Omdurman Islamic University

Faculty of Engineering

Electrical & Electronic Engineering
(4th year)

Signal Processing and Systems

Lecturer
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Module 5

Continuous-Time Fourier Transform (CTFT)

Course Description (Part 1)

- Module 1: Introduction to signals and systems.
- Module 2: Continuous-Time (CT) Signals and Systems
- Module 3: Continuous-Time Linear Time-Invariant (LTI) Systems
- Module 4: Continuous-Time Fourier Series (CTFS)
- Module 5: Continuous-Time Fourier Transform (CTFT)
- Module 6: Laplace Transform (LT)

Course Description (Part2)

- Module 1: Introduction to Digital signal Processing.
- Module 2: Analogue to Digital conversion, Sampling, Quantization
- Module 3-1: Digital signal and systems .
- Module 3-2: LTI systems described by difference equations.
- Module 4-1: Discrete Time Fourier Transform.
- Module 4-2: Fast Fourier Transforms (FFT).

Course Description (Part2)

- Module 5: Z Transform
- Module 6: Basic Filtering Types
- Module 7: FIR Filters design, implementation.
- Module 8: IIR Filters design, implementation.

Part 1

Continuous-Time Fourier Transform (CTFT)

Motivation for the Fourier Transform

- Fourier series provide an extremely useful representation for periodic signals.
- Often, however, we need to deal with signals that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The Fourier transform can be used to represent both periodic and aperiodic signals.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Section 5.1

Fourier Transform

Development of the Fourier Transform

- The Fourier series is an extremely useful signal representation.
- Unfortunately, this signal representation can only be used for periodic signals, since a Fourier series is inherently periodic.
- Many signals are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can be applied to aperiodic signals.
- By viewing an aperiodic signal as the limiting case of a periodic signal with period T where $T \to \infty$, we can use the Fourier series to develop a more general signal representation that can be used for both aperiodic and periodic signals.
- This more general signal representation is called the Fourier transform.

CT Fourier Transform (CTFT)

- The (CT) Fourier transform of the signal x, denoted $\mathcal{F}\{x\}$ or X, is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

- The preceding equation is sometimes referred to as Fourier transform analysis equation (or forward Fourier transform equation).
- The inverse Fourier transform of X, denoted $\mathcal{F}^{-1}\{X\}$ or x, is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega.$$

- The preceding equation is sometimes referred to as the Fourier transform synthesis equation (or inverse Fourier transform equation).
- As a matter of notation, to denote that a signal x has the Fourier transform X, we write $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$.
- A signal x and its Fourier transform X constitute what is called a Fourier transform pair.

Section
5.2

Convergence Properties of the Fourier Transform

- Consider an arbitrary signal x.
- The signal x has the Fourier transform representation $\tilde{x}$ given by

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

- Now, we need to concern ourselves with the convergence properties of this representation.
- In other words, we want to know when $\tilde{x}$ is a valid representation of x.
- Since the Fourier transform is essentially derived from Fourier series, the convergence properties of the Fourier transform are closely related to the convergence properties of Fourier series.

- If a signal x is *continuous* and *absolutely integrable* (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$) and the Fourier transform X of x is absolutely integrable (i.e., $\int_{-\infty}^{\infty} |X(\omega)|\, d\omega < \infty$), then the Fourier transform representation of x converges *pointwise* (i.e., $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] e^{j\omega t} d\omega$ for all t).
- Since, in practice, we often encounter signals with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value.

Convergence of the Fourier Transform: Finite-Energy Case

- If a signal x is of *finite energy* (i.e., $\int_{-\infty}^{\infty} |x(t)|^2\,dt < \infty$), then its Fourier transform representation converges in the *MSE sense*.
- In other words, if x is of finite energy, then the energy E in the difference signal $\tilde{x} - x$ is zero; that is,

$$E = \int_{-\infty}^{\infty} |\tilde{x}(t) - x(t)|^2\,dt = 0.$$

- Since, in situations of practice interest, the finite-energy condition in the above theorem is often satisfied, the theorem is frequently applicable.
- It is important to note, however, that the condition $E = 0$ does not necessarily imply $\tilde{x}(t) = x(t)$ for all t.
- Thus, the above convergence result does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier transform representation converges.

Convergence of the Fourier Transform: Dirichlet Case

- The Dirichlet conditions for the signal x are as follows:
  1. The signal x is *absolutely integrable* (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$).
  2. On any finite interval, x has a finite number of maxima and minima (i.e., x is of *bounded variation*).
  3. On any finite interval, x has a *finite number of discontinuities* and each discontinuity is itself *finite*.
- If a signal x satisfies the *Dirichlet conditions*, then:
  1. The Fourier transform representation $\tilde{x}$ converges pointwise everywhere to x, except at the points of discontinuity of x.
  2. At each point $t = t_a$ of discontinuity, the Fourier transform representation $\tilde{x}$ converges to

$$\tilde{x}(t_a) = \frac{1}{2}\left[x(t_a^+) + x(t_a^-)\right],$$

     where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal x on the left- and right-hand sides of the discontinuity, respectively.
- Since most signals tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier transform representation at every point, this result is often very useful in practice.

Section 5.3

Properties of the Fourier Transform

Properties of the (CT) Fourier Transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency-Domain Convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}(t/T)$	$\lvert T\rvert\operatorname{sinc}(T\omega/2)$
9	$\frac{\lvert B\rvert}{\pi}\operatorname{sinc}Bt$	$\operatorname{rect}\frac{\omega}{2B}$
10	$e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{\lvert T\rvert}{2}\operatorname{sinc}^2(T\omega/4)$

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$a_1x_1(t)+a_2x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} a_1X_1(\omega)+a_2X_2(\omega),$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the linearity property of the Fourier transform.

Translation (Time-Domain Shifting)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$,
  then
  $$x(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$
  where t_0 is an arbitrary real constant.
- This is known as the translation (or time-domain shifting) property of the Fourier transform.

Modulation (Frequency-Domain Shifting)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$e^{j\omega_0 t}x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega - \omega_0),$$

  where ω_0 is an arbitrary real constant.
- This is known as the modulation (or frequency-domain shifting) property of the Fourier transform.

Dilation (Time- and Frequency-Domain Scaling)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

  where a is an arbitrary nonzero real constant.
- This is known as the dilation (or time/frequency-scaling) property of the Fourier transform.

Conjugation

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$,
  then
  $$x^*(t) \overset{\text{CTFT}}{\longleftrightarrow} X^*(-\omega).$$
- This is known as the conjugation property of the Fourier transform.

Duality

- If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$,
  then $$X(t) \stackrel{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega)$$
- This is known as the duality property of the Fourier transform.
- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by
$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta) e^{-j\theta\lambda} d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) e^{j\theta\lambda} d\theta.$$
- That is, the forward and inverse Fourier transform equations are identical except for a *factor of* 2π and *different sign* in the parameter for the exponential function.
- Although the relationship $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform of $x(t)$, the duality property allows us to indirectly infer the Fourier transform of $X(t)$. Consequently, the duality property can be used to effectively *double* the number of Fourier transform pairs that we know.

- If $x_1(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1 * x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

- This is known as the convolution (or time-domain convolution) property of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

Multiplication (Frequency-Domain Convolution)

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{2\pi}X_1 * X_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)d\theta.$$

- This is known as the multiplication (or frequency-domain convolution) property of the Fourier transform.
- In other words, multiplication in the time domain becomes convolution in the frequency domain (up to a scale factor of 2π).
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

Differentiation

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$\frac{dx(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\omega X(\omega).$$
- This is known as the differentiation property of the Fourier transform.
- Differentiation in the time domain becomes multiplication by $j\omega$ in the frequency domain.
- Of course, by repeated application of the above property, we have that $\left(\frac{d}{dt}\right)^n x(t) \overset{\text{CTFT}}{\longleftrightarrow} (j\omega)^n X(\omega)$.
- The above suggests that the Fourier transform might be a useful tool when working with differential (or integro-differential) equations.

