\partial g_{ik}} + \frac{\partial \mathbf{P}}{\partial g_{ik}}\mathbf{A}_c\right)\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\mathbf{\Lambda}\left(\frac{\partial \mathbf{A}_c^T}{\partial g_{ik}}\mathbf{P} + \mathbf{P}\frac{\partial \mathbf{A}_c}{\partial g_{ik}} + \frac{\partial \mathbf{W}(\mathbf{G})}{\partial g_{ik}}\right)\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\mathbf{\Lambda}\frac{\partial \mathbf{A}_c^T}{\partial g_{ik}}\mathbf{P} + \mathbf{\Lambda}\frac{\partial \mathbf{A}_c^T}{\partial g_{ik}}\mathbf{P} + \mathbf{\Lambda}\frac{\partial \mathbf{W}(\mathbf{G})}{\partial g_{ik}}\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\mathbf{\Lambda}\left(2\frac{\partial \mathbf{A}_c^T}{\partial g_{ik}}\mathbf{P} + \frac{\partial \mathbf{W}(\mathbf{G})}{\partial g_{ik}}\right)\right]
\end{aligned}
\tag{17.170}
$$

The last expression shows that we can avoid to solve the above set of Lyapunov equations. The MATLAB pseudo-code is reported below.

MATLAB code 17.2

```
G0 = ...
g0 = reshape(G0,nu*ny,1)
options = optimset('GradObj','on','Display','iter',...)
g = fminunc(@objg,g0,options)
G = reshape(g,nu,ny)

function [J,gradJ] = objg(g)
G = rebuild(g)
Ac = A-Bu*G*Cy
Wg = Cz*Wzz*Cz + Cy'*G'*Wuu*G*Cy
Lambda = lyap(Ac,X0)
P = lyap(Ac',Wg)
J = (1/2)*trace(P*X0)
gradJ = (1/2)*trace(Lambda*(2*Adg'*P+Wdg))
```

17.12 Guidelines for the selection of weighting matrices in the LQR design

17.12.1 Eigenvalue shift (prescribed degree of stability)

Cost function:

$$J = \frac{1}{2}\int_0^\infty e^{2\alpha t}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt \qquad (\alpha > 0) \tag{17.171}$$

The above can be also written as

$$J = \frac{1}{2}\int_0^\infty \left(e^{\alpha t}\mathbf{x}^T\mathbf{Q}\mathbf{x}e^{\alpha t} + e^{\alpha t}\mathbf{u}^T\mathbf{R}\mathbf{u}e^{\alpha t}\right)dt \tag{17.172}$$

Define:

$$
\begin{aligned}
\hat{\mathbf{x}} &= \mathbf{x}e^{\alpha t} \\
\hat{\mathbf{u}} &= \mathbf{u}e^{\alpha t}
\end{aligned}
\tag{17.173}
$$

The corresponding dynamics is given by

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \dot{\mathbf{x}}(t)e^{\alpha t} + \mathbf{x}(t)\alpha e^{\alpha t} \\ &= (\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u})\, e^{\alpha t} + \mathbf{x}(t)\alpha e^{\alpha t} \\ &= (\mathbf{A} + \alpha\mathbf{I})\,\mathbf{x}e^{\alpha t} + \mathbf{B}_u\mathbf{u}e^{\alpha t} \\ &= (\mathbf{A} + \alpha\mathbf{I})\,\hat{\mathbf{x}}(t) + \mathbf{B}_u\hat{\mathbf{u}}(t)\end{aligned} \tag{17.174}$$

The cost function to be minimised is

$$J = \frac{1}{2}\int_0^\infty \left(\hat{\mathbf{x}}^T\mathbf{Q}\hat{\mathbf{x}} + \hat{\mathbf{u}}^T\mathbf{R}\hat{\mathbf{u}}\right) dt \tag{17.175}$$

