Omdurman Islamic University

Faculty of Engineering

Electrical & Electronic Engineering

(4th year)

Signal Processing and Systems

Lecturer

FAWAZ FATHI

Module 6

Laplace Transform (LT)

Course Description (Part 1)

- Module 1: Introduction to signals and systems.
- Module 2: Continuous-Time (CT) Signals and Systems
- Module 3: Continuous-Time Linear Time-Invariant (LTI) Systems
- Module 4: Continuous-Time Fourier Series (CTFS)
- Module 5: Continuous-Time Fourier Transform (CTFT)
- Module 6: Laplace Transform (LT)

Course Description (Part2)

- Module 1: Introduction to Digital signal Processing.
- Module 2: Analogue to Digital conversion, Sampling, Quantization
- Module 3-1: Digital signal and systems .
- Module 3-2: LTI systems described by difference equations.
- Module 4-1: Discrete Time Fourier Transform.
- Module 4-2: Fast Fourier Transforms (FFT).

Course Description (Part2)

- Module 5: Z Transform
- Module 6: Basic Filtering Types
- Module 7: FIR Filters design, implementation.
- Module 8: IIR Filters design, implementation.

Part 1

Laplace Transform (LT)

Motivation Behind the Laplace Transform

- Another important mathematical tool in the study of signals and systems is known as the Laplace transform.
- The Laplace transform can be viewed as a *generalization of the Fourier transform*.
- Due to its more general nature, the Laplace transform has a number of *advantages* over the Fourier transform.
- First, the Laplace transform representation exists for some signals that do not have Fourier transform representations. So, we can handle a *larger class of signals* with the Laplace transform.
- Second, since the Laplace transform is a more general tool, it can provide *additional insights* beyond those facilitated by the Fourier transform.

- Earlier, we saw that complex exponentials are eigenfunctions of LTI systems.
- In particular, for a LTI system $\mathcal{H}$ with impulse response h, we have that

$$\mathcal{H}\{e^{st}\} = H(s)e^{st} \quad \text{where} \quad H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$

- Previously, we referred to H as the system function.
- As it turns out, H is the Laplace transform of h.
- Since the Laplace transform has already appeared earlier in the context of LTI systems, it is clearly a useful tool.
- Furthermore, as we will see, the Laplace transform has many additional uses.

Section 6.1

Laplace Transform

(Bilateral) Laplace Transform

- The (bilateral) Laplace transform of the function x, denoted $\mathcal{L}\{x\}$ or X, is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt.$$

- The inverse Laplace transform of X, denoted $\mathcal{L}^{-1}\{X\}$ or x, is then given by

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds,$$

  where $\operatorname{Re}\{s\} = \sigma$ is in the ROC of X. (Note that this is a *contour integration*, since s is complex.)

- We refer to x and X as a Laplace transform pair and denote this relationship as

$$x(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s).$$

- In practice, we do not usually compute the inverse Laplace transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

Bilateral and Unilateral Laplace Transforms

- Two different versions of the Laplace transform are commonly used:
  1. the *bilateral* (or *two-sided*) Laplace transform; and
  2. the *unilateral* (or *one-sided*) Laplace transform.
- The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions.
- As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the *lower limit of integration*.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0.
- For the most part, we will focus our attention primarily on the bilateral Laplace transform.
- We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations.
- Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean *bilateral* Laplace transform.

Relationship Between Laplace and Fourier Transforms

- Let X and X_{F} denote the Laplace and (CT) Fourier transforms of x, respectively.
- The function $X(s)$ evaluated at $s = j\omega$ (where ω is real) yields $X_{\mathrm{F}}(\omega)$. That is,

$$X(s)|_{s=j\omega} = X_{\mathrm{F}}(\omega).$$

- Due to the preceding relationship, the Fourier transform of x is sometimes written as $X(j\omega)$.
- The function $X(s)$ evaluated at an arbitrary complex value $s = \sigma + j\omega$ (where $\sigma = \mathrm{Re}\{s\}$ and $\omega = \mathrm{Im}\{s\}$) can also be expressed in terms of a Fourier transform involving x. In particular, we have

$$X(s)|_{s=\sigma+j\omega} = X'_{\mathrm{F}}(\omega),$$

where X'_{F} is the (CT) Fourier transform of $x'(t) = e^{-\sigma t}x(t)$.
- So, in general, the Laplace transform of x is the Fourier transform of an exponentially-weighted version of x.
- Due to this weighting, the Laplace transform of a signal may exist when the Fourier transform of the same signal does not.