Frequency-Domain Differentiation

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$tx(t) \overset{\text{CTFT}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega).$$
- This is known as the frequency-domain differentiation property of the Fourier transform.

Integration

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$
- This is known as the *integration property* of the Fourier transform.
- Whereas differentiation in the time domain corresponds to *multiplication* by $j\omega$ in the frequency domain, integration in the time domain is associated with *division* by $j\omega$ in the frequency domain.
- Since integration in the time domain becomes division by $j\omega$ in the frequency domain, integration can be easier to handle in the frequency domain.
- The above property suggests that the Fourier transform might be a useful tool when working with integral (or integro-differential) equations.

- Recall that the energy of a signal x is given by $\int_{-\infty}^{\infty} |x(t)|^2\,dt$.
- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

(i.e., the energy of x and energy of X are equal up to a factor of 2π).
- This relationship is known as Parseval's relation.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform *preserves energy* (up to a scale factor).

Even and Odd Symmetry

- For a signal x with Fourier transform X, the following assertions hold:

$$x \text{ is even} \Leftrightarrow X \text{ is even; and}$$
$$x \text{ is odd} \Leftrightarrow X \text{ is odd.}$$

- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

- A signal x is *real* if and only if its Fourier transform X satisfies

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

(i.e., X has *conjugate symmetry*).

- Thus, for a real-valued signal, the portion of the graph of a Fourier transform for negative values of frequency ω is *redundant*, as it is completely determined by symmetry.
- From properties of complex numbers, one can show that $X(\omega) = X^*(-\omega)$ is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega)$$

(i.e., $|X(\omega)|$ is *even* and $\arg X(\omega)$ is *odd*).

- Note that x being real does *not* necessarily imply that X is real.

Fourier Transform of Periodic Signals

- The Fourier transform can be generalized to also handle periodic signals.
- Consider a periodic signal x with period T and frequency $\omega_0 = \frac{2\pi}{T}$
- Define the signal x_T as

$$x_T(t) = \begin{cases} x(t) & \text{for } -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(i.e., $x_T(t)$ is equal to $x(t)$ over a single period and zero elsewhere).
- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_T denote the Fourier transforms of x and x_T, respectively.
- The following relationships can be shown to hold:

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0),$$

$$a_k = \frac{1}{T} X_T(k\omega_0), \quad \text{and} \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0).$$

Fourier Transform of Periodic Signals (Continued)

- The Fourier series coefficient sequence a_k is produced by sampling X_T at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$.
- The Fourier transform of a periodic signal can only be nonzero at integer multiples of the fundamental frequency.

Section 5.4

Fourier Transform and Frequency Spectra of Signals

Frequency Spectra of Signals

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on signals.
- That is, instead of viewing a signal as having information distributed with respect to *time* (i.e., a function whose domain is time), we view a signal as having information distributed with respect to *frequency* (i.e., a function whose domain is frequency).
- The Fourier transform of a signal x provides a means to *quantify* how much information x has at different frequencies.
- The distribution of information in a signal over different frequencies is referred to as the *frequency spectrum* of the signal.

Fourier Transform and Frequency Spectra

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\omega)$ expressed in *polar form* as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j[\omega t+\arg X(\omega)]}d\omega.$$

- In effect, the quantity $|X(\omega)|$ is a *weight* that determines how much the complex sinusoid at frequency ω contributes to the integration result x.
- Perhaps, this can be more easily seen if we express the above integral as the *limit of a sum*, derived from an approximation of the integral using the area of rectangles, as shown on the next slide. [Recall that $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x\to 0}\sum_{k=-\infty}^{\infty} \Delta x f(k\Delta x)$.]

Fourier Transform and Frequency Spectra (Continued 1)

- Expressing the integral (from the previous slide) as the *limit of a sum*, we obtain

$$x(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega \cdot |X(\omega')| \cdot e^{j[\omega' t + \arg X(\omega')]},$$

where $\omega' = k\Delta\omega$.

- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\omega' = k\Delta\omega$ that has had its *amplitude scaled* by a factor of $|X(\omega')|$ and has been *time shifted* by an amount that depends on $\arg X(\omega')$.
- For a given $\omega' = k\Delta\omega$ (which is associated with the kth term in the summation), the larger $|X(\omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega' t}$ will be, and therefore the larger the contribution the kth term will make to the overall summation.
- In this way, we can use $|X(\omega')|$ as a *measure* of how much information a signal x has at the frequency ω'.

Fourier Transform and Frequency Spectra (Continued 2)

- The Fourier transform X of the signal x is referred to as the frequency spectrum of x.
- The magnitude $|X(\omega)|$ of the Fourier transform X is referred to as the magnitude spectrum of x.
- The argument $\arg X(\omega)$ of the Fourier transform X is referred to as the phase spectrum of x.
- Since the Fourier transform is a function of a real variable, a signal can potentially have information at any real frequency.
- Earlier, we saw that for periodic signals, the Fourier transform can only be nonzero at integer multiples of the fundamental frequency.
- So, the Fourier transform and Fourier series give a consistent picture in terms of frequency spectra.
- Since the frequency spectrum is complex (in the general case), it is *usually represented using two plots*, one showing the magnitude spectrum and one showing the phase spectrum.

Frequency Spectra of Real Signals

- Recall that, for a *real* signal x, the Fourier transform X of x satisfies

$$X(\omega) = X^*(-\omega)$$

(i.e., X is *conjugate symmetric*), which is equivalent to

$$|X(\omega)| = |X(-\omega)| \qquad \text{and} \qquad \arg X(\omega) = -\arg X(-\omega).$$

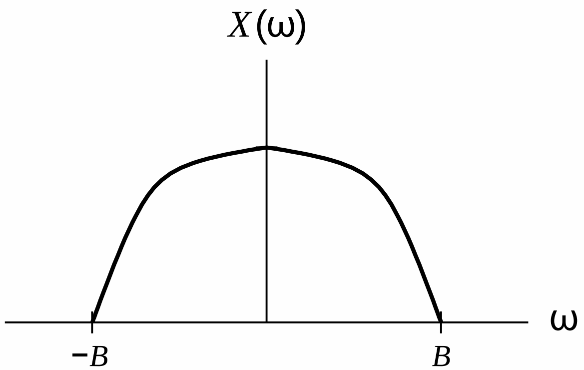
- Since $|X(\omega)| = |X(-\omega)|$, the magnitude spectrum of a *real* signal is always *even*.
- Similarly, since $\arg X(\omega) = -\arg X(-\omega)$, the phase spectrum of a *real* signal is always *odd*.
- Due to the symmetry in the frequency spectra of real signals, we typically *ignore negative frequencies* when dealing with such signals.
- In the case of signals that are complex but not real, frequency spectra do not possess the above symmetry, and *negative frequencies become important*.

Bandwidth

- A signal x with Fourier transform X is said to be bandlimited if, for some nonnegative real constant B, the following condition holds:

$$X(\omega) = 0 \text{ for all } \omega \text{ satisfying } |\omega| > B.$$

- In the context of real signals, we usually refer to B as the bandwidth of the signal x.
- The (real) signal with the Fourier transform X shown below has bandwidth B.



- One can show that a signal *cannot be both time limited and bandlimited*. (This follows from the time/frequency scaling property of the Fourier transform.)