17.12.2 Implicit model following

Desired closed-loop behaviour:

$$\dot{\mathbf{x}}_d(t) = \mathbf{A}_d\mathbf{x}(t) \tag{17.176}$$

The performance is defined as

$$\mathbf{z}(t) = \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_d(t) \tag{17.177}$$

Accordingly, the cost function is given by

$$J = \frac{1}{2}\int_0^\infty \left[(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} - \mathbf{A}_d\mathbf{x})^T\,\mathbf{W}_{zz}\,(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} - \mathbf{A}_d\mathbf{x}) + \mathbf{u}^T\mathbf{W}_{uu}\mathbf{u}\right] dt \tag{17.178}$$

which can be written in the usual form as

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{S}\mathbf{u} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \tag{17.179}$$

where

$$\begin{aligned}\mathbf{Q} &= (\mathbf{A} - \mathbf{A}_d)^T\,\mathbf{W}_{zz}\,(\mathbf{A} - \mathbf{A}_d) \\ \mathbf{S} &= (\mathbf{A} - \mathbf{A}_d)^T\,\mathbf{W}_{zz}\mathbf{B}_u \\ \mathbf{R} &= \mathbf{B}_u^T\mathbf{W}_{zz}\mathbf{B}_u + \mathbf{W}_{uu}\end{aligned} \tag{17.180}$$

17.12.3 Frequency shaping

Parseval's theorem:

$$\int_{-\infty}^{+\infty} \mathbf{z}^T(t)\mathbf{z}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^*(j\omega)\mathbf{z}(j\omega)d\omega \tag{17.181}$$

Proof:

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathbf{z}^T(t)\mathbf{z}(t)dt &= \int_{-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^T(j\omega)e^{j\omega t}d\omega\,\mathbf{z}(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^T(j\omega)\int_{-\infty}^{+\infty} \mathbf{z}(t)e^{-(-j\omega t)}dt\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^T(j\omega)\mathbf{z}(-j\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^T(-j\omega)\mathbf{z}(j\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{z}^*(j\omega)\mathbf{z}(j\omega)d\omega\end{aligned}$$

The cost function can be written as

$$\begin{aligned}J &= \frac{1}{2}\int_{-\infty}^{+\infty} \left[\mathbf{z}^T(t)\mathbf{z}(t) + \rho\mathbf{u}^T(t)\mathbf{u}(t)\right] dt \\ &= \frac{1}{4\pi}\int_{-\infty}^{+\infty} \left[\mathbf{z}^*(j\omega)\mathbf{z}(j\omega) + \rho\mathbf{u}^*(j\omega)\mathbf{u}(j\omega)\right] d\omega\end{aligned} \tag{17.182}$$

Frequency shaping of the performance and control effort is achieved by introducing appropriate filters as follows

$$\hat{\mathbf{z}}(j\omega) = \mathbf{W}_z(j\omega)\mathbf{z}(j\omega) \tag{17.183}$$

and

$$\mathbf{u}(j\omega) = \mathbf{W}_u(j\omega)\hat{\mathbf{u}}(j\omega) \tag{17.184}$$

Inclusion of the above filtering actions requires to define the cost function as

$$J = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \left[\hat{\mathbf{z}}^*(j\omega)\hat{\mathbf{z}}(j\omega) + \rho\hat{\mathbf{u}}^*(j\omega)\hat{\mathbf{u}}(j\omega)\right] d\omega \tag{17.185}$$

LQR design is performed in the time-domain. Therefore, the filters represented by the frequency response functions $\mathbf{W}_z(j\omega)$ and $\mathbf{W}_u(j\omega)$ must be realized by suitable state-space models as follows

$$\text{Filter on performance:} \qquad \begin{aligned} \dot{\mathbf{x}}_z(t) &= \hat{\mathbf{A}}_z\mathbf{x}_z(t) + \hat{\mathbf{B}}_z\mathbf{z}(t) \\ \hat{\mathbf{z}}(t) &= \hat{\mathbf{C}}_z\mathbf{x}_z(t) + \hat{\mathbf{D}}_z\mathbf{z}(t) \end{aligned} \tag{17.186}$$