Section 6.2

Region of Convergence (ROC)

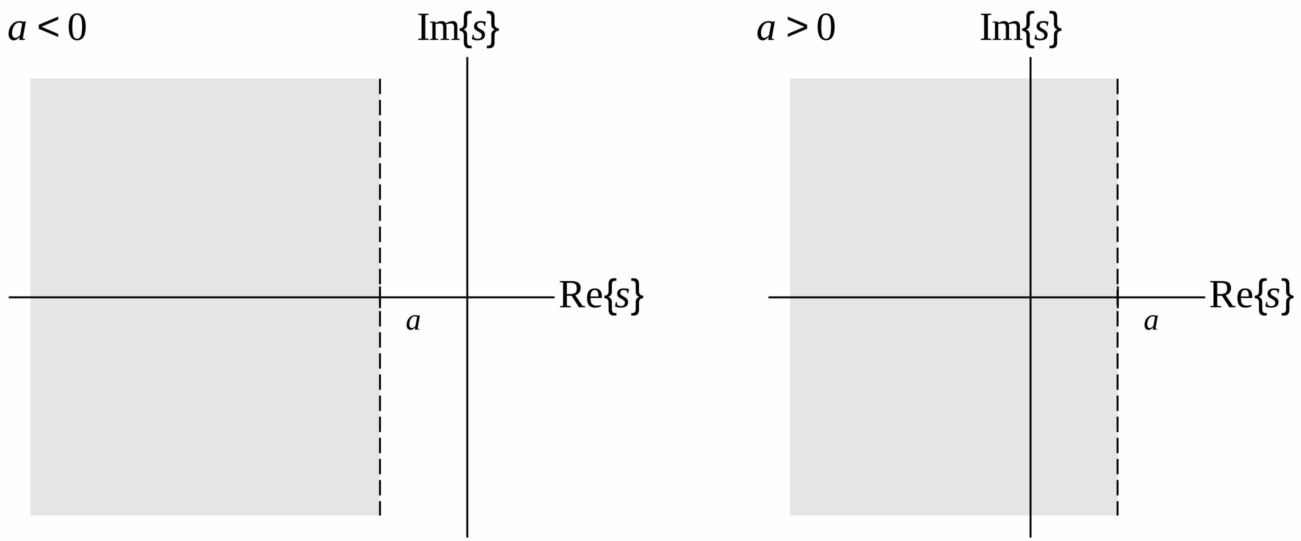
Left-Half Plane (LHP)

- The set R of all complex numbers s satisfying

$$\text{Re}\{s\} < a$$

for some real constant a is said to be a left-half plane (LHP).

- Some examples of LHPs are shown below.



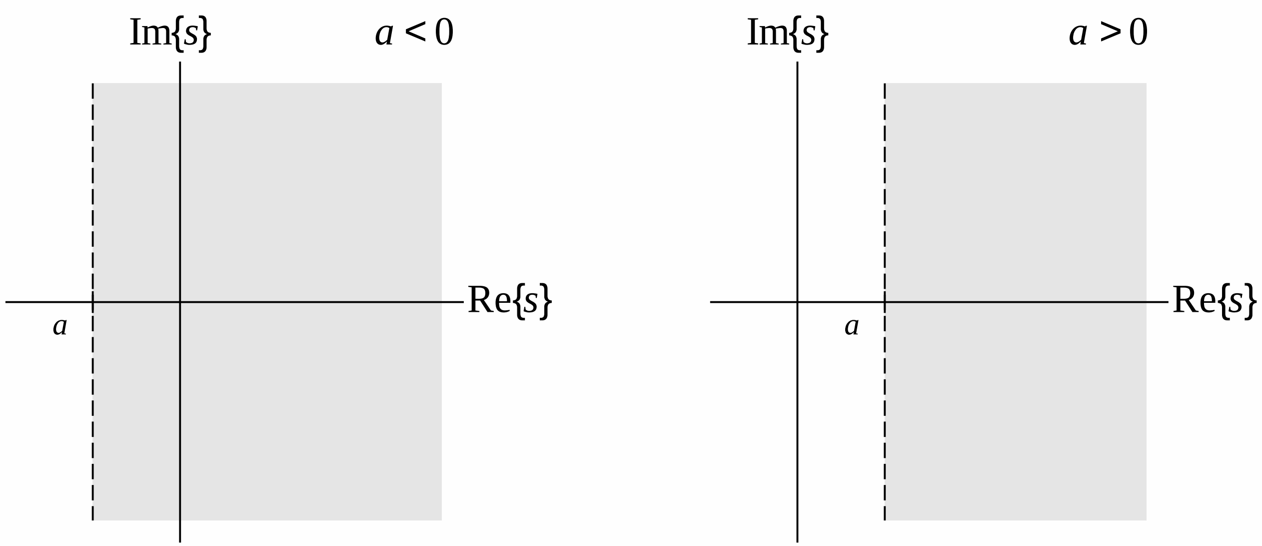
Right-Half Plane (RHP)