Section
5.5

Fourier Transform and LTI Systems

Frequency Response of LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, we have that

$$Y(\omega) = X(\omega)H(\omega).$$

- The function H is called the frequency response of the system.
- A LTI system is *completely characterized* by its frequency response H.
- The above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output signals.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

Frequency Response of LTI Systems (Continued 1)

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$.
- The quantity $|H(\omega)|$ is called the **magnitude response** of the system.
- The quantity $\arg H(\omega)$ is called the **phase response** of the system.
- Since $Y(\omega) = X(\omega)H(\omega)$, we trivially have that

$$|Y(\omega)| = |X(\omega)||H(\omega)| \qquad \text{and} \qquad \arg Y(\omega) = \arg X(\omega) + \arg H(\omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.

Frequency Response of LTI Systems (Continued 2)

- Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is *real*, then

$$|H(\omega)| = |H(-\omega)| \qquad \text{and} \qquad \arg H(\omega) = -\arg H(-\omega)$$

(i.e., the magnitude response $|H(\omega)|$ is *even* and the phase response $\arg H(\omega)$ is *odd*).

Block Diagram Representations of LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X , Y , and H denote the Fourier transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the frequency domain as shown below.

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)$$

- Since a LTI system is completely characterized by its frequency response, we typically label the system with this quantity.

Frequency Response and Differential Equation Representations of LTI Systems

- Many LTI systems of practical interest can be represented using an *Nth-order linear differential equation with constant coefficients*.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t) \quad \text{where} \quad M \leq N.$$

- Let h denote the impulse response of the system, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

- Observe that, for a system of the form considered above, the frequency response is a *rational function*.

Section 5.6

Application: Circuit Analysis

Resistors

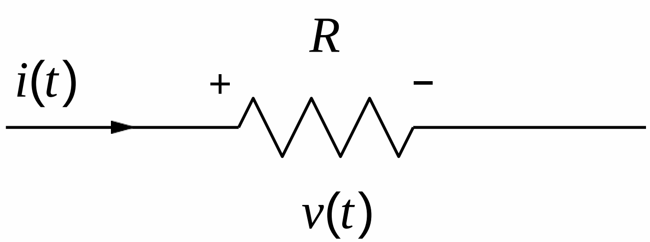
- A resistor is a circuit element that opposes the flow of electric current.
- A resistor with resistance R is governed by the relationship

$$v(t) = Ri(t) \qquad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

  where v and i respectively denote the voltage across and current through the resistor as a function of time.
- In the frequency domain, the above relationship becomes

$$V(\omega) = RI(\omega) \qquad \left(\text{or equivalently, } I(\omega) = \tfrac{1}{R}V(\omega)\right),$$

  where V and I denote the Fourier transforms of v and i, respectively.
- In circuit diagrams, a resistor is denoted by the symbol shown below.



Inductors

- An inductor is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor with inductance L is governed by the relationship

$$v(t) = L\frac{d}{dt}i(t) \qquad \left(\text{or equivalently, } i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau\right),$$

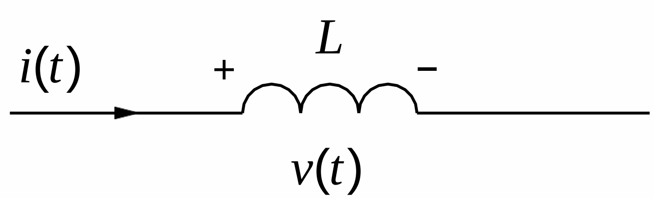
where v and i respectively denote the voltage across and current through the inductor as a function of time.

- In the frequency domain, the above relationship becomes

$$V(\omega) = j\omega L I(\omega) \qquad \left(\text{or equivalently, } I(\omega) = \frac{1}{j\omega L}V(\omega)\right),$$

where V and I denote the Fourier transforms of v and i, respectively. In

- circuit diagrams, an inductor is denoted by the symbol shown below.



- A capacitor is a circuit element that stores electric charge.
- A capacitor with capacitance C is governed by the relationship

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \left(\text{or equivalently, } i(t) = C\frac{d}{dt}v(t)\right),$$

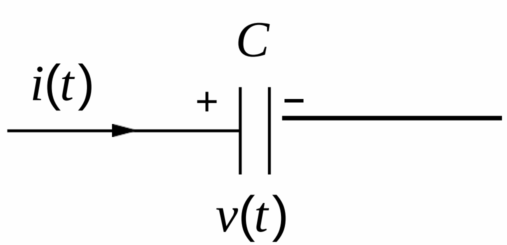
where v and i respectively denote the voltage across and current through the capacitor as a function of time.

- In the frequency domain, the above relationship becomes

$$V(\omega) = \frac{1}{j\omega C} I(\omega) \quad \left(\text{or equivalently, } I(\omega) = j\omega C V(\omega)\right),$$

where V and I denote the Fourier transforms of v and i, respectively. In

- circuit diagrams, a capacitor is denoted by the symbol shown below.



- The Fourier transform is a very useful tool for circuit analysis.
- The utility of the Fourier transform is partly due to the fact that the *differential/integral* equations that describe inductors and capacitors are much simpler to express in the Fourier domain than in the time domain.

Section 5.7

Application: Filtering

Filtering

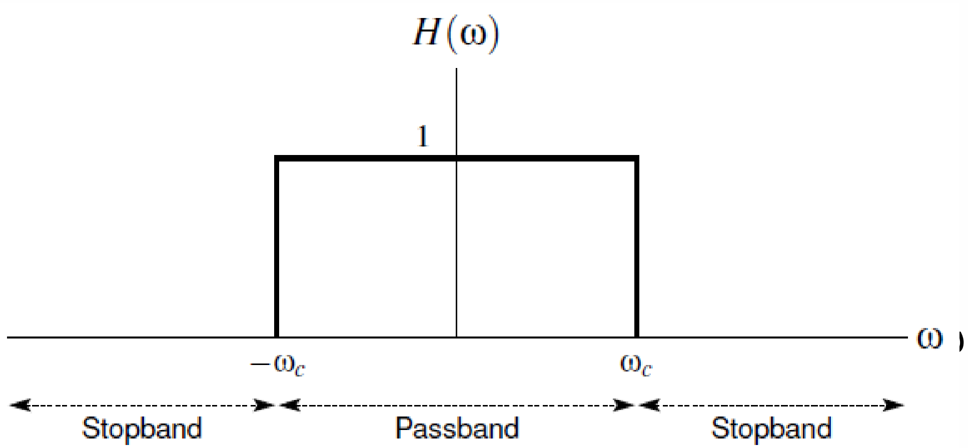
- In many applications, we want to *modify the spectrum* of a signal by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a signal is called filtering.
- A system that performs a filtering operation is called a filter.
- Many types of filters exist.
- Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

Ideal Lowpass Filter

- An ideal lowpass filter eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* H of the form

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

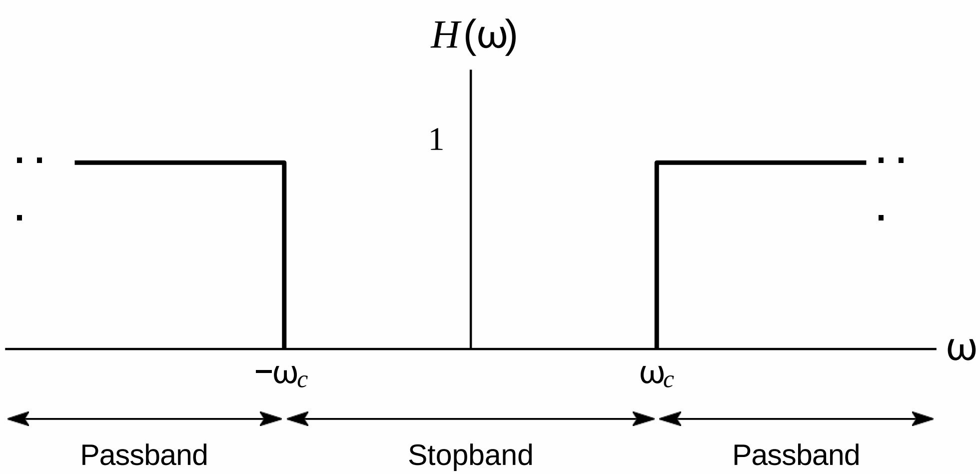
  where ω_c is the cutoff frequency.
- A plot of this frequency response is given below.