$$\text{Filter on control:} \qquad \begin{aligned} \dot{\mathbf{x}}_u(t) &= \hat{\mathbf{A}}_u\mathbf{x}_u(t) + \hat{\mathbf{B}}_u\hat{\mathbf{u}}(t) \\ \mathbf{u}(t) &= \hat{\mathbf{C}}_u\mathbf{x}_u(t) + \hat{\mathbf{D}}_u\hat{\mathbf{u}}(t) \end{aligned} \tag{17.187}$$

where

$$\mathbf{W}_z(j\omega) = \hat{\mathbf{C}}_z\left(j\omega\mathbf{I} - \hat{\mathbf{A}}_z\right)^{-1}\hat{\mathbf{B}}_z + \hat{\mathbf{D}}_z \tag{17.188}$$

and

$$\mathbf{W}_u(j\omega) = \hat{\mathbf{C}}_u\left(j\omega\mathbf{I} - \hat{\mathbf{A}}_u\right)^{-1}\hat{\mathbf{B}}_u + \hat{\mathbf{D}}_u \tag{17.189}$$

Augmented plant:

$$\begin{aligned} \dot{\mathbf{x}}_{\text{aug}}(t) &= \mathbf{A}_{\text{aug}}\mathbf{x}_{\text{aug}}(t) + \mathbf{B}_{\text{aug}}\hat{\mathbf{u}}(t) \\ \hat{\mathbf{z}}(t) &= \mathbf{C}_{\text{aug}}\mathbf{x}_{\text{aug}}(t) + \mathbf{D}_{\text{aug}}\hat{\mathbf{u}}(t) \end{aligned} \tag{17.190}$$

where

$$\mathbf{x}_{\text{aug}}(t) = \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{x}_z(t) \\ \mathbf{x}_u(t) \end{Bmatrix} \tag{17.191}$$

$$\mathbf{A}_{\text{aug}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{B}_u\hat{\mathbf{C}}_u \\ \hat{\mathbf{B}}_z\mathbf{C}_z & \hat{\mathbf{A}}_z & \hat{\mathbf{B}}_z\mathbf{D}_{zu}\hat{\mathbf{C}}_z \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{A}}_u \end{bmatrix} \qquad \mathbf{B}_{\text{aug}} = \begin{bmatrix} \mathbf{B}_u\hat{\mathbf{D}}_u \\ \hat{\mathbf{B}}_z\mathbf{D}_{zu}\hat{\mathbf{D}}_u \\ \hat{\mathbf{B}}_u \end{bmatrix} \tag{17.192}$$

$$\mathbf{C}_{\text{aug}} = \begin{bmatrix} \hat{\mathbf{D}}_z\mathbf{C}_z & \hat{\mathbf{C}}_z & \hat{\mathbf{D}}_z\mathbf{D}_{zu}\hat{\mathbf{C}}_u \end{bmatrix} \qquad \mathbf{D}_{\text{aug}} = \hat{\mathbf{D}}_z\mathbf{D}_{zu}\hat{\mathbf{D}}_u \tag{17.193}$$

17.12.4 Sensitivity weighted LQR

Cost function:

$$J = \frac{1}{2}\int_0^{\infty} \left[\mathbf{x}^T\hat{\mathbf{Q}}\mathbf{x} + \mathbf{u}^T\hat{\mathbf{R}}\mathbf{u} + \left(\frac{\partial\mathbf{x}}{\partial\alpha}\right)^T \mathbf{W}_{\alpha\alpha}\left(\frac{\partial\mathbf{x}}{\partial\alpha}\right)\right] dt \tag{17.194}$$

Derivative of the state equation:

$$\frac{\partial\dot{\mathbf{x}}}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u}\right) = \frac{\partial\mathbf{A}}{\partial\alpha}\mathbf{x} + \mathbf{A}\frac{\partial\mathbf{x}}{\partial\alpha} + \frac{\partial\mathbf{B}_u}{\partial\alpha}\mathbf{u} \tag{17.195}$$