- The set R of all complex numbers s satisfying

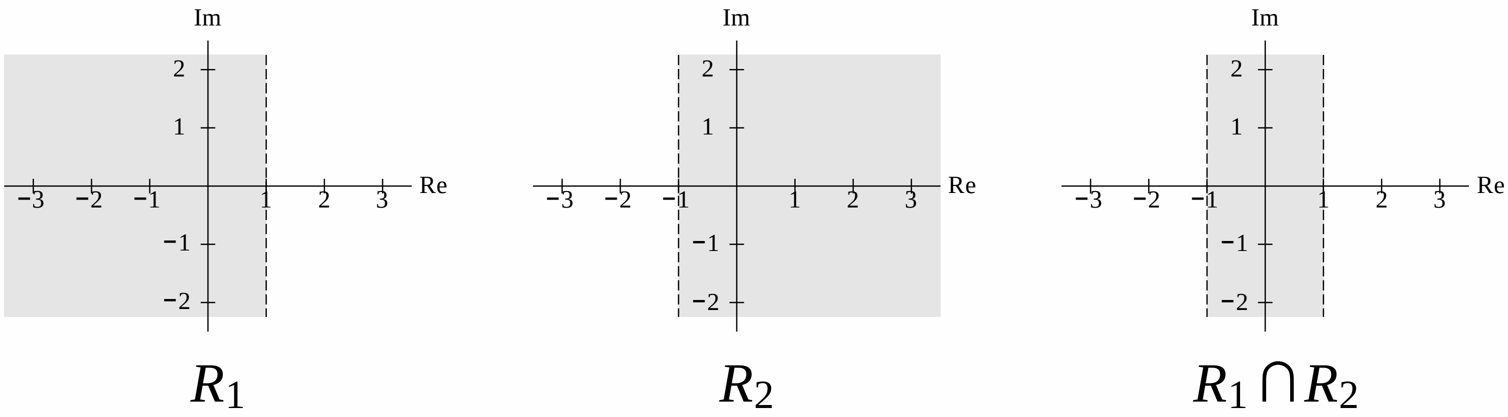
$$\operatorname{Re}\{s\} > a$$

for some real constant a is said to be a right-half plane (RHP).

- Some examples of RHPs are shown below.



- For two sets A and B, the intersection of A and B, denoted $A \cap B$, is the set of all points that are in both A and B.
- An illustrative example of set intersection is shown below.



- As we saw earlier, for a signal x, the complete specification of its Laplace transform X requires not only an algebraic expression for X, but also the ROC associated with X.
- Two very different signals can have the same algebraic expressions for X.
- Now, we examine some of the constraints on the ROC (of the Laplace transform) for various classes of signals.

Properties of the ROC

1. The ROC of the Laplace transform X consists of *strips parallel to the imaginary axis* in the complex plane.
2. If the Laplace transform X is a *rational* function, the ROC *does not contain any poles*, and the ROC is *bounded by poles or extends to infinity*.
3. If the signal x is *finite duration* and its Laplace transform $X(s)$ converges for some value of s, then $X(s)$ converges for *all values* of s (i.e., the ROC is the entire complex plane).
4. If the signal x is *right sided* and the (vertical) line $\operatorname{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}\{x\}$, then all values of s for which $\operatorname{Re}\{s\} > \sigma_0$ must also be in the ROC (i.e., the ROC contains a *RHP* including $\operatorname{Re}\{s\} = \sigma_0$).
5. If the signal x is *left sided* and the (vertical) line $\operatorname{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}\{x\}$, then all values of s for which $\operatorname{Re}\{s\} < \sigma_0$ must also be in the ROC (i.e., the ROC contains a *LHP* including $\operatorname{Re}\{s\} = \sigma_0$).

6. If the signal x is *two sided* and the (vertical) line $\operatorname{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}\{x\}$, then the ROC will consist of a *strip* in the complex plane that includes the line $\operatorname{Re}\{s\} = \sigma_0$.
7. If the Laplace transform X of the signal x is *rational* (with at least one pole), then:
   1. If x is *right sided*, the ROC of X is to the right of the rightmost pole of X (i.e., the *RHP* to the *right of the rightmost pole*).
   2. If x is *left sided*, the ROC of X is to the left of the leftmost pole of X (i.e., the *LHP* to the *left of the leftmost pole*).