- An ideal highpass filter eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* H of the form

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

  where ω_c is the cutoff frequency.
- A plot of this frequency response is given below.

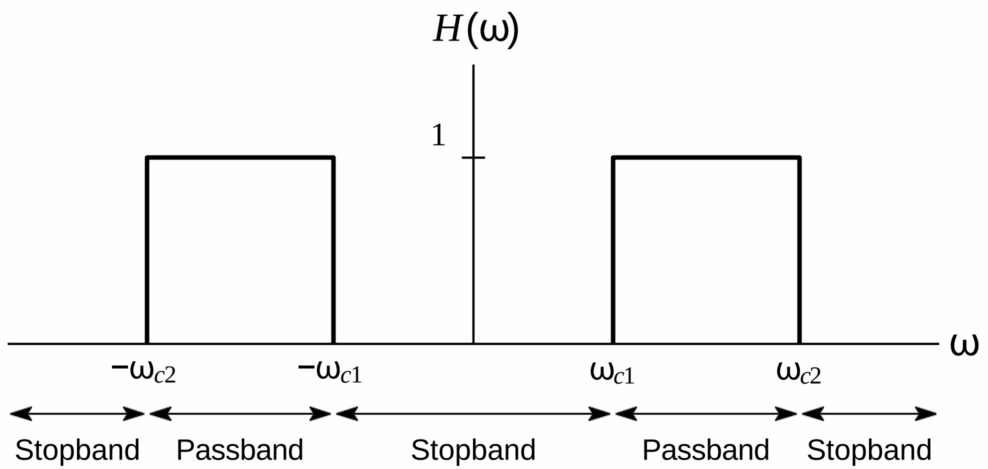


Ideal Bandpass Filter

- An ideal bandpass filter eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* H of the form

$$H(\omega) = \begin{cases} 1 & \text{for } \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.



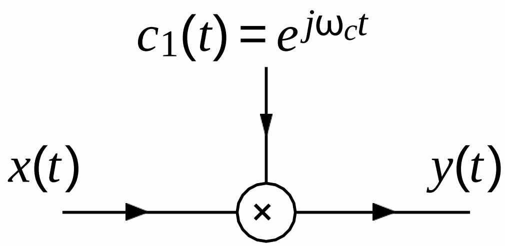
Section 5.8

Application: Amplitude Modulation (AM)

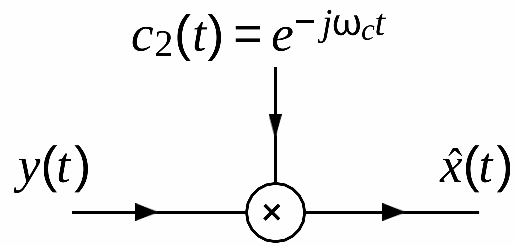
Motivation for Amplitude Modulation (AM)

- In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal.
- For example, voice/audio signals typically have information in the range of 0 to 22 kHz.
- Often, it is not practical to transmit such a signal using its original frequency range.
- Two potential problems with such an approach are:
  1. interference; and
  2. constraints on antenna length.
- Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference.
- Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals.
- For the preceding reasons, we often need to change the frequency range associated with a signal before transmission.

Trivial Amplitude Modulation (AM) System



Transmitter



Receiver

- The transmitter is characterized by

$$y(t) = e^{j\omega_c t} x(t) \quad \Longleftrightarrow \quad Y(\omega) = X(\omega - \omega_c).$$

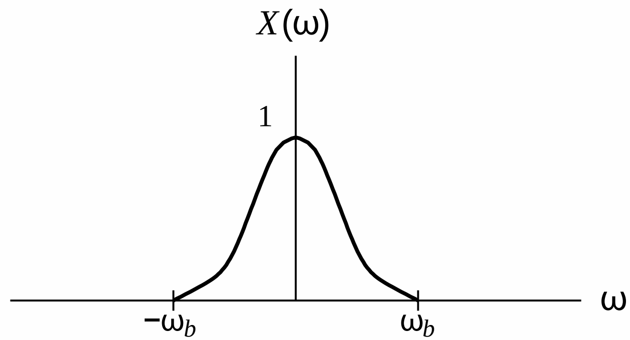
- The receiver is characterized by

$$\hat{x}(t) = e^{-j\omega_c t} y(t) \quad \Longleftarrow \quad \hat{X}(\omega) = Y(\omega + \omega_c).$$

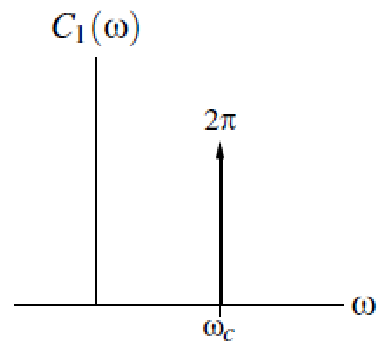
- $\Rightarrow$

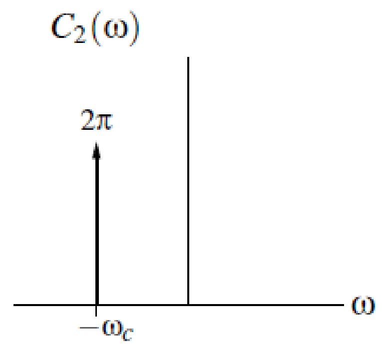
Clearly, $\hat{x}(t) = e^{j\omega_c t} e^{-j\omega_c t} x(t) = x(t)$.

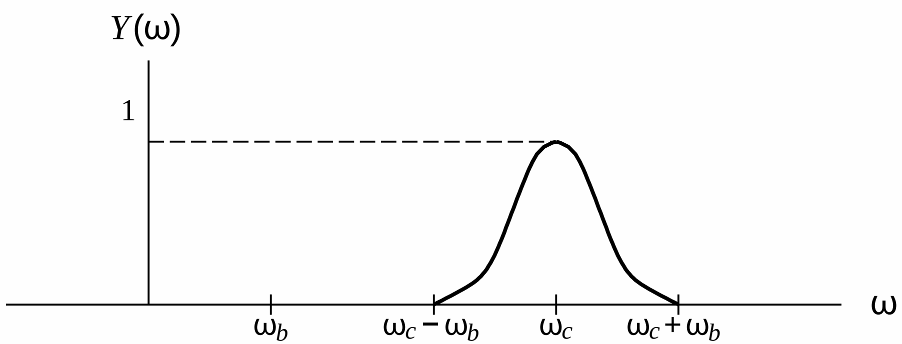
Trivial Amplitude Modulation (AM) System: Example