Steady-state solution:

$$\frac{\partial\dot{\mathbf{x}}}{\partial\alpha} = \mathbf{0} \tag{17.196}$$

Accordingly,

$$\frac{\partial \mathbf{x}}{\partial \alpha} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \alpha}\mathbf{x} - \mathbf{A}^{-1}\frac{\partial \mathbf{B}_u}{\partial \alpha}\mathbf{u} \tag{17.197}$$

and

$$\left(\frac{\partial \mathbf{x}}{\partial \alpha}\right)^T = -\mathbf{x}^T\frac{\partial \mathbf{A}^T}{\partial \alpha}\mathbf{A}^{-T} - \mathbf{u}^T\frac{\partial \mathbf{B}_u^T}{\partial \alpha}\mathbf{A}^{-T} \tag{17.198}$$

The cost function can be written as

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{S}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \tag{17.199}$$

where

$$\begin{aligned}
\mathbf{Q} &= \hat{\mathbf{Q}} + \frac{\partial \mathbf{A}^T}{\partial \alpha}\mathbf{A}^{-T}\mathbf{W}_{\alpha\alpha}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \alpha} \\
\mathbf{S} &= \frac{\partial \mathbf{A}^T}{\partial \alpha}\mathbf{A}^{-T}\mathbf{W}_{\alpha\alpha}\mathbf{A}^{-1}\frac{\partial \mathbf{B}_u}{\partial \alpha} \\
\mathbf{R} &= \hat{\mathbf{R}} + \frac{\partial \mathbf{B}_u^T}{\partial \alpha}\mathbf{A}^{-T}\mathbf{W}_{\alpha\alpha}\mathbf{A}^{-1}\frac{\partial \mathbf{B}_u}{\partial \alpha}
\end{aligned} \tag{17.200}$$

17.13 Finite-horizon optimal control

The *finite-horizon* optimal control problem of a LTI system is defined as the solution of the control input vector $\mathbf{u}(t)$ which is capable of minimizing the following cost functional

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt + \frac{1}{2}\mathbf{x}_f^T\mathbf{Z}\mathbf{x}_f \tag{17.201}$$

subjected to initial state vector

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

and to the constraint imposed by the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

where t_f is the specified final time and $\mathbf{x}_f$ is a free (not prescribed) final state vector. It is assumed that

- $\mathbf{Q}$ is a symmetric nonnegative matrix
- $\mathbf{R}$ is a symmetric positive definite matrix
- $\mathbf{Z}$ is a symmetric nonnegative matrix

Note that the above cost function differ from what presented for the LQR design in two aspects. First, it is observed that the optimal tradeoff between regulation performance and control effort is sought in a finite time interval $[t_0, t_f]$. Second, a terminal cost is added to introduce a penalty on the final system state. The optimization problem at hand is to determine an optimal input signal with the goal of maintaining the state trajectory *close* to the equilibrium at the origin while expending *moderate* control effort.

The solution of the optimization problem can be obtained by a variational approach. The constrained minimization is related to the following cost functional

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + 2\boldsymbol{\lambda}^T\left(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} - \dot{\mathbf{x}}\right)\right] dt + \frac{1}{2}\mathbf{x}_f^T\mathbf{Z}\mathbf{x}_f \tag{17.202}$$

where $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. The necessary condition for the optimum is that the variation of J vanishes, i.e.,