- Some of the preceding properties are *redundant* (e.g., properties 1, 2, 4, and 5 imply property 7).
- Since every function can be classified as one of finite duration, left sided but not right sided, right sided but not left sided, or two sided, we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of a LHP, RHP, vertical strip, the entire complex plane, or the empty set. Thus, the ROC must be a *connected region*.

Section
6.3

Properties of the Laplace Transform

Properties of the Laplace Transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\mathrm{Re}\{s_0\}$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathrm{Re}\{s\} > 0\}$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Laplace Transform Pairs

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\operatorname{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\operatorname{Re}\{s\} < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\operatorname{Re}\{s\} > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\operatorname{Re}\{s\} < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\operatorname{Re}\{s\} > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\operatorname{Re}\{s\} < -a$
8	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\operatorname{Re}\{s\} > -a$
9	$-t^n e^{-at}u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\operatorname{Re}\{s\} < -a$
10	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\operatorname{Re}\{s\} > 0$
11	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\operatorname{Re}\{s\} > 0$
12	$[e^{-at}\cos\omega_0 t]u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}\{s\} > -a$
13	$[e^{-at}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}\{s\} > -a$

Linearity

- If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

  $$a_1x_1(t)+a_2x_2(t) \overset{\text{LT}}{\longleftrightarrow} a_1X_1(s)+a_2X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2,$$

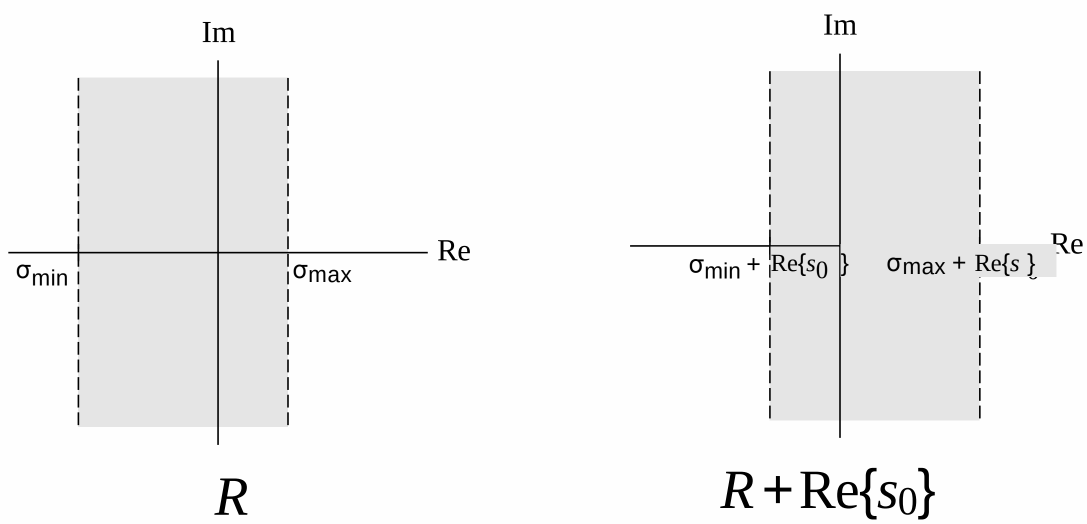
  where a_1 and a_2 are arbitrary complex constants.
- This is known as the linearity property of the Laplace transform.
- The ROC always contains the intersection but could be larger (in the case that pole-zero cancellation occurs).

Time-Domain Shifting

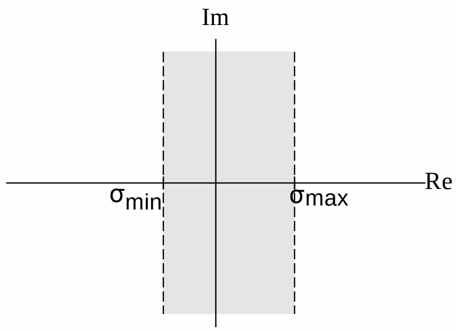
- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then
$$x(t - t_0) \overset{\text{LT}}{\longleftrightarrow} e^{-st_0}X(s) \quad \text{with ROC } R,$$
where t_0 is an arbitrary real constant.
- This is known as the time-domain shifting property of the Laplace transform.