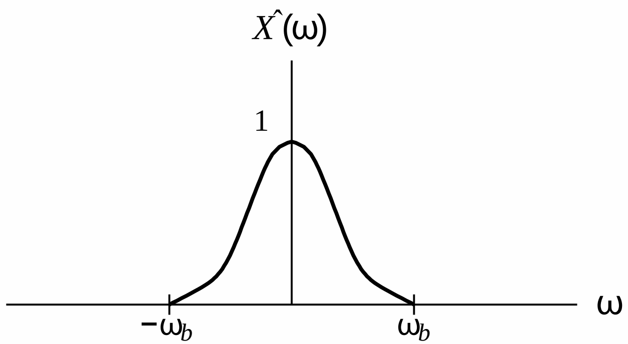
Transmitter Input





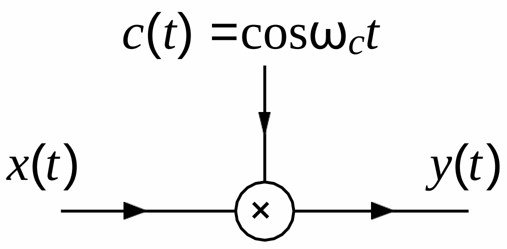


Transmitter Output



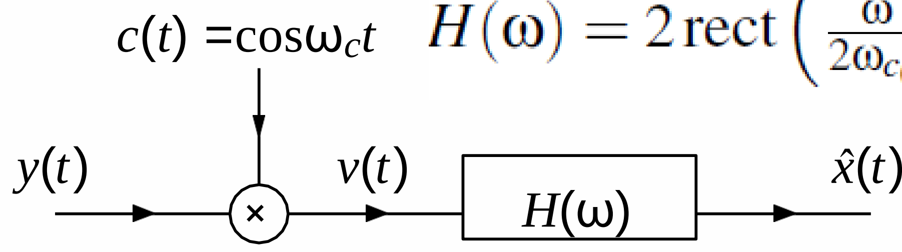
Receiver Output

Double-Sideband Suppressed-Carrier (DSB-SC) AM



Transmitter

$$H(\omega) = 2\,\mathrm{rect}\left(\frac{\omega}{2\omega_{c_0}}\right)$$



Receiver

- Suppose that $X(\omega) = 0$ for all $\omega \notin [-\omega_b, \omega_b]$.
- The transmitter is characterized by

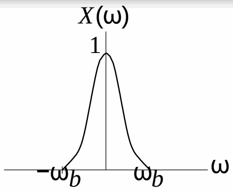
$$Y(\omega) = \tfrac{1}{2}\left[X(\omega+\omega_c) + X(\omega-\omega_c)\right].$$

- The receiver is characterized by

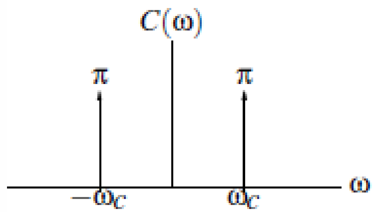
$$\hat{X}(\omega) = \left[Y(\omega+\omega_c) + Y(\omega-\omega_c)\right]\mathrm{rect}\left(\frac{\omega}{2\omega_{c_0}}\right).$$

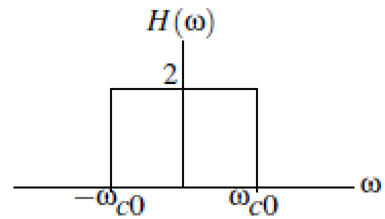
- If $\omega_b < \omega_{c_0} < 2\omega_c - \omega_b$, we have $\hat{X}(\omega) = X(\omega)$ (implying $\hat{x}(t) = x(t)$).