$$\delta J = 0 \tag{17.203}$$

The variation is given by

$$\begin{aligned}\delta J &= \int_{t_0}^{t_f} \left[\delta\mathbf{x}^T\mathbf{Q}\mathbf{x} + \delta\mathbf{u}^T\mathbf{R}\mathbf{u} + \delta\boldsymbol{\lambda}^T\left(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} - \dot{\mathbf{x}}\right) + \boldsymbol{\lambda}^T\left(\mathbf{A}\delta\mathbf{x} + \mathbf{B}_u\delta\mathbf{u} - \delta\dot{\mathbf{x}}\right)\right] dt + \delta\mathbf{x}_f^T\mathbf{Z}\mathbf{x}_f \\ &= \int_{t_0}^{t_f} \left[\delta\mathbf{x}^T\mathbf{Q}\mathbf{x} + \delta\mathbf{u}^T\mathbf{R}\mathbf{u} + \delta\boldsymbol{\lambda}^T\left(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} - \dot{\mathbf{x}}\right) + \delta\mathbf{x}^T\mathbf{A}^T\boldsymbol{\lambda} + \delta\mathbf{u}^T\mathbf{B}_u^T\boldsymbol{\lambda} - \delta\dot{\mathbf{x}}^T\boldsymbol{\lambda}\right] dt + \delta\mathbf{x}_f^T\mathbf{Z}\mathbf{x}_f \\ &= \int_{t_0}^{t_f} \left[\delta\mathbf{x}^T\mathbf{Q}\mathbf{x} + \delta\mathbf{u}^T\mathbf{R}\mathbf{u} + \delta\boldsymbol{\lambda}^T\left(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} - \dot{\mathbf{x}}\right) + \delta\mathbf{x}^T\mathbf{A}^T\boldsymbol{\lambda} + \delta\mathbf{u}^T\mathbf{B}_u^T\boldsymbol{\lambda}\right] dt \\ &\quad - \delta\mathbf{x}^T\boldsymbol{\lambda}\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \delta\mathbf{x}^T\dot{\boldsymbol{\lambda}} dt + \delta\mathbf{x}_f^T\mathbf{Z}\mathbf{x}_f\end{aligned} \tag{17.204}$$

Since the state vector is prescribed at initial time t_0, i.e., $\mathbf{x}(t_0) = \mathbf{x}_0$, the corresponding variation is equal to zero. Therefore, we can write

$$\delta J = \int_{t_0}^{t_f} \left[\delta\mathbf{x}^T\left(\mathbf{Q}\mathbf{x} + \mathbf{A}^T\boldsymbol{\lambda} + \dot{\boldsymbol{\lambda}}\right) + \delta\mathbf{u}^T\left(\mathbf{R}\mathbf{u} + \mathbf{B}_u^T\boldsymbol{\lambda}\right) + \delta\boldsymbol{\lambda}^T\left(\mathbf{A}\mathbf{x} + \mathbf{B}_u\mathbf{u} - \dot{\mathbf{x}}\right)\right] + \delta\mathbf{x}_f^T\left(\mathbf{Z}\mathbf{x}_f - \boldsymbol{\lambda}_f\right) \tag{17.205}$$

where $\boldsymbol{\lambda}_f = \boldsymbol{\lambda}(t_f)$. Since all the variations are arbitrary, the optimum is achieved if

$$\begin{aligned}&\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) \\ &\dot{\boldsymbol{\lambda}}(t) = -\mathbf{A}^T\boldsymbol{\lambda}(t) - \mathbf{Q}\mathbf{x}(t) \\ &\mathbf{R}\mathbf{u}(t) + \mathbf{B}_u^T\boldsymbol{\lambda}(t) = \mathbf{0} \\ &\mathbf{Z}\mathbf{x}_f = \boldsymbol{\lambda}_f\end{aligned} \tag{17.206}$$

From the third equation we obtain the optimal control as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}_u^T\boldsymbol{\lambda}(t) \tag{17.207}$$

where we notice that $\mathbf{R}$ must be positive definite so that its inverse exists. By inserting such optimal solution into the first equation of the set (17.206) we can write the following differential equations

$$\begin{aligned}&\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\boldsymbol{\lambda}(t) \\ &\dot{\boldsymbol{\lambda}}(t) = -\mathbf{Q}\mathbf{x}(t) - \mathbf{A}^T\boldsymbol{\lambda}(t)\end{aligned} \tag{17.208}$$

which is subjected to the following conditions

$$\begin{aligned}&\mathbf{x}(t_0) = \mathbf{x}_0 \\ &\boldsymbol{\lambda}(t_f) = \mathbf{Z}\mathbf{x}_f\end{aligned} \tag{17.209}$$