Laplace-Domain Shifting

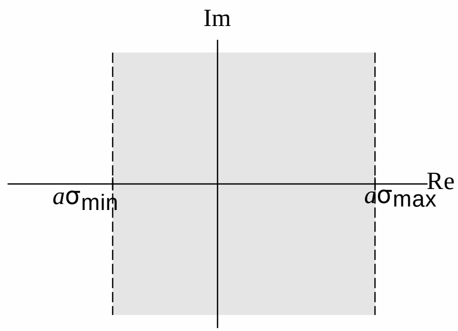
- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then
$$e^{s_0 t}x(t) \overset{\text{LT}}{\longleftrightarrow} X(s-s_0) \text{ with ROC } R+\operatorname{Re}\{s_0\},$$
where s_0 is an arbitrary complex constant.
- This is known as the Laplace-domain shifting property of the Laplace transform.
- As illustrated below, the ROC R is *shifted* right by $\operatorname{Re}\{s_0\}$.



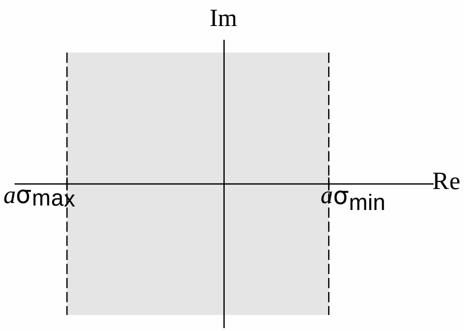
- If $x(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then
$$x(at) \stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \text{ with ROC } R_1 = aR,$$
where a is a nonzero real constant.
- This is known as the (time-domain/Laplace-domain) scaling property of the Laplace transform.
- As illustrated below, the ROC R is *scaled* and *possibly flipped* left to right.



R



aR, $a > 0$



aR, $a < 0$

Conjugation

- If $x(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x^*(t) \stackrel{\text{LT}}{\longleftrightarrow} X^*(s^*) \text{ with ROC } R.$$

- This is known as the conjugation property of the Laplace transform.

Time-Domain Convolution

- If $x_1(t) \xleftrightarrow{\text{LT}} X_1(s)$ with ROC R_1 and $x_2(t) \xleftrightarrow{\text{LT}} X_2(s)$ with ROC R_2, then

$$x_1 * x_2(t) \xleftrightarrow{\text{LT}} X_1(s)X_2(s) \text{ with ROC containing } R_1 \cap R_2.$$

- This is known as the time-domain convolution property of the Laplace transform.
- The ROC always contains the intersection but can be larger than the intersection (if pole-zero cancellation occurs).
- Convolution in the time domain becomes *multiplication* in the Laplace domain.
- Consequently, it is often much easier to work with LTI systems in the Laplace domain, rather than the time domain.

Time-Domain Differentiation

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then
$$\frac{dx(t)}{dt} \overset{\text{LT}}{\longleftrightarrow} sX(s) \text{ with ROC containing } R.$$
- This is known as the time-domain differentiation property of the Laplace transform.
- The ROC always contains R but can be larger than R (if pole-zero cancellation occurs).
- Differentiation in the time domain becomes *multiplication by s* in the Laplace domain.
- Consequently, it can often be much easier to work with differential equations in the Laplace domain, rather than the time domain.

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then
$$-tx(t) \overset{\text{LT}}{\longleftrightarrow} \frac{dX(s)}{ds} \text{ with ROC } R.$$
- This is known as the Laplace-domain differentiation property of the Laplace transform.

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then
$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s}X(s)$$
with ROC containing $R \cap \{\operatorname{Re}\{s\} > 0\}$.
- This is known as the time-domain integration property of the Laplace transform.
- The ROC always contains at least $R \cap \{\operatorname{Re}\{s\} > 0\}$ but can be larger (if pole-zero cancellation occurs).
- Integration in the time domain becomes *division by s* in the Laplace domain.
- Consequently, it is often much easier to work with integral equations in the Laplace domain, rather than the time domain.

Initial Value Theorem

- For a function x with Laplace transform X, if x is *causal* and contains *no impulses or higher order singularities at the origin*, then

$$x(0^+) = \lim_{s \to \infty} sX(s),$$

where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t.

- This result is known as the initial value theorem.

Final Value Theorem

- For a function x with Laplace transform X, if x is *causal* and $x(t)$ has a *finite limit* as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

- This result is known as the final value theorem.
- Sometimes the initial and final value theorems are useful for checking for errors in Laplace transform calculations. For example, if we had made a mistake in computing $X(s)$, the values obtained from the initial and final value theorems would most likely disagree with the values obtained directly from the original expression for $x(t)$.

Section 6.4

Determination of Inverse Laplace Transform

- Recall that the inverse Laplace transform x of X is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds,$$

where $\mathrm{Re}\{s\} = \sigma$ is in the ROC of X.

