Omdurman Islamic University

Faculty of Engineering

Electrical & Electronic Engineering (4th year)

Signal Processing and Systems

Lecturer

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Module 4

Continuous-Time Fourier Series (CTFS)

Course Description (Part 1)

- Module 1: Introduction to signals and systems.
- Module 2: Continuous-Time (CT) Signals and Systems
- Module 3: Continuous-Time Linear Time-Invariant (LTI) Systems
- Module 4: Continuous-Time Fourier Series (CTFS)
- Module 5: Continuous-Time Fourier Transform (CTFT)
- Module 6: Laplace Transform (LT)

Course Description (Part2)

- Module 1: Introduction to Digital signal Processing.
- Module 2: Analogue to Digital conversion, Sampling, Quantization
- Module 3-1: Digital signal and systems .
- Module 3-2: LTI systems described by difference equations.
- Module 4-1: Discrete Time Fourier Transform.
- Module 4-2: Fast Fourier Transforms (FFT).

Course Description (Part2)

- Module 5: Z Transform
- Module 6: Basic Filtering Types
- Module 7: FIR Filters design, implementation.
- Module 8: IIR Filters design, implementation.

Part 1

Continuous-Time Fourier Series (CTFS)

Introduction

- The Fourier series is a representation for *periodic* signals.
- With a Fourier series, a signal is represented as a *linear combination of complex sinusoids*.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- For example, complex sinusoids are continuous and differentiable. They are also easy to integrate and differentiate.
- Perhaps, most importantly, complex sinusoids are *eigenfunctions* of LTI systems.

Section 4.1

Fourier Series

Harmonically-Related Complex Sinusoids

- A set of complex sinusoids is said to be harmonically related if there exists some constant ω_0 such that the fundamental frequency of each complex sinusoid is an integer multiple of ω_0.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(t) = e^{jk\omega_0 t} \qquad \text{for all integer } k.$$

- The fundamental frequency of the kth complex sinusoid ϕ_k is $k\omega_0$, an integer multiple of ω_0.
- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic.
- More specifically, a linear combination of these complex sinusoids is periodic with period $T = 2\pi/\omega_0$.

CT Fourier Series

- A periodic complex signal x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Such a representation is known as (the complex exponential form of) a (CT) Fourier series, and the c_k are called Fourier series coefficients.
- The above formula for x is often referred to as the Fourier series synthesis equation.
- The terms in the summation for $k = K$ and $k = -K$ are called the Kth harmonic components, and have the fundamental frequency $K\omega_0$.
- To denote that a signal x has the Fourier series coefficient sequence c_k, we write

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k.$$

- The periodic signal x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ has the Fourier series coefficients c_k given by

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

where $\int_T$ denotes integration over an arbitrary interval of length T (i.e., one period of x).

- The above equation for c_k is often referred to as the Fourier series analysis equation.

- Consider the periodic signal x with the Fourier series coefficients c_k.
- If x is real, then its Fourier series can be rewritten in two other forms, known as the combined trigonometric and trigonometric forms.
- The combined trigonometric form of a Fourier series has the appearance

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k),$$

  where $\theta_k = \arg c_k$.
- The trigonometric form of a Fourier series has the appearance

$$x(t) = c_0 + \sum_{k=1}^{\infty} [\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t],$$

  where $\alpha_k = 2\operatorname{Re} c_k$ and $\beta_k = -2\operatorname{Im} c_k$.
- Note that the trigonometric forms contain only *real* quantities.

Section 4.2

Convergence Properties of Fourier Series

- Since a Fourier series can have an infinite number of terms, and an infinite sum may or may not converge, we need to consider the issue of convergence.
- That is, when we claim that a periodic signal $x(t)$ is equal to the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, is this claim actually correct?
- Consider a periodic signal x that we wish to represent with the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Let x_N denote the Fourier series truncated after the Nth harmonic components as given by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

- Here, we are interested in whether $\lim_{N\to\infty} x_N(t)$ is equal (in some sense) to $x(t)$.