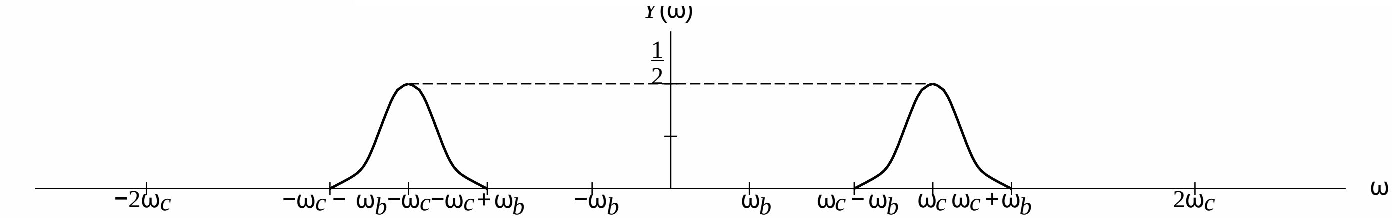
DSB-SC AM: Example



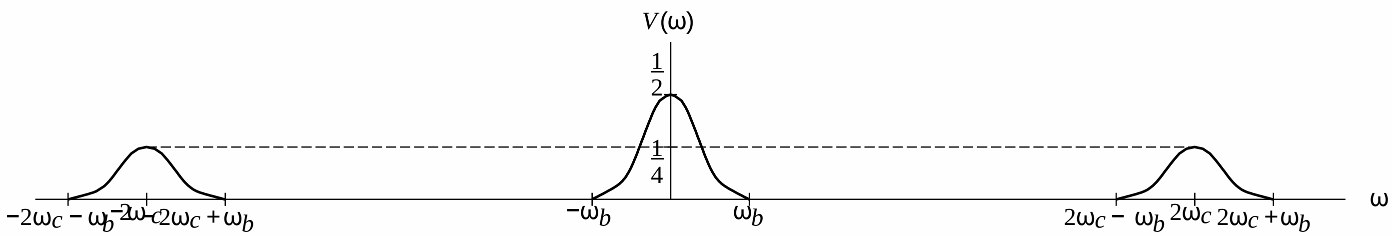
Transmitter Input

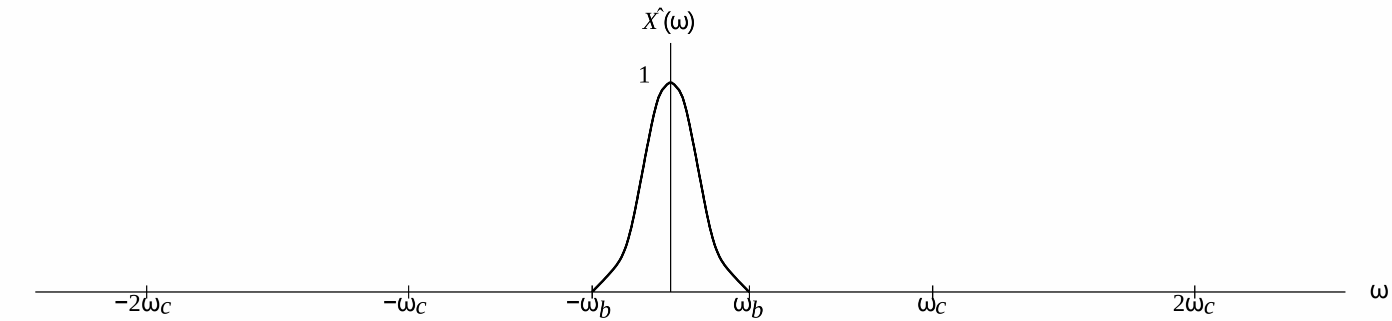






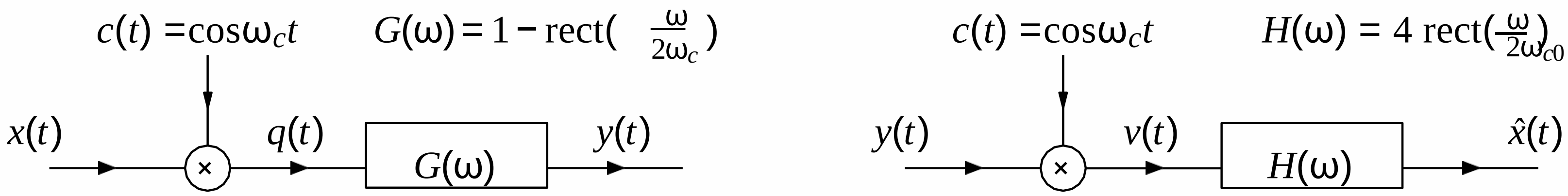
Transmitter Output





Receiver Output

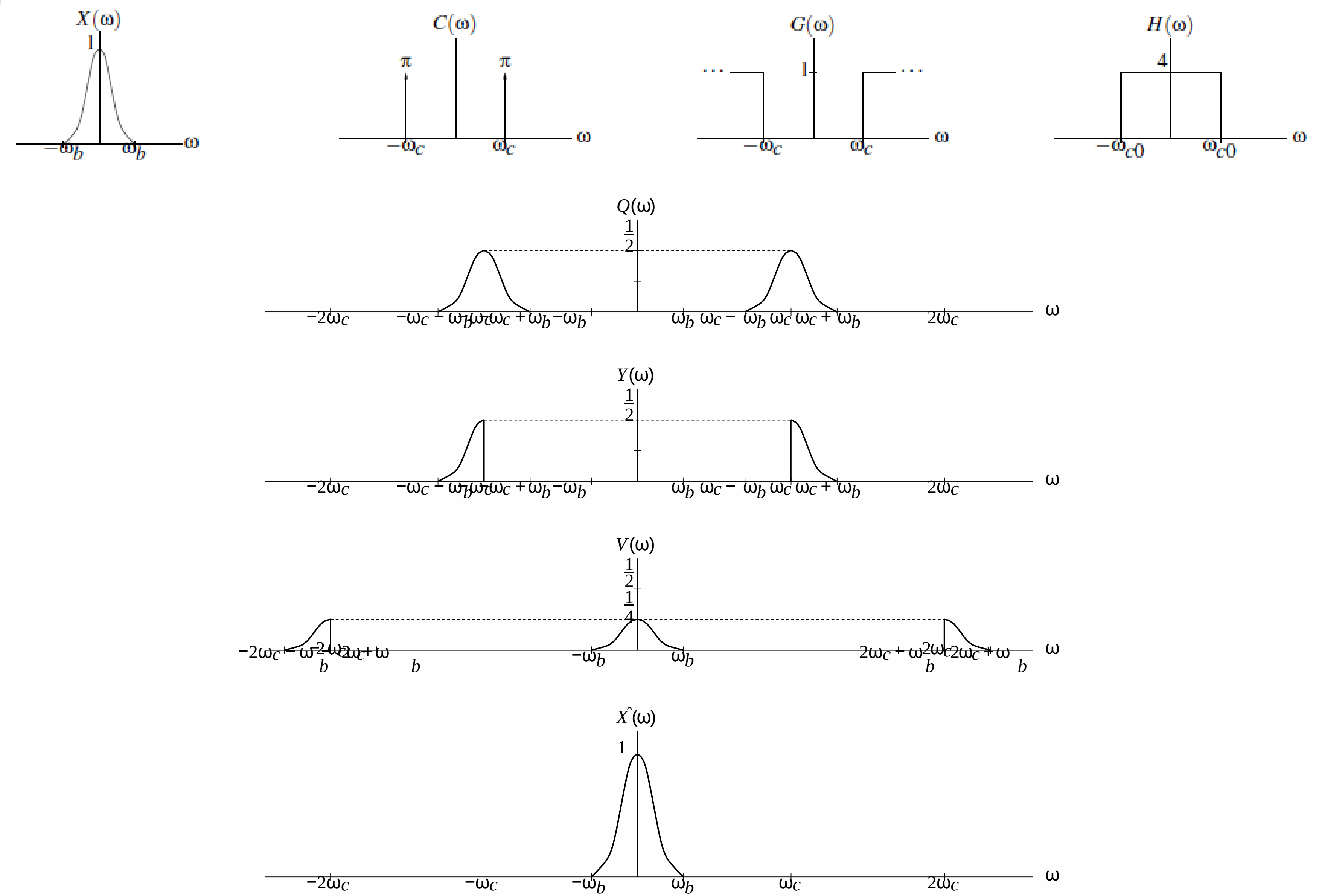
Single-Sideband Suppressed-Carrier (SSB-SC) AM



Transmitter   Receiver

- The basic analysis of the SSB-SC AM system is similar to the DSB-SC AM system.
- SSB-SC AM requires half as much bandwidth for the transmitted signal as DSB-SC AM.

SSB-SC AM: Example



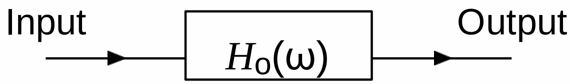
Section
5.9

Application: Equalization

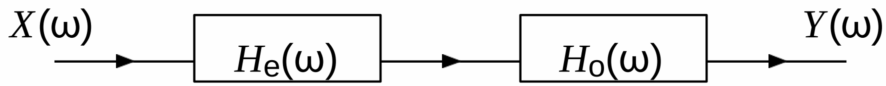
Equalization

- Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand.
- As a result, we would like to change the frequency response of the system to be something more desirable.
- This process of modifying the frequency response in this way is referred to as equalization. [Essentially, equalization is just a filtering operation.]
- Equalization is used in many applications.
- In real-world *communication systems*, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel.
- In *audio applications*, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, equalization can be used to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

Equalization (Continued)



Original System



New System with Equalization

- Let H_o denote the frequency response of *original* system (i.e., without equalization).
- Let H_d denote the *desired* frequency response.
- Let H_e denote the frequency response of the *equalizer*.
- The new system with equalization has frequency response

$$H_{new}(\omega) = H_e(\omega)H_o(\omega).$$

- By choosing $H_e(\omega) = H_d(\omega)/H_o(\omega)$, the new system with equalization will have the frequency response

$$H_{new}(\omega) = [H_d(\omega)/H_o(\omega)]\,H_o(\omega) = H_d(\omega).$$

- In effect, by using an equalizer, we can obtain a new system with the frequency response that we desire.

Section 5.10

Application: Sampling and Interpolation

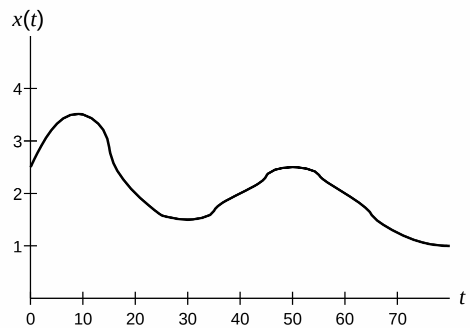
Periodic Sampling

- Although sampling can be performed in many different ways, the most commonly used scheme is periodic sampling.
- With this scheme, a sequence y of samples is obtained from a continuous-time signal x according to the relation

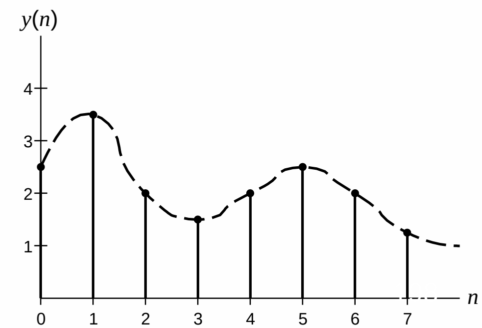
$$y(n) = x(nT) \qquad \text{for all integer } n,$$

where T is a positive real constant.

- As a matter of terminology, we refer to T as the sampling period, and $\omega_s = 2\pi/T$ as the (angular) sampling frequency.
- An example of periodic sampling is shown below, where the original continuous-time signal x has been sampled with *sampling period $T = 10$*, yielding the sequence y.



Original Signal



Sampled Signal

- The sampling process is not generally invertible.
- In the absence of any constraints, a continuous-time signal cannot usually be uniquely determined from a sequence of its equally-spaced samples.
- Consider, for example, the continuous-time signals x_1 and x_2 given by

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

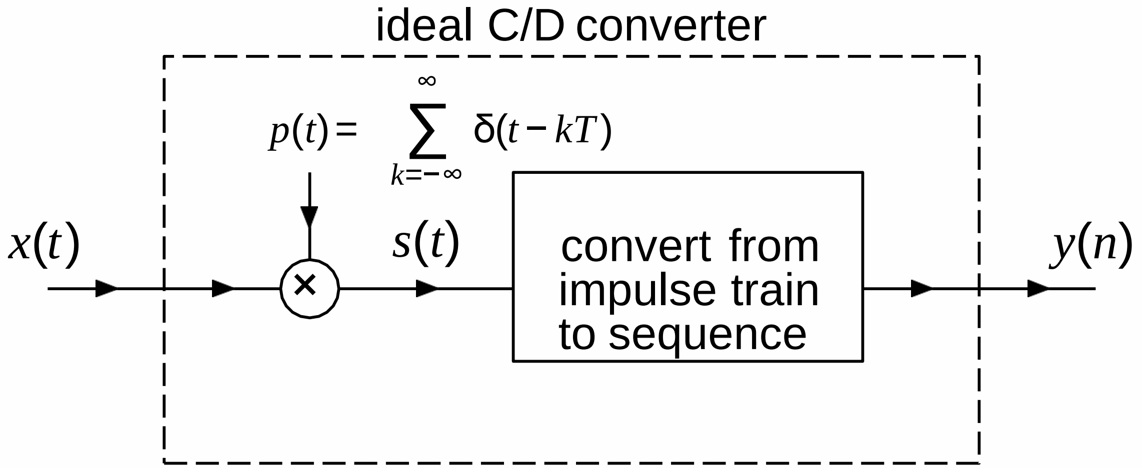
- If we sample each of these signals with the sampling period $T = 1$, we obtain the respective sequences

$$y_1(n) = x_1(nT) = x_1(n) = 0 \quad \text{and}$$
$$y_2(n) = x_2(nT) = \sin(2\pi n) = 0.$$

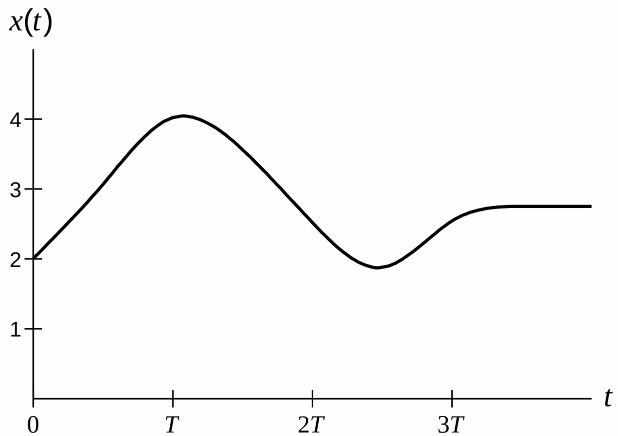
- Thus, $y_1(n) = y_2(n)$ for all n, although $x_1(t) \neq x_2(t)$ for all noninteger t.
- Fortunately, under certain circumstances, a continuous-time signal can be recovered exactly from its samples.

Model of Sampling

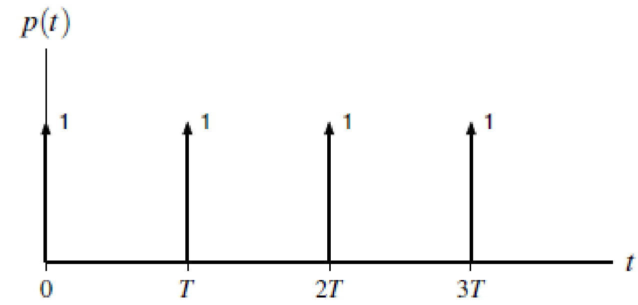
- An impulse train is a signal of the form $v(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - kT)$, where a_k and T are real constants (i.e., $v(t)$ consists of weighted impulses spaced apart by T).
- For the purposes of analysis, sampling with sampling period T and frequency $\omega_s = \frac{2\pi}{T}$ can be modelled as shown below.



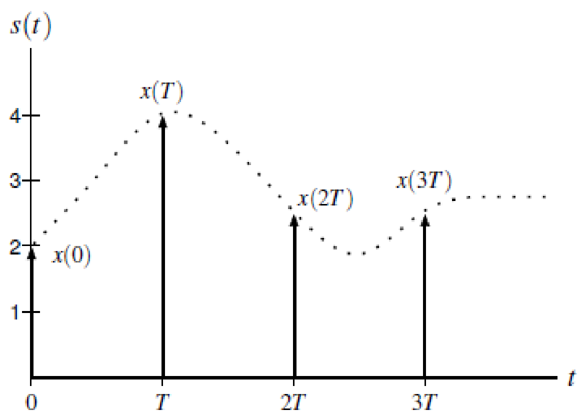
- The sampling of a continuous-time signal x to produce a sequence y consists of the following two steps (in order):
  1. Multiply the signal x to be sampled by a periodic impulse train p, yielding the impulse train s.
  2. Convert the impulse train s to a sequence y, by forming a sequence from the weights of successive impulses in s.



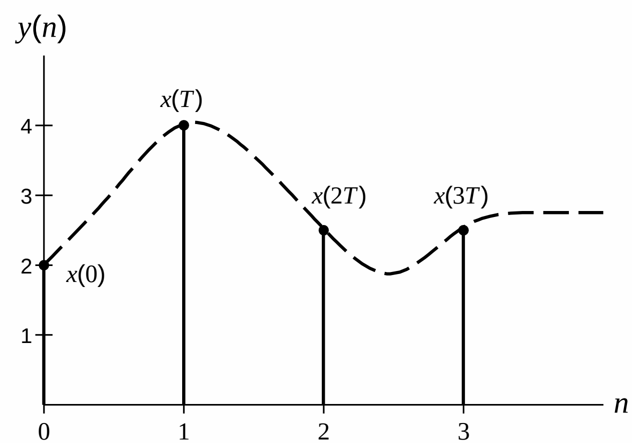
Input Signal (Continuous-Time)



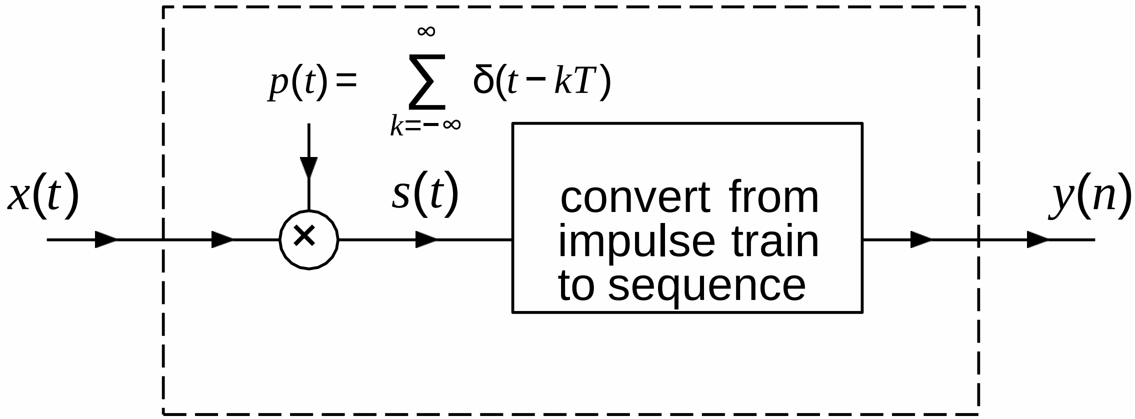
Periodic Impulse Train



Impulse-Sampled Signal *(Continuous-Time)*



Output Sequence *(Discrete-Time)*



- In the time domain, the impulse-sampled signal s is given by

$$s(t) = x(t)p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

- In the Fourier domain, the preceding equation becomes

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

- Thus, the spectrum of the impulse-sampled signal s is a scaled sum of an infinite number of *shifted copies* of the spectrum of the original signal x.

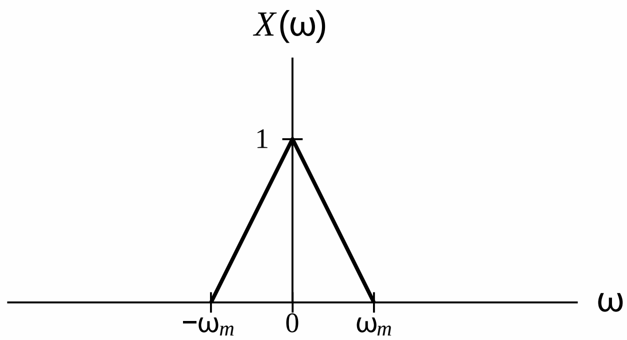
Model of Sampling: Aliasing

- Consider frequency spectrum S of the impulse-sampled signal s given by

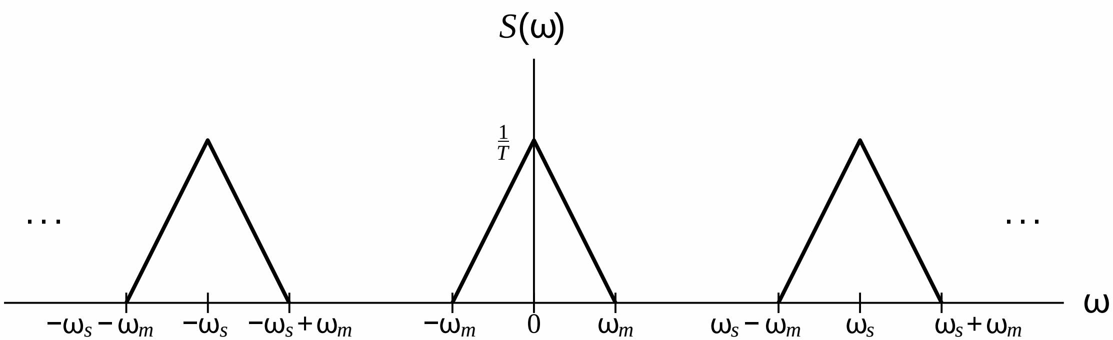
$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

- The function S is a scaled sum of an infinite number of *shifted copies* of X.
- Two distinct behaviors can result in this summation, depending on ω_s and the bandwidth of x.
- In particular, the nonzero portions of the different shifted copies of X can either:
  1. overlap; or
  2. not overlap.
- In the case where overlap occurs, the various shifted copies of X add together in such a way that the original shape of X is lost. This phenomenon is known as aliasing.
- When aliasing occurs, the original signal x cannot be recovered from its samples in y.

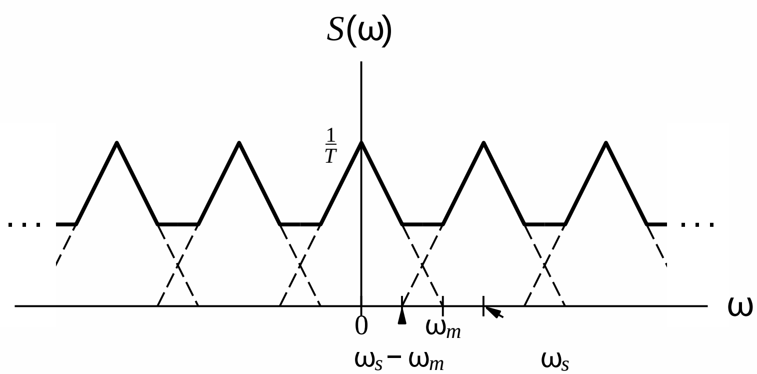
Model of Sampling: Aliasing (Continued)



Spectrum of Input Signal
(Bandwidth ω_m)



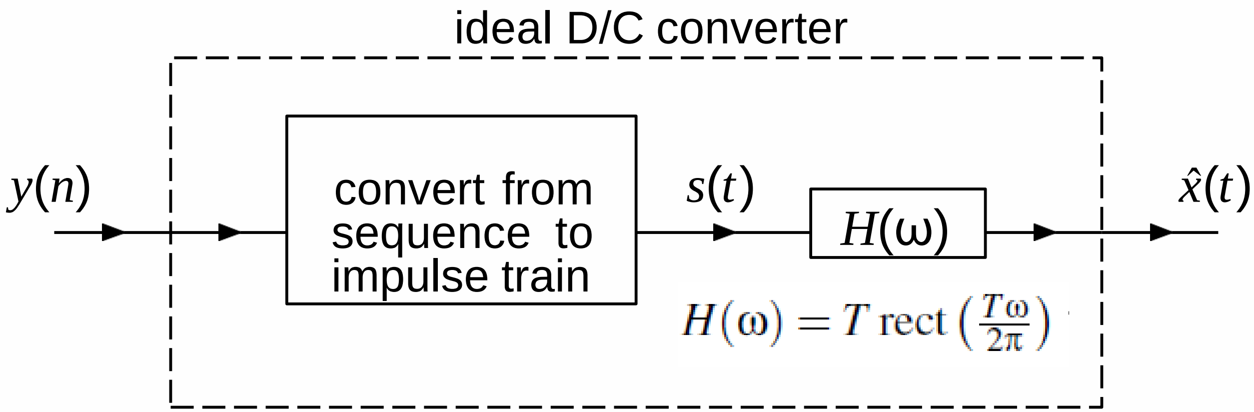
Spectrum of Impulse-Sampled Signal:
No Aliasing Case
($\omega_s > 2\omega_m$)



Spectrum of Impulse-Sampled Signal:
Aliasing Case
($\omega_s \leq 2\omega_m$)

Model of Interpolation

- For the purposes of analysis, interpolation can be modelled as shown below.



- The inverse Fourier transform h of H is $h(t) = \operatorname{sinc}(\pi t/T)$.
- The reconstruction of a continuous-time signal x from its sequence y of samples (i.e., bandlimited interpolation) consists of the following two steps (in order):
  1. Convert the sequence y to the impulse train s, by using the elements in the sequence as the weights of successive impulses in the impulse train.
  2. Apply a lowpass filter to s to produce $\hat{x}$.
- The lowpass filter is used to eliminate the extra copies of the original signal's spectrum present in the spectrum of the impulse-sampled signal s.

- In more detail, the reconstruction process proceeds as follows.
- First, we convert the sequence y to the impulse train s to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - nT).$$

- Then, we filter the resulting signal s with the lowpass filter having impulse response h, yielding

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} y(n)\operatorname{sinc}\left(\tfrac{\pi}{T}(t - nT)\right).$$

- **Sampling Theorem.** Let x be a signal with Fourier transform X, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., x is bandlimited to frequencies $[-\omega_M, \omega_M]$). Then, x is uniquely determined by its samples $y(n) = x(nT)$ for all integer n, if

$$\omega_s > 2\omega_M,$$

where $\omega_s = 2\pi/T$. The preceding inequality is known as the Nyquist condition. If this condition is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y(n)\operatorname{sinc}(\tfrac{\pi}{T}(t-nT)),$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y(n)\operatorname{sinc}(\tfrac{\omega_s}{2}t-\pi n).$$

- We call $\omega_s/2$ the Nyquist frequency and $2\omega_M$ the Nyquist rate.