- Unfortunately, the above contour integration can often be *quite tedious* to compute.
- Consequently, we do not usually compute the inverse Laplace transform directly using the above equation.
- For rational functions, the inverse Laplace transform can be more easily computed using *partial fraction expansions*.
- Using a partial fraction expansion, we can express a rational function as a sum of lower-order rational functions whose inverse Laplace transforms can typically be found in tables.

Section
6.5

Laplace Transform and LTI Systems

System Function of LTI Systems

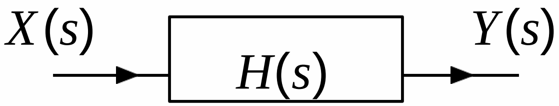
- Consider a LTI system with input x, output y, and impulse response h. Let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, the system is characterized in the Laplace domain by

$$Y(s) = X(s)H(s).$$

- As a matter of terminology, we refer to H as the system function (or transfer function) of the system (i.e., the system function is the Laplace transform of the impulse response).
- When viewed in the Laplace domain, a LTI system forms its output by multiplying its input with its system function.
- A LTI system is *completely characterized* by its system function H.
- If the ROC of H includes the imaginary axis, then $H(s)|_{s=j\omega}$ is the *frequency response* of the LTI system.

Block Diagram Representations of LTI Systems

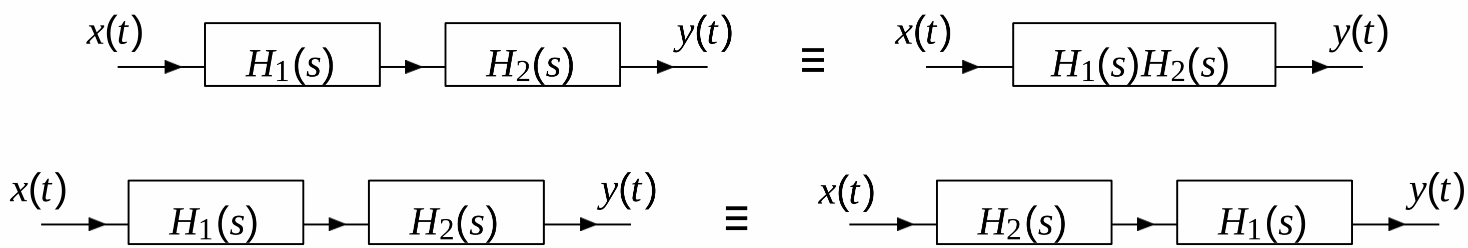
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the Laplace domain as shown below.



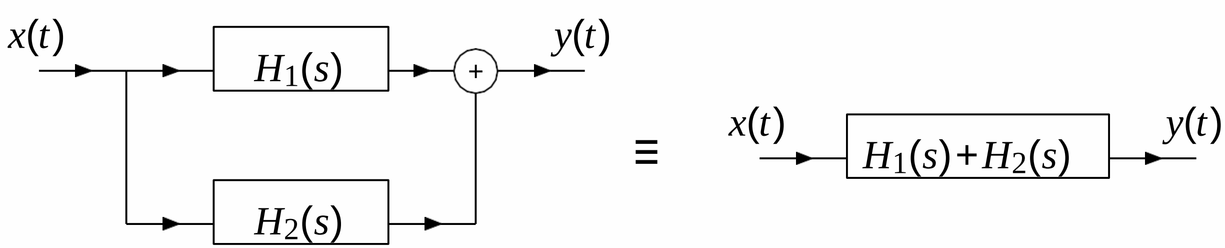
- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

Interconnection of LTI Systems

- The *series* interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with system function $H = H_1H_2$. That is, we have the equivalences shown below.



- The *parallel* interconnection of the LTI systems with impulse responses H_1 and H_2 is a LTI system with the system function $H = H_1 + H_2$. That is, we have the equivalence shown below.



- If a LTI system is *causal*, its impulse response is causal, and therefore *right sided*. From this, we have the result below.
- **Theorem.** The ROC associated with the system function of a *causal* LTI system is a *right-half plane* or the entire complex plane.
- In general, the *converse* of the above theorem is *not necessarily true*. That is, if the ROC of the system function is a RHP or the entire complex plane, it is not necessarily true that the system is causal.
- If the system function is *rational*, however, we have that the converse does hold, as indicated by the theorem below.
- **Theorem.** For a LTI system with a *rational* system function H, *causality* of the system is *equivalent* to the ROC of H being the *right-half plane* to the right of the rightmost pole or, if H has no poles, the entire complex plane.

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is *BIBO stable* if and only if the ROC of its system function H includes the (entire) *imaginary axis* (i.e., $\mathrm{Re}\{s\} = 0$).
- **Theorem.** A *causal* LTI system with a (proper) *rational* system function H is BIBO stable if and only if all of the poles of H lie in the left half of the plane (i.e., all of the poles have *negative real parts*).