The differential problem, known as Hamiltonian system, can be put in matrix form as

$$\begin{Bmatrix}\dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}}\end{Bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{E} \\ -\mathbf{Q} & -\mathbf{A}^T\end{bmatrix}\begin{Bmatrix}\mathbf{x} \\ \boldsymbol{\lambda}\end{Bmatrix} \tag{17.210}$$

where $\mathbf{E} = \mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T$. The vector $\boldsymbol{\lambda}$ is also called *costate*. The final condition on the costate $\boldsymbol{\lambda}_f$ together with the initial condition on the state $\mathbf{x}_0$ and the Hamiltonian system of equations (17.210) form a two-point boundary value problem.

The above problem can be formulated as a closed-loop optimal control solution. The goal is to write the optimal control law, which is written in Eq. (17.207) as a function of the costate $\boldsymbol{\lambda}(t)$, in terms of the state vector $\mathbf{x}(t)$. This can be done by examining the final condition on $\boldsymbol{\lambda}_f$, which relates the final costate in terms of the final state. Similarly, we may like to connect the costate with the state not only at the final time but for the complete time horizon $[t_0, t_f]$. Thus, let us assume

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t) \tag{17.211}$$

where the time-variant $\mathbf{P}(t)$ matrix is yet to be determined. Accordingly, the optimal control input is expressed as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t)\mathbf{x}(t) \tag{17.212}$$

which is now a full state feedback control. Note that it can be also written as

$$\mathbf{u}(t) = -\mathbf{G}(t)\mathbf{x}(t) \tag{17.213}$$

where $\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}$ is the time-variant gain matrix. Differentiating Eq. (17.211) with respect to time yields

$$\dot{\boldsymbol{\lambda}} = \dot{\mathbf{P}}\mathbf{x} + \mathbf{P}\dot{\mathbf{x}} \tag{17.214}$$

Using the equations of the Hamiltonian system, we get

$$-\mathbf{Q}\mathbf{x}(t) - \mathbf{A}^T\mathbf{P}(t)\mathbf{x}(t) = \dot{\mathbf{P}}(t)\mathbf{x}(t) + \mathbf{P}(t)\left[\mathbf{A}\mathbf{x}(t) - \mathbf{E}\mathbf{P}(t)\mathbf{x}(t)\right] \tag{17.215}$$

which must hold for any value of $\mathbf{x}(t)$. This clearly means that the matrix $\mathbf{P}(t)$ satisfies the matrix differential equation

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t) + \mathbf{Q} = \mathbf{0} \tag{17.216}$$

which is known as *differential Riccati equation*. Comparing the final condition on the costate with the transformation (17.211), we have the final condition on $\mathbf{P}(t)$ as

$$\mathbf{P}(t_f) = \mathbf{Z} \tag{17.217}$$

Thus, the Riccati equation (17.216) is to be solved *backward* in time from the final condition (17.217) to obtain the solution $\mathbf{P}(t)$ for the entire interval $[t_0, t_f]$.

The matrix $\mathbf{P}(t)$ is a symmetric matrix. Indeed, by taking the transpose of the Riccati equation, due to the symmetry of the matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{Z}$, we can write

$$\dot{\mathbf{P}}^T(t) + \mathbf{P}^T(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}^T(t) - \mathbf{P}^T(t)\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}^T(t) + \mathbf{Q} = \mathbf{0} \tag{17.218}$$

with the final condition

$$\mathbf{P}^T(t_f) = \mathbf{Z} \tag{17.219}$$

Therefore, it is noticed that *both* $\mathbf{P}$ and $\mathbf{P}^T$ are solutions of the *same* differential equation and both satisfy the same final condition. It follows that $\mathbf{P} = \mathbf{P}^T$.

The optimum value of the cost functional can be obtained by putting the optimal feedback control solution (17.212) into the expression of J as follows

$$J_{\text{opt}} = \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}^T\left(\mathbf{Q} + \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}\right)\mathbf{x}\right]dt + \frac{1}{2}\mathbf{x}_f^T\mathbf{Z}\mathbf{x}_f \tag{17.220}$$

Since the closed-loop dynamics is governed by

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A} - \mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t)\right]\mathbf{x}(t) \tag{17.221}$$

we can write

$$\begin{aligned}
\int_{t_0}^{t_f} \frac{d}{dt}\left(\mathbf{x}^T\mathbf{P}\mathbf{x}\right)dt &= \mathbf{x}^T(t_f)\mathbf{P}(t_f)\mathbf{x}(t_f) - \mathbf{x}^T(t_0)\mathbf{P}(t_0)\mathbf{x}(t_0) \\
&= \int_{t_0}^{t_f}\left(\dot{\mathbf{x}}\mathbf{P}\mathbf{x} + \mathbf{x}^T\dot{\mathbf{P}}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}}\right)dt \\
&= \int_{t_0}^{t_f}\left[\mathbf{x}^T\left(\mathbf{A}^T - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\right)\mathbf{P}\mathbf{x} + \mathbf{x}^T\left(-\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P} - \mathbf{Q} + \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}\right)\mathbf{x}\right. \\
&\qquad \left. + \mathbf{x}^T\mathbf{P}\left(\mathbf{A} - \mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}\right)\mathbf{x}\right]dt \\
&= -\int_{t_0}^{t_f}\mathbf{x}^T\left(\mathbf{Q} + \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}\right)\mathbf{x}dt \\
&= -2J_{\text{opt}} + \mathbf{x}^T(t_f)\mathbf{Z}\mathbf{x}(t_f)
\end{aligned}$$

Therefore, we have

$$J_{\text{opt}} = \frac{1}{2}\mathbf{x}^T(t_0)\mathbf{P}(t_0)\mathbf{x}(t_0) \tag{17.222}$$

which is given by a quadratic form involving initial state and the differential Riccati equation solution evaluated at the initial time.

17.14 Optimal tracking

The function of the optimal control law presented thus far is to hold the state vector near zero, that is to guarantee closed-loop stability. Another fundamental optimal design problem is to control a system so that a specified output $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ follows a given nonzero reference trajectory $\mathbf{r}(t)$. An example is controlling a spacecraft to follow a desired step input command (e.g., change in attitude). This is called *optimal tracking*. For this purpose, the performance index is modified as follows

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(\mathbf{e}^T\mathbf{Q}\mathbf{e} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt + \frac{1}{2}\mathbf{e}_f^T\mathbf{Z}\mathbf{e}_f \tag{17.223}$$

where

$$\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{r}(t) \tag{17.224}$$

and $\mathbf{e}_f = \mathbf{e}(t_f) = \mathbf{y}(t_f) - \mathbf{r}(t_f)$. By following the same procedure used before (the proof is left to the reader), it can be shown that the solution of the problem yields the following optimal control law

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t)\mathbf{x}(t) - \mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{s}(t) \tag{17.225}$$

where $\mathbf{P}$ is the solution of the following Riccati equation

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}(t) + \mathbf{C}^T\mathbf{Q}\mathbf{C} = \mathbf{0} \tag{17.226}$$

with the final condition

$$\mathbf{P}(t_f) = \mathbf{C}^T\mathbf{Z}\mathbf{C} \tag{17.227}$$

and the feedforward signal $\mathbf{s}(t)$ is the solution of the following differential equation

$$\dot{\mathbf{s}}(t) + \left(\mathbf{A}^T - \mathbf{P}\mathbf{B}_u\mathbf{R}^{-1}\mathbf{B}_u^T\right)\mathbf{s}(t) - \mathbf{C}^T\mathbf{Q}\mathbf{r}(t) = \mathbf{0} \tag{17.228}$$

with the final condition

$$\mathbf{s}(t_f) = \mathbf{C}^T\mathbf{Z}\mathbf{r}(t_f) \tag{17.229}$$

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