Convergence of Fourier Series (Continued)

- The *error* in approximating $x(t)$ by $x_N(t)$ is given by

$$e_N(t) = x(t) - x_N(t),$$

and the corresponding *mean-squared error (MSE)* (i.e., energy of the error) is given by

$$E_N = \frac{1}{T}\int_T |e_N(t)|^2\,dt.$$

- If $\lim_{N\to\infty} e_N(t) = 0$ for all t (i.e., the error goes to zero at every point), the Fourier series is said to converge pointwise to $x(t)$.
- If convergence is pointwise and the rate of convergence is the same everywhere, the convergence is said to be uniform.
- If $\lim_{N\to\infty} E_N = 0$ (i.e., the energy of the error goes to zero), the Fourier series is said to converge to x in the MSE sense.
- Pointwise convergence implies MSE convergence, but the converse is not true. Thus, pointwise convergence is a much stronger condition than MSE convergence.

Convergence of Fourier Series: Continuous Case

- If a periodic signal x is *continuous* and its Fourier series coefficients c_k are *absolutely summable* (i.e., $\sum_{k=-\infty}^{\infty} |c_k| < \infty$), then the Fourier series representation of x converges *uniformly* (i.e., pointwise at the same rate everywhere).
- Since, in practice, we often encounter signals with discontinuities (e.g., a square wave), the above result is of somewhat limited value.

Convergence of Fourier Series: Finite-Energy Case

- If a periodic signal x has *finite energy* in a single period (i.e., $\int_T |x(t)|^2 dt < \infty$), the Fourier series converges in the *MSE* sense.
- Since, in situations of practice interest, the finite-energy condition in the above theorem is typically satisfied, the theorem is usually applicable.
- It is important to note, however, that MSE convergence (i.e., $E = 0$) does not necessarily imply pointwise convergence (i.e., $\tilde{x}(t) = x(t)$ for all t).
- Thus, the above convergence theorem does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier series converges.

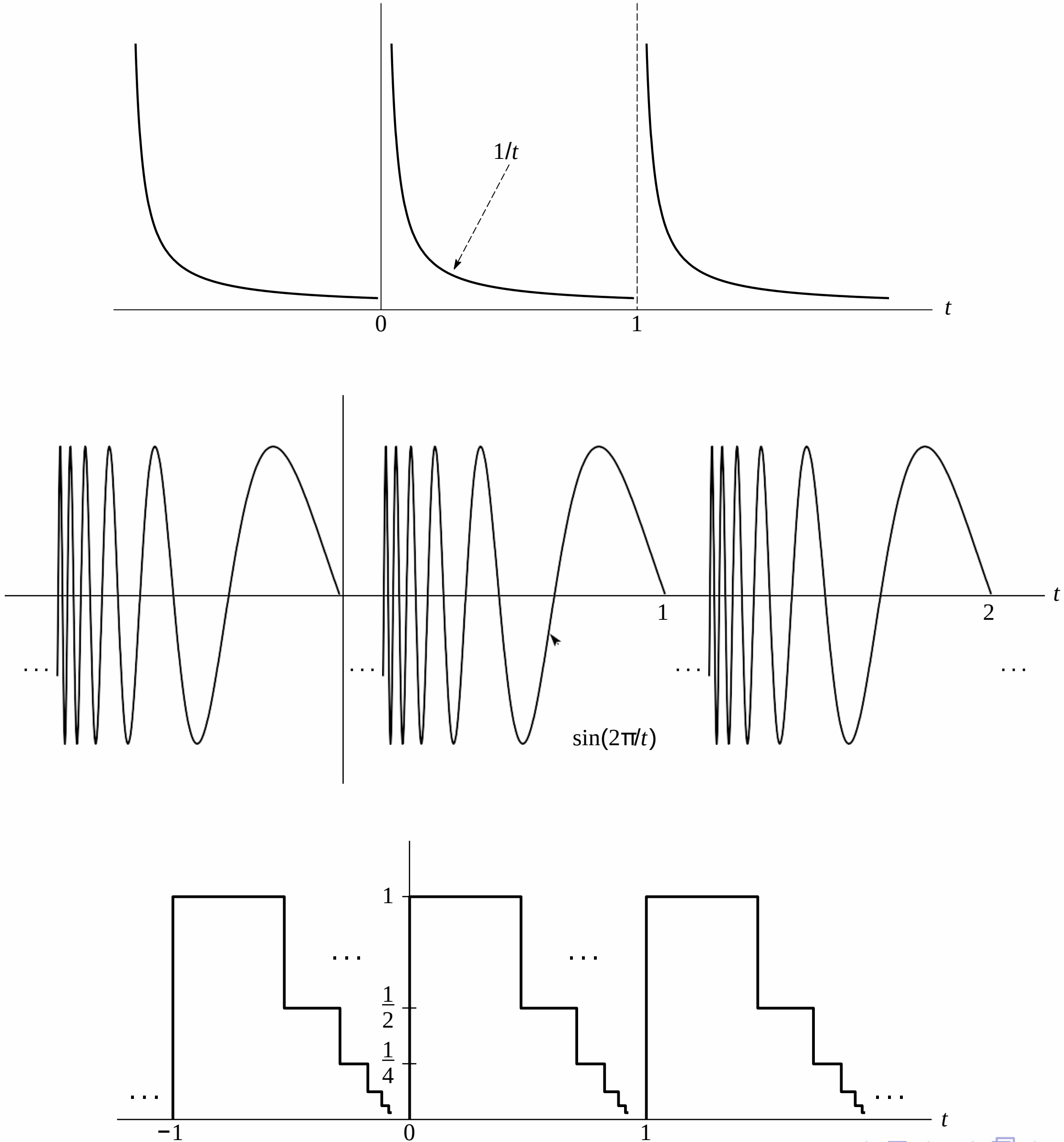
- The Dirichlet conditions for the periodic signal x are as follows:
  1. Over a single period, x is *absolutely integrable* (i.e., $\int_T |x(t)|\,dt < \infty$)
  2. Over a single period, x has a finite number of maxima and minima (i.e., x is of *bounded variation*).
  3. Over any finite interval, x has a *finite number of discontinuities*, each of which is *finite*.
- If a periodic signal x satisfies the *Dirichlet conditions*, then:
  1. The Fourier series converges pointwise everywhere to x, except at the points of discontinuity of x.
  2. At each point $t = t_a$ of discontinuity of x, the Fourier series $\tilde{x}$ converges to

$$\tilde{x}(t_a) = \frac{1}{2}\left[x(t_a^-) + x(t_a^+)\right],$$

where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal x on the left- and right-hand sides of the discontinuity, respectively.

- Since most signals tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier series at every point, this result is often very useful in practice.

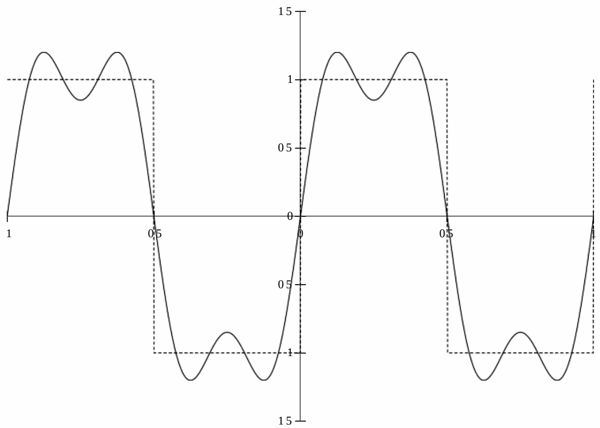
Examples of Functions Violating the Dirichlet Conditions



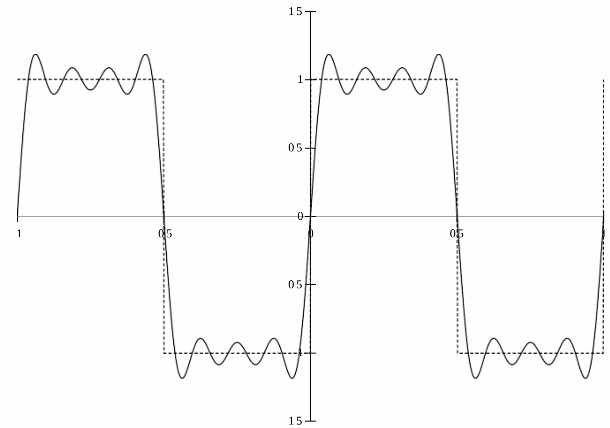
Gibbs Phenomenon

- In practice, we frequently encounter signals with discontinuities.
- When a signal x has discontinuities, the Fourier series representation of x does not converge uniformly (i.e., at the same rate everywhere).
- The rate of convergence is much slower at points in the vicinity of a discontinuity.
- Furthermore, in the vicinity of a discontinuity, the truncated Fourier series x_N exhibits ripples, where the peak amplitude of the ripples does not seem to decrease with increasing N.
- As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N, the peak amplitude of the ripples remains approximately constant.
- This behavior is known as Gibbs phenomenon.
- The above behavior is one of the weaknesses of Fourier series (i.e., Fourier series converge very slowly near discontinuities).

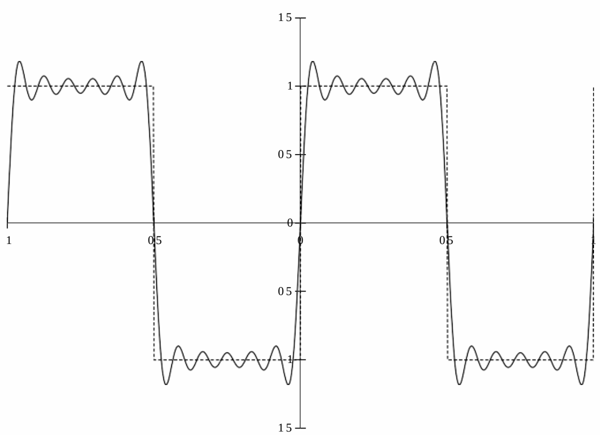
Gibbs Phenomenon: Periodic Square Wave Example



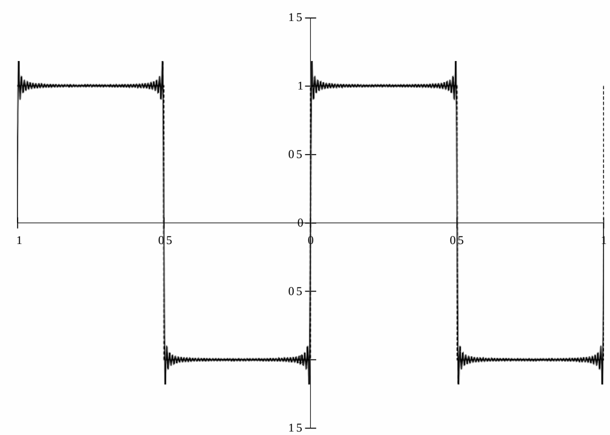
Fourier series truncated after the 3rd harmonic components



Fourier series truncated after the 7th harmonic components



Fourier series truncated after the 11th harmonic components



Fourier series truncated after the 101th harmonic components

Section
4.3

Properties of Fourier Series

Properties of (CT) Fourier Series

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$ x	a^*_{-k}
Even symmetry	even	a even
Odd symmetry	x odd	a odd
Real	$x(t)$ real	$a_k = a^*{}_{-k}$

Property	
Parseval's relation	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$

Linearity

- Let x and y be two periodic signals with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$,
then
$$\alpha x(t) + \beta y(t) \overset{\text{CTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$
where α and β are complex constants.
- That is, a linear combination of signals produces the same linear combination of their Fourier series coefficients.

Time Shifting (Translation)

- Let x denote a periodic signal with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(t - t_0) \overset{\text{CTFS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} c_k = e^{-jk(2\pi/T)t_0} c_k,$$

  where t_0 is a real constant.
- In other words, time shifting a periodic signal changes the argument (but not magnitude) of its Fourier series coefficients.

Time Reversal (Reflection)

- Let x denote a periodic signal with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(-t) \overset{\text{CTFS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversal of a signal results in a time reversal of its Fourier series coefficients.

Conjugation

- For a T-periodic function x with Fourier series coefficient sequence c, the following properties hold:

$$x^*(t) \overset{\text{CT FS}}{\longleftrightarrow} c^*_{-k}$$

- In other words, conjugating a signal has the effect of time reversing and conjugating the Fourier series coefficient sequence.

Even and Odd Symmetry

- For a T-periodic function x with Fourier series coefficient sequence c, the following properties hold:

$$x \text{ is even} \Leftrightarrow c \text{ is even; and}$$
$$x \text{ is odd} \Leftrightarrow c \text{ is odd.}$$

- In other words, the even/odd symmetry properties of x and c always match.

- A signal x is *real* if and only if its Fourier series coefficient sequence c satisfies

$$c_k = c^*_{-k} \quad \text{for all } k$$

(i.e., c has *conjugate symmetry*).

- Thus, for a real-valued signal, the negative-indexed Fourier series coefficients are *redundant*, as they are completely determined by the nonnegative-indexed coefficients.
- From properties of complex numbers, one can show that $c_k = c^*_{-k}$ is equivalent to

$$|c_k| = |c_{-k}| \qquad \text{and} \qquad \arg c_k = -\arg c_{-k}$$

(i.e., $|c_k|$ is *even* and $\arg c_k$ is *odd*).

- Note that x being real does *not* necessarily imply that c is real.

Other Properties of Fourier Series

- For a T-periodic function x with Fourier-series coefficient sequence c, the following properties hold:
  1. c_0 is the average value of x over a single period;
  2. x is real and even $\Leftrightarrow$ c is real and even; and
  3. x is real and odd $\Leftrightarrow$ c is purely imaginary and odd.

Section
4.4

Fourier Series and Frequency Spectra

A New Perspective on Signals: The Frequency Domain

- The Fourier series provides us with an entirely new way to view signals.
- Instead of viewing a signal as having information distributed with respect to *time* (i.e., a function whose domain is time), we view a signal as having information distributed with respect to *frequency* (i.e., a function whose domain is frequency).
- This so called frequency-domain perspective is of fundamental importance in engineering.
- Many engineering problems can be solved *much more easily* using the frequency domain than the time domain.
- The Fourier series coefficients of a signal x provide a means to *quantify* how much information x has at different frequencies.
- The distribution of information in a signal over different frequencies is referred to as the *frequency spectrum* of the signal.

Fourier Series and Frequency Spectra

- To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the signal x, it is helpful to write the Fourier series with the c_k expressed in *polar form* as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} |c_k| \, e^{j(k\omega_0 t + \arg c_k)}.$$

- Clearly, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has been *amplitude scaled* by a factor of $|c_k|$ and *time-shifted* by an amount that depends on $\arg c_k$.
- For a given k, the *larger $|c_k|$* is, the larger is the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the *larger the contribution* the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation.
- In this way, we can use $|c_k|$ as a *measure* of how much information a signal x has at the frequency $k\omega_0$.

Fourier Series and Frequency Spectra (Continued)

- The Fourier series coefficients c_k are referred to as the frequency spectrum of x.
- The magnitudes $|c_k|$ of the Fourier series coefficients are referred to as the magnitude spectrum of x.
- The arguments $\arg c_k$ of the Fourier series coefficients are referred to as the phase spectrum of x.
- Normally, the spectrum of a signal is plotted against frequency $k\omega_0$ instead of k.
- Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, the frequency spectrum is *discrete* in the independent variable (i.e., frequency).
- Due to the general appearance of frequency-spectrum plot (i.e., a number of vertical lines at various frequencies), we refer to such spectra as line spectra.

- Recall that, for a *real* signal x, the Fourier series coefficient sequence c satisfies

$$c_k = c^*_{-k}$$

(i.e., c is *conjugate symmetric*), which is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}.$$

- Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a *real* signal is always *even*.
- Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a *real* signal is always *odd*.
- Due to the symmetry in the frequency spectra of real signals, we typically *ignore negative frequencies* when dealing with such signals.
- In the case of signals that are complex but not real, frequency spectra do not possess the above symmetry, and *negative frequencies become important*.

Section 4.5

Fourier Series and LTI Systems

Frequency Response

- Recall that a LTI system $\mathcal{H}$ with impulse response h is such that $\mathcal{H}\{e^{st}\} = H(s)e^{st}$, where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$. (That is, complex exponentials are *eigenfunctions* of LTI systems.)
- Since a complex sinusoid is a *special case* of a complex exponential, we can reuse the above result for the special case of complex sinusoids.
- For a LTI system $\mathcal{H}$ with impulse response h and a complex sinusoid $e^{j\omega t}$ (where ω is a real constant),

$$\mathcal{H}\{e^{j\omega t}\} = H(j\omega)e^{j\omega t},$$

where

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt.$$

- That is, $e^{j\omega t}$ is an *eigenfunction* of a LTI system and $H(j\omega)$ is the corresponding *eigenvalue*.
- We refer to $H(j\omega)$ as the frequency response of the system $\mathcal{H}$.

Fourier Series and LTI Systems

- Consider a LTI system with input x, output y, and frequency response $H(j\omega)$.
- Suppose that the T-periodic input x is expressed as the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{where } \omega_0 = 2\pi/T.$$

- Using our knowledge about the *eigenfunctions* of LTI systems, we can conclude

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t}.$$

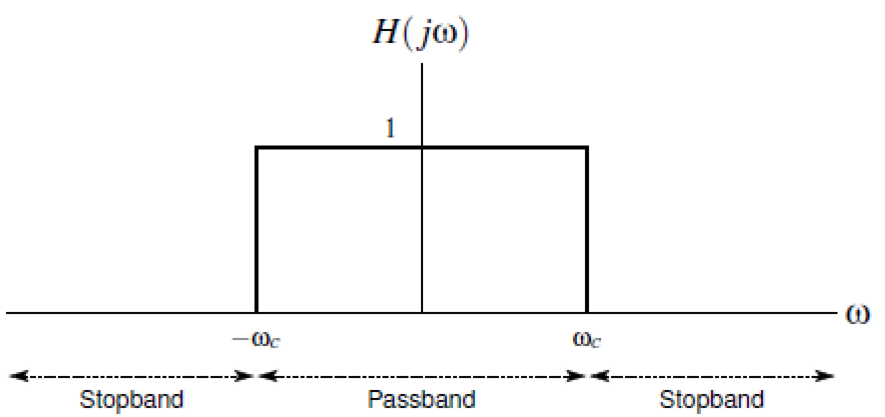
- Thus, if the input x to a LTI system is a Fourier series, the output y is also a Fourier series. More specifically, if $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$ then $y(t) \overset{\text{CTFS}}{\longleftrightarrow} H(jk\omega_0)c_k$.
- The above formula can be used to determine the output of a LTI system from its input in a way that *does not require convolution*.

- In many applications, we want to *modify the spectrum* of a signal by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a signal is called filtering.
- A system that performs a filtering operation is called a filter.
- Many types of filters exist.
- Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

- An ideal lowpass filter eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency.
- A plot of this frequency response is given below.



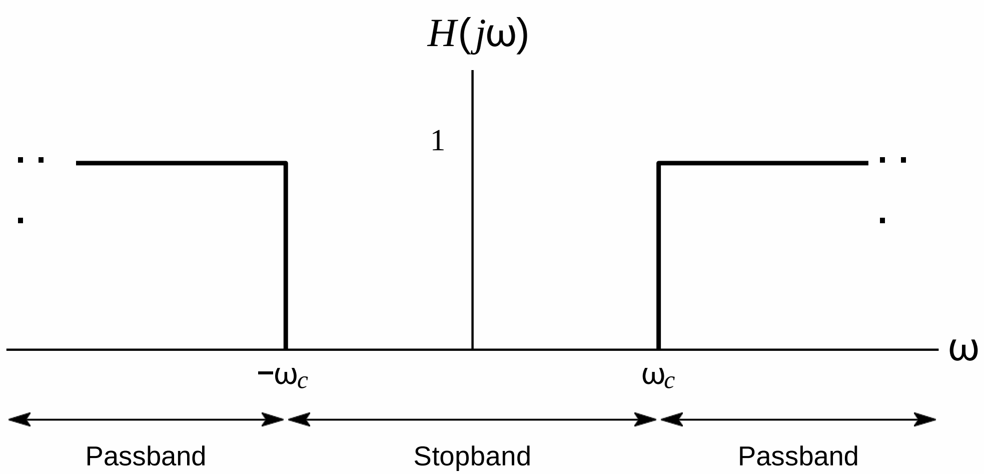
Ideal Highpass Filter

- An ideal highpass filter eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency.

- A plot of this frequency response is given below.



Ideal Bandpass Filter

- An ideal bandpass filter eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.