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(s)H_{\text{inv}}(s) = 1,$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(s) = \frac{1}{H(s)}.$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is *not necessarily unique*.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in *one specific choice* of inverse system (due to these additional constraints of stability and/or causality).

System Function and Differential Equation Representations of LTI Systems

- Many LTI systems of practical interest can be represented using an *Nth-order linear differential equation with constant coefficients*.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t) \quad \text{where} \quad M \le N.$$

- Let h denote the impulse response of the system, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

- Observe that, for a system of the form considered above, the system function is always *rational*.

Section 6.6

Application: Circuit Analysis

Resistors

- A resistor is a circuit element that opposes the flow of electric current.
- A resistor with resistance R is governed by the relationship

$$v(t) = Ri(t) \qquad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

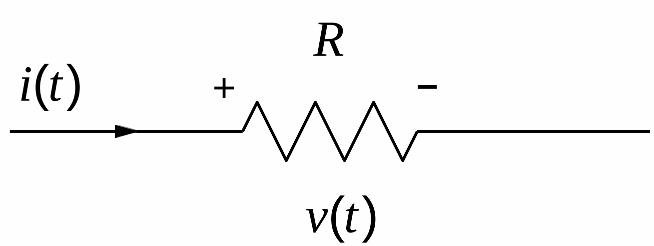
where v and i respectively denote the voltage across and current through the resistor as a function of time.

- In the Laplace domain, the above relationship becomes

$$V(s) = RI(s) \qquad \left(\text{or equivalently, } I(s) = \tfrac{1}{R}V(s)\right),$$

where V and I denote the Laplace transforms of v and i, respectively.

- In circuit diagrams, a resistor is denoted by the symbol shown below.



Inductors

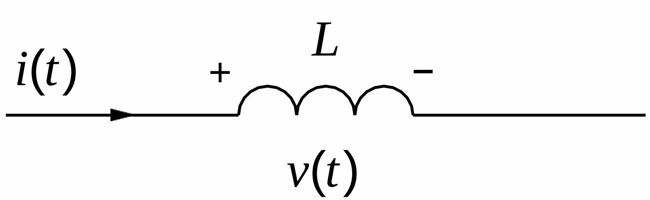
- An inductor is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor with inductance L is governed by the relationship

$$v(t) = L\frac{d}{dt}i(t) \qquad \left(\text{or equivalently, } i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau\right),$$

  where v and i respectively denote the voltage across and current through the inductor as a function of time.
- In the Laplace domain, the above relationship becomes

$$V(s) = sLI(s) \qquad \left(\text{or equivalently, } I(s) = \frac{1}{sL}V(s)\right),$$

  where V and I denote the Laplace transforms of v and i, respectively.
- In circuit diagrams, an inductor is denoted by the symbol shown below.



Capacitors

- A capacitor is a circuit element that stores electric charge.
- A capacitor with capacitance C is governed by the relationship

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \quad \left(\text{or equivalently, } i(t) = C\frac{d}{dt}v(t)\right),$$

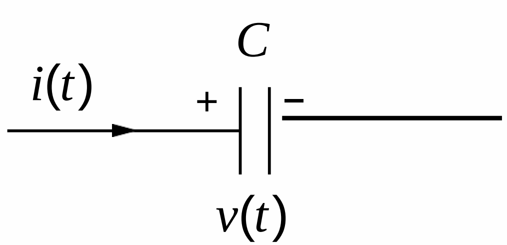
where v and i respectively denote the voltage across and current through the capacitor as a function of time.

- In the Laplace domain, the above relationship becomes

$$V(s) = \frac{1}{sC}I(s) \qquad (\text{or equivalently, } I(s) = sCV(s)),$$

where V and I denote the Laplace transforms of v and i, respectively.

- In circuit diagrams, a capacitor is denoted by the symbol shown below.



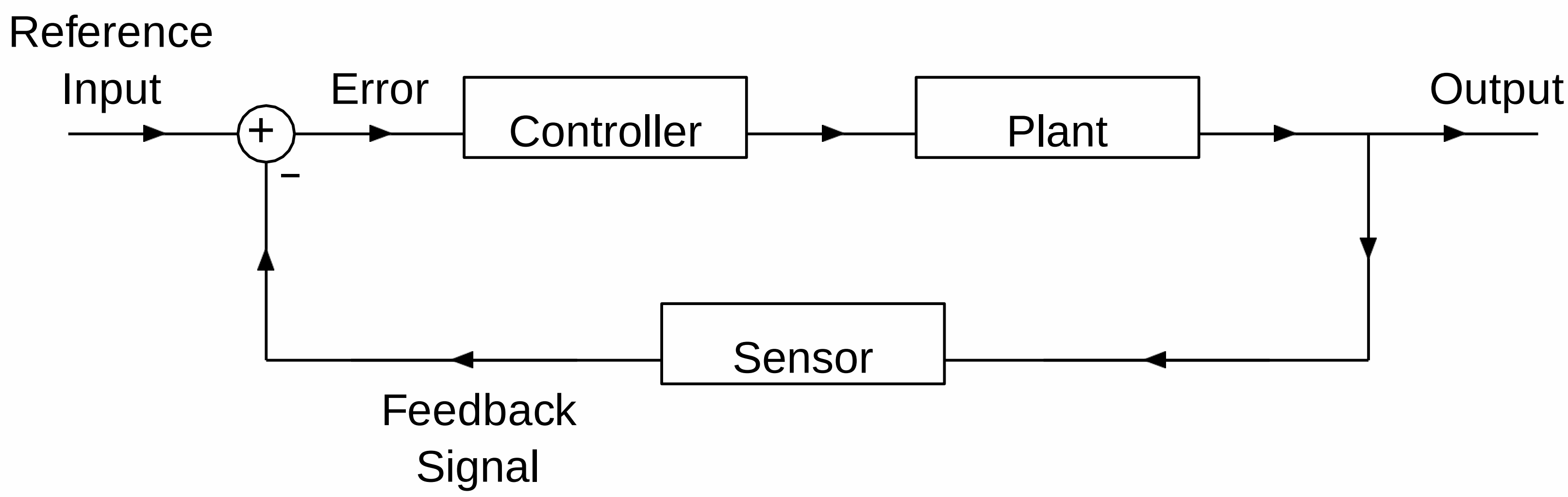
Circuit Analysis

- The Laplace transform is a very useful tool for circuit analysis.
- The utility of the Laplace transform is partly due to the fact that the *differential/integral* equations that describe inductors and capacitors are much simpler to express in the Laplace domain than in the time domain.

Section 6.7

Application: Analysis of Control Systems

Feedback Control Systems



- input: *desired value* of the quantity to be controlled
- output: *actual value* of the quantity to be controlled
- error: *difference* between the desired and actual values
- plant: system to be controlled
- sensor: device used to measure the actual output
- controller: device that monitors the error and changes the input of the plant with the goal of forcing the error to zero

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the Laplace domain than in the time domain.
- Therefore, the Laplace domain is extremely useful for the stability analysis of systems.

Section 6.8

Unilateral Laplace Transform

Unilateral Laplace Transform

- The unilateral Laplace transform of the signal x, denoted $\mathcal{UL}\{x\}$ or X, is defined as

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt.$$

- The unilateral Laplace transform is related to the bilateral Laplace transform as follows:

$$\mathcal{UL}\{x\}(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt = \int_{-\infty}^{\infty} x(t)u(t)e^{-st}\,dt = \mathcal{L}\{xu\}(s).$$

- In other words, the unilateral Laplace transform of the signal x is simply the bilateral Laplace transform of the signal xu.
- Since $\mathcal{UL}\{x\} = \mathcal{L}\{xu\}$ and xu is always a *right-sided* signal, the ROC associated with $\mathcal{UL}\{x\}$ is always a *right-half plane*.
- For this reason, we often *do not explicitly indicate the ROC* when working with the unilateral Laplace transform.

- With the unilateral Laplace transform, the same inverse transform equation is used as in the bilateral case.
- The unilateral Laplace transform is *only invertible for causal signals*. In particular, we have

$$
\begin{aligned}
\mathcal{UL}^{-1}\{\mathcal{UL}\{x\}\}(t) &= \mathcal{UL}^{-1}\{\mathcal{L}\{xu\}\}(t) \\
&= \mathcal{L}^{-1}\{\mathcal{L}\{xu\}\}(t) \\
&= x(t)u(t) \\
&= \begin{cases} x(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}
\end{aligned}
$$

- For a noncausal signal x, we can only recover x for $t > 0$.

Unilateral Laplace Transform (Continued 2)

- Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties.
- Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways.

Properties of the Unilateral Laplace Transform

Property	Time Domain	Laplace Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$
Time/Frequency-Domain Scaling	$x(at)$, $a>0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1 * x_2(t)$, x_1 and x_2 are causal	$X_1(s)X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$
Time-Domain Integration	$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Unilateral Laplace Transform Pairs

Pair	$x(t),\ t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

Solving Differential Equations Using the Unilateral Laplace Transform

- Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations.
- One common